Free Boundary Problem for a Gas Bubble in a Liquid, and Exponential Stability of the Manifold of Spherically Symmetric Equilibria

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Abstract

We consider the dynamics of a gas bubble immersed in an incompressible fluid of fixed temperature, and focus on the relaxation of an expanding and contracting spherically symmetric bubble due to thermal effects. We study two models, both systems of PDEs with an evolving free boundary: the full mathematical model and an approximate model, arising, for example, in the study of sonoluminescence. For fixed physical parameters (surface tension of the gas–liquid interface, liquid viscosity, thermal conductivity of the gas, etc.), both models share a family of spherically symmetric equilibria, smoothly parametrized by the mass of the gas bubble. Our main result concerns the approximate model. We prove the nonlinear asymptotic stability of the manifold of equilibria with respect to small spherically symmetric perturbations. The rate of convergence is exponential in time. To prove this result we first prove a weak form of nonlinear asymptotic stability –with no explicit rate of time-decay– using the energy dissipation law, and then, via a center manifold analysis, bootstrap the weak time-decay to exponential time-decay. We also study the uniqueness of the family of spherically symmetric equilibria within each model. The family of spherically symmetric equilibria captures all spherically symmetric equilibria of the approximate system. However within the full model, this family is embedded in a larger family of spherically symmetric solutions. For the approximate system, we prove that all equilibrium bubbles are spherically symmetric, by an application of Alexandrov's theorem on closed surfaces of constant mean curvature.

Contents

1. Introduction

This paper considers a free boundary problem for the dynamics of a gas bubble immersed in a liquid. The bubble occupies a bounded and simply connected subset of $\mathbb{R}^3$, denoted $\Omega(t)$. The gas within the bubble is a compressible fluid characterized by its density, velocity, pressure and temperature, as well as constitutive relations relating these variables and the specific entropy. The surrounding liquid is assumed to be incompressible and is described by its velocity, pressure and temperature. The gas inside the bubble and liquid outside the bubble are coupled at the boundary by kinematic and stress-balance equations. Section 2 contains the full mathematical formulation of the liquid/gas model. We are interested in the long time evolution of the coupled bubble/liquid system for initial conditions which are near a spherically symmetric equilibrium.

Energy dissipation plays an important role in the bubble/fluid dynamics. Generally, there are three mechanisms for energy dissipation of bubbles [34, p. 175]: *radiation damping* (of sound waves toward infinity for the compressible fluid case),

thermal damping (transfer of energy from the gas into the fluid via thermal conduction) and *viscous damping*. We consider an approximation to the full liquid/gas dynamics in which thermal damping is the dominant dissipation mechanism; viscous damping is comparatively negligible, and there is no radiation damping due to sound wave emission because the liquid surrounding the gas bubble is assumed incompressible.

The model we study is an asymptotic model introduced by Prosperetti [49]. In [49] the linearized problem was studied by means of Laplace transform, and under various simplifying approximations, linear asymptotic stability of spherical equilibria is argued. When the liquid is inviscid on the liquid–gas interface, the Prosperetti model coincides with the approximate model derived by Biro and Velázquez in [6] based on the parameter regimes of sonoluminescence experiments [3–5]. We present the asymptotic model of [6,49] in Sect. 3. In this model, the gas pressure, gas density and gas temperature all vary and are related via the ideal gas equation of state. Solutions which are spherically symmetric are determined by a reduced free boundary problem (5.1a)–(5.1c): a quasilinear parabolic PDE (nonlinear diffusion) for the density $\rho_g(r, t)$ in the gas bubble region, $0 \leq r \leq R(t)$, coupled to a second-order nonlinear ODE for the bubble radius, $R(t)$; see Sect. 5. Local-in-time well-posedness in Hölder spaces was proved for the initial value problem in [6].

1.1. Main Results

1.1.1. Exponential Stability of the Spherical Equilibrium Bubble   The system (5.1a)–(5.1c), in which spherical symmetry is imposed, has an equilibrium solution for any prescribed gas bubble mass. In [6] these spherically symmetric equilibria were proved to be *Lyapunov stable*. That is, a small spherically symmetric perturbation of the spherical equilibrium bubble of the same mass will evolve, under the dynamics (5.1a)–(5.1c), as a spherically symmetric solution which remains near the equilibrium bubble for all $t > 0$.

The collection of all such spherical equilibrium bubbles forms a smooth manifold of spherically symmetric equilibria parameterized by the bubble mass; see Sect. 4. Our main result, Theorem 6.7, is the nonlinear asymptotic stability, with an exponential rate of time-decay, of the manifold of spherically symmetric equilibria with respect to small spherically symmetric perturbations:

> *For sufficiently small spherically symmetric initial data perturbations of any given spherical equilibrium bubble, the evolving bubble shape and surrounding liquid relax, as time advances, toward a spherical equilibrium bubble exponentially fast.*

The equilibrium bubble evolving from the perturbed initial data are typically different from the given equilibrium bubble unless the perturbed initial data has the same mass as the given equilibrium bubble. More precisely, the equilibrium bubble radius and uniform gas density, which emerge as $t \to \infty$, are determined by the bubble mass of the initial data. These results are stated in detail in Theorem 6.7. The proof of Theorem 6.7 consists of the following two major steps.

Step 1. Building on the well-posedness and Lyapunov stability work of Biro-Velázquez [6], we first prove a weak form of asymptotic stability –with no rate of time-decay– using the energy dissipation law. Detailed statements are presented in Proposition 8.1 (asymptotic stability of a fixed equilibrium relative to small mass preserving perturbations) and Theorem 6.5 (asymptotic stability of the manifold of equilibria relative to arbitrary small perturbations). Theorem 6.5 is a consequence of Proposition 8.1 and the continuity of functionals.

Step 2. We bootstrap the weak asymptotic stability of Step 1 to obtain quantitative exponential asymptotic stability, Theorem 6.7. In particular, we show that the manifold of spherically symmetric equilibria is an attracting center manifold in Sect. 9.6. This requires analysis of spectrum of the linearized operator (Proposition 9.3), a proof of an exponential time-decay estimate on a codimension one subspace associated with the manifold of equilibria (Proposition F.2), and estimates of nonlinear terms (Proposition 9.7). To implement this program we must extend the standard center manifold analysis to a class of fully nonlinear autonomous dynamical system equipped with *a priori* estimates coming from Step 1; see Appendix F.

1.1.2. Symmetry of all Equilibrium Bubbles of the Asymptotic Model   We also study the general version of (5.1a)–(5.1c), not constrained by the assumption of spherical symmetry, i.e. the asymptotic model (3.1)–(3.3). We show that all equilibrium bubbles of (3.1)–(3.3) must be spherically symmetric. The detailed result is presented in Part (1) of Proposition 4.3.

1.2. General Context of Our Work and Relation to Other Physical Models

The dynamics of gas bubbles immersed in a liquid play an important role in fundamental and applied physics and in engineering applications. Examples include underwater explosion bubbles [27], bubble jetting [30,47], seismic wave-producing bubbles in magma [54], bubbles at the ocean surface [39], sonochemistry [59], sonoluminescence [8,52]. Engineering and industrial examples include microfluidics [64], ultrasonic cavitation cleaning [44,58], and applications of ultrasound cavitation bubbles such as medical imaging [19], shock wave lithotripsy (ESWL) [9,26,29,35,36], tissue ablation [13,14,55], oncology and cardiology [37]. For a discussion of these and other applications of bubble dynamics, see the excellent review articles [33,38,50] and the book [34], and references cited therein.

The study of bubble dynamics was initiated in 1917 by Lord Rayleigh [53] during his work with Royal Navy to investigate cavitation damage on ship propellers. He derived an equation for the radial oscillations of a spherically symmetric gas bubble in an incompressible, inviscid liquid with surface tension and examined the pressure prediction during the collapse of a spherical bubble. Over several decades his work was refined and developed by numerous researchers. The Rayleigh-Plesset equation [46] is a second-order nonlinear ODE for the bubble radius. J.B. Keller and collaborators [15,27,28] incorporated the effect of liquid compressibility on the bubble dynamics and incorporated sound radiation from the oscillating bubble. These models have been extensively used in modeling and studied by asymptotic analytical and numerical and methods; see, for example,

[18,20,21,45,48,50,57,60–62,65] and references therein. These models all impose isothermal or adiabatic approximations in which the gas obeys polytropic equation of state (pressure × a power of the volume is equal to a constant). Over the course of bubble oscillations, there are periods where the isothermal assumption and hence an adiabatic pressure volume law is valid (expansion), and periods over which the isothermal approximation is violated; strong compression, as in sonoluminescence experiments. Numerous works compare the two approximations and find a balance between them, e.g. [6,24,49].

The model we study (3.1)–(3.3), or more specifically its spherically symmetric reduction, (5.1a)–(5.1c), was introduced by Prosperetti [49], as an asymptotic approximation of the full liquid/gas-bubble system (2.1)–(2.4), in which the gas pressure, density and temperature are related by an ideal gas law. Neither an isothermal nor adiabatic assumption is made. Over the years, researchers have extensively investigated the model using various approximation techniques. For instance, in [66], quadratic and biquadratic approximations were employed to transform the full PDE model into a simplified ODE model, leading to a significant reduction in computational costs during simulation. The model studied by Biro and Velázquez in [6] reduces to that of Prosperetti [49] when the liquid viscosity, μ_l, is assumed to be zero on the bubble interface. The article [6] studies, in the spherically symmetric setting: (i) local well-posedness in the space-time Hölder space, (ii) global well-posedness for initial data near a spherically symmetric equilibrium, and (iii) Lyapunov stability of the equilibrium relative to small mass-preserving perturbations. At the heart of their stability result is an energy dissipation identity and a coercivity estimate (lower bound) on the energy around the equilibrium, showing that spherically symmetric equilibria are constrained local minimizers.

Our work extends the result of [6] in the following directions:

(1) We consider a more general model, i.e., the Prosperetti model [49], which incorporates liquid viscosity, $\mu_l \geq 0$, on the liquid-bubble interface.
(2) We construct a manifold of spherically symmetric equilibria parametrized by the mass of gas bubble (Proposition 4.1).
(3) We show that equilibrium bubbles of the approximate model are spherically symmetric provided $\mu_l \neq 0$ (Part (1) of Proposition 4.3).
(4) We extend the conditional Lyapunov stability result of [6] to Lyapunov stability relative to arbitrary spherically symmetric perturbations which are small.
(5) Most significantly, we prove asymptotic stability of the manifold of equilibria (Theorem 6.5) with an exponential rate of convergence (Theorem 6.7). Our analysis demonstrates that the equilibrium gas-bubble, which emerges as $t \to \infty$, is determined by the initial data (and prescribed parameters of the model); it is the equilibrium bubble on the manifold of spherical states having the same mass as the initial (perturbed) bubble data.
(6) We also study the persistence (structural stability) of the asymptotic stability result, Theorem 6.5, under a far-field time-dependent pressure, $p_\infty(t)$; see Corollary 6.4 and Corollary 6.6.

Finally, we remark that there is an analogy of the present study with the asymptotic stability of coherent structures that the equilibrium state is determined by

initial data in other nonlinear diffusive dynamical systems, e.g. volume-preserving geometric flows in the Mullins-Sekerka model [16], smoothed out shock profiles in viscous perturbations of hyperbolic conservation laws [25], traveling front solutions in nonlinear reaction diffusion dynamics [7,41,42], and spatial uniform equilibrium in two dimensional chemotaxis-fluid model [63].

1.3. Some Future Directions and Open Problems in the Context of the Current and Closely Related Models

1. *Time-periodically expanding and contracting bubble oscillations.* The far-field liquid pressure $p_\infty(t)$, is an external forcing term in our free boundary problem. In this present work, it is prescribed to be either the constant ($p_\infty(t) \equiv p_{\infty,*}$) or such that $p_\infty(t) - p_{\infty,*}$ is small and decaying to zero sufficiently rapidly as $t \to \infty$. It is also of interest to study the bubble dynamics when the far-field pressure $p_\infty(t)$ is time-periodic, for example of the form $p_\infty(t) = p_{\infty,*} + A\cos(\omega t)$, corresponding to far-field periodic acoustic forcing as in physical experiments [3–5]. In a forthcoming paper [32], we prove that for sufficiently small forcing amplitude, A, there exists a unique asymptotically stable $2\pi/\omega$-time periodic spherically symmetric pulsating bubble solution.

2. *Uniqueness of the spherically symmetric equilibria.* Spherically symmetric equilibria of the asymptotic model (3.1)–(3.3) are uniquely characterized in Part (2) of Proposition 4.3. In the context of the general evolution for the asymptotic model, any equilibrium bubble is necessarily spherical (Part (1) of Proposition 4.3). However, there exist non-trivial (rotational) equilibrium gas flows inside the equilibrium spherical bubble (Remark 4.4). In other words, the spherically symmetric equilibria are not unique within the asymptotic model. Under what circumstances is the family spherically symmetric equilibria are unique? Certainly, the above rotational equilibrium gas flows are ruled out if only seek gas flows which are irrotational. But are the spherically symmetric equilibria are unique within another closely related model? We expect that adding a gas viscosity term in the stress balance equation (3.3b) can help us exclude the case of non-trivial equilibrium gas flow in an equilibrium spherical bubble.

3. *Nonspherically symmetric dynamics. Are spherically symmetric bubbles stable against small perturbations, unconstrained by symmetry?* The main asymptotic stability result of the present paper requires spherically symmetric perturbations which are small. It is then natural to ask: *Is the manifold of spherically symmetric equilibria asymptotically stable relative to small arbitrary (non-spherically symmetric) perturbations in the approximate system* (3.1)–(3.3) *(or in the full system* (2.1)–(2.4))? We expect that surface tension plays an important role in rounding out bubbles during the evolution.

A related question was studied in a model of a spherical *polytropic* gas bubble in a *compressible* and *non-viscous* liquid [12,56]. In this case, the damping mechanism is acoustic radiation of waves to spatial infinity, rather than thermal diffusion. In [56] it is proved that the spherically symmetric bubble is linearly asymptotically stable relative to general (not necessarily spherical) perturbations. In the weakly

compressible regime, the sharp exponential decay rate of perturbations was determined in [12,56] and is proved to be governed by very high angular momentum *shape mode* deformations of the bubble. The energy of these shape modes is transferred very slowly to the surrounding compressible liquid and is radiated to infinity via acoustic waves.

Finally, we list some related broader open questions.

(1) *Well-posedness* of the full liquid/gas model (2.1)–(2.4) for general data appears to be open.
(2) *Nonuniqueness of spherical equilibria of the full liquid/gas model.* Even the classification of equilibrium solutions of (2.1)–(2.4) appears to be non-trivial. It is not known, for example, whether there are non-spherical equilibria. And, while in the approximate system (3.1)–(3.3), all symmetric equilibria are spatially uniform, this is not the case for (2.1)–(2.4); within the class of spherically symmetric equilibria of (2.1)–(2.4) there are solutions with spatially non-uniform temperature profiles (Remark 4.2). Some of these equilibria have a singularity in the gas temperature at the origin. It would be an interesting and challenging mathematical problem to investigate the dynamics for perturbations of such equilibria and to see whether they may participate in singularity formation for some classes of smooth initial conditions.
(3) *Radiation condition for the full liquid/gas model.* The radiation condition (4.2) for the liquid temperature is required in proving the uniqueness of the spherical equilibrium solutions in (4.3) of a specified mass. *Can one ensure that this radiation condition holds for the time-evolution if it is imposed on the initial data?* The proof would require an a priori regularity of solutions to a free boundary problem of parabolic equations with time-dependent boundary condition.
(4) *Global energy minimizer.* For the asymptotic model we have shown that the equilibrium $(\rho_*, R_*, \dot{R}_* = 0)$ is a conditional local energy minimizer of the total energy $\mathcal{E}_{\text{total}}$ (Definition 7.1), constrained to fixed mass. Our analysis relies on the Taylor expansion of the energy near an equilibrium. *Is the equilibrium bubble solution of mass M a global minimizer of $\mathcal{E}_{\text{total}}$ relative to arbitrarily large spherically symmetric deformations of mass M?*

1.4. Notation and Conventions

(1) $B_R = \{x \in \mathbb{R}^3 : |x| < R\}$
(2) For a function $f(r)$ defined for $0 < r < R$, we set $\overline{f}(x) = f(|x|)$ for $x \in B_R$ and denote

$$\|f\|_{C_r^{2+2\alpha}} = \|\overline{f}\|_{C_x^{2+2\alpha}} = \max_{|\beta|\leq 2} \sup_{x \in B_R} |D^\beta \overline{f}(x)| + \sup_{\substack{x \neq y \\ x,y \in B_R}} \frac{|D^2\overline{f}(x) - D^2\overline{f}(y)|}{|x - y|^{2\alpha}}.$$

(3) Denote $\nabla_r f = \nabla_x \overline{f}$ and $\Delta_r f = \Delta_x \overline{f} = \frac{1}{r^2}\partial_r(r^2 \partial_r f)$ the radial part of the Laplace operator in $\mathbb{R}^3$.
(4) For a state variable, such as the density ρ, if it corresponds to value of a constant equilibrium solution, then we denote it by ρ_*, and similarly for the values of other equilibrium state variables.

(5) If **u**, **v** and **w** are vector fields, then $\mathbf{u} \cdot \nabla\mathbf{v} \cdot \mathbf{w} = [(\mathbf{u} \cdot \nabla)\,\mathbf{v}] \cdot \mathbf{w}$.

(6) If $A = (A_{ij})$ and $B = (B_{ij})$, the $A : B = \mathrm{tr}(AB^\top) = \sum_{i,j} A_{ij} B_{ij}$.

(7) $\nabla\mathbf{u} = (\partial_i u_j)$; $|\nabla\mathbf{u}|^2 \equiv \sum_{i,j} \partial_i u_j \partial_i u_j = \mathrm{tr}\left(\nabla\mathbf{u}(\nabla\mathbf{u})^\top\right)$.

(8) To a function $f(t)$, defined for $t \geq 0$, we introduce the Laplace transform and its inverse

$$\tilde{f}(\tau) = \int_0^\infty e^{-t\tau} f(t)\mathrm{d}t, \quad \text{where} \quad f(t) = \frac{1}{2\pi i} \int_{\Gamma_a} e^{\tau t} \tilde{f}(\tau)\mathrm{d}\tau. \quad (1.1)$$

Here, $\Gamma_a = \{\tau : \mathrm{Re}\tau = a\}$ where a is chosen such that $\tilde{f}(\tau)$ is analytic on the set $\{\tau : \mathrm{Re}\tau > a\}$.

(9) We introduce the normalized radial-Dirichlet eigenfunctions and eigenvalues in the unit ball B_1, i.e.

$$-\Delta_y \phi_j = \lambda_j \phi_j, \quad \phi_j|_{y=1} = 0, \quad 1 = \int_{B_1} \phi_j^2(|x|)\,\mathrm{d}x = 4\pi \int_0^1 \phi_j^2(y) y^2\,\mathrm{d}y.$$

$$\phi_j(y) = \frac{\sin(j\pi y)}{\sqrt{2\pi}\, y}, \quad j = 1, 2, \ldots, \quad \lambda_j = (j\pi)^2, \quad j = 1, 2, \ldots.$$
$$(1.2)$$

2. Gas Bubble in an Incompressible Liquid; the Complete Mathematical Formulation

In this section we first discuss the complete mathematical description of a gas bubble immersed in an incompressible liquid with constant surface tension. We then present in Sect. 3 the asymptotic approximation studied in [6] in which thermal diffusion is the key dissipation mechanism. In Sect. 4, we derive an explicit family of our spherically symmetric equilibrium solutions of both of the full and approximate systems.

Equations for the liquid. Let $\mathbf{v}_l(x, t)$ denote the liquid velocity, $p_l(x, t)$ the liquid pressure and $T_l(x, t)$ is the liquid temperature. We assume that the dynamics of the liquid outside the bubble is described by the incompressible (constant density) Navier–Stokes equations

$$\left.\begin{aligned}
\partial_t(\rho_l \mathbf{v}_l) + \mathrm{div}(\rho_l \mathbf{v}_l \otimes \mathbf{v}_l) &= \mathrm{div}\mathbb{T}_l, \\
\mathrm{div}\mathbf{v}_l &= 0, \\
\rho_l c_l(\partial_t T_l + \mathbf{v}_l \cdot \nabla T_l) &= \mathrm{div}(\kappa_l \nabla T_l) + \mathbb{S}_l : \nabla\mathbf{v}_l.
\end{aligned}\right\} \quad \text{in } \mathbb{R}^3 \setminus \Omega(t), t > 0,
\begin{aligned}
&(2.1a) \\
&(2.1b) \\
&(2.1c)
\end{aligned}$$

The stress tensor, $\mathbb{T}_l$ and viscous stress tensor, $\mathbb{S}_l$, are given, respectively, by:

$$\mathbb{T}_l = -p_l \mathbb{I} + \mathbb{S}_l(\mathbf{v}_l) \quad \text{and} \quad \mathbb{S}_l(\mathbf{v}_l) = 2\mu_l \mathbb{D}(\mathbf{v}_l), \quad \mathbb{D}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top).$$

Here, $(\mathbf{u} \otimes \mathbf{v})_{ij} = u_i v_j$ denotes the tensor product of vectors and $\mathbb{A} : \mathbb{B} = \sum_{i,j=1}^{3} A_{ij} B_{ij} = \mathrm{tr}(AB^{\top})$, where $\mathbb{A} = (A)_{i,j=1}^{3}$, $\mathbb{B} = (B)_{i,j=1}^{3}$. The equations depend on parameters: $\rho_l > 0$, the density of the liquid, $\mu_l \geq 0$, the dynamic viscosity of the liquid, c_l, the specific heat of the liquid, and κ_l, the thermal conductivity of the liquid. Equations (2.1a) and (2.1b) express, respectively, balance of momentum and conservation of mass. These are coupled to equation (2.1c), which governs the temperature field in the liquid.

Equations for the gas. The gas within the bubble is assumed to be a compressible fluid, characterized by its velocity $\mathbf{v}_g(x,t)$, pressure $p_g(x,t)$, density $\rho_g(x,t)$, temperature $T_g(x,t)$, and entropy per unit mass (specific entropy) $s(x,t)$, with the assumption of the ideal gas law relating p_g, T_g and ρ_g. The governing equations are the viscous, compressible Navier–Stokes equations

$$
\left.
\begin{aligned}
\partial_t \rho_g + \mathrm{div}(\rho_g \mathbf{v}_g) &= 0, && \text{(2.2a)}\\
\partial_t(\rho_g \mathbf{v}_g) + \mathrm{div}(\rho_g \mathbf{v}_g \otimes \mathbf{v}_g) &= \mathrm{div}\mathbb{T}_g, && \text{(2.2b)}\\
\rho_g T_g \left(\partial_t s + \mathbf{v}_g \cdot \nabla s \right) &= \mathrm{div}(\kappa_g \nabla T_g) + \mathbb{S}_g : \nabla \mathbf{v}_g, && \text{(2.2c)}\\
p_g &= \mathscr{R}_g T_g \rho_g, && \text{(2.2d)}\\
s &= c_v \log\left(\frac{p_g}{\rho_g^{\gamma}} \right), && \text{(2.2e)}
\end{aligned}
\right\} \ \text{in } \Omega(t),\ t > 0,
$$

where

$$
\mathbb{T}_g = -p_g \mathbb{I} + 2\mu_g \left(\mathbb{D}(\mathbf{v}_g) - \frac{1}{3}(\mathrm{div}\mathbf{v}_g)\mathbb{I} \right) + \zeta_g(\mathrm{div}\mathbf{v}_g)\mathbb{I}
$$

is the stress tensor of the gas in which $\mu_g > 0$ and ζ_g are the dynamic viscosity and the bulk viscosity for the gas, respectively, and

$$
\mathbb{S}_g = 2\mu_g \left(\mathbb{D}(\mathbf{v}_g) - \frac{1}{3}(\mathrm{div}\mathbf{v}_g)\mathbb{I} \right) + \zeta_g(\mathrm{div}\mathbf{v}_g)\mathbb{I}
$$

is the viscous tensor of gas. The constant κ_g is the thermal conductivity of the gas. The constant $\mathscr{R}_g$ is the specific gas constant, the ratio of the ideal gas constant to the molar mass. The constant

$$
\gamma \equiv 1 + \frac{\mathscr{R}_g}{c_v} = \frac{c_p}{c_v} > 1 \tag{2.3}
$$

is called the adiabatic constant. Here, c_p denotes the heat capacity at constant pressure and c_v denotes the heat capacity at constant volume. Equations (2.2a) and (2.2b) are the equations of motion and continuity, respectively, of a compressible fluid. Equation (2.2c) is the entropy equation. Equation (2.2d) is the equation of state (Boyle's law) for ideal gases. Equation (2.2e) is a consequence of the second law of thermodynamics, (2.2d), and Joule's second law for ideal gases.

Boundary conditions at the liquid/gas interface. Let the bubble surface, $\partial\Omega(t)$, be given in spherical coordinates

$$\boldsymbol{\omega} = \boldsymbol{\omega}(\theta, \varphi, t)$$

and let $\mathbf{n}$ denote the outward pointing unit outer normal on $\partial\Omega(t)$. The boundary conditions on $\partial\Omega(t)$ are

$$\mathbf{v}_l(\boldsymbol{\omega}, t) \cdot \hat{\mathbf{n}} = \mathbf{v}_g(\boldsymbol{\omega}, t) \cdot \hat{\mathbf{n}} = \partial_t \boldsymbol{\omega} \cdot \hat{\mathbf{n}}, \tag{2.4a}$$

$$\hat{\mathbf{n}} \cdot \mathbb{T}_l - \hat{\mathbf{n}} \cdot \mathbb{T}_g = \sigma \hat{\mathbf{n}} (\nabla_S \cdot \hat{\mathbf{n}}), \left. \right\} \text{ on } \partial\Omega(t), t > 0, \tag{2.4b}$$

$$T_g = T_l, \tag{2.4c}$$

where $\nabla_S \cdot$ denotes the surface divergence, and $\sigma > 0$ is the surface tension of the liquid - gas interface, here assumed to be a constant. Equation (2.4a) is the kinematic boundary condition; the normal velocity of the material point on the bubble surface moves with the normal velocity of both the gas and the liquid. Equation (2.4b) is the stress balance equation. Equation (2.4c) means the temperature is continuous across the interface. A detailed derivation of the fundamental equations of fluid dynamics is presented, for example, in [17,43].

The system (2.1)–(2.4) depends on the

$$\text{physical parameters: } \nu_l = \mu_l/\rho_l, \rho_l, c_l, \kappa_l, \mu_g, \kappa_g, \mathscr{R}_g, \gamma, c_v, \zeta_g, \sigma, \tag{2.5}$$

where ν_l, the kinematic viscosity, is a nonnegative constant and all other parameters are all strictly positive constants. We assume these parameters to be prescribed and fixed. Furthermore, one prescribes:

$$\text{the far-field liquid velocity } \mathbf{v}_\infty(t) := \lim_{|x| \to \infty} \mathbf{v}_l(x, t),$$

$$\text{the far-field liquid pressure } p_\infty(t) := \lim_{|x| \to \infty} p_l(x, t), and$$

$$\text{the far-field liquid temperature } T_\infty(t) := \lim_{|x| \to \infty} T_l(x, t).$$

Here, we assume that the far-field pressure in the liquid, $p_\infty(t)$, is spatially uniform and is a small perturbation of a positive constant $p_{\infty,*}$:

$$p_\infty \in C_t^{1+\alpha}(\mathbb{R}_+), \quad |p_\infty(t) - p_{\infty,*}| + \|\partial_t p_\infty\|_{L_t^1(\mathbb{R}_+)} \leq \eta_0,$$

$$p_\infty(t) \to p_{\infty,*} \text{ as } t \to \infty,$$

where $\eta_0 > 0$ is some small number to be chosen later. We also assume that the far-field liquid temperature is a constant, T_∞, and that the far-field liquid velocity vanishes:

$$T_\infty(t) \equiv T_\infty, \qquad \mathbf{v}_\infty(t) \equiv \mathbf{0}.$$

For fixed physical parameters (2.5), $p_\infty(t)$, and T_∞, the system (2.1)–(2.4) governs the time-evolution of the

state variables in the liquid: $\mathbf{v}_l(x, t), p_l(x, t), T_l(x, t),$

state variables in the gas: $\rho_g(x, t), \mathbf{v}_g(x, t), p_g(x, t), T_g(x, t), s(x, t),$ and

the gas bubble region, $\Omega(t) \subset \mathbb{R}^3$.

To this we add constitutive relations (2.2d)–(2.2e), which enable us to express T_g and s in terms of ρ_g, $\mathbf{v}_g$ and p_g. Moreover, since the liquid pressure p_l solves the exterior Dirichlet boundary-value problem of Poisson equation

$$-\Delta p_l = \rho_l \nabla \mathbf{v}_l : (\nabla \mathbf{v}_l)^\top \ \text{ in } \mathbb{R}^3 \setminus \Omega(t),$$

with the boundary conditions

$$p_l = \hat{\mathbf{n}} \cdot \mathbb{S}_l \cdot \hat{\mathbf{n}} - \hat{\mathbf{n}} \cdot \mathbb{T}_g \cdot \hat{\mathbf{n}} - \sigma \nabla_S \cdot \hat{\mathbf{n}} \ \text{ on } \partial\Omega(t), \qquad \lim_{|x|\to\infty} p_l(x,t) = p_\infty(t),$$

$p_l - p_\infty(t)$ satisfies

$$
\begin{aligned}
-\Delta(p_l - p_\infty(t)) &= \rho_l \nabla \mathbf{v}_l : (\nabla \mathbf{v}_l)^\top \ \text{ in } \mathbb{R}^3 \setminus \Omega(t), \\
p_l - p_\infty(t) &= \hat{\mathbf{n}} \cdot \mathbb{S}_l \cdot \hat{\mathbf{n}} - \hat{\mathbf{n}} \cdot \mathbb{T}_g \cdot \hat{\mathbf{n}} - \sigma \nabla_S \cdot \hat{\mathbf{n}} \\
&\quad - p_\infty(t) \ \text{ on } \partial\Omega(t), \qquad \lim_{|x|\to\infty} p_l(x,t) - p_\infty(t) = 0.
\end{aligned}
$$

For suitable $\nabla\mathbf{v}_l$ with sufficient decay at spatial infinity, $p_l - p_\infty(t)$ can be expressed by means layer potentials as

$$
\begin{aligned}
p_l(x,t) - p_\infty(t) = {} & \int_{\mathbb{R}^3 \setminus \Omega(t)} G(x,y;t) \left[\rho_l \nabla \mathbf{v}_l : (\nabla \mathbf{v}_l)^\top \right](y)\, \mathrm{d}y \\
& - \int_{\partial\Omega(t)} \nabla_y G(x,y;t) \cdot \hat{\mathbf{n}} \left[\hat{\mathbf{n}} \cdot \mathbb{S}_l \cdot \hat{\mathbf{n}} - \hat{\mathbf{n}} \cdot \mathbb{T}_g \cdot \hat{\mathbf{n}} - \sigma \nabla_S \cdot \hat{\mathbf{n}} - p_\infty(t) \right](y)\, \mathrm{d}S_y,
\end{aligned}
$$

where $G(x,y;t)$ is the Green's function for the exterior domain $\mathbb{R}^3 \setminus \Omega(t)$. Then p_l, for $x \notin \Omega(t)$, can be expressed in terms $\mathbf{v}_l$ by

$$
\begin{aligned}
p_l(x,t) = {} & p_\infty(t) + \int_{\mathbb{R}^3 \setminus \Omega(t)} G(x,y;t) \left[\rho_l \nabla \mathbf{v}_l : (\nabla \mathbf{v}_l)^\top \right](y)\, \mathrm{d}y \\
& - \int_{\partial\Omega(t)} \nabla_y G(x,y;t) \cdot \hat{\mathbf{n}} \left[\hat{\mathbf{n}} \cdot \mathbb{S}_l \cdot \hat{\mathbf{n}} - \hat{\mathbf{n}} \cdot \mathbb{T}_g \cdot \hat{\mathbf{n}} - \sigma \nabla_S \cdot \hat{\mathbf{n}} - p_\infty(t) \right](y)\, \mathrm{d}S_y.
\end{aligned}
$$

Therefore, (2.1)–(2.4) can be reduced to a problem for the unknown liquid and gas state variables, and the region filled with gas:

$$\mathbf{v}_l(x,t), \ T_l(x,t), \ \rho_g(x,t), \ \mathbf{v}_g(x,t), \ p_g(x,t), \ \Omega(t).$$

Initial data. To solve for the evolution given by the full liquid/gas bubble system (2.1)–(2.4), we must prescribe initial conditions for the state variables:

$$\mathbf{v}_l(\cdot,0), \ T_l(\cdot,0), \ \rho_g(\cdot,0), \ \mathbf{v}_g(\cdot,0), \ p_g(\cdot,0), \tag{2.6}$$

and for the bubble shape at time $t = 0$:

$$\Omega(t)\Big|_{t=0} = \Omega(0). \tag{2.7}$$

We assume the compatibility conditions for the initial data, i.e., they satisfy (2.1)–(2.4) at $t = 0$. In particular, $\mathrm{div}\,\mathbf{v}_l(\cdot,0) = 0$.

3. An Asymptotic Approximation to (2.1)–(2.4)

In this paper we work with the following approximation of Prosperetti [49] (see also Biro–Velázquez [6, Appendix A]) to the full liquid-bubble system (2.1)–(2.4):

$$\partial_t \mathbf{v}_l = \nu_l \Delta \mathbf{v}_l - \mathbf{v}_l \cdot \nabla \mathbf{v}_l - \frac{1}{\rho_l} \nabla p_l, \tag{3.1a}$$

$$\left.\begin{array}{l} \mathrm{div}\mathbf{v}_l = 0, \\[4pt] T_l(x,t) = T_\infty, \quad \text{a prescribed constant,} \end{array}\right\} \text{in } \mathbb{R}^3 \setminus \Omega(t), t > 0, \tag{3.1b}$$

$$T_l(x,t) = T_\infty, \quad \text{a prescribed constant,} \tag{3.1c}$$

where $\nu_l = \frac{\mu_l}{\rho_l} \geq 0$ is the kinematic viscosity of the liquid,

$$\partial_t \rho_g + \mathrm{div}(\rho_g \mathbf{v}_g) = 0, \tag{3.2a}$$

$$p_g = p_g(t), \tag{3.2b}$$

$$\rho_g T_g \left(\partial_t s + \mathbf{v}_g \cdot \nabla s \right) = \mathrm{div}(\kappa_g \nabla T_g), \tag{3.2c}$$

$$p_g = \mathscr{R}_g T_g \rho_g, \tag{3.2d}$$

$$s = c_v \log \left(\frac{p_g}{\rho_g^\gamma} \right), \tag{3.2e}$$

in $\Omega(t), t > 0,$

and

$$\mathbf{v}_l(\boldsymbol{\omega}, t) \cdot \hat{\mathbf{n}} = \mathbf{v}_g(\boldsymbol{\omega}, t) \cdot \hat{\mathbf{n}} = \partial_t \boldsymbol{\omega} \cdot \hat{\mathbf{n}}, \tag{3.3a}$$

$$p_g \hat{\mathbf{n}} - p_l \hat{\mathbf{n}} + 2\mu_l \hat{\mathbf{n}} \cdot \mathbb{D}(\mathbf{v}_l) = \sigma \hat{\mathbf{n}}(\nabla_S \cdot \hat{\mathbf{n}}), \tag{3.3b}$$

$$T_g = T_\infty, \tag{3.3c}$$

on $\partial \Omega(t), t > 0,$

with the far-field conditions

$$\lim_{|x| \to \infty} \mathbf{v}_l(x,t) = \mathbf{0}, \qquad \lim_{|x| \to \infty} p_l(x,t) = p_\infty(t), \qquad \lim_{|x| \to \infty} T_l(x,t) = T_\infty. \tag{3.4}$$

This model reduces to that of [6, Appendix A] for the special case when $\mu_l = 0$ in (3.3b). Equation (3.3b) is the Young–Laplace boundary condition; the jump in pressure at the liquid–gas interface is equal to the surface tension, σ, times the mean curvature $H = \frac{1}{2} \nabla_S \cdot \hat{\mathbf{n}}$. The approximate system (3.1)–(3.3) depends on the

$$\text{physical parameters: } \nu_l, \rho_l, \kappa_g, \mathscr{R}_g, \gamma, c_v, \sigma. \tag{3.5}$$

For fixed physical parameters (3.5), $p_\infty(t)$, and T_∞, the approximate system (3.1)–(3.3) governs the time-evolution of the state variables in the liquid:

$$\mathbf{v}_l(x,t), \ p_l(x,t)$$

and in the gas

$$\rho_g(x,t), \ \mathbf{v}_g(x,t), \ p_g(t), \ T_g(x,t), \ s(x,t)$$

and $\Omega(t)$. We show below in Appendix B that the system (3.2) can be reduced to a single equation (B.6) for ρ_g depending only on Ω and p_g. Thus, (3.1)–(3.3) can be reduced to a problem with unknowns

$$\mathbf{v}_l(x, t), \ \rho_g(x, t), \ p_g(t), \ \Omega(t).$$

As for the initial conditions to prescribe for the approximate system (3.1)–(3.3), we need

$$\mathbf{v}_l(\cdot, 0), \ \rho_g(\cdot, 0) \tag{3.6}$$

for state variables, and (2.7) for the bubble shape at time $t = 0$. We do not prescribe initial data for p_g since it can be derived from $p_l(\cdot, 0)$ and $\Omega(0)$ via (3.3b).

In this article the approximate system (3.1)–(3.3) is considered under the assumption of spherical symmetry. More precisely, we assume for the system (3.1)–(3.3) that $\Omega(t)$ is a sphere, $\mathbf{v}_l, \mathbf{v}_g$ are spherically symmetric, p_l, ρ_g, T_g, s are radial. Recall that a vector field $\mathbf{u} : \mathbb{R}^3 \to \mathbb{R}^3$ is spherically symmetric if $\mathbf{u} = u(r)\hat{\mathbf{r}}$, $r = |x|$ and $\hat{\mathbf{r}} = x/|x|$, and a scalar function $f : \mathbb{R}^3 \to \mathbb{R}$ is radial if $f = f(r)$. In this setting, (3.1)–(3.3) reduces to the system (5.1a)–(5.1c) for $\rho(r, t)$, $R(t)$. Under the assumption of spherical symmetry, (5.1a)–(5.1c) is well-posed locally in time (Theorem 5.3), and well-posed globally in time for initial data which is close to the equilibrium (see [6, Theorem 4.1] and Theorem 6.3).

4. Spherically Symmetric Equilibrium Solutions

Both the full liquid/gas model (2.1)–(2.4) and the asymptotic model (3.1)–(3.3) share a family of spherically symmetric equilibrium (time-independent) solutions. Let B_R denote the open ball in $\mathbb{R}^3$ of radius R which is centered at the origin. Suppose a gas of density $\rho_g(x)$ occupies the region B_R. Then, the mass of the bubble is given by

$$\text{Mass}[\rho_g, R] \equiv \int_{B_R} \rho_g(x) dx. \tag{4.1}$$

Below, we investigate spherical symmetric equilibrium solutions of both (2.1)–(2.4) and the approximation system (3.1)–(3.3). We prove that the spherically symmetric equilibrium gas bubble of the approximate system (3.1)–(3.3), and of the original system (2.1)–(2.4) with additional conditions, is, up to spatial translation of its center, uniquely determined by its total mass. Moreover, we prove that equilibrium bubbles of the approximate system (3.1)–(3.3) are spherical by applying the Alexandrov's theorem on closed constant-mean-curvature (CMC) surfaces. In an equilibrium spherical bubble of the approximate system (3.1)–(3.3), there exists a nontrivial, e.g. rotational, equilibrium gas flow (see Remark 4.4).

Proposition 4.1. (Spherically symmetric equilibria of the original system *(2.1)–(2.4)*) *Fix a constant $p_{\infty,*} > 0$. Assume the radiation condition for liquid temperature:*

$$T_l(|x|) = T_\infty + o(|x|^{-1}), \quad |x| \to \infty. \tag{4.2}$$

Then, there is a smooth map from values of the bubble mass to equilibrium radii, R_*:

$$M \in (0, \infty) \mapsto R_*[M],$$

such that any regular (non-singular) spherical equilibrium solution of (2.1)–(2.4) (for fixed parameters (2.5)) of bubble mass M is expressible as:

$$\mathbf{v}_{l,*} = \mathbf{0}, \qquad p_{l,*} = p_{\infty,*}, \qquad \Omega_* = B_{R_*[M]}, \tag{4.3a}$$

$$\rho_{g,*}[M] = \frac{1}{\mathscr{R}_g T_\infty}\left(p_{\infty,*} + \frac{2\sigma}{R_*[M]}\right), \quad \mathbf{v}_{g,*} = \mathbf{0}, \quad p_{g,*}[M] = p_{\infty,*} + \frac{2\sigma}{R_*[M]}, \tag{4.3b}$$

$$T_{g,*} = T_{l,*} = T_\infty, \quad s_* = c_v \log\left((\mathscr{R}_g T_\infty)^\gamma \left(p_{\infty,*} + \frac{2\sigma}{R_*[M]}\right)^{1-\gamma}\right). \tag{4.3c}$$

The proof of Proposition 4.1 is given in Appendix A.

Remark 4.2. The radiation condition (4.2) and the regularity assumption in part (1) are necessary for the uniqueness of the spherical equilibrium solutions $T_{l,*}$ and $T_{g,*}$ in (4.3). In fact, without such hypotheses there exists a two parameter family of spherical equilibrium solutions $T_{l,*}$ and $T_{g,*}$ of (2.1)–(2.4) given by:

$$T_{l,*}(r) = T_\infty - a_1/r, r \in [R_*, \infty), \text{ and}$$
$$T_{g,*}(r) = T_\infty - a_1/R_* + a_2(1/R_* - 1/r), r \in [0, R*),$$

where $a_1, a_2 \in \mathbb{R}$ are arbitrary.

Proposition 4.3. (Equilibria of the approximate system (3.1)–(3.3)]) *Fix a constant $p_{\infty,*} > 0$.*

(1) Equilibrium bubbles of (3.1)–(3.3) are spherical: Let $(\mathbf{v}_{l,*}, p_{l,*}, \partial\Omega_*, p_{g,*}, \mathbf{v}_{g,*}, \rho_{g,*}, \cdots)$ be a C^2 steady-state solution of (3.1)–(3.3) with $\lim_{|x|\to\infty} p_{l,*}(x) = p_{\infty,*}$. Assume $\mu_l \neq 0$ and $\sigma \neq 0$ in (3.3b). Suppose that $\lim_{|x|\to\infty} \mathbf{v}_{l,*}(x) = O(|x|^{-2})$ and $\lim_{|x|\to\infty} \nabla\mathbf{v}_{l,*}(x) = \mathbb{O}$. Then $\mathbf{v}_{l,*} = \mathbf{0}$, $p_{l,*} = p_{\infty,*}$ and Ω_* is a sphere. Moreover, $\rho_{g,*}$ is constant and $\mathrm{div}\mathbf{v}_{g,*} = 0$.

(2) Spherically symmetric equilibria of (3.1)–(3.3): The reduced/asymptotic model (3.1)–(3.3) shares the family of spherically symmetric equilibria displayed in (4.3). Furthermore, any spherical equilibrium solution of (3.1)–(3.3) is uniquely determined by its total mass as (4.3). No radiation condition (4.2) or regularity assumption is required.

(3) The mappings $M \in (0, \infty) \mapsto R_*[M]$ and $\rho_*[M]$, where $\rho_* := \rho_{g,*}$, arising in Proposition 4.1 are continuous (even smooth).

Remark 4.4. Part (1) of Proposition 4.3 is the uniqueness of (4.3a) for the equilibrium liquid flow and bubble shape. It does not imply the uniqueness of (4.3b)–(4.3c) for the gas phase. Indeed, replacing $\mathbf{v}_{g,*} = \mathbf{0}$ in (4.3b) with any non-trivial solenoidal vector field in B_{R_*} yields another steady state solution to the approximate

system (3.1)–(3.3). Recall that a vector field $\mathbf{u}$ in Ω is solenoidal if $\mathrm{div}\mathbf{u} = 0$ and $\mathbf{u} \cdot \hat{\mathbf{n}}|_{\partial\Omega} = 0$. One can choose, for example, $\mathbf{v}_{g,*}(x_1, x_2, x_3) = (-x_2, x_1, 0)$. The example is ruled out by the spherically symmetric assumption in Part (1) of Proposition 4.3. Other possible way to show the uniqueness of (4.3b)–(4.3c) is to impose irrotational assumption. The nonuniqueness is due to the absence of viscosity for the gas in the approximated stress balance equation (3.3b).

Remark 4.5. When the liquid is inviscid, i.e., $\mu_l = \nu_l = 0$, in Part (1) of Proposition 4.3, we still expect $\mathbf{v}_{l,*} \equiv \mathbf{0}$ because of the far-field condition $\lim_{|x|\to\infty} \mathbf{v}_{l,*}(x) = \mathbf{0}$. In this case, a Liouville-type theorem for the stationary Euler equations in a three-dimensional exterior domain with slip boundary condition is needed. However, such Liouville-type result is unavailable to our best knowledge. For survey on related problems, see, for example, [22, I.2.1] and [2].

Proof of Proposition 4.3. We first prove Part (1) concerning the uniqueness of the equilibrium (4.3a) of the approximate system (3.1)–(3.3). Note that steady-state solutions of (3.1)–(3.3) solve

$$0 = \nu_l \Delta \mathbf{v}_{l,*} - \mathbf{v}_{l,*} \cdot \nabla \mathbf{v}_{l,*} - \frac{1}{\rho_l} \nabla p_{l,*}, \tag{4.4a}$$

$$\mathrm{div}\mathbf{v}_{l,*} = 0, \qquad\qquad\qquad \text{in } \mathbb{R}^3 \setminus \Omega_*, \tag{4.4b}$$

$$T_{l,*}(x) = T_\infty, \quad \text{a prescribed constant}, \tag{4.4c}$$

$$\mathrm{div}(\rho_{g,*}\mathbf{v}_{g,*}) = 0, \tag{4.5a}$$

$$p_{g,*}(x) = p_{g,*}, \quad \text{a constant}, \tag{4.5b}$$

$$\rho_{g,*}T_{g,*}\left(\mathbf{v}_{g,*} \cdot \nabla s_*\right) = \mathrm{div}(\kappa_g \nabla T_{g,*}), \tag{4.5c}$$

$$p_{g,*} = \mathscr{R}_g T_{g,*}\rho_{g,*}, \qquad\qquad \text{in } \Omega_*, \tag{4.5d}$$

$$s_* = c_v \log\left(\frac{p_{g,*}}{\rho_{g,*}^\gamma}\right), \tag{4.5e}$$

$$\mathbf{v}_{l,*} \cdot \hat{\mathbf{n}} = \mathbf{v}_{g,*} \cdot \hat{\mathbf{n}} = 0, \tag{4.6a}$$

$$p_{g,*}\hat{\mathbf{n}} - p_{l,*}\hat{\mathbf{n}} + 2\mu_l\hat{\mathbf{n}} \cdot \mathbb{D}(\mathbf{v}_{l,*}) = \sigma\hat{\mathbf{n}}(\nabla_S \cdot \hat{\mathbf{n}}), \quad \text{on } \partial\Omega_*, \tag{4.6b}$$

$$T_{g,*} = T_\infty, \tag{4.6c}$$

and the far-field velocity and pressure are

$$\lim_{|x|\to\infty} \mathbf{v}_{l,*}(x) = \mathbf{0}, \qquad \lim_{|x|\to\infty} p_{l,*}(x) = p_{\infty,*}. \tag{4.7}$$

For $r > 0$ sufficiently large, multiplying the equation (4.4a) by $\mathbf{v}_{l,*}$, integrating over $B_r \setminus \Omega_*$, using integration by parts formula and $\mathrm{div}\,\mathbf{v}_{l,*} = 0$, we obtain

$$
\begin{aligned}
0 &= \nu_l \int_{B_r \setminus \Omega_*} \Delta \mathbf{v}_{l,*} \cdot \mathbf{v}_{l,*}\, dx - \int_{B_r \setminus \Omega_*} \mathbf{v}_{l,*} \cdot \nabla \mathbf{v}_{l,*} \cdot \mathbf{v}_{l,*}\, dx - \frac{1}{\rho_l} \int_{B_r \setminus \Omega_*} \nabla p_{l,*} \cdot \mathbf{v}_{l,*}\, dx \\
&= -\nu_l \int_{B_r \setminus \Omega_*} |\nabla \mathbf{v}_{l,*}|^2\, dx + \nu_l \int_{\partial \Omega_*} (-\hat{\mathbf{n}}) \cdot \nabla \mathbf{v}_{l,*} \cdot \mathbf{v}_{l,*}\, dS \\
&\quad - \int_{\partial \Omega_*} \frac{|\mathbf{v}_{l,*}|^2}{2} \mathbf{v}_{l,*} \cdot (-\hat{\mathbf{n}})\, dS - \frac{1}{\rho_l} \int_{\partial \Omega_*} p_{l,*} \mathbf{v}_{l,*} \cdot (-\hat{\mathbf{n}})\, dS \\
&\quad + \nu_l \int_{\partial B_r} \hat{\mathbf{n}}_{\partial B_r} \cdot \nabla \mathbf{v}_{l,*} \cdot \mathbf{v}_{l,*}\, dS - \int_{\partial B_r} \frac{|\mathbf{v}_{l,*}|^2}{2} \mathbf{v}_{l,*} \cdot \hat{\mathbf{n}}_{\partial B_r}\, dS \\
&\quad - \frac{1}{\rho_l} \int_{\partial B_r} p_{l,*} \mathbf{v}_{l,*} \cdot \hat{\mathbf{n}}_{\partial B_r}\, dS.
\end{aligned}
\tag{4.8}
$$

The third and fourth terms of (4.8) on the right hand side vanish since $\mathbf{v}_{l,*} \cdot \hat{\mathbf{n}}|_{\partial \Omega_*} = 0$. Consider now the last three terms in (4.8). The first two tend to zero as $r \to \infty$ using the hypotheses that $\mathbf{v}_{l,*}(x) = O(|x|^{-2})$ and $\lim_{|x| \to \infty} \nabla \mathbf{v}_{l,*}(x) = \mathbb{O}$. Finally, the last term in (4.8) also tends to zero as $r \to \infty$. Indeed, since $\lim_{|x| \to \infty} \mathbf{v}_{l,*}(x) = O(|x|^{-2})$,

$$
\begin{aligned}
&\frac{1}{\rho_l} \lim_{r \to \infty} \int_{\partial B_r} p_{l,*} \mathbf{v}_{l,*} \cdot \hat{\mathbf{n}}\, dS \\
&= \frac{p_{\infty,*}}{\rho_l} \lim_{r \to \infty} \int_{\partial B_r} \mathbf{v}_{l,*} \cdot \hat{\mathbf{n}}_{\partial B_r}\, dS + \frac{1}{\rho_l} \lim_{r \to \infty} \int_{\partial B_r} (p_{l,*} - p_{\infty,*}) \mathbf{v}_{l,*} \cdot \hat{\mathbf{n}}_{\partial B_r}\, dS \\
&= \left(\frac{p_{\infty,*}}{\rho_l} \int_{\mathbb{R}^3 \setminus \Omega_*} \mathrm{div}\,\mathbf{v}_{l,*}\, dx - \frac{1}{\rho_l} p_{\infty,*} \int_{\partial \Omega_*} \mathbf{v}_{l,*} \cdot (-\hat{\mathbf{n}})\, dS \right) + 0 = 0
\end{aligned}
$$

since $\mathbf{v}_{l,*}$ is solenoidal: $\mathrm{div}\,\mathbf{v}_{l,*} = 0$ and $\mathbf{v}_{l,*} \cdot \hat{\mathbf{n}}|_{\partial \Omega_*} = 0$. Thus, by taking $r \to \infty$, (4.8) becomes

$$
0 = -\nu_l \int_{\mathbb{R}^3 \setminus \Omega_*} |\nabla \mathbf{v}_{l,*}|^2\, dx - \nu_l \int_{\partial \Omega_*} \hat{\mathbf{n}} \cdot \nabla \mathbf{v}_{l,*} \cdot \mathbf{v}_{l,*}\, dS.
\tag{4.9}
$$

Multiplying the stress balance equation (4.6b) by $\mathbf{v}_{l,*}$ and using $\mathbf{v}_{l,*} \cdot \hat{\mathbf{n}} = 0$ yield $\hat{\mathbf{n}} \cdot \mathbb{D}(\mathbf{v}_{l,*}) \cdot \mathbf{v}_{l,*}|_{\partial \Omega_*} = 0$ since $\mu_l \neq 0$. Using the expression of the deformation tensor $\mathbb{D}(\mathbf{v}_{l,*}) = (\nabla \mathbf{v}_{l,*} + (\nabla \mathbf{v}_{l,*})^\top)/2$,

$$
\begin{aligned}
0 = 2\hat{\mathbf{n}} \cdot \mathbb{D}(\mathbf{v}_{l,*}) \cdot \mathbf{v}_{l,*} &= \hat{\mathbf{n}} \cdot \left(\nabla \mathbf{v}_{l,*} + (\nabla \mathbf{v}_{l,*})^\top \right) \cdot \mathbf{v}_{l,*} \\
&= \hat{\mathbf{n}} \cdot \nabla \mathbf{v}_{l,*} \cdot \mathbf{v}_{l,*} + \mathbf{v}_{l,*} \cdot \nabla \mathbf{v}_{l,*} \cdot \hat{\mathbf{n}}.
\end{aligned}
\tag{4.10}
$$

Using (4.10), (4.9) can be written as

$$
0 = -\nu_l \int_{\mathbb{R}^3 \setminus \Omega_*} |\nabla \mathbf{v}_{l,*}|^2\, dx + \nu_l \int_{\partial \Omega_*} \mathbf{v}_{l,*} \cdot \nabla \mathbf{v}_{l,*} \cdot \hat{\mathbf{n}}\, dS.
\tag{4.11}
$$

Since $-\hat{\mathbf{n}}$ is the outward normal of $\mathbb{R}^3 \setminus \Omega_*$ on $\partial\Omega_*$, by (4.11) and the divergence theorem,

$$
\begin{aligned}
0 &= -\nu_l \int_{\mathbb{R}^3\setminus\Omega_*} |\nabla\mathbf{v}_{l,*}|^2 \, dx - \nu_l \int_{\partial\Omega_*} \mathbf{v}_{l,*} \cdot \nabla\mathbf{v}_{l,*} \cdot (-\hat{\mathbf{n}}) \, dS \\
&= -\nu_l \int_{\mathbb{R}^3\setminus\Omega_*} |\nabla\mathbf{v}_{l,*}|^2 \, dx - \nu_l \int_{\mathbb{R}^3\setminus\Omega_*} \mathrm{div}(\mathbf{v}_{l,*} \cdot \nabla\mathbf{v}_{l,*}) \, dx \\
&= -\nu_l \int_{\mathbb{R}^3\setminus\Omega_*} |\nabla\mathbf{v}_{l,*}|^2 \, dx - \nu_l \int_{\mathbb{R}^3\setminus\Omega_*} \nabla\mathbf{v}_{l,*} : (\nabla\mathbf{v}_{l,*})^\top \, dx, \\
&\quad \text{where } A : B := \sum_{i,j} A_{ij} B_{ij} = \mathrm{tr}(AB^\top), \\
&= -\nu_l \int_{\mathbb{R}^3\setminus\Omega_*} \nabla\mathbf{v}_{l,*} : \left(\nabla\mathbf{v}_{l,*} + (\nabla\mathbf{v}_{l,*})^\top\right) dx \\
&= -\nu_l \int_{\mathbb{R}^3\setminus\Omega_*} \mathrm{tr}\left[\nabla\mathbf{v}_{l,*}\left(\nabla\mathbf{v}_{l,*} + (\nabla\mathbf{v}_{l,*})^\top\right)\right] dx.
\end{aligned}
$$

For any square matrix A, decomposing into symmetric and anti-symmetric parts we have:
$A(A + A^\top) = \frac{1}{2}(A + A^\top)^2 + \frac{1}{2}(A - A^\top)(A + A^\top)$. Linearity of the trace and the identities $\mathrm{tr}(AC) = \mathrm{tr}(CA)$ and $\mathrm{tr}(A) = \mathrm{tr}(A^\top)$, then imply $\mathrm{tr}\left[A(A + A^\top)\right] = \frac{1}{2}\mathrm{tr}\left[(A + A^\top)^2\right] = \frac{1}{2}|A + A^\top|^2$ since $A + A^\top$ is symmetric. Hence,

$$
\frac{1}{2} \int_{\mathbb{R}^3\setminus\Omega_*} |\nabla\mathbf{v}_{l,*} + (\nabla\mathbf{v}_{l,*})^\top|^2 \, dx = 0.
$$

Therefore, $\nabla\mathbf{v}_{l,*} + (\nabla\mathbf{v}_{l,*})^\top \equiv 0$ in $\mathbb{R}^3 \setminus \Omega_*$. Integrating directly the equation $\partial_i(\mathbf{v}_{l,*})_j + \partial_j(\mathbf{v}_{l,*})_i = 0$, $i, j = 1, 2, 3$, we obtain that for some vector $\mathbf{v}_0, \boldsymbol{\omega} \in \mathbb{R}^3$ and some point $x_0 \in \mathbb{R}^3$

$$
\mathbf{v}_{l,*}(x) = \mathbf{v}_0 + \boldsymbol{\omega} \times (x - x_0).
$$

This implies $\mathbf{v}_{l,*} \equiv \mathbf{0}$ since $\mathbf{v}_{l,*}(x) \to \mathbf{0}$ as $|x| \to 0$. Since $\mathbf{v}_{l,*} \equiv \mathbf{0}$, (4.4a) implies that $\nabla p_{l,*} = \mathbf{0}$, and so $p_{l,*} \equiv p_\infty$ is a constant. Since $\mathbb{D}(\mathbf{v}_{l,*}) = (\nabla\mathbf{v}_{l,*} + (\nabla\mathbf{v}_{l,*})^\top)/2 = 0$, the stress balance equation (4.6b) becomes

$$
p_{g,*} - p_{l,*} = \sigma\nabla_S \cdot \hat{\mathbf{n}} \quad \text{on } \partial\Omega_*.
$$

Since both $p_{g,*}$ and $p_{l,*}$ are constant, $\partial\Omega_*$ is a closed constant-mean-curvature (CMC) surface. By Alexandrov's Theorem [1], Ω_* must be a sphere.

We now deal with the system (4.5) for the gas. Plugging (4.5e) into (4.5c) and using (4.5b) and (4.5d), we have

$$
\frac{\kappa_g}{\gamma c_v} \Delta\left(\frac{1}{\rho_{g,*}}\right) = -\mathbf{v}_{g,*} \cdot \nabla\log\rho_{g,*}. \tag{4.12}
$$

Integrating the above equation over Ω_*, applying integration by parts formula, and using the boundary condition (4.6a), we derive

$$\int_{\Omega_*} \Delta \left(\frac{1}{\rho_{g,*}} \right) dx = 0.$$

In view of (4.5d) and (4.6c), $\rho_{g,*}|_{\partial \Omega_*}$ is a constant. So

$$0 = \int_{\Omega_*} \Delta \left(\frac{1}{\rho_{g,*}} \right) dx = \int_{\partial \Omega_*} \nabla \left(\frac{1}{\rho_{g,*}} \right) \cdot \hat{\mathbf{n}} \, dS = -\frac{1}{(\rho_{g,*}|_{\partial \Omega_*})^2} \int_{\partial \Omega_*} \nabla \rho_{g,*} \cdot \hat{\mathbf{n}} \, dS,$$

implying $\int_{\partial \Omega_*} \nabla \rho_{g,*} \cdot \hat{\mathbf{n}} \, dS = 0$. Hence, for any $f, g \in C^\infty$,

$$\int_{\Omega_*} g(\rho_{g,*}) \Delta \left(f(\rho_{g,*}) \right) dx$$

$$= -\int_{\Omega_*} \nabla \left(g(\rho_{g,*}) \right) \cdot \nabla \left(f(\rho_{g,*}) \right) dx + \int_{\partial \Omega_*} g(\rho_{g,*}) \nabla \left(f(\rho_{g,*}) \right) \cdot \hat{\mathbf{n}} \, dS$$

$$= -\int_{\Omega_*} g'(\rho_{g,*}) f'(\rho_{g,*}) |\nabla \rho_{g,*}|^2 \, dx + g(\rho_{g,*}|_{\partial \Omega_*}) f'(\rho_{g,*}|_{\partial \Omega_*}) \int_{\partial \Omega_*} \nabla \rho_{g,*} \cdot \hat{\mathbf{n}} \, dS$$

$$= -\int_{\Omega_*} g'(\rho_{g,*}) f'(\rho_{g,*}) |\nabla \rho_{g,*}|^2 \, dx. \tag{4.13}$$

Moreover, using the steady-state continuity equation (4.5a) and the boundary condition (4.6a), we have for any $h \in C^\infty$ that

$$\int_{\Omega_*} h(\rho_{g,*}) \text{div} \mathbf{v}_{g,*} \, dx = -\int_{\Omega_*} \nabla \left(h(\rho_{g,*}) \right) \cdot \mathbf{v}_{g,*} \, dx + \int_{\partial \Omega_*} h(\rho_{g,*}) \mathbf{v}_{g,*} \cdot \hat{\mathbf{n}} \, dS$$

$$= -\int_{\Omega_*} h'(\rho_{g,*}) \nabla \rho_{g,*} \cdot \mathbf{v}_{g,*} \, dx = \int_{\Omega_*} h'(\rho_{g,*}) \rho_{g,*} \text{div} \mathbf{v}_{g,*} \, dx,$$

or equivalently,

$$0 = \int_{\Omega_*} \left(h(\rho_{g,*}) - h'(\rho_{g,*}) \rho_{g,*} \right) \text{div} \mathbf{v}_{g,*} \, dx.$$

Using (4.5a), (4.12), and (4.13) with $g(\rho) = h'(\rho) - h''(\rho)\rho$, $f(\rho) = 1/\rho$, we obtain

$$0 = -\int_{\Omega_*} \left(h(\rho_{g,*}) - h'(\rho_{g,*}) \rho_{g,*} \right) \frac{\nabla \rho_{g,*}}{\rho_{g,*}} \cdot \mathbf{v}_{g,*} \, dx$$

$$= -\int_{\Omega_*} \left(h(\rho_{g,*}) - h'(\rho_{g,*}) \rho_{g,*} \right) \nabla \log \rho_{g,*} \cdot \mathbf{v}_{g,*} \, dx$$

$$= \frac{\kappa_g}{\gamma c_v} \int_{\Omega_*} \left(h(\rho_{g,*}) - h'(\rho_{g,*}) \rho_{g,*} \right) \Delta \left(\frac{1}{\rho_{g,*}} \right) dx = -\frac{\kappa_g}{\gamma c_v} \int_{\Omega_*} \frac{h''(\rho_{g,*})}{\rho_{g,*}} |\nabla \rho_{g,*}|^2 \, dx.$$

Simply choose $h(\rho) = \rho^3$ to derive $\int_{\Omega_*} |\nabla \rho_{g,*}|^2 \, dx = 0$. This implies $\rho_{g,*}$ is constant and thus $\text{div} \mathbf{v}_{g,*} = \mathbf{0}$ by (4.5a). Part (1) of Proposition 4.3 is asserted.

To derive the spherically symmetric equilibria of the approximate system (3.1)–(3.3), we make use of Proposition 5.1 (below), which presents a reduction of (3.1)–(3.3), in the spherically symmetric case, to an equivalent system for ρ and R, where $\rho = \rho_g$; see (5.1a)–(5.1c) below. All other state variables may be derived from these;

see Remark 5.2. It therefore suffices to seek time-independent solutions of (5.1a)–(5.1c). Setting $\partial_t \rho = \partial_t R = 0$ we obtain from (5.1a) and (5.1b) that $R(t) \equiv R_*$ (constant equilibrium radius) and

$$\Delta \log \rho = 0 \quad \text{in } B_{R_*}, \quad \text{and } \partial_r \rho(R_*) = 0.$$

Therefore, $\rho(r) \equiv \rho_*$ for $0 \leq r \leq R_*$ (constant equilibrium density). Evaluating (5.1c) at $r = R_*$ and using that $\rho(R_*) = \rho_*$ we conclude

$$\rho_* = \frac{1}{\mathscr{R}_g T_\infty} \left(p_{\infty,*} + \frac{2\sigma}{R_*} \right).$$

The mass of the equilibrium gas bubble of density ρ_* and radius R_* is given by

$$M = \int_{B_{R_*}} \rho_* \, dx = \frac{4\pi}{3} \rho_* R_*^3.$$

Therefore, for fixed mass M, the steady state (ρ_*, R_*) is determined by the simultaneous algebraic equations:

$$\frac{4\pi}{3} \rho_* R_*^3 = M, \tag{4.14a}$$

$$\mathscr{R}_g T_\infty \rho_* = p_{\infty,*} + \frac{2\sigma}{R_*}. \tag{4.14b}$$

Therefore, the equilibrium radius R_* is given by a solution to the cubic equation

$$p_{\infty,*} R_*^3 + 2\sigma R_*^2 - \frac{3\mathscr{R}_g T_\infty M}{4\pi} = 0. \tag{4.15}$$

It is readily seen that for each fixed $M > 0$, the cubic (4.15) has a unique positive root R_*. This choice of R_* determines the equilibria gas density and, via the relation $p_g = \mathscr{R}_g T_g \rho_g$, the gas pressure:

$$\rho_* = \frac{1}{\mathscr{R}_g T_\infty} \left(p_{\infty,*} + \frac{2\sigma}{R_*} \right), \quad p_* = \mathscr{R}_g T_\infty \rho_* = p_{\infty,*} + \frac{2\sigma}{R_*}.$$

Once we obtain the equilibrium (ρ_*, R_*, p_*), we can recover, using the formulas in (5.5), the corresponding steady state solution to the system (3.1)–(3.3) for the gas velocity $\mathbf{v}_g$, the gas temperature T_g, the specific entropy s of the gas, the liquid velocity $\mathbf{v}_l$, and the liquid pressure p_l:

$$\mathbf{v}_{g,*} = \mathbf{v}_{l,*} = \mathbf{0}, \quad T_{g,*} = T_\infty, \quad s_* = c_v \log \left(\frac{p_*}{\rho_*^\gamma} \right), \quad p_{l,*} = p_{\infty,*}.$$

Summarizing, we have derived the spherically symmetric equilibrium stated in (4.3). This proves Part (2). Part (3) of Proposition 4.3 follows from the smooth dependence of the simple roots of a given polynomial on its coefficients. This completes the proof of Proposition 4.3. □

Remark 4.6. The equilibrium radius R_* can be expressed explicitly in terms of $p_{\infty,*}, \sigma, \mathscr{R}_g, T_\infty, M$ by using the solution formula for the cubic equation (4.15).

Below, in Proposition 5.1, we shall reduce the study of spherically symmetric solutions to a closed system of equations for the gas density, $\rho(r, t)$ and the bubble radius $R(t)$, together with a condition on $\rho(R(t), t)$, the gas density at the free boundary. Our proof of asymptotic stability is carried out in this setting. We shall use the following continuity result:

Proposition 4.7. *Fix a constant* $p_{\infty,*} > 0$. *Fix a density-radius pair* $(\rho_0(x), R_0) \in L^\infty \times \mathbb{R}_+$ *and let* $M_0 = \mathrm{Mass}[\rho_0, R_0]$ *denote the mass of the corresponding bubble. Let* $(\rho_*[M_0], R_*[M_0])$ *denote the equilibrium radius and density, given by Proposition 4.3, for which* $\mathrm{Mass}[\rho_*[M_0], R_*[M_0]] = M_0 = \mathrm{Mass}[\rho_0, R_0]$. *Then, any equilibrium* $(\rho_*[M_*], R_*[M_*])$, $M_* > 0$, *close to* $(\rho_0(x), R_0)$ *in* $L^\infty \times \mathbb{R}$ *is also close to* $(\rho_*[M_0], R_*[M_0])$. *Even more strongly, there is a constant* $C = C(R_*, \rho_0, R_0) > 0$ *such that*

$$|R_*[M_0] - R_*[M_*]| + |\rho_*[M_0] - \rho_*[M_*]|$$
$$\leq C \left(|R_0 - R_*[M_*]| + \|\rho_0 - \rho_*[M_*]\|_{L^\infty(B_{R_0})} \right). \qquad (4.16)$$

Proof. Let $(\rho_0(x), R_0)$, M_0, and $(\rho_*[M_*], R_*[M_*])$ be as hypothesized. We first bound the difference $|R_*[M_0] - R_*[M_*]|$. The equilibrium radii $R_*[M_*]$ and $R_*[M_0]$ satisfy cubic equations with mass parameters M_* and M_0, respectively:

$$p_{\infty,*} R_*[M_*]^3 + 2\sigma R_*[M_*]^2 - \frac{3\mathscr{R}_g M_* T_\infty}{4\pi} = 0 \text{ and}$$

$$p_{\infty,*} R_*[M_0]^3 + 2\sigma R_*[M_0]^2 - \frac{3\mathscr{R}_g M_0 T_\infty}{4\pi} = 0.$$

Taking the difference of these two equations gives:

$$p_{\infty,*}(R_*[M_*] - R_*[M_0])(R_*[M_*]^2 + R_*[M_*]R_*[M_0] + R_*[M_0]^2)$$
$$+ 2\sigma(R_*[M_*] - R_*[M_0])(R_* + R_*[M_0]) - \frac{3\mathscr{R}_g T_\infty}{4\pi}(M_* - M_0) = 0,$$

and therefore

$$|R_*[M_0] - R_*[M_*]|$$
$$= \frac{3\mathscr{R}_g T_\infty |M_0 - M_*|}{4\pi \left[p_{\infty,*}(R_*[M_*]^2 + R_*[M_*]R_*[M_0] + R_*[M_0]^2) + 2\sigma(R_*[M_*] + R_*[M_0]) \right]}. \qquad (4.17)$$

Bounding $|R_*[M_0] - R_*[M_*]|$ therefore reduces to bounding $|M_0 - M_*|$. Expanding M_0 about M_* we have:

$$M_0 = \int_{B_{R_0}} \rho_0 = \int_{B_{R_0}} \rho_*[M_*] + \int_{B_{R_0}} (\rho_0 - \rho_*[M_*])$$

$$= \frac{4\pi}{3} R_0^3 \rho_*[M_*] + \int_{B_{R_0}} (\rho_0 - \rho_*[M_*])$$

$$= M_* + \frac{4\pi}{3}(R_0^3 - R_*[M_*]^3)\rho_*[M_*] + \int_{B_{R_0}} (\rho_0 - \rho_*[M_*]).$$

Therefore,

$$|M_0 - M_*| \le \frac{4\pi}{3} \rho_*[M_*](R_0^2 + R_0 R_*[M_*] + R_*[M_*]^2)|R_0 - R_*[M_*]|$$
$$+ \frac{4\pi}{3} R_0^3 \|\rho_0 - \rho_*[M_*]\|_{L^\infty(B_{R_0})}. \tag{4.18}$$

The bounds (4.18) and (4.17) imply that $|R_*[M_0] - R_*[M_*]|$ satisfies the bound (4.16).

Finally, we bound the difference $|\rho_*[M_0] - \rho_*[M_*]|$. Taking the difference of the relations $\frac{4\pi}{3}\rho_*[M_0]R_*[M_0]^3 = M_0$ and $\frac{4\pi}{3}\rho_*[M_*]R_*[M_*]^3 = M_*$, we have

$$\frac{4\pi}{3} R_*[M_0]^3 \rho_*[M_0] - \frac{4\pi}{3} R_*[M_*]^3 \rho_*[M_*] = M_0 - M_*.$$

Therefore,

$$\frac{4\pi}{3} R_*[M_0]^3 (\rho_*[M_0] - \rho_*[M_*])$$
$$= (M_0 - M_*) + \frac{4\pi}{3} \left[R_*[M_0]^2 + R_*[M_0]R_*[M_*] + R_*[M_*]^2 \right] (R_*[M_0] - R_*[M_*]). \tag{4.19}$$

The bound on $|\rho_*[M_0] - \rho_*[M_*]|$ now follows by estimating (4.19) using the bounds (4.18) and (4.17). $\square$

5. Reduction of the Asymptotic Model to a System for $\rho(r, t)$ and $R(t)$

The main purpose of this article is to study the stability of the spherically symmetric equilibrium (4.3) of the approximation system (3.1)–(3.3). The perturbations we consider are spherically symmetric and hence we work with the following reduction of the initial value problem:

Proposition 5.1. *Any sufficiently regular spherically symmetric solution of (3.1)–(3.3) can be constructed from a solution of the following reduced system of equations for the gas density $\rho_g(r, t) \equiv \rho(r, t)$, for $0 \le r \le R(t)$, and the bubble radius $R(t)$, together with a boundary condition on $\rho(r, t)$ at the free boundary $r = R(t)$:*

$$\partial_t \rho(r, t) = \frac{\kappa}{\gamma c_v} \Delta_r \log \rho(r, t) + \frac{1}{\gamma} \frac{\partial_t p(t)}{p(t)} \left(\frac{1}{3} r \partial_r \rho(r, t) + \rho(r, t) \right), \quad 0 \le r \le R(t), \ t > 0, \tag{5.1a}$$

$$\dot{R}(t) = -\frac{\kappa}{\gamma c_v} \frac{\partial_r \rho(R(t), t)}{(\rho(R(t), t))^2} - \frac{R(t)}{3\gamma} \frac{\partial_t p(t)}{p(t)}, \quad t > 0, \tag{5.1b}$$

$$\rho(R(t), t) = \frac{1}{\mathscr{R}_g T_\infty} \left[p_\infty(t) + \frac{2\sigma}{R(t)} + 4\mu_l \frac{\dot{R}}{R} + \rho_l \left(R(t)\ddot{R}(t) + \frac{3}{2}(\dot{R}(t))^2 \right) \right], \quad t > 0, \tag{5.1c}$$

with initial data $\rho(\cdot, 0)$, $R(0)$, $\dot{R}(0)$. *Here,* $p = p(t) = p_g(t)$ *(gas pressure) and* $\rho(R(t), t)$ *are related through the constitutive relation*

$$p(t) = \mathscr{R}_g T_\infty \rho(R(t), t), \quad t > 0, \tag{5.2}$$

Note that (5.1c) and (5.2) imply

$$p(t) - p_\infty(t) - \frac{2\sigma}{R(t)} - 4\mu_l \frac{\dot{R}}{R} = \rho_l \left(R(t) \ddot{R}(t) + \frac{3}{2} (\dot{R}(t))^2 \right). \tag{5.3}$$

The system (5.1a)–(5.1c) depends on the

$$\text{physical parameters:} \kappa = \kappa_g, \mathscr{R}_g, \gamma, c_v, \sigma. \tag{5.4}$$

The proof of Proposition 5.1 is given in Appendix B. The calculations also yields the following expressions for all state variables:

Proposition 5.2. *Denote the radial components of the gas and liquid velocities by* $v_g(r, t)$ *and* $v_l(r, t)$, *respectively. Given a solution* $(\rho(r, t), R(t))$ *to the system (5.1a)–(5.1c), we can reconstruct a spherically symmetric solution* $(v_l, p_l, \rho_g, v_g, p_g, T_g, s)$ *to the system (3.1)–(3.3) in terms of* ρ *and* R *by*

$$
\begin{aligned}
\Omega(t) &= B_{R(t)}, & t &> 0, \\
\rho_g(r, t) &= \rho(r, t), & 0 &\leq r \leq R(t), \, t > 0, \\
p_g(t) &= \mathscr{R}_g T_\infty \rho(R(t), t), & t &> 0, \\
v_g(r, t) &= \frac{\kappa}{\gamma c_v} \partial_r \left(\frac{1}{\rho(r, t)} \right) - \frac{\partial_t p_g(t)}{p_g(t)} \frac{r}{3\gamma}, & 0 &\leq r \leq R(t), \, t > 0, \\
T_g(r, t) &= \frac{p_g(t)}{\mathscr{R}_g \rho(r, t)}, & 0 &\leq r \leq R(t), \, t > 0, \\
s(r, t) &= c_v \log \left(\frac{p_g(t)}{(\rho(r, t))^\gamma} \right), & 0 &\leq r \leq R(t), \, t > 0, \\
v_l(r, t) &= \frac{(R(t))^2 \dot{R}(t)}{r^2}, & r &\geq R(t), \, t > 0, \\
p_l(r, t) &= p_\infty(t) + \rho_l \left(\frac{2R(t)(\dot{R}(t))^2 + (R(t))^2 \ddot{R}(t)}{r} - \frac{(R(t))^4 (\dot{R}(t))^2}{2r^4} \right), & r &\geq R(t), \, t > 0.
\end{aligned}
\tag{5.5}
$$

When $\mu_l = 0$ in (5.1c), to study well-posedness [6], Biro and Velázquez mapped, by a change of variables, the free boundary problem on $B_{R(t)}$: (5.1a)–(5.1c) to a problem on the fixed domain B_1 as

$$\partial_t \overline{\rho}(y, t) = \frac{\kappa}{\gamma c_v} \frac{1}{R^2} \Delta_y \log \overline{\rho}(y, t) + \frac{1}{\gamma} \frac{\partial_t p(t)}{p(t)} \left(\frac{1}{3} y \partial_y \overline{\rho}(y, t) + \overline{\rho}(y, t) \right), \quad 0 \leq y \leq 1, \, t > 0, \tag{5.6a}$$

$$\dot{R}(t) = -\frac{\kappa}{\gamma c_v} \frac{1}{R} \frac{\partial_y \overline{\rho}(1, t)}{(\overline{\rho}(1, t))^2} - \frac{R(t)}{3\gamma} \frac{\partial_t p(t)}{p(t)}, \quad t > 0, \tag{5.6b}$$

$$\overline{\rho}(1, t) = \frac{1}{\mathscr{R}_g T_\infty} \left[p_\infty(t) + \frac{2\sigma}{R(t)} + 4\mu_l \frac{\dot{R}}{R} + \rho_l \left(R(t) \ddot{R}(t) + \frac{3}{2} (\dot{R}(t))^2 \right) \right], \quad t > 0. \tag{5.6c}$$

In this setting they proved the local well-posedness for the free boundary problem (5.1a)–(5.1c). The proof is based on the derivation of a priori Schauder estimates, application of a Leray–Schauder fixed point argument and the classical regularity

theory for quasilinear parabolic equations; see, for example, [31, Chapter V. Theorem 6.1]. We extend their result to the general case involving liquid viscosity on the free boundary, which implies local well-posedness of the system (3.1)–(3.3) in the spherically symmetric case.

Theorem 5.3. (Local in time well-posedness) *Consider the initial value problem for (5.1a)–(5.1c) with initial radius $R(0) = R_0 > 0$. and initial density $\rho_0 \in C^{2+2\alpha}([0, R(0)])$, $0 < \alpha < \frac{1}{2}$. Suppose also that for some $\eta > 0$, $\rho_0(r) \geq \eta$ for $0 \leq r \leq R_0$. Then, there exists $\delta = \delta(\|\rho_0\|_{C^{2+2\alpha}})$ such that the free boundary problem (5.1a)–(5.1c) has a unique solution satisfying*

$$\rho \in C_t^{1+\alpha}([0, \delta]; C_r^{2+2\alpha}([0, R(t)))),$$
$$R \in C^{3+\alpha}[0, \delta].$$

Proof. The proof is essentially the same as the proof for the case when $\mu_l = 0$ in [6, Theorem 3.1]. The only difference is that, for the case when $\mu > 0$, an extra viscous term $4\mu_l \dot{R}/R$ needs to be added to the right hand side of [6, (3.18)]. Since $4\mu_l \dot{R}/R$ is analytic in R and $\dot{R}$ for $R \neq 0$, one can follow the same procedure in the proof of [6, Theorem 3.1]–deriving an a priori Schauder estimates and applying Leray–Schauder fixed point theorem along with the regularity theory for quasilinear parabolic equations—to conclude the same local well-posedness result. We omit the proof and refer the reader to [6].   □

6. Dynamic Stability of Spherical Bubble

In Sect. 6.1 we recall the results in [6] on conditional Lyapunov stability of spherical symmetric equilibria, that is Lyapunov stability relative to small perturbations of the same bubble mass. We then, in Theorem 6.3, extend this result to Lyapunov stability relative to arbitrary small perturbations. Then, in Sect. 6.2 we state Theorem 6.5, the result of asymptotic stability. The proof is presented in subsequent sections.

6.1. Lyapunov Stability

In [6, Theorem 4.1], Biro and Velázquez established the global well-posedness of the free boundary problem (5.1a)–(5.1c), $\mu_l = 0$ in (5.1c), when the initial data is sufficiently close to a given spherically symmetric equilibrium and has the same mass as the mass of the equilibrium solution.

When $\mu_l > 0$ in (5.1c) the extra viscous term on the boundary leads to the extra term: $-16\pi \mu_l R(t)(\dot{R}(t))^2$ on the right hand side of the energy dissipation law (7.4). Hence, the key bound [6, (4.41)] still holds, and thus, their proof also applies. In other words, we have that the spherical equilibrium are Lyapunov stable relative to mass preserving perturbations. Introduce the norm

$$\Big| \big(\rho_1(\cdot, t) - \rho_2(\cdot, t), R_1(t) - R_2(t), \dot{R}_1(t) - \dot{R}_2(t) \big) \Big|$$
$$\equiv \| \overline{\rho}_1(\cdot, t) - \overline{\rho}_2(\cdot, t) \|_{C_y^{2+2\alpha}(B_1)}$$
$$+ |R_1(t) - R_2(t)| + \big| \dot{R}_1(t) - \dot{R}_2(t) \big|, \quad \overline{\rho}_i(y, t) = \rho_i(R_i(t)y, t), \ i = 1, 2.$$
$$(6.1)$$

In [6] it is shown that given $\varepsilon_0 > 0$, there exist $\eta_0 = \eta_0(\varepsilon_0) > 0$ such that

$$\Big| \big(\rho_0 - \rho_*[M_0], R_0 - R_*[M_0], \dot{R}_0 \big) \Big| \le \eta_0, \tag{6.2}$$

where $M_0 = \mathrm{Mass}[\rho_0, R_0]$, then for all $t > 0$

$$\Big| \big(\rho(\cdot, t) - \rho_*[M_0], R(t) - R_*[M_0], \dot{R}(t) \big) \Big| \le \varepsilon_0. \tag{6.3}$$

Remark 6.1. We note that the smallness of initial radial velocity, $\dot{R}(0)$, is not explicitly assumed in [6, Theorem 4.1]. The smallness is needed to control the kinetic energy, KE_l, and higher derivatives of R.

Remark 6.2. The proof of [6, Theorem 4.1] gives a better regularity and control than $|R(t) - R_*| \le \varepsilon_0$ and $|\dot{R}(t)| \le \varepsilon_0$ which was stated in [6, Theorem 4.1],. In fact, we obtain

$$\| R - R_* \|_{C_t^{3+\alpha}(\mathbb{R}_+)} \le \varepsilon_0. \tag{6.4}$$

Note further from (5.2), that

$$\big\| p_g \big\|_{C_t^{1+\alpha}(\mathbb{R}_+)} \le \varepsilon_0. \tag{6.5}$$

Using the continuity of functionals, we now extend the conditional Lyapunov stability result [6, Theorem 4.1] to Lyapunov stability relative to arbitrary small perturbations. Specifically, we prove the Lyapunov stability of the manifold of equilibria to the system (5.1a)–(5.1c)

$$\mathcal{M}_* = \big\{ (\rho_*[M], R_*[M], \dot{R}_* = 0) : 0 < M < \infty \big\}, \tag{6.6}$$

where $\rho_*[M], R_*[M]$ are given in Proposition 4.3.

We introduce the distance of the state defined by $(\rho(\cdot, t), R(t), \dot{R}(t))$ to the manifold of equilibria:

$$\mathrm{dist}((\rho(\cdot, t), R(t), \dot{R}(t)), \mathcal{M}_*)$$
$$\equiv \inf \Big\{ \Big| \big(\rho(\cdot, t) - \rho_*, R(t) - R_*, \dot{R}(t) - \dot{R}_* \big) \Big| : (\rho_*, R_*, \dot{R}_*) \in \mathcal{M}_* \Big\}$$
$$= \inf_{0 < M < \infty} \Big| \big(\rho(\cdot, t) - \rho_*[M], R(t) - R_*[M], \dot{R}(t) \big) \Big|.$$

Theorem 6.3. (Lyapunov stability) *Consider the time evolution equation (5.1a)–(5.1c) with $p_\infty(t) \equiv p_{\infty,*}$. Let $\varepsilon_0 > 0$ be arbitrary. There exists $\eta_0 > 0$ such if the initial data $\rho_0(r)$, R_0, and $\dot{R}_0$ satisfies*

$$\mathrm{dist}((\rho_0, R_0, \dot{R}_0), \mathcal{M}_*) \le \eta_0,$$

then $(\rho(r, t), R(t))$, the global in time solution of the initial value problem (5.1a)–(5.1c), satisfies

$$\mathrm{dist}((\rho(\cdot, t), R(t), \dot{R}(t)), \mathcal{M}_*) \le \varepsilon_0, \quad \text{for all } t > 0.$$

Proof. The proof is a consequence of [6, Theorem 4.1] and Proposition 4.7. Namely, assuming $(\rho_*[M_*], R_*[M_*], \dot{R}_* = 0) \in \mathcal{M}_*$ is close to $(\rho_0, R_0, \dot{R}_0)$, Proposition 4.7 implies that there is a unique $(\rho_*[M_0], R_*[M_0])$ such that $\mathrm{Mass}[\rho_*[M_0], R_*[M_0]] = M_0 = \mathrm{Mass}[\rho_0, R_0]$, and

$$|R_*[M_0] - R_*[M_*]| + |\rho_*[M_0] - \rho_*[M_*]|$$
$$\leq C \left(|R_0 - R_*[M_*]| + \|\rho_0 - \rho_*[M_*]\|_{L^\infty(B_{R_0})} \right) \qquad (6.7)$$
$$\leq C \left(|R_0 - R_*[M_*]| + \|\rho_0 - \rho_*[M_*]\|_{C_r^{2+2\alpha}(B_{R_0})} \right).$$

Hence,

$$\|\rho_0 - \rho_*[M_0]\|_{C_r^{2+2\alpha}} + |R_0 - R_*[M_0]|$$
$$= \|\rho_0 - \rho_*[M_*] + \rho_*[M_*] - \rho_*[M_0]\|_{C_r^{2+2\alpha}} + |R_0 - R_*[M_*] + R_*[M_*] - R_*[M_0]|$$
$$\leq \|\rho_0 - \rho_*[M_*]\|_{C_r^{2+2\alpha}} + |\rho_*[M_*] - \rho_*[M_0]| + |R_0 - R_*[M_*]| + |R_*[M_*] - R_*[M_0]|$$
$$\leq C' \left(\|\rho_0 - \rho_*[M_*]\|_{C_r^{2+2\alpha}} + |R_0 - R_*[M_*]| \right).$$

Therefore, choosing $\|\rho_0 - \rho_*[M_*]\|_{C_r^{2+2\alpha}} + |R_0 - R_*[M_*]|$ and $\dot{R}_0$ sufficiently small we conclude from [6, Theorem 4.1] ((6.2) implies (6.3)) that

$$\mathrm{dist}((\rho(\cdot, t), R(t), \dot{R}(t)), \mathcal{M}_*) \leq \|\rho(\cdot, t) - \rho_*[M_0]\|_{C_r^{2+2\alpha}} + |R(t) - R_*[M_0]| + |\dot{R}(t)|$$
$$\leq \varepsilon_0.$$

This completes the proof. □

The proof of the Lyapunov stability in [6, Theorem 4.1] relies on a coercive energy estimate ([6, Lemma 4.2]), for the case of constant external far-field pressure. In Appendix C we prove an extension of this energy estimate, Theorem 7.5, which enables us to generalize Theorem 6.3.

Corollary 6.4. *The conclusions of Theorem 6.3 hold provided we choose η_0 sufficiently small and so that the following additional conditions are satisfied:*

$$|p_\infty(t) - p_{\infty,*}| \leq \eta_0, \qquad \|\partial_t p_\infty\|_{L_t^1(\mathbb{R}_+)} \leq \eta_0.$$

6.2. Nonlinear Asymptotic Stability with No Decay Rate

The first goal of this article is to study the asymptotic stability of the family of spherically symmetric equilibria of the approximate system (3.1)–(3.3) against small spherically symmetric perturbations. This is a consequence of the following result on asymptotic stability for the reduced system (5.1a)–(5.1c).

Theorem 6.5. (Asymptotic stability of the manifold of spherically symmetric equilibria) *Fix parameters (5.4) and set $p_\infty(t) = p_{\infty,*}$ in the system (5.1a)–(5.1c).*

(1) There exist a constant $\eta > 0$ such that if $\mathrm{dist}((\rho_0, R_0, \dot{R}_0), \mathcal{M}_*) \leq \eta$, then

$$\mathrm{dist}((\rho(\cdot, t), R(t), \dot{R}(t)), \mathcal{M}_*) \to 0 \ \text{ as } t \to +\infty.$$

(2) *More precisely, let $M_* > 0$ and $(\rho_*[M_*], R_*[M_*])$ as Proposition 8.1, there exist constants $\eta_1 > 0$ and $C_1 > 0$ such that the following holds for all $0 < \eta \le \eta_1$: Consider initial data $\rho_0(r)$, R_0, $\dot{R}_0$, an <u>arbitrary</u> small perturbation of $(\rho_*[M_*], R_*[M_*], \dot{R}_* = 0)$:*

$$\big| \left(\rho_0 - \rho_*[M_*], R_0 - R_*[M_*], \dot{R}_0 \right) \big| \le \eta. \qquad (6.8)$$

Let $M_0 = \int_{B_{R_0}} \rho_0$ denote the initial bubble mass. In general, $M_0 \ne M_*$, however by Proposition 4.7 the corresponding points on the manifold of equilibria are close:

$$|\rho_*[M_0] - \rho_*[M_*]| \le C_1 \eta \quad \text{and} \quad |R_*[M_0] - R_*[M_*]| \le C_1 \eta.$$

Let $(\rho(r, t), R(t))$ denote the global in time solution of the free boundary problem (5.1a)–(5.1c) with initial data satisfying (6.8). Then, as $t \to +\infty$

$$\text{dist}((\rho(\cdot, t), R(t), \dot{R}(t)), (\rho_*[M_0], R_*[M_0], \dot{R}_* = 0)) \to 0, \qquad (6.9)$$

and $|\ddot{R}(t)| + |\dddot{R}(t)| \to 0$ as $t \to +\infty$.

(3) *The convergence of (ρ, R) in (6.9) is sufficient to imply the convergence of the quantities in Proposition 5.2 to their equilibrium values. Therefore, the spherical equilibrium (4.3) of the system (3.1)–(3.3) is asymptotically stable.*

It is simple to generalize Theorem 6.5, the asymptotic stability for the model of constant external far-field pressure p_∞, to the following result for the case that $p_\infty(t)$ is a small perturbation of a constant.

Corollary 6.6. *The conclusions of Theorem 6.5 hold provided we choose the constant $\eta_0 > 0$ sufficiently small and such the following conditions on the asymptotically constant far-field pressure hold:*

$$p_\infty \in C_t^{1+\alpha}(\mathbb{R}_+), \quad |p_\infty(t) - p_{\infty,*}| + \|\partial_t p_\infty\|_{L_t^1(\mathbb{R}_+)} \le \eta_0, \quad p_\infty(t) \to p_{\infty,*} \text{ as } t \to \infty.$$
$$(6.10)$$

Strategy of the Proof of Theorem 6.5. By a continuity argument, the proof of asymptotic stability relative to arbitrary small perturbations, can be reduced to Proposition 8.1 on asymptotic stability relative to perturbations of a spherical equilibrium which have the same bubble mass. At the heart of the proof of Proposition 8.1 is

(1) the time-integrability over $[0, \infty)$ of the energy dissipation rate: $\int_{B_{R(t)}} |T_g(\cdot, t)|^{-2} |\nabla T_g(\cdot, t)|^2$ and

(2) the coercive energy estimate, Theorem 7.5, which expresses that the spherical equilibrium of an arbitrary specified mass is a local minimizer of the total energy relative to spherically symmetric perturbations of the same mass; see [6, Lemma 4.2].

By the equation of state, (3.2d), $\int_{B_{R(t)}} |\rho_g(\cdot, t)|^{-2} |\nabla \rho_g(\cdot, t)|^2$ is time-integrable over $[0, \infty)$. This implies convergence of $\rho(\cdot, t)$, to an equilbrium density. With control of the density, $\rho(r, t)$, and in particular $\rho(R(t), t)$, we obtain that $\dot{R}(t) \to 0$ as $t \to \infty$, from (5.1b), the equation for the motion of the boundary. The limiting constant values of ρ_g and R satisfy the system (4.14) and it follows that these correspond to the unique spherically symmetric equilibrium of the given initial mass. We present the detailed proofs in Sect. 8.

6.3. Main Result: Exponential Rate of Convergence Toward the Manifold of Equilibria

The ultimate goal of this article is to show that, within the approximate system (3.1)–(3.3), the manifold of the spherically symmetric equilibria, given in (4.3) and parametrized by the bubble mass M is nonlinearly and exponentially asymptotically stable with respect to small spherically symmetric perturbations.

Theorem 6.7. *Assume constant pressure at infinity, $p_\infty(t) \equiv p_{\infty,*}$. Then, the manifold of the equilibria $\mathcal{M}_*$ (defined in (6.6)) of the free boundary problem (5.1) is (locally) nonlinearly exponentially stable. Specifically, there exist a constant $\eta > 0$ such that if $\mathrm{dist}\left((\rho_0(\cdot), R(t), \dot{R}(t)), \mathcal{M}_*\right) \leq \eta$, then for some $\overline{\beta} > 0$*

$$\mathrm{dist}\left((\rho(\cdot, t), R(t), \dot{R}(t)), \mathcal{M}_*\right) = O\left(e^{-\overline{\beta}t}\right) \quad \text{as } t \to +\infty.$$

Strategy of the Proof of Theorem 6.7. The detailed proof of Theorem 6.7 is presented in Sect. 9 and makes use of ideas from center manifold theory; see, e.g. [10]. First, we rewrite the quasi-linear parabolic partial differential equation in the free boundary problem (5.1) as an infinite-dimensional dynamical system by means of Dirichlet eigenfunction decomposition in Proposition 9.1. The linearized operator has a neutral direction (zero eigenvalue) associated with the manifold of equilibrium and all its other spectrum is contained in the open left half plane, and bounded away from the imaginary axis; Proposition 9.3, Appendix D. Subject to a codimension one constraint, the linearized flow satisfies an exponential time-decay bound; Proposition 9.5. Next, we develop a center manifold analysis to obtain an exponential rate of convergence of the solution toward the center manifold of spherically symmetric equilibria.

We note that an obstacle to applying the standard center manifold framework to our stability problem is that our infinite dimensional dynamical system (9.5) is quasilinear and autonomous: $\dot{\mathbf{w}} = \mathcal{L}\mathbf{w} + \mathcal{N}^1(\mathbf{w})\dot{\mathbf{w}} + \mathcal{N}^0(\mathbf{w})$ (or (5.1) equivalently). The system is nonlinear, and linear in $\dot{\mathbf{w}}$. We are unaware of a center manifold formulation admitting direct application to our free boundary problem. Hence in Appendix F, we develop an applicable general approach for a class of fully nonlinear autonomous systems equipped with "weak" *a priori* bounds; see Proposition F.2. The *a priori* information on the regularity and weak time-decay of $\dot{\mathbf{w}}$ (Theorem 6.5) enables us, via Proposition F.2, to prove exponential convergence to a center manifold.

7. Conservation of Mass and Energy Dissipation

Solutions of the free boundary problem (3.1)–(3.3) satisfy conservation of mass and an energy dissipation law. These play a central role in the Lyapunov stability theory of [6] and in our asymptotic stability theory. To derive these statements one makes use of the following technical results.

Given a smooth velocity field, $\mathbf{v} : (x, t) \mapsto \mathbf{v}(x, t) \in \mathbb{R}^3$, define $X^t(\alpha)$ to be the particle trajectory map given by the solution of the initial value problem: $\dot{X}^t(\alpha) = \mathbf{v}\left(X^t(\alpha), t\right)$, $X^0(\alpha) = \alpha$. The mapping $\alpha \mapsto X^t(\alpha)$ is smooth and invertible for all t sufficiently small. For an open subset $\Omega \subset \mathbb{R}^2$, let $X^t(\Omega) = \{X^t(\alpha) : \alpha \in \Omega\}$. We first recall the transport formula that gives the rate of change of a function in a domain transported with the fluid.

Proposition 7.1. ([40, Proposition 1.3]) *Let Ω denote an open, bounded domain with a smooth boundary. Then for any smooth function $f(x, t)$,*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{X^t(\Omega)} f \, \mathrm{d}x = \int_{X^t(\Omega)} [\partial_t f + \mathrm{div}(f\mathbf{v})] \, \mathrm{d}x.$$

With the aid of this transport formula, we have the following lemma which is used to compute the time-evolution of the mass and energies on a time-varying spatial domain, $\Omega(t)$.

Lemma 7.2. *If $\partial_t \rho + \mathrm{div}(\rho\mathbf{v}) = 0$ in $\Omega(t)$ and $\mathbf{v}(\boldsymbol{\omega}(t), t) = \dot{\boldsymbol{\omega}}(t)$ for $\boldsymbol{\omega}(t) \in \partial\Omega(t)$, then*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega(t)} \rho\phi \, \mathrm{d}x = \int_{\Omega(t)} \rho \frac{\mathrm{D}\phi}{\mathrm{D}t} \, \mathrm{d}x$$

for any smooth function ϕ, where $\frac{\mathrm{D}}{\mathrm{D}t}$ is the material derivative of f given by

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \partial_t f + \mathbf{v} \cdot \nabla f.$$

Proof of Lemma 7.2. Since the boundary $\partial\Omega(t)$ moves along the particle-trajectory mapping X of the velocity field $\mathbf{v}$, $\Omega(t) = X(\Omega(0), t)$. By the transport formula Proposition 7.1,

$$\frac{d}{dt} \int_{\Omega(t)} \rho\phi \, \mathrm{d}x = \int_{\Omega(t)} [\partial_t(\rho\phi) + \mathrm{div}_x(\rho\phi\mathbf{v})] \, \mathrm{d}x.$$

Using the continuity equation, the right hand side of above equation becomes

$$\int_{\Omega(t)} [\rho\partial_t\phi + \rho\mathbf{v} \cdot \nabla\phi] \, \mathrm{d}x.$$

This proves the lemma. □

7.1. Conservation of Mass

By taking $\phi \equiv 1$ in Lemma 7.2 we have

Proposition 7.3. (Bubble mass conservation) *Let X^t be the particle-trajectory mapping associated with $\mathbf{v}_g$. Denote by $\Omega_0 \subset \mathbb{R}^3$ the bubble region at time $t = 0$, assumed to have a smooth boundary, and $\Omega(t) = X^t(\Omega_0)$. Let $\rho = \rho_g$ denote a $C_t^{1+\alpha}([0, \infty); C_x^{2+2\alpha}(\Omega(t))), 0 < \alpha < \frac{1}{2}$ solution of (2.2a) (or equivalently (3.2a)) with $\mathbf{v}_g$ satisfying (2.4a) (or equivalently (3.3a)) and initial data $\rho_0 \in C^{2+2\alpha}(\Omega_0)$. Then, the mass of the bubble is constant in time:*

$$\int_{\Omega(t)} \rho(x, t)\, dx = \int_{\Omega_0} \rho_0(x)\, dx \quad t > 0. \tag{7.1}$$

7.2. Energy Dissipation Law

Definition 7.1. (The total energy) Consider the case of spherically symmetric solutions of (3.1)–(3.3). The total energy of the system is given by

$$\mathcal{E}_{\text{total}}(t) = FE(t) + KE_l(t) + U_{g-l}(t) + PV_{p_\infty}(t), \tag{7.2}$$

where the total energy is made up of the following components:

(1) $FE(t)$, the Helmholtz free energy:

$$
\begin{aligned}
FE(t) &= c_v \int_{B_R} \rho_g T_g\, dx - T_\infty \int_{B_R} \rho_g s\, dx \\
&= \frac{4\pi c_v}{3\mathscr{R}_g} p_g R^3 - T_\infty \int_{B_R} \rho_g s\, dx \\
&= \frac{4\pi c_v}{3\mathscr{R}_g} p_g R^3 - c_v T_\infty M_0 \log p_g + c_v \gamma T_\infty \int_{B_R} \rho_g \log \rho_g\, dx,
\end{aligned}
\tag{7.3}
$$

where $M_0 = \text{Mass}[\rho_g, R]$, and the second and the last equalities hold by (3.2d) and (3.2e), respectively.

(2) $KE_l(t)$, the kinetic energy of the liquid:

$$KE_l(t) = \frac{1}{2} \int_{\mathbb{R}^3 \setminus B_{R(t)}} \rho_l |\mathbf{v}_l|^2\, dx = 2\pi \rho_l [R(t)]^3\, [\dot{R}(t)]^2,$$

(3) $U_{g-l}(t)$, the surface energy of the liquid–gas interface

$$U_{g-l}(t) = \sigma \int_{\partial B_{R(t)}} dS = 4\pi\sigma\, [R(t)]^2,$$

(4) $PV_{p_\infty}(t)$, the energy contributed by the work done by the external sound field.

$$PV_{p_\infty}(t) = |B_{R(t)}|\, p_\infty(t) = \frac{4\pi}{3} [R(t)]^3 p_\infty(t).$$

The energy functional (7.2) is at the heart of the stability analysis. Its importance is clear from the following result on energy dissipation, proved in [6] (using Lemma 7.2) for the system (5.1) (equivalently (3.1)–(3.3) under the assumption of spherical symmetry) and $p_\infty = 1$. We state and prove a mild generalization to the case of a time-dependent pressure at $p_\infty(t)$. We shall use the abbreviated notation: $\rho = \rho_g, p = p_g, T = T_g$ and $\kappa = \kappa_g$.

Proposition 7.4. (Energy dissipation law) *Assume that* $(\rho(r, t), R(t), p(t) = \mathcal{R}_g T(r, t)\rho(r, t))$ *is a solution of (5.1), or equivalently (3.1)–(3.3) under the assumption of spherical symmetry.*

Then,

$$\frac{d}{dt}\mathcal{E}_{total}(t) = -\kappa T_\infty \int_{B_{R(t)}} \frac{|\nabla_r T(|x|, t)|^2}{T^2(|x|, t)}\,dx$$
$$- 16\pi \mu_l R(t)(\dot{R}(t))^2 + \frac{4\pi}{3}R^3(t)\partial_t p_\infty(t). \tag{7.4}$$

Proof. By Lemma 7.2, differentiating $\mathcal{E}_{total}$ with respect to t using the third line of (7.3) yields

$$\frac{d}{dt}\mathcal{E}_{total} = \frac{4\pi c_v}{3\mathcal{R}_g}\left(\partial_t p_g R^3 + 3 p_g R^2 \dot{R}\right) - T_\infty \int_{B_R} \rho_g \frac{Ds}{Dt}\,dx$$
$$+ 4\pi \rho_l R^2 \dot{R}\left(R\ddot{R} + \frac{3}{2}\dot{R}^2\right) + 8\pi \sigma R\dot{R} + 4\pi R^2 \dot{R}\ p_\infty(t) + \frac{4\pi}{3}R^3\partial_t p_\infty. \tag{7.5}$$

Consider the second term on the right hand side of (7.5). Using (3.2c), integrating by parts, and (3.3c), we obtain

$$\int_{B_R} \rho_g \frac{Ds}{Dt}\,dx = \kappa_g \int_{B_R} \frac{\Delta_r T_g}{T_g}\,dx = \kappa_g \int_{B_R} \frac{|\nabla_r T_g|^2}{T_g^2}\,dx + \frac{4\pi \kappa_g R^2}{T_\infty}\partial_r T_g(R(t), t)$$
$$= \kappa_g \int_{B_R} \frac{|\nabla_r T_g|^2}{T_g^2}\,dx - \frac{4\pi \kappa_g R^2 p_g(t)}{\mathcal{R}_g T_\infty}\frac{\partial_r \rho_g(R(t), t)}{\rho_g^2(R(t), t)}, \tag{7.6}$$

where the last equality follows from the constitutive relation $T_g = p_g(\mathcal{R}_g \rho_g)^{-1}$.

For the third term on the right hand side of (7.5) we use (5.3)

$$\rho_l \left(R(t)\ddot{R}(t) + \frac{3}{2}(\dot{R}(t))^2\right) = p_g(t) - p_\infty(t) - \frac{2\sigma}{R(t)} - 4\mu_l \frac{\dot{R}}{R}. \tag{7.7}$$

Substituting (7.6) and (7.7) into (7.5) we obtain

$$\frac{d}{dt}\mathcal{E}_{total} = \frac{4\pi c_v}{3\mathcal{R}_g}\partial_t p_g R^3 + 4\pi\left(\frac{c_v}{\mathcal{R}_g} + 1\right)p_g R^2 \dot{R} - T_\infty \kappa_g \int_{B_R} \frac{|\nabla_r T_g|^2}{T_g^2}\,dx$$
$$+ 4\pi \kappa_g R^2 p_g(t)\frac{1}{\mathcal{R}_g}\frac{\partial_r \rho_g(R(t), t)}{\rho_g^2(R(t), t)} - 16\pi \mu_l R(t)(\dot{R}(t))^2 + \frac{4\pi}{3}R^3\partial_t p_\infty. \tag{7.8}$$

or

$$\frac{d}{dt}\mathcal{E}_{\text{total}} = -T_\infty \kappa_g \int_{B_R} \frac{|\nabla_r T_g|^2}{T_g^2}\, dx - 16\pi \mu_l R(t)(\dot{R}(t))^2 + \frac{4\pi}{3} R^3 \partial_t p_\infty \tag{7.9}$$
$$+ 4\pi R^2 p_g \left[\frac{c_v}{\mathcal{R}_g} \frac{1}{3} \frac{\partial_t p_g}{p_g} R + \left(1 + \frac{c_v}{\mathcal{R}_g}\right)\dot{R} + \frac{\kappa_g}{\mathcal{R}_g} \frac{\partial_r \rho_g}{\rho_g^2}\right].$$

Finally we claim that the expression in the square brackets in (7.9) vanishes:

$$\mathcal{I}(r,t) \equiv \frac{c_v}{\mathcal{R}_g} \frac{1}{3} \frac{\partial_t p_g}{p_g} R + \left(1 + \frac{c_v}{\mathcal{R}_g}\right)\dot{R} + \frac{\kappa_g}{\mathcal{R}_g} \frac{\partial_r \rho_g}{\rho_g^2} = 0, \tag{7.10}$$

from which Proposition 7.4 follows. To prove (7.10) note that the relation $\gamma = 1 + \frac{\mathcal{R}_g}{c_v}$ (see (2.3)) implies

$$\mathcal{I}(r,t) = \frac{1}{\gamma - 1} \frac{1}{3} \frac{\partial_t p_g}{p_g} R + \frac{\gamma}{\gamma - 1}\dot{R} + \frac{\kappa_g}{\mathcal{R}_g} \frac{\partial_r \rho_g}{\rho_g}$$
$$= \frac{\gamma}{\gamma - 1}\left(\frac{1}{3\gamma} \frac{\partial_t p_g}{p_g} R + \dot{R} + \frac{\kappa_g}{\mathcal{R}_g} \frac{\gamma - 1}{\gamma} \frac{\partial_r \rho_g}{\rho_g^2}\right) \tag{7.11}$$

Next, we use (5.1b) to simplify (7.12). This yields

$$\mathcal{I}(r,t) = \frac{\kappa_g}{\gamma - 1} \frac{1}{\mathcal{R}_g}\left(-\frac{\mathcal{R}_g}{c_v} + \gamma - 1\right) \frac{\partial_r \rho_g}{\rho_g^2} = 0 \tag{7.12}$$

The proof of Proposition 7.4 is now complete. □

7.3. Coercivity Energy Estimate

To prove the global existence of solutions and Lyapunov stability, the authors in [6] considered the energy $\mathcal{E}_{\text{total}}$ defined in (7.2) for $p_\infty(t) \equiv 1$, and used the energy dissipation formula (7.4). By expanding the energy $\mathcal{E}_{\text{total}}$ at the steady state energy up to quadratic terms, they derived the coercivity estimate of the perturbed energy from the steady state energy in [6, Lemma 4.2]. We generalize [6, Lemma 4.2] to the following result for the case of general (nonstationary) external far-field pressure $p_\infty(t)$ whose proof is similar to that of [6, Lemma 4.2] and is in Appendix C for the reader's convenience.

Theorem 7.5. *Given positive constants $\rho_*, R_*, p_{\infty,*}$. Assume that there exists a constant $v > 1$ such that*

$$v^{-1} \leq \rho(r) \leq v, \tag{7.13}$$
$$v^{-1} \leq R \leq v, \tag{7.14}$$
$$Mass[\rho, R] = Mass[\rho_*, R_*], \tag{7.15}$$

where $Mass[\rho, R]$ is given in (4.1). Let

$$\varrho(y) = \rho(Ry) - \rho_*, \mathcal{R} = R - R_*, \dot{\mathcal{R}} = \dot{R}, \text{ and } \mathcal{P}_\infty = p_\infty - p_{\infty,*}.$$

Then,

(a)
$$\mathcal{E}_{\text{total}} - \mathcal{E}_* \geq \frac{c_v T_\infty R_*^3 |B_1|}{2\rho_*} \left(\varrho(1) - \frac{1}{|B_1|} \int_{B_1} \varrho \right)^2 + 2\pi \rho_l R_*^3 \dot{\mathcal{R}}^2$$

$$+ \frac{R_*^3}{\rho_*^2} \left(\frac{p_{\infty,*}}{2} + \frac{2\sigma}{3R_*} \right) \int_{B_1} \varrho^2$$

$$- 4\pi R_*^2 |\mathcal{P}_\infty| |\mathcal{R}| - \frac{R_*^3}{4\pi \rho_*^2} |\mathcal{P}_\infty| |B_1| \int_{B_1} \varrho^2$$

$$+ O\left(|\mathcal{R}|^3 + |\varrho(1)|^3 + \left(\int_{B_1} |\varrho| \right)^3 \right), \tag{7.16}$$

where $\mathcal{E}_*$ is $\mathcal{E}_{\text{total}}$ evaluated at $(\rho_*, R_*, \dot{R}_* = 0)$. Furthermore,
(2) there exist constants $\Theta > 0$, $\delta_0 \in (0, 1]$ depending only on $\text{Mass}[\rho_*, R_*]$, T_∞, and v such that if $\|\rho - \rho_*\|_{L^\infty(B_R)} + |p_\infty - p_{\infty,*}| \leq \delta_0$, then

$$\mathcal{E}_{\text{total}} - \mathcal{E}_* \geq \Theta \left(\int_{B_R} (\rho(|x|) - \rho_*)^2 \, dx \right). \tag{7.17}$$

Comments on Theorem 7.5

(1) Fix $p_\infty = p_{\infty,*}$. Then, the coercive energy estimate of Theorem 7.5 implies that, relative to perturbations of the same bubble mass, the total energy $\mathcal{E}_{\text{total}}$ is locally convex around the equilibrium $(\rho_*, R_*, \dot{R}_* = 0)$ and that the equilibrium $(\rho_*, R_*, \dot{R}_* = 0)$ is the unique local minimizer of the total energy $\mathcal{E}_{\text{total}}$.
(2) The estimate (7.17) is a lower bound for the functional $(\rho, R, \dot{R}) \mapsto \mathcal{E}_{\text{total}}[\rho, R, \dot{R}]$. It does not depend on $(\rho, R, \dot{R})$ being a solution of the evolution equations.
(3) Theorem 7.5 applies to all positive solutions (ρ_*, R_*) of the algebraic system (4.14).

Remark 7.6. (Surface tension versus thermal diffusion) In the spherically symmetric approximate model we study, the surface tension σ does not play a role in the relaxation of the bubble to equilibrium. One expects it to play a role in the rounding out of non-spherical bubble deformations, which are not under consideration here. Although typically surface tension σ is positive for liquid/gas interface, our analysis for asymptotic stability applies to $\sigma = 0$ or even some negative range of σ. In fact, the cubic equation (4.15) admits a unique positive solution for all $\sigma \in \mathbb{R}$. Besides, the equilibrium energy $\mathcal{E}_*$ remains the conditional minimizer as long as the coefficient of the third term on the right of (7.16) is positive by the coercive energy estimate (7.17). It is equivalent to $\sigma > -3R_* p_{\infty,*}/4$.

On the other hand, the thermal conductivity of gas, κ_g, and far-field liquid temperature, T_∞, both play a role in energy dissipation (7.4). Our analysis fails when either of these two parameters vanishes. However, the case when $T_\infty = 0$ is excluded since it would lead to a solution that is singular everywhere: $p_g(t) = \mathcal{R}_g T_\infty \rho_g(R(t), t) = 0$, and $s = c_v \log(p_g/(\rho_g)^\gamma) = -\infty$. Therefore, the only physical parameter that plays a role in the damping mechanism is the thermal conductivity of gas, κ_g. Indeed, $\kappa_g/(\gamma c_v)$ is the diffusion coefficient of the parabolic PDE (5.1a) that forces the gas density to distribute uniformly inside the bubble and causes the energy dissipation. This then leads to the thermal damping mechanism of the bubble radius by the ODE of bubble radius.

8. Nonlinear Asymptotic Stability: Proof of Theorem 6.5

In this section, we prove Theorem 6.5. We show that family (manifold) of sphericlly symmetric equilibrium states is asymptotically stable. Our first step is to prove the following:

8.1. Asymptotic Stability of Equilibria with Respect to Small Mass-Preserving Perturbations

We begin with proving the asymptotic stability of a fixed equilibrium relative to mass preserving perturbations. To this end, we make use of Theorem 7.5 and the temporal integrability of the right hand side of (7.4).

Proposition 8.1. (Asymptotic stability of a fixed equilibrium relative to mass preserving perturbations) *Fix parameters (5.4) and set $p_\infty = p_{\infty,*}$ in the system (5.1a)–(5.1c). For arbitrary fixed $M_0 > 0$, let $(\rho_*[M_0], R_*[M_0])$ denote the unique spherically symmetric equilibrium with bubble mass M_0 given in Proposition 4.3, i.e. $M_0 = \frac{4}{3}\pi\rho_*[M_0](R_*[M_0])^3$. There exists $\eta_0 > 0$, such that the following holds for all $0 < \eta \le \eta_0$:*
Let $(\rho(r,t), R(t))$ be the global in time solution of the free boundary problem (5.1a)–(5.1c) with initial data $\rho(r,0) = \rho_0(r)$, $R(0) = R_0$, $\dot{R}(0) = \dot{R}_0$ which is a mass preserving *and small perturbation of $(\rho_*[M_0], R_*[M_0])$, i.e.*

$$M_0 = \int_{R_0} \rho_0 \quad \text{(mass preserving perturbation)}$$

and

$$\left| \left(\rho_0 - \rho_*[M_0], R_0 - R_*[M_0], \dot{R}_0 \right) \right| \le \eta. \tag{8.1}$$

Then, as $t \to +\infty$

$$\left| \left(\rho(\cdot,t) - \rho_*[M_0], R(t) - R_*[M_0], \dot{R}(t) \right) \right| \to 0, \tag{8.2}$$

where the norm $|\cdot|$ is defined in (6.1). Moreover, $|\ddot{R}(t)| + |\dddot{R}(t)| \to 0$ as $t \to +\infty$.

Proof. Consider a fixed equilibrium $(\rho_*, R_*, \dot{R}_* = 0)$ and a nearby (non-constant) initial condition $(\rho_0, R_0, \dot{R}_0)$ (see the hypothesis (8.1)) and such that

$$\int_{B_{R_0}} \rho_0 = \int_{B_{R_*}} \rho_* = \frac{4\pi}{3} R_*^3 \rho_* = M_0.$$

Hence,

$$\int_{B_{R(t)}} \rho(\cdot,t) = \frac{4\pi}{3} R_*^3 \rho_*, \quad \text{for all } t \ge 0. \tag{8.3}$$

First recall that $p_g = \mathscr{R}_g T_g \rho_g$ (see (2.2d)) and hence

$$\frac{\nabla_r T}{T} = \nabla_r \log T = \nabla_r \left(\log p - \log(\mathscr{R}_g \rho) \right) = -\frac{\nabla_r \rho}{\rho}, \qquad T = T_g, \ p = p_g, \ \rho = \rho_g.$$

Together with the energy dissipation relation (7.4), we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{\text{total}}(t) &= -\kappa T_\infty \int_{B_{R(t)}} \frac{|\nabla_r T|^2}{T^2} - 16\pi \mu_l R \dot{R}^2 \\
&= -\kappa T_\infty \int_{B_{R(t)}} \frac{|\nabla_r \rho|^2}{\rho^2} - 16\pi \mu_l R \dot{R}^2,
\end{aligned}
\tag{8.4}
$$

where $\mathcal{E}_{\text{total}}(t)$ is given in Definition 7.1:

$$
\mathcal{E}_{\text{total}}(t) = FE(t) + 4\pi\sigma(R(t))^2 + \frac{4\pi}{3}R^3(t) + 2\pi\rho_l \dot{R}^2(t) R^3(t).
$$

Integrating (8.4) with respect time we obtain

$$
\begin{aligned}
\kappa T_\infty \int_0^t \int_{B_{R(\tau)}} \frac{|\nabla_r \rho(|x|,\tau)|^2}{\rho(|x|,\tau)^2}\,\mathrm{d}x\mathrm{d}\tau + 16\pi\mu_l \int_0^t R(\tau)(\dot{R}(\tau))^2\,\mathrm{d}\tau \\
= \mathcal{E}_{\text{total}}(0) - \mathcal{E}_{\text{total}}(t).
\end{aligned}
\tag{8.5}
$$

Applying now the key coercive lower bound on $\mathcal{E}_{\text{total}}$ (Theorem 7.5, and in particular (7.17)) gives

$$
\begin{aligned}
\kappa T_\infty \int_0^t \int_{B_{R(\tau)}} \frac{|\nabla_r \rho(|x|,\tau)|^2}{\rho(|x|,\tau)^2}\,\mathrm{d}x\mathrm{d}\tau &+ 16\pi\mu_l \int_0^t R(\tau)(\dot{R}(\tau))^2\,\mathrm{d}\tau \\
&\leq \mathcal{E}_{\text{total}}(0) - \mathcal{E}_* - (\mathcal{E}_{\text{total}}(t) - \mathcal{E}_*) \\
&\leq \mathcal{E}_{\text{total}}(0) - \mathcal{E}_* - \Theta\left(\int_{B_R}(\rho - \rho_*)^2\,\mathrm{d}x\right) \\
&\leq \mathcal{E}_{\text{total}}(0) - \mathcal{E}_*.
\end{aligned}
$$

By the regularity of $\rho(x,t)$ and $R(t)$, we have that $\int_{B_{R(t)}} \frac{|\nabla_r \rho(|x|,\tau)|^2}{\rho(|x|,\tau)^2}\,\mathrm{d}x$ and $R(t)(\dot{R}(t))^2$ are uniformly continuous. Recall the following alternative form of Barbalat's lemma:

$$
\begin{aligned}
&\textit{Suppose } \int_0^t f(\tau)\,\mathrm{d}\tau \textit{ has a finite limit as } t\to\infty. \\
&\textit{If f(t) is uniformly continuous function, then } \lim_{t\to\infty} f(t) = 0.
\end{aligned}
\tag{8.6}
$$

By the above Barbalat's lemma, we conclude that

$$
\int_{B_{R(t)}} \frac{|\nabla_r \rho(|x|,t)|^2}{\rho(|x|,t)^2}\,\mathrm{d}x \in L_t^1((0,\infty)) \quad \text{and hence} \quad \lim_{t\to\infty} \int_{B_R} \frac{|\nabla_r \rho(|x|,t)|^2}{\rho(|x|,t)^2}\,\mathrm{d}x = 0,
\tag{8.7}
$$

and, if $\mu_l > 0$,

$$
R(t)(\dot{R}(t))^2 \in L_t^1((0,\infty)) \quad \text{and hence} \quad \lim_{t\to\infty} R(t)(\dot{R}(t))^2 = 0.
$$

We next change variables, via $|x| = r = Ry$, to transform integrals over $B_{R(t)}$ into integrals over B_1. Noting that $\overline{\rho}(y, t) = \rho(Ry, t)$, we get

$$\int_{B_{R(t)}} \frac{|\nabla_r \rho(|x|, t)|^2}{\rho(|x|, t)^2} \, dx = R(t) \int_{B_1} \frac{|\nabla_y \overline{\rho}(y, t)|^2}{\overline{\rho}(y, t)^2} \, dy.$$

Since that $R(t)$ is bounded away from zero and that $\overline{\rho}(y, t)$ is bounded from above (see (6.3), (7.13)) we have

$$\lim_{t \to \infty} \int_{B_1} |\nabla_y \overline{\rho}(y, t)|^2 \, dy = 0. \tag{8.8}$$

Using the interpolation Lemma D.1 with $k = 0$, $n = 3$, $p = 2$, $m = 1$, $\gamma = 2\alpha$, and $s = 2$,

$$\left\| \nabla_y \overline{\rho} \right\|_{L^\infty(B_1)} \leq C_1 \left\| \nabla_y \overline{\rho} \right\|_{L^2(B_1)}^{\frac{2}{5}} \left\| \nabla_y \overline{\rho} \right\|_{C_y^{1+2\alpha}(B_1)}^{\frac{3}{5}} + C_2 \left\| \nabla_y \overline{\rho} \right\|_{L^2(B_1)}$$

where $\left\| \nabla_y \overline{\rho} \right\|_{C_y^{1+2\alpha}(B_1)} \sim \|\rho\|_{C_r^{2+2\alpha}(B_R)}$ is uniformly bounded by (6.3). Therefore,

$$\nabla_y \overline{\rho}(\cdot, t) \to 0 \text{ uniformly in } B_1 \text{ as } t \to \infty.$$

and by (6.3)

$$\nabla_r \rho(\cdot, t) \to 0 \text{ uniformly in } B_{R(t)} \text{ as } t \to \infty. \tag{8.9}$$

Furthermore, by the Poincaré inequality and (8.8)

$$\int_{B_1} \left(\overline{\rho}(|y|, t) - \frac{1}{|B_1|} \int_{B_1} \overline{\rho}(|z|, t) \, dz \right)^2 dy \lesssim \int_{B_1} |\nabla_y \overline{\rho}|^2 \, dy \to 0 \text{ as } t \to \infty. \tag{8.10}$$

Moreover, by (8.3) (conservation of mass)

$$\frac{1}{|B_1|} \int_{B_1} \overline{\rho}(|z|, t) \, dz = \left(\frac{R_*}{R(t)} \right)^3 \rho_*. \tag{8.11}$$

and hence,

$$\int_{B_1} \left(\overline{\rho}(\cdot, t) - \left(\frac{R_*}{R(t)} \right)^3 \rho_* \right)^2 \to 0 \text{ as } t \to \infty. \tag{8.12}$$

In fact we claim that

$$\overline{\rho}(\cdot, t) - \left(\frac{R_*}{R(t)} \right)^3 \rho_* \to 0 \text{ uniformly on } B_1. \tag{8.13}$$

Indeed, applying interpolation Lemma D.1, with $k = 0$, $n = 3$, $p = 2$, $m = 2$, $\gamma = 2\alpha$, and $s = 2$, we have

$$
\left\| \overline{\rho}(\cdot, t) - \left(\frac{R_*}{R(t)} \right)^3 \rho_* \right\|_{L^\infty(B_1)}
$$

$$
\leq C_1 \left\| \overline{\rho}(\cdot, t) - \left(\frac{R_*}{R(t)} \right)^3 \rho_* \right\|_{L^2(B_1)}^{\frac{4}{7}} \left\| \overline{\rho}(\cdot, t) - \left(\frac{R_*}{R(t)} \right)^3 \rho_* \right\|_{C_y^{2+2\alpha}(B_1)}^{\frac{3}{7}}
$$

$$
+ C_2 \left\| \overline{\rho}(\cdot, t) - \left(\frac{R_*}{R(t)} \right)^3 \rho_* \right\|_{L^2(B_1)}.
$$

Since by (6.3), we have that $\left\| \overline{\rho}(\cdot, t) - \left(\frac{R_*}{R(t)} \right)^3 \rho_* \right\|_{C_y^{2+2\alpha}(B_1)} \sim$ $\left\| \rho(\cdot, t) - \left(\frac{R_*}{R(t)} \right)^3 \rho_* \right\|_{C_r^{2+2\alpha}(B_R)}$ is uniformly bounded, we have (8.13) and hence

$$
\overline{\rho}(\cdot, t)(R(t))^3 \to \rho_* R_*^3 \quad \text{uniformly in } B_1 \text{ as } t \to \infty. \tag{8.14}
$$

We next show that as $t \to \infty$, we have $\dot{R}$, $\ddot{R}$, and $\dddot{R} \to 0$.

Since $p(t) = \mathscr{R}_g T_\infty \overline{\rho}(y = 1, t)$ (using (3.2) and $T(r = 1, t) = T_\infty$), we have $p(t)(R(t))^3 \to \mathscr{R}_g T_\infty \rho_* R_*^3$ as $t \to \infty$. By the regularity of $p(t)$ and $R(t)$, (6.4) and (6.5), we have that $\frac{d}{dt}(p(t)(R(t))^3)$ is uniformly continuous. Recall Barbalat's lemma:

Suppose $f(t) \in C^1(a, \infty)$ and $\lim_{t \to \infty} f(t) = \alpha$, with $|\alpha| < \infty$.

If $f'(t)$ is uniformly continuous, then $\lim_{t \to \infty} f'(t) = 0$.

By the Barbalat's lemma, $\frac{d}{dt}(p(t)(R(t))^3) \to 0$ as $t \to \infty$. That is, $\partial_t p(t)(R(t))^3 + 3p(t)(R(t))^2 \dot{R}(t) \to 0$ as $t \to \infty$, and therefore since $\rho(\cdot, t)$ and $R(t)$ are uniformly bounded away from zero,

$$
\frac{\partial_t p(t)}{3p(t)} R(t) + \dot{R} \to 0 \quad \text{as } t \to \infty,
$$

Sending $t \to \infty$ in (5.1b) yields

$$
\lim_{t \to \infty} \dot{R}(t) = \lim_{t \to \infty} \left[-\frac{\kappa}{\gamma c_v} \frac{\partial_r \rho(R(t), t)}{(\rho(R(t), t))^2} - \frac{R}{3\gamma} \frac{\partial_t p}{p} \right]
$$

$$
= \lim_{t \to \infty} \left[-\frac{\kappa}{\gamma c_v} \frac{\partial_r \rho(R(t), t)}{(\rho(R(t), t))^2} - \frac{1}{\gamma} \left(\frac{R}{3} \frac{\partial_t p}{p} + \dot{R}(t) \right) + \frac{1}{\gamma} \dot{R}(t) \right]
$$

$$
= \gamma^{-1} \lim_{t \to \infty} \dot{R}(t).
$$

Hence, $(1 - \gamma^{-1}) \lim_{t \to \infty} \dot{R}(t) = 0$, and since $\gamma \neq 1$ we have $\lim_{t \to \infty} \dot{R}(t) = 0$. Further application of Barbalat's lemma yields $\ddot{R}(t)$, $\dddot{R}(t) \to 0$ as $t \to \infty$.

We next prove $R(t) \to R_*$ along a sequence of time $t_k \to \infty$. Since $R(t)$ is bounded in t, there is a sequence $\{t_k\}$ along which $R(t_k) \to R_{**}$ as $k \to \infty$ for some $R_{**} > 0$. Furthermore, by (8.13) $\overline{\rho}(y, t_k)$ converges uniformly on B_1 to a limit ρ_{**} as $k \to \infty$. By (5.1c), since $\dot{R}(t)$ and $\ddot{R}(t)$ tend to zero as $t \to \infty$, we obtain

$$\rho_{**} = \frac{1}{\mathscr{R}_g T_\infty} \left(p_{\infty,*} + \frac{2\sigma}{R_{**}} \right).$$

Further, passing to the limit in (8.3), we obtain

$$\rho_{**} R_{**}^3 = \rho_* R_*^3.$$

Hence, (ρ_{**}, R_{**}) satisfies the algebraic system (4.14) characterizing the unique spherically symmetric equilibrium of with bubble mass $(4\pi/3) R_*^3 \rho_*$. We conclude that $(\rho_{**}, R_{**}) = (\rho_*, R_*)$. Furthermore, $p(t_k) \to p_*$ as $k \to \infty$, where $p_* = \mathscr{R}_g T_\infty \rho_* = p_{\infty,*} + \frac{2\sigma}{R_*}$.

The above discussion establishes $(\overline{\rho}(y, t), R(t))$ converge to (ρ_*, R_*) along the sequence $\{t_k\}$. To prove this limit holds as $t \to \infty$ we return to the energy dissipation law. Since $\int_{B_{R(\tau)}} \frac{|\nabla \rho(|x|, \tau)|^2}{\rho(|x|, \tau)^2} \, dx$ and $R(\tau)(\dot{R}(\tau))^2$ are integrable on $[0, \infty)$ (see (8.7)) we have, from the identity

$$\mathcal{E}_{\text{total}}(t) = -\kappa T_\infty \int_0^t \int_{B_{R(\tau)}} \frac{|\nabla_r \rho(|x|, \tau)|^2}{\rho(|x|, \tau)^2} \, dx d\tau - 16\pi \mu_l \int_0^t R(\tau)(\dot{R}(\tau))^2 \, d\tau$$

$$+ \mathcal{E}_{\text{total}}(0), \tag{8.15}$$

which follows from rearranging terms in (8.5), that $\lim_{t \to \infty} \mathcal{E}_{\text{total}}(t)$ exists. Since $\mathcal{E}_{\text{total}}(t_k) \to \mathcal{E}_*$ as $k \to \infty$, we have $\mathcal{E}_{\text{total}}(t) \to \mathcal{E}_*$ as $t \to \infty$. By Theorem 7.5,

$$\lim_{t \to \infty} \int_{B_{R(t)}} (\rho(\cdot, t) - \rho_*)^2 \leq \Theta^{-1} \lim_{t \to \infty} (\mathcal{E}_{\text{total}}(t) - \mathcal{E}_*) = 0.$$

Using the interpolation Lemma D.1 with $k = 0$, $n = 3$, $p = 2$, $m = 2$, and $\gamma = 2\alpha$, we obtain

$$\|\rho(\cdot, t) - \rho_*\|_{L^\infty(B_{R(t)})} \leq C_1 \|\rho(\cdot, t) - \rho_*\|_{L^2(B_{R(t)})}^{\frac{4}{7}} \|\rho(\cdot, t) - \rho_*\|_{C_r^{2+2\alpha}(B_{R(t)})}^{\frac{3}{7}}$$

$$+ C_2 \|\rho(\cdot, t) - \rho_*\|_{L^2(B_{R(t)})} \to 0 \text{ as } t \to \infty. \tag{8.16}$$

Thus, $\rho(x, t) \to \rho_*$ uniformly on $B_{R(t)}$.

From the uniform convergence of $\rho(x, t)$ to ρ_*, we will finally conclude that $R(t) \to R_*$. By conservation of mass,

$$\int_{B_{R(t)}} (\rho(\cdot, t) - \rho_*) = \int_{B_{R_0}} \rho_0 - \frac{4\pi \rho_*}{3} R^3(t) = \frac{4\pi \rho_*}{3} (R_*^3 - R^3(t)), \tag{8.17}$$

which implies

$$|R(t) - R_*| \le \frac{3}{4\pi\rho_*(R^2(t) + R(t)R_* + R_*^2)}|B_{R(t)}|^{\frac{1}{2}}\left(\int_{B_{R(t)}}|\rho(\cdot,t) - \rho_*|^2\right)^{\frac{1}{2}}$$

$$\le C\left(\int_{B_{R(t)}}|\rho(\cdot,t) - \rho_*|^2\right)^{\frac{1}{2}} \tag{8.18}$$

It follows from (8.16) and (8.18) that $R(t) \to R_*$ as $t \to \infty$.

Finally, since we have $\|\rho(\cdot,t) - \rho_*\|_{L_r^\infty} + \|\nabla_r\rho(\cdot,t)\|_{L_r^\infty} + |R(t) - R_*| + |\dot{R}(t)| + |\ddot{R}(t)| + |\dddot{R}(t)| \to 0$ as $t \to +\infty$, the same bootstrap argument in the end of the proof of [6, Theorem 4.1] yields

$$\|\rho(\cdot,t) - \rho_*\|_{C_r^{2+2\alpha}} \to 0 \text{ as } t \to \infty,$$

completing the proof of Proposition 8.1. □

8.2. *Proof of Theorem 6.5; Asymptotic Stability of the Family of Spherically Symmetric Equilibria Relative to Arbitrary Small Perturbations*

We are in the position to prove Theorem 6.5. The proof is similar to that of Theorem 6.3. We now assume that $(\rho_0, R_0, \dot{R}_0)$ is an <u>arbitrary</u> sufficiently small perturbation of some fixed equilibrium $(\rho_*[M_*], R_*[M_*], \dot{R}_* = 0)$, in the sense of (6.8). By Proposition 4.7 there is a unique $(\rho_*[M_0], R_*[M_0])$ such that $\text{Mass}[\rho_*[M_0], R_*[M_0]] = M_0 = \text{Mass}[\rho_0, R_0]$, and

$$|R_*[M_0] - R_*[M_*]| + |\rho_*[M_0] - \rho_*[M_*]|$$
$$\le C\left(|R_0 - R_*[M_*]| + \|\rho_0 - \rho_*[M_*]\|_{L^\infty(B_{R_0})}\right) \tag{8.19}$$
$$\le C\left(|R_0 - R_*[M_*]| + \|\rho_0 - \rho_*[M_*]\|_{C_r^{2+2\alpha}(B_{R_0})}\right).$$

Hence,

$$\|\rho_0 - \rho_*[M_0]\|_{C_r^{2+2\alpha}} + |R_0 - R_*[M_0]|$$
$$= \|\rho_0 - \rho_*[M_*] + \rho_*[M_*] - \rho_*[M_0]\|_{C_r^{2+2\alpha}} + |R_0 - R_*[M_*] + R_*[M_*] - R_*[M_0]|$$
$$\le \|\rho_0 - \rho_*[M_*]\|_{C_r^{2+2\alpha}} + |\rho_*[M_*] - \rho_*[M_0]| + |R_0 - R_*[M_*]| + |R_*[M_*] - R_*[M_0]|$$
$$\le C'\left(\|\rho_0 - \rho_*[M_*]\|_{C_r^{2+2\alpha}} + |R_0 - R_*[M_*]|\right).$$

Therefore, choosing $\|\rho_0 - \rho_*[M_*]\|_{C_r^{2+2\alpha}} + |R_0 - R_*[M_*]|$ and $\dot{R}_0$ sufficiently small we conclude from Proposition 8.1 that $(\rho(\cdot,t), R, \dot{R})$ converges to $(\rho_*[M_0], R_*[M_0], \dot{R}_* = 0)$ in the norm $\|\cdot\|$ as $t \to \infty$. This completes the proof of Theorem 6.5.

9. Exponential Decay in Nonlinear Bubble Oscillations; $\mathcal{M}_*$, as an Attracting Center Manifold: Proof of the Main Result Theorem 6.7

In this section, we use a weak form of the asymptotic stability result in Theorem 6.5 to promote the result of exponential rate of convergence toward the manifold of equilibria, Theorem 6.7. We carry out the analysis in line with center manifold theorem for a dynamical system in the phase space ℓ^2. To this end, we first convert the free boundary problem (5.1)–(5.2) into an infinite-dimensional dynamical system, (9.5), in Proposition 9.1. Next, we interpret the decay result from Theorem 6.5 as an *a priori* estimate in Proposition 9.2. We then discuss the spectrum of the linear operator in the dynamical system (9.5) and find that there is a neutral mode and decaying modes (Proposition 9.3). In light of this, we decompose the solution of the system a slow decaying part (9.32a) and a fast decaying part (9.32a). Moreover, we show that the manifold of equilibria is a center manifold of the system (9.5) in Lemma 9.6. We derive an estimate of the nonlinearity in Proposition 9.7. With the *a priori* estimate and the nonlinear estimate in hand, we apply Proposition F.2 to conclude Theorem 6.7.

9.1. A Dynamical System Formulation of the Free Boundary Problem (5.1)–(5.2)

To investigate the solution near a given equilibrium solution, we derive two equivalent systems of (5.1)–(5.2) below.

Proposition 9.1. *Given $(\rho_*, R_*, \dot{R}_* = 0) \in \mathcal{M}_*$. Let (ρ, R) denote a solution of the free boundary problem (5.1)–(5.2) with $p_\infty(t) \equiv p_{\infty,*}$. Decompose*

$$\rho(R(t)y, t) = \rho_* + u(y, t) + z(t), \quad z(t) = \rho(R(t), t) - \rho_*, \quad R(t) = R_* + \mathcal{R}(t). \tag{9.1}$$

Then,

(1) Equations (5.1)–(5.2) are equivalent to the following system for $(u, z, \mathcal{R})$ with zero-Dirichlet boundary condition

$$\partial_t u = \overline{\kappa} \Delta_y u - \left(1 - \frac{1}{\gamma}\right) \dot{z} + F, \quad 0 \leq y \leq 1, \quad u(1, t) = 0, \quad t > 0, \tag{9.2a}$$

$$\dot{\mathcal{R}} = -\frac{R_* \overline{\kappa}}{\rho_*} \partial_y u(1, t) - \frac{R_*}{3\gamma \rho_*} \dot{z} + G, \quad t > 0, \tag{9.2b}$$

$$z(t) = \frac{1}{\mathscr{R}_g T_\infty} \left(-\frac{2\sigma}{R_*^2} \mathcal{R} + \frac{4\mu_l}{R_*} \dot{\mathcal{R}} + \rho_l R_* \ddot{\mathcal{R}}\right) + H, \quad t > 0, \tag{9.2c}$$

where

$$\overline{\kappa} = \frac{\kappa}{\gamma c_v R_*^2 \rho_*}, \tag{9.3}$$

$$F = \frac{\kappa}{\gamma c_v} \left[\frac{1}{(R_* + \mathcal{R})^2 (\rho_* + u + z)} - \frac{1}{R_*^2 \rho_*} \right] \Delta_y u \tag{9.4a}$$
$$- \frac{\kappa}{\gamma c_v} \frac{|\nabla_y u|^2}{(R_* + \mathcal{R})^2 (\rho_* + u + z)^2} + \frac{1}{\gamma} \frac{\dot{z}}{\rho_* + z} \left(\frac{1}{3} y \partial_y u + u \right),$$

$$G = -\frac{\kappa}{\gamma c_v} \left[\frac{1}{(R_* + \mathcal{R})(\rho_* + z)^2} - \frac{1}{R_* \rho_*^2} \right] \partial_y u(1, t) - \frac{\mathcal{R}\dot{z}}{3\gamma(\rho_* + z)}$$
$$+ \frac{R_*}{3\gamma} \frac{z\dot{z}}{\rho_*(\rho_* + z)}, \tag{9.4b}$$

$$H = \frac{1}{\mathscr{R}_g T_\infty} \left[-\frac{\mathcal{R}}{R_*(R_* + \mathcal{R})} \left(-\frac{2\sigma}{R_*} \mathcal{R} + 4\mu_l \dot{\mathcal{R}} \right) + \rho_l \left(\mathcal{R}\ddot{\mathcal{R}} + \frac{3}{2} \dot{\mathcal{R}}^2 \right) \right]. \tag{9.4c}$$

(2) Let $c_j = c_j(t)$ denote the jth coefficients in the radial-Dirichlet-eigenfunction decomposition of u as in (9.6).

Then, (9.2) is further equivalent to the following infinite-dimensional dynamical system for $\mathbf{w} = (z, \mathcal{R}, \dot{\mathcal{R}}, c_1, c_2, \cdots)^\top$

$$\dot{\mathbf{w}} = \mathcal{L}\mathbf{w} + \mathcal{N}(\mathbf{w}, \dot{\mathbf{w}}) = \mathcal{L}\mathbf{w} + \mathcal{N}^1(\mathbf{w})\dot{\mathbf{w}} + \mathcal{N}^0(\mathbf{w}), \tag{9.5}$$

where $\mathcal{L}$ is a linear operator given in (9.13), $\mathcal{N}(\mathbf{w}, \dot{\mathbf{w}})$ is defined in (9.14), and $\mathcal{N}^1(\mathbf{w})$ and $\mathcal{N}^0(\mathbf{w})$ are given in (9.18).

Below, we shall study the operator $\mathcal{L}$ acting in ℓ^2 with dense domain consisting of vectors $\mathbf{w} = (z, \mathcal{R}, \dot{\mathcal{R}}, c_1, c_2, \cdots)^\top \in \ell^2$, such that $\sum_{j \geq 1} j^4 |c_j|^2 < \infty$.

Proof of Proposition 9.1. To begin with, plugging (9.1) into the equation (5.1)–(5.2) and grouping linear and nonlinear terms, one directly obtain (9.2).

Expand u in terms of the radial-Dirichlet eigenfunctions as

$$u(y, t) = \sum_{j=1}^\infty c_j(t) \phi_j(y), \tag{9.6}$$

where ϕ_j, $j = 1, 2, \ldots$, are defined in (1.2). Plugging (9.6) into (9.2a),

$$\sum_{j=1}^\infty \dot{c}_j(t) \phi_j(y) = -\overline{\kappa} \sum_{j=1}^\infty \lambda_j c_j(t) \phi_j(y) - \frac{\gamma - 1}{\gamma} \dot{z}(t) + F. \tag{9.7}$$

Taking inner product of (9.7) in $L^2(B_1)$ with $\phi_k(y)$, $k = 1, 2, \ldots$, one has

$$\dot{c}_k(t) = -\overline{\kappa} \lambda_k c_k(t) - \Gamma_k \dot{z}(t) + F_k, \tag{9.8}$$

where

$$\Gamma_k = \frac{\gamma - 1}{\gamma} \int_{B_1} \phi_k(|x|) \, dx = \frac{2\sqrt{2}(\gamma - 1)}{\sqrt{\pi} \gamma} \frac{(-1)^{k-1}}{k}, \tag{9.9}$$

and $F_k = \int_{B_1} F\phi_k \, dx$.

Since ϕ_j is explicitly given in (1.2), it is readily to compute $\partial_y \phi_j(1) = \sqrt{\frac{\pi}{2}} (-1)^j j$. Then (9.2b) becomes

$$\dot{\mathcal{R}} = -\frac{R_* \overline{\kappa}}{\rho_*} \sum_{j=1}^{\infty} c_j(t) \partial_y \phi_j(1) - \frac{R_*}{3\gamma\rho_*} \dot{z} + G = \sum_{j=1}^{\infty} c_j(t) \Lambda_j - \frac{R_*}{3\gamma\rho_*} \dot{z} + G,$$

$$(9.10)$$

where

$$\Lambda_j = -\frac{R_* \overline{\kappa}}{\rho_*} \sqrt{\frac{\pi}{2}} (-1)^j j. \tag{9.11}$$

Thus, (9.10), (9.2c), and (9.8) form the infinite-dimensional dynamical system

$$
\begin{pmatrix} \frac{R_*}{3\gamma\rho_*} & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ \Gamma_1 & 0 & 0 & 1 & 0 & \cdots \\ \Gamma_2 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}
\begin{bmatrix} z \\ \mathcal{R} \\ \dot{\mathcal{R}} \\ c_1 \\ c_2 \\ \vdots \end{bmatrix}'
=
\begin{pmatrix} 0 & 0 & 0 & \Lambda_1 & \Lambda_2 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ \frac{\mathscr{R}_g T_\infty}{\rho_l R_*} & \frac{2\sigma}{\rho_l R_*^3} & -\frac{4\mu_l}{\rho_l R_*^2} & 0 & 0 & \cdots \\ 0 & 0 & 0 & -\overline{\kappa}\lambda_1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & -\overline{\kappa}\lambda_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}
\begin{bmatrix} z \\ \mathcal{R} \\ \dot{\mathcal{R}} \\ c_1 \\ c_2 \\ \vdots \end{bmatrix}
$$

$$
+
\begin{bmatrix} G \\ 0 \\ -\frac{\mathscr{R}_g T_\infty}{\rho_l R_*} H \\ F_1 \\ F_2 \\ \vdots \end{bmatrix},
$$

where Γ_k and Λ_j are given in (9.9) and (9.11), respectively, $F_k = \int_{B_1} F\phi_k \, dx$, and $F = F(\mathbf{w}, \dot{\mathbf{w}})$, $G = G(\mathbf{w}, \dot{\mathbf{w}})$, $H = H(\mathbf{w}, \dot{\mathbf{w}})$ are defined in (9.4a)–(9.4a) with $u = \sum_{k=1}^{\infty} c_k \phi_k$. The inverse of the matrix on the left hand side above is

$$
\begin{pmatrix} \frac{R_*}{3\gamma\rho_*} & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ \Gamma_1 & 0 & 0 & 1 & 0 & \cdots \\ \Gamma_2 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^{-1}
=
\begin{pmatrix} \frac{3\gamma\rho_*}{R_*} & -\frac{3\gamma\rho_*}{R_*} & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ -\Gamma_1 \frac{3\gamma\rho_*}{R_*} & \Gamma_1 \frac{3\gamma\rho_*}{R_*} & 0 & 1 & 0 & \cdots \\ -\Gamma_2 \frac{3\gamma\rho_*}{R_*} & \Gamma_2 \frac{3\gamma\rho_*}{R_*} & 0 & 0 & 1 & \cdots \\ \vdots & & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.
$$

Left-multiplying the inverse on both sides, we obtain

$$
\begin{bmatrix} z \\ \mathcal{R} \\ \dot{\mathcal{R}} \\ c_1 \\ c_2 \\ \vdots \end{bmatrix}' =
\begin{pmatrix}
0 & 0 & -\frac{3\gamma\rho_*}{R_*} & \Lambda_1\frac{3\gamma\rho_*}{R_*} & \Lambda_2\frac{3\gamma\rho_*}{R_*} & \cdots \\
0 & 0 & 1 & 0 & 0 & \cdots \\
\frac{\mathscr{R}_g T_\infty}{\rho_l R_*} & \frac{2\sigma}{\rho_l R_*^3} & -\frac{4\mu_l}{\rho_l R_*^2} & 0 & 0 & \cdots \\
0 & 0 & \Gamma_1\frac{3\gamma\rho_*}{R_*} & -\Gamma_1\Lambda_1\frac{3\gamma\rho_*}{R_*}-\overline{\kappa}\lambda_1 & -\Gamma_1\Lambda_2\frac{3\gamma\rho_*}{R_*} & \cdots \\
0 & 0 & \Gamma_2\frac{3\gamma\rho_*}{R_*} & -\Gamma_2\Lambda_1\frac{3\gamma\rho_*}{R_*} & -\Gamma_2\Lambda_2\frac{3\gamma\rho_*}{R_*}-\overline{\kappa}\lambda_2 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddot{s}
\end{pmatrix}
\begin{bmatrix} z \\ \mathcal{R} \\ \dot{\mathcal{R}} \\ c_1 \\ c_2 \\ \vdots \end{bmatrix}
$$

$$
+ \begin{bmatrix}
\frac{3\gamma\rho_*}{R_*}G \\
0 \\
-\frac{\mathscr{R}_g T_\infty}{\rho_l R_*}H \\
-\Gamma_1\frac{3\gamma\rho_*}{R_*}G + F_1 \\
-\Gamma_2\frac{3\gamma\rho_*}{R_*}G + F_2 \\
\vdots
\end{bmatrix}.
$$

It can be written in the form

$$
\dot{\mathbf{w}} = \mathcal{L}\mathbf{w} + \mathcal{N}(\mathbf{w}, \dot{\mathbf{w}}), \tag{9.12}
$$

where $\mathbf{w} = (z, \mathcal{R}, \dot{\mathcal{R}}, c_1, c_2, \dots)^\top$,

$$
\mathcal{L} =
\begin{pmatrix}
0 & 0 & -\frac{3\gamma\rho_*}{R_*} & \Lambda_1\frac{3\gamma\rho_*}{R_*} & \Lambda_2\frac{3\gamma\rho_*}{R_*} & \cdots \\
0 & 0 & 1 & 0 & 0 & \cdots \\
\frac{\mathscr{R}_g T_\infty}{\rho_l R_*} & \frac{2\sigma}{\rho_l R_*^3} & -\frac{4\mu_l}{\rho_l R_*^2} & 0 & 0 & \cdots \\
0 & 0 & \Gamma_1\frac{3\gamma\rho_*}{R_*} & -\Gamma_1\Lambda_1\frac{3\gamma\rho_*}{R_*}-\overline{\kappa}\lambda_1 & -\Gamma_1\Lambda_2\frac{3\gamma\rho_*}{R_*} & \cdots \\
0 & 0 & \Gamma_2\frac{3\gamma\rho_*}{R_*} & -\Gamma_2\Lambda_1\frac{3\gamma\rho_*}{R_*} & -\Gamma_2\Lambda_2\frac{3\gamma\rho_*}{R_*}-\overline{\kappa}\lambda_2 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddot{s}
\end{pmatrix}, \tag{9.13}
$$

$$
\mathcal{N}(\mathbf{w}, \dot{\mathbf{w}}) =
\begin{bmatrix}
\frac{3\gamma\rho_*}{R_*}G \\
0 \\
-\frac{\mathscr{R}_g T_\infty}{\rho_l R_*}H \\
-\Gamma_1\frac{3\gamma\rho_*}{R_*}G + F_1 \\
-\Gamma_2\frac{3\gamma\rho_*}{R_*}G + F_2 \\
\vdots
\end{bmatrix}. \tag{9.14}
$$

Write $\mathcal{N}(\mathbf{w}, \dot{\mathbf{w}}) = \mathcal{N}(\mathbf{w}, \mathbf{p})$, in which

$$
\mathbf{p} = \dot{\mathbf{w}} = (\dot{z}, \dot{\mathcal{R}}, \ddot{\mathcal{R}}, \dot{c}_1, \dot{c}_2, \dots)^\top
$$
$$
=: (a, \mathcal{S}, \mathcal{U}, d_1, d_2, \dots)^\top.
$$

Then

$$
F(\mathbf{w}, \mathbf{p}) = \frac{\kappa}{\gamma c_v} \left[\frac{1}{(R_* + \mathcal{R})^2 (\rho_* + u + z)} - \frac{1}{R_*^2 \rho_*} \right] \sum_{j=1}^{\infty} (-\lambda_j) c_j \phi_j
$$
$$
- \frac{\kappa}{\gamma c_v} \frac{|\nabla_y u|^2}{(R_* + \mathcal{R})^2 (\rho_* + u + z)^2}
$$
$$
+ \frac{1}{\gamma} \frac{a}{\rho_* + z} \left(\frac{1}{3} y \sum_{j=1}^{\infty} c_j \partial_y \phi_j + \sum_{j=1}^{\infty} c_j \phi_j \right), \tag{9.15a}
$$

where $u = \sum_{j=1}^{\infty} c_j \phi_j$

$$
G(\mathbf{w}, \mathbf{p}) = -\frac{\kappa}{\gamma c_v} \left[\frac{1}{(R_* + \mathcal{R})(\rho_* + z)^2} - \frac{1}{R_* \rho_*^2} \right]
$$
$$
\sum_{j=1}^{\infty} \sqrt{\frac{\pi}{2}} (-1)^j j c_j - \frac{\mathcal{R} a}{3\gamma (\rho_* + z)}
$$
$$
+ \frac{R_*}{3\gamma} \frac{za}{\rho_* (\rho_* + z)}, \tag{9.15b}
$$
$$
H(\mathbf{w}, \mathbf{p}) = \frac{1}{\mathscr{R}_g T_\infty} \left[-\frac{\mathcal{R}}{R_* (R_* + \mathcal{R})} \left(-\frac{2\sigma}{R_*} \mathcal{R} + 4\mu_l \dot{\mathcal{R}} \right) + \rho_l \left(\mathcal{R} \mathcal{U} + \frac{3}{2} \dot{\mathcal{R}}^2 \right) \right]. \tag{9.15c}
$$

It is easily see from above that

$$
F(\mathbf{w}, \mathbf{p}) = \langle \mathbf{F}^1(\mathbf{w}), \mathbf{p} \rangle + F^0(\mathbf{w}), \quad G(\mathbf{w}, \mathbf{p}) = \langle \mathbf{G}^1(\mathbf{w}), \mathbf{p} \rangle + G^0(\mathbf{w}),
$$
$$
H(\mathbf{w}, \mathbf{p}) = \langle \mathbf{H}^1(\mathbf{w}), \mathbf{p} \rangle + H^0(\mathbf{w}),
$$

where $\langle \cdot, \cdot \rangle$ denotes inner product in the Hilbert space ℓ^2 and

$$
\mathbf{F}^1(\mathbf{w}) = \left(\frac{1}{\gamma} \frac{1}{\rho_* + z} \left(\frac{1}{3} y \sum_{j=1}^{\infty} c_j \partial_y \phi_j + \sum_{j=1}^{\infty} c_j \phi_j \right), 0, 0, 0, 0, \cdots \right)^\top,
$$
$$
F^0(\mathbf{w}) = \frac{\kappa}{\gamma c_v} \left[\frac{1}{(R_* + \mathcal{R})^2 (\rho_* + u + z)} - \frac{1}{R_*^2 \rho_*} \right] \sum_{j=1}^{\infty} (-\lambda_j) c_j \phi_j
$$
$$
- \frac{\kappa}{\gamma c_v} \frac{|\nabla_y u|^2}{(R_* + \mathcal{R})^2 (\rho_* + u + z)^2},
$$
$$
\mathbf{G}^1(\mathbf{w}) = \left(-\frac{\mathcal{R}}{3\gamma (\rho_* + z)} + \frac{R_* z}{3\gamma \rho_* (\rho_* + z)}, 0, 0, 0, 0, \cdots \right)^\top,
$$
$$
G^0(\mathbf{w}) = -\frac{\kappa}{\gamma c_v} \left[\frac{1}{(R_* + \mathcal{R})(\rho_* + z)^2} - \frac{1}{R_* \rho_*^2} \right] \sum_{j=1}^{\infty} \sqrt{\frac{\pi}{2}} (-1)^j j c_j,
$$

$$\mathbf{H}^1(\mathbf{w}) = \left(0, 0, \frac{\rho_l \mathcal{R}}{\mathscr{R}_g T_\infty}, 0, 0, \cdots\right)^\top,$$

$$H^0(\mathbf{w}) = \frac{1}{\mathscr{R}_g T_\infty}\left[-\frac{\mathcal{R}}{R_*(R_* + \mathcal{R})}\left(-\frac{2\sigma}{R_*}\mathcal{R} + 4\mu_l \dot{\mathcal{R}}\right) + \rho_l \frac{3}{2}\dot{\mathcal{R}}^2\right]. \quad (9.16)$$

Therefore, the nonlinearity $\mathcal{N}(\mathbf{w}, \mathbf{p})$ takes the form

$$\mathcal{N}(\mathbf{w}, \mathbf{p}) = \mathcal{N}^1(\mathbf{w})\mathbf{p} + \mathcal{N}^0(\mathbf{w}), \quad (9.17)$$

where

$$\mathcal{N}^1(\mathbf{w}) = \begin{pmatrix} \frac{3\gamma\rho_*}{R_*}\mathbf{G}^1(\mathbf{w}) \\ \mathbf{0} \\ -\frac{\mathscr{R}_g T_\infty}{\rho_l R_*}\mathbf{H}^1(\mathbf{w}) \\ -\Gamma_1 \frac{3\gamma\rho_*}{R_*}\mathbf{G}^1(\mathbf{w}) + \mathbf{F}_1^1(\mathbf{w}) \\ -\Gamma_2 \frac{3\gamma\rho_*}{R_*}\mathbf{G}^1(\mathbf{w}) + \mathbf{F}_2^1(\mathbf{w}) \\ \vdots \end{pmatrix}, \quad \mathcal{N}^0(\mathbf{w}) = \begin{bmatrix} \frac{3\gamma\rho_*}{R_*}G^0(\mathbf{w}) \\ 0 \\ -\frac{\mathscr{R}_g T_\infty}{\rho_l R_*}H^0(\mathbf{w}) \\ -\Gamma_1 \frac{3\gamma\rho_*}{R_*}G^0(\mathbf{w}) + F_1^0(\mathbf{w}) \\ -\Gamma_2 \frac{3\gamma\rho_*}{R_*}G^0(\mathbf{w}) + F_2^0(\mathbf{w}) \\ \vdots \end{bmatrix}$$

$$(9.18)$$

in which $F_j^0(\mathbf{w}) = \int_{B_1} F^0(\mathbf{w})\phi_j \, dx$ and

$$\mathbf{F}_j^1(\mathbf{w}) = \int_{B_1} \mathbf{F}^1(\mathbf{w})\phi_j \, dx$$

$$= \left(\frac{1}{\gamma}\frac{1}{\rho_* + z}\int_{B_1}\left(\frac{1}{3}y\sum_{k=1}^\infty c_k\partial_y\phi_k + \sum_{k=1}^\infty c_k\phi_k\right)\phi_j \, dx, 0, 0, 0, 0, \cdots\right)^\top.$$

Therefore, (9.12) is of the form (9.5). This completes the proof of Proposition 9.1.
□

Proposition 9.2. *Given* $(\rho_*, R_*, \dot{R}_* = 0) \in \mathcal{M}_*$. *Consider* $(\rho_0, R_0, \dot{R}_0) \in C_r^{2+2\alpha}$ $(B_{R_0}) \times \mathbb{R}_+ \times \mathbb{R}$.

(1) Let $\mathbf{w}(0) = (z(0), \mathcal{R}(0), \dot{\mathcal{R}}(0), c_1(0), c_2(0), \ldots)^\top$, *where* $z(0) = \rho_0(R_0) - \rho_*$, $\mathcal{R}(0) = R_0 - R_*$, *and* $c_k(0) = \int_{B_1}(\rho_0(R_0 y) - \rho_0(R_0) - \rho_*)\phi_k(y)dy$, ϕ_k *is defined in (1.2), as in (9.1).*
Then,

$$\mathbf{w}(0) = (z(0), \mathcal{R}(0), \dot{\mathcal{R}}(0), c_1(0), c_2(0), \ldots)^\top \in \ell^2, \quad \{j^2 c_j(0)\}_{j=1}^\infty \in \ell^2.$$

(2) Suppose $\left|\left(\rho_0 - \rho_*, R_0 - R_*, \dot{R}_0 - \dot{R}_*\right)\right|$ *is sufficiently small. Let* $(\rho, R) \in$ $C_{r,t}^{2+2\alpha, 1+\alpha}(B_{R(t)} \times [0, \infty)) \times C_t^{3+\alpha}$ *be the solution of (5.1)–(5.2) with the initial data* $(\rho_0, R_0, \dot{R}_0)$, *obtained in Theorem 6.5, such that* $\left|\left(\rho(\cdot, t) - \rho_{**}, R(t) - R_{**}, \dot{R}(t)\right)\right| + |\ddot{R}(t)| + |\dddot{R}(t)| \to 0$ *as* $t \to \infty$ *for some* $(\rho_{**}, R_{**}, 0) \in \mathcal{M}_*$. *Let* $\mathbf{w}$ *be the corresponding solution of the infinite-dimensional dynamical system (9.5) in Proposition 9.1.*
Then, we have the a priori bounds for $\mathbf{w}$: $\{j^2 c_j\}_{j=1}^\infty \in \ell^2$, *and, as* $t \to \infty$,
$\dot{\mathbf{w}}(t) \to \mathbf{0}$ *and* $\mathbf{w}(t) \to \mathbf{w}_*$ *in* ℓ^2, *where* $\mathbf{w}_* = (\rho_{**} - \rho_*, R_{**} - R_*, 0, 0, 0, \ldots)^\top.$

Proof. First of all, setting $u_0(y) = \rho_0(R_0 y) - \rho_0(R_0) - \rho_*$, we have $u_0(y) \in C_y^{2+2\alpha}(B_1) \subset \{u \in L^2(B_1) : \Delta u \in L^2(B_1)\}$, $y = R_0^{-1}r$. Thus,

$$\infty > \int_{B_1} (-\Delta u_0)^2 \, dx = \sum_{j,k=1}^{\infty} c_j(0)\lambda_j c_k(0)\lambda_k(0) \int_{B_1} \phi_j \phi_k \, dx = \sum_{j=1}^{\infty} \lambda_j^2 (c_j(0))^2$$

$$= \pi^4 \sum_{j=1}^{\infty} j^4 (c_j(0))^2. \tag{9.19}$$

So $\{j^2 c_j(0)\}_{j=1}^{\infty} \in \ell^2$. This proves Part (1) of the proposition.

For Part (1), by Proposition 9.1 (5.1)–(5.2) is equivalent to (9.2). Let $(u, z, \mathcal{R})$ be the corresponding solution of (9.2). Since $u(\cdot, t) \in C_y^{2+2\alpha} \subset \{u \in L^2 : \Delta u \in L^2\}$, from the same argument as in (9.19) one has $\{j^2 c_j\}_{j=1}^{\infty} \in \ell^2$. From the convergence of $(\rho, R, \dot{R})$ to $(\rho_*, R_*, 0)$, it is obvious that $\mathbf{w}(t) \to \mathbf{w}_*$ in ℓ^2 as $t \to \infty$. It remains to prove the convergence $\dot{\mathbf{w}}(t) \to \mathbf{0}$, as $t \to \infty$. It follows from the equations (5.1a) and (5.1b) in the original system that, as $t \to \infty$, $\partial_t \rho \to 0$ or, equivalently, $|\partial_t u| + |\dot{z}| \to 0$. Thus, $\dot{\mathbf{w}}(t) \to \mathbf{0}$ in ℓ^2 as $t \to \infty$. This proves Proposition 9.2.

$\qquad \square$

9.2. Spectral Analysis of the Linear Operator

We now study the spectrum of the linear operator $\mathcal{L}$, defined in (9.13), acting in the space ℓ^2. Recall the definition of Laplace transform and its inverse in (1.1). Taking Laplace transform of the linear system $\dot{\mathbf{w}} = \mathcal{L}\mathbf{w}$, one derive $(\mathcal{L} - \tau I)\widetilde{\mathbf{w}}(\tau) = -\mathbf{w}(0)$, where I is the identity operator. Denote the spectrum of $\mathcal{L}$ by

$$\sigma(\mathcal{L}) = \{\tau \in \mathbb{C} : \mathcal{L} - \tau I \text{ is non-bijective on } \ell^2\}.$$

Then, formally $\sigma(\mathcal{L})$ consists of all the poles of $\widetilde{\mathbf{w}}(\tau)$.

Proposition 9.3. *Let $\mathcal{L}$ the the linear operator defined in (9.13). Then*

(1)
$$\sigma(\mathcal{L}) = \{0\} \cup \{\tau \in \mathbb{C} : Q(\tau) = 0\},$$

where $\tau = 0$ has multiplicity one, and $Q(\tau)$ is a meromorphic function defined by

$$Q(\tau) = \frac{1}{\mathscr{R}_g T_\infty} \left(\frac{4\pi}{3\gamma} + \frac{8(\gamma - 1)}{\pi\gamma} \sum_{j=1}^{\infty} \frac{\pi^2 \overline{\kappa}}{\pi^2 \overline{\kappa} j^2 + \tau} \right) \left(\rho_l R_* \tau^2 + \frac{4\mu_l}{R_*}\tau - \frac{2\sigma}{R_*^2} \right)$$

$$+ 4\pi \frac{\rho_*}{R_*}. \tag{9.20}$$

(2) There exists $\beta > 0$ such that if $\tau \neq 0$ is in $\sigma(\mathcal{L})$, the spectrum of $\mathcal{L}$, then $\mathrm{Re}(\tau) \leq -\beta < 0$. Moreover, there exists a constant $C = C(\beta)$ such that $\left\| (\mathcal{L} - \tau I)^{-1} \right\|_{\ell^2 \to \ell^2} \leq C(\beta)$ for all $\tau \neq 0$ with $\mathrm{Re}(\tau) > -\beta$.

(3) A negative upper bound for $-\beta$ in terms of physical parameters is displayed in (E.1) of Appendix D.

Proof. In the same spirit of Proposition 9.1, the linear system $\dot{\mathbf{w}} = \mathcal{L}\mathbf{w}$ is equivalent to the following linear version of (9.2):

$$\partial_t u = \overline{\kappa}\Delta_y u - \left(1 - \frac{1}{\gamma}\right)\dot{z}, \quad 0 \le y \le 1, \qquad u(1,t) = 0, \quad t > 0, \quad (9.21a)$$

$$\dot{\mathcal{R}} = -\frac{\overline{\kappa}R_*}{\rho_*}\partial_y u(1,t) - \frac{R_*}{3\gamma\rho_*}\dot{z}, \quad t > 0. \quad (9.21b)$$

$$z(t) = \frac{1}{\mathscr{R}_g T_\infty}\left(-\frac{2\sigma}{R_*^2}\mathcal{R} + \frac{4\mu_l}{R_*}\dot{\mathcal{R}} + \rho_l R_*\ddot{\mathcal{R}}\right), \quad t > 0. \quad (9.21c)$$

Similar to the proof in Proposition 9.1, using the eigenfunction decomposition (9.6) in (9.21a) and testing the equation against ϕ_k, we have

$$\dot{c}_k(t) = -\overline{\kappa}\lambda_k c_k(t) - \Gamma_k \dot{z}(t). \quad (9.22)$$

Taking Laplace transform of (9.22), we have

$$-c_k(0) + \tau\widetilde{c}_k(\tau) = -\overline{\kappa}\lambda_k\widetilde{c}_k(\tau) - \Gamma_k\left(-z(0) + \tau\widetilde{z}(\tau)\right),$$

or

$$\widetilde{c}_k(\tau) = \frac{c_k(0) + \Gamma_k z(0)}{\overline{\kappa}\lambda_k + \tau} - \frac{\Gamma_k \tau}{\overline{\kappa}\lambda_k + \tau}\widetilde{z}(\tau). \quad (9.23)$$

Using the eigenfunction decomposition (9.6) in (9.21b), we get

$$\dot{\mathcal{R}} = \sum_{j=1}^{\infty} c_j(t)\Lambda_j - \frac{R_*}{3\gamma\rho_*}\dot{z}, \quad (9.24)$$

where Λ_j is defined in (9.11). Taking Laplace transform of (9.24), we deduce

$$-\mathcal{R}(0) + \tau\widetilde{\mathcal{R}}(\tau) = \sum_{j=1}^{\infty}\widetilde{c}_j(\tau)\Lambda_j - \frac{R_*}{3\gamma\rho_*}(-z(0) + \tau\widetilde{z}(\tau)). \quad (9.25)$$

Taking Laplace transform of (9.21c), we derive

$$\mathscr{R}_g T_\infty\widetilde{z}(\tau) = -\frac{2\sigma}{R_*^2}\widetilde{R}(\tau) + \frac{4\mu_l}{R_*}\left(-\mathcal{R}(0) + \tau\widetilde{R}(\tau)\right)$$

$$+ \rho_l R_*\left(-\dot{\mathcal{R}}(0) - \tau\mathcal{R}(0) + \tau^2\widetilde{R}(\tau)\right),$$

or

$$\left(\rho_l R_*\tau^2 + \frac{4\mu_l}{R_*}\tau - \frac{2\sigma}{R_*^2}\right)\widetilde{\mathcal{R}}(\tau) - \mathscr{R}_g T_\infty\widetilde{z}(\tau) = \rho_l R_*\left(\dot{\mathcal{R}}(0) + \tau\mathcal{R}(0)\right). \quad (9.26)$$

Replacing the $\widetilde{c}_k$'s and $\widetilde{z}$ in (9.25) using (9.23) and (9.26), multiplying the equation by $4\pi \frac{\rho_*}{R_*}$, using the identity $\Gamma_j \Lambda_j = \frac{2(\gamma-1)\overline{\kappa} R_*}{\gamma \rho_*}$, we obtain

$$\tau Q(\tau)\widetilde{\mathcal{R}}(\tau) = \text{DATA}(\tau), \tag{9.27}$$

where $Q(\tau)$ is as in (9.20), and $\text{DATA}(\tau)$ is analytic for all $\tau \in \mathbb{C}$ with $\tau \neq -\overline{\kappa}\lambda_j = -\pi^2\overline{\kappa} j^2$, $j = 1, 2, \ldots$, and is defined as

$$\text{DATA}(\tau) = 4\pi \frac{\rho_*}{R_*}\left[\frac{\rho_l R_* \tau}{\mathscr{R}_g T_\infty}\left(\sum_{j=1}^\infty \frac{\Gamma_j \Lambda_j}{\overline{\kappa}\lambda_j + \tau} + \frac{R_*}{3\gamma \rho_*}\right)\left(\dot{\mathcal{R}}(0) + \tau \mathcal{R}(0)\right)\right.$$

$$\left. + \sum_{j=1}^\infty \frac{c_j(0) + \Gamma_j z(0)}{\overline{\kappa}\lambda_j + \tau} + \frac{R_*}{3\gamma \rho_*}z(0) + \mathcal{R}(0)\right].$$

It then follows from (9.27), (9.26), and (9.23), that

$$\widetilde{z}(\tau) = \frac{1}{\mathscr{R}_g T_\infty}\left[\frac{\rho_l R_* \tau^2 + \frac{4\mu_l}{R_*}\tau - \frac{2\sigma}{R_*^2}}{\tau Q(\tau)}\text{DATA}(\tau) - \rho_l R_*\left(\dot{\mathcal{R}}(0) + \tau \mathcal{R}(0)\right)\right],$$

$$\widetilde{\mathcal{R}}(\tau) = \frac{1}{\tau Q(\tau)}\text{DATA}(\tau), \qquad \widetilde{\dot{\mathcal{R}}}(\tau) = -\mathcal{R}(0) + \frac{1}{Q(\tau)}\text{DATA}(\tau),$$

$$\widetilde{c}_k(\tau) = \frac{c_k(0) + \Gamma_k z(0)}{\overline{\kappa}\lambda_k + \tau}$$

$$- \frac{\Gamma_k \tau}{\overline{\kappa}\lambda_k + \tau}\frac{1}{\mathscr{R}_g T_\infty}\left[\frac{\rho_l R_* \tau^2 + \frac{4\mu_l}{R_*}\tau - \frac{2\sigma}{R_*^2}}{\tau Q(\tau)}\text{DATA}(\tau) - \rho_l R_*\left(\dot{\mathcal{R}}(0) + \tau \mathcal{R}(0)\right)\right],$$

which amounts to $\widetilde{\mathbf{w}}(\tau) = (\mathcal{L} - \tau I)^{-1}\widetilde{\mathbf{w}}(0)$ for all $\tau \in \mathbb{C}$ with $\tau \neq -\pi^2\overline{\kappa} j^2$, $j = 1, 2, \ldots$, and $\tau Q(\tau) \neq 0$. By Lemma E.1, there exists $\beta > 0$ such that $\text{Re}(\tau) < -\beta$ for all τ satisfying $Q(\tau) = 0$. Thus, $\left\|(\mathcal{L} - \tau I)^{-1}\right\|_{\ell^2 \to \ell^2} \leq C(\beta)$ for all $\tau \neq 0$ with $\text{Re}(\tau) > -\beta$. This completes the proof of the proposition. $\quad\square$

Proposition 9.4. *The linear operator $\mathcal{L}$ defined in (9.13) has a one-dimensional kernel* $\ker \mathcal{L} = \text{span}(\mathbf{b})$.

$$\mathbf{b} = \left(-\frac{2\sigma}{\mathscr{R}_g T_\infty R_*^2}, 1, 0, 0, 0, \ldots\right)^\top, \tag{9.28}$$

Moreover, $\mathbf{b}^\dagger \in \text{coker}\,\mathcal{L}$ *where*

$$\mathbf{b}^\dagger = \left(\frac{4\pi}{3}, 4\pi\frac{\rho_*}{R_*}, 0, \frac{\gamma}{\gamma-1}\Gamma_1, \frac{\gamma}{\gamma-1}\Gamma_2, \ldots\right). \tag{9.29}$$

Proof. It is a direct consequence of Proposition 9.3 that $\dim \ker \mathcal{L} = 1$ since $\tau = 0$ has multiplicity one. Moreover, it is easy to check that $\mathcal{L}\mathbf{b} = \mathbf{0}$. So $\ker \mathcal{L} = \text{span}(\mathbf{b})$. On the other hand, one can check directly $\mathbf{b}^\dagger \mathcal{L} = \mathbf{0}$ by using the identity $\sum_{j=1}^\infty \Gamma_j^2 = \frac{4(\gamma-1)^2\pi}{3\gamma^2}$. We note that $\mathbf{b}^\dagger$ satisfies the linearized constant mass constraint

$$\frac{\gamma}{\gamma-1}\sum_{j=1}^\infty \Gamma_j c_j = \int_{B_1} u = -\frac{4\pi}{3}z - 4\pi\frac{\rho_*}{R_*}\mathcal{R}. \tag{9.30}$$

$\square$

Using $\mathscr{R}_g T_\infty \rho_* = p_{\infty,*} + 2\sigma/R_*$,

$$\langle \mathbf{b}^\dagger, \mathbf{b} \rangle = \frac{4\pi p_{\infty,*}}{3\mathscr{R}_g T_\infty R_*} + \frac{8\pi \rho_*}{3R_*}.$$

Normalize $\mathbf{b}^\dagger$ as

$$\mathbf{b}_0^\dagger = K\mathbf{b}^\dagger, \qquad K := \left(\frac{4\pi p_{\infty,*}}{3\mathscr{R}_g T_\infty R_*} + \frac{8\pi \rho_*}{3R_*} \right)^{-1}, \tag{9.31}$$

so that $\langle \mathbf{b}_0^\dagger, \mathbf{b} \rangle = 1$.

9.3. Toward a Center Manifold Formulation

Denote $X = \ker \mathcal{L} = \mathrm{span}(\mathbf{b})$, and let Y be the orthogonal complement of X in ℓ^2. Decompose

$$\mathbf{w} = \mathbf{x} + \mathbf{y}, \qquad \mathbf{x} = Q_1\mathbf{w} := \langle \mathbf{b}_0^\dagger, \mathbf{w} \rangle \, \mathbf{b}, \quad \mathbf{y} = Q_2\mathbf{w} := \mathbf{w} - Q_1\mathbf{w}.$$

Since $\mathcal{L}\mathbf{b} = \mathbf{b}^\dagger \mathcal{L} = \mathbf{0}$, we have

$$Q_1\mathcal{L}\mathbf{w} = \mathcal{L}Q_1\mathbf{w} = \mathbf{0}, \qquad Q_2\mathcal{L}\mathbf{w} = \mathcal{L}Q_2\mathbf{w} = \mathcal{L}\mathbf{w}$$

In particular, $\mathcal{L}Q_2$ is the restriction of $\mathcal{L}$ on Y. Then we derive, from (9.5), a system of $(\mathbf{x}, \mathbf{y})$

$$\dot{\mathbf{x}} = Q_1\mathcal{N}(\mathbf{x} + \mathbf{y}, \dot{\mathbf{x}} + \dot{\mathbf{y}}) = Q_1\left[\mathcal{N}^1(\mathbf{x} + \mathbf{y})[\dot{\mathbf{x}} + \dot{\mathbf{y}}] \right] + Q_1\mathcal{N}^0(\mathbf{x} + \mathbf{y}), \tag{9.32a}$$

$$\dot{\mathbf{y}} = \mathcal{L}\mathbf{y} + Q_2\mathcal{N}(\mathbf{x} + \mathbf{y}, \dot{\mathbf{x}} + \dot{\mathbf{y}}) = \mathcal{L}\mathbf{y} + Q_2\left[\mathcal{N}^1(\mathbf{x} + \mathbf{y})[\dot{\mathbf{x}} + \dot{\mathbf{y}}] \right] + Q_2\mathcal{N}^0(\mathbf{x} + \mathbf{y}). \tag{9.32b}$$

In order to apply the center manifold analysis developed in Appendix F, we check the setup of the system (9.32) in the following proposition.

Proposition 9.5. *The subspaces X and Y are $\mathcal{L}$-invariant and $e^{\mathcal{L}t}$-invariant, respectively. Moreover,*

$$\left\| e^{\mathcal{L}t} Q_2 \right\|_{\ell^2 \to \ell^2} \le ce^{-\beta t}, \quad t \ge 0, \tag{9.33}$$

for some $c > 0$, where β given in (E.1).

Proof. Obviously, X is $\mathcal{L}$-invariant since $X = \ker \mathcal{L}$. To show Y is $e^{\mathcal{L}t}$-invariant, let $\mathbf{y}_0 \in Y$ and fix $\mathbf{x} \in X, \mathbf{x} \ne \mathbf{0}$, and $\mathbf{y} \in Y, \mathbf{y} \ne \mathbf{0}$. Decompose $e^{\mathcal{L}t}\mathbf{y}_0 = \alpha_{\mathbf{x}}(t)\mathbf{x} + \alpha_{\mathbf{y}}(t)\mathbf{y}$. Since $\frac{d}{dt}(e^{\mathcal{L}t}\mathbf{y}_0) = \mathcal{L}(e^{\mathcal{L}t}\mathbf{y}_0)$ and $\mathbf{x} \in X = \ker \mathcal{L}$, $\dot{\alpha}_{\mathbf{x}}(t)\mathbf{x} + \dot{\alpha}_{\mathbf{y}}(t)\mathbf{y} = \alpha_{\mathbf{y}}(t)\mathcal{L}\mathbf{y}$ which implies $\dot{\alpha}_{\mathbf{x}}(t) = 0$. But since $\alpha_{\mathbf{x}}(0) = 0$ in view of $\mathbf{y}_0 \in Y$, we have $\alpha_{\mathbf{x}} \equiv 0$. So we deduce $e^{\mathcal{L}t}\mathbf{y}_0 = \alpha_{\mathbf{y}}(t)\mathbf{y} \in Y$ and conclude that Y is $e^{\mathcal{L}t}$-invariant. Since $\mathrm{Range}(Q_2) = Y = X^\perp$, $X = \ker \mathcal{L}$, the operator estimate (9.33) then follows from the spectrum analysis in Proposition 9.3 and the Gearhart-Prüss theorem [23,51] for C_0 semigroups on Hilbert spaces. This proves the proposition.

$\square$

The system (9.32) is equivalent to those systems in Proposition 9.1, namely, (9.2) and (9.5) as well as the original free boundary problem (5.1)–(5.2).

A curve $\mathbf{y} = h(\mathbf{x})$, defined for $|\mathbf{x}|$ small, is said to be an *invariant manifold* for (9.32) if the solution $(\mathbf{x}(t), \mathbf{y}(t))$ of (9.32) starting from $(\mathbf{x}(0), h(\mathbf{x}(0)))$ satisfies $\mathbf{y}(t) = h(\mathbf{x}(t))$. A *center manifold* for (9.32) is an invariant manifold that is tangent to X space at the origin.

9.4. Manifold of Equilibria, $\mathcal{M}_*$, as a Local Center Manifold Through (ρ_*, R_*)

In the following lemma, we show that the manifold of equilibria $\mathcal{M}_*$ (given in (6.6)) is a local center manifold for (9.32).

Lemma 9.6. *Let $\rho_{**}(\alpha)$ and $R_{**}(\alpha)$ satisfy*

$$\mathscr{R}_g T_\infty \rho_{**}(\alpha) = p_{\infty,*} + \frac{2\sigma}{R_{**}(\alpha)}, \qquad R_{**}(\alpha) = \frac{-\alpha + \sqrt{\alpha^2 + 4R_*^2}}{2}. \tag{9.34}$$

Then $(\rho_{**}(0), R_{**}(0)) = (\rho_*, R_*)$ and

$$\mathbf{y} = h(\mathbf{x}) = h(\alpha\mathbf{b}) := \begin{bmatrix} \rho_{**}(\alpha) - \rho_* \\ R_{**}(\alpha) - R_* \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \tag{9.35}$$

is a local center manifold for the system (9.32). Specifically, for α small enough the dynamics on the center manifold is trivial. That is, for $(\mathbf{x}(t), \mathbf{y}(t)) = (\mathbf{x}(t), h(\mathbf{x}(t)))$, (9.32a) becomes

$$\dot{\alpha}(t) = 0. \tag{9.36}$$

Proof. We first show that $\mathbf{y} = h(\mathbf{x})$ is an invariant manifold for (9.32). For $\mathbf{x}(t) = \alpha(t)\mathbf{b}$,

$$\mathbf{x} + h(\mathbf{x}) = \begin{bmatrix} -\frac{2\sigma}{\mathscr{R}_g T_\infty R_*^2}\alpha + \rho_{**}(\alpha) - \rho_* \\ \alpha + R_{**}(\alpha) - R_* \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \qquad \dot{\mathbf{x}} + h'(\mathbf{x})\dot{\mathbf{x}} = \dot{\alpha} \begin{bmatrix} -\frac{2\sigma}{\mathscr{R}_g T_\infty R_*^2} + \rho'_{**}(\alpha) \\ 1 + R'_{**}(\alpha) \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}.$$

Thus, for $\mathbf{F}^1$, $\mathbf{G}^1$, $\mathbf{H}^1$ and F^0, G^0, H^0 defined in (9.16) we have $\mathbf{F}^1(\mathbf{x} + h(\mathbf{x})) = \mathbf{0}$, $F^0(\mathbf{x} + h(\mathbf{x})) = 0$,

$$\mathbf{G}^1(\mathbf{x} + h(\mathbf{x})) = (J(\alpha), 0, 0, 0, 0, \cdots)^\top, \qquad G^0(\mathbf{x} + h(\mathbf{x})) = 0,$$

where

$$J(\alpha) = \frac{-\left(\rho_* + \frac{2\sigma}{\mathscr{R}_g T_\infty R_*}\right)\alpha - \rho_* R_{**}(\alpha) + R_* \rho_{**}(\alpha)}{3\gamma \rho_* \left(-\frac{2\sigma}{\mathscr{R}_g T_\infty R_*^2}\alpha + \rho_{**}(\alpha)\right)}, \tag{9.37}$$

and

$$\mathbf{H}^1(\mathbf{x} + h(\mathbf{x})) = \left(0, 0, \frac{\rho_l(\alpha + R_{**}(\alpha) - R_*)}{\mathscr{R}_g T_\infty}, 0, 0, \cdots\right)^\top,$$

$$H^0(\mathbf{x} + h(\mathbf{x})) = \frac{2\sigma}{\mathscr{R}_g T_\infty R_*^2} \frac{(\alpha + R_{**}(\alpha) - R_*)^2}{\alpha + R_{**}(\alpha)}.$$

Thus, by (9.18) we get

$$\mathcal{N}^1(\mathbf{x} + h(\mathbf{x}))\left[\dot{\mathbf{x}} + h'(\mathbf{x})\dot{\mathbf{x}}\right] = \left(-\frac{2\sigma}{\mathscr{R}_g T_\infty R_*^2} + \rho'_{**}(\alpha)\right) \frac{3\gamma \rho_*}{R_*} J(\alpha)\dot{\alpha} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -\Gamma_1 \\ -\Gamma_2 \\ \vdots \end{bmatrix},$$

and

$$\mathcal{N}^0(\mathbf{x} + h(\mathbf{x})) = \begin{bmatrix} 0 \\ 0 \\ -\frac{2\sigma}{\rho_l R_*^3} \frac{(\alpha + R_{**}(\alpha) - R_*)^2}{\alpha + R_{**}(\alpha)} \\ 0 \\ 0 \\ \vdots \end{bmatrix}.$$

Therefore, $(\mathbf{x}, h(\mathbf{x}))$ solves (9.32a) if and only if, by using the identity $\sum_{j=1}^\infty \Gamma_j^2 = \frac{4(\gamma-1)^2\pi}{3\gamma^2}$,

$$\dot{\alpha}\mathbf{b} = \dot{\mathbf{x}} = Q_1\left[\mathcal{N}^1(\mathbf{x} + h(\mathbf{x}))[\dot{\mathbf{x}} + h'(\mathbf{x})\dot{\mathbf{x}}]\right] + Q_1\mathcal{N}^0(\mathbf{x} + h(\mathbf{x}))$$

$$= \langle \mathbf{b}_0^\dagger, \mathcal{N}^1(\mathbf{x} + h(\mathbf{x}))[\dot{\mathbf{x}} + h'(\mathbf{x})\dot{\mathbf{x}}]\rangle\mathbf{b} + \langle \mathbf{b}_0^\dagger, Q_1\mathcal{N}^0(\mathbf{x} + h(\mathbf{x}))\rangle\mathbf{b}$$

$$= K\left\{\left(-\frac{2\sigma}{\mathscr{R}_g T_\infty R_*^2} + \rho'_{**}(\alpha)\right) \frac{3\gamma \rho_*}{R_*} J(\alpha)\dot{\alpha}\left(\frac{4\pi}{3} - \frac{\gamma}{\gamma-1}\sum_{j=1}^\infty \Gamma_j^2\right) + 0\right\}\mathbf{b}$$

$$= K\frac{4\pi}{3\gamma}\left(-\frac{2\sigma}{\mathscr{R}_g T_\infty R_*^2} + \rho'_{**}(\alpha)\right) \frac{3\gamma \rho_*}{R_*} J(\alpha)\dot{\alpha}\mathbf{b},$$

which is equivalent to

$$\dot{\alpha}\left\{1 + \frac{4\pi K}{3\gamma}\left(-\frac{2\sigma}{\mathscr{R}_g T_\infty R_*^2} + \rho'_{**}(\alpha)\right) \frac{3\gamma \rho_*}{R_*} J(\alpha)\right\} = 0.$$

Since $J(\alpha) \to 0$ as $\alpha \to 0$, the above equation yields $\dot{\alpha} = 0$ for all α sufficiently small. This shows that the dynamics on the local center manifold is trivial.

We now verify equation (9.32b). Note that $(\mathbf{x}, h(\mathbf{x}))$ solves (9.32b) if and only if

$$h'(\mathbf{x})\dot{\mathbf{x}} = \mathcal{L}h(\mathbf{x}) + Q_2\left[\mathcal{N}^1(\mathbf{x} + h(\mathbf{x})[\dot{\mathbf{x}} + h'(\mathbf{x})\dot{\mathbf{x}}]\right] + Q_2\mathcal{N}^0(\mathbf{x} + h(\mathbf{x}).$$
(9.38)

Since $Q_1\left[\mathcal{N}^1(\mathbf{x} + h(\mathbf{x}))[\dot{\mathbf{x}} + h'(\mathbf{x})\dot{\mathbf{x}}]\right] = \mathbf{0}$ for sufficiently small α and $Q_1\mathcal{N}^0(\mathbf{x} + h(\mathbf{x})) \equiv 0$, we have $Q_2\left[\mathcal{N}^1(\mathbf{x} + h(\mathbf{x})[\dot{\mathbf{x}} + h'(\mathbf{x})\dot{\mathbf{x}}]\right] = \mathcal{N}^1(\mathbf{x}+h(\mathbf{x})[\dot{\mathbf{x}}+h'(\mathbf{x})\dot{\mathbf{x}}]$ for sufficiently small α and $Q_2\mathcal{N}^0(\mathbf{x} + h(\mathbf{x})) \equiv \mathcal{N}^0(\mathbf{x} + h(\mathbf{x}))$. Thus, for sufficiently small α, (9.38) becomes

$$h'(\mathbf{x})\dot{\mathbf{x}} = \mathcal{L}h(\mathbf{x}) + \mathcal{N}^1(\mathbf{x} + h(\mathbf{x})[\dot{\mathbf{x}} + h'(\mathbf{x})\dot{\mathbf{x}}] + N^0(\mathbf{x} + h(\mathbf{x})),$$

or equivalently,

$$\dot{\alpha}\begin{bmatrix} \rho'_{**}(\alpha) \\ R'_{**}(\alpha) \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{\mathscr{R}_g T_\infty}{\rho_l R_*}(\rho_{**}(\alpha) - \rho_*) + \frac{2\sigma}{\rho_l R_*^3}(R_{**}(\alpha) - R_*) \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

$$+ \left(-\frac{2\sigma}{\mathscr{R}_g T_\infty R_*^2} + \rho'_{**}(\alpha)\right) J(\alpha)\dot{\alpha}\begin{bmatrix} 1 \\ 0 \\ 0 \\ -\Gamma_1 \\ -\Gamma_2 \\ \vdots \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ -\frac{2\sigma}{\rho_l R_*^3}\frac{(\alpha + R_{**}(\alpha) - R_*)^2}{\alpha + R_{**}(\alpha)} \\ 0 \\ 0 \\ \vdots \end{bmatrix}.$$

Since $\dot{\alpha} = 0$ for α sufficiently small, the above equation is further equivalent to

$$0 = \frac{\mathscr{R}_g T_\infty}{\rho_l R_*}(\rho_{**}(\alpha) - \rho_*) + \frac{2\sigma}{\rho_l R_*^3}(R_{**}(\alpha) - R_*) - \frac{2\sigma}{\rho_l R_*^3}\frac{(\alpha + R_{**}(\alpha) - R_*)^2}{\alpha + R_{**}(\alpha)}.$$

Using (9.34) and $\mathscr{R}_g T_\infty \rho_* = p_{\infty,*} + 2\sigma/R_*$, we derive from above a quadratic equation:

$$R_{**}^2 + \alpha R_{**} - R_*^2 = 0,$$

for which the solution is $R_{**}(\alpha) = \left(-\alpha \pm \sqrt{\alpha^2 + 4R_*^2}\right)/2$. In view of the condition $R_{**}(0) = R_*$, we choose the plus sign:

$$R_{**}(\alpha) = \frac{-\alpha + \sqrt{\alpha^2 + 4R_*^2}}{2}.$$

Finally, we verify that $\mathbf{y} = h(\mathbf{x})$ is tangent to X at the origin. Differentiating (9.34) with respect to α, evaluating at $\alpha = 0$ and using $R_{**}(0) = R_*$, we get $\mathscr{R}_g T_\infty \rho'_{**}(0) = -(2\sigma/R_*^2)R'_{**}(0)$. So,

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\bigg|_{\alpha=0} h(\alpha\mathbf{b}) = \begin{bmatrix} \rho'_{**}(0) \\ R'_{**}(0) \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} -\frac{2\sigma}{\mathscr{R}_g T_\infty R_*^2} R'_{**}(0) \\ R'_{**}(0) \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} = R'_{**}(0)\mathbf{b} \in X.$$

This completes the proof of Lemma 9.6. $\quad\square$

9.5. Nonlinear Estimates

In order to apply Proposition F.2, we derive the following estimates for the nonlinear terms in the dynamical system (9.5).

Proposition 9.7. *Let $\mathcal{N}(\mathbf{w}, \mathbf{p})$ be given in (9.14). Let $\mathbf{w} = (z, \mathcal{R}, \dot{\mathcal{R}}, c_1, c_2, \ldots)^\top$ be obtained from the solution $(\overline{\rho}, R) \in C_{y,t}^{2+2\alpha, 1+\alpha} \times C_t^{3+\alpha}$ of (5.6) by means of (9.1) and (9.6). Then $\mathcal{N}(\mathbf{w}, \mathbf{p}) \in \ell^2$. Moreover, $\mathcal{N}(\mathbf{0}, \mathbf{p}) = \mathbf{0}$ for all $\mathbf{p}$ and $\partial_\mathbf{w}\mathcal{N}(\mathbf{0}, \mathbf{0}) = \partial_\mathbf{p}\mathcal{N}(\mathbf{0}, \mathbf{0}) = \mathbf{O}$.*

Proof. We first show that $\mathcal{N}(\mathbf{w}, \mathbf{p}) \in \ell^2$. Note that $G(\mathbf{w}, \mathbf{p})$ is well-defined since $\{jc_j\}_{j=1}^\infty \in \ell^1$: $\sum j|c_j| = \sum j^{-1}j^2|c_j| \leq \left(\sum j^{-2}\right)^{1/2}\left(\sum j^4 c_j^2\right)^{1/2} < \infty$. Since $\{\Gamma_j \sim (-1)^{-j}/j\}_{j=1}^\infty \in \ell^2$, it remains to show $\{F_j(\mathbf{w}, \mathbf{p})\}_{j=1}^\infty \in \ell_j^2$. Indeed, using $|\nabla u(y, t)|^2 \leq E$, where $E > 0$ is a constant independent of y and t, we have

$$F_j(\mathbf{w}, \mathbf{p}) = \int_{B_1} F(\mathbf{w}, \mathbf{p})\phi_j \, \mathrm{d}x$$

$$= O(1)\left[j^2 c_j + \Gamma_j + \left(\frac{1}{3}\sum_{k=1}^\infty c_k \int_{B_1} y\partial_y\phi_k\phi_j \, \mathrm{d}x + \sum_{k=1}^\infty c_k \int_{B_1} \phi_k\phi_j \, \mathrm{d}x\right)\right]$$

Since $\int_{B_1} \phi_k\phi_j \, \mathrm{d}x = \delta_{kj}$ and

$$\int_{B_1} y\partial_y\phi_k\phi_j \, \mathrm{d}x = 4\pi \int_0^1 \left(\frac{k\pi \cos(k\pi y)}{\sqrt{2\pi}y} - \frac{\sin(k\pi y)}{\sqrt{2\pi}y^2}\right)\frac{\sin(j\pi y)}{\sqrt{2\pi}y}y^3 \, \mathrm{d}y$$

$$= 2k\pi \int_0^1 \cos(k\pi y)\sin(j\pi y)y \, \mathrm{d}y - 2\int_0^1 \sin(k\pi y)\sin(j\pi y) \, \mathrm{d}y$$

$$= \begin{cases} -\frac{1}{2} - 1 = -\frac{3}{2}, & \text{if } j = k, \\ k\left(\frac{(-1)^{k-j}}{k-j} - \frac{(-1)^{k+j}}{k+j}\right) = (-1)^{k+j}\frac{2jk}{k^2-j^2}, & \text{if } j \neq k, \end{cases} \tag{9.39}$$

we have

$$F_j(\mathbf{w}, \mathbf{p}) = O(1) \left[j^2 c_j + j^{-1} + \left(\frac{1}{2} c_j + \sum_{k \neq j} c_k (-1)^{k+j} \frac{2jk}{k^2 - j^2} \right) \right],$$

where the first three terms are obviously in ℓ^2. For the last term, by Minkowski inequality,

$$\sum_{j=1}^{\infty} \left(\sum_{k \neq j} c_k (-1)^{k+j} \frac{2jk}{k^2 - j^2} \right)^2$$

$$\leq \left[\sum_{k=1}^{\infty} \left(\sum_{j \neq k} c_k^2 \left(\frac{2jk}{k^2 - j^2} \right)^2 \right)^{\frac{1}{2}} \right]^2 \sim \left[\sum_{k=1}^{\infty} k |c_k| \left(\sum_{j \neq k} \frac{j^2}{(k^2 - j^2)^2} \right)^{\frac{1}{2}} \right]^2$$

$$\lesssim \left[\sum_{k=1}^{\infty} k |c_k| \right]^2 < \infty, \quad \text{since } \{ j c_j \}_{j=1}^{\infty} \in \ell^1,$$

where we've used

$$\sum_{j \neq k} \frac{j^2}{(k^2 - j^2)^2} \sim \sum_{j \neq k} \left(\frac{1}{k - j} - \frac{1}{k + j} \right)^2$$

$$= \sum_{j \neq k} \left[\frac{1}{(k - j)^2} - \frac{2}{k^2 - j^2} + \frac{1}{(k + j)^2} \right]$$

$$\leq \sum_{j \neq k} \frac{1}{(k - j)^2} - 2 \sum_{j \neq k} \frac{1}{k^2 - j^2} + \sum_{j=1}^{\infty} \frac{1}{j^2}$$

$$= \left(\sum_{j=1}^{k-1} \frac{1}{(k - j)^2} + \sum_{j=k+1}^{\infty} \frac{1}{(k - j)^2} \right) + \frac{3}{2k^2} + \frac{\pi^2}{6}$$

$$\leq \left(\sum_{j=1}^{\infty} \frac{1}{j^2} + \sum_{j=1}^{\infty} \frac{1}{j^2} \right) + \frac{3}{2} + \frac{\pi^2}{6} = \frac{3}{2} + \frac{3\pi^2}{6}, \quad \text{independent of } k.$$

Therefore, $\{ F_j(\mathbf{w}, \mathbf{p}) \}_{j=1}^{\infty} \in \ell_j^2$, and thus $\mathcal{N}(\mathbf{w}, \mathbf{p}) \in \ell^2$.

It is readily see from (9.15a)–(9.15c) that $\mathcal{N}(\mathbf{0}, \mathbf{p}) = \mathbf{0}$ for all $\mathbf{p}$. To estimate the derivative, we adopt the form $\mathcal{N}(\mathbf{w}, \mathbf{p}) = \mathcal{N}^1(\mathbf{w}) \mathbf{p} + \mathcal{N}^0(\mathbf{w})$ derived in (9.17). Note that from (9.16), $|F^1(\mathbf{w})| + |G^1(\mathbf{w})| + |H^1(\mathbf{w})| = O(\|\mathbf{w}\|)$ and $|F^0(\mathbf{w})| + |G^0(\mathbf{w})| + |H^0(\mathbf{w})| = O(\|\mathbf{w}\|^2)$. Thus, $\mathcal{N}^1(\mathbf{0}) = \mathbf{O}, \mathcal{N}^0(\mathbf{0}) = \mathbf{0}$, and $\partial_{\mathbf{w}} \mathcal{N}^0(\mathbf{0}) = \mathbf{O}$ by (9.18). Therefore, we have $\mathcal{N}(\mathbf{0}, \mathbf{p}) = \mathcal{N}^1(\mathbf{0}) \mathbf{p} + \mathcal{N}^0(\mathbf{0}) = \mathbf{0}$ for all $\mathbf{p}$, $\partial_{\mathbf{w}} \mathcal{N}(\mathbf{0}, \mathbf{0}) = \partial_{\mathbf{w}} \mathcal{N}^0(\mathbf{0}) = \mathbf{O}, \partial_{\mathbf{p}} \mathcal{N}(\mathbf{0}, \mathbf{0}) = \mathcal{N}^1(\mathbf{0}) = \mathbf{O}$, completing the proof of the proposition. $\square$

9.6. Proof of Theorem 6.7; Nonlinear Exponential Stability of the Manifold of Equilibria $\mathcal{M}_*$

The proof of Theorem 6.7 is based on a center manifold analysis of the system (9.32) adapted from a quasi-linear PDE with *a priori* estimates; see Proposition F.2. Proposition F.2 requires that we verify i) a Lipschitz estimate for the nonlinear term, ii) a linear exponential decay estimate, iii) existence of a local center manifold, and iv) an ℓ^2 *a priori* bound on $\dot{\mathbf{w}}$.

 i) The nonlinear Lipschitz estimate follows from the expression of the nonlinear term in (9.16) and is proved in Proposition 9.7.
 ii) Linear exponential decay follows from Proposition 9.5.
 iii) The existence of a local center manifold follows from the identification of the manifold of equilibria $\mathcal{M}_*$ as a local center manifold shown in Lemma 9.6.
 iv) The *a priori* estimate follows from Proposition 9.2.

With these observations in place, we argue as follows.

Recall the free boundary problem (5.1) is equivalent to the system (5.6) in the fixed domain B_1, and it is also equivalent to the dynamical system (9.5) by Proposition 9.1. From the asymptotic stability result in Theorem 6.5, we have, as $t \to \infty$, that $\mathbf{w}(t) \to \mathbf{w}_*$, where $\mathbf{w}_*$ is described in Proposition 9.2. Also, one has $|\ddot{R}(t)| + |\dddot{R}(t)| \to 0$ in Theorem 6.5. Moreover, it follows from the equation (5.6a) that $\partial_t \overline{\rho} \to 0$ uniformly in B_1. So $\dot{\mathbf{w}}(t) \to \mathbf{0}$ in ℓ^2 as $t \to \infty$, and thus we may assume that $\sup_{t \geq 0} \|\dot{\mathbf{w}}(t)\|$ is arbitrarily small.

We have shown in Lemma 9.6 that the equation on the local center manifold is trivial, (9.36). Therefore, by applying Proposition F.2 to the system (9.12) with the nonlinear estimates Proposition 9.7, we conclude that the convergence of $\mathbf{w}(t) \to \mathbf{w}_*$ is exponentially fast in ℓ^2.

This amounts to the exponential convergence of $(u, z, \mathcal{R})$ to some $(u_{**}, z_{**}, \mathcal{R}_{**})$, where $(\rho_* + u_{**} + z_{**}, R_* + \mathcal{R}_{**}, 0)$ lies on the manifold of equilibria $\mathcal{M}_*$, in the following L^2 sense: for some $\beta_0 > 0$

$$\int_{B_1} (u - u_{**})^2 \, \mathrm{d}x + (z - z_{**})^2 + (\mathcal{R} - \mathcal{R}_{**})^2 + \dot{\mathcal{R}}^2 = O\left(e^{-2\beta_0 t}\right) \quad \text{as } t \to \infty.$$
(9.40)

From time-decay in L^2 norms to time-decay of L^∞ norms of gradients. We next want to use the L^2 bound of (9.40) to obtain gradient bounds in L^∞. For this we make use of the following inequality which is based on the equation and an interpolation inequality.

Lemma 9.8. *Assume that $(u, z, \mathcal{R})$ is the solution of the system (9.2) and that F is given in (9.4a). Then for any constant u_{**}*

$$\frac{1}{2} \frac{d}{dt} \int_{B_1} |\nabla_y u|^2 \, \mathrm{d}x \leq -\frac{\overline{\kappa}}{2} \frac{\left(\int_{B_1} |\nabla_y u|^2 \, \mathrm{d}x + 4\pi u_{**} \partial_y u(1, t)\right)^2}{\int_{B_1} (u - u_{**})^2 \, \mathrm{d}x}$$

$$+ \frac{1}{2\overline{\kappa}} \int_{B_1} F^2 \, \mathrm{d}x + 4\pi \left(1 - \frac{1}{\gamma}\right) \dot{z} \partial_y u(1, t).$$
(9.41)

Proof. Taking gradient of (9.2a), we have $\nabla_y(\partial_t u) = \overline{\kappa}\nabla_y\Delta_y u + \nabla_y F$. Thus, we have

$$
\frac{1}{2}\frac{d}{dt}\int_{B_1}|\nabla_y u|^2\,dx
$$

$$
= \int_{B_1}\nabla_y u\cdot\nabla_y(\partial_t u)\,dx = \overline{\kappa}\int_{B_1}\nabla_y u\cdot\nabla_y\Delta_y u\,dx + \int_{B_1}\nabla_y u\cdot\nabla_y F\,dx
$$

$$
= -\overline{\kappa}\int_{B_1}(\Delta_y u)^2\,dx - \int_{B_1}(\Delta_y u)F\,dx + \overline{\kappa}\int_{\partial B_1}\Delta_y u(\nabla_y u\cdot\hat{\mathbf{n}})\,dS
$$

$$
+ \int_{\partial B_1}F(\nabla_y u\cdot\hat{\mathbf{n}})\,dS
$$

$$
= -\overline{\kappa}\int_{B_1}(\Delta_y u)^2\,dx - \int_{B_1}(\Delta_y u)F\,dx + 4\pi(\overline{\kappa}\Delta_y u(1,t)+F(1,t))\partial_y u(1,t)
$$

Since $u(1,t)=0$ implies $\partial_t u(1,t)=0$, evaluating (9.2a) at the boundary yields

$$
\overline{\kappa}\Delta_y u(1,t) + F(1,t) = \left(1-\frac{1}{\gamma}\right)\dot{z}(t).
$$

Substituting the above equation for $\overline{\kappa}\Delta_y u(1,t) - F(1,t)$ and using Young's inequality, we obtain

$$
\frac{1}{2}\frac{d}{dt}\int_{B_1}|\nabla_y u|^2\,dx = -\overline{\kappa}\int_{B_1}(\Delta_y u)^2\,dx - \int_{B_1}(\Delta_y u)F\,dx + 4\pi\left(1-\frac{1}{\gamma}\right)\dot{z}\partial_y u(1,t)
$$
$$
\le -\overline{\kappa}\int_{B_1}(\Delta_y u)^2\,dx + \frac{\overline{\kappa}}{2}\int_{B_1}(\Delta_y u)^2\,dx + \frac{1}{2\overline{\kappa}}\int_{B_1}F^2\,dx + 4\pi\left(1-\frac{1}{\gamma}\right)\dot{z}\partial_y u(1,t). \tag{9.42}
$$

Now, since $\int_{B_1}|\nabla_y u|^2\,dy = -\int_{B_1}(u-u_{**})\Delta_y u\,dy - 4\pi u_{**}\partial_y u(1,t)$, by Hölder's inequality we have the interpolation inequality

$$
\left(\int_{B_1}|\nabla_y u|^2\,dx + 4\pi u_{**}\partial_y u(1,t)\right)^2 \le \int_{B_1}(u-u_{**})^2\,dx\cdot\int_{B_1}(\Delta_y u)^2\,dx.
$$

Using the interpolation inequality for $\int_{B_1}(\Delta_y u)^2\,dx$ in (9.42), the lemma follows. $\quad\square$

With Lemma 9.8 in hand, we are able to obtain a decay rate of $\nabla_y u$ from the decay rate of u.

Proposition 9.9. *Let $(u, z, \mathcal{R})$ be the solution of the the system (9.2) with the convergence rate in (9.40), then*

$$
\left\|\nabla_y u(\cdot,t)\right\|_{L^\infty(B_1)} = O\left(e^{-\frac{\beta_0}{10}t}\right).
$$

Proof. First recall Proposition 9.1 that the system (9.2) is equivalent to (5.1)–(5.2).

(1) It follows from the Lyapunov stability result for (5.1)–(5.2) in Theorem 6.3 that $u \in C_{y,t}^{2+2\alpha,1+\alpha}$ and $\dot{z} \in C_t^\alpha$. Hence, $\frac{d}{dt}\int_{B_1}|\nabla_y u|^2\,dx$ is bounded in t.

(2) Moreover, the asymptotic stability result in Theorem 6.5 implies that $\dot{z} \to 0$ and $\left\| \nabla_y u(\cdot, t) \right\|_{L^\infty(B_1)} \to 0$ as $t \to \infty$. It also implies that $\int_{B_1} F^2(x, t) \, \mathrm{d}x \to 0$ as $t \to \infty$, where F given in (9.4a).

(3) Since $z(t) - z_{**} = O(e^{-\beta_0 t})$, $\int_0^t (z(\tau) - z_{**}) e^{(\beta_0/2)\tau} \, \mathrm{d}\tau$ has a finite limit as $t \to \infty$ and $(z(t) - z_{**}) e^{(\beta_0/2)t}$ is uniformly continuous for $t > 0$. By the Barbalat's lemma (8.6), $\frac{d}{dt} \left[(z(t) - z_{**}) e^{(\beta_0/2)t} \right] \to 0$ as $t \to \infty$. This implies $\dot{z}(t) e^{(\beta_0/2)t} + (\beta_0/2) (z(t) - z_{**}) e^{(\beta_0/2)t} \to 0$ as $t \to \infty$. In particular, we have $\dot{z} = O(e^{-(\beta_0/2)t})$. It then follows form (9.2b) and $\dot{\mathcal{R}} = O(e^{-\beta_0 t})$ that $\partial_y u(1, t) = O(e^{-(\beta_0/2)t})$.

We claim that the quotient

$$\frac{\left(\int_{B_1} |\nabla_y u|^2 \, \mathrm{d}x + 4\pi u_{**} \partial_y u(1, t) \right)^2}{\int_{B_1} (u - u_{**})^2 \, \mathrm{d}x} \tag{9.43}$$

is uniformly bounded in time. Indeed, if the quotient (9.43) was unbounded, in view of (1) and (2) the right hand side of (9.41) would become $-\infty$ as $t \to \infty$ while the left hand side is bounded, violating the inequality (9.41) in Lemma 9.8.

Since that $\int_{B_1} (u - u_{**})^2 \, \mathrm{d}x$ decays exponentially at the rate $2\beta_0$ as in (9.40) and that the quotient (9.43) is bounded,

$$\int_{B_1} |\nabla_y u|^2 \, \mathrm{d}x + 4\pi u_{**} \partial_y u(1, t) = O(e^{-\beta_0 t}).$$

Thus, (3) implies $\int_{B_1} |\nabla_y u|^2 \, \mathrm{d}x = O(e^{-(\beta_0/2)t})$. By the interpolation inequality in Lemma D.1,

$$\left\| \nabla_y u \right\|_{L^\infty(B_1)} \le C_1 \left\| \nabla_y u \right\|_{L^2(B_1)}^{\frac{2}{5}} \left\| \nabla_y u \right\|_{C_y^{1+2\alpha}(B_1)}^{\frac{3}{5}} + C_2 \left\| \nabla_y u \right\|_{L^2(B_1)} = O \left(e^{-\frac{\beta_0}{10} t} \right).$$

This proves the proposition. □

Using Proposition 9.9, we derive the exponential decay of $u - u_{**}$ in the higher norm $\| \cdot \|_{W^{1,\infty}}$ from L^2 convergence. By bootstrapping the argument in the proof of Proposition 9.9, we derive the exponential decay of u in $C_y^{2+2\alpha}$. This amounts to the exponential rate of convergence of $(\rho, R, \dot{R})$ toward the center manifold $\mathcal{M}_*$ in $|\cdot|$ and completes the proof of Theorem 6.7. □

Appendix A: Spherically Symmetric Equilibria of the Full Liquid/Gas Model

In this appendix, we prove Proposition 4.1. That is, we show that (4.3) is the unique regular spherically symmetric equilibrium solution to the system (2.1)–(2.4) under the radiation condition (4.2) for T_l.

Proof of Proposition 4.1. We consider the full liquid/gas model (2.1)–(2.4) and prove Proposition 4.1. Steady-state solutions of (2.1)–(2.4) solve

$$0 = \nu_l \Delta \mathbf{v}_{l,*} - \mathbf{v}_{l,*} \cdot \nabla \mathbf{v}_{l,*} - \frac{1}{\rho_{l,*}} \nabla p_{l,*}, \tag{A.1a}$$

$$\left. \mathrm{div}\, \mathbf{v}_{l,*} = 0, \right\} \text{ in } \mathbb{R}^3 \setminus \Omega_*, \tag{A.1b}$$

$$\rho_l c_l \mathbf{v}_{l,*} \cdot \nabla T_{l,*} = \kappa_l \Delta T_{l,*} + 2\mu_l \mathbb{D}(\mathbf{v}_{l,*}) : \nabla \mathbf{v}_{l,*}, \tag{A.1c}$$

$$\mathrm{div}(\rho_{g,*} \mathbf{v}_{g,*}) = 0, \tag{A.2a}$$

$$\rho_{g,*} \mathbf{v}_{g,*} \cdot \nabla \mathbf{v}_{g,*} = \mu_g \Delta \mathbf{v}_{g,*} - \nabla p_{g,*}, \tag{A.2b}$$

$$\rho_{g,*} T_{g,*} \mathbf{v}_{g,*} \cdot \nabla s_* = \kappa_g \Delta T_{g,*} + 2\mu_g \mathbb{D}(\mathbf{v}_{g,*}) : \nabla \mathbf{v}_{g,*} \tag{A.2c}$$

$$\left. - \left(\frac{2}{3} \mu_g - \zeta_g \right) (\mathrm{div}\, \mathbf{v}_{g,*})^2, \right\} \text{ in } \Omega_*,$$

$$p_{g,*} = \mathscr{R}_g T_{g,*} \rho_{g,*}, \tag{A.2d}$$

$$s_* = c_v \log \left(\frac{p_{g,*}}{\rho_{g,*}^\gamma} \right), \tag{A.2e}$$

$$\mathbf{v}_{l,*}(\boldsymbol{\omega}, t) \cdot \hat{\mathbf{n}} = \mathbf{v}_{g,*}(\boldsymbol{\omega}, t) \cdot \hat{\mathbf{n}} = 0,$$

$$\hat{\mathbf{n}} \cdot \left(-p_{l,*} \mathbb{I} + 2\mu_l \mathbb{D}(\mathbf{v}_{l,*}) \right)$$

$$\left. - \hat{\mathbf{n}} \cdot \left[-p_{g,*} \mathbb{I} + 2\mu_g \left(\mathbb{D}(\mathbf{v}_{g,*}) - \frac{1}{3}(\mathrm{div}\, \mathbf{v}_{g,*})\mathbb{I} \right) + \zeta_g (\mathrm{div}\, \mathbf{v}_{g,*})\mathbb{I} \right] = \sigma \hat{\mathbf{n}}(\nabla_S \cdot \mathbf{n}), \right\} \text{ on } \partial \Omega_*.$$

$$T_{g,*} = T_{l,*},$$

$$\tag{A.3a}$$
$$\tag{A.3b}$$

$$\tag{A.3c}$$

For the spherically symmetric case, (A.1b) we have $\mathbf{v}_{l,*}(x) = v_{l,*}(r)\frac{x}{r}$. Therefore, $0 = \operatorname{div}\mathbf{v}_{l,*} = \partial_r v_{l,*}(r) + (2/r)v_{l,*}(r)$ and hence

$$\frac{1}{r^2}\partial_r(r^2 v_{l,*}(r)) = 0, \quad r \geq R_*. \tag{A.4}$$

Therefore,

$$v_{l,*}(r) = \frac{a}{r^2}, \quad r \geq R_*, \tag{A.5}$$

for constant a. But the boundary condition (A.3a) implies $v_{l,*}(R_*) = 0$. So $a = 0$, and thus $v_{l,*} \equiv 0$. Therefore, (A.1a) becomes $\nabla p_{l,*} = 0$. So the pressure $p_{l,*}$ is a constant and equal to its value at infinity, $p_{\infty,*}$.

For the gas velocity $v_{g,*}$, (A.2a) becomes

$$\frac{1}{r^2}\partial_r(\rho_{g,*}v_{g,*}) = 0, \quad 0 \leq r \leq R_*,$$

which implies $\rho_{g,*}v_{g,*}$ is a constant. Again, the boundary condition (A.3a) implies $v_{g,*}(R_*) = 0$. So $\rho_{g,*}v_{g,*} \equiv 0$. But $\rho_{g,*} \neq 0$ since otherwise s_* is singular in (A.2e). Therefore, $\mathbf{v}_{g,*} \equiv \mathbf{0}$ and thus (A.2b) becomes $\nabla p_{g,*} = 0$. So $p_{g,*}$ is a constant. Moreover, by $v_{l,*} = v_{g,*} \equiv 0$, (A.3b) yields $-p_{l,*} + p_{g,*} = \frac{2\sigma}{R_*}$. So $p_{g,*} = p_{\infty,*} + \frac{2\sigma}{R_*}$.

For the equations of the temperatures, due to $v_{l,*} = 0$ (A.1c) becomes $\Delta T_{l,*} = 0$ in $\mathbb{R}^3 \backslash B_{R_*}$, or, in spherical coordinates,

$$\frac{1}{r^2}\partial_r(r^2\partial_r T_{l,*}) = 0, \quad r \geq R_*,$$

which implies

$$\partial_r T_{l,*} = \frac{a_1}{r^2}, \quad r \geq R_*$$

for some constant a_1. Integrating over (r, ∞) we get

$$T_{l,*}(r) = T_\infty - \frac{a_1}{r}, \quad r \geq R_*.$$

By the radiation condition (4.2), $T_{l,*}(r) = T_\infty + o(r^{-1})$ as $r \to \infty$. This gives $a_1 = 0$ and thus $T_{l,*} \equiv T_\infty$. On the other hand, (A.2c) becomes $\Delta T_{g,*} = 0$ in B_{R_*} since $v_{g,*} = 0$. Since $T_{g,*}$ is regular, $T_{g,*} \equiv T_{g,*}(R_*) = T_\infty$ by the maximum principle.

For the gas density $\rho_{g,*}$, by (A.2d) $\rho_{g,*} = \frac{p_{g,*}}{\mathscr{R}_g T_\infty} = \frac{1}{\mathscr{R}_g T_\infty}\left(p_{\infty,*} + \frac{2\sigma}{R_*}\right)$. Due to the conservation of mass (7.1),

$$M := \int_{B_{R_0}} \rho_0 = \lim_{t\to\infty}\int_{B_{R(t)}} \rho_g(\cdot, t) = \frac{4\pi}{3}\rho_{g,*}R_*^3,$$

where $(\rho_0(x), R_0)$, $\rho_0(x) \geq 0$, $R_0 > 0$, is the initial data. Therefore, the steady state $(\rho_{g,*}, R_*)$ can be obtained by solving

$$\left.\begin{array}{l} \dfrac{4\pi}{3}\rho_{g,*}R_*^3 = M, \\[3mm] \rho_{g,*} = \dfrac{1}{\mathscr{R}_g T_\infty}\left(p_{\infty,*} + \dfrac{2\sigma}{R_*}\right). \end{array}\right\}$$

(A.6a)

(A.6b)

In particular, the stationary radius R_* satisfies the cubic equation

$$p_{\infty,*}R_*^3 + 2\sigma R_*^2 - \frac{3\mathscr{R}_g T_\infty M}{4\pi} = 0. \tag{A.7}$$

It is readily seen that for any $M > 0$, the cubic equation has a unique positive root $R_*[M]$. This proves Proposition 4.1. □

Appendix B: Derivation of the Reduced System for $\rho(r, t)$ and $R(t)$: Proof of Proposition 5.1

Considering the uniformity of the pressure p_g in (3.2b), we can eliminate T_g by plugging (3.2d) into (3.2c) and deduce

$$\partial_t s + \mathbf{v}_g \cdot \nabla s = \kappa_g \Delta\left(\frac{1}{\rho_g}\right). \tag{B.1}$$

Plugging (3.2e) into the left hand side of (B.1) and using (3.2b), we have

$$c_v\left\{\frac{\partial_t p_g}{p_g} - \frac{\gamma}{\rho_g}\left[\partial_t \rho_g + \mathbf{v}_g \cdot \nabla \rho_g\right]\right\} = \kappa_g \Delta\left(\frac{1}{\rho_g}\right). \tag{B.2}$$

Using (3.2a) in (B.2), we obtain

$$c_v\left\{\frac{\partial_t p_g}{p_g} + \gamma\,\mathrm{div}\mathbf{v}_g\right\} = \kappa_g \Delta\left(\frac{1}{\rho_g}\right), \tag{B.3}$$

Therefore, the system (3.2) is reduced to

$$\left.\begin{array}{l} \partial_t \rho_g + \mathrm{div}(\rho_g \mathbf{v}_g) = 0, \\[3mm] \dfrac{\partial_t p_g}{p_g} = \dfrac{\kappa}{c_v}\Delta\left(\dfrac{1}{\rho_g}\right) - \gamma\,\mathrm{div}\mathbf{v}_g, \end{array}\right\} \text{ in } \Omega(t), t > 0.$$

(B.4a)

(B.4b)

Expanding the term $\mathrm{div}(\rho_g \mathbf{v}_g)$ in (B.4a) and substituting $\mathrm{div}\mathbf{v}_g$ using (B.4b), and using the elementary identity

$$\rho_g \Delta\left(\frac{1}{\rho_g}\right) = -\Delta \log \rho_g + \frac{|\nabla \rho_g|^2}{\rho_g^2}, \tag{B.5}$$

we get

$$\partial_t \rho_g = \frac{\kappa}{\gamma c_v} \Delta \log \rho_g - \frac{\kappa}{\gamma c_v} \frac{|\nabla \rho_g|^2}{\rho_g^2} - \mathbf{v}_g \cdot \nabla \rho_g + \frac{\partial_t p_g}{\gamma p_g} \rho_g. \tag{B.6}$$

Assuming the bubble is a sphere $B_{R(t)}$ and solutions are spherically symmetric, and recalling we denoted the radial components of the gas and liquid velocity by $v_g(r,t)$ and $v_l(r,t)$, respectively, the systems (3.1) and (B.4) become

$$\left. \begin{aligned} \partial_t v_l = v_l \left(\Delta_r v_l - \frac{2v_l}{r^2} \right) - v_l \partial_r v_l - \frac{1}{\rho_l} \partial_r p_l, \\ \frac{1}{r^2} \partial_r (r^2 v_l) = 0, \end{aligned} \right\} \text{ for } r \geq R(t), t > 0, \tag{B.7a}$$
$$\tag{B.7b}$$

and

$$\left. \begin{aligned} \partial_t \rho_g + \frac{1}{r^2} \partial_r (\rho_g r^2 v_g) = 0, \\ \frac{\partial_t p_g}{p_g} = \frac{\kappa}{c_v} \frac{1}{r^2} \partial_r \left(r^2 \partial_r \left(\frac{1}{\rho_g} \right) \right) - \gamma \frac{1}{r^2} \partial_r (r^2 v_g), \end{aligned} \right\} \text{ for } 0 \leq r \leq R(t), t > 0, \tag{B.8a}$$
$$\tag{B.8b}$$

and the boundary condition (3.3) becomes

$$\left. \begin{aligned} v_l(R(t), t) = v_g(R(t), t) = \dot{R}(t), \\ p_g(t) - p_l(R(t), t) + 2\mu_l \partial_r v_l(R(t), t) = \frac{2\sigma}{R(t)}, \\ T(R(t), t) = T_\infty, \end{aligned} \right\} \text{ for } t > 0. \tag{B.9a}$$
$$\tag{B.9b}$$
$$\tag{B.9c}$$

The liquid velocity and pressure (v_l, p_l) can be directly solved in terms of $R(t)$, $\dot{R}(t)$, the liquid pressure $p_l(R(t), t)$ on the bubble wall, and the far-field liquid pressure $p_\infty(t) := p_l(r = \infty, t)$. In fact, the incompressibility condition (B.7b) and the kinematic boundary condition (B.9a) imply

$$v_l(r, t) = \frac{(R(t))^2 \dot{R}(t)}{r^2}, \quad r \geq R(t), \ t > 0. \tag{B.10}$$

Plugging Equation (B.10) into Equation (B.7a), we have

$$\frac{2R\dot{R}^2 + R^2\ddot{R}}{r^2} = v_l \left(\frac{2R^2\dot{R}}{r^4} - 2\frac{R^2\dot{R}}{r^4} \right) + 2\frac{R^4\dot{R}^2}{r^5} - \frac{1}{\rho_l} \partial_r p_l, \quad r \geq R(t), \ t > 0. \tag{B.11}$$

Note that the diffusion term in (B.11) vanishes. So the reduction using spherical symmetry assumption also works for Euler equation, i.e., we can take $v_l = 0$ in (3.1a). Integrating Equation (B.11) over $r > R(t)$, we deduce

$$p_l(r, t) = p_\infty(t) + \rho_l \left(\frac{2R(t)(\dot{R}(t))^2 + (R(t))^2 \ddot{R}(t)}{r} - \frac{(R(t))^4 (\dot{R}(t))^2}{2r^4} \right), \quad r \geq R(t), \ t > 0. \tag{B.12}$$

In particular, on the boundary the liquid pressure is

$$p_l(R(t), t) = p_\infty(t) + \rho_l \left(\frac{3}{2} \dot{R}^2 + R\ddot{R} \right), \quad t > 0.$$

Moreover, (B.10) implies $\partial_r v_l(r, t) = -2(R(t))^2 \dot{R}(t)/r^3$ so that

$$\partial_r v_l(R(t), t) = -2 \frac{\dot{R}(t)}{R(t)}.$$

This implies

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{1}{\rho_l} \left(p_g(t) - p_\infty(t) - \frac{2\sigma}{R} - 4\mu_l \frac{\dot{R}}{R} \right), \quad t > 0, \quad \text{(B.13)}$$

where the Young–Laplace boundary condition (B.9b) has been used.
For the gas dynamics in the bubble, by integrating (B.8b) in r, the radial component of the gas velocity v_g can be expressed in terms of $\rho_g(r, t)$, $\partial_r \rho_g(r, t)$, $p_g(t)$, and $\partial_t p_g(t)$. To be more precise,

$$v_g(r, t) = \frac{\kappa}{\gamma c_v} \partial_r \left(\frac{1}{\rho_g(r, t)} \right) - \frac{\partial_t p_g(t)}{p_g(t)} \frac{r}{3\gamma}, \quad 0 \le r \le R(t), \ t > 0.$$

$$\text{(B.14)}$$

Using (B.14) we can eliminate $\mathbf{v}_g$ in (B.6) and obtain

$$\partial_t \rho_g = \frac{\kappa}{\gamma c_v} \Delta_r \log \rho_g + \frac{\partial_t p_g}{3\gamma p_g} r \partial_r \rho_g + \frac{\partial_t p_g}{\gamma p_g} \rho_g, \quad 0 \le r \le R(t), \ t > 0,$$

$$\text{(B.15)}$$

where $\Delta_r f = \frac{1}{r^2} \partial_r (r^2 \partial_r f)$ for spherically symmetric functions f. From the boundary condition (B.9c) for the gas temperature and the equation of state (3.2d),

$$p_g(t) = \mathscr{R}_g \rho_g(R(t), t) T_\infty. \quad \text{(B.16)}$$

Taking time derivative of (B.16) we obtain

$$\frac{\partial_t p_g}{p_g} = \frac{\partial_t \rho_g(R(t), t)}{\rho_g(R(t), t)} + \frac{\partial_r \rho_g(R(t), t)}{\rho_g(R(t), t)} \dot{R}(t). \quad \text{(B.17)}$$

Evaluating (B.14) at $r = R(t)$ and using the kinematic boundary condition (B.9a) it follows that

$$\dot{R}(t) = -\frac{\kappa}{\gamma c_v} \frac{\partial_r \rho_g(R(t), t)}{(\rho_g(R(t), t))^2} - \frac{R(t)}{3\gamma} \frac{\partial_t p_g}{p_g}. \quad \text{(B.18)}$$

For the boundary data for the gas density, we use (B.16) and (B.13) to deduce

$$\rho_g(R(t), t) = \frac{1}{\mathscr{R}_g T_\infty} \left[p_\infty + \frac{2\sigma}{R} + \rho_l \left(R\ddot{R} + \frac{3}{2}(\dot{R})^2 \right) \right]. \quad \text{(B.19)}$$

Collecting the results (B.15), (B.16), (B.18), (B.19), we conclude that, under the spherical symmetry assumption, the system (3.1)–(3.3) is reduced to a system of $(\rho(r, t), R(t))$:

$$\partial_t \rho = \frac{\kappa}{\gamma c_v} \Delta_r \log \rho + \frac{\partial_t p}{3\gamma p} r \partial_r \rho + \frac{\partial_t p}{\gamma p} \rho, \quad 0 \leq r \leq R(t), \ t > 0,$$

$$\tag{B.20}$$

$$p(t) = \mathscr{R}_g T_\infty \rho(R(t), t), \quad t > 0, \tag{B.21}$$

$$\dot{R}(t) = -\frac{\kappa}{\gamma c_v} \frac{\partial_r \rho(R(t), t)}{(\rho(R(t), t))^2} - \frac{R(t)}{3\gamma} \frac{\partial_t p}{p}, \quad t > 0, \tag{B.22}$$

$$\rho(R(t), t) = \frac{1}{\mathscr{R}_g T_\infty} \left[p_\infty + \frac{2\sigma}{R} + 4\mu_l \frac{\dot{R}}{R} + \rho_l \left(R\ddot{R} + \frac{3}{2}(\dot{R})^2 \right) \right], \quad t > 0, \tag{B.23}$$

where $\rho \equiv \rho_g$, $p \equiv p_g$, $\kappa = \kappa_g$. This is the reduced system (5.1a)–(5.1c).

Appendix C: A Perspective on Coercive Energy Estimate of Biro–Velázquez, and an Extension

In this appendix, we prove Theorem 7.5, which extends the coercivity estimate of Biro-Velázquez to the case where $p_\infty - p_{\infty,*}$ is small in norm.

Proof of Theorem 7.5. Let us recall the total energy

$$\mathcal{E}_{\text{total}} = FE + KE_l + U_{g-l} + PV_{p_\infty},$$

where

$$FE = \frac{4\pi c_v}{3\mathscr{R}_g} pR^3 - c_v T_\infty M_0 \log p + c_v \gamma T_\infty \int_{B_R} \rho \log \rho, \quad M_0 = \text{Mass}[\rho, R], \tag{C.1a}$$

$$KE_l = 2\pi \rho_l R^3 \dot{R}^2, \tag{C.1b}$$

$$U_{g-l} = 4\pi \sigma R^2, \tag{C.1c}$$

$$PV_{p_\infty} = \frac{4\pi}{3} R^3 p_\infty. \tag{C.1d}$$

The energy is a functional of state variables, which are defined on a deforming regime, B_R. We fix the region to be B_1 by setting $x = Ry$, where $y \in B_1$. Defining $\overline{\rho}(y) = \rho(Rr)$ and using the constitutive relation $p = \mathscr{R}_g T_\infty \rho(R) = \mathscr{R}_g T_\infty \overline{\rho}(1)$ we have that

$$FE = \frac{4\pi c_v T_\infty}{3} \overline{\rho}(1) R^3 - c_v T_\infty M_0 \log(\mathscr{R}_g T_\infty) - c_v T_\infty M_0 \log \overline{\rho}(1)$$

$$+ c_v \gamma T_\infty R^3 \int_{B_1} \overline{\rho} \log \overline{\rho}.$$

Thus, $\mathcal{E}_{\text{total}}$ is a functional of $(\overline{\rho}, R, \dot{R})$:

$$
\begin{aligned}
\mathcal{E}_{\text{total}} &= \mathcal{E}_{\text{total}}[\overline{\rho}, R, \dot{R}] \\
&= \frac{4\pi c_v T_\infty}{3} \overline{\rho}(1) R^3 - c_v T_\infty M_0 \log(\mathscr{R}_g T_\infty) - c_v T_\infty M_0 \log \overline{\rho}(1) \\
&\quad + c_v \gamma T_\infty R^3 \int_{B_1} \overline{\rho} \log \overline{\rho} \\
&\quad + 2\pi \rho_l R^3 \dot{R}^2 + 4\pi \sigma R^2 + \frac{4\pi}{3} R^3 p_\infty.
\end{aligned}
$$

We set $\overline{\rho} = \rho_* + \varrho$, $R = R_* + \mathcal{R}$ and expand the total energy $\mathcal{E}_{\text{total}}[\overline{\rho}, R, \dot{R}]$ at $(\rho_*, R_*, \dot{R}_* = 0)$ along the mass preserving hypersurface $M_0 = \text{Mass}[\rho, R]$:

$$
\mathcal{E}_{\text{total}}[\overline{\rho}, R, \dot{R}] = \mathcal{E}_* + d\mathcal{E}_*[\varrho, \mathcal{R}, \dot{\mathcal{R}}] + \frac{1}{2} d^2 \mathcal{E}_*[\varrho, \mathcal{R}, \dot{\mathcal{R}}] + O(|(\varrho, \mathcal{R}, \dot{\mathcal{R}})|^3), \quad \text{where}
$$

$$
\begin{aligned}
\mathcal{E}_* &= \mathcal{E}_{\text{total}}[\rho_*, R_*, \dot{R}_* = 0] \\
d\mathcal{E}_*[\varrho, \mathcal{R}, \dot{\mathcal{R}}] &= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathcal{E}_{\text{total}}(\rho_*, R_* + \varepsilon \mathcal{R}, \varepsilon \dot{\mathcal{R}}) \\
d^2 \mathcal{E}_*[\varrho, \mathcal{R}, \dot{\mathcal{R}}] &= \frac{d^2}{d\varepsilon^2}\bigg|_{\varepsilon=0} \mathcal{E}_{\text{total}}(\rho_*, R_* + \varepsilon \mathcal{R}, \varepsilon \dot{\mathcal{R}})
\end{aligned}
\tag{C.2}
$$

To expand along the mass preserving hypersurface, we first use $M_0 = \int_{B_R} \rho \, dx$ to rewrite

$$
\begin{aligned}
R^3 \int_{B_1} \overline{\rho} \log \overline{\rho} &= \int_{B_R} \rho \log \rho = \int_{B_R} \rho \log \rho_* + \int_{B_R} \rho \log \frac{\rho}{\rho_*} \\
&= \log \rho_* \int_{B_R} \rho + \int_{B_R} \rho + \int_{B_R} \rho \left(\log \frac{\rho}{\rho_*} - 1 \right) \\
&= M_0 \log \rho_* + M_0 + R^3 \int_{B_1} \overline{\rho} \left(\log \frac{\overline{\rho}}{\rho_*} - 1 \right),
\end{aligned}
$$

giving the following expression for the total energy:

$$
\begin{aligned}
\mathcal{E}_{\text{total}}[\overline{\rho}, R, \dot{R}] &= \frac{4\pi c_v T_\infty}{3} \overline{\rho}(1, t) R^3 - c_v T_\infty M_0 \log(\mathscr{R}_g T_\infty) - c_v T_\infty M_0 \log \overline{\rho}(1, t) \\
&\quad + c_v \gamma T_\infty M_0 \log \rho_* + c_v \gamma T_\infty M_0 + c_v \gamma T_\infty R^3 \int_{B_1} \overline{\rho} \left(\log \frac{\overline{\rho}}{\rho_*} - 1 \right) \\
&\quad + 2\pi \rho_l R^3 \dot{R}^2 + 4\pi \sigma R^2 + \frac{4\pi}{3} R^3 p_\infty.
\end{aligned}
$$

To expand the logarithmic terms we note that for $z_* \neq 0$ and $|z - z_*| < \frac{1}{2}|z_*|$:

$$
\left| \left(z \left(\log \left(\frac{z}{z_*} \right) - 1 \right) \right) - \left(-z_* + \frac{1}{2z_*}(z - z_*)^2 \right) \right| \leq \frac{2}{|z_*|} |z - z_*|^3
\tag{C.3}
$$

$$
\left| \log z - \left(\log z_* + \frac{1}{z_*}(z - z_*) - \frac{1}{2z_*^2}(z - z_*)^2 \right) \right| \leq \frac{2}{3|z_*|^3} |z - z_*|^3
\tag{C.4}
$$

Applying (C.3) and (C.4) we have

$$
\mathcal{E}_{\text{total}}[\overline{\rho}, R, \dot{R}] = \frac{4\pi c_v T_\infty}{3} \overline{\rho}(1, t) R^3 - c_v T_\infty M_0 \log(\mathscr{R}_g T_\infty)
$$

$$
- c_v T_\infty M_0 \left(\log \rho_* + \frac{1}{\rho_*} \varrho(1) - \frac{1}{2\rho_*^2} \varrho(1)^2 \right) + O\left(\varrho(1)^3 \right)
$$

$$
+ c_v \gamma T_\infty M_0 \log \rho_* + c_v \gamma T_\infty M_0
$$

$$
+ c_v \gamma T_\infty R^3 \left(-\frac{4\pi}{3} \rho_* + \frac{1}{2\rho_*} \int_{B_1} \varrho^2 \right) + O\left(R^3 \int_{B_1} |\varrho|^3 \right)
$$

$$
+ 2\pi \rho_l R^3 \dot{R}^2 + 4\pi \sigma R^2 + \frac{4\pi}{3} R^3 p_\infty.
$$

Rearranging and simplifying gives

$$
\mathcal{E}_{\text{total}}[\overline{\rho}, R, \dot{R}] = -c_v T_\infty M_0 \log(\mathscr{R}_g T_\infty) + c_v(\gamma - 1) T_\infty M_0 \log \rho_* + c_v \gamma T_\infty M_0
$$

$$
+ \frac{4\pi c_v T_\infty}{3} \overline{\rho}(1) R^3 + c_v T_\infty M_0 \left(-\frac{1}{\rho_*} \varrho(1) + \frac{1}{2\rho_*^2} \varrho(1)^2 \right) + O\left(\varrho(1)^3 \right)
$$

$$
+ c_v \gamma T_\infty R^3 \left(-\frac{4\pi}{3} \rho_* + \frac{1}{2\rho_*} \int_{B_1} \varrho^2 \right) + O\left(R^3 \int_{B_1} |\varrho|^3 \right)
$$

$$
+ 2\pi \rho_l R^3 \dot{R}^2 + 4\pi \sigma R^2 + \frac{4\pi}{3} R^3 p_\infty. \tag{C.5}
$$

Verification that $d\mathcal{E}_*[\varrho, \mathcal{R}, \dot{\mathcal{R}}] = 0$ **when** $p_\infty = p_{\infty,*}$. Starting with (C.5) we calculate:

$$
d\mathcal{E}_*[\varrho, \mathcal{R}, \dot{\mathcal{R}}] = \frac{4\pi c_v T_\infty}{3} \left(3\rho_* R_*^2 \mathcal{R} + R_*^3 \varrho(1) \right) - \frac{c_v T_\infty}{\rho_*} \left(\frac{4\pi}{3} \rho_* R_*^3 \right) \varrho(1)
$$

$$
- 4\pi c_v \gamma T_\infty R_*^2 \rho_* \mathcal{R} + 8\pi \sigma R_* \mathcal{R} + 4\pi R_*^2 p_\infty \mathcal{R}
$$

$$
= 4\pi R_*^2 \left(c_v T_\infty \rho_*(1 - \gamma) + \frac{2\sigma}{R_*} + p_\infty \right) \mathcal{R}
$$

$$
= 4\pi R_*^2 \left(-\mathscr{R}_g T_\infty \rho_* + \frac{2\sigma}{R_*} + p_\infty \right) \mathcal{R} \quad \left[\gamma - 1 = \frac{\mathscr{R}_g}{c_v} \text{ by (2.3)} \right]
$$

$$
= 4\pi R_*^2 \mathcal{P}_\infty \mathcal{R} \quad \left[\text{by (4.14b)} \right], \tag{C.6}
$$

where $\mathcal{P}_\infty = p_\infty - p_{\infty,*}$. It is readily to see that $d\mathcal{E}_*[\varrho, \mathcal{R}, \dot{\mathcal{R}}] = 0$ when $p_\infty = p_{\infty,*}$.

Computation of $\frac{1}{2} d^2 \mathcal{E}[\varrho, \mathcal{R}, \dot{\mathcal{R}}]$.
From (C.5) we compute the quadratic terms:

$$
\frac{1}{2} d^2 \mathcal{E}[\varrho, \mathcal{R}, \dot{\mathcal{R}}] = 4\pi c_v T_\infty \left(\rho_* R_* \mathcal{R}^2 + R_*^2 \varrho(1) \mathcal{R} \right) + \frac{c_v T_\infty M_0}{2\rho_*^2} \varrho(1)^2
$$

$$
+ \frac{c_v \gamma T_\infty R_*^3}{2\rho_*} \int_{B_1} \varrho^2 - 4\pi c_v \gamma \rho_* T_\infty R_* \mathcal{R}^2
$$

$$
+ 2\pi \rho_l R_*^3 \dot{\mathcal{R}}^2 + 4\pi \sigma \mathcal{R}^2 + 4\pi R_* p_\infty \mathcal{R}^2. \tag{C.7}
$$

Next, using that the perturbed bubble is assumed to have mass equal to $M_0 = \text{Mass}(\rho_*, R_*)$, we express the cross-term just above in terms of a quadratic expression in ϱ as follows:

$$M_0 = R^3 \int_{B_1} \overline{\rho} = (R_* + \mathcal{R})^3 \int_{B_1} (\rho_* + \varrho)$$

$$= M_0 + 4\pi R_*^2 \rho_* \mathcal{R} + R_*^3 \int_{B_1} \varrho + O\left(\mathcal{R}^2 + \left(\int_{B_1} \varrho\right)^2\right)$$

and therefore

$$\mathcal{R} = -\frac{R_*}{4\pi \rho_*} \int_{B_1} \varrho + O\left(\mathcal{R}^2 + \left(\int_{B_1} \varrho\right)^2\right). \tag{C.8}$$

Substitution of (C.8) into (C.7) we obtain a leading expression entirely in terms of the perturbed density ϱ. We list the various terms that we rewrite exclusively in terms of ϱ:

$$4\pi c_v T_\infty \rho_* R_* \mathcal{R}^2 = \frac{c_v T_\infty \rho_* R_*^3}{4\pi \rho_*^2} \left(\int_{B_1} \varrho\right)^2 + O\left(|\mathcal{R}|^3 + \left(\int_{B_1} |\varrho|\right)^3\right)$$

$$4\pi c_v T_\infty R_*^2 \varrho(1) \mathcal{R} = -\frac{c_v T_\infty R_*^3}{\rho_*} \varrho(1) \int_{B_1} \varrho + O\left(|\mathcal{R}|^3 + |\varrho(1)|^3 + \left(\int_{B_1} |\varrho|\right)^3\right)$$

$$-4\pi c_v \gamma \rho_* T_\infty R_* \mathcal{R}^2 = -\frac{c_v \gamma T_\infty R_*^3}{4\pi \rho_*} \left(\int_{B_1} \varrho\right)^2 + O\left(|\mathcal{R}|^3 + \left(\int_{B_1} |\varrho|\right)^3\right)$$

$$4\pi \left(\sigma + R_* p_\infty\right) \mathcal{R}^2 = \frac{1}{4\pi \rho_*^2} \left(\frac{\sigma}{R_*} + p_\infty\right) R_*^3 \left(\int_{B_1} \varrho\right)^2 + O\left(|\mathcal{R}|^3 + \left(\int_{B_1} |\varrho|\right)^3\right).$$

Inserting these expressions into (C.7), we obtain

$$\frac{1}{2} d^2 \mathcal{E}[\varrho, \mathcal{R}, \dot{\mathcal{R}}] = \frac{c_v T_\infty M_0}{2\rho_*^2} \varrho(1)^2 + \frac{c_v \gamma T_\infty R_*^3}{2\rho_*} \int_{B_1} \varrho^2 + 2\pi \rho_l R_*^3 \dot{\mathcal{R}}^2$$

$$+ \frac{R_*^3}{4\pi \rho_*^2} \left(c_v(1-\gamma) T_\infty \rho_* + \frac{\sigma}{R_*} + p_\infty\right) \left(\int_{B_1} \varrho\right)^2 - \frac{c_v T_\infty R_*^3}{\rho_*} \varrho(1) \int_{B_1} \varrho$$

$$+ O\left(|\mathcal{R}|^3 + |\varrho(1)|^3 + \left(\int_{B_1} |\varrho|\right)^3\right) \tag{C.9}$$

The coefficient of the fourth term on the right of (C.9) can be simplified using the relation $1 - \gamma = -\mathcal{R}_g / c_v$ and the relation $\mathcal{R}_g T_\infty \rho_* = p_{\infty,*} + 2\sigma/R_*$ between the equilibrium density and bubble radius:

$$c_v(1-\gamma) T_\infty \rho_* + \frac{\sigma}{R_*} + p_\infty = -\mathcal{R}_g T_\infty \rho_* + \frac{\sigma}{R_*} + p_\infty = -\frac{\sigma}{R_*} + \mathcal{P}_\infty,$$

where $\mathcal{P}_\infty = p_\infty - p_{\infty,*}$. Thus,

$$
\begin{aligned}
\frac{1}{2} d^2 \mathcal{E}[\varrho, \mathcal{R}, \dot{\mathcal{R}}] = {} & \frac{c_v T_\infty M_0}{2\rho_*^2} \varrho(1)^2 + \frac{c_v \gamma T_\infty R_*^3}{2\rho_*} \int_{B_1} \varrho^2 + 2\pi \rho_l R_*^3 \dot{\mathcal{R}}^2 \\
& - \frac{\sigma R_*^2}{4\pi \rho_*^2} \left(\int_{B_1} \varrho \right)^2 + \frac{R_*^3}{4\pi \rho_*^2} \mathcal{P}_\infty \left(\int_{B_1} \varrho \right)^2 - \frac{c_v T_\infty R_*^3}{\rho_*} \varrho(1) \int_{B_1} \varrho \\
& + O \left(|\mathcal{R}|^3 + |\varrho(1)|^3 + \left(\int_{B_1} |\varrho| \right)^3 \right)
\end{aligned}
\tag{C.10}
$$

Using that $M_0 = \rho_* R_*^3 |B_1|$, we may rewrite (C.10) as

$$
\begin{aligned}
& \frac{1}{2} d^2 \mathcal{E}[\varrho, \mathcal{R}, \dot{\mathcal{R}}] \\
& = \frac{c_v T_\infty R_*^3 |B_1|}{2} \left(\frac{\varrho(1)^2}{\rho_*} - 2 \frac{\varrho(1)}{\rho_*} \frac{1}{|B_1|} \int_{B_1} \varrho \right) + \frac{c_v \gamma T_\infty R_*^3}{2\rho_*} \int_{B_1} \varrho^2 + 2\pi \rho_l R_*^3 \dot{\mathcal{R}}^2 \\
& \quad - \frac{\sigma R_*^2}{4\pi \rho_*^2} \left(\int_{B_1} \varrho \right)^2 + \frac{R_*^3}{4\pi \rho_*^2} \mathcal{P}_\infty \left(\int_{B_1} \varrho \right)^2 + O \left(|\mathcal{R}|^3 + |\varrho(1)|^3 + \left(\int_{B_1} |\varrho| \right)^3 \right)
\end{aligned}
\tag{C.11}
$$

or

$$
\begin{aligned}
\frac{1}{2} d^2 \mathcal{E}[\varrho, \mathcal{R}, \dot{\mathcal{R}}] = {} & \frac{c_v T_\infty R_*^3 |B_1|}{2\rho_*} \left(\varrho(1) - \frac{1}{|B_1|} \int_{B_1} \varrho \right)^2 + \frac{c_v \gamma T_\infty R_*^3}{2\rho_*} \int_{B_1} \varrho^2 + 2\pi \rho_l R_*^3 \dot{\mathcal{R}}^2 \\
& - \left[\frac{\sigma R_*^2}{4\pi \rho_*^2} + \frac{c_v T_\infty R_*^3}{2\rho_* |B_1|} \right] \left(\int_{B_1} \varrho \right)^2 + \frac{R_*^3}{4\pi \rho_*^2} \mathcal{P}_\infty \left(\int_{B_1} \varrho \right)^2 \\
& + O \left(|\mathcal{R}|^3 + |\varrho(1)|^3 + \left(\int_{B_1} |\varrho| \right)^3 \right)
\end{aligned}
\tag{C.12}
$$

By the Cauchy-Schwarz inequality $\left(\int_{B_1} \varrho \right)^2 \leq |B_1| \int_{B_1} \varrho^2$ and therefore

$$
\begin{aligned}
\frac{1}{2} d^2 \mathcal{E}[\varrho, \mathcal{R}, \dot{\mathcal{R}}] \geq {} & \frac{c_v T_\infty R_*^3 |B_1|}{2\rho_*} \left(\varrho(1) - \frac{1}{|B_1|} \int_{B_1} \varrho \right)^2 + 2\pi \rho_l R_*^3 \dot{\mathcal{R}}^2 \\
& + \left(\frac{c_v \gamma T_\infty R_*^3}{2\rho_*} - \left[\frac{\sigma R_*^2 |B_1|}{4\pi \rho_*^2} + \frac{c_v T_\infty R_*^3}{2\rho_*} \right] \right) \int_{B_1} \varrho^2 + \frac{R_*^3}{4\pi \rho_*^2} \mathcal{P}_\infty \left(\int_{B_1} \varrho \right)^2 \\
& + O \left(|\mathcal{R}|^3 + |\varrho(1)|^3 + \left(\int_{B_1} |\varrho| \right)^3 \right)
\end{aligned}
\tag{C.13}
$$

Finally, we find for the constant in (C.13) that

$$
\frac{c_v \gamma T_\infty R_*^3}{2\rho_*} - \left[\frac{\sigma R_*^2 |B_1|}{4\pi \rho_*^2} + \frac{c_v T_\infty R_*^3}{2\rho_*} \right] = \frac{R_*^3}{\rho_*^2} \left(\frac{p_{\infty,*}}{2} + \frac{2\sigma}{3 R_*} \right).
\tag{C.14}
$$

This follows, yet again, from the relations $\gamma - 1 = \mathcal{R}_g / c_v$ and $\mathcal{R}_g T_\infty \rho_* = p_{\infty,*} + 2\sigma / R_*$. For the term involving $\mathcal{P}_\infty$, using the Cauchy-Schwarz inequality $\left(\int_{B_1} \varrho \right)^2 \leq |B_1| \int_{B_1} \varrho^2$,

$$
\mathcal{P}_\infty \left(\int_{B_1} \varrho \right)^2 \geq -|\mathcal{P}_\infty| |B_1| \int_{B_1} \varrho^2.
$$

Summarizing

$$
\begin{aligned}
\mathcal{E}_{\text{total}} - \mathcal{E}_* \geq{} & \frac{c_v T_\infty R_*^3 |B_1|}{2\rho_*} \left(\varrho(1) - \frac{1}{|B_1|} \int_{B_1} \varrho \right)^2 \\
& + 2\pi \rho_l R_*^3 \dot{\mathcal{R}}^2 + \frac{R_*^3}{\rho_*^2} \left(\frac{p_{\infty,*}}{2} + \frac{2\sigma}{3R_*} \right) \int_{B_1} \varrho^2 \\
& - 4\pi R_*^2 |\mathcal{P}_\infty| |\mathcal{R}| - \frac{R_*^3}{4\pi \rho_*^2} |\mathcal{P}_\infty| |B_1| \int_{B_1} \varrho^2 \\
& + O\left(|\mathcal{R}|^3 + |\varrho(1)|^3 + \left(\int_{B_1} |\varrho| \right)^3 \right),
\end{aligned}
\tag{C.15}
$$

where all explicit terms are non-negative except for the terms involving $\mathcal{P}_\infty$ which can be made small since $|\mathcal{P}_\infty| = |p_\infty - p_{\infty,*}| \leq \delta_0$.

We now conclude the proof by bounding the error term in (C.15) from above by a sufficiently small constant times $\int_{B_1} \varrho^2$.

For the fourth term on the right hand side of (C.15), using (C.8), in terms of the perturbed density ϱ,

$$
-4\pi R_*^2 |\mathcal{P}_\infty| |\mathcal{R}| \geq -\frac{R_*^3}{\rho_*} |\mathcal{P}_\infty| \int_{B_1} |\varrho| - C_0 |\mathcal{P}_\infty| \left(\int_{B_1} \varrho \right)^2
$$

for some constant $C_0 > 0$. Since $|\varrho| \leq \delta_0 \leq 1$ and $|\mathcal{P}_\infty| = |p_\infty - p_{\infty,*}| \leq \delta_0$, by the Cauchy-Schwarz inequality $\left(\int_{B_1} \varrho \right)^2 \leq |B_1| \int_{B_1} \varrho^2$

$$
-4\pi R_*^2 |\mathcal{P}_\infty| |\mathcal{R}| \geq -\frac{R_*^3}{\rho_*} |\mathcal{P}_\infty| \int_{B_1} \varrho^2 - C_0 |\mathcal{P}_\infty| |B_1| \int_{B_1} \varrho^2 \geq -C_1 \delta_0 \int_{B_1} \varrho^2
$$

for some constant $C_1 > 0$.

Now we estimate the cubic term in the third line on the right hand side of (C.15). Since $M_0 = \text{Mass}[\rho, R]$,

$$
\int_{B_R} (\rho - \rho_*)\, dx = \int_{B_R} \varrho\, dx = M_0 - \frac{4\pi R^3}{3} \rho_* = \frac{4\pi R_*^3}{3} \rho_* - \frac{4\pi R^3}{3} \rho_*,
$$

or

$$
\frac{4\pi \rho_*}{3} (R^3 - R_*^3) = -\int_{B_R} (\rho - \rho_*)\, dx,
\tag{C.16}
$$

which implies

$$
\begin{aligned}
|\mathcal{R}| = |R - R_*| &\leq \frac{3}{4\pi \rho_* (R^2 + R R_* + R_*^2)} |B_R|^{\frac{1}{2}} \left(\int_{B_R} |\rho - \rho_*|^2\, dx \right)^{\frac{1}{2}} \\
&\leq C_2 \left(\int_{B_R} |\rho - \rho_*|^2\, dx \right)^{\frac{1}{2}},
\end{aligned}
\tag{C.17}
$$

where $C_2 > 0$ depends only on ν, M_0, T_∞. We now control $|\varrho(1,t)|^3$ by the first and the third terms on the right hand side of (C.15). Indeed,

$$
|\varrho(1)| = \left| \left(\varrho(1) - \frac{1}{|B_1|} \int_{B_1} \varrho \right) + \frac{1}{|B_1|} \int_{B_1} \varrho \right|
$$

$$
\leq \left| \varrho(1) - \frac{1}{|B_1|} \int_{B_1} \varrho \right| + \frac{1}{|B_1|^{\frac{1}{2}}} \left(\int_{B_1} \varrho^2 \right)^{\frac{1}{2}}
$$

$$
\leq C_3 \left\{ \left| \varrho(1) - \frac{1}{|B_1|} \int_{B_1} \varrho \right| + \left(\int_{B_1} \varrho^2 \right)^{\frac{1}{2}} \right\}
$$

for some $C_3 > 0$ depending only on ν, M_0, T_∞. Since $|\varrho(1)| = |\rho(R) - \rho_*| \leq \delta_0$,

$$
|\varrho(1)|^3 = |\varrho(1)||\varrho(1)|^2 \leq \delta_0 C_3 \left\{ \left| \varrho(1) - \frac{1}{|B_1|} \int_{B_1} \varrho \right| + \left(\int_{B_1} \varrho^2 \right)^{\frac{1}{2}} \right\}^2 \tag{C.18}
$$

$$
\leq 2\delta_0 C_3 \left| \varrho(1) - \frac{1}{|B_1|} \int_{B_1} \varrho \right|^2 + 2\delta_0 C_3 \int_{B_1} \varrho^2.
$$

Using (C.17) and (C.18), one has

$$
O\left(|\mathcal{R}|^3 + |\varrho(1)|^3 + \left(\int_{B_1} |\varrho| \right)^3 \right)
$$

$$
\geq -C \left(\int_{B_R} (\rho - \rho_*)^2 \right)^{\frac{3}{2}} - C\delta_0 \left| \varrho(1) - \frac{1}{|B_1|} \int_{B_1} \varrho \right|^2 - C\delta_0 \int_{B_1} \varrho^2 - C \int_{B_1} |\varrho|^3
$$

for some $C > 0$ depending only on ν, M_0, T_∞.

Consequently, using $|\mathcal{P}_\infty| = |p_\infty - p_{\infty,*}| \leq \delta_0$, (C.15) can be further computed as

$$
\mathcal{E}_{\text{total}} - \mathcal{E}_* \geq -C_1 \delta_0 \int_{B_1} \varrho^2 + \left(\frac{c_v T_\infty R_*^3 |B_1|}{2\rho_*} - C\delta_0 \right) \left(\varrho(1) - \frac{1}{|B_1|} \int_{B_1} \varrho \right)^2
$$

$$
+ \frac{R_*^3}{\rho_*} \left(\frac{p_{\infty,*}}{2} + \frac{2\sigma}{3R_*} \right) \int_{B_1} \varrho^2
$$

$$
- C\delta_0 |B_R|^{\frac{1}{2}} \int_{B_R} (\rho - \rho_*)^2 \, dx - 2C\delta_0 \int_{B_1} \varrho^2 - \delta_0 \frac{R_*^3}{4\pi\rho_*} |B_1| \int_{B_1} \varrho^2
$$

$$
\geq \Theta \left(\int_{B_R} (\rho - \rho_*)^2 \right)
$$

for some constant $\Theta > 0$, provided $\delta_0 > 0$ is sufficiently small. Note that we've used $\int_{B_1} \varrho^2 \, dy = R^{-3} \int_{B_R} (\rho - \rho_*)^2 \, dx$ in which $R^{-3} \geq \nu^3$ above. This completes the proof of Theorem 7.5.

Appendix D: An Interpolation Lemma

Lemma D.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $k < m$, and $0 < \gamma \le 1$. For $u \in C^\infty(\Omega)$,*

$$\left\| \nabla^k u \right\|_{L^\infty(\Omega)} \le C_1 \|u\|_{L^p(\Omega)}^\lambda \|u\|_{C^{m,\gamma}(\Omega)}^{1-\lambda} + C_2 \|u\|_{L^s(\Omega)}$$

for arbitrary $s \ge 1$, where $-\frac{k}{n} = \frac{\lambda}{p} - (1-\lambda)\frac{m}{n}$, and the constants C_1, C_2 depend on the domain Ω and on s in addition to the other parameters.

Proof. By Gagliardo–Nirenberg interpolation inequality,

$$\left\| \nabla^k u \right\|_{L^\infty(\Omega)} \le C_1 \|u\|_{L^p(\Omega)}^\lambda \left\| \nabla^m u \right\|_{L^\infty(\Omega)}^{1-\lambda} + C_2 \|u\|_{L^s(\Omega)}$$

for arbitrary $s \ge 1$, where

$$0 = \frac{k}{n} - \frac{m}{n}(1-\lambda) + \frac{\lambda}{p},$$

and the constants C_1, C_2 depend on the domain Ω and on s in addition to the other parameters. The lemma then follows since $\|\nabla^m u\|_{L^\infty(\Omega)} \le \|u\|_{C^{m,\gamma}(\Omega)}$. $\quad\square$

Appendix E: Estimate of the Exponential Decay Rate β in the Linearized System

In this appendix, we prove parts (2) and (3) of Theorem 9.3. In particular, we investigate the location of the roots of the meromorphic function $Q(\tau)$ defined in (9.20) which corresponds to the spectrum of the linear operator $\mathcal{L}$.

Lemma E.1. *There exists a negative upper bound for the real parts of the roots of the meromorphic function $Q(\tau)$ in (9.20). More precisely, there exists $\beta > 0$ such that $\xi < -\beta$ for all roots $\tau = \xi + i\eta$ of $Q(\tau)$. The constant β can be chosen as*

$$\beta = \min\left\{ \left(1 - \sqrt{\frac{1 - \frac{1}{\gamma}}{\frac{3p_{\infty,*}R_* + 6\sigma}{2p_{\infty,*}R_* + 6\sigma} - \frac{1}{\gamma}}} \right) \pi^2 \overline{\kappa}, \ \sqrt{\frac{\mathscr{R}_g T_\infty \rho_*}{\rho_l R_*^2}}, \right.$$

$$\frac{2\mu_l}{\rho_l R_*^2} + \mathbb{1}_{\Delta \le 0} \frac{\mathscr{R}_g T_\infty \rho_*}{\pi^4 \overline{\kappa} \rho_l R_*^2} \left(1 - \frac{1}{\gamma} \right) \left[\frac{\pi^4}{90} + O\left(\left(\frac{1}{\pi^2 \overline{\kappa}} \sqrt{\frac{\mathscr{R}_g T_\infty \rho_*}{\rho_l R_*^2}} \right)^{3/2} \right) \right]$$

$$\left. - \mathbb{1}_{\Delta > 0} \frac{\sqrt{\Delta}}{2\rho_l R_*} \right\},$$

(E.1)

in which

$$\Delta := \left(\frac{4\mu_l}{R_*} \right)^2 - 8\rho_l \mathscr{R}_g T_\infty \rho_*.$$

Proof of Lemma E.1. Let $\tau = \xi + i\eta$ be a root of $Q(\tau)$, i.e., $Q(\tau) = 0$. Plugging $\tau = \xi + i\eta, \xi \in \mathbb{R}, \eta \in \mathbb{R}$, into (9.20), we have

$$Q(\xi + i\eta) = \frac{1}{\mathscr{R}_g T_\infty} \left(\Xi_1 + i H_1\right)\left(\Xi_2 + i H_2\right) + 4\pi \frac{\rho_*}{R_*},$$

where

$$\begin{aligned}
\Xi_1 &= \frac{4\pi}{3\gamma} + \frac{8(\gamma - 1)}{\pi\gamma} \sum_{j=1}^{\infty} \frac{\pi^2 \overline{\kappa} \left(\pi^2 \overline{\kappa} j^2 + \xi\right)}{\left(\pi^2 \overline{\kappa} j^2 + \xi\right)^2 + \eta^2}, \\
H_1 &= -\frac{8(\gamma - 1)}{\pi\gamma} \sum_{j=1}^{\infty} \frac{\pi^2 \overline{\kappa} \eta}{\left(\pi^2 \overline{\kappa} j^2 + \xi\right)^2 + \eta^2}, \\
\Xi_2 &= \rho_l R_* \left(\xi^2 - \eta^2\right) + \frac{4\mu_l}{R_*} \xi - \frac{2\sigma}{R_*^2}, \\
H_2 &= \rho_l R_*(2\xi\eta) + \frac{4\mu_l}{R_*} \eta.
\end{aligned}$$

(E.2)

Setting real and imaginary parts of Q equal to zero, we obtain

$$\begin{aligned}
\text{real part:} \quad & \frac{1}{\mathscr{R}_g T_\infty} \left(\Xi_1 \Xi_2 - H_1 H_2\right) + 4\pi \frac{\rho_*}{R_*} = 0, \\
\text{imaginary part:} \quad & \frac{1}{\mathscr{R}_g T_\infty} \left(\Xi_1 H_2 + H_1 \Xi_2\right) = 0.
\end{aligned}$$

(E.3)

The real part in (E.3) reads

$$\begin{aligned}
0 &= \frac{1}{\mathscr{R}_g T_\infty} \left[\left(\frac{4\pi}{3\gamma} + \frac{8(\gamma - 1)}{\pi\gamma} \sum_{j=1}^{\infty} \frac{\pi^2 \overline{\kappa} \left(\pi^2 \overline{\kappa} j^2 + \xi\right)}{\left(\pi^2 \overline{\kappa} j^2 + \xi\right)^2 + \eta^2}\right) \left(\rho_l R_* \left(\xi^2 - \eta^2\right) + \frac{4\mu_l}{R_*} \xi - \frac{2\sigma}{R_*^2}\right) \right. \\
&\quad \left. + \frac{8(\gamma - 1)}{\pi\gamma} \sum_{j=1}^{\infty} \frac{\pi^2 \overline{\kappa} \eta^2}{\left(\pi^2 \overline{\kappa} j^2 + \xi\right)^2 + \eta^2} \left(\rho_l R_*(2\xi) + \frac{4\mu_l}{R_*}\right) \right] + 4\pi \frac{\rho_*}{R_*} \\
&= \frac{1}{\mathscr{R}_g T_\infty} \left[\left(\frac{4\pi}{3\gamma} + \frac{8(\gamma - 1)}{\pi\gamma} \sum_{j=1}^{\infty} \frac{\pi^4 \overline{\kappa}^2 j^2}{\left(\pi^2 \overline{\kappa} j^2 + \xi\right)^2 + \eta^2}\right) \left(\rho_l R_* \left(\xi^2 - \eta^2\right) + \frac{4\mu_l}{R_*} \xi - \frac{2\sigma}{R_*^2}\right) \right. \\
&\quad \left. + \frac{8(\gamma - 1)}{\pi\gamma} \sum_{j=1}^{\infty} \frac{\pi^2 \overline{\kappa}}{\left(\pi^2 \overline{\kappa} j^2 + \xi\right)^2 + \eta^2} \left(\rho_l R_* \xi(\xi^2 + \eta^2) + \frac{4\mu_l}{R_*}(\xi^2 + \eta^2) - \frac{2\sigma}{R_*^2} \xi\right) \right] + 4\pi \frac{\rho_*}{R_*}.
\end{aligned}$$

(E.4)

When $\eta \neq 0$, the imaginary part in (E.3) reads

$$
\begin{aligned}
0 &= \left(\frac{4\pi}{3\gamma} + \frac{8(\gamma-1)}{\pi\gamma} \sum_{j=1}^{\infty} \frac{\pi^2\overline{\kappa}\left(\pi^2\overline{\kappa}j^2 + \xi\right)}{\left(\pi^2\overline{\kappa}j^2 + \xi\right)^2 + \eta^2} \right) \left(\rho_l R_*(2\xi) + \frac{4\mu_l}{R_*} \right) \\
&\quad - \frac{8(\gamma-1)}{\pi\gamma} \sum_{j=1}^{\infty} \frac{\pi^2\overline{\kappa}}{\left(\pi^2\overline{\kappa}j^2 + \xi\right)^2 + \eta^2} \left(\rho_l R_*(\xi^2 - \eta^2) + \frac{4\mu_l}{R_*}\xi - \frac{2\sigma}{R_*^2} \right) \\
&= \left(\frac{4\pi}{3\gamma} + \frac{8(\gamma-1)}{\pi\gamma} \sum_{j=1}^{\infty} \frac{\pi^4\overline{\kappa}^2 j^2}{\left(\pi^2\overline{\kappa}j^2 + \xi\right)^2 + \eta^2} \right) \left(\rho_l R_*(2\xi) + \frac{4\mu_l}{R_*} \right) \\
&\quad + \frac{8(\gamma-1)}{\pi\gamma} \sum_{j=1}^{\infty} \frac{\pi^2\overline{\kappa}}{\left(\pi^2\overline{\kappa}j^2 + \xi\right)^2 + \eta^2} \left(\rho_l R_*(\xi^2 + \eta^2) + \frac{2\sigma}{R_*^2} \right).
\end{aligned}
\tag{E.5}
$$

For $\eta \neq 0$, the equation (E.5) implies

$$
\rho_l R_*(2\xi) + \frac{4\mu_l}{R_*} < 0,
\tag{E.6}
$$

which gives

$$
\xi < -\frac{2\mu_l}{\rho_l R_*^2} \leq 0, \qquad \eta \neq 0.
\tag{E.7}
$$

The equation (E.5) also implies

$$
\begin{aligned}
& \frac{8(\gamma-1)}{\pi\gamma} \sum_{j=1}^{\infty} \frac{\pi^2\overline{\kappa}}{\left(\pi^2\overline{\kappa}j^2 + \xi\right)^2 + \eta^2} \\
&= -\left(\frac{4\pi}{3\gamma} + \frac{8(\gamma-1)}{\pi\gamma} \sum_{j=1}^{\infty} \frac{\pi^4\overline{\kappa}^2 j^2}{\left(\pi^2\overline{\kappa}j^2 + \xi\right)^2 + \eta^2} \right) \frac{\rho_l R_*(2\xi) + \frac{4\mu_l}{R_*}}{\rho_l R_*(\xi^2 + \eta^2) + \frac{2\sigma}{R_*^2}}.
\end{aligned}
\tag{E.8}
$$

Plugging (E.8) into the real part (E.4), we derive

$$
\begin{aligned}
0 &= \frac{1}{\mathscr{R}_g T_\infty} \left(\frac{4\pi}{3\gamma} + \frac{8(\gamma-1)}{\pi\gamma} \sum_{j=1}^{\infty} \frac{\pi^4\overline{\kappa}^2 j^2}{\left(\pi^2\overline{\kappa}j^2 + \xi\right)^2 + \eta^2} \right) \left[\rho_l R_* \left(\xi^2 - \eta^2\right) + \frac{4\mu_l}{R_*}\xi - \frac{2\sigma}{R_*^2} \right. \\
&\quad \left. - \frac{\left(\rho_l R_*(2\xi) + \frac{4\mu_l}{R_*}\right)\left(\rho_l R_*\xi(\xi^2 + \eta^2) + \frac{4\mu_l}{R_*}(\xi^2 + \eta^2) - \frac{2\sigma}{R_*^2}\xi\right)}{\rho_l R_*(\xi^2 + \eta^2) + \frac{2\sigma}{R_*^2}} \right] + 4\pi\frac{\rho_*}{R_*} \\
&=: \frac{1}{\mathscr{R}_g T_\infty} \left(\frac{4\pi}{3\gamma} + \frac{8(\gamma-1)}{\pi\gamma} \sum_{j=1}^{\infty} \frac{\pi^4\overline{\kappa}^2 j^2}{\left(\pi^2\overline{\kappa}j^2 + \xi\right)^2 + \eta^2} \right) \Lambda + 4\pi\frac{\rho_*}{R_*},
\end{aligned}
\tag{E.9}
$$

where Λ is the square bracket on the right hand side of the first equation. A straightforward calculation shows that

$$\left(\rho_l R_*(\xi^2 + \eta^2) + \frac{2\sigma}{R_*^2}\right)\Lambda$$

$$= -\rho_l^2 R_*^2(\xi^2 + \eta^2)^2 - \frac{4\mu_l}{R_*}\left(\rho_l R_*(2\xi) + \frac{4\mu_l}{R_*}\right)(\xi^2 + \eta^2)$$

$$+ \frac{2\sigma}{R_*^2}\left(2\rho_l R_*\xi^2 + 2\frac{4\mu_l}{R_*}\xi - 2\rho_l R_*\eta^2\right) - \left(\frac{2\sigma}{R_*^2}\right)^2$$

$$> -\rho_l^2 R_*^2(\xi^2 + \eta^2)^2 - \frac{2\sigma}{R_*^2}2\rho_l R_*(\xi^2 + \eta^2) - \left(\frac{2\sigma}{R_*^2}\right)^2$$

$$= -\left(\rho_l R_*(\xi^2 + \eta^2) + \frac{2\sigma}{R_*^2}\right)^2,$$

where we've used (E.6) and so $2\rho_l R_*\xi^2 + 2\frac{4\mu_l}{R_*}\xi = \xi\left(\rho_l R_*(2\xi) + 2\frac{4\mu_l}{R_*}\right) > -2\rho_l R_*\xi^2$ in the last inequality. This implies

$$\Lambda > -\left(\rho_l R_*(\xi^2 + \eta^2) + \frac{2\sigma}{R_*^2}\right). \tag{E.10}$$

Using (E.10) in (E.9), we get

$$0 > -\frac{1}{\mathscr{R}_g T_\infty}\left(\frac{4\pi}{3\gamma} + \frac{8(\gamma - 1)}{\pi\gamma}\sum_{j=1}^\infty \frac{\pi^4\bar\kappa^2 j^2}{\left(\pi^2\bar\kappa j^2 + \xi\right)^2 + \eta^2}\right)\left(\rho_l R_*(\xi^2 + \eta^2) + \frac{2\sigma}{R_*^2}\right) \tag{E.11}$$

$$+ 4\pi\frac{\rho_*}{R_*}.$$

Suppose

$$\xi \geq -\theta\pi^2\bar\kappa, \tag{E.12}$$

where $\theta \in (0, 1)$, to be chosen. Then $\xi \geq -\theta\pi^2\bar\kappa j^2$ for all $j = 1, 2, \ldots$. We further assume that

$$\xi \geq -\sqrt{\frac{2p_{\infty,*}}{\rho_l R_*^2} + \frac{4\sigma}{\rho_l R_*^3} - \eta^2}, \tag{E.13}$$

provided $\eta^2 \leq \frac{2p_{\infty,*}}{\rho_l R_*^2} + \frac{4\sigma}{\rho_l R_*^3}$, so that $\rho_l R_*(\xi^2 + \eta^2) + \frac{2\sigma}{R_*^2} \leq \frac{2p_{\infty,*}}{R_*} + \frac{6\sigma}{R_*^2}$. Then (E.11) gives

$$0 > -\frac{1}{\mathscr{R}_g T_\infty}\left(\frac{4\pi}{3\gamma} + \frac{8(\gamma - 1)}{\pi\gamma}\frac{1}{(1 - \theta)^2}\sum_{j=1}^\infty \frac{1}{j^2}\right)\left(\frac{2p_{\infty,*}}{R_*} + \frac{6\sigma}{R_*^2}\right) + 4\pi\frac{\rho_*}{R_*}.$$

Using $\sum_{j=1}^\infty j^{-2} = \pi^2/6$, one has

$$3\mathscr{R}_g T_\infty\frac{\rho_*}{R_*} < \left[\frac{1}{\gamma} + \left(1 - \frac{1}{\gamma}\right)\frac{1}{(1 - \theta)^2}\right]\left(\frac{2p_{\infty,*}}{R_*} + \frac{6\sigma}{R_*^2}\right),$$

or, equivalently, using $\mathscr{R}_g T_\infty \rho_* = p_{\infty,*} + 2\sigma/R_*$,

$$\theta > 1 - \sqrt{\frac{1 - \frac{1}{\gamma}}{\frac{3p_{\infty,*}R_* + 6\sigma}{2p_{\infty,*}R_* + 6\sigma} - \frac{1}{\gamma}}}.$$

We simply choose

$$\theta = 1 - \sqrt{\frac{1 - \frac{1}{\gamma}}{\frac{3p_{\infty,*}R_* + 6\sigma}{2p_{\infty,*}R_* + 6\sigma} - \frac{1}{\gamma}}} \in (0, 1)$$

to reach a contradiction to (E.12) and (E.13). Therefore, we have for $\eta \in \mathbb{R}$ with $\eta \neq 0$ and $\eta^2 \leq \frac{2p_{\infty,*}}{\rho_l R_*^2} + \frac{4\sigma}{\rho_l R_*^3}$ that

$$\xi < -\min\left\{ \left(1 - \sqrt{\frac{1 - \frac{1}{\gamma}}{\frac{3p_{\infty,*}R_* + 6\sigma}{2p_{\infty,*}R_* + 6\sigma} - \frac{1}{\gamma}}}\right)\pi^2\overline{\kappa},\ \sqrt{\frac{2p_{\infty,*}}{\rho_l R_*^2} + \frac{4\sigma}{\rho_l R_*^3} - \eta^2} \right\}.$$

Combining (E.7), for $\eta \in \mathbb{R}$ with $\eta \neq 0$ and $\eta^2 \leq \frac{p_{\infty,*}}{\rho_l R_*^2} + \frac{2\sigma}{\rho_l R_*^3}$ we have

$$\xi < -\max\left\{ \frac{2\mu_l}{\rho_l R_*^2},\ \min\left\{ \left(1 - \sqrt{\frac{1 - \frac{1}{\gamma}}{\frac{3p_{\infty,*}R_* + 6\sigma}{2p_{\infty,*}R_* + 6\sigma} - \frac{1}{\gamma}}}\right)\pi^2\overline{\kappa},\ \sqrt{\frac{p_{\infty,*}}{\rho_l R_*^2} + \frac{2\sigma}{\rho_l R_*^3}} \right\} \right\}.$$

$$\text{(E.14)}$$

Now, we consider the case $\eta^2 > \frac{p_{\infty,*}}{\rho_l R_*^2} + \frac{2\sigma}{\rho_l R_*^3}$. Since $\eta \neq 0$, the imaginary part in (E.3) gives the identity $\Xi_1 = -\frac{H_1}{H_2}\Xi_2$. Using this identity in the real part in (E.3), we derive

$$4\pi \frac{\rho_*}{R_*}\mathscr{R}_g T_\infty H_2 = H_1\left(\Xi_2^2 + H_2^2\right),$$

which implies

$$4\pi \frac{\rho_*}{R_*}\mathscr{R}_g T_\infty\left(\rho_l R_*(2\xi) + \frac{4\mu_l}{R_*}\right)$$
$$= -\frac{8(\gamma - 1)}{\pi\gamma}\sum_{j=1}^{\infty}\frac{\pi^2\overline{\kappa}}{\left(\pi^2\overline{\kappa}j^2 + \xi\right)^2 + \eta^2}\left(\Xi_2^2 + H_2^2\right). \tag{E.15}$$

To find a positive lower bound for $\Xi_2^2 + H_2^2$, note that

$$\Xi_2^2 + H_2^2 = \left|\rho_l R_*\tau^2 + \frac{4\mu_l}{R_*}\tau - \frac{2\sigma}{R_*^2}\right|^2 = \rho_l^2 R_*^2\,|\tau - \tau_+|^2\,|\tau - \tau_-|^2,$$

where

$$\tau_\pm = \frac{-\frac{4\mu_l}{R_*} \pm \sqrt{\left(\frac{4\mu_l}{R_*}\right)^2 + 4\rho_l R_*\frac{2\sigma}{R_*^2}}}{2\rho_l R_*}.$$

are on the real axis. By the triangular inequality $|\tau - \tau_\pm| > |\eta|$, and so

$$\Xi_2^2 + H_2^2 > \rho_l^2 R_*^2 \eta^4.$$

Therefore, (E.15) yields

$$4\pi \frac{\rho_*}{R_*} \mathscr{R}_g T_\infty \left(\rho_l R_*(2\xi) + \frac{4\mu_l}{R_*} \right) < -\frac{8(\gamma-1)}{\pi\gamma} \sum_{j=1}^\infty \frac{\pi^2 \overline{\kappa}}{\pi^4 \overline{\kappa}^2 j^4 + \eta^2} \rho_l^2 R_*^2 \eta^4.$$

Since $\eta^2 > \frac{p_{\infty,*}}{\rho_l R_*^2} + \frac{2\sigma}{\rho_l R_*^3}$ and $\frac{a^2}{\pi^4 \overline{\kappa}^2 j^4 + a}$ is increasing in a for $a = \eta^2 \geq 0$, we further derive

$$4\pi \frac{\rho_*}{R_*} \mathscr{R}_g T_\infty \left(\rho_l R_*(2\xi) + \frac{4\mu_l}{R_*} \right)$$

$$< -\frac{8(\gamma-1)}{\pi\gamma} \sum_{j=1}^\infty \frac{\pi^2 \overline{\kappa}}{\pi^4 \overline{\kappa}^2 j^4 + \left(\frac{p_{\infty,*}}{\rho_l R_*^2} + \frac{2\sigma}{\rho_l R_*^3} \right)} \rho_l^2 R_*^2 \left(\frac{p_{\infty,*}}{\rho_l R_*^2} + \frac{2\sigma}{\rho_l R_*^3} \right)^2 \quad \text{(E.16)}$$

$$= -\frac{8(\gamma-1)}{\pi\gamma} \frac{1}{\pi^2 \overline{\kappa}} \sum_{j=1}^\infty \frac{1}{j^4 + B^2} \rho_l^2 R_*^2 \left(\frac{p_{\infty,*}}{\rho_l R_*^2} + \frac{2\sigma}{\rho_l R_*^3} \right)^2,$$

where

$$B := \frac{1}{\pi^2 \overline{\kappa}} \sqrt{\frac{p_{\infty,*}}{\rho_l R_*^2} + \frac{2\sigma}{\rho_l R_*^3}}.$$

Using

$$\sum_{j=1}^\infty \frac{1}{j^4 + B^2} = \frac{e^{\frac{i\pi}{4}} \pi \cot\left(e^{\frac{i\pi}{4}} \pi \sqrt{B}\right) + e^{\frac{3i\pi}{4}} \pi \cot\left(e^{\frac{3i\pi}{4}} \pi \sqrt{B}\right)}{4B^{3/2}} - \frac{1}{2B^2}$$

$$= \frac{\frac{2\pi}{\sqrt{2}} \left(\frac{\cot(\pi\sqrt{B/2})\,\mathrm{csch}^2(\pi\sqrt{B/2}) + \csc^2(\pi\sqrt{B/2})\coth(\pi\sqrt{B/2})}{\cot^2(\pi\sqrt{B/2}) + \coth^2(\pi\sqrt{B/2})} \right)}{4B^{3/2}} - \frac{1}{2B^2}$$

$$= \frac{\frac{2\pi}{\sqrt{2}} \left(\frac{1}{\pi\sqrt{B/2}} + \frac{4}{45}\left(\pi\sqrt{B/2}\right)^3 + O\left(\left(\pi\sqrt{B/2}\right)^6\right) \right)}{4B^{\frac{3}{2}}} - \frac{1}{2B^2}$$

$$= \frac{\pi^4}{90} + O\left(B^{3/2}\right) \quad \text{for } B \ll 1,$$

we have from (E.16) that

$$4\pi \frac{\rho_*}{R_*} \mathscr{R}_g T_\infty \left(\rho_l R_*(2\xi) + \frac{4\mu_l}{R_*} \right)$$

$$< -\frac{8(\gamma-1)}{\pi\gamma} \frac{1}{\pi^2 \overline{\kappa}} \left(\frac{\pi^4}{90} + O\left(\left(\frac{1}{\pi^2 \overline{\kappa}} \sqrt{\frac{p_{\infty,*}}{\rho_l R_*^2} + \frac{2\sigma}{\rho_l R_*^3}} \right)^{3/2} \right) \right) \rho_l^2 R_*^2 \left(\frac{p_{\infty,*}}{\rho_l R_*^2} + \frac{2\sigma}{\rho_l R_*^3} \right)^2.$$

Consequently, we have for $\eta^2 > \frac{p_{\infty,*}}{\rho_l R_*^2} + \frac{2\sigma}{\rho_l R_*^3}$ that

$$\xi < -\frac{2\mu_l}{\rho_l R_*^2} - \frac{\mathscr{R}_g T_\infty \rho_*}{\pi^4 \overline{\kappa} \rho_l R_*^2} \left(1 - \frac{1}{\gamma}\right) \left(\frac{\pi^4}{90} + O\left(\left(\frac{1}{\pi^2 \overline{\kappa}} \sqrt{\frac{p_{\infty,*}}{\rho_l R_*^2} + \frac{2\sigma}{\rho_l R_*^3}}\right)^{3/2}\right)\right),$$

(E.17)

where $\mathscr{R}_g T_\infty \rho_* = p_{\infty,*} + 2\sigma/R_*$ is used.

It remains to consider the case $\eta = 0$. We first show that $\xi < 0$. Suppose for the sake of contradiction that $\xi \geq 0$ then

$$Q(\xi) = \frac{1}{\mathscr{R}_g T_\infty} \left(\frac{4\pi}{3\gamma} + \frac{8(\gamma - 1)}{\pi\gamma} \sum_{j=1}^{\infty} \frac{\pi^2 \overline{\kappa}}{\pi^2 \overline{\kappa} j^2 + \xi}\right) \left(\rho_l R_* \xi^2 + \frac{4\mu_l}{R_*} \xi\right)$$

$$- \frac{2\sigma}{\mathscr{R}_g T_\infty R_*^2} \left(\frac{4\pi}{3\gamma} + \frac{8(\gamma - 1)}{\pi\gamma} \sum_{j=1}^{\infty} \frac{\pi^2 \overline{\kappa}}{\pi^2 \overline{\kappa} j^2 + \xi}\right) + 4\pi \frac{\rho_*}{R_*},$$

where the second line is greater than

$$- \frac{2\sigma}{\mathscr{R}_g T_\infty R_*^2} \left(\frac{4\pi}{3\gamma} + \frac{8(\gamma - 1)}{\pi\gamma} \sum_{j=1}^{\infty} \frac{\pi^2 \overline{\kappa}}{\pi^2 \overline{\kappa} j^2}\right) + 4\pi \frac{\rho_*}{R_*} = \frac{8\pi\rho_*}{3R_*} + \frac{4\pi p_{\infty,*}}{3\mathscr{R}_g T_\infty R_*} > 0.$$

Here the identities $\sum_{j=1}^{\infty} j^{-2} = \frac{\pi^2}{6}$ and $\mathscr{R}_g T_\infty \rho_* = p_{\infty,*} + \frac{2\sigma}{R_*}$ are used. This yields $Q(\xi) > 0$, which contradicts to the fact that $Q(\tau) = 0$. Thus, we have $\xi < 0$ for $\eta = 0$.

Now we search for a negative upper bound for ξ when $\eta = 0$. Suppose

$$\xi \geq -\theta_0 \pi^2 \overline{\kappa},$$

(E.18)

where $0 < \theta_0 < 1$ to be chosen. Suppose further that

$$\xi > \frac{-\frac{4\mu_l}{R_*} + \sqrt{\Delta}}{2\rho_l R_*} \quad \text{if } \Delta := \left(\frac{4\mu_l}{R_*}\right)^2 - 4\rho_l R_* \left(\frac{2p_{\infty,*}}{R_*} + \frac{4\sigma}{R_*^2}\right) > 0 \text{ (E.19)}$$

such that

$$\rho_l R_* \xi^2 + \frac{4\mu_l}{R_*} \xi - \frac{2\sigma}{R_*^2} \geq -\frac{2p_{\infty,*}}{R_*} - \frac{6\sigma}{R_*^2}.$$

(E.20)

Note that the inequality (E.20) always holds when $\Delta \leq 0$. Then

$$0 = Q(\xi) = \frac{1}{\mathscr{R}_g T_\infty} \left(\frac{4\pi}{3\gamma} + \frac{8(\gamma - 1)}{\pi\gamma} \sum_{j=1}^{\infty} \frac{\pi^2 \overline{\kappa}}{\pi^2 \overline{\kappa} j^2 + \xi}\right) \left(\rho_l R_* \xi^2 + \frac{4\mu_l}{R_*} \xi - \frac{2\sigma}{R_*^2}\right)$$

$$+ 4\pi \frac{\rho_*}{R_*}$$

$$> -\frac{1}{\mathscr{R}_g T_\infty} \left(\frac{4\pi}{3\gamma} + \frac{8(\gamma - 1)}{\pi\gamma(1 - \theta_0)} \sum_{j=1}^{\infty} \frac{\pi^2 \overline{\kappa}}{\pi^2 \overline{\kappa} j^2}\right) \left(\frac{2p_{\infty,*}}{R_*} + \frac{6\sigma}{R_*^2}\right) + 4\pi \frac{\rho_*}{R_*}$$

$$= -\frac{1}{\mathscr{R}_g T_\infty} \left(\frac{4\pi}{3\gamma} + \frac{4\pi}{3}\left(1 - \frac{1}{\gamma}\right) \frac{1}{1 - \theta_0}\right) \left(\frac{2p_{\infty,*}}{R_*} + \frac{6\sigma}{R_*^2}\right) + 4\pi \frac{\rho_*}{R_*}$$

So

$$4\pi \mathscr{R}_g T_\infty \frac{\rho_*}{R_*} < \left(\frac{4\pi}{3\gamma} + \frac{4\pi}{3} \left(1 - \frac{1}{\gamma} \right) \frac{1}{1 - \theta_0} \right) \left(\frac{2p_{\infty,*}}{R_*} + \frac{6\sigma}{R_*^2} \right),$$

or, equivalently,

$$\theta_0 > 1 - \frac{1 - \dfrac{1}{\gamma}}{\dfrac{3\mathscr{R}_g T_\infty \frac{\rho_*}{R_*}}{\dfrac{2p_{\infty,*}}{R_*} + \dfrac{6\sigma}{R_*^2}} - \dfrac{1}{\gamma}} = 1 - \frac{1 - \dfrac{1}{\gamma}}{\dfrac{3p_{\infty,*}R_* + 6\sigma}{2p_{\infty,*}R_* + 6\sigma} - \dfrac{1}{\gamma}},$$

where $\mathscr{R}_g T_\infty \rho_* = p_{\infty,*} + \frac{2\sigma}{R_*}$ has been used in the last equation. We then choose

$$\theta_0 = 1 - \frac{1 - \dfrac{1}{\gamma}}{\dfrac{3p_{\infty,*}R_* + 6\sigma}{2p_{\infty,*}R_* + 6\sigma} - \dfrac{1}{\gamma}} \in (0, 1)$$

to reach a contradiction to (E.18) and (E.19). Hence we derive for $\eta = 0$

$$\xi < \begin{cases} -\min\left\{ \left(1 - \dfrac{1 - \dfrac{1}{\gamma}}{\dfrac{3p_{\infty,*}R_* + 6\sigma}{2p_{\infty,*}R_* + 6\sigma} - \dfrac{1}{\gamma}} \right) \pi^2 \overline{\kappa}, \dfrac{\dfrac{4\mu_l}{R_*} - \sqrt{\Delta}}{2\rho_l R_*} \right\} & \text{if } \Delta > 0, \\[2em] -\left(1 - \dfrac{1 - \dfrac{1}{\gamma}}{\dfrac{3p_{\infty,*}R_* + 6\sigma}{2p_{\infty,*}R_* + 6\sigma} - \dfrac{1}{\gamma}} \right) \pi^2 \overline{\kappa} & \text{if } \Delta \leq 0. \end{cases} \tag{E.21}$$

Combining the upper bounds (E.14), (E.17), and (E.21) for different cases of η and using the identity $\mathscr{R}_g T_\infty \rho_* = p_{\infty,*} + \frac{2\sigma}{R_*}$, the lemma follows. □

Appendix F: Rate of Convergence of Slow Solutions Approaching to Center Manifold for a Class of Fully Nonlinear Autonomous Systems

As mentioned in the paragraph below Theorem 6.7, there are several obstacles preventing us from direct applying center manifold theorem to prove the exponential decay in nonlinear bubble oscillations. One of which is the quasilinear character of the problem (9.5). For this purpose, we develop in this appendix a geometric theory for a class of fully nonlinear autonomous systems which covers the quasilinear system (9.5).

We study a larger class of fully nonlinear autonomous systems of the form $\dot{\mathbf{w}} = \mathcal{L}\mathbf{w} + \mathcal{N}(\mathbf{w}, \dot{\mathbf{w}})$ that includes the quasilinear autonomous system (9.5) of our concern. Assuming that the solution of such equation converges toward a given center manifold and that the time derivative of the solution is sufficiently small for all time, we prove that the convergence rate is exponential. The proof is an adaptation

of [10, Sections 2.4 and 6.3], where the existence and stability of center manifold for semilinear equations are established.

Setup of the fully nonlinear autonomous system, assumptions on the solution, and the center manifold. Let Z be a Banach space with norm $\|\cdot\|$. We consider the evolution equation

$$\dot{\mathbf{w}} = \mathcal{L}\mathbf{w} + \mathcal{N}(\mathbf{w}, \dot{\mathbf{w}}), \quad \mathbf{w}(0) \in Z, \tag{F.1}$$

where $\mathcal{N}(\mathbf{w}, \mathbf{p}) : Z \times Z \to Z$ has a uniformly continuous second derivative with $\mathcal{N}(\mathbf{0}, \mathbf{p}) = \mathbf{0}$ and $\partial_{(\mathbf{w}, \mathbf{p})}\mathcal{N}(\mathbf{0}, \mathbf{0}) = \mathbf{O}$.

Assume

(i) $Z = X \oplus Y$ where X is finite dimensional and Y is closed.

(ii) X is $\mathcal{L}$-invariant and that if $\mathcal{A} := \mathcal{L}|_X$, then the real parts of the eigenvalues of $\mathcal{A}$ are all zeros.

(iii) Y is $e^{\mathcal{L}t}$-invariant. Let Q_1 be a projection on X (not necessarily along Y) and $Q_2 := I - Q_1$. For some positive constants b, c,

$$\left\| e^{\mathcal{L}t} Q_2 \right\| \le c e^{-bt}, \quad t \ge 0. \tag{F.2}$$

Let $\mathbf{w}$ be a solution of (F.1). Decompose $\mathbf{w}$ into $\mathbf{w} = \mathbf{x} + \mathbf{y}$ where $\mathbf{x} = Q_1\mathbf{w}$ and $\mathbf{y} = Q_2\mathbf{w}$. Let $\mathcal{B} = Q_2\mathcal{L}$. Then equation (9.12) can be written as

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathcal{A}\mathbf{x} + f(\mathbf{x}, \mathbf{y}, \dot{\mathbf{x}}, \dot{\mathbf{y}}), \\
\dot{\mathbf{y}} &= \mathcal{B}\mathbf{y} + g(\mathbf{x}, \mathbf{y}, \dot{\mathbf{x}}, \dot{\mathbf{y}}),
\end{aligned} \tag{F.3}$$

where

$$f(\mathbf{x}, \mathbf{y}, \dot{\mathbf{x}}, \dot{\mathbf{y}}) = Q_1\mathcal{N}(\mathbf{x} + \mathbf{y}, \dot{\mathbf{x}} + \dot{\mathbf{y}}), \quad g(\mathbf{x}, \mathbf{y}, \dot{\mathbf{x}}, \dot{\mathbf{y}}) = Q_2\mathcal{N}(\mathbf{x} + \mathbf{y}, \dot{\mathbf{x}} + \dot{\mathbf{y}}).$$

A curve $\mathbf{y} = h(\mathbf{x})$, defined for $|\mathbf{x}|$ small, is said to be an *invariant manifold* for (F.3) if the solution $(\mathbf{x}(t), \mathbf{y}(t))$ of (F.3) through $(\mathbf{x}(0), h(\mathbf{x}(0)))$ satisfies $\mathbf{y}(t) = h(\mathbf{x}(t))$. A *center manifold* for (F.3) is an invariant manifold that is tangent to X space at the origin.

By the assumption on the nonlinearity $\mathcal{N}(\mathbf{w}, \dot{\mathbf{w}})$, there exists a continuous function $k(\varepsilon)$ with $k(0) = 0$ such that

$$\begin{aligned}
&\|f(\mathbf{x}, \mathbf{y}, \dot{\mathbf{x}}, \dot{\mathbf{y}})\| + \|g(\mathbf{x}, \mathbf{y}, \dot{\mathbf{x}}, \dot{\mathbf{y}})\| \le \varepsilon k(\varepsilon), \\
&\|f(\mathbf{x}, \mathbf{y}, \dot{\mathbf{x}}, \dot{\mathbf{y}}) - f(\mathbf{x}', \mathbf{y}', \dot{\mathbf{x}}', \dot{\mathbf{y}}')\| \le k(\varepsilon)\left[\|\mathbf{x} - \mathbf{x}'\| + \|\mathbf{y} - \mathbf{y}'\| + \|\dot{\mathbf{x}} - \dot{\mathbf{x}}'\| + \|\dot{\mathbf{y}} - \dot{\mathbf{y}}'\|\right], \\
&\|g(\mathbf{x}, \mathbf{y}, \dot{\mathbf{x}}, \dot{\mathbf{y}}) - g(\mathbf{x}', \mathbf{y}', \dot{\mathbf{x}}', \dot{\mathbf{y}}')\| \le k(\varepsilon)\left[\|\mathbf{x} - \mathbf{x}'\| + \|\mathbf{y} - \mathbf{y}'\| + \|\dot{\mathbf{x}} - \dot{\mathbf{x}}'\| + \|\dot{\mathbf{y}} - \dot{\mathbf{y}}'\|\right],
\end{aligned} \tag{F.4}$$

for all $\mathbf{x}, \mathbf{x}' \in X, \mathbf{y}, \mathbf{y}' \in Y$ and all $\dot{\mathbf{x}}, \dot{\mathbf{x}}', \dot{\mathbf{y}}, \dot{\mathbf{y}}' \in Z$ with $\|(\mathbf{x}, \mathbf{y})\|, \|(\dot{\mathbf{x}}, \dot{\mathbf{y}})\|, \|(\mathbf{x}', \mathbf{y}')\|, \|(\dot{\mathbf{x}}', \dot{\mathbf{y}}')\| < \varepsilon$.

Let $\mathcal{M}$ be a center manifold for (F.3) given by $\mathbf{y} = h(\mathbf{x})$. If we substitute $\mathbf{y}(t) = h(\mathbf{x}(t))$ into (F.3) we obtain

$$h'(\mathbf{x})\left[\mathcal{A}\mathbf{x} + f(\mathbf{x}, h(\mathbf{x}), \dot{\mathbf{x}}, h'(\mathbf{x})\dot{\mathbf{x}})\right] = \mathcal{B}h(\mathbf{x}) + g(\mathbf{x}, h(\mathbf{x}), \dot{\mathbf{x}}, h'(\mathbf{x})\dot{\mathbf{x}}). \tag{F.5}$$

The equation on the center manifold is given by

$$\dot{\mathbf{u}} = \mathcal{A}\mathbf{u} + f(\mathbf{u}, h(\mathbf{u}), \dot{\mathbf{u}}, h'(\mathbf{u})\dot{\mathbf{u}}). \tag{F.6}$$

We assume that $\mathbf{w}(t)$ converges to some element in $\mathcal{M}$, as $t \to \infty$, and that $\sup_{t \geq 0} \|\dot{\mathbf{w}}(t)\|$ is sufficiently small.

Rate of convergence to the center manifold. The following lemma describes that the trajectory shadows the center manifold and corresponds to [10, Lemma 2.4.1].

Lemma F.1. *Let $(\mathbf{x}(t), \mathbf{y}(t))$ be a solution of (F.3) with $\|(\mathbf{x}(0), \mathbf{y}(0))\|$ and $\|(\dot{\mathbf{x}}(t), \dot{\mathbf{y}}(t))\|$, for all $t \geq 0$, sufficiently small. Then there exist positive C_1 and β_1 such that*

$$\|\mathbf{y}(t) - h(\mathbf{x}(t))\| \leq C_1 e^{-\beta_1 t} \|\mathbf{y}(0) - h(\mathbf{x}(0))\|$$

for all $t \geq 0$.

Proof. Let $(\mathbf{x}(t), \mathbf{y}(t))$ be a solution of (F.3) with $(\mathbf{x}(0), \mathbf{y}(0))$ sufficiently small. Let $\mathbf{z}(t) = \mathbf{y}(t) - h(\mathbf{x}(t))$, then

$$\dot{\mathbf{z}} = \mathcal{B}\mathbf{z} + \mathcal{R}(\mathbf{x}, \mathbf{z}, \dot{\mathbf{x}}, \dot{\mathbf{z}}) \tag{F.7}$$

where

$$\mathcal{R}(\mathbf{x}, \mathbf{z}, \dot{\mathbf{x}}, \dot{\mathbf{z}}) = h'(\mathbf{x}) \left[f(\mathbf{x}, h(\mathbf{x}), \dot{\mathbf{x}}, h'(\mathbf{x})\dot{\mathbf{x}}) - f(\mathbf{x}, \mathbf{z} + h(\mathbf{x}), \dot{\mathbf{x}}, \dot{\mathbf{z}} + h'(\mathbf{x})\dot{\mathbf{x}}) \right] \\ + g(\mathbf{x}, \mathbf{z} + h(\mathbf{x}), \dot{\mathbf{x}}, \dot{\mathbf{z}} + h'(\mathbf{x})\dot{\mathbf{x}}) - g(\mathbf{x}, h(\mathbf{x}), \dot{\mathbf{x}}, h'(\mathbf{x})\dot{\mathbf{x}}). \tag{F.8}$$

Using the hypotheses of f and g and the bounds on h,

$$\|\mathcal{R}(\mathbf{x}, \mathbf{z}, \dot{\mathbf{x}}, \dot{\mathbf{z}})\| \leq \|h'(\mathbf{x})\| \left(\| f(\mathbf{x}, h(\mathbf{x}), \dot{\mathbf{x}}, h'(\mathbf{x})\dot{\mathbf{x}}) - f(\mathbf{x}, \mathbf{z} + h(\mathbf{x}), \dot{\mathbf{x}}, h'(\mathbf{x})\dot{\mathbf{x}}) \| \right.$$
$$+ \| f(\mathbf{x}, \mathbf{z} + h(\mathbf{x}), \dot{\mathbf{x}}, h'(\mathbf{x})\dot{\mathbf{x}}) - f(\mathbf{x}, \mathbf{z} + h(\mathbf{x}), \dot{\mathbf{x}}, \dot{\mathbf{z}} + h'(\mathbf{x})\dot{\mathbf{x}}) \| \left. \right)$$
$$+ \| g(\mathbf{x}, \mathbf{z} + h(\mathbf{x}), \dot{\mathbf{x}}, \dot{\mathbf{z}} + h'(\mathbf{x})\dot{\mathbf{x}}) - g(\mathbf{x}, h(\mathbf{x}), \dot{\mathbf{x}}, \dot{\mathbf{z}} + h'(\mathbf{x})\dot{\mathbf{x}}) \| \tag{F.9}$$
$$+ \| g(\mathbf{x}, h(\mathbf{x}), \dot{\mathbf{x}}, \dot{\mathbf{z}} + h'(\mathbf{x})\dot{\mathbf{x}}) - g(\mathbf{x}, h(\mathbf{x}), \dot{\mathbf{x}}, h'(\mathbf{x})\dot{\mathbf{x}}) \|$$
$$\leq \delta(\varepsilon) \left[\|\mathbf{z}\| + \|\dot{\mathbf{z}}\| \right],$$

if $\|\mathbf{z}\|, \|\dot{\mathbf{z}}\| < \varepsilon$, for some continuous function $\delta(\varepsilon)$ with $\delta(0) = 0$. Using (F.2) we obtain, from (F.7),

$$\|\mathbf{z}(t)\| \leq c e^{-bt} \|\mathbf{z}(0)\| + c\delta(\varepsilon) \int_0^t e^{-b(t-s)} \left[\|\mathbf{z}(s)\| + \|\dot{\mathbf{z}}(s)\| \right] ds. \tag{F.10}$$

Using (F.9) in (F.7) one has

$$\|\dot{\mathbf{z}}\| \leq \|\mathcal{B}\| \|\mathbf{z}\| + \|\mathcal{R}(\mathbf{x}, \mathbf{z}, \dot{\mathbf{x}}, \dot{\mathbf{z}})\| \leq \|\mathcal{B}\| \|\mathbf{z}\| + \delta(\varepsilon) \left[\|\mathbf{z}\| + \|\dot{\mathbf{z}}\| \right],$$

and so

$$\|\dot{\mathbf{z}}\| \leq C_0 \|\mathbf{z}\|, \qquad C_0 = (1 - \delta(\varepsilon))^{-1} \left(\|\mathcal{B}\| + \delta(\varepsilon) \right). \tag{F.11}$$

Therefore, (F.10) yields

$$e^{bt} \|\mathbf{z}(t)\| \leq c \|\mathbf{z}(0)\| + c\delta(\varepsilon)(1 + C_0) \int_0^t e^{bs} \|\mathbf{z}(s)\| ds \tag{F.12}$$

By Gronwall's lemma,

$$e^{bt} \|\mathbf{z}(t)\| \leq c \|\mathbf{z}(0)\| e^{c\delta(\varepsilon)(1+C_0)t}.$$

The lemma follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition F.2. *Suppose that the zero solution of (F.6) is Lyapunov stable. Let $(\mathbf{x}(t), \mathbf{y}(t))$ be a solution of (F.3). There exists $\varepsilon > 0$ such that if $\|(\mathbf{x}(0), \mathbf{y}(0))\| < \varepsilon$ and if $\|(\dot{\mathbf{x}}(t), \dot{\mathbf{y}}(t))\| < \varepsilon$ for all $t \geq 0$, then there exists a solution $\mathbf{u}(t)$ of (F.6) such that as $t \to \infty$,*

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{u}(t) + O(e^{-b_1 t}), \\ \mathbf{y}(t) &= h(\mathbf{u}(t)) + O(e^{-b_1 t}), \end{aligned} \tag{F.13}$$

where $b_1 = \min(b, \beta_1)$, b and β_1 are given in the assumption (F.2) and in Lemma F.1, respectively.

Proof. The proof is based on that of [10, Theorem 2.4.2]. Let $(\mathbf{x}(t), \mathbf{y}(t))$ be a solution of (F.3). Since the zero solution of (F.6) is Lyapunov stable, solutions $\mathbf{u}(t)$ of (F.6) are Lyapunov stable if $\mathbf{u}(0)$ is sufficiently small. Let $\mathbf{u}(t)$ be a solution of (F.6) with $\mathbf{u}(0)$ sufficiently small. Let $\mathbf{z}(t) = \mathbf{y}(t) - h(\mathbf{x}(t))$, $\boldsymbol{\phi}(t) = \mathbf{x}(t) - \mathbf{u}(t)$. Then

$$\dot{\mathbf{z}} = \mathcal{B}\mathbf{z} + \mathcal{R}(\boldsymbol{\phi} + \mathbf{u}, \mathbf{z}, \dot{\boldsymbol{\phi}} + \dot{\mathbf{u}}, \dot{\mathbf{z}}), \tag{F.14a}$$

$$\dot{\boldsymbol{\phi}} = \mathcal{A}\boldsymbol{\phi} + \mathcal{V}(\boldsymbol{\phi}, \mathbf{z}, \dot{\boldsymbol{\phi}}, \dot{\mathbf{z}}), \tag{F.14b}$$

where $\mathcal{R}$ is defined in (F.8) and

$$\begin{aligned} \mathcal{V}(\boldsymbol{\phi}, \mathbf{z}, \dot{\boldsymbol{\phi}}, \dot{\mathbf{z}}) &= f(\mathbf{u}+\boldsymbol{\phi}, \mathbf{z} + h(\mathbf{u}+\boldsymbol{\phi}), \dot{\mathbf{u}}+\dot{\boldsymbol{\phi}}, \dot{\mathbf{z}} + h'(\mathbf{u}+\boldsymbol{\phi})(\dot{\mathbf{u}}+\dot{\boldsymbol{\phi}})) \\ &\quad - f(\mathbf{u}, h(\mathbf{u}), \dot{\mathbf{u}}, h'(\mathbf{u})\dot{\mathbf{u}}). \end{aligned}$$

We now formulate (F.14a)–(F.14b) as a fixed point problem. Let $\mathscr{X}$ be the set of continuous differentiable functions $\boldsymbol{\phi} : [0, \infty) \to X$ of (F.14b) with $\|\boldsymbol{\phi}(t)e^{at}\| \leq 1$ and $\|\dot{\boldsymbol{\phi}}(t)e^{at}\| \leq a$ for all $t \geq 0$, where $a = b/2$ in which b is defined in (F.2). We define the norm $\|\boldsymbol{\phi}\|_{\mathscr{X}} = \sup\{\|\boldsymbol{\phi}(t)e^{at}\| + \|\dot{\boldsymbol{\phi}}(t)e^{at}\| : t \geq 0\}$. By the assumption on the operator $\mathcal{A}$, we can decompose $\mathcal{A}$ into $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$ where

$$\left\| e^{\mathcal{A}_1 t}\mathbf{x} \right\| = \|\mathbf{x}\|, \qquad \|\mathcal{A}_2\mathbf{x}\| \leq (b/4)\|\mathbf{x}\|, \tag{F.15}$$

where b is defined by (F.2). Then (F.14b) can be written as

$$\dot{\boldsymbol{\phi}} = \mathcal{A}_1\boldsymbol{\phi} + \left[\mathcal{A}_2\boldsymbol{\phi} + \mathcal{V}(\boldsymbol{\phi}, \mathbf{z}, \dot{\boldsymbol{\phi}}, \dot{\mathbf{z}})\right],$$

and $\boldsymbol{\phi}(\infty) = 0$ for $\boldsymbol{\phi} \in \mathscr{X}$. Let $\mathbf{z}(t)$ be a given solution of (F.14a). By Duhamel's formula, a solution $\boldsymbol{\phi} \in \mathscr{X}$ of (F.14b) must satisfy

$$\boldsymbol{\phi}(t) = -\int_t^\infty e^{\mathcal{A}_1(t-s)}\left[\mathcal{A}_2\boldsymbol{\phi}(s) + \mathcal{V}(\boldsymbol{\phi}(s), \mathbf{z}(s), \dot{\boldsymbol{\phi}}(s), \dot{\mathbf{z}}(s))\right] ds.$$

Thus, a solution $\boldsymbol{\phi} \in \mathscr{X}$ of (F.14b) is a fixed point of the mapping T that defined by

$$(T\boldsymbol{\phi})(t) = -\int_t^\infty e^{\mathcal{A}_1(t-s)}\left[\mathcal{A}_2\boldsymbol{\phi}(s) + \mathcal{V}(\boldsymbol{\phi}(s), \mathbf{z}(s), \dot{\boldsymbol{\phi}}(s), \dot{\mathbf{z}}(s))\right] ds. \tag{F.16}$$

Using the bounds on f, g, h, and the fact that $\mathcal{R}(\mathbf{x}, \mathbf{0}, \dot{\mathbf{x}}, \mathbf{0}) = \mathbf{0}$ and $\mathcal{V}(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}) = \mathbf{0}$, there is a continuous function $k(\varepsilon)$ with $k(0) = 0$ such that if $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2 \in X$, $\mathbf{z}_1, \mathbf{z}_2 \in Y$, and $\dot{\boldsymbol{\phi}}_1, \dot{\boldsymbol{\phi}}_2, \dot{\mathbf{z}}_1, \dot{\mathbf{z}}_2 \in Z$ with $\left\| (\boldsymbol{\phi}_i, \mathbf{z}_i) \right\|$, $\left\| (\dot{\boldsymbol{\phi}}_i, \dot{\mathbf{z}}_i) \right\| < \varepsilon$, $i = 1, 2$, then

$$
\begin{aligned}
&\left\| \mathcal{R}(\boldsymbol{\phi}_1, \mathbf{z}_1, \dot{\boldsymbol{\phi}}_1, \dot{\mathbf{z}}_1) - \mathcal{R}(\boldsymbol{\phi}_2, \mathbf{z}_2, \dot{\boldsymbol{\phi}}_2, \dot{\mathbf{z}}_2) \right\| \\
&\qquad \leq k(\varepsilon) \left[\|(\mathbf{z}_1, \dot{\mathbf{z}}_1)\| \left(\|\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2\| + \|\dot{\boldsymbol{\phi}}_1 - \dot{\boldsymbol{\phi}}_2\| \right) + \|\mathbf{z}_1 - \mathbf{z}_2\| + \|\dot{\mathbf{z}}_1 - \dot{\mathbf{z}}_2\| \right], \\
&\left\| \mathcal{V}(\boldsymbol{\phi}_1, \mathbf{z}_1, \dot{\boldsymbol{\phi}}_1, \dot{\mathbf{z}}_1) - \mathcal{V}(\boldsymbol{\phi}_2, \mathbf{z}_2, \dot{\boldsymbol{\phi}}_2, \dot{\mathbf{z}}_2) \right\| \leq k(\varepsilon) \left[\|\mathbf{z}_1 - \mathbf{z}_2\| + \|\dot{\mathbf{z}}_1 - \dot{\mathbf{z}}_2\| + \|\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2\| + \|\dot{\boldsymbol{\phi}}_1 - \dot{\boldsymbol{\phi}}_2\| \right].
\end{aligned}
\tag{F.17}
$$

By the same argument as in the proof of Lemma F.1, one has

$$
\|\mathbf{z}(t)\| \leq C_1 \|\mathbf{z}(0)\| \, e^{-\beta_1 t}, \tag{F.18}
$$

where

$$
\beta_1 = b - c k(\varepsilon) (1 + C_0), \qquad C_0 = (1 - k(\varepsilon))^{-1} (\|\mathcal{B}\| + k(\varepsilon)).
$$

Using (F.15), we obtain, from (F.16), that

$$
\begin{aligned}
\|T\boldsymbol{\phi}(t)\| &\leq \int_t^\infty \frac{b}{4} |\boldsymbol{\phi}(s)| \, \mathrm{d}s + \int_t^\infty \left\| \mathcal{V}(\boldsymbol{\phi}(s), \mathbf{z}(s), \dot{\boldsymbol{\phi}}(s), \dot{\mathbf{z}}(s)) \right\| \mathrm{d}s \\
&= \frac{a}{4} \int_t^\infty \|\boldsymbol{\phi}(s)\| \, \mathrm{d}s + \int_t^\infty \left\| \mathcal{V}(\boldsymbol{\phi}(s), \mathbf{z}(s), \dot{\boldsymbol{\phi}}(s), \dot{\mathbf{z}}(s)) - \mathcal{V}(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}) \right\| \mathrm{d}s \\
&\leq \frac{e^{-at}}{2} + k(\varepsilon) \int_t^\infty \left[\|\boldsymbol{\phi}(s)\| + \|\mathbf{z}(s)\| + \left\| \dot{\boldsymbol{\phi}}(s) \right\| + \|\dot{\mathbf{z}}(s)\| \right] \mathrm{d}s,
\end{aligned}
\tag{F.19}
$$

where we've used $\mathcal{V}(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}) = \mathbf{0}$ in the second equation and (F.17) in the last inequality. In view of (F.11) in the proof of Lemma F.1 and (F.18),

$$
\|\mathbf{z}(s)\| + \|\dot{\mathbf{z}}(s)\| \leq (1 + C_0) \|\mathbf{z}(s)\| \leq (1 + C_0) C_1 \|\mathbf{z}(0)\| \, e^{-\beta_1 s} =: C_2 \, e^{-\beta_1 s}. \tag{F.20}
$$

Together with the hypothesis $\boldsymbol{\phi} \in \mathscr{X}$, we derive

$$
\|T\boldsymbol{\phi}(t)\| \leq \frac{e^{-at}}{2} + k(\varepsilon) \int_t^\infty \left[(1 + a)e^{-as} + C_2 e^{-\beta_1 s} \right] \mathrm{d}s \leq e^{-at}
$$

for ε sufficiently small such that $\beta_1 = b - c k(\varepsilon) (1 + C_0) \geq b/2 = a$ and $k(\varepsilon) \leq \min\{(1 + a)^{-1}, C_2^{-1}\}/2$. To estimate $\frac{d}{dt} T\boldsymbol{\phi}$, we compute

$$
\begin{aligned}
\left\| \frac{d}{dt} T\boldsymbol{\phi}(t) \right\| &= \left\| A_2 \boldsymbol{\phi}(t) + V(\boldsymbol{\phi}(t), \mathbf{z}(t), \dot{\boldsymbol{\phi}}(t), \dot{\mathbf{z}}(t)) \right\| \\
&\leq \frac{a}{2} \|\boldsymbol{\phi}(t)\| + k(\varepsilon) \left(\|\boldsymbol{\phi}(t)\| + \|\mathbf{z}(t)\| + \left\| \dot{\boldsymbol{\phi}}(t) \right\| + \|\dot{\mathbf{z}}(t)\| \right) \\
&\leq \left(\frac{a}{2} + k(\varepsilon)(1 + a) \right) e^{-at} + k(\varepsilon) C_2 e^{-\beta_1 t} \leq a e^{-at}
\end{aligned}
$$

for ε sufficiently small. This proves T maps $\mathscr{X}$ into $\mathscr{X}$.

We now show that T is a contraction on $\mathscr{X}$. Let $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2 \in \mathscr{X}$ and let $\mathbf{z}_1, \mathbf{z}_2$ be the corresponding solutions of (F.14a) with $\mathbf{z}_i(0) = \mathbf{z}_0$, $i = 1, 2$. We first estimate $\mathbf{v}(t) = \mathbf{z}_1(t) - \mathbf{z}_2(t)$. From (F.14a) and (F.17),

$$\|\mathbf{v}(t)\| \le ck(\varepsilon) \int_0^t e^{-b(t-s)} \left[\|(\mathbf{z}_1(s), \dot{\mathbf{z}}_1(s))\| \left(\|\boldsymbol{\phi}_1(s) - \boldsymbol{\phi}_2(s)\| + \|\dot{\boldsymbol{\phi}}_1(s) - \dot{\boldsymbol{\phi}}_2(s)\| \right) + \|\mathbf{v}(s)\| + \|\dot{\mathbf{v}}(s)\| \right] ds.$$
(F.21)

Since

$$\dot{\mathbf{v}} = \mathcal{B}\mathbf{v} + \mathcal{R}(\boldsymbol{\phi}_1 + \mathbf{u}, \mathbf{z}_1, \dot{\boldsymbol{\phi}}_1 + \dot{\mathbf{u}}, \dot{\mathbf{z}}_1) - \mathcal{R}(\boldsymbol{\phi}_2 + \mathbf{u}, \mathbf{z}_2, \dot{\boldsymbol{\phi}}_2 + \dot{\mathbf{u}}, \dot{\mathbf{z}}_2),$$

using (F.17), we get

$$\|\dot{\mathbf{v}}\| \le \|\mathcal{B}\| \, \|v\| + k(\varepsilon) \left[\|(\mathbf{z}_1, \dot{\mathbf{z}}_1)\| \left(\|\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2\| + \|\dot{\boldsymbol{\phi}}_1 - \dot{\boldsymbol{\phi}}_2\| \right) + \|\mathbf{v}\| + \|\dot{\mathbf{v}}\| \right].$$

So

$$\|\dot{\mathbf{v}}\| \le (1 - k(\varepsilon))^{-1} \left[(\|\mathcal{B}\| + k(\varepsilon)) \|\mathbf{v}\| + k(\varepsilon) \|(\mathbf{z}_1, \dot{\mathbf{z}}_1)\| \left(\|\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2\| + \|\dot{\boldsymbol{\phi}}_1 - \dot{\boldsymbol{\phi}}_2\| \right) \right],$$

implying

$$\|\dot{\mathbf{v}}(s)\| \le C_3 \|\mathbf{v}(s)\| + k_1(\varepsilon) \|(\mathbf{z}_1, \dot{\mathbf{z}}_1)\| \, \|\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2\|_{\mathscr{X}} \, e^{-as}, \qquad k_1(0) = 0,$$

for some $C_3 > 0$. Together with $\|(\mathbf{z}_1(s), \dot{\mathbf{z}}_1(s))\| \le C_2 e^{-\beta_1 s}$, which is followed by (F.20), (F.21) implies

$$\|\mathbf{v}(t)\| \le ck(\varepsilon) \int_0^t e^{-b(t-s)} \left[(1 + k_1(\varepsilon)) C_2 e^{-\beta_1 s} \|\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2\|_{\mathscr{X}} \, e^{-as} + (1 + C_3) \|\mathbf{v}(s)\| \right] ds$$

$$\le C_4 k(\varepsilon) \|\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2\|_{\mathscr{X}} \, e^{-bs} + ck(\varepsilon)(1 + C_3) \int_0^t e^{-b(t-s)} \|\mathbf{v}(s)\| \, ds$$

for ε sufficiently small, where $C_4 > 0$ is a constant. By Gronwall's lemma,

$$\|\mathbf{v}(t)\| \le C_4 k(\varepsilon) \|\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2\|_{\mathscr{X}} \, e^{-\beta_2 t}, \qquad \beta_2 = b - ck(\varepsilon)(1 + C_3). \quad \text{(F.22)}$$

Using (F.15) and (F.22),

$$\|T\boldsymbol{\phi}_1(t) - T\boldsymbol{\phi}_2(t)\|$$

$$\le \int_t^\infty \left[\frac{a}{2} \|\boldsymbol{\phi}_1(s) - \boldsymbol{\phi}_2(s)\| + \|\mathcal{R}(\boldsymbol{\phi}_1(s), \mathbf{z}_1(s), \dot{\boldsymbol{\phi}}_1(s), \dot{\mathbf{z}}_1(s)) - \mathcal{R}(\boldsymbol{\phi}_2(s), \mathbf{z}_2(s), \dot{\boldsymbol{\phi}}_2(s), \dot{\mathbf{z}}_2(s))\| \right]$$

$$\le \frac{a}{2} \int_t^\infty \|\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2\|_{\mathscr{X}} \, e^{-as} \, ds$$

$$\quad + k(\varepsilon) \int_t^\infty \left[\|(\mathbf{z}_1(s), \dot{\mathbf{z}}_1(s))\| \left(\|\boldsymbol{\phi}_1(s) - \boldsymbol{\phi}_2(s)\| + \|\dot{\boldsymbol{\phi}}_1(s) - \dot{\boldsymbol{\phi}}_2(s)\| \right) + \|\mathbf{v}(s)\| + \|\dot{\mathbf{v}}(s)\| \right] ds$$

$$\le \frac{1}{2} \|\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2\|_{\mathscr{X}}$$

$$\quad + k(\varepsilon) \int_t^\infty \left[(1 + k_1(\varepsilon)) C_2 e^{-\beta_1 s} \|\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2\|_{\mathscr{X}} \, e^{-as} + (1 + C_3) C_4 k(\varepsilon) \|\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2\|_{\mathscr{X}} \, e^{-\beta_2 s} \right] ds$$

$$\le \alpha \|\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2\|_{\mathscr{X}}, \qquad \alpha < 1,$$

for α sufficiently small. This shows that T is a contraction and, hence, has a unique fixed point.

Note that $T = T_{\mathbf{z}}$ and the above analysis proves that $T_{\mathbf{z}}$ has a unique fixed point in $\mathscr{X}$ provided $\mathbf{z}$ and $\dot{\mathbf{z}}$ are sufficiently small. Denote $\boldsymbol{\phi} \in \mathscr{X}$ the fixed point of the contraction $T_{\mathbf{z}}$, where $\mathbf{z} = \mathbf{y} - h(\mathbf{x})$. Let $\mathbf{u}(0) = \mathbf{x}(0) - \boldsymbol{\phi}(0)$. Then let $\mathbf{u}(t)$ be the solution of (F.6) evolving from $\mathbf{u}(0)$. Then $(\mathbf{x}(t), \mathbf{y}(t))$ can be decomposed as in $\mathbf{x}(t) = \mathbf{u}(t) + \boldsymbol{\phi}(t)$ and $\mathbf{y} = h(\mathbf{x}(t)) + \mathbf{z}(t)$. In view of the fact that $\boldsymbol{\phi} \in \mathscr{X}$ and Lemma F.1, the asymptotic limits in (F.13) follow, which completing the proof of Proposition F.2. $\quad\square$

Appendix G: Asymptotic Expansion of the Local Center Manifold

In this appendix, we check the expression of local center manifold in Lemma 9.6 by asymptotic expansion.

If we substitute $\mathbf{y} = h(\mathbf{x})$ into (9.32b), we see that the center manifold can be obtained by solving

$$h'(\mathbf{x})\left[Q_1\mathcal{N}(\mathbf{x} + h(\mathbf{x}), \dot{\mathbf{x}} + h'(\mathbf{x})\dot{\mathbf{x}})\right] = \mathcal{L}h(\mathbf{x}) + Q_2\mathcal{N}(\mathbf{x} + h(\mathbf{x}), \dot{\mathbf{x}} + h'(\mathbf{x})\dot{\mathbf{x}}).$$

We want to show the manifold of equilibria $\mathcal{M}_*$ is a center manifold by showing it satisfies the above equation. On the manifold of equilibria $\mathcal{M}_*$, $\dot{\mathbf{x}} = 0$. So we need to check

$$h'(\mathbf{x})\left[Q_1\mathcal{N}(\mathbf{x} + h(\mathbf{x}), \mathbf{0})\right] = \mathcal{L}h(\mathbf{x}) + Q_2\mathcal{N}(\mathbf{x} + h(\mathbf{x}), \mathbf{0}). \qquad \text{(G.1)}$$

Note that

$$\mathcal{N}(\mathbf{x} + h(\mathbf{x}), \mathbf{0}) = \mathcal{N}^0(\mathbf{x} + h(\mathbf{x})) = \begin{bmatrix} 0 \\ 0 \\ -\frac{\mathscr{R}_g T_\infty}{\rho_l R_*} H^0(\mathbf{x} + h(\mathbf{x})) \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ -\frac{2\sigma}{\rho_l R_*^3} \frac{(\alpha + R_{**}(\alpha) - R_*)^2}{\alpha + R_{**}(\alpha)} \\ 0 \\ 0 \\ \vdots \end{bmatrix}.$$

So $Q_1\mathcal{N}(\mathbf{x} + h(\mathbf{x}), \mathbf{0}) = \mathbf{0}$ and $\mathcal{N}(\mathbf{x} + h(\mathbf{x}), \mathbf{0}) = \mathcal{N}(\mathbf{x} + h(\mathbf{x}), \mathbf{0})$. It suffices to check $\mathcal{L}h(\mathbf{x}) + Q_2\mathcal{N}(\mathbf{x} + h(\mathbf{x}), \mathbf{0}) = \mathbf{0}$ in (G.1). Taylor expand $h(\mathbf{x}) = h(\alpha\mathbf{b})$ at

$\alpha = 0$, one gets

$$h(\mathbf{x}) = \begin{bmatrix} \rho_{**}(\alpha) - \rho_* \\ R_{**}(\alpha) - R_* \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} \rho_{**,1}\alpha + \rho_{**,2}\alpha^2 + \rho_{**,3}\alpha^3 + \cdots \\ R_{**,1}\alpha + R_{**,2}\alpha^2 + R_{**,3}\alpha^3 + \cdots \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix},$$

where $\rho_{**,m} = (m!)^{-1}\rho_{**}^{(m)}(0)$ and $R_{**,m} = (m!)^{-1}R_{**}^{(m)}(0)$ in which $\rho_{**}^{(m)}(0)$ and $R_{**}^{(m)}(0)$ are the mth derivatives of ρ_{**} and R_{**} at $\alpha = 0$, respectively. Then $\mathcal{L}h(\mathbf{x}) + \mathcal{N}(\mathbf{x} + h(\mathbf{x}), \mathbf{0}) = \mathbf{0}$ is equivalent to

$$\sum_{m=1}^{\infty} \left(\frac{\mathcal{R}_g T_\infty}{\rho_l R_*} \rho_{**,m} + \frac{2\sigma}{\rho_l R_*^3} R_{**,m} \right) \alpha^m - \frac{2\sigma}{\rho_l R_*^3} \frac{(\alpha + R_{**}(\alpha) - R_*)^2}{\alpha + R_{**}(\alpha)} = 0. \tag{G.2}$$

For $m = 1$, we differentiate (9.34) with respect to α, evaluate at $\alpha = 0$ and use $R_{**}(0) = R_*$, we get $\mathcal{R}_g T_\infty \rho_{**}'(0) = -(2\sigma/R_*^2)R_{**}'(0)$. So we have

$$\frac{\mathcal{R}_g T_\infty}{\rho_l R_*} \rho_{**,1} + \frac{2\sigma}{\rho_l R_*^3} R_{**,1} = 0.$$

For $m = 2$, we first expand

$$\frac{(\alpha + R_{**}(\alpha) - R_*)^2}{\alpha + R_{**}(\alpha)} = \frac{(1 + R_{**}'(0))^2}{R_*}\alpha^2 + \cdots.$$

Then the coefficient of α^2 in (G.2) is

$$\frac{\mathcal{R}_g T_\infty}{\rho_l R_*} \rho_{**,2} + \frac{2\sigma}{\rho_l R_*^3} R_{**,2} - \frac{2\sigma}{\rho_l R_*^4} \left(1 + R_{**,1} \right)^2. \tag{G.3}$$

By differentiating (9.34) with respect to α twice, evaluating at $\alpha = 0$ and using $R_{**}(0) = R_*$, we have $\mathcal{R}_g T_\infty \rho_{**}''(0) = (4\sigma/R_*^3)(R_{**}'(0))^2 - (2\sigma/R_*^2)R_{**}''(0)$. So

$$\frac{\mathcal{R}_g T_\infty}{\rho_l R_*} \rho_{**,2} + \frac{2\sigma}{\rho_l R_*^3} R_{**,2} = (2!)^{-1} \frac{4\sigma}{\rho_l R_*^4}(R_{**,1})^2.$$

In order to have the term (G.3) for $m = 2$ vanishing, we require

$$0 = \frac{2\sigma}{\rho_l R_*^4} R_{**,1}^2 - \frac{2\sigma}{\rho_l R_*^4}(1 + R_{**,1})^2,$$

whose solution is $R_{**,1} = -1/2$.

For $m = 3$, we further expand, using $R_{**,1} = -1/2$, that

$$\frac{(\alpha + R_{**}(\alpha) - R_*)^2}{\alpha + R_{**}(\alpha)} = \frac{(1 + R_{**,1})^2}{R_*}\alpha^2 + \left(\frac{1}{2R_*}R_{**}''(0) - \frac{1}{8R_*^2} \right)\alpha^3 + \cdots$$

Then the coefficient of α^3 in (G.2) is

$$\frac{\mathscr{R}_g T_\infty}{\rho_l R_*}\rho_{**,3} + \frac{2\sigma}{\rho_l R_*^3}R_{**,3} - \frac{2\sigma}{\rho_l R_*^3}\left(\frac{1}{2R_*}R_{**}''(0) - \frac{1}{8R_*^2}\right). \tag{G.4}$$

By differentiating (9.34) with respect to α three times, evaluating at $\alpha = 0$ and using $R_{**}(0) = R_*$, we get $\mathscr{R}_g T_\infty \rho_{**}'''(0) = -(12\sigma/R_*^4)(R_{**}'(0))^3 + (12\sigma/R_*^3)R_{**}'(0)R_{**}''(0) - (2\sigma/R_*^2)R_{**}'''(0)$. So we have

$$\frac{\mathscr{R}_g T_\infty}{\rho_l R_*}\rho_{**,3} = -\frac{2\sigma}{\rho_l R_*^5}R_{**,1}^3 + \frac{4\sigma}{\rho_l R_*^4}R_{**,1}R_{**,2} - \frac{2\sigma}{\rho_l R_*^3}R_{**,3}.$$

In order to have the term (G.4) for $m = 3$ vanishing, we require

$$0 = -\frac{2\sigma}{\rho_l R_*^5}R_{**,1}^3 + \frac{4\sigma}{\rho_l R_*^4}R_{**,1}R_{**,2} - \frac{2\sigma}{\rho_l R_*^3}R_{**,3} + \frac{2\sigma}{\rho_l R_*^3}R_{**,3}$$
$$- \frac{\sigma}{\rho_l R_*^3}\left(\frac{1}{R_*}R_{**}''(0) - \frac{1}{4R_*^2}\right),$$

where the third term cancels the fourth. Using $R_{**,1} = -1/2$ and $R_{**}''(0) = 2R_{**,2}$,

$$0 = \frac{\sigma}{4\rho_l R_*^5} - \frac{2\sigma}{\rho_l R_*^4}R_{**,2} - \frac{2\sigma}{\rho_l R_*^4}R_{**,2} + \frac{\sigma}{4\rho_l R_*^5}$$

for which the solution is $R_{**,2} = 1/(8R_*)$. Thus,

$$R_{**}(\alpha) = R_* - \frac{1}{2}\alpha + \frac{1}{8R_*}\alpha^2 + \cdots = \frac{-\alpha + \sqrt{\alpha^2 + 4R_*^2}}{2},$$

which coincides with the center manifold expression in (9.34).

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