



Products of manifolds with fibered corners

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Abstract

Manifolds with fibered corners arise as resolutions of stratified spaces, as ‘many-body’ compactifications of vector spaces, and as compactifications of certain moduli spaces including those of non-abelian Yang–Mills–Higgs monopoles, among other settings. However, Cartesian products of manifolds with fibered corners do not generally have fibered corners themselves and thus fail to reflect the appropriate structure of products of the underlying spaces in the above settings. Here, we determine a resolution of the Cartesian product of fibered corners manifolds by blow-up which we call the ‘ordered product,’ which leads to a well-behaved category of fibered corners manifolds in which the ordered product satisfies the appropriate universal property. In contrast to the usual category of manifolds with corners, this category of fibered corners not only has all finite products, but all finite transverse fiber products as well, and we show in addition that the ordered product is a natural product for wedge (aka incomplete edge) metrics and quasi-fibered boundary metrics, a class which includes QAC and QALE metrics.

Keywords Fibered corners · Stratified space · Many-body space · Wedge metric · Quasi-fibered boundary metric

1 Introduction

There is a large body of work in geometric analysis in which non-compact and/or singular spaces are compactified and/or resolved to manifolds with corners, on which detailed analysis, especially of asymptotic expansions at boundary faces, gives precise results not readily available by other means. The manifolds with corners associated with *products* of such spaces are naturally of interest, not least because the typical approach in such problems is to consider operators via their distributional Schwartz kernels. Particularly when there are many boundary faces involved, the identification of the ‘correct’ version of the product

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can be a subtle problem which may involve seemingly arbitrary choices and can often seem somewhat ad hoc. However, in many (if not most) of these problems, the manifolds with corners additionally admit a natural *fibred corners* structure in which each boundary hypersurface is the total space of a fiber bundle comparable in a precise sense to its neighboring bundles at the corners. We show here that consideration of this additional structure leads to a remarkably well-behaved theory of products and fiber products.

Fibred corners structures arise in particular in two settings: The first setting is the resolution of (*smoothly*) *stratified spaces* [1, 3–5], which are often equipped with *wedge* (or ‘*iterated incomplete edge*’) *metrics*, Riemannian metrics degenerating conically along the strata in an iterated fashion. The second setting is typified in simplest form by *many-body spaces*, which are vector spaces which have been radially compactified and subsequently blown up along the boundaries of a family of linear subspaces [6, 21, 31]. Euclidean metrics on the original vector spaces become *quasi-asymptotically conic* (QAC) metrics on the associated many-body spaces, and this asymptotic geometry may be generalized to the manifold setting in the form of QAC manifolds [12] and even more generally in the form of ‘quasi-fibred boundary’ (QFB, aka Φ) manifolds [10], extending the *scattering* and *fibred boundary* structures of [28] and [27] on manifolds with boundary, respectively, as well as the *quasi-asymptotically locally Euclidean* (QALE) metrics introduced by Joyce [8, 17, 19]. Important examples of QFB manifolds include the compactifications of the hyperKähler moduli spaces of $SU(2)$ monopoles of arbitrary charge [13]. Roughly speaking, spaces arising in the first setting are modeled (iteratively and in parameterized fashion) on the small ends of cones, while those arising in the second setting are modeled on the large ends.

While a fair amount has been written about manifolds with fibred corners, this article establishes some aspects of an associated *category*, in particular the existence of products and fiber products in this category. The results presented here will have applications to the construction of various pseudodifferential calculi associated with manifolds with fibred corners. In particular, these results are used in an essential way in [24] and [23], the authors’ work on elliptic pseudodifferential operator analysis and L^2 -cohomology of QFB manifolds. In fact, it will be shown in a forthcoming work that a wide variety of the known pseudodifferential calculi adapted to various geometric settings can be constructed within the category of manifolds with fibred corners introduced below, and that doing so simplifies and clarifies many of their features. This suggests that the fibred corners category may be the appropriate setting in which to answer a question posed by Richard Melrose, namely to axiomatize and subsequently classify all ‘generalized products,’ meaning the sequences of spaces $X_1, X_2, X_3, \dots$ and maps between these spaces which satisfy certain properties enjoyed by the projection maps and diagonal inclusions when X_1, X_2 , and X_3 are the respective single, double, and triple spaces of a geometric pseudodifferential calculus.

In more detail, a *fibred corners structure*¹ on a compact (not necessarily connected) manifold with corners X consists of a locally trivial fiber bundle structure $\phi_G : G \rightarrow B_G$ (hereafter simply called a *fibration*) on each boundary hypersurface G (again not necessarily connected), the base B_G and typical fiber F_G of which are also manifolds with corners (and in fact fibred corners), satisfying a comparability condition wherever two hypersurfaces intersect (see Definition 4.1).

Among other things this determines a partial order on the set $\mathcal{M}_1(X)$ of boundary hypersurfaces, in which $G < G'$ whenever $G \cap G' \neq \emptyset$ and G has a strictly coarser fibration (meaning that G is associated with a smaller stratum when X is associated with a stratified

¹ Various known also as a ‘resolution structure,’ ‘iterated boundary fibration structure,’ or simply ‘iterated structure’ [3, 5, 11]

space). As a matter of notation, we write $G \sim G'$ when G and G' are comparable, meaning $G = G'$, $G < G'$, or $G' < G$.

A key observation, seemingly overlooked in the literature, is that this structure and associated partial order extends to the larger set $\mathcal{M}_{1,0}(X) = \mathcal{M}_1(X) \cup \{X\}$ of *principal faces*, meaning the boundary hypersurfaces along with the interior, provided the interior itself is equipped with a fibration $\phi_X : X \rightarrow B_X$ similarly comparable to the ϕ_G . The two most common situations arise by taking one of the two trivial fibration structures on X :

- If $B_X = X$ and $\phi_X = 1$, making $X \in \mathcal{M}_{1,0}(X)$ maximal in the order, we say X is *interior maximal*, denoted by $X = X_{\max}$, which turns out to be natural in the setting of stratified spaces and wedge metrics.
- If $B_X = *$ is a single point and ϕ_X is the constant map, making $X \in \mathcal{M}_{1,0}(X)$ minimal in the order, we say X is *interior minimal*, denoted by $X = X_{\min}$, which turns out to be natural in the setting of many-body spaces and QFB manifolds.

We treat the general case of an arbitrary fibration on X below, obtaining results for interior maximal and interior minimal spaces by specialization. Note that every manifold with fibered corners admits both wedge and QFB metrics.

Even with interior fibrations specified, the Cartesian product $X \times Y$ is not generally a manifold with fibered corners in any natural way, since the product fibrations are not generally comparable where they meet. Instead, the product in the fibered corners category is the *ordered product*, defined for interior maximal manifolds with fibered corners by the iterated blow-up

$$X_{\max} \tilde{\times} Y_{\max} = [X \times Y; \mathcal{M}_1(X) \times \mathcal{M}_1(Y)]$$

of all codimension 2 corners of the form $G \times H$, where $G \in \mathcal{M}_1(X)$ and $H \in \mathcal{M}_1(Y)$, taken in any order consistent with the partial order on the product $\mathcal{M}_1(X) \times \mathcal{M}_1(Y)$; in other words $G' \times H'$ must be blown-up prior to $G \times H$ whenever $G' < G$ and $H' < H$, and any order satisfying this condition leads the same space up to natural diffeomorphism.

On the other hand, for interior minimal manifolds with fibered corners, the ordered product is defined by the opposite order blow-up

$$X_{\min} \tilde{\times} Y_{\min} = [X \times Y; \widehat{\mathcal{M}}_1(X) \times \widehat{\mathcal{M}}_1(Y)]$$

where $\widehat{\mathcal{M}}_1(X)$ denotes the partially ordered set $\mathcal{M}_1(X)$ with order reversed; in other words $G' \times H'$ must be blown up prior to $G \times H$ whenever $G' > G$ and $H' > H$. In general, these are not diffeomorphic, i.e., $X_{\max} \tilde{\times} Y_{\max} \not\cong X_{\min} \tilde{\times} Y_{\min}$, whenever one of the factors has a corner of codimension at least 2. Both products are in fact special cases of the general ordered product

$$X \tilde{\times} Y = [X \times Y; \mathcal{M}_1^<(X) \times \mathcal{M}_1^<(Y), \widehat{\mathcal{M}}_1^>(X) \times \widehat{\mathcal{M}}_1^>(Y)], \quad (1.1)$$

where $\mathcal{M}_1^{\geq}(X) = \{G \in \mathcal{M}_1(X) : G \geq X\}$. Our main results about the ordered product are summarized as follows.

Theorem (Theorems 4.13, 4.8, 6.6, 6.6) *For manifolds X and Y with fibered corners:*

- The ordered product $X \tilde{\times} Y$ is naturally a manifold with fibered corners, with principal faces $L(G, H) \in \mathcal{M}_{1,0}(X \tilde{\times} Y)$ identified with those pairs $(G, H) \in \mathcal{M}_{1,0}(X) \times \mathcal{M}_{1,0}(Y)$ which are comparable to (X, Y) in the product order, with $L(G, H) < L(G', H')$ if $(G, H) < (G', H')$. In particular, $L(G, H)$ and $L(G', H')$ are disjoint if $(G, H) \approx (G', H')$.*

(b) For interior maximal spaces, the fibrations of $X_{\max} \tilde{\times} Y_{\max}$ have the form

$$\begin{array}{ccc} F_H \longrightarrow L(X, H) & F_G \longrightarrow L(G, Y) & F_G \tilde{\star}_{\max} F_H \longrightarrow L(G, H) \\ \downarrow \phi_{X,H} & \downarrow \phi_{G,Y} & \text{or} \quad \downarrow \phi_{G,H} \\ X \tilde{\times} B_H, & B_G \tilde{\times} Y, & B_G \tilde{\times} B_H, \end{array} \quad (1.2)$$

for $H \in \mathcal{M}_{1,0}(Y)$ and $G \in \mathcal{M}_{1,0}(X)$ in the first and second cases, and $(G, H) \in \mathcal{M}_1(X) \times \mathcal{M}_1(Y)$ in the third case, and where $\tilde{\star}$ denotes the join product discussed below (see also Definition 3.8).

(c) For interior minimal spaces, the fibrations of $X_{\min} \tilde{\times} Y_{\min}$ have the form

$$\begin{array}{ccc} X \tilde{\times} F_H \longrightarrow L(X, H) & F_G \tilde{\times} Y \longrightarrow L(G, Y) & F_G \tilde{\times} F_H \longrightarrow L(G, H) \\ \downarrow \phi_{X,H} & \downarrow \phi_{G,Y} & \text{or} \quad \downarrow \phi_{G,H} \\ B_H, & B_G, & B_G \tilde{\times}_{\min} B_H. \end{array} \quad (1.3)$$

For the structure of the fibrations on a general ordered product (1.1) specializing to (1.2) and (1.3), we refer to Theorem 4.13.

(d) $X \tilde{\times} Y$ satisfies the universal property of the product in a category of manifolds with fibered corners and appropriate morphisms (see below), namely any morphisms $f : W \rightarrow X$ and $g : W \rightarrow Y$ factor through a unique morphism $W \rightarrow X \tilde{\times} Y$ forming a commutative diagram

$$\begin{array}{ccc} & W & \\ f \swarrow & \downarrow \exists! & \searrow g \\ X & \longleftarrow X \tilde{\times} Y \longrightarrow & Y \end{array}$$

with the lifted projections $\pi_X : X \tilde{\times} Y \rightarrow X$ and $\pi_Y : X \tilde{\times} Y \rightarrow Y$. In particular, as a consequence of this universal property, the ordered product is associative and commutative up to unique isomorphism.

(e) If g_X and g_Y are wedge metrics (resp. QFB metrics) on $X_{\max}$ and $Y_{\max}$ (resp. $X_{\min}$ and $Y_{\min}$), then $g_X + g_Y$ lifts to a wedge metric (resp. QFB metric) on $X_{\max} \tilde{\times} Y_{\max}$ (resp. $X_{\min} \tilde{\times} Y_{\min}$).

Some remarks:

- The fibered corners structure on X induces a fibered corners structure on each boundary hypersurface G as well as its fiber F_G and base B_G , with respect to which F_G is interior minimal and B_G is interior maximal. In particular, the base spaces in (1.2) are ordered products of interior maximal manifolds, while the fibers in (1.3) are ordered products of interior minimal manifolds.
- We do not assume that the spaces X , G , F_G or B_G are connected; in particular, it often arises in practice that $\phi_G : G \rightarrow B_G$ is a fibration with fiber F_G having disjoint components.
- The join product $\tilde{\star}$ appearing in the fibers in (1.2), so named for the fact that its associated stratified space is the topological join of the stratified spaces of its factors (see Corollary 5.2), may be identified as the iterated blow-up

$$\begin{aligned} F_G \tilde{\star}_{\max} F_H &= [F_G \times F_H \times I; \mathcal{M}_1(F_G) \times F_H \times \{0\}, \\ &\quad F_G \times \mathcal{M}_1(F_H) \times \{1\}, \\ &\quad \mathcal{M}_1(F_G) \times \mathcal{M}_1(F_H) \times I], \quad I = [0, 1], \end{aligned} \quad (1.4)$$

with blow-ups taken in any order consistent with the partial orders on $\mathcal{M}_1(F_G)$, $\mathcal{M}_1(F_H)$, and $\mathcal{M}_1(F_G) \times \mathcal{M}_1(F_H)$. The join product appearing the bases in (1.3) is given by a similar formula but with orders reversed:

$$\begin{aligned} B_G \tilde{\smile}_{\min} B_H &= [B_G \times B_H \times I; \overleftarrow{\mathcal{M}}_1(B_G) \times B_H \times \{0\}, \\ &\quad B_G \times \overleftarrow{\mathcal{M}}_1(B_H) \times \{1\}, \\ &\quad \overleftarrow{\mathcal{M}}_1(B_G) \times \overleftarrow{\mathcal{M}}_1(B_H) \times I], \end{aligned} \quad (1.5)$$

In contrast to the ordered product, there is in fact a remarkably non-trivial diffeomorphism $X \tilde{\smile}_{\max} Y \cong X \tilde{\smile}_{\min} Y$ between the minimal and maximal join products which is proved in §7, from which it follows that the associated stratified space to (1.5) is also the topological join of the stratified spaces of the factors. A general version of the join (see Definition 3.8) which appears in the boundary fibrations for a general ordered product (1.1) is somewhat more complicated and depends on the place of X in the order on $\mathcal{M}_{1,0}(F_G)$ and $\mathcal{M}_{1,0}(B_G)$, viewed as ordered subsets of $\mathcal{M}_{1,0}(X)$.

- The boundary fibrations (1.3) in the interior minimal case (and more generally whenever X and Y are not both maximal) depend on equivalence classes of boundary defining functions on X and Y , where two defining functions are equivalent if their ratio is basic (i.e., constant on fibers) over each boundary hypersurface (see Definition 4.4). Such a choice of equivalence classes of boundary defining functions is therefore part of the data for X and Y as objects in the general category of manifolds with fibered corners (unless all spaces are taken to be interior maximal). This equivalence relation also arises naturally in the consideration of QFB metrics (see Sect. 6).
- The morphisms referenced in part (d) are taken to be those interior b-maps $f : X \rightarrow Y$ which are
 - *simple*, meaning all boundary exponents are 0 or 1,
 - *b-normal*, in particular boundary hypersurfaces map either to boundary hypersurfaces of the target or to the interior, but never to faces of higher codimension, so f determines a map $f_{\sharp} : \mathcal{M}_{1,0}(X) \rightarrow \mathcal{M}_{1,0}(Y)$ between sets of principal faces,
 - *ordered*, meaning $f_{\sharp} : \mathcal{M}_{1,0}(X) \rightarrow \mathcal{M}_{1,0}(Y)$ is a map of ordered sets,
 - *fibered*, meaning f restricts over each $G \in \mathcal{M}_{1,0}(X)$ to a map of fiber bundles

$$\begin{array}{ccc} G & \xrightarrow{f} & H = f_{\sharp}(G) \\ \downarrow \phi_G & & \downarrow \phi_H \\ B_G & \xrightarrow{f_G} & B_H \end{array} \quad \text{for some } f_G$$

- *consistent* with the equivalence classes of boundary defining functions on X and Y (see Definition 4.4 for a precise statement), except in the case that all spaces are interior maximal.

In contrast to the usual category of manifolds with corners and b-maps,² the fibered corners category has all transverse fiber products:

Theorem (Theorems 3.12, 3.13 and Proposition 4.14) *If $f : X \rightarrow Z$ and $g : Y \rightarrow Z$ are fibered corners morphisms which are b-transverse, meaning that ${}^b d f_* {}^b T_x X + {}^b d g_* {}^b T_y Y =$*

² As shown in [18], the suitably transverse fiber product of manifolds with corners exists as a manifold with ‘generalized corners,’ a larger category which has products and transverse fiber products, but not typically a manifold with ordinary corners.

${}^bT_z Z$ whenever $f(x) = g(y) = z$, then

$$X \tilde{\times}_Z Y = \overline{\{(x, y) \in X^\circ \times Y^\circ : f(x) = g(y)\}} \subset X \tilde{\times} Y$$

is a manifold with fibered corners, embedded as a p -submanifold in the ordered product and satisfying the universal property of the fiber product in the category of fibered corners. The fibrations on $X \tilde{\times}_Z Y$, consistent with the restriction of from those of $X \tilde{\times} Y$, have the form

$$\begin{array}{ccc} F_G \tilde{\times}_{F_K} F_H & \longrightarrow & G \tilde{\times}_K H \\ & \downarrow \phi_{G,H} & \text{where } K = f_\#(G) = g_\#(H) \in \mathcal{M}_{1,0}(Z) \\ & B_G \tilde{\times}_{B_K} B_H & \end{array}$$

except for the case that $G \in \mathcal{M}_1(X)$ and $H \in \mathcal{M}_1(Y)$ are proper boundary hypersurfaces with $f_\#(G) = g_\#(H) = Z \in \mathcal{M}_0(Z)$, in which case the fibrations have a more complicated form

$$\begin{array}{ccc} F_G \tilde{\star}_{\max, Z} F_H & \longrightarrow & X \tilde{\times}_Z Y \cap L(G, H) \\ & \downarrow \phi_{G,H} & \text{if } X = X_{\max}, Y = Y_{\max}, \text{ or} \\ & B_G \tilde{\times}_{B_Z} B_H & \\ \\ F_G \tilde{\times}_{F_Z} F_H & \longrightarrow & X \tilde{\times}_Z Y \cap L(G, H) \\ & \downarrow \phi_{G,H} & \text{if } X = X_{\min}, Y = Y_{\min}. \\ & B_G \tilde{\star}_{\min, Z} B_H & \end{array}$$

In the notation, $F_G \tilde{\star}_{\max, Z} F_H$ denotes the closure of $F_G^\circ \times_Z F_H^\circ \times I$ in (1.4) and $B_G \tilde{\star}_{\min, Z} B_H$ denotes the closure of $B_G^\circ \times_Z B_H^\circ \times I$ in (1.5); the general case (X and Y not necessarily minimal or maximal) is covered in Theorem 3.13. In particular, for any morphism $f : X \rightarrow Y$, the graph $\text{Gr}(f) = X \tilde{\times}_Y Y$ is realizable as a fiber product; thus, f factors canonically within the category as the inclusion of a p -submanifold $X \cong \text{Gr}(f) \hookrightarrow X \tilde{\times} Y$ and a b -fibration $X \tilde{\times} Y \rightarrow Y$.

The ordered product realizes and generalizes known products from two different settings mentioned previously. The first is the product of *smoothly stratified spaces*, which are topological spaces $\mathcal{X} = \bigsqcup s_i$ decomposed into disjoint manifolds s_i of varying dimension called *strata*, with some conditions on how the strata fit together (see Sect. 5.1) amounting to an iterative, parameterized conic degeneration of each stratum onto the next. There is a well-known equivalence between a smoothly stratified space $\mathcal{X}$ on the one hand, and a manifold with fibered corners X on the other, wherein $\mathcal{X}$ is obtained by collapsing the fibers of the boundary fibrations of X , while X is recovered from $\mathcal{X}$ by iteratively resolving the strata by a kind of blow-up which resolves cones to cylinders [1, 3].

Theorem (Theorem 5.1) *If X and Y are manifolds with fibered corners associated with smoothly stratified spaces $\mathcal{X} = \bigsqcup_i s_i$ and $\mathcal{Y} = \bigsqcup_j s'_j$, then the interior maximal ordered product $X_{\max} \tilde{\times} Y_{\max}$ is the manifold with fibered corners associated with the product $\mathcal{X} \times \mathcal{Y} = \bigsqcup_{i,j} s_i \times s'_j$ of stratified spaces.*

The second is the product of *many-body spaces*, which are compactifications

$$M(V) = [\overline{V}; \{\partial \overline{S} : S \in \mathcal{S}_V\}]$$

of finite-dimensional real vector spaces V obtained by iteratively blowing up the radial compactification $\overline{V}$ along the boundaries of a finite set $\mathcal{S}_V$ of linear subspaces. Such a space is naturally an interior minimal manifold with fibered corners.

Theorem (Theorem 5.3) *There is a natural isomorphism $M(V)_{\min} \tilde{\times} M(W)_{\min} \cong M(V \times W) = \overline{[V \times W; \{\partial(\overline{S \times S'}) : S \in \mathcal{S}_V, S' \in \mathcal{S}_W\}]}]$ between the interior minimal ordered product of $M(V)$ and $M(W)$ and the many-body space of the product $V \times W$.*

An overview of the paper is as follows. We first recall some notation and background on manifolds with corners in Sect. 2, including some of the theory of *generalized blow-up* from [22] (even though all blow-ups here are ordinary ones), which is used throughout to simplify the proofs of our results. As it happens, a number of properties of the ordered product do not actually depend on the fibration structures on the principal faces of the factors, but rather only on the induced order among these principal faces. For this reason, and since it may have more general applications, we first develop much of the theory in a larger category of *ordered corners* in Sect. 3 in which manifolds are equipped only with a suitable orderings of their principal faces and morphisms are consistent with these. The properties of products in this ordered corners category are proved in Sect. 3.1, and the properties of fiber products are proved in Sect. 3.3. In Sect. 3.2, we give a detailed analysis of the product structure of boundary hypersurfaces of $X \tilde{\times} Y$ in preparation for the fibered corners category; a key role is played by so-called *compressed projection maps* which in a certain sense extend the boundary fibrations to fibrations involving a cone over the fiber or base, depending on context. We then review fibered corners structures in Sect. 4 and obtain the results stated as parts (a)–(d) of the main product theorem above as well as the fiber product theorem in Sect. 4.1. The connection to stratified spaces and many-body spaces is discussed in Sects. 5.1 and 5.2, respectively. In Sect. 6, we introduce the geometric structures encoded by the wedge and Φ tangent bundles, proving the product result for Φ structures in Sect. 6.1 and for wedge structures in Sect. 6.2. In fact, we give two proofs each of the product results for wedge and QFB metrics, one based on a direct analysis of the respective tangent bundles, and another alternate proof based on examination of the form of the lifted product metrics. In Sect. 7, we prove the diffeomorphism between the minimal and maximal join products, which apart from its conceptual relevance in this paper gives an unusual example of a diffeomorphism of blow-ups of manifolds with corners which is non-trivial on the interior. Finally, Appendix A contains a short exposition of some technical results concerning tubular neighborhood structures with nice properties and an Ehresmann lemma in the fibered corners category.

2 Background

We briefly recall some of the important notions from the theory of manifolds with corners used below, though we assume the reader is already somewhat familiar; for more complete references and/or more leisurely introductions, see [15, 16, 29, 30].

A *manifold with corners* is a Hausdorff space X locally diffeomorphic³ to open sets in $\mathbb{R}_+^n$, where $\mathbb{R}_+ = [0, \infty)$. With a few notable exceptions, such as the cones and normal models of hypersurfaces considered below, all manifolds with corners in this article are assumed to be compact unless context makes it clear otherwise. Every point has a well defined codimension, given by the number of vanishing $\mathbb{R}_+$ factors in any chart, and the closure of a maximal (connected) set of points with a fixed codimension is a *boundary face*, with the set of boundary faces of codimension k denoted by $\mathcal{M}_k(X)$; in particular $\mathcal{M}_1(X)$ is the set of *boundary hypersurfaces*. We write $\mathcal{M}(X) = \bigcup_{k \geq 0} \mathcal{M}_k(X)$ for the set of all boundary faces. The *interior*, X° , of X is the set of points with codimension 0, and the *depth*

³ i.e., locally homeomorphic with transition maps given by diffeomorphisms.

of X is the maximum codimension occurring on X , or equivalently the maximum number of boundary hypersurfaces that have a non-empty mutual intersection. We assume boundary hypersurfaces (and therefore all boundary faces) are embedded; in particular each boundary face is again a manifold with corners. A function $\rho_G : X \rightarrow \mathbb{R}_+$ is *boundary defining* for $G \in \mathcal{M}_1(X)$ if $\rho_G^{-1}(0) = G$ and $d\rho_G \neq 0$ on G ; the ratio ρ_G/ρ'_G of two boundary defining functions for G is a strictly positive smooth function.

Note that we do not require that a manifold with corners is connected, nor do we strictly require that boundary faces are connected, despite how they have just been defined! Indeed, it will be useful below to allow certain unions of disjoint boundary hypersurfaces to be considered as a single hypersurface, or what is called in [5] a *collective boundary hypersurface*. This identification of collective boundary hypersurfaces (i.e., which components are to be considered to belong to ‘the same’ hypersurface) constitutes additional data on the manifold with corners, and in the event that such data has been specified, we abuse notation by using $\mathcal{M}_1(X)$ to refer to the set of collective boundary hypersurfaces; in the absence of such a specification $\mathcal{M}_1(X)$ consists by default of connected components only. By contrast, we adopt the convention that $X \in \mathcal{M}_0(X)$ is always considered as a single face even when X is not connected; thus $\mathcal{M}_0(X) = \{X\}$ is always a singleton.

The b-vector fields $\mathcal{V}_b(X) \subset \mathcal{V}(X) = C^\infty(X; TX)$ are by definition the Lie subalgebra of smooth vector fields tangent to all boundary faces. These form a locally free sheaf of constant rank, defining the b-tangent vector bundle

$${}^bTX \rightarrow X \quad \text{by} \quad \mathcal{V}_b(X) = C^\infty(X; {}^bTX),$$

spanned in local coordinates $(x_1, \dots, x_k, y_1, \dots, y_l) \in \mathbb{R}_+^k \times \mathbb{R}^l$ by $x_i \partial_{x_i}$ and ∂_{y_j} . Over $x_i = 0$, the b-tangent vector $x_i \partial_{x_i} = \rho_{G_i} \partial_{\rho_{G_i}}$ is actually independent of the choice of boundary defining coordinate x_i . The inclusion $\mathcal{V}_b(X) \subset \mathcal{V}(X)$ induces a natural bundle map ${}^bTX \rightarrow TX$ which is an isomorphism over the interior; at $p \in X$, the kernel defines the b-normal space ${}^bN_p E$, where E is the unique boundary face whose interior contains p . Taken together, these form a vector subbundle, bNE , of bTX over the interior of E which extends by continuity to the whole of E . In fact ${}^bNE \rightarrow E$ is trivialized globally by the frame $\{\rho_{G_i} \partial_{\rho_{G_i}}\}$, where ρ_{G_i} are any boundary defining functions for the hypersurfaces $G_1, \dots, G_k$ whose intersection has E as a connected component.

An *interior b-map* (hereafter simply a b-map) $f : X \rightarrow Y$ is a smooth map such that, for any sets $\{\rho_G : G \in \mathcal{M}_1(X)\}$ and $\{\rho_H : H \in \mathcal{M}_1(Y)\}$ of boundary defining functions for X and Y ,

$$f^*(\rho_H) = a_H \prod_{G \in \mathcal{M}_1(X)} \rho_G^{e(H,G)}, \quad \text{for every } H \in \mathcal{M}_1(Y) \quad (2.1)$$

where $a_H > 0$ is smooth and strictly positive and the *boundary exponents* $e(H, G) \in \mathbb{Z}_+$ are nonnegative integers. We will say that a b-map f is *rigid* with respect to fixed sets of boundary defining functions on X and Y if each a_H in (2.1) is identically 1; while this is not a standard notion, it turns out to be useful below.

Returning to standard concepts, a b-map f is said to be *simple* if each $e(H, G) \in \{0, 1\}$. Under an interior b-map f , every boundary face $E \in \mathcal{M}_k(X)$ is mapped into a unique smallest face $F \in \mathcal{M}_l(Y)$ (determined by the condition that $f(E^\circ) \subset F^\circ$), an assignment we denote by

$$f_\# : \mathcal{M}(X) \rightarrow \mathcal{M}(Y). \quad (2.2)$$

In general, the differential of a b-map extends by continuity from the interior to a bundle map

$${}^bdf_* : {}^bTX \rightarrow {}^bTY \quad (2.3)$$

over f , and this b -differential restricts over $E \in \mathcal{M}(X)$ to a bundle map

$${}^bdf_* : {}^bNE \rightarrow {}^bNF, \quad \text{where } F = f_\#(E) \in \mathcal{M}(Y). \quad (2.4)$$

Indeed, (2.4) is determined on the frame $\{\rho_G \partial_{\rho_G}\}$ by

$$\rho_G \partial_{\rho_G} \mapsto \sum_H e(H, G) \rho_H \partial_{\rho_H}, \quad (2.5)$$

where $e(H, G) \in \mathbb{Z}_+$ are the boundary exponents of f .

A map f is said to be a b -submersion (resp. b -immersion) if (2.3) is surjective (resp. injective) on fibers, and is said to be b -normal provided each map (2.4) is surjective on fibers; note that the latter condition holds if and only if it holds for all hypersurfaces $E \in \mathcal{M}_1(X)$. In particular a b -normal map cannot map a boundary hypersurface into any face of codimension 2 or more, so (2.2) restricts to a map

$$f_\# : \mathcal{M}_1(X) \rightarrow \mathcal{M}_{1,0}(Y) \quad \text{if } f \text{ is } b\text{-normal}$$

where $\mathcal{M}_{1,0}(Y) = \mathcal{M}_1(Y) \cup \mathcal{M}_0(Y) = \mathcal{M}_1(Y) \cup \{Y\}$ is the set of *principal faces*, notation and terminology which is frequently employed below. An equivalent condition to b -normality is that for each $G \in \mathcal{M}_1(X)$, there is at most one $H \in \mathcal{M}_1(Y)$ for which the boundary exponent $e(H, G)$ is nonzero.

A b -normal map can be *rigidified* in the sense that, for any set of boundary defining functions on Y , there is a set of boundary defining functions on X with respect to which f is rigid; indeed, starting with an arbitrary set $\{\rho_G : G \in \mathcal{M}_1(X)\}$, this is achieved by replacing ρ_G by $b_G \rho_G$, where $b_G = \prod_{H \in \mathcal{M}_1(Y)} (a_H^{-1})^{g(G,H)}$ and $g(\bullet, \bullet)$ forms a right inverse for $e(\bullet, \bullet)$ viewed as a $|f_\#(\mathcal{M}_1(X))| \times |\mathcal{M}_1(X)|$ matrix. In particular, a b -normal map can be put locally into the normal coordinate form:

$$f : (x, y) \mapsto (x^e, \bar{y}(x, y)) = (\bar{x}, \bar{y}) \quad (2.6)$$

where $(\bar{x}, \bar{y}) \in \mathbb{R}_+^{k'} \times \mathbb{R}^{l'}$ are coordinates centered at $f(p) \in Y$ and $(x, y) \in \mathbb{R}_+^k \times \mathbb{R}^l$ are coordinates centered at $p \in X$, and where $\bar{x} = x^e$ is shorthand for $\bar{x}_j = \prod_i x_i^{e(j,i)}$, $1 \leq j \leq k'$.

Remark We will occasionally employ the exponential notation $x^e = (\prod_{i=1}^k x_i^{e(1,i)}, \dots, \prod_{i=1}^k x_i^{e(n,i)})$ where $x = (x_1, \dots, x_k)$ and $e = [e(j, i)]$ is an $n \times k$ matrix. In this convention, $(x^e)^g = x^{ge}$, where ge is the usual matrix product of g and e .

A b -normal b -submersion is said to be a b -fibration, and such a map can always be written in the local normal form

$$f : (x, y) \mapsto (x^e, \text{pr}_{l'} y), \quad (2.7)$$

where $\text{pr}_{l'} : (y_1, \dots, y_l) \rightarrow (y_1, \dots, y_{l'})$ denotes the projection onto the first $l' \leq l$ coordinates. The restriction of a b -fibration to any boundary face $E \in \mathcal{M}(X)$ is again a b -fibration $f : E \rightarrow f_\#(E)$ onto its image in Y . A simple b -fibration with the property that each boundary hypersurface H of Y has a unique hypersurface $G \in f_\#^{-1}(H)$ in X which maps to it is in fact a fiber bundle of manifolds with corners; indeed in this case it follows from the local

normal form (2.7) (in which e is a projection matrix up to permutation) that f is a surjective submersion of compact manifolds with corners, and the standard argument for manifolds then shows that it is a fiber bundle.

A subset $P \subset X$ is a p -submanifold if it is covered by coordinate charts (x, y) in which it is locally defined by the vanishing of $\text{codim}(P)$ of the coordinates. Examples include boundary faces themselves.

The *blow-up* of a closed p -submanifold $P \subset X$ is the space

$$[X; P] = (X \setminus P) \cup S_+P$$

where the *front face* of the blow-up, S_+P , consists of the inward pointing spherical normal bundle of P , the fiber at $p \in P$ of which consists of normal vectors to P at p which have unit length with respect to any norm and point inward, with smooth structure generated by polar coordinates normal to P . This is equipped with a canonical b-map $\beta : [X; P] \rightarrow X$ (which is not generally b-normal), given by the bundle projection $S_+P \rightarrow P$ at the front face and the identity elsewhere. If $S \subset X$ is another (connected) set, the *lift* of S to $[X; P]$ is defined by $\beta^{-1}(S)$ if $S \subset P$, and the closure of $S \setminus P$ in $[X; P]$ otherwise, and provided S lifts to a p -submanifold in $[X; P]$ (abusively denoted by the same letter), the iterated blow-up

$$[X; P, S] := [[X; P]; S]$$

is well defined. In particular, every boundary face lifts to a p -submanifold under a blow-up, and the iterated blow-up of any number of boundary faces is always well defined.

Three additional topics from manifolds with corners will be used below. As these are either new or relatively unknown, we devote a subsection to each for ease of reference.

2.1 Normal models

For $G \in \mathcal{M}_1(X)$, we define $\nu : N_+G \rightarrow G$ to be the inward pointing normal (not b-normal!) bundle to G , which will be referred to as the *normal model* of G in X . That is to say, $NG = TX|_G/ TG$, which is a non-canonically trivial line bundle with well-defined inward pointing subspace N_+G , taken to include the zero section, with an action by $(0, \infty)$ fixing the zero section G_0 which we identify with G itself. It is convenient to extend this to the case $G = X \in \mathcal{M}_0(X)$ as well, in which case N_+X is the unique 0-dimensional bundle $N_+X \equiv X_0 \rightarrow X$, identified with X itself, with trivial $(0, \infty)$ action. If $f : X \rightarrow Y$ is a b-normal b-map, then the differential descends to a well-defined $(0, \infty)$ -equivariant map

$$df : N_+G \rightarrow N_+H, \quad H = f_\sharp(G) \in \mathcal{M}_{1,0}(Y) \quad (2.8)$$

where again $N_+Y = Y$ itself in case $f_\sharp(G) = Y$. In this latter case $df = \nu^*(f|_G)$ factors as the composition of the bundle projection $\nu : N_+G \rightarrow G$ with $f|_G : G \rightarrow Y$. This consideration applies in particular to smooth functions, with $f \in C^\infty(X; \mathbb{R})$ regarded as a b-map $f : X \rightarrow \mathbb{R}$ with $f_\sharp(G) = \mathbb{R} \in \mathcal{M}_0(\mathbb{R})$, lifting to $df = \nu^*f : N_+G \rightarrow \mathbb{R}$. In contrast, boundary defining functions on X , regarded as b-maps $\rho_{G'} : X \rightarrow [0, \infty)$, lift to N_+G as $d\rho_{G'} = \nu^*(\rho_{G'}|_G)$ for $G' \neq G$ and $d\rho_G : N_+G \rightarrow N_+\{0\} = [0, \infty)$ (with non-trivial $(0, \infty)$ action) for G itself. It follows from this observation that (2.8) is a b-normal b-map of non-compact manifolds with corners serving as an infinitesimal model for f itself near $G \subset X$, and we frequently use such models below.

2.2 Generalized blow-up

We will make use of some of the theory of *generalized blow-up* developed in [22], and we now recall the relevant details here, keeping the treatment as self-contained as possible. We begin with the observation that within each b-normal space ${}^bN_p F$, $F \in \mathcal{M}(X)$, there is a freely generated *monoid* (meaning a commutative semigroup with identity)

$${}^bM_p F := \mathbb{Z}_+ \langle \rho_1 \partial_{\rho_1}, \dots, \rho_k \partial_{\rho_k} \rangle,$$

which is well defined by the independence of the b-normal vectors $\rho_i \partial_{\rho_i}$ of the choices $\rho_i = \rho_{G_i}$ of boundary defining functions, and which is likewise independent of $p \in F$, so we just denote this monoid by bMF from now on and denote the generator associated with the boundary hypersurface $G_i = \{\rho_i = 0\}$ simply by

$$g_i := \rho_i \partial_{\rho_i}.$$

Whenever $F \subset E$, the normal monoid bME (which is generated by fewer of the same g_i) includes as a *face* of bMF , which, up to reordering, is just the inclusion

$${}^bME \cong \mathbb{Z}_+ \langle g_1, \dots, g_l, 0, \dots, 0 \rangle \subset \mathbb{Z}_+ \langle g_1, \dots, g_l, g_{l+1}, \dots, g_k \rangle \cong {}^bMF. \quad (2.9)$$

Since the coefficients of (2.4) are nonnegative integers, the b-differential of a b-map $f : X \rightarrow Y$ induces well-defined monoid homomorphisms⁴

$${}^bdf_* : {}^bMF \rightarrow {}^bMf_*(F), \quad \text{for each } E \in \mathcal{M}(X), \quad (2.10)$$

which are consistent with the inclusions (2.9). Note that if f is both simple and b-normal, then in (2.10) each generator of bMF maps to a generator of ${}^bMf_*(F)$, and conversely: The condition that generators map to generators in (2.10) implies that f is simple and b-normal.

In the particular case that $f = \beta : \tilde{X} = [X; E_1, \dots, E_n] \rightarrow X$ is the blow-down map of an iterated blow-up of boundary faces $\{E_i\} \subset \mathcal{M}(X)$ of X , the images under (2.10) of the monoids from $\tilde{X}$ form a complete decomposition of each monoid bMF of the target, in the sense that bMF is a union of such image monoids ${}^bdf_*({}^bM\tilde{F})$, $\tilde{F} \in \mathcal{M}(\tilde{X})$, meeting along boundary faces. Such decompositions constitute what is called a *refinement* in [20, 22].

For example, if $E = G_1 \cap \dots \cap G_l$, with ${}^bME = \mathbb{Z}_+ \langle g_1, \dots, g_l \rangle$, then for any $F \subseteq E$, the refinement of bMF induced by the blow-up $\beta : [X; E] \rightarrow X$ consists of the l submonoids of bMF in which precisely one of the generators $\{g_1, \dots, g_l\}$ is replaced by $g_1 + \dots + g_l$; this corresponds to the geometric process of ‘barycentric subdivision’ of bME , considered as a face of bMF . Examples are illustrated in Fig. 1.

As summarized below, the monoids of the refinement actually characterize the iterated blow-up $[X; E_1, \dots, E_n]$ completely up to diffeomorphism, and an arbitrary refinement determines what is called a *generalized blow-up*,⁵ meaning a manifold with corners $\tilde{X}$ and a suitable blow-down map $\beta : \tilde{X} \rightarrow X$, the images of the induced monoid homomorphisms (2.10) of which constitute the given refinement. In particular, the boundary hypersurfaces of $\tilde{X}$ and their incidence relations (meaning their intersection properties) can be simply read off from the monoids of the refinement: Hypersurfaces are in bijection with generators (i.e.,

⁴ Meaning maps intertwining addition which send 0 to 0.

⁵ While the refinements considered here arise from ordinary iterated boundary blow-up, the general theory in [22] accounts for inhomogeneous blow-ups, as well as refinements which do not arise from any classical iterated boundary blow-up, hence the term ‘generalized blow-up.’ The paper [20] extends the notion of refinement and generalized blow-up to the category [18] of manifolds with generalized corners.

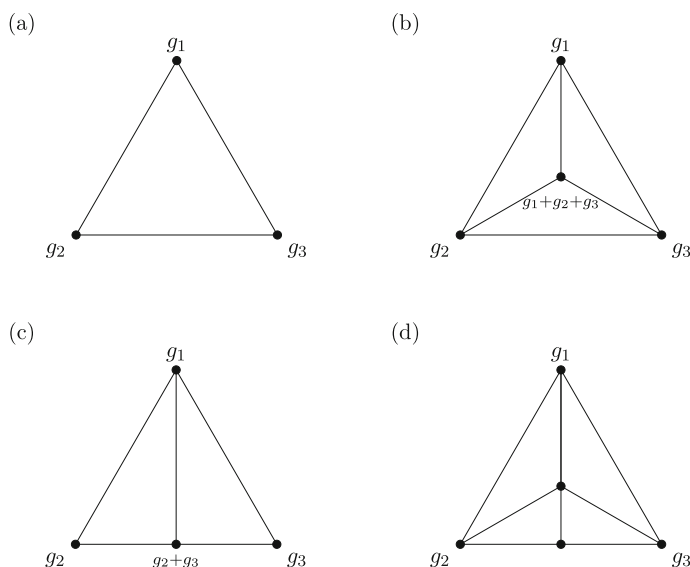


Fig. 1 A schematic depiction of **a** the monoid $\mathbb{Z}_+ \langle g_1, g_2, g_3 \rangle$ associated with a boundary face $E_{123} = G_1 \cap G_2 \cap G_3$, **b** the refinement associated with the blow-up $[X; E_{123}]$, **c** the refinement associated with the blow-up $[X; E_{23}]$, and **d** the refinement associated with their common resolution, the iterated blow-up $[X; E_{123}, E_{23}] \cong [X; E_{23}, E_{123}]$

one-dimensional monoids) and intersect if the associated generators together generate a two-dimensional monoid in the refinement. The main results we use below are the following, which assert the characterization of an iterated boundary blow-up by its monoid refinement, as well as a criterion for lifting b-maps under such a blow-up.

Theorem 2.1 ([22, Theorem 6.3], also [20, Theorem 3.7]) *Let $[Y; F_1, \dots, F_n]$ be an iterated blow-up of boundary faces $F_1, \dots, F_n \in \mathcal{M}(Y)$ of a manifold with corners Y , with blow down map $\beta : [Y; F_1, \dots, F_n] \rightarrow Y$, and let $\mathcal{R}$ denote the associated collection of monoid refinements (given by ${}^b d\beta_*({}^b M\tilde{F}) \subset {}^b M F$ for $\tilde{F} \in \mathcal{M}([Y; F_1, \dots, F_n])$ and $F = \beta_*(\tilde{F}) \in \mathcal{M}(Y)$). Then,*

- (a) $\mathcal{R}$ determines a manifold with corners $\tilde{Y}$ and along with an interior b-map $\beta' : \tilde{Y} \rightarrow Y$, restricting to a diffeomorphism $\beta' : \tilde{Y}^\circ \cong Y^\circ$ of interiors, such that
- (b) if $f : X \rightarrow Y$ is an interior b-map with the property that every monoid homomorphism ${}^b d f_* : {}^b M E \rightarrow {}^b M f_*(E)$, $E \in \mathcal{M}(X)$, factors through a monoid in $\mathcal{R}$, then f lifts to a unique interior b-map $\tilde{f} : X \rightarrow \tilde{Y}$ such that $f = \beta' \circ \tilde{f}$:

$$\begin{array}{ccc} & \tilde{Y} & \\ \tilde{f} \nearrow & & \downarrow \beta \\ X & \xrightarrow{f} & Y \end{array}$$

- (c) In particular, the blow-down map $\beta : [Y; F_1, \dots, F_n] \rightarrow Y$ factors through a unique diffeomorphism $\tilde{\beta} : [Y; F_1, \dots, F_n] \cong \tilde{Y}$ identifying $\tilde{Y}$ with the iterated blow-up itself, and if $[Y; F_{\sigma(1)}, \dots, F_{\sigma(n)}]$ is an iterated blow-up of the same boundary faces in a different order (i.e., for a permutation σ of $\{1, \dots, n\}$), then as long as the two associated refinements of the monoids of Y agree, there is a canonical diffeomorphism $[Y; F_1, \dots, F_n] \cong [Y; F_{\sigma(1)}, \dots, F_{\sigma(n)}]$ extending the identity in the interior.

Though we refer to [22] for the full proof of these results, we give a sketch here for the convenience of the unfamiliar reader. First, the space $\tilde{Y}$ may be constructed directly from the monoid refinement of Y as follows. Locally, if $(x, y) \in \mathbb{R}_+^l \times \mathbb{R}^{n-l}$ are coordinates on Y centered at an interior point of a face $F \in \mathcal{M}_l(Y)$, then for each monoid of dimension l in the refinement of bMF , let $v = [v(i, j)] \in \text{Mat}(l \times l, \mathbb{Z}_+)$ denote the matrix of the inclusion map with respect to the generators (which is unimodular and in particular invertible) and construct a coordinate patch $(t, y) \in \mathbb{R}_+^l \times \mathbb{R}^{n-l}$ for $\tilde{Y}$ with local blow-down map

$$\beta : (t, y) \mapsto (x, y) = (t^v, y). \quad (2.11)$$

In particular $t = x^{v^{-1}}$ gives ‘blow-up coordinates’ on $\tilde{Y}$ as rational combinations of the x . For adjacent monoids in the refinement, associated with matrices v and v' , respectively, the transition maps

$$(t', y) \mapsto (t, y) = (t^{v^{-1}v'}, y)$$

are seen to be diffeomorphisms on open subsets where some of the variables (associated with those generators which are not common to both monoids) are strictly positive, and glue together naturally with respect to coordinate transitions on Y itself to determine $\tilde{Y}$ and the blow down map $\beta : \tilde{Y} \rightarrow Y$ whose boundary exponents are the $v(i, j)$.

For the lifting result, suppose f is given locally by

$$f : (x', y') \rightarrow (x, y) = (ax'^\mu, y)$$

where $y = y(x', y')$, and $ax'^\mu = (a_1(x', y') \prod_j x_j^{\mu(1,j)}, \dots, a_l(x', y') \prod_j x_j^{\mu(l,j)})$ with $a_i > 0$, so that the matrix $\mu = [\mu(i, j)] \in \text{Mat}(l \times k, \mathbb{Z}_+)$ of boundary exponents represents the associated monoid homomorphism ${}^bdf_* : {}^bME \rightarrow {}^bMF$ (with $E \in \mathcal{M}_k(X)$ given locally by $\{x' = 0\}$). The condition that the latter factors through some monoid in the refinement means that $\mu = v\tilde{\mu}$ for some v as above, and then,

$$\tilde{f} : (x', y') \rightarrow (t, y) = (a^{v^{-1}}x'^{\tilde{\mu}}, y) \quad (2.12)$$

gives the local coordinate expression for the unique lift $\tilde{f} : X \rightarrow \tilde{Y}$; it is straightforward to check that this is functorial and in particular behaves well with respect to changes of coordinates.

2.3 Rational combinations of boundary defining functions

A *rational combination* of boundary defining functions $\{\rho_i\}$ on a space X associated with boundary hypersurfaces $G_1, \dots, G_n$ is defined to be a product of the form

$$\sigma = \prod_j \rho_j^{a_j}, \quad a_j \in \mathbb{Z}. \quad (2.13)$$

It will be of interest to know where in X such a combination determines a smooth function locally, and more importantly, when σ lifts globally to a smooth function under a blow-up $\beta : \tilde{X} \rightarrow X$. These properties may be determined by considering the logarithmic differential

$${}^b d\sigma = d \log \sigma = \sum_j a_j \frac{d\rho_j}{\rho_j}$$

of σ , which induces a $\mathbb{Z}$ -linear map

$${}^b d\sigma : {}^bMF \rightarrow \mathbb{Z}$$

on the monoid of each boundary face $F \in \mathcal{M}(X)$, determined on generators by ${}^b\mathrm{d}\sigma(g_i) = a_i$.

Since F is covered by coordinate charts with boundary defining coordinates x_i associated with the generators $g_i \in {}^bMF$ in which σ has the local coordinate expression $\sigma(x, y) = b(x, y) \prod_i x_i^{a_i}$ where $b > 0$, it follows that σ determines a smooth function $\sigma : U \supset F^\circ \rightarrow [0, \infty)$ for some neighborhood U of F° if and only if each $a_i = {}^b\mathrm{d}\sigma(g_i) \geq 0$, or equivalently if ${}^b\mathrm{d}\sigma({}^bMF) \subset \mathbb{Z}_+$. Likewise, σ^{-1} determines a smooth function $\sigma^{-1} : U \subset F^\circ \rightarrow [0, \infty)$ if and only if ${}^b\mathrm{d}\sigma({}^bMF) \subset \mathbb{Z}_-$. Moreover, σ (resp. σ^{-1}) vanishes on a hypersurface $G_j \cap U$ if and only if ${}^b\mathrm{d}\sigma(g_j) > 0$ (resp. ${}^b\mathrm{d}\sigma(g_j) < 0$), since this corresponds to σ having a local expression involving a term $x_j^{a_j}$ with $a_j > 0$ (resp. $a_j < 0$). Note that in either case (in which ${}^b\mathrm{d}\sigma$ has a constant sign on bMF), σ can be regarded as a b -map $\sigma : U \rightarrow [0, \infty]$, with $[0, \infty]$ considered as a one-dimensional compact manifold with two boundary hypersurfaces $\{0\}$ and $\{\infty\}$.

Now if $\beta : \tilde{X} = [X; F_1, \dots, F_n] \rightarrow X$ is an iterated boundary blow-up, the behavior of the pullback $\beta^*\sigma$ of σ to $\tilde{X}$ can be analyzed using the fact that ${}^b\mathrm{d}(\beta^*\sigma)(m) = {}^b\mathrm{d}\sigma(\beta_*m)$ for $m \in {}^bM\tilde{F}$, $\tilde{F} \in \mathcal{M}(\tilde{X})$, from which we obtain the following result.

Proposition 2.2 *A rational combination σ lifts to a smooth b -map $\beta^*\sigma : \tilde{X} = [X; F_1, \dots, F_n] \rightarrow [0, \infty]$ if and only if ${}^b\mathrm{d}\sigma$ takes a constant sign on each monoid in the refinement associated with $\tilde{X}$, and in this case $\beta^*\sigma$ (resp. $\beta^*\sigma^{-1}$) vanishes on $\tilde{G} \in \mathcal{M}(\tilde{X})$ provided ${}^b\mathrm{d}\sigma(\tilde{g}) > 0$ (resp. ${}^b\mathrm{d}\sigma(\tilde{g}) < 0$), where $\tilde{g}$ is the generator in the refinement associated with the hypersurface $\tilde{G} \in \mathcal{M}_1(\tilde{X})$.*

Note that boundary defining functions on a blow-up $\tilde{Y}$ of Y are given locally by rational combinations of boundary defining functions from Y , as follows from writing $t = x^{v^{-1}}$ in (2.11). Moreover, if a map $f : X \rightarrow Y$ is locally rigid and satisfies the hypotheses of Theorem 2.1, then its lift $f : X \rightarrow \tilde{Y}$ is rigid with respect to the rational boundary defining functions on $\tilde{Y}$ as follows from (2.12).

It is evident from Proposition 2.2 that a set of the form $\{\sigma = 1\} \subset X$ (or more generally $\{\sigma = c\}$ for $c \in (0, \infty)$) lifts to an interior p -submanifold of a blow-up $\tilde{X}$ whenever σ lifts to be smooth on $\tilde{X}$, and by iteration, a set of the form $\{\sigma_1 = \dots = \sigma_r = 1\}$ lifts to an interior p -submanifold of $\tilde{X}$ if each of the σ_i lifts to be smooth; equivalently each ${}^b\mathrm{d}\sigma_i$ has a constant sign on each monoid associated with the blow-up $\tilde{X}$.

On the other hand, $\{\sigma_1 = \dots = \sigma_r = 1\}$ may lift to a p -submanifold of $\tilde{X}$ even when none of the individual rational combinations σ_i lift to be smooth, as in the following result (a specialization of [22, Prop. 10.3]) that will be used below. Note that the property that ${}^b\mathrm{d}\sigma$ has a constant sign on a monoid $M = \mathbb{Z}_+ \langle g_1, \dots, g_l \rangle$ is equivalent to the property that the intersection of M with the subspace $\ker {}^b\mathrm{d}\sigma$ is a face (possibly $\{0\}$) of M , meaning a submonoid generated by a (possibly empty) subset of $\{g_1, \dots, g_l\}$. Indeed, this is the case if and only if the remaining generators of M all lie entirely within one or the other of the half-spaces determined by ${}^b\mathrm{d}\sigma > 0$ or ${}^b\mathrm{d}\sigma < 0$.

Proposition 2.3 ([22, Prop. 10.3]) *Let σ_i be rational combinations of boundary defining functions on X and $c_i \in (0, \infty)$ for $1 \leq i \leq r$. Then, the set $\{\sigma_1 = c_1, \dots, \sigma_r = c_r\} \subset X$ lifts to an interior p -submanifold of the iterated blow-up $\tilde{X} = [X; F_1, \dots, F_n]$ provided that each monoid M in the refinement associated with $\tilde{X}$ intersects the subspace $\bigcap_{i=1}^r \ker {}^b\mathrm{d}\sigma_i$ along a face of M .*

Proof The key observation is that the equations $\{\log \sigma_i = \log c_i : 1 \leq i \leq r\}$ may be replaced by equivalent $\mathbb{Z}$ -linear combinations. Thus for any given monoid M in the refinement associated with $\tilde{X}$, the hypothesis that M meets $\bigcap_{i=1}^r \ker {}^b\mathrm{d}\sigma_i$ along a face of M means that the

$\log \sigma_i$ may be replaced by $\mathbb{Z}$ -linear combinations $\log \tilde{\sigma}_i = \sum_j b_{ij} \log \sigma_j$ with the property that $\text{bd} \tilde{\sigma}_i = \sum_j b_{ij} \text{bd} \sigma_j$ has constant sign on M , and then, it follows from Proposition 2.2 that the set lifts to a p -submanifold in any local coordinates on $\tilde{X}$ associated with M . $\square$

As an example, in Fig. 1, neither of the rational combinations ρ_1/ρ_2 or ρ_1/ρ_3 lifts to be smooth in any of the depicted blow-ups, yet the set $\{\rho_1/\rho_2 = \rho_1/\rho_3 = 1\}$ lifts to a p -submanifold in the blow-ups depicted in (b) and (d), since $\ker \text{bd}(\rho_1/\rho_2) \cap \ker \text{bd}(\rho_1/\rho_3)$ is the span of $g_1 + g_2 + g_3$.

3 Ordered corners

Definition 3.1 The category of *manifolds with ordered corners* is as follows:

- An *object* is a manifold with corners X equipped with a partial order on the set

$$\mathcal{M}_{1,0}(X) := \mathcal{M}_1(X) \cup \mathcal{M}_0(X) = \mathcal{M}_1(X) \cup \{X\}$$

of principal faces, with the property that every pair of faces in $\mathcal{M}_{1,0}(X)$ which meet are comparable with respect to the order; equivalently, incomparable faces must be disjoint. In particular every element of $\mathcal{M}_{1,0}(X)$ is comparable to $X \in \mathcal{M}_0(X)$. Recall that we use the notation

$$G \sim G' \quad \text{if } G = G', G < G', \text{ or } G' < G$$

to denote comparable elements.

- A *morphism* is a simple, b -normal, interior b -map $f : X \rightarrow Y$ which is *ordered*, meaning that the induced map

$$f_{\sharp} : \mathcal{M}_{1,0}(X) \rightarrow \mathcal{M}_{1,0}(Y) \quad (3.1)$$

is order preserving.

Remark • As noted in Sect. 2, we allow for the possibility that $\mathcal{M}_1(X)$ consists of collective boundary hypersurfaces, that is, unions of disjoint components which are identified as a single element of $\mathcal{M}_1(X)$. This always applies to $X \in \mathcal{M}_0(X)$ if X is disconnected, recalling that by convention we always require $\mathcal{M}_0(X) = \{X\}$ to be a singleton.

- We do not require that a pair of faces be comparable *if and only if* they are non-disjoint; it is sometimes convenient to consider orders in which possibly disjoint faces are deemed comparable, such as a total order; nevertheless, for any admissible order on $\mathcal{M}_{1,0}(X)$ there is a minimal suborder with the property that elements are comparable if and only if they are non-disjoint.

We will often be interested in the case that $X \in \mathcal{M}_0(X)$ is either maximal or minimal, in which case we say X is respectively *interior maximal* or *interior minimal*. Given an order on $\mathcal{M}_1(X)$ satisfying the requirements of Definition 3.1, we denote by $X_{\max}$ or $X_{\min}$ the manifold with ordered corners in which the interior is made maximal or minimal, respectively. We introduce the notation

$$\mathcal{M}_1^<(X) = \{G \in \mathcal{M}_1(X) : G < X\}, \quad \mathcal{M}_1^>(X) = \{G \in \mathcal{M}_1(X) : G > X\},$$

with respect to which $\mathcal{M}_{1,0}(X)$ decomposes as a disjoint union

$$\mathcal{M}_{1,0}(X) = \mathcal{M}_1^<(X) \cup \{X\} \cup \mathcal{M}_1^>(X).$$

Note that $X \in \mathcal{M}_0(X)$ forms a base point for the ordered set $\mathcal{M}_{1,0}(X)$, in that it is comparable to every other element and is preserved by the map $f_{\sharp} : \mathcal{M}_{1,0}(X) \rightarrow \mathcal{M}_{1,0}(Y)$ associated with a morphism $f : X \rightarrow Y$ (as f is interior); thus, the association $X \mapsto \mathcal{M}_{1,0}(X)$ is a functor from the category of manifolds with ordered corners to the category of *pointed ordered sets*. Note also that $f_{\sharp}(\mathcal{M}_1^<(X)) \subset \mathcal{M}_1^<(Y) \cup \{Y\}$ and $f_{\sharp}(\mathcal{M}_1^>(X)) \subset \mathcal{M}_1^>(Y) \cup \{Y\}$, with notation as above.

3.1 Products

Products of manifolds with ordered corners can be motivated by considering the functor $X \mapsto \mathcal{M}_{1,0}(X)$. The product of $\mathcal{M}_{1,0}(X)$ and $\mathcal{M}_{1,0}(Y)$ as ordered sets is the Cartesian product $\mathcal{M}_{1,0}(X) \times \mathcal{M}_{1,0}(Y)$ equipped with the product order, depicted schematically as follows:

$$\begin{array}{ccccc} \mathcal{M}_1^<(X) \times \mathcal{M}_1^>(Y) & \rightarrow & X \times \mathcal{M}_1^>(Y) & \rightarrow & \mathcal{M}_1^>(X) \times \mathcal{M}_1^>(Y) \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{M}_1^<(X) \times Y & \longrightarrow & X \times Y & \longrightarrow & \mathcal{M}_1^>(X) \times Y \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{M}_1^<(X) \times \mathcal{M}_1^<(Y) & \rightarrow & X \times \mathcal{M}_1^<(Y) & \rightarrow & \mathcal{M}_1^>(X) \times \mathcal{M}_1^<(Y) \end{array} \quad (3.2)$$

Notation Convention 3.2 Here and below, we employ the following diagrammatic notation conventions for products of ordered sets: An arrow of the form $A \times B \rightarrow A \times B'$ between products with a single factor in common (and $B \cap B' = \emptyset$) means that $(a, b) < (a, b')$ for each $a \in A, b \in B$, and $b' \in B'$. On the other hand, an arrow of the form $A \times B' \rightarrow A' \times B$ with no common factor on both sides (and $A \cap A' = B \cap B' = \emptyset$) means that $(a, b') < (a', b)$ for every $a \in A, a' \in A', b \in B$, and $b' \in B'$.

However, the product in the category of *pointed ordered sets* is the subset of pairs which are comparable to the base point $(X, Y) \in \mathcal{M}_0(X) \times \mathcal{M}_0(Y)$:

$$\begin{array}{ccccc} & & X \times \mathcal{M}_1^>(Y) & \rightarrow & \mathcal{M}_1^>(X) \times \mathcal{M}_1^>(Y) \\ & & \uparrow & & \uparrow \\ \mathcal{M}_1^<(X) \times Y & \longrightarrow & X \times Y & \longrightarrow & \mathcal{M}_1^>(X) \times Y \\ \uparrow & & \uparrow & & \\ \mathcal{M}_1^<(X) \times \mathcal{M}_1^<(Y) & \rightarrow & X \times \mathcal{M}_1^<(Y) & \rightarrow & \end{array} \quad (3.3)$$

It is easy to see that the Cartesian product $X \times Y$ cannot be a product in the ordered corners category in general; its set $\mathcal{M}_{1,0}(X \times Y)$ of boundary hypersurfaces may be identified with the subset of $\mathcal{M}_{1,0}(X) \times \mathcal{M}_{1,0}(Y)$ depicted by

$$\begin{array}{ccccc} & & X \times \mathcal{M}_1^>(Y) & & \\ & & \uparrow & & \\ \mathcal{M}_1^<(X) \times Y & \longrightarrow & X \times Y & \longrightarrow & \mathcal{M}_1^>(X) \times Y \\ & & \uparrow & & \\ & & X \times \mathcal{M}_1^<(Y) & & \end{array}$$

and hypersurfaces in $\mathcal{M}_1^>(X) \times Y$ and $X \times \mathcal{M}_1^>(Y)$ meet yet are incomparable in the product order (and similarly for faces in $\mathcal{M}_1^<(X) \times Y$ and $X \times \mathcal{M}_1^<(Y)$). There is therefore no natural way to put an ordered corners structure on $X \times Y$ such that both projection maps $X \times Y \rightarrow X$

and $X \times Y \rightarrow Y$ are ordered. On the other hand, faces in $\mathcal{M}_1^<(X) \times Y$ (resp. $\mathcal{M}_1^>(X) \times Y$) are comparable to those in $X \times \mathcal{M}_1^>(Y)$ (resp. $X \times \mathcal{M}_1^<(Y)$), so in the case that one of X or Y is interior minimal, while the other is interior maximal, $X \times Y$ is naturally a manifold with ordered corners when equipped with the product order, and as we will see below, it satisfies the universal property of the product in this case.

Remark It is also possible to equip $X \times Y$ with one of the two *lexicographic* orders rather than the product order, which gives it an ordered structure with respect to which one of the projections (but generally not the other) is ordered; we will not make use of this.

Definition 3.3 The *ordered product* $X \tilde{\times} Y$ is the iterated blow-up

$$\begin{aligned} X \tilde{\times} Y &= [X \times Y; \mathcal{M}_1^<(X) \times \mathcal{M}_1^<(Y), \tilde{\mathcal{M}}_1^>(X) \times \tilde{\mathcal{M}}_1^>(Y)] \\ &\cong [X \times Y; \tilde{\mathcal{M}}_1^>(X) \times \tilde{\mathcal{M}}_1^>(Y), \mathcal{M}_1^<(X) \times \mathcal{M}_1^<(Y)] \end{aligned} \quad (3.4)$$

where the blow-up is performed in any order consistent with the partial order on the products, and where here $\tilde{\mathcal{M}}_1^>(\bullet)$ denotes the ordered set $\mathcal{M}_1^>(\bullet)$ with the opposite order.

Theorem 3.4 The space $X \tilde{\times} Y$ in (3.4) is a well-defined manifold with ordered corners, with $\mathcal{M}_{1,0}(X \tilde{\times} Y)$ isomorphic to the ordered set $\{(G, H) \in \mathcal{M}_{1,0}(X) \times \mathcal{M}_{1,0}(Y) : (G, H) \sim (X, Y)\}$ as depicted in (3.3).

It is the product of X and Y in the category of manifolds with ordered corners. More precisely, the projections of $X \times Y$ onto X and Y lift to morphisms (in fact ordered b-fibrations) $X \tilde{\times} Y \rightarrow X$ and $X \tilde{\times} Y \rightarrow Y$, and $X \tilde{\times} Y$ satisfies the universal property that if W is a manifold with ordered corners and $f : W \rightarrow X$ and $g : W \rightarrow Y$ are morphisms, then there exists a unique morphism $W \rightarrow X \tilde{\times} Y$ making the following diagram commute:

$$\begin{array}{ccccc} & & W & & \\ & f \swarrow & \downarrow \exists! & \searrow g & \\ X & \longleftarrow & X \tilde{\times} Y & \longrightarrow & Y \end{array}$$

Corollary 3.5 If $X = X_{\min}$ and $Y = Y_{\max}$ are interior minimal and maximal, respectively, then the Cartesian product $X \times Y \equiv X \tilde{\times} Y$ is already the product of X and Y in the category of manifolds with ordered corners.

Proof of Theorem 3.4 While it is possible to prove these results by local coordinate computations, the sheer number of coordinate charts on $X \tilde{\times} Y$ makes this tedious. Instead, we employ the generalized blow-up machinery discussed in Sect. 2.2 to identify the monoid refinement of $X \tilde{\times} Y$ as a generalized blow-up of $X \times Y$, from which its ordered corners structure and universal property are easily derived. We will see that the refinement consists of monoids which are freely generated by sums of generators $g + h$ forming maximally ordered chains in $\mathcal{M}_{1,0}(X) \times \mathcal{M}_{1,0}(Y)$, under the identification of hypersurfaces $G \in \mathcal{M}_1(X)$ and $H \in \mathcal{M}_1(Y)$ with their associated monoid generators $g \in {}^bMG$ and $h \in {}^bMH$, and with X and Y identified with 0.

Thus, consider an arbitrary corner in $X \times Y$. By relabeling if necessary, we may assume this has the form

$$(G_{-m'} \cap \cdots \cap G_{-1} \cap G_1 \cap \cdots \cap G_m) \times (H_{-n'} \cap \cdots \cap H_{-1} \cap H_1 \cap \cdots \cap H_n)$$

for totally ordered chains $G_{-m'} < \cdots < G_{-1} < X < G_1 < \cdots < G_m$ and $H_{-n'} < \cdots < H_{-1} < Y < H_1 < \cdots < H_n$ of hypersurfaces of X and Y , respectively, which, since we

work locally near the corner, we may assume constitute all of the hypersurfaces of X and Y . The monoid associated with this corner of $X \times Y$ has the form

$$\begin{aligned} \mathbb{Z}_+ \langle g_{-m'}, \dots, g_{-1}, g_1, \dots, g_m \rangle \times \mathbb{Z}_+ \langle h_{-n'}, \dots, h_{-1}, h_1, \dots, h_n \rangle \\ = \mathbb{Z}_+ \langle g_{-m'}, \dots, g_{-1}, g_1, \dots, g_m, h_{-n'}, \dots, h_{-1}, h_1, \dots, h_n \rangle \end{aligned} \quad (3.5)$$

where we denote the generators associated with the hypersurfaces by lower case letters. Identifying $g_0 := 0 \in \mathbb{Z}_+ \langle g_{-m'}, \dots, g_m \rangle$ and $h_0 := 0 \in \mathbb{Z}_+ \langle h_{-n'}, \dots, h_n \rangle$ with X and Y , respectively, determines an identification between the ordered set $\mathcal{M}_{1,0}(X) \times \mathcal{M}_{1,0}(Y)$ and the elements $\{g_i + h_j : -m \leq i \leq m, -n' \leq j \leq n\} \subset \mathbb{Z}_+ \langle g_{-m'}, \dots, g_m, h_{-n'}, \dots, h_n \rangle$ and therefore determines a partial order on the latter set. In other words, we regard g_i and h_j as $g_i + 0 = g_i + h_0$ and $0 + h_j = g_0 + h_j$, respectively and equip the sums (where we fix the order of addition) with the product order induced by

$$\begin{aligned} g_{-m'} < \dots < g_{-1} < 0 = g_0 < g_1 < \dots < g_m, \quad \text{and} \\ h_{-n'} < \dots < h_{-1} < 0 = h_0 < h_1 < \dots < h_n, \end{aligned} \quad (3.6)$$

Assuming temporarily that X and Y are minimal in $\mathcal{M}_{1,0}(X)$ and $\mathcal{M}_{1,0}(Y)$, so $m' = n' = 0$, consider the iterated blow-up $[X \times Y; \overleftarrow{\mathcal{M}}_1^>(X) \times \overleftarrow{\mathcal{M}}_1^>(Y)]$. The first blow-up of $G_m \times H_n$ subdivides (3.5) into the pair of monoids

$$\mathbb{Z}_+ \langle g_1, \dots, g_m + h_n, h_1, h_2, \dots, h_n \rangle \quad \text{and} \quad \mathbb{Z}_+ \langle g_1, \dots, g_m, h_1, h_2, \dots, g_m + h_n \rangle.$$

In particular, the incomparable pair $\{g_m, h_n\} = \{g_m + 0, 0 + h_n\}$ is replaced by either of the comparable pairs $\{g_m + h_n, g_m + 0\}$ or $\{g_m + h_n, 0 + h_n\}$, and in the resulting monoids, the generator $g_m + h_n$ is comparable to all other generators $g_i = g_i + 0$, $i < m$ and $h_j = 0 + h_j$, $j < n$. Proceeding through the rest of the iterated blow-up of $\overleftarrow{\mathcal{M}}_1^>(X) \times \overleftarrow{\mathcal{M}}_1^>(Y)$ (in any order consistent with the product order) has the effect of subsequently subdividing each remaining monoid with incomparable generators into two new monoids by replacing the highest indexed incomparable pair $\{g_i, h_j\}$ with one of the comparable pairs $\{g_i + h_j, g_i + 0\}$ or $\{g_i + h_j, 0 + h_j\}$, at the end of which process (3.5) is replaced by the collection of monoids generated by totally ordered chains among

$$\{g_i + h_j : 0 \leq i \leq m, 0 \leq j \leq n\}.$$

Note that had we instead blown up in the opposite order $\mathcal{M}_1^>(X) \times \mathcal{M}_1^>(Y)$, we would have started with the blow-up of $G_1 \times H_1$, in which $\{g_1 + 0, 0 + h_1\}$ is replaced by either $\{g_1 + h_1, g_1 + 0\}$ or $\{g_1 + h_1, 0 + h_1\}$; however, the sum $g_1 + h_1$ is then incomparable to the remaining generators $g_i + 0$, $i > 0$ and $0 + h_j$, $j > 0$ in the resulting monoids, an incomparability which is not resolved by any subsequent blow-up. In particular, the iterated blow-up $[X \times Y; \mathcal{M}_1^>(X) \times \mathcal{M}_1^>(Y)]$ (or indeed the iterated blow-up in any other order besides $\overleftarrow{\mathcal{M}}_1^>(X) \times \overleftarrow{\mathcal{M}}_1^>(Y)$) fails to obtain an ordered corners structure consistent with (3.3).

Likewise, assume temporarily that X and Y are maximal in $\mathcal{M}_{1,0}(X)$ and $\mathcal{M}_{1,0}(Y)$, so $m = n = 0$, and consider the iterated blow-up $[X \times Y, \mathcal{M}_1^<(X) \times \mathcal{M}_1^<(Y)]$. Now the first blow-up of $G_{-m'} \times H_{-n'}$ subdivides (3.5) into the pair of monoids

$$\begin{aligned} \mathbb{Z}_+ \langle g_{-m'} + h_{-n'}, g_{-m'+1}, \dots, g_{-1}, h_{-n'}, h_{-n'+1}, \dots, h_{-1} \rangle \\ \text{and} \quad \mathbb{Z}_+ \langle g_{-m'}, g_{-m'+1}, \dots, g_{-1}, g_{-m'} + h_{-n'}, h_{-n'+1}, \dots, h_{-1} \rangle, \end{aligned}$$

in which the incomparable pair $\{g_{-m'} + 0, 0 + h_{-n'}\}$ is replaced by either of the comparable pairs $\{g_{-m'} + h_{-n'}, g_{-m'} + 0\}$ or $\{g_{-m'} + h_{-n'}, 0 + h_{-n'}\}$, with the result that the sum $g_{-m'} + h_{-n'}$ is comparable to all other generators $g_i = g_i + 0$, $i > -m'$ and $h_j = 0 + h_j$, $j > -n'$.

$-n'$. Proceeding through the rest of the iterated blow-up of $\mathcal{M}_1^<(X) \times \mathcal{M}_1^<(Y)$ has the effect of iteratively subdividing each remaining monoid with incomparable generators g_i and h_j , $-m' \leq i \leq -1$ and $-n' \leq j \leq -1$ into two new monoids, by replacing its lowest indexed incomparable pair $\{g_i, h_j\}$ with one of the comparable pairs $\{g_i + h_j, g_i + 0\}$ or $\{g_i + h_j, 0 + h_j\}$. At the end of this process (3.5) is replaced by the collection of monoids generated by maximal totally ordered chains among the generators

$$\{g_i + h_j : -m' \leq i \leq 0, -n' \leq j \leq 0\}. \quad (3.7)$$

Again, had we begun instead by blowing up $G_{-1} \times H_{-1}$, then generator pairs $\{g_{-1} + 0, 0 + h_{-1}\}$ would be replaced by either $\{g_{-1} + h_{-1}, g_{-1} + 0\}$ or $\{g_{-1} + h_{-1}, 0 + h_{-1}\}$, leaving $g_{-1} + h_{-1}$ incomparable to generators $g_i + 0 = g_i + h_0$, $i < -1$ and $0 + h_j$, $j < -1$.

In the general case, since the monoids generated by totally ordered chains decompose as products of factors with generators indexed by $i, j < 0$ on the one hand and factors with generators indexed by $i, j > 0$ on the other, it follows from Theorem 2.1 that an equivalent blow-up is obtained by combining the above procedures in either order; in other words $[X \times Y; \mathcal{M}_1^<(X) \times \mathcal{M}_1^<(Y), \widehat{\mathcal{M}}_1^>(X) \times \widehat{\mathcal{M}}_1^>(Y)] \cong [X \times Y; \widehat{\mathcal{M}}_1^>(X) \times \widehat{\mathcal{M}}_1^>(Y), \mathcal{M}_1^<(X) \times \mathcal{M}_1^<(Y)]$, as both are associated with the same refinement of monoids.

Since $X \widetilde{\times} Y$ has boundary hypersurfaces associated with the one-dimensional monoids $\mathbb{Z}_+ \langle g_i + h_j \rangle$ in the refinement, with hypersurfaces meeting if and only if their associated monoids generate a two-dimensional monoid, it follows immediately that $X \widetilde{\times} Y$ admits an ordered corners structure with $\mathcal{M}_{1,0}(X \widetilde{\times} Y)$ order isomorphic to the pointed product

$$\{(G, H) \in \mathcal{M}_{1,0}(X) \times \mathcal{M}_{1,0}(Y) : (G, H) \sim (X, Y)\}.$$

To prove the universal property, suppose that an arbitrary corner in W has the local form $F_{-l'} \cap \cdots \cap F_{-1} \cap F_1 \cap \cdots \cap F_l$ for a totally ordered chain $F_{-l'} < \cdots < F_{-1} < F_0 := W < F_1 < \cdots < F_l$ of boundary hypersurfaces of W , with associated monoid $\mathbb{Z}_+ \langle f_{-l'}, \dots, f_{-1}, f_1, \dots, f_l \rangle$.

Given ordered morphisms $f : W \rightarrow X$ and $g : W \rightarrow Y$, with respect to which this corner of W maps into the corner $G_{-m'} \cap \cdots \cap G_{-1} \cap G_1 \cap \cdots \cap G_m$ of X and $H_{-n'} \cap \cdots \cap H_{-1} \cap H_1 \cap \cdots \cap H_n$ of Y , it follows that the monoid homomorphism

$$\mathbb{Z}_+ \langle f_{l'}, \dots, f_{-1}, f_1, \dots, f_l \rangle \rightarrow \mathbb{Z}_+ \langle g_{-m'}, \dots, g_{-1}, g_1, \dots, g_m \rangle$$

is determined on generators by a map of pointed ordered sets

$$\begin{aligned} &\{f_{-l'} < \cdots < f_{-1} < f_0 := 0 < f_1 < \cdots < f_l\} \\ &\rightarrow \{g_{-m'} < \cdots < g_{-1} < g_0 := 0 < g_1 < \cdots < g_m\}, \end{aligned}$$

which is encoded by an assignment $f_i \mapsto g_{\alpha(i)}$ for an increasing sequence $\alpha(-l') \leq \cdots \leq \alpha(l)$ with $\alpha(0) = 0$. Similarly, the homomorphism $\mathbb{Z}_+ \langle f_{-l'}, \dots, f_{-1}, f_1, \dots, f_l \rangle \rightarrow \mathbb{Z}_+ \langle h_{-n'}, \dots, h_{-1}, h_1, \dots, h_n \rangle$ is determined by an assignment $f_i \mapsto h_{\beta(i)}$ for an increasing sequence $\beta(-l') \leq \cdots \leq \beta(l)$ with $\beta(0) = 0$. It follows that the product homomorphism

$$\mathbb{Z}_+ \langle f_{-l'}, \dots, f_l \rangle \rightarrow \mathbb{Z}_+ \langle g_{-m'}, \dots, g_m, h_{-n'}, \dots, h_n \rangle$$

is determined by

$$f_i \mapsto g_{\alpha(i)} + h_{\beta(i)}$$

and hence its image is the submonoid generated by the totally ordered chain $g_{\alpha(-l')} + h_{\beta(-l')} \leq \cdots \leq 0 \leq \cdots \leq g_{\alpha(l)} + h_{\beta(l)}$ within (3.7), which is precisely a monoid in the refinement discussed above (or a face thereof). As a consequence of Theorem 2.1, the map $W \rightarrow X \times Y$

factors uniquely through the blow-up $X \tilde{\times} Y$, and moreover, the map $W \rightarrow X \tilde{\times} Y$ is simple and b-normal since generators are mapped to generators in the associated monoid homomorphisms, completing the proof. $\square$

Remark As is evident from the proof, the order of blow-ups of corners necessary to obtain an order structure consistent with (3.2) follows the rule of thumb that elements further away from the base point $(X, Y) \in \mathcal{M}_0(X) \times \mathcal{M}_0(Y)$ are to be blown up prior to those which are closer to it.

The boundary hypersurfaces of $X \tilde{\times} Y$ are the lifts and/or front faces of the corners $G \times H \subset X \times Y$ for $(G, H) \in \mathcal{M}_{1,0}(X) \times \mathcal{M}_{1,0}(Y)$, and we denote these by

$$L(G, H) := \text{lift of } G \times H \text{ in } \mathcal{M}_{1,0}(X \tilde{\times} Y).$$

In the next section we determine the structure of these boundary hypersurfaces.

3.2 Boundary hypersurfaces

We show below (and it is straightforward to see) that a face of the form $L(X, H)$ or $L(G, Y)$ of $X \tilde{\times} Y$ is essentially an ordered product $X \tilde{\times} H$ or $G \tilde{\times} Y$, respectively, at least when X and Y are both either maximal or minimal. On the other hand, when both $G \in \mathcal{M}_1(X)$ and $H \in \mathcal{M}_1(Y)$ are proper hypersurfaces, $L(G, H)$ is the (lifted) front face of the blow-up of the codimension 2 corner $G \times H$, and as such has dimension strictly larger than that of the product of G and H . One description of $L(G, H)$, as an iterated blow-up of $G \times H \times I$, where $I = [0, 1]$, is obtained by considering the restriction to $G \times H$ of the sequence of steps in the iterated blow-up $X \tilde{\times} Y \rightarrow X \times Y$; we record this below as Proposition 3.6. Far more important than this description, however, is the description of the *product structure* of $L(G, H)$ obtained when G and H decompose into Cartesian products, as is the case locally when the spaces have fibered corners, and we devote the remainder of the section to this latter situation.

Proposition 3.6 *Let $(G, H) \in \mathcal{M}_1(X) \times \mathcal{M}_1(Y)$ and decompose $\mathcal{M}_1(G)$ into the following sets*

$$\begin{aligned} \mathcal{M}_1^{<,<}(G) &= \{G' \cap G : G' < G, G' < X\}, & \mathcal{M}_1^{<,>}(G) &= \{G' \cap G : G' < G, G' > X\}, \\ \mathcal{M}_1^{>,<}(G) &= \{G' \cap G : G' > G, G' < X\}, & \text{and } \mathcal{M}_1^{>,>}(G) &= \{G' \cap G : G' > G, G' > X\}, \end{aligned}$$

with a similar decomposition for $\mathcal{M}_1(H)$. The boundary hypersurface $L(G, H)$ of $X \tilde{\times} Y$ has the following form.

- *If $(G, H) \in \mathcal{M}_1^<(X) \times \mathcal{M}_1^<(Y)$, then*

$$\begin{aligned} L(G, H) &\cong [G \times H \times I; \\ &\quad \mathcal{M}_1^{<,<}(G) \times \mathcal{M}_1^{<,<}(H) \times I, \\ &\quad \mathcal{M}_1^{>,<}(G) \times H \times \{0\}, G \times \mathcal{M}_1^{>,<}(H) \times \{1\}, \\ &\quad \mathcal{M}_1^{>,<}(G) \times \mathcal{M}_1^{>,<}(H) \times I, \\ &\quad \mathcal{M}_1^{>,>}(G) \times \mathcal{M}_1^{>,>}(H) \times I] \end{aligned} \quad (3.8)$$

- If $(G, H) \in \mathcal{M}_1^>(X) \times \mathcal{M}_1^>(Y)$, then

$$\begin{aligned} L(G, H) \cong [G \times H \times I; \\ \wideleftarrow{\mathcal{M}}_1^{>, >}(G) \times \wideleftarrow{\mathcal{M}}_1^{>, >}(H) \times I, \\ \wideleftarrow{\mathcal{M}}_1^{<, >}(G) \times H \times \{0\}, G \times \wideleftarrow{\mathcal{M}}_1^{<, >}(H) \times \{1\}, \\ \wideleftarrow{\mathcal{M}}_1^{<, <}(G) \times \wideleftarrow{\mathcal{M}}_1^{<, <}(H) \times I, \\ \mathcal{M}_1^{<, <}(G) \times \mathcal{M}_1^{<, <}(H) \times I] \end{aligned} \quad (3.9)$$

- Finally, if $G = X \in \mathcal{M}_0(X)$ or $H = Y \in \mathcal{M}_0(Y)$, then

$$\begin{aligned} L(X, H) \cong [X \times H; \mathcal{M}_1^{<, <}(X) \times \mathcal{M}_1^{<, <}(H), \wideleftarrow{\mathcal{M}}_1^{>, >}(X) \times \wideleftarrow{\mathcal{M}}_1^{>, >}(H)] \quad \text{or} \\ L(G, Y) \cong [G \times Y; \mathcal{M}_1^{<, <}(G) \times \mathcal{M}_1^{<, <}(Y), \wideleftarrow{\mathcal{M}}_1^{>, >}(G) \times \wideleftarrow{\mathcal{M}}_1^{>, >}(Y)]. \end{aligned} \quad (3.10)$$

In particular, if X and Y are both interior maximal or both interior minimal, these reduce to the ordered products $X \times H$ and $G \times Y$, respectively.

Proof Suppose that $(G, H) < (X, Y)$ and consider the effect of the blow-up $X \times Y \rightarrow X \times Y$ on the boundary face $G \times H$. Only blow-ups of the form $G' \times H'$ for $(G', H') \sim (G, H)$ need to be considered since the others are either disjoint or induce a blow-up of a boundary hypersurface of $G \times H$, having no effect.

First, the blow-ups of $G' \times H'$ for $(G', H') < (G, H)$ induce the blow-up $[G \times H; \mathcal{M}_1^{<, <}(G) \times \mathcal{M}_1^{<, <}(H)]$. Note that the blow-ups of $G \times H'$ and $G' \times H$ for $G' < G$ and $H' < H$ induce blow-ups of hypersurfaces in $G \times H$ and have no effect. Next comes the blow-up of (the lift of) $G \times H$ itself, introducing the product with an interval: $[G \times H; \mathcal{M}_1^{<, <}(G) \times \mathcal{M}_1^{<, <}(H)] \times I \cong [G \times H \times I; \mathcal{M}_1^{<, <}(G) \times \mathcal{M}_1^{<, <}(H) \times I]$. Next come the blow-up of faces $G \times H'$ and $G' \times H$ for $G < G' < X$ and $H < H' < Y$ which induce the blow-ups $\wideleftarrow{\mathcal{M}}_1^{<, <}(G) \times H \times \{0\}$ and $G \times \wideleftarrow{\mathcal{M}}_1^{<, <}(H) \times \{1\}$, respectively. Next come the blow-ups of $G' \times H'$ for $G < G' < X$ and $H < H' < Y$, which induce blow-ups of $\mathcal{M}_1^{<, <}(G) \times \mathcal{M}_1^{<, <}(H) \times I$, and finally come the blow-ups (in reverse order) of $G' \times H'$ for $G' > X$ and $H' > Y$, which induce the blow-ups of $\wideleftarrow{\mathcal{M}}_1^{>, >}(G) \times \wideleftarrow{\mathcal{M}}_1^{>, >}(H) \times I$.

The case that $(G, H) > (X, Y)$ is similar, proceeding in the order indicated on the second line of (3.4). The case of (X, H) (or (G, Y)) is also similar, with the omission of the blow-up of $X \times H$ itself (since it is already codimension 1), as well as the omission of the blow-ups of the faces $G \times H'$ for $G < X$ and $H' > H$ or $G > X$ and $H' < H$ since these are incomparable as noted above. $\square$

We say that a boundary hypersurface G is *product-type* if it is a product, and hence an ordered product by Corollary 3.5, $G = F \times B = F_{\min} \times B_{\max}$ of an interior minimal manifold $F = F_{\min}$ and an interior maximal manifold $B = B_{\max}$. As noted, this holds locally when X is a manifold with fibered corners, where G is a fiber bundle $G \rightarrow B$ with fiber F (hence our choice of notation). Moreover, this is always true locally in general; indeed, a manifold with ordered corners can always be decomposed locally as the product of an interior minimal and an interior maximal manifold. For a product-type hypersurface, the ordered corners structures on G , F and B are identified with the following subsets of $\mathcal{M}_{1,0}(X)$:

$$\begin{aligned} \mathcal{M}_{1,0}(G) &\cong \{G' \in \mathcal{M}_1(X) : G' \sim G\}, & \partial_{G'} G &= G \cap G' \leftrightarrow G' \\ \mathcal{M}_{1,0}(F) &\cong \{G' \in \mathcal{M}_1(X) : G' \geq G\}, & \partial_{G'} F &= (F \times *) \cap G' \leftrightarrow G' \\ \mathcal{M}_{1,0}(B) &\cong \{G' \in \mathcal{M}_1(X) : G' \leq G\}, & \partial_{G'} B &= (* \times B) \cap G' \leftrightarrow G' \end{aligned} \quad (3.11)$$

and we note that $\mathcal{M}_1^<(G) = F \times \mathcal{M}_1(B)$, while $\mathcal{M}_1^>(G) = \mathcal{M}_1(F) \times B$.

Fix a product-type hypersurface $G = F \times B \in \mathcal{M}_1(X)$ and consider the ordered corners structure of the normal model $N_+G \cong F \times B \times [0, \infty)$ as a model for X near G . When $G < X$, $\mathcal{M}_{1,0}(N_+G)$ has the order structure

$$\begin{array}{ccc} & & \mathcal{M}_1^>(F) \times B \times [0, \infty) \\ & \nearrow & \uparrow \\ F \times B \times 0 & \longrightarrow & F \times B \times [0, \infty) \\ & \searrow & \uparrow \\ & & \mathcal{M}_1^<(F) \times B \times [0, \infty) \\ & & \uparrow \\ & & F \times \mathcal{M}_1(B) \times [0, \infty) \end{array} \quad (3.12)$$

while for $G > X$, $\mathcal{M}_{1,0}(N_+G)$ has the order structure

$$\begin{array}{ccc} & & \mathcal{M}_1(F) \times B \times [0, \infty) \\ & \nearrow & \uparrow \\ & & F \times \mathcal{M}_1^>(B) \times [0, \infty) \\ & \nwarrow & \uparrow \\ F \times B \times 0 & \longleftarrow & F \times B \times [0, \infty) \\ & \nwarrow & \uparrow \\ & & F \times \mathcal{M}_1^<(B) \times [0, \infty) \end{array} \quad (3.13)$$

where we decompose $\mathcal{M}_1(F) = \mathcal{M}_1^<(F) \cup \mathcal{M}_1^>(F)$ or $\mathcal{M}_1(B) = \mathcal{M}_1^<(B) \cup \mathcal{M}_1^>(B)$ into subsets of elements which are above or below X in the order with respect to the identifications (3.11).

Discarding the factor B in the first case or F in the second leads to the following ‘relative cone’ construction, which plays an important role below.

Definition 3.7 Let $G = F \times B$ be a product-type hypersurface of a manifold with ordered corners X . If $G < X$, the *cone on F relative to X* is the ordered corners manifold $C_X(F) := F \times [0, \infty)$ with the order on $\mathcal{M}_{1,0}(C_X(F))$ depicted by the following diagram:

$$\begin{array}{ccc} & & \mathcal{M}_1^>(F) \times [0, \infty) \\ & \nearrow & \uparrow \\ F \times 0 & \longrightarrow & F \times [0, \infty) \\ & \searrow & \uparrow \\ & & \mathcal{M}_1^<(F) \times [0, \infty) \end{array} \quad (3.14)$$

Thus, $F \times 0$ is minimal, while $F \times [0, \infty)$ plays the role of X in the order. Likewise, if $G > X$, the *cone on B relative to X* is the ordered corners manifold $C_X(B) = B \times [0, \infty)$ with the order on $\mathcal{M}_{1,0}(C_X(B))$ depicted by the following diagram:

$$\begin{array}{ccc} & & \mathcal{M}_1^>(B) \times [0, \infty) \\ & \nwarrow & \uparrow \\ B \times 0 & \longleftarrow & B \times [0, \infty) \\ & \nwarrow & \uparrow \\ & & \mathcal{M}_1^<(B) \times [0, \infty) \end{array} \quad (3.15)$$

We call $F_0 := F \times 0$ or $B_0 := B \times 0$ the *end* of the cone. If X is either maximal or minimal, then so is $C_X(B)$ or $C_X(F)$; in this case we use the alternate notation $C_{\max}(F)$ or $C_{\min}(B)$, respectively.

Definition 3.8 If $G = F_G \times B_G$ and $H = F_H \times B_H$ are product-type hypersurfaces of X and Y , respectively, then the *relative join* of F_G and F_H for $(G, H) < (X, Y)$, or B_G and B_H for $(G, H) > (X, Y)$, is the boundary hypersurface of the product of relative cones associated with the ends:

$$\begin{aligned} F_G \tilde{\star}_{X,Y} F_H &:= L(F_G \times 0, F_H \times 0) \subset C_X(F_G) \tilde{\times} C_Y(F_H), \\ B_G \tilde{\star}_{X,Y} B_H &:= L(B_G \times 0, B_H \times 0) \subset C_X(B_G) \tilde{\times} C_Y(B_H). \end{aligned}$$

In view of Proposition 3.6 these spaces can be expressed as iterated blow-ups

$$\begin{aligned} F_G \tilde{\star}_{X,Y} F_H &\cong [F_G \times F_H \times I; \\ &\quad \mathcal{M}_1^{<X}(F_G) \times F_H \times \{0\}, F_G \times \mathcal{M}_1^{<Y}(F_H) \times \{1\}, \\ &\quad \mathcal{M}_1^{<X}(F_G) \times \mathcal{M}_1^{<Y}(F_H) \times I, \\ &\quad \tilde{\mathcal{M}}_1^{>X}(F_G) \times \tilde{\mathcal{M}}_1^{>Y}(F_H) \times I], \\ B_G \tilde{\star}_{X,Y} B_H &\cong [B_G \times B_H \times I; \\ &\quad \tilde{\mathcal{M}}_1^{>X}(B_G) \times B_H \times \{0\}, B_G \times \tilde{\mathcal{M}}_1^{>Y}(B_H) \times \{1\}, \\ &\quad \tilde{\mathcal{M}}_1^{>X}(B_G) \times \tilde{\mathcal{M}}_1^{>Y}(B_H) \times I, \\ &\quad \mathcal{M}_1^{<X}(B_G) \times \mathcal{M}_1^{<Y}(B_H) \times I], \end{aligned}$$

specializing to (1.4) and (1.5) in the case that X and Y are maximal or minimal, respectively.

Note that, in view of (3.12) and (3.14), or (3.13) and (3.15), the projection maps

$$\begin{aligned} \text{pr}_B : N_+G &= F \times B \times [0, \infty) \rightarrow B \quad \text{for } G < X, \text{ or} \\ \text{pr}_F : N_+G &= F \times B \times [0, \infty) \rightarrow F \quad \text{for } G > X \end{aligned} \quad (3.16)$$

are ordered morphisms. In contrast, neither of the other factor projection maps

$$\begin{aligned} N_+G &= F \times B \times [0, \infty) \rightarrow F \times [0, \infty) = C_X(F) \quad \text{for } G < X, \text{ or} \\ N_+G &= F \times B \times [0, \infty) \rightarrow B \times [0, \infty) = C_X(B) \quad \text{for } G > X \end{aligned}$$

are ordered morphisms, since in the first case, faces of the form $F \times \mathcal{M}_1(B) \times [0, \infty)$, which lie below those of the form $\mathcal{M}_1(F) \times B \times [0, \infty)$ relative to the order, are mapped to $F \times [0, \infty)$, which is situated above faces of the form $\mathcal{M}_1(F) \times [0, \infty)$ in the order on $C_X(F)$, with a similar order violation in the second case.

The remedies for this are the following ‘compressed projections,’ defined relative to the differentials, in the sense of §2.1, of a choice of boundary defining functions on X , lifted to N_+G . These compressed projections play an important role at various points throughout the remainder of the paper.

Lemma 3.9 *Let $G = F \times B$ be a product-type hypersurface of X , and fix a set of boundary defining functions $\{\rho_{G'}\}$ on X , with differentials $d\rho_{G'} = v^*\rho_{G'}$ for $G' \neq G$ and $t := d\rho_G : N_+G \rightarrow [0, \infty)$ (which we assume coincides with the projection to $[0, \infty)$ with respect to the trivialization $N_+G \cong F \times B \times [0, \infty)$) on N_+G . Set $\rho_{\leq G} = t \prod_{G' < G} v^*\rho_{G'}$ and $\rho_{\geq G} = t \prod_{G' > G} v^*\rho_{G'}$ on N_+G .*

Then, the compressed projections

$$\begin{aligned}\tilde{\text{pr}}_{C_X(F)} &:= \text{pr}_F \times \rho_{\leq G} : N_+G \rightarrow C_X(F), & (f, b, t) &\mapsto (f, t \prod_{G' < G} \rho_{G'}(b, f)), & \text{for } G < X, \text{ and} \\ \tilde{\text{pr}}_{C_X(B)} &:= \text{pr}_B \times \rho_{\geq G} : N_+G \rightarrow C_X(B), & (f, b, t) &\mapsto (b, t \prod_{G' > G} \rho_{G'}(b, f)), & \text{for } G > X\end{aligned}\quad (3.17)$$

are morphisms of ordered corners.

Proof The maps are clearly simple and b-normal and satisfy the ordered condition since in the first case all hypersurfaces of the form $F \times \mathcal{M}_1(B) \times [0, \infty)$ are mapped to the minimal hypersurfaces $F \times 0$ and in the second case all hypersurfaces of the form $\mathcal{M}_1(F) \times B \times [0, \infty)$ are mapped to the maximal hypersurface $B \times 0$. $\square$

We are now in a position to state the main result about product-type hypersurfaces, which identifies the product-type structure of their lifts in $X \tilde{\times} Y$.

Theorem 3.10 *For product-type boundary hypersurfaces $G = F_G \times B_G \in \mathcal{M}_1(X)$ and $H = F_H \times B_H \in \mathcal{M}_1(Y)$, the hypersurface $L(G, H) \in \mathcal{M}_1(X \tilde{\times} Y)$, which is canonically diffeomorphic to $L(G_0, H_0) \in \mathcal{M}_1(N_+G \tilde{\times} N_+H)$, has a product-type structure as follows.*

- If $(G, H) < (X, Y)$, then

$$L(G, H) \cong (F_G \tilde{\star}_{X,Y} F_H)_{\min} \times (B_G \tilde{\times} B_H)_{\max} \quad (3.18)$$

with the projections coinciding with the restriction to $L(G, H) \subset N_+G \tilde{\times} N_+H$ of the lifts $\tilde{\text{pr}}_{C_X(F_G)} \tilde{\times} \tilde{\text{pr}}_{C_Y(F_H)} : N_+G \tilde{\times} N_+H \rightarrow C_X(F_G) \tilde{\times} C_Y(F_H)$ and $\text{pr}_{B_G} \tilde{\times} \text{pr}_{B_H} : N_+G \tilde{\times} N_+H \rightarrow B_G \tilde{\times} B_H$, respectively.

- If $(G, H) > (X, Y)$, then

$$L(G, H) \cong (F_G \tilde{\times} F_H)_{\min} \times (B_G \tilde{\star}_{X,Y} B_H)_{\max}. \quad (3.19)$$

with the projections coinciding with the restriction to $L(G, H) \subset N_+G \tilde{\times} N_+H$ of the lifts $\text{pr}_{F_G} \tilde{\times} \text{pr}_{F_H} : N_+G \tilde{\times} N_+H \rightarrow F_G \tilde{\times} F_H$ and $\tilde{\text{pr}}_{C_X(B_G)} \tilde{\times} \tilde{\text{pr}}_{C_Y(B_H)} : N_+G \tilde{\times} N_+H \rightarrow C_X(B_G) \tilde{\times} C_Y(B_H)$, respectively.

Proof To establish the canonical diffeomorphism $L(G, H) \cong L(G_0, H_0)$, we note that the normal differentials of the lifted projections $X \tilde{\times} Y \rightarrow X$ and $X \tilde{\times} Y \rightarrow Y$ determine ordered morphisms $N_+L(G, H) \rightarrow N_+G$ and $N_+L(G, H) \rightarrow N_+H$ which by the universal property for the ordered product factor through a unique morphism $N_+L(G, H) \rightarrow N_+G \tilde{\times} N_+H$, and this restricts to a morphism $L(G, H) \cong L(G, H)_0 \rightarrow L(G_0, H_0)$. This map is clearly a diffeomorphism since G (resp. H) has diffeomorphic neighborhoods in X and N_+G (resp. Y and N_+H). Thus, it suffices from now on to replace X and Y with N_+G and N_+H , respectively.

We consider in detail the case $(G, H) > (X, Y)$ as it has the most relevance in §4; the case that $(G, H) < (X, Y)$ is similar. From the universal property of the ordered product applied to the ordered morphisms (3.16) and (3.17), we obtain morphisms

$$\begin{aligned}\text{pr}_{F_G} \tilde{\times} \text{pr}_{F_H} &: N_+G \tilde{\times} N_+H \rightarrow F_G \tilde{\times} F_H, \quad \text{and} \\ \tilde{\text{pr}}_{C_X(B_G)} \tilde{\times} \tilde{\text{pr}}_{C_Y(B_H)} &: N_+G \tilde{\times} N_+H \rightarrow C_X(B_G) \tilde{\times} C_Y(B_H).\end{aligned}$$

As morphisms these are simple and b-normal, and they are also easily seen to be b-surjective (for example, using [16, Lemmas 2.5 and 2.7] applied to the b-fibrations $N_+G \times N_+H \rightarrow$

$F_G \times F_H$ and $N_+G \times N_+H \rightarrow C_X(B_G) \times C_Y(B_H)$, noting that all blow-ups of the domain are either lifts of blow-ups of the target or are transversal. In particular, they are b-fibrations. We claim that the restrictions

$$\begin{aligned} \text{pr}_{F_G} \tilde{\times} \text{pr}_{F_H} : L(G, H) &\rightarrow F_G \tilde{\times} F_H, \quad \text{and} \\ \tilde{\text{pr}}_{C_X(B_G)} \tilde{\times} \tilde{\text{pr}}_{C_Y(B_H)} : L(G, H) &\rightarrow B_G \tilde{\star}_{X,Y} B_H \subset C_X(B_G) \tilde{\times} C_Y(B_H) \end{aligned} \quad (3.20)$$

of these b-fibrations to the boundary hypersurface $L(G, H)$ are in fact transverse surjective submersions of manifolds with corners, from which it follows that the product map

$$\begin{aligned} (\text{pr}_{F_G} \tilde{\times} \text{pr}_{F_H}) \times (\tilde{\text{pr}}_{C_X(B_G)} \tilde{\times} \tilde{\text{pr}}_{C_Y(B_H)}) : L(G, H) \\ \rightarrow (F_G \tilde{\times} F_H) \times (B_G \tilde{\star}_{X,Y} B_H) \end{aligned} \quad (3.21)$$

is a surjective submersion of compact manifolds with corners of the same dimension. This latter map cannot be a non-trivial cover or else its descent to $G \times H \rightarrow (F_G \times F_H) \times (B_G \times B_H)$ would be a non-trivial cover as well; hence, it must be a diffeomorphism. Moreover, since $F_G \tilde{\times} F_H$ is interior minimal and $B_G \tilde{\star}_{X,Y} B_H$ is interior maximal, the target in (3.21) is an ordered product and the diffeomorphism is an isomorphism of ordered corners manifolds.

To show that (3.20) are surjective submersions we note first of all that they are simple b-fibrations (being restrictions to a boundary hypersurface of simple b-fibrations), and proceed to show that they have the property discussed in Sect. 2, that each boundary hypersurface of the target is mapped onto by a unique boundary hypersurface of the source. The boundary hypersurfaces of $F_G \tilde{\times} F_H$ are indexed by (G', H') for $G' \geq G$ and $H' \geq H$ (except for (G, H) itself, which corresponds to the interior) and consist of the lifts of $\partial_{G'} F_G \times \partial_{H'} F_H$. Each of these is mapped onto uniquely by the face of $L(G, H)$ given by its intersection with $L(G', H')$ in $N_+G \tilde{\times} N_+H$, and all other boundary hypersurfaces of $L(G, H)$ map to the interior of $F_G \tilde{\times} F_H$. Regarding the second map, boundary hypersurfaces of $B_G \tilde{\star}_{X,Y} B_H$ can be grouped into four types:

First are those indexed by (G', H') for $G' \leq X$ and $H' \leq Y$, which arise as the intersection of $B_G \tilde{\star}_{X,Y} B_H$ with the lift of $(\partial_{G'} B_G \times [0, \infty)) \times (\partial_{H'} B_H \times [0, \infty))$ in the ordered product of cones, and each of these is mapped onto uniquely by the boundary hypersurface of $L(G, H)$ given by its intersection with $L(G', H')$.

Second are those indexed by (G', H') for $X \leq G' < G$ or $Y \leq H' < H$ (except (X, Y) itself), which similarly arise as the intersection of $B_G \tilde{\star}_{X,Y} B_H$ with the lift of $(\partial_{G'} B_G \times [0, \infty)) \times (\partial_{H'} B_H \times [0, \infty))$, and again each of these is mapped onto uniquely by the boundary hypersurface of $L(G, H)$ given by its intersection with $L(G', H')$.

Third are those associated with (G, Y) or (X, H) , given by the intersection of $B_G \tilde{\star}_{X,Y} B_H$ with the lift of $(B_G \times 0) \times (B_H \times [0, \infty))$ or $(B_G \times [0, \infty)) \times (B_H \times 0)$, respectively. While the latter boundary hypersurfaces of $C_X(B_G) \tilde{\times} C_Y(B_H)$ typically have many preimage hypersurfaces in $N_+G \tilde{\times} N_+H$, namely $L(G', Y)$ for each $G' \geq G$ or $L(X, H')$ for each $H' \geq H$, of these only $L(G, Y)$ and $L(X, H)$ are comparable to $L(G, H)$ itself, so the restriction $L(G, H) \rightarrow B_G \tilde{\star}_{X,Y} B_H$ has unique preimages of these boundary hypersurfaces.

Last are the boundary hypersurfaces associated with (G, H') for $Y < H' < H$ and (G', H) for $X < G' < G$, given by the intersection of $B_G \tilde{\star}_{X,Y} B_H$ with the lift of $(B_G \times 0) \times (\partial_{H'} B_H \times [0, \infty))$ or $(\partial_{G'} B_G \times [0, \infty)) \times (B_H \times \{0\})$, respectively, and each of these is mapped onto uniquely by the boundary hypersurface of $L(G, H)$ given by its intersection with $L(G, H')$ or $L(G', H)$, respectively. All other boundary hypersurfaces of $L(G, H)$, given by its intersection with $L(G', H')$ for $G' \geq G$ and $H' \geq H$, are mapped to the interior of $B_G \tilde{\star}_{X,Y} B_H$.

Thus, we have shown that (3.20) are surjective submersions and it remains to see that they are transverse (i.e., the kernels of their pointwise differentials intersect trivially). As this is a local property, we can consider local coordinates, and we can essentially ignore any interior (meaning non-boundary defining) coordinates, since these are unaffected by boundary blow-ups and are carried along as products locally; moreover, transversality is clear for interior coordinates since they decompose into factors belonging to the interiors of the factors $F_\bullet$ or $C_\bullet(B_\bullet)$, respectively. On the other hand, for a simple surjective submersion, the behavior of the boundary defining coordinates is entirely combinatorial: The kernel of the differential of $L(G, H) \rightarrow F_G \times F_H$ is spanned by the coordinate vector fields of boundary defining coordinates for hypersurfaces mapping to the interior of the target and likewise for $L(G, H) \rightarrow B_G \times_{X,Y} B_H$. As we have just shown above that these sets of boundary hypersurfaces are complementary, transversality follows, completing the proof.

The proof of the analogous product-type structure of $L(G, H)$ for $(G, H) < (X, Y)$ is essentially the same but with orders reversed, so we omit the details. $\square$

3.3 Fiber products

We now show that the ordered corners category contains fiber products of appropriately transverse maps. In general, recall that interior b-maps $f : X \rightarrow Z$ and $g : Y \rightarrow Z$ are said to be *b-transverse* [18, 22] provided that whenever $f(x) = g(y) = z$,

$${}^bdf_* {}^bT_x X + {}^bdg_* {}^bT_y Y = {}^bT_z Z. \quad (3.22)$$

Such maps are transverse in the ordinary sense over the interiors of X , Y , and Z , where the b-tangent bundles coincide with the ordinary tangent bundles, and here, it is a standard result that

$$X^\circ \times_Z Y^\circ = \{(x, y) \in X^\circ \times Y^\circ : f(x) = g(y)\} \subset X^\circ \times Y^\circ \quad (3.23)$$

is a smoothly embedded submanifold. On the other hand, the closure of this set in the Cartesian product $X \times Y$ is typically singular, and even when it can be given a smooth manifold with corners structure, it is almost never a p-submanifold apart from extremely trivial cases. The main result below says that the closure of (3.23) in the *ordered* product $X \times Y$ is well behaved and satisfies the universal property of the fiber product.

Remark In [22], the notion of a ‘binomial subvariety’ is defined, generalizing sets like the closure of (3.23) in $X \times Y$, and the authors determine a monoidal condition under which such a subvariety lifts to be smooth under a generalized blow-up of the ambient manifold, with applications to fiber products more specifically. In a related but slightly different direction, Joyce in [18] develops the differential topology of ‘manifolds with generalized corners,’ an intrinsic structure that can be considered on a binomial subvariety without reference to the ambient manifold, and which always contains b-transverse fiber products; in particular the closure of (3.23) in $X \times Y$ can always be given a manifold with generalized corners structure when f and g are b-transverse. Finally, in [20] the theory of generalized blow-up is extended to manifolds with generalized corners, which could be used to characterize resolutions of the singular fiber product. While these theories apply to the present context, they involve quite a bit of machinery, so we opt for a self-contained treatment that can be understood independently.

Lemma 3.11 ([22, Prop. 11.3]) *If $f : X \rightarrow Z$ and $g : Y \rightarrow Z$ are b-transverse interior b-maps, then the restrictions $f|_E : E \rightarrow G$ and $g|_F : F \rightarrow G$ are also b-transverse for every*

pair of boundary faces $E \in \mathcal{M}(X)$ and $F \in \mathcal{M}(Y)$ such that $f_{\sharp}(E) = g_{\sharp}(F) = G \in \mathcal{M}(Z)$.

Proof The short exact sequence ${}^bN_X E \rightarrow {}^bT_X X \rightarrow {}^bT_X E$ for $E \in \mathcal{M}(X)$ along with similar sequences for Y and Z fit into a commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & {}^bN_X E & \rightarrow & {}^bT_X X & \rightarrow & {}^bT_X E \rightarrow 0 \\ & & {}^bdf_* \downarrow & & {}^bdf_* \downarrow & & \downarrow {}^bd(f|_E)_* \\ 0 & \rightarrow & {}^bN_Z G & \rightarrow & {}^bT_Z Z & \rightarrow & {}^bT_Z G \rightarrow 0 \\ & & {}^bdg_* \uparrow & & {}^bdg_* \uparrow & & \uparrow {}^bd(g|_F)_* \\ 0 & \rightarrow & {}^bN_Y F & \rightarrow & {}^bT_Y Y & \rightarrow & {}^bT_Y F \rightarrow 0 \end{array}$$

in light of which it follows that ${}^bd(f|_E)_* {}^bT_X E + {}^bd(g|_F)_* {}^bT_Y F = {}^bT_Z G$. $\square$

Theorem 3.12 If $f : X \rightarrow Z$ and $g : Y \rightarrow Z$ are ordered morphisms which are b -transverse, then the lift of (3.23) to $X \tilde{\times} Y$ is a smooth interior p -submanifold

$$X \tilde{\times}_Z Y := \overline{\{(x, y) \in X^\circ \times Y^\circ : f(x) = g(y)\}} \subset X \tilde{\times} Y. \quad (3.24)$$

Moreover, $X \tilde{\times}_Z Y$ satisfies the universal property of the fiber product in the category of ordered corners: Whenever there are morphisms $W \rightarrow X$ and $W \rightarrow Y$ commuting with the morphisms to Z , then there exists a unique morphism $W \rightarrow X \tilde{\times}_Z Y$ fitting in the commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\quad} & Y \\ \downarrow & \searrow \exists! & \downarrow \\ X & \xrightarrow{\quad} & Z \end{array} \quad \begin{array}{c} X \tilde{\times}_Z Y \end{array} \quad (3.25)$$

Proof To show that (3.24) is a p -submanifold, we work locally in coordinates $(x, y) \in \mathbb{R}_+^{l'} \times \mathbb{R}^{m'}$ for X , $(x', y') \in \mathbb{R}_+^{l'} \times \mathbb{R}^{m'}$ for Y , and $(x'', y'') \in \mathbb{R}_+^{l''} \times \mathbb{R}^{m''}$ for Z , taken in normal form (2.6) for the b -normal maps f and g so that

$$\begin{aligned} f(x, y) &= (x^v, a(x, y)) = (x'', y'') \quad \text{and} \\ g(x', y') &= ((x')^\mu, b(x', y')) = (x'', y'') \end{aligned}$$

By Lemma 3.11, the maps $y \mapsto a(0, y)$ and $y' \mapsto b(0, y')$ are transverse in the ordinary sense, from which it follows that $d(a_j - b_j)$, $1 \leq j \leq m''$, are independent over $(x, x') = (0, 0)$ and hence in a suitably small neighborhood thereof. Thus, $\bar{y}_j = a_j(x, y) - b_j(x', y')$, $1 \leq j \leq m''$, can be taken as the first m'' of $m + m'$ local coordinates $(x, x', \bar{y}) \in \mathbb{R}_+^{l'} \times \mathbb{R}_+^{l'} \times \mathbb{R}^{m+m'}$ on $X \times Y$ in which (3.23) takes the form

$$\left\{ \frac{(x)^{v_i}}{(x')^{\mu_i}} = 1, \bar{y}_j = 0 : 1 \leq i \leq l'', 1 \leq j \leq m'' \right\},$$

and it remains only to show that $\left\{ \frac{(x)^{v_i}}{(x')^{\mu_i}} = 1 : 1 \leq i \leq l'' \right\}$ lifts to an interior p -submanifold.

According to Proposition 2.3, this is the case provided that each monoid $M \subset \mathbb{Z}_+ \langle g_1, \dots, g_l, h_1, \dots, h_{l'} \rangle$, $g_i = x_i \partial_{x_i}$, $h_j = x'_j \partial_{x'_j}$ associated with the blow-up $X \tilde{\times} Y$ meets the subspace $\bigcap_{i=1}^{l''} \ker(v_i - \mu_i)$ along a face of M . However, this subspace is spanned explicitly by all sums of the form $g_i + h_j$ such that $f_{\sharp}(G_i) = g_{\sharp}(H_j) \in \mathcal{M}_1(Z)$, and since M is generated by a totally ordered chain $\{g_i + h_j\}$, this intersection is precisely the face of

M generated by those $g_i + h_j$ with $\text{bdf}(g_i) = \text{bdg}(h_j)$. It follows that $X \tilde{\times}_Z Y$ is an interior p-submanifold of $X \tilde{\times} Y$.

To see that it satisfies the universal property, suppose that W admits morphisms to X and Y forming a commutative diagram

$$\begin{array}{ccc} W & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

Then, W factors uniquely through a morphism to $X \tilde{\times} Y$, with W° mapping into the lift of $X^\circ \times_{Z^\circ} Y^\circ$. By continuity, the image of W in $X \tilde{\times} Y$ lies in the p-submanifold $X \tilde{\times}_Z Y$, giving the diagram (3.25). $\square$

Theorem 3.13 *The boundary hypersurfaces of $X \tilde{\times}_Z Y$ are given by its intersection with the boundary faces $L(G, H)$ of $X \tilde{\times} Y$ such that $f_\sharp(G) = g_\sharp(H) \in \mathcal{M}_{1,0}(Z)$, and have the following form:*

- If $G \in \mathcal{M}_1(X)$ and $H \in \mathcal{M}_1(Y)$ are proper hypersurfaces with $f_\sharp(G) = g_\sharp(H) = Z \in \mathcal{M}_0(Z)$, then

$$(X \tilde{\times}_Z Y) \cap L(G, H) \cong \overline{\{G^\circ \times_Z H^\circ \times I\}} \subset L(G, H), \quad (3.26)$$

meaning the lift of $G \times_Z H \times I$ within $L(G, H)$, considered as a blow-up of $G \times H \times I$ as in (3.8) or (3.9).

- In all other cases,

$$(X \tilde{\times}_Z Y) \cap L(G, H) \cong G \tilde{\times}_K H \quad (3.27)$$

is naturally identified with the fiber product of G and H over $K = f_\sharp(G) = g_\sharp(H) \in \mathcal{M}_{1,0}(Z)$.

Remark Taking $Z = *$ recovers $X \tilde{\times}_Z Y = X \tilde{\times} Y$ with the structure of its boundary faces.

Proof As an interior p-submanifold of $X \tilde{\times} Y$, the boundary hypersurfaces of $X \tilde{\times}_Z Y$ are precisely its interior intersections with the hypersurfaces $L(G, H)$ of $X \tilde{\times} Y$, and we observe that $X \tilde{\times}_Z Y$ can only meet the interior of $L(G, H)$ provided that $f_\sharp(G) = g_\sharp(H) = K \in \mathcal{M}_{1,0}(Z)$. In particular $\mathcal{M}_{1,0}(X \tilde{\times}_Z Y) \cong \mathcal{M}_{1,0}(X) \times_{\mathcal{M}_{1,0}(Z)} \mathcal{M}_{1,0}(Y)$ as pointed ordered sets.

In the first case, if $G \in \mathcal{M}_1(X)$ and $H \in \mathcal{M}_1(Y)$ are proper hypersurfaces mapping to the interior of Z , then it follows that none of the rational boundary defining function equations defining $X \tilde{\times}_Z Y$ involve boundary defining functions of G or H . Considering the steps of the blow-up $X \tilde{\times} Y$ as in the proof of Proposition 3.6 and the lift of $\{(x, y) : f(x) = g(y)\}$ at each step, it follows in particular that in the blow-up of $G \times H$ itself, the lift meets the front face in the set $\{G^\circ \times_Z H^\circ \times I\}$, i.e., it is independent of the interval I since boundary defining functions for G or H may be taken as coordinates along this interval. With this observation, (3.26) follows along the lines of the proof of Proposition 3.6.

In the other cases, we show that $L(G, H)_K := X \tilde{\times}_Z Y \cap L(G, H)$ can be identified with the fiber product $G \tilde{\times}_K H$. If $G = X$, then $K = Z$ by necessity, and since $L(X, H) \cong X \tilde{\times} H$ in this case, it is straightforward to see that the lift of $\{(x, y) \in X \times Y : f(x) = g(y)\}$ to $X \tilde{\times} Y$ meets $L(X, H)$ in a set equivalent to the lift of $\{(x, y) \in X \times H : f(x) = g(y)\}$ to $X \tilde{\times} H$. A similar consideration applies to the case that $H = Y$.

Finally, if G , H , and K are all proper hypersurfaces, then our strategy is to show that $L(G, H)_K$ satisfies the universal property of the fiber product $G \tilde{\times}_K H$, namely that every commutative diagram formed by the outermost square of

$$\begin{array}{ccc} F & \xrightarrow{\psi} & H \\ \downarrow \phi & \searrow \exists! & \downarrow g \\ & L(G, H)_K & \\ \downarrow & \swarrow f & \downarrow \\ G & \xrightarrow{\quad} & K \end{array} \quad (3.28)$$

determines a unique morphism $F \rightarrow L(G, H)_K$ filling in the remainder of the diagram. It suffices to replace X , Y , and Z by the normal models N_+G , N_+H and N_+K , respectively. We proceed by first showing that the outer square of (3.28) can be ‘thickened out’ to a diagram of ordered corners morphisms of the form

$$\begin{array}{ccc} F \times \mathbb{R}_+ & \xrightarrow{\tilde{\psi}} & N_+H \\ \downarrow \tilde{\phi} & & \downarrow dg \\ N_+G & \xrightarrow{df} & N_+K \end{array} \quad (3.29)$$

uniquely up to an automorphism of $F \times \mathbb{R}_+$ which restricts to the identity on F , whence the restriction of the map $F \times \mathbb{R}_+ \rightarrow N_+G \tilde{\times}_{N_+K} N_+H$ to $F \times \{0\}$ provides the required unique morphism characterizing $L(G, H)_K$ as the fiber product $G \tilde{\times}_K H$.

Indeed, with respect to trivializations of the normal models, $df : N_+G \rightarrow N_+K$ has the form

$$\begin{aligned} N_+G &\cong G \times \mathbb{R}_+ \xrightarrow{df} K \times \mathbb{R}_+ \cong N_+K \\ (p, t) &\mapsto (f(p), t\sigma(p)) \end{aligned}$$

where $\sigma = \prod_{G' \in f_{\sharp}^{-1}(K)} \rho_{G'}$ is a product of boundary defining functions on G for those faces $\partial_{G'}G$ which map to K itself. Similarly, $dg : N_+H \rightarrow N_+K$ has the form $(p', t') \mapsto (g(p'), t'\sigma'(p'))$ for a product $\sigma' = \prod_{H' \in g_{\sharp}^{-1}(K)} \rho_{H'}$ on H .

Fixing boundary defining functions on F , it follows from commutativity of (3.28) that $\phi^*\sigma = a_G \prod_{F': F' \mapsto K} \rho_{F'}$ and $\psi^*\sigma' = a_H \prod_{F': F' \mapsto K} \rho_{F'}$ for the same set $\{\rho_{F'} : F' \mapsto K\}$ of boundary defining functions on F , namely those associated with hypersurfaces mapped to the interior of K by $\psi \circ g = \phi \circ f$ in (3.28), but generally different positive functions $a_G, a_H \in C^\infty(F; (0, \infty))$. But then (3.29) commutes if we define

$$\begin{aligned} \tilde{\phi} : F \times \mathbb{R}_+ \ni (q, t) &\mapsto (\phi(q), a_G^{-1}(q)t) \in G \times \mathbb{R}_+ \cong N_+G \\ \text{and } \tilde{\psi} : F \times \mathbb{R}_+ \ni (q, t) &\mapsto (\psi(q), a_H^{-1}(q)t) \in H \times \mathbb{R}_+ \cong N_+H \end{aligned}$$

Moreover, these are the only possible extensions of ϕ and ψ up to an $(0, \infty)$ -equivariant automorphism of $F \times \mathbb{R}_+$ of the form $(q, t) \mapsto (q, b(q)t)$ for $b \in C^\infty(F; (0, \infty))$, which in particular restricts to the identity on $F \times \{0\}$, so the restriction $F \times \{0\} \rightarrow L(G, H)_K$ of the map $F \times \mathbb{R}_+ \rightarrow N_+G \tilde{\times}_{N_+K} N_+H$ induced by any such choice is unique, giving the characterization of $L(G, H)_K$ as the fiber product $G \tilde{\times}_K H$. Note that the above strategy fails if $K = Z$, since in this case the maps $N_+G \cong G \times \mathbb{R}_+ \rightarrow Z$ and $N_+H \cong H \times \mathbb{R}_+ \rightarrow Z$ are respectively independent of $t \in \mathbb{R}_+$, and while extensions $\tilde{\phi} : F \times \mathbb{R}_+ \rightarrow N_+G$ and $\tilde{\psi} : H \times \mathbb{R}_+ \rightarrow N_+H$ may be defined, they are certainly not unique. $\square$

As an application of the previous result, note that if $f : X \rightarrow Y$ is any morphism, then the fiber product of f with the identity $1 : Y \rightarrow Y$ (which is b-transversal to every b-map) gives an embedding of the *graph* $\text{Gr}(f) = X \tilde{\times}_Y Y \subset X \tilde{\times} Y$ of f as a p-submanifold of $X \tilde{\times} Y$. In particular this shows that every morphism can be factored as a p-submanifold inclusion $X \hookrightarrow \text{Gr}(f) \subset X \tilde{\times} Y$ and a b-submersion $X \tilde{\times} Y \rightarrow Y$.

Another application, which will be important in the forthcoming work [24], is the following. Let $\rho_X = \prod_{G \in \mathcal{M}_1(X)} \rho_G$ and $\rho_Y = \prod_{H \in \mathcal{M}_1(Y)} \rho_H$ be total boundary defining functions on interior minimal manifolds X and Y with ordered corners, respectively. These constitute ordered corners morphisms $\rho_X : X \rightarrow [0, \infty)_{\min}$ and $\rho_Y : Y \rightarrow [0, \infty)_{\min}$, with target given the interior minimal ordered corners structure, and these are b-transverse in a sufficiently small neighborhood of the boundaries of X and Y where $d\rho_X \neq 0$ and $d\rho_Y \neq 0$.

Corollary 3.14 *In the situation above, the fiber product $\{\rho_X/\rho_Y = 1\} \subset X \tilde{\times} Y$ is a well-defined p-submanifold sufficiently near the boundary of $X \tilde{\times} Y$, meeting $L(G, H)$ in a p-submanifold of the form $G \tilde{\times}_{\{0\}} H = G \tilde{\times}_{\{0\}} H$ for each pair $(G, H) \in \mathcal{M}_1(X) \times \mathcal{M}_1(Y)$.*

4 Fibered corners

We now turn to the consideration of manifolds with fibered corners.

Definition 4.1 (c.f. [11]) A *fibered corners structure* (also known as a ‘resolution structure’ [5] or an ‘iterated boundary fibration structure’ [3]) on a manifold with corners X consists of the structure of a fiber bundle⁶ of manifolds with corners

$$\phi_G : G \rightarrow B_G$$

for each (collective) boundary hypersurface $G \in \mathcal{M}_1(X)$, with typical fiber⁷ denoted F_G , satisfying the condition that whenever $G \cap G' \neq \emptyset$ then $\dim(B_G) \neq \dim(B_{G'})$; moreover, if $\dim(B_G) < \dim(B_{G'})$, say, then $\phi_{G'}$ constitutes a coarser fibration on $G \cap G'$ compared to ϕ_G in the following sense:

- ϕ_G maps $G \cap G'$ surjectively onto B_G (in other words, $G \cap G'$ is ‘horizontal’ with respect to ϕ_G), whereas $G \cap G' = \phi_{G'}^{-1}(\partial_G B_{G'})$ for a boundary hypersurface $\partial_G B_{G'}$ of $B_{G'}$ (in other words, $G \cap G'$ is ‘vertical’ with respect to $\phi_{G'}$), and
- $\partial_G B_{G'}$ has a fiber bundle structure $\phi_{GG'} : \partial_G B_{G'} \rightarrow B_G$ such that the following diagram commutes:

$$\begin{array}{ccc} G \cap G' & \xrightarrow{\phi_{G'}} & \partial_G B_{G'} \\ & \searrow \phi_G & \downarrow \phi_{GG'} \\ & & B_G. \end{array} \quad (4.1)$$

This induces an ordering on $\mathcal{M}_1(X)$ in the sense of Definition 3.1 (to be extended to $\mathcal{M}_{1,0}(X)$ momentarily) defined by

$$G < G' \iff G \cap G' \neq \emptyset \text{ and } \dim(B_G) < \dim(B_{G'}). \quad (4.2)$$

⁶ In fact, by Corollary A.6 it is sufficient for ϕ_G to be a fibration, i.e., a proper surjective submersion; by an Ehresmann lemma for manifolds with fibered corners (Proposition A.5), such a map is locally trivial.

⁷ If B_G is disconnected, different connected components may have different fibers.

We require here in addition the structure of a fiber bundle $\phi_X : X \rightarrow B_X$ on X itself, with fiber F_X . Of course $X \cap G = G \neq \emptyset$ for every $G \in \mathcal{M}_1(X)$, and in order to extend the compatibility condition above, there are two possibilities. We require either:

- $G = X \cap G$ is the restriction of ϕ_X over a proper boundary hypersurface $\partial_G B_X$ of B_X , which fibers over B_G forming a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\phi_X} & \partial_G B_X \\ & \searrow \phi_G & \downarrow \phi_{GX} \\ & & B_G \end{array}$$

in which case we set $G < X$, or else

- G is trivially the restriction of ϕ_G over the ‘honorary boundary hypersurface’ $\partial_X B_G = B_G$, which fibers over B_X forming a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\phi_G} & \partial_X B_G = B_G \\ & \searrow \phi_X & \downarrow \phi_{XG} \\ & & B_X \end{array}$$

in which case we set $X < G$.

This extends the order on $\mathcal{M}_1(X)$ to the set of principal faces $\mathcal{M}_{1,0}(X)$, and in particular every manifold with fibered corners is naturally a manifold with ordered corners in the sense of Definition 3.1. In the second case above, we relax the strict dimension comparison on the base manifolds to require merely that $\dim(B_X) \leq \dim(B_G)$, to account for the possibility that $\phi_{XG} = 1$ has trivial fibers (which may occur). As a matter of notation, we denote the fiber of $\phi_{GG'} : \partial_G B_{G'} \rightarrow B_G$ by

$$E_{G,G'} := \phi_{GG'}^{-1}(*).$$

Remark

- For the definition we do not necessarily require that boundary hypersurfaces are connected; as in [5] we allow G to be a ‘collective boundary hypersurface,’ i.e., a union of disjoint connected boundary hypersurfaces. One reason for this is that a fiber F_G , a fibered corners manifold in its own right, may have distinct boundary hypersurfaces which are the restrictions to the fiber of a single connected boundary hypersurface of the total space, as in the example of a finite width Möbius strip fibering over a circle. Another reason comes from the connection to stratified spaces as discussed in Sect. 5.1, in which the B_G and F_G are closely related to the strata and associated links, respectively, of a stratified space. In the latter theory one often wants to consider stratifications in which the links, and hence the F_G , may be disconnected; for example, a torus pinched along a non-bounding circle has a singular stratum whose link consists of two disjoint circles, and the associated manifold with fibered corners is a cylinder with both boundary components considered as a single collective boundary hypersurface fibering over a point.

Note that if B_G is disconnected, then different components of the fibration G may have non-diffeomorphic typical fibers; unfortunately our notational convention, in which we denote the fiber systematically by F_G , does not support this scenario. Rather than complicate the notation further, we leave it to the reader to make the necessary adjustments.

- In the existing literature it is typical to only consider fibrations only on the proper boundary hypersurfaces, and in the absence of an explicit fibration on X itself, there are two canonical choices: namely ϕ_X can be either of the trivial fibrations: $\phi_X = 1 : X \rightarrow X$, making X interior maximal, or $\phi_X : X \rightarrow *$, making X interior minimal. We denote these by $X_{\max}$ and $X_{\min}$, respectively.

In Appendix A, we characterize interior fibrations $\phi_X : X \rightarrow B_X$ as certain surjective submersions in the category of fibered corners manifolds: see Definition A.3 and Corollary A.6.

As a locally trivial fiber bundle, the boundary hypersurfaces of G (given by $G \cap G'$) come in two types, either the preimage with respect to ϕ_G of a boundary face $\partial_{G'} B_G$ of the base when $G' < G$, or a subbundle of ϕ_G with fiber a (possibly disconnected) boundary face of $\partial_{G'} F_G$ when $G' > G$. From this the next result immediately follows.

Proposition 4.2 *For each $G \in \mathcal{M}_1(X)$, F_G , B_G , and G itself all inherit fibered corners structures as follows:*

- $\mathcal{M}_{1,0}(B_G) \cong \{G' \in \mathcal{M}_1(X) : G' \leq G\}$, with fibrations $\phi_{G'G} : \partial_{G'} B_G \rightarrow B_{G'}$ and the identity fibration on B_G itself. In particular B_G is interior maximal.
- $\mathcal{M}_{1,0}(F_G) \cong \{G' \in \mathcal{M}_1(X) : G' \geq G\}$, with fibrations $\phi_{G'} : \partial_{G'} F_G \rightarrow E_{G,G'}$ (obtained by restricting (4.1) over $*$ in B_G), and the trivial fibration $F_G \rightarrow *$ on the interior. In particular F_G is interior minimal.
- $\mathcal{M}_{1,0}(G) \cong \{G' \in \mathcal{M}_1(X) : G' \sim G\}$, with fibrations $\phi_{G'} : \partial_{G'} G := G \cap G' \rightarrow B_{G'}$ for $G' \leq G$ and $\phi_{G'} : \partial_{G'} G := G \cap G' \rightarrow \partial_G B_{G'}$ for $G' > G$.

In particular every $G \in \mathcal{M}_1(X)$ is locally a product-type hypersurface in the sense of §3.2, given locally by the ordered product $G \stackrel{loc}{=} F_G \times B_G = F_G \times B_G$.

Prior to defining morphisms of manifolds with fibered corners, we first consider some natural conditions that may be imposed.

Definition 4.3 Let X and Y be manifolds with fibered corners. We say a b-map $f : X \rightarrow Y$ is *fibered* if the restriction of f to each $G \in \mathcal{M}_{1,0}(X)$ is a map of fiber bundles: Thus,

$$\begin{array}{ccc} G & \xrightarrow{f|_G} & H = f_{\#}(G) \\ \phi_G \downarrow & & \downarrow \phi_H \\ B_G & \xrightarrow{f_G} & B_H \end{array} \quad (4.3)$$

for some map $f_G : B_G \rightarrow B_H$, which, as a consequence of the fibered property of f for each $G' < G$, is itself fibered.

This condition is natural from the point of view of stratified spaces (see Sect. 5.1), since a fibered map $f : X \rightarrow Y$ descends to a well-defined map $\bar{f} : \mathcal{X} \rightarrow \mathcal{Y}$ of the associated stratified spaces, obtained by collapsing the fibers of the respective boundary fibrations of X and Y . As it turns out, this condition alone is not sufficient to determine a category of fibered manifolds containing products; we need to impose some additional structure, in support of which we introduce the following definitions.

Definition 4.4 Let X be a manifold with fibered corners.

- A smooth function $u \in C^\infty(X; \mathbb{R})$ is said to be *basic* at $G \in \mathcal{M}_1(X)$ if its restriction to G is pulled back from B_G : $u|_G = \phi_G^* u_G$ for some $u_G \in C^\infty(B_G; \mathbb{R})$; equivalently, u is constant on fibers of G . A function which is basic at every proper boundary hypersurface $G \in \mathcal{M}_1(X)$ will be simply called *basic*.

- (b) A boundary defining function ρ_G for G is *good* if it is basic at G' for every $G' > G$; in particular $\rho_G|_{G'}$ is the pullback of a boundary defining function for $\partial_G B_{G'}$ on $B_{G'}$. Note that it is not possible for ρ_G to be basic at $G' < G$.
- (c) Two boundary defining functions ρ_G and ρ'_G for the same $G \in \mathcal{M}_1(X)$ are *fc-equivalent* if the ratio ρ_G/ρ'_G is basic (constant on fibers of each $G' \in \mathcal{M}_1(X)$). This is an equivalence relation. Note that for good boundary defining functions, ρ_G/ρ'_G is automatically basic at every $G' > G$, so it is enough to require that the ratio be basic at each $G' \leq G$.
- (d) A simple, b-normal b-map $f: X \rightarrow Y$ is *compatible* with given fc-equivalence classes of boundary defining functions on X and Y , respectively, provided that for some (and hence any) representatives $\{\rho_G: G \in \mathcal{M}_1(X)\}$ and $\{\rho_H: H \in \mathcal{M}_1(Y)\}$,

$$f^*(\rho_H) = a_H \prod_{G \in f_{\#}^{-1}(H)} \rho_G \implies a_H \in C^\infty(X; (0, \infty)) \text{ is basic for every } H \in \mathcal{M}_1(Y).$$

Equivalently, as a consequence of the normal form for b-normal maps discussed in Sect. 2, a map f is compatible if and only if it is rigid with respect to some representative boundary defining functions for the equivalence classes on X and Y .

Lemma 4.5 *An fc-equivalence class of good boundary defining functions for $G' \in \mathcal{M}_1(X)$ on X induces an fc-equivalence class of good boundary defining functions on each $G \in \mathcal{M}_1(X)$ such that $G \cap G' \neq \emptyset$, and likewise on B_G if $G' < G$ and F_G if $G' > G$.*

Proof For G and F_G , the equivalence class is given by restriction of representative elements. For B_G , it follows by definition that $\rho_{G'}|_G = \phi_G^* \rho'_{G'}$ for a boundary defining function $\rho'_{G'}$ for $\partial_{G'} B_G$ on B_G ; moreover, the ratio of two such fc-equivalent boundary defining functions for G' is pulled back from $B_{G''}$ when restricted over any $G'' < G$; hence, they descend to fc-equivalent boundary defining functions for $\partial_{G'} B_G$ on B_G . $\square$

It is a consequence of Proposition A.4 (see also [5, Prop. 3.7] or [11, Lem. 1.4]) that good boundary defining functions always exist. What is more, having fixed fc-equivalence classes of good boundary defining functions, local coordinates on X can always be chosen of the form

$$(y_{-n'}, x_{-n'}, y_{-n'+1}, x_{-n'+1}, \dots, y_0, y_1, x_1, y_2, x_2, \dots, x_n, y_{n+1}) \quad (4.4)$$

where the x_i are representative local boundary defining functions for the hypersurfaces G_i forming a maximal chain $G_{-n'} < \dots < G_{-1} < X < G_1 < \dots < G_n$ and y_i are coordinates in $\mathbb{R}^{k_i}$, such that the boundary fibration $\phi_i = \phi_{G_i}$ for $-n' \leq i \leq n$ is given by the projection

$$\phi_i: (y_{-n'}, x_{-n'}, \dots, x_{-1}, y_0, y_1, x_1, y_2, x_2, \dots, x_n, y_{n+1})|_{x_i=0} \mapsto (y_{-n'}, x_{-n'}, \dots, y_i) \quad (4.5)$$

including the case $i = 0$ which corresponds to ϕ_X . This is proved as Corollary A.7. In particular $(y_{-n'}, x_{-n'}, \dots, y_i)$ are coordinates for B_{G_i} and $(y_{i+1}, x_{i+1}, \dots, y_{n+1})$ are coordinates for F_{G_i} . We say such coordinates are in *standard form*.

Remark When X is maximal, we may opt to enumerate hypersurfaces and coordinates with positive instead of negative integers, so using coordinates $(y_1, x_1, \dots, x_n, y_{n+1})$ in which ϕ_i is still given by

$$\phi_i: (y_1, x_1, \dots, y_i, 0, y_{i+1}, \dots, y_n, x_n, y_{n+1}) \mapsto (y_1, x_1, \dots, y_i).$$

We are now in a position to define the category of manifolds with fibered corners in which the ordered product will be shown to be the categorical product.

Definition 4.6 The category of manifolds with fibered corners is defined as follows.

- An *object* is a manifold with fibered corners X equipped with an fc-equivalence class of good boundary defining functions for each $G \in \mathcal{M}_1(X)$.
- A *morphism* is an ordered morphism $f : X \rightarrow Y$ in the sense of Definition 3.1 (i.e., a simple, interior, b-normal b-map for which $f_\sharp : \mathcal{M}_{1,0}(X) \rightarrow \mathcal{M}_{1,0}(Y)$ is ordered) which is fibered in the sense of Definition 4.3 and compatible with the fc-equivalence classes of boundary defining functions in the sense of Definition 4.4.

The following result is a simple consequence of unwinding definitions.

Lemma 4.7 For a manifold with fibered corners X , the fibration $\phi_G : G \rightarrow B_G$ for each $G \in \mathcal{M}_{1,0}(X)$ is a morphism of fibered corners manifolds.

Remark Note that the condition that the boundary defining functions on X are good is equivalent to the rigidity of each boundary fibration $\phi_G : G \rightarrow B_G$, $G \in \mathcal{M}_1(X)$.

4.1 Products and fiber products

We now proceed to show that the ordered product $X \tilde{\times} Y$, equipped with a suitable set of fibrations and fc-equivalence classes of boundary defining functions, is also a product in the category of manifolds with fibered corners.

The fibrations on $L(G, H)$ for $(G, H) \leq (X, Y)$ are straightforward. Indeed, the composite maps $L(G, H) \rightarrow G \rightarrow B_G$ and $L(G, H) \rightarrow H \rightarrow B_H$ are ordered morphisms, so the universal property of $B_G \tilde{\times} B_H$ as a manifold with ordered corners implies that there is a unique consistent map

$$\phi_{G,H} : L(G, H) \rightarrow B_G \tilde{\times} B_H, \quad (G, H) \leq (X, Y).$$

and it then follows from a local application of Theorem 3.10 that this is in fact a fiber bundle with typical fiber the join $F_G \tilde{\star}_{X,Y} F_H$ (or simply $F_G \tilde{\times} F_H$ if $G = X$ or $H = Y$).

Moreover, by the universal property of products in the ordered corners category, it follows that whenever W is a manifold with fibered corners with morphisms $f : W \rightarrow X$ and $g : W \rightarrow Y$ thereby inducing an ordered morphism $W \rightarrow X \tilde{\times} Y$, the restriction of this morphism to any $E \in \mathcal{M}_{1,0}(W)$ with $E \leq W$ fits into a naturally commutative diagram

$$\begin{array}{ccc} E & \longrightarrow & L(G, H) \\ \downarrow \phi_E & & \downarrow \phi_{L(G,H)} \\ B_E & \longrightarrow & B_G \tilde{\times} B_H \end{array} \quad G = f_\sharp(E), \quad H = g_\sharp(E),$$

with the map $B_E \rightarrow B_G \tilde{\times} B_H$ induced by $f_E : B_E \rightarrow B_G$ and $g_E : B_E \rightarrow B_H$. In other words, the map $W \rightarrow X \tilde{\times} Y$ is canonically fibered over each $E \leq W \in \mathcal{M}_{1,0}(W)$.

Note that none of the above makes any reference to fc-equivalence classes of boundary defining functions on X , Y , or W . In fact, it follows from this observation that, provided we restrict consideration to *interior maximal* manifolds with fibered corners, we can define a weaker category in which the objects are merely required to have fibered corners (without specifying equivalence classes of boundary defining functions) and morphisms are merely required to be ordered and fibered; what we have just shown is that $X \tilde{\times} Y$ is a product in this category:

Theorem 4.8 *The ordered product is a categorical product in the category whose objects are interior maximal manifolds with fibered corners and whose morphisms are ordered morphisms in the sense of Definition 3.1 which are fibered in the sense of Definition 4.3.*

However, when the manifolds are not interior maximal, the extra data (of equivalence classes of boundary defining functions and compatible maps) is required to get a theory with products, as we now show. Note that while there is always a canonical map $L(G, H) \rightarrow B_G \times B_H$ consistent with the composite projections $L(G, H) \rightarrow B_G$ and $L(G, H) \rightarrow B_H$, this is not a fiber bundle when $(G, H) > (X, Y)$, as can be seen from Theorem 3.10. Instead, we expect $L(G, H)$ for $(G, H) > (X, Y)$ to fiber over the join $B_G \tilde{*}_{X,Y} B_H$. To define these fibrations we consider the normal models N_+G for $G \in \mathcal{M}_1(X)$.

Definition 4.9 Let $G \in \mathcal{M}_1(X)$. The fibered corners structure on the normal model N_+G consists of fibrations given by the differentials of those from X :

$$\begin{aligned} d\phi_{G'} : N_+G|_{G' \cap G} &\rightarrow N_+\partial_G B_{G'}, & \text{for } G' > G, \\ d\phi_{G'} : N_+G|_{G' \cap G} &\rightarrow B_{G'} & \text{for } G' \leq G, \end{aligned}$$

where the latter factors as the composite $N_+G \rightarrow G \rightarrow \partial_{G'} B_G \rightarrow B_{G'}$. In particular, the interior of N_+G is equipped with the fibration $d\phi_X : N_+G \rightarrow \partial_X B_G = B_G \rightarrow B_X$ in case $G > X$ and $N_+G \rightarrow N_+\partial_G B_X$ in case $G < X$. To keep the notation uncluttered, when $G' = G$ we denote $d\phi_G$ simply by $\phi_G : N_+G \rightarrow B_G$. The order structure associated with these fibrations is consistent with (3.12) and (3.13).

Fc-equivalence classes of good boundary defining functions on X induce equivalence classes of $(0, \infty)$ -equivariant defining functions on N_+G by taking their normal derivatives as in §2.1; for notational convenience we shall simply refer to these by the same notation as the boundary defining functions on X from which they are derived.

We also equip the relative cone $C_X(B_G)$ of Definition 3.7 with a fibered corners structure.

Definition 4.10 Let $G > X$. The fibered corners structure on $C_X(B_G) = B_G \times [0, \infty)$ consists of fibrations

$$B_G \times \{0\} \rightarrow B_G, \quad \partial_{G'} B_G \times [0, \infty) \rightarrow B_{G'}, \quad C_X(B_G) = B_G \times [0, \infty) \rightarrow B_X$$

given by the obvious maps coming from the boundary fibrations $\phi_{G'/G} : \partial_{G'} B_G \rightarrow B_{G'}$ and using $\phi_{XG} : \partial_X B_G = B_G \rightarrow B_X$ on B_G itself (rather than the identity fibration) on the interior $B_G \times [0, \infty)$. The ordered structure induced by these fibrations is consistent with Definition 3.7. Fc-equivalence classes of good boundary defining functions on $C_X(B_G)$ are given by pulling back $\partial_{G'} B_G$ defining functions from B_G for the boundary faces $\partial_{G'} B_G \times [0, \infty)$ and by the projection to $[0, \infty)$ for $B_G \times 0$.

Though we will not use it immediately, this is an appropriate moment to record the fibered corners structure of $C_X(F_G)$ for $G < X$:

Definition 4.11 Let $G < X$. The fibered corners structure on $C_X(F_G) = F_G \times [0, \infty)$ consists of boundary fibrations

$$C_X(F_G) \rightarrow E_{G,X} \times [0, \infty) \quad F_G \times \{0\} \rightarrow *, \quad \partial_{G'} F_G \times [0, \infty) \rightarrow E_{G,G'} \times [0, \infty)$$

where we recall that $E_{G,G'} = \phi_{GG'}^{-1}(*)$, the fiber of $\phi_{GG'} : \partial_G B_{G'} \rightarrow B_G$, forms the base of the boundary fibration on $\partial_{G'} F_G$, and here we use the fibration induced by ϕ_X instead

of ϕ_G on the interior $F_G \times [0, \infty)$ itself. The ordered structure induced by these boundary fibrations is consistent with the order structure in Definition 3.7.

Remark The fibered corners structure induced on $B_G \tilde{\star}_{X,Y} B_H$ as a boundary hypersurface of $C_X(B_G) \tilde{\times} C_Y(B_H)$ in the case that $(G, H) > (X, Y)$ is the same as the one induced by considering it as the base of the hypersurface $L(G, H)$ of $X \tilde{\times} Y$, and likewise, the fibered corners structure on $F_G \tilde{\star}_{X,Y} F_H$ as a boundary hypersurface of $C_X(F_G) \tilde{\times} C_Y(F_H)$ in the case that $(G, H) < (X, Y)$ is the same as the one induced by considering it as a fiber of $L(G, H)$ in $X \tilde{\times} Y$.

The key observation leading to a well-defined theory of products in the fibered corners category is the following result, in which the compressed projection defined in Sect. 3.2 amounts to a natural thickening of the boundary fibrations $\phi_G : G \rightarrow B_G$ to morphisms $\tilde{\phi}_G : N_+G \rightarrow C_X(B_G)$.

Lemma 4.12 *Let $G \in \mathcal{M}_1^>(X)$ and fix a choice of representative boundary defining functions $\{\rho_{G'}\}$ on X , denoting their lifts to N_+G by the same notation.*

(a) *The compressed projection*

$$\tilde{\phi}_G := \tilde{\text{pr}}_{C_X(B_G)} : N_+G \rightarrow C_X(B_G) \quad v \mapsto (\phi_G(v), \rho_{\geq G}(v)) \quad (4.6)$$

is a fibered corners morphism extending $\phi_G : G \rightarrow B_G$, and is independent of the choice of representatives $\{\rho_{G'}\}$ up to an automorphism of $C_X(B_G)$.

(b) *The map (4.6) is functorial in the following sense: Given a morphism $f : X \rightarrow Y$ sending $G > X$ to $H > Y$, there exists a morphism $\tilde{f}_G : C_X(B_G) \rightarrow C_Y(B_H)$ extending $f_G : B_G \rightarrow B_H$ such that the following diagram commutes:*

$$\begin{array}{ccc} N_+G & \xrightarrow{df} & N_+H \\ \tilde{\phi}_G \downarrow & & \downarrow \tilde{\phi}_H \\ C_X(B_G) & \xrightarrow{\tilde{f}_G} & C_Y(B_H) \end{array}$$

Proof The map (4.6) is clearly simple, b-normal, and compatible with the fc-equivalence classes of boundary defining functions on N_+G and $C_X(B_G)$. The fact that it is fibered follows from the commutative diagrams

$$\begin{array}{ccccc} N_+G|_{G \cap G'} & \xrightarrow{\tilde{\phi}_G} & \partial_{G'} B_G \times [0, \infty) & N_+G|_{G \cap G'} & \xrightarrow{\tilde{\phi}_G} & B_G \times \{0\} & G \times \{0\} & \xrightarrow{\tilde{\phi}_G = \phi_G} & B_G \times \{0\} \\ \downarrow d\phi_{G'} & & \downarrow \phi_{G'} & \downarrow d\phi_{G'} & & \downarrow \phi_G & \downarrow \phi_G & & \downarrow 1 \\ B_{G'} & \xrightarrow{1} & B_{G'} & N_+ \partial_G B_{G'} & \xrightarrow{d\phi_{GG'}} & B_G & B_G & \xrightarrow{1} & B_G \end{array}$$

for $G' < G$, $G' > G$, and $G' = G$, respectively. Suppose now that $\{\rho'_{G'}\}$ is another choice of fc-equivalent boundary defining functions on X , defining another such map $\tilde{\phi}'_G : N_+G \rightarrow C_X(B_G)$. Recall that for $G' \neq G$, the normal derivative $d\rho'_{G'} = v^* \rho'_{G'}$ on N_+G is the pullback of the restriction of $\rho'_{G'}$ to G ; hence, the ratio $d\rho'_{G'}/d\rho_{G'} = v^*(\rho'_{G'})/v^*(\rho_{G'})$ is pulled back from B_G by fc-equivalence. Likewise, the $(0, \infty)$ invariant ratio $d\rho'_G/d\rho_G$ is pulled back from B_G , so it follows that $\rho'_{\geq G} = \phi_G^* a \rho_{\geq G}$ for some $a \in C^\infty(B_G; (0, \infty))$, and hence $\tilde{\phi}'_G = \alpha \circ \tilde{\phi}_G$ for the automorphism $\alpha : C_X(B_G) \rightarrow C_X(B_G)$ defined by $\alpha : (b, t) \mapsto (b, a(b)t)$. In particular $\tilde{\phi}_G$ is well defined up to automorphism of the target.

For part (b), we may use representative boundary defining functions with respect to which f is rigid, and then df pulls back $\rho_{\geq H}$ to a function of the form

$$(df)^* \rho_{\geq H} = \rho_{\geq G_H} = \sigma \rho_{\geq G},$$

where $G_H = \min \{G' : f_{\sharp}(G') = H\} \leq G$ and $\sigma = \prod_{G_H \leq G' < G} \rho_{G'}$ is a product of boundary defining functions for those G' such that $G_H \leq G' < G$. As a product of good boundary defining functions for $G' < G$, it follows that $\sigma = \phi_G^* \sigma' \in \phi_G^* C^\infty(B_G)$ is basic. Then, the map

$$\tilde{f}_G : C_X(B_G) \rightarrow C_Y(B_H), \quad (b, t) \mapsto (f_G(b), \sigma'(b)t)$$

has the required property in view of the following commutative diagram:

$$\begin{array}{ccc} v & \xrightarrow{df} & df(v) \\ \tilde{\phi}_G \downarrow & & \downarrow \tilde{\phi}_H \\ & & (\phi_H(df(v)), \rho_{\geq H}(df(v))) \\ & & \parallel \\ (\phi_G(v), \rho_{\geq G}(v)) & \xrightarrow{\tilde{f}_G} & (f_G(\phi_G(v)), \sigma'(\phi_G(v))\rho_{\geq G}(v)) \end{array}$$

□

Theorem 4.13 *If X and Y are manifolds with fibered corners, then $X \tilde{\times} Y$ has a canonical fibered corners structure with fibrations of the form*

$$\begin{array}{ccc} F_G \tilde{\times} F_Y & \xrightarrow{\quad} & L(G, Y) \quad F_X \tilde{\times} F_H \xrightarrow{\quad} L(X, H) \\ \downarrow \phi_{G,Y} & & \downarrow \phi_{X,H} \\ B_G \tilde{\times} B_Y & & B_X \tilde{\times} B_H \end{array} \quad (4.7)$$

$$\begin{array}{ccc} F_G \tilde{\star}_{X,Y} F_H & \xrightarrow{\quad} & L(G, H) \\ \downarrow \phi_{G,H} & & \downarrow \phi_{G,H} \\ B_G \tilde{\times} B_H & & \end{array} \quad \text{for } (G, H) < (X, Y) \quad (4.8)$$

$$\begin{array}{ccc} F_G \tilde{\times} F_H & \xrightarrow{\quad} & L(G, H) \\ \downarrow \phi_{G,H} & & \downarrow \phi_{G,H} \\ B_G \tilde{\star}_{X,Y} B_H & & \end{array} \quad \text{for } (G, H) > (X, Y) \quad (4.9)$$

and fc-equivalence classes of good boundary defining functions given locally by rational combinations of boundary defining functions from X and Y . The ordered product satisfies the universal property of a product in the category of manifolds with fibered corners, meaning that if $f : W \rightarrow X$ and $g : W \rightarrow Y$ are fibered corners morphisms, then the associated morphism $W \rightarrow X \tilde{\times} Y$ of ordered corners is in fact a morphism of fibered corners.

Proof The fibrations on $L(G, H)$ for $(G, H) \leq (X, Y)$, and the fibered property of $W \rightarrow X \tilde{\times} Y$ with respect to these fibrations has been discussed above. The fibrations for $(G, H) > (X, Y)$ are defined as follows. First, we use the canonical diffeomorphism $L(G, H) \cong L(G_0, H_0) \subset N_+G \tilde{\times} N_+H$ as in the proof of Theorem 3.10 to replace $X \tilde{\times} Y$ by $N_+G \tilde{\times} N_+H$. Then, it follows from Lemma 4.12 and Theorem 3.4 that there is a natural morphism (of ordered corners) $N_+G \tilde{\times} N_+H \rightarrow C_X(B_G) \tilde{\times} C_Y(B_H)$ restricting to a fiber bundle $L(G, H) \rightarrow B_G \tilde{\star}_{X,Y} B_H$ with fiber $F_G \tilde{\times} F_H$, and this defines $\phi_{G,H}$, with respect to which the lifted projections $X \tilde{\times} Y \rightarrow X$ and $X \tilde{\times} Y \rightarrow Y$ are clearly fibered.

If $f : W \rightarrow X$ and $g : W \rightarrow Y$ are fibered corners morphisms with the property that $f_{\sharp}(E) = G \in \mathcal{M}_1(X)$ and $g_{\sharp}(E) = H \in \mathcal{M}_1(Y)$ for $E \in \mathcal{M}_1(W)$ with $E > W$ (and hence $G > X$ and $H > Y$), then it follows again from Lemma 4.12 and Theorem 3.4 that there is a commutative diagram of ordered corners morphisms

$$\begin{array}{ccc} N_+E & \longrightarrow & N_+G \tilde{\times} N_+H \\ \downarrow & & \downarrow \\ C_W(B_E) & \rightarrow & C_X(B_G) \tilde{\times} C_Y(B_H) \end{array} \quad \text{restricting to} \quad \begin{array}{ccc} E & \longrightarrow & L(G, H) \\ \downarrow & & \downarrow \\ B_E & \rightarrow & B_G \tilde{\star}_{X,Y} B_H \end{array}$$

and so the map $W \rightarrow X \tilde{\times} Y$ is fibered.

The equivalence classes of boundary defining functions on $X \tilde{\times} Y$ are defined locally by rational combinations of boundary defining functions from X and Y . More precisely, if $\{\rho_G\}$ and $\{\rho_H\}$ are representative boundary defining functions on X and Y , respectively, then on a neighborhood in $X \tilde{\times} Y$ which meets a totally ordered set of boundary hypersurfaces $L(G_i, H_j)$, the linear system

$${}^b d\sigma(g_i + h_j) = \begin{cases} 1, & \text{if } (g_i, h_j) = (g, h), \\ 0, & \text{otherwise} \end{cases}$$

fixes the exponents of $\sigma = \prod_i \rho_{G_i}^{a_i} \prod_j \rho_{H_j}^{b_j}$ uniquely so that it constitutes a local boundary defining function for $L(G, H)$ on such a neighborhood in view of Proposition 2.2. Indeed, the coefficients $\{a_i, b_j\}$ are precisely those for the basis of ${}^b N^*F$, $F = \bigcap_{i,j} L(G_i, H_j)$ which is dual to the basis $\{g_i + h_j\}$ of ${}^b NF$; since the latter is unimodular, it follows that $a_i, b_j \in \{-1, 0, 1\}$. (And in fact either all the a_i are nonnegative, while the b_j are non-positive, or vice versa.) As fc-equivalence is a local property, the equivalence class of boundary defining functions on $X \tilde{\times} Y$ which are locally fc-equivalent to rational combinations as above is well defined.

The fact that these rational boundary defining functions are good is equivalent to the rigidity of $L(G, H) \rightarrow B_G \tilde{\star}_{X,Y} B_H$ for $(G, H) > (X, Y)$ (respectively $L(G, H) \rightarrow B_G \tilde{\times} B_H$ for $(G, H) < (X, Y)$). To see this, first note that $N_+G \rightarrow C_X(B_G)$ (resp. $N_+G \rightarrow B_G$) is rigid by definition and the fact that boundary defining functions on X are good, and hence $N_+G \tilde{\times} N_+H \rightarrow C_X(B_G) \times C_Y(B_H)$ (resp. $N_+G \tilde{\times} N_+H \rightarrow B_G \times B_H$) is rigid with respect to rational boundary defining functions on the domain. It then follows from the local coordinate structure of the unique lift $N_+G \tilde{\times} N_+H \rightarrow C_X(B_G) \tilde{\times} C_Y(B_H)$ (resp. $N_+G \tilde{\times} N_+H \rightarrow B_G \tilde{\times} B_H$) that this lift is rigid with respect to rational boundary defining functions on the domain and codomain, and restricting this to $L(G, H)$ gives the desired result.

The projections $X \tilde{\times} Y \rightarrow X$ and $X \tilde{\times} Y \rightarrow Y$ are compatible with the fc-equivalence classes on $X \tilde{\times} Y$ essentially by definition, and the fact that $W \rightarrow X \tilde{\times} Y$ is compatible follows from the general fact that the lift to a generalized blow-up as in Theorem 2.1.(b) of a locally rigid map is rigid with respect to rational local boundary defining functions. $\square$

We next address the fibered corners structure of the fiber products considered in Sect. 3.3.

Proposition 4.14 *Suppose $f : X \rightarrow Z$ and $g : Y \rightarrow Z$ are morphisms of fibered corners manifolds which are b-transverse. Then, $X \tilde{\times}_Z Y$ obtains a canonical fibered corners structure, with fibrations on the principal faces $L(G, H)_Z := L(G, H) \cap X \tilde{\times}_Z Y$ as follows:*

- For $(G, H) < (X, Y)$ with $G \in \mathcal{M}_1(X)$, $H \in \mathcal{M}_1(Y)$ and $f_{\sharp}(G) = g_{\sharp}(H) = Z$, the fibration has the form

$$\begin{array}{ccc} F_G \tilde{\times}_{X,Y,Z} F_H & \longrightarrow & L(G, H)_Z \\ & \downarrow & \\ & B_G \tilde{\times}_{B_Z} B_H & \end{array}$$

where $F_G \tilde{\times}_{X,Y,Z} F_H$ is the boundary hypersurface of $C_X(F_G) \tilde{\times}_{F_Z} C_Y(F_H)$ of the form (3.26).

- For $(G, H) > (X, Y)$ with $G \in \mathcal{M}_1(X)$, $H \in \mathcal{M}_1(Y)$ and $f_{\sharp}(G) = g_{\sharp}(H) = Z$, the fibration has the form

$$\begin{array}{ccc} F_G \tilde{\times}_{F_Z} F_H & \longrightarrow & L(G, H)_Z \\ & \downarrow & \\ & B_G \tilde{\times}_{X,Y,Z} B_H & \end{array}$$

where $B_G \tilde{\times}_{X,Y,Z} B_H$ is the boundary hypersurface of $C_X(B_G) \tilde{\times}_{B_Z} C_Y(B_H)$ of the form (3.26).

- In all other cases, the fibration has the form

$$\begin{array}{ccc} F_G \tilde{\times}_{F_K} F_H & \longrightarrow & L(G, H)_Z = G \tilde{\times}_K H \\ & \downarrow & \\ & B_G \tilde{\times}_{B_K} B_H & \end{array} \quad \text{where } K = f_{\sharp}(G) = g_{\sharp}(H) \in \mathcal{M}_1(Z).$$

Moreover, $X \tilde{\times}_Z Y$ satisfies the universal property of the fiber product in the category of manifolds with fibered corners.

Proof Suppose first that $G < X$ and $H < Y$ with $f_{\sharp}(G) = g_{\sharp}(H) = Z \in \mathcal{M}_0(Z)$, and replace X and Y by N_+G and N_+H , respectively. From the commutative diagram

$$\begin{array}{ccccc} N_+G & \longrightarrow & Z & \longleftarrow & N_+H \\ & & \downarrow & & \downarrow \\ B_G & \longrightarrow & B_Z & \longleftarrow & B_H \end{array}$$

it follows that $N_+G \tilde{\times}_Z N_+H$ admits a natural map to $B_G \tilde{\times}_{B_Z} B_H$, and restricting over a point in the latter shows that the fiber may be identified with $C_X(F_G) \tilde{\times}_{F_Z} C_Y(F_H)$, as it satisfies the relevant universal property. Restriction to the face $L(G, H)_Z \subset N_+G \tilde{\times}_Z N_+H$ shows that this fibers over $B_G \tilde{\times}_{B_Z} B_H$ with fiber $F_G \tilde{\times}_{X,Y,Z} F_H$, using the characterization of the associated principal face of $C_X(F_G) \tilde{\times}_{F_Z} C_Y(F_H)$ from Theorem 3.13.

In a similar manner, if $G > X$ and $H > Y$ are such that $f_{\sharp}(G) = g_{\sharp}(H) = Z$, then the commutative diagram

$$\begin{array}{ccccc} N_+G & \longrightarrow & Z & \longleftarrow & N_+H \\ & \downarrow & \downarrow & & \downarrow \\ C_X(B_G) & \longrightarrow & B_Z & \longleftarrow & C_Y(B_H) \end{array}$$

obtained from Lemma 4.12 shows that $N_+G \tilde{\times}_Z N_+H$ admits a natural map to $C_X(B_G) \tilde{\times}_{B_Z} C_Y(B_H)$, restricting over a point of which shows that the fiber may be identified with $F_G \tilde{\times}_{F_Z} F_H$. The restriction to $L(G, H)_Z$ maps over the face $B_G \tilde{\times}_{X,Y,Z} B_H \subset C_X(B_G) \tilde{\times}_{B_Z} C_Y(B_H)$, giving the result in this case.

In all other cases, $L(G, H)_Z \cong G \tilde{\times}_K H$, where $K = f_{\sharp}(G) = g_{\sharp}(H) \in \mathcal{M}_1(Z)$, fits into the diagram

$$\begin{array}{ccccc} & & G \tilde{\times}_K H & & \\ & \swarrow & \downarrow & \searrow & \\ G & \xrightarrow{\quad} & K & \xleftarrow{\quad} & H \\ \downarrow & & \downarrow & & \downarrow \\ B_G & \longrightarrow & B_K & \longleftarrow & B_H \end{array}$$

and consequently maps naturally to $B_G \tilde{\times}_{B_K} B_H$, and restriction over a point in the latter shows that the fiber satisfies the universal property of $F_G \tilde{\times}_{F_K} F_H$, proving the result in these cases.

Note that in all cases, the fibration just described coincides with the restriction to $L(G, H)_Z$ of the fibration $\phi_{G,H}$ on $L(G, H)$. This is clear in all cases, except perhaps the last case when $(G, H) > (X, Y)$, but in that case the diagram of relevant maps

$$\begin{array}{ccccc} N_+G & \longrightarrow & N_+K & \longleftarrow & N_+H \\ \downarrow & & \downarrow & & \downarrow \\ C_X(B_G) & \rightarrow & C_Z(B_K) & \leftarrow & C_Y(B_H) \end{array}$$

leads to the natural map $N_+G \tilde{\times}_{N_+K} N_+H \rightarrow C_X(B_G) \tilde{\times}_{C_Z(B_K)} C_Y(B_H)$, and arguing as in the proof of Theorem 3.13 shows that the relevant boundary face of $C_X(B_G) \tilde{\times}_{C_Z(B_K)} C_Y(B_H)$ satisfies the universal property of $B_G \tilde{\times}_{B_K} B_H$, to which the restriction to $L(G, H)_Z$ maps.

The fc-equivalence classes of boundary defining functions of $X \tilde{\times}_Z Y$ are inherited by restriction from the classes on $X \tilde{\times} Y$, and the fact that these are good with respect to the fibrations described amounts to rigidity of all maps just described with respect to the fc-equivalence classes on N_+G , N_+H , and so on. $\square$

Returning to the application discussed in Corollary 3.14 in the fibered corners setting yields the following.

Corollary 4.15 *If X and Y are manifolds with fibered corners with representative total boundary defining functions $\rho_X = \prod_{G \in \mathcal{M}_1(X)} \rho_G : X \rightarrow [0, \infty)_{\min}$ and $\rho_Y = \prod_{H \in \mathcal{M}_1(Y)} \rho_H : Y \rightarrow [0, \infty)_{\min}$, then the fiber product*

$$X \tilde{\times}_{[0, \infty)_{\min}} Y = \{\rho_X / \rho_Y = 1\} \subset X \tilde{\times} Y$$

is a p -submanifold in a sufficiently small neighborhood of the boundary (where $d\rho_X \neq 0$ and $d\rho_Y \neq 0$), and has fibrations of the form

$$\begin{array}{c} F_G \tilde{\times} F_H \longrightarrow L(G, H) \cap \{\rho_X / \rho_Y = 1\} \\ \downarrow \phi_{G,H} \\ B_G \tilde{\times} B_H \end{array}$$

with fiber $F_G \tilde{\times} F_H = (F_G)_{\min} \tilde{\times} (F_H)_{\min}$ and base $B_G \tilde{\times} B_H = (B_G)_{\max} \tilde{\times} (B_H)_{\max}$.

5 Relation to other theories

In this section we discuss how products of fibered corners manifolds relate and/or generalize products in other categories.

5.1 Smoothly stratified spaces

We briefly recall the definition of smoothly stratified spaces, referring to [1, 25] for a more complete account. A *stratified space* is a topological space $\mathcal{X}$ decomposed as a disjoint union $\mathcal{X} = \bigsqcup_i s_i$ of locally closed subspaces s_i called *strata*, each of which is itself a manifold of some dimension, satisfying the *frontier condition* that $s_i \cap \overline{s_j} \neq \emptyset$ if and only if $s_i \subset \overline{s_j}$, which defines a partial order $s_i < s_j$. In particular, there is a maximal *principal stratum* $s_0 = \mathcal{X}^\circ$ which is dense, with the rest referred to as the *singular strata*. The closed strata $\overline{s_i}$ are again stratified spaces with strata $\{s_j : s_j < s_i\}$. A *smoothly stratified space* (aka ‘Thom-Mather stratified space’) is a stratified space $\mathcal{X}$ along with *control data* consisting of open neighborhoods $\mathcal{T}_i \supset s_i$ in $\mathcal{X}$ called *tubes*, equipped with retractions $r_i : \mathcal{T}_i \rightarrow s_i$ and distance functions $\rho_i : \mathcal{T}_i \rightarrow \mathbb{R}_+$ such that $s_i = \rho_i^{-1}(0)$ and satisfying the conditions that whenever $s_i < s_j$,

$$\rho_i \circ r_j = \rho_i, \quad r_i \circ r_j = r_i, \quad \text{and} \quad (r_i, \rho_i) : \mathcal{T}_i \cap s_j \rightarrow s_i \times (0, \infty) \text{ is a submersion.}$$

As a consequence of Thom’s isotopy lemma for smoothly stratified spaces (see [25, 3.9.2]), the $\mathcal{T}_i$ are locally trivial bundles of cones over s_i , the links of which are stratified spaces of lower depth.

Associated with each interior maximal manifold with fibered corners X is a smoothly stratified space $\mathcal{X}$ obtained by iteratively collapsing the fibers over each boundary hypersurface in reverse order (starting with the largest base and ending with the smallest). The strata of $\mathcal{X}$ are the smooth manifolds $s_G = B_G^\circ$, for $G \in \mathcal{M}_{1,0}(X)$, with principal smooth stratum $\mathcal{X}^\circ$, and the control data are given by the passage to the quotient of tubular neighborhoods of the boundary hypersurfaces, their associated retractions, and the images of (good) boundary defining functions. The closure $\overline{s_G} = B_G$ is the stratified space obtained by fiber collapse of the fibered corners manifold B_G itself, and the partial order $s_G < s_{G'}$ on strata defined by $s_G \subset \overline{s_{G'}}$ coincides with the order $G < G'$ induced by the fibered corners structure. The link of the cone bundle $\mathcal{T}_G \rightarrow s_G$ is the stratified space $\mathcal{F}_G$ obtained by collapsing the fibers of the boundary hypersurfaces of F_G .

Conversely, as shown in [1, 3] (a result originating in [32]; see also [25, 3.9.4]), every smoothly stratified space can be resolved to a manifold with fibered corners by iteratively replacing the cone bundle $\mathcal{T}_G$ of each singular stratum with an associated cylinder bundle in order from smallest stratum to largest, with the tubes $\mathcal{T}_G$ resolving to tubular neighborhoods of the boundary hypersurfaces, and distance functions resolving to boundary defining functions.

Remark While this constitutes an equivalence of objects between smoothly stratified spaces on the one hand, and (interior maximal) manifolds with fibered corners on the other hand, correspondingly little has been written about morphisms. In [3], Albin et. al. show that the resolution procedure above is functorial with respect to *isomorphisms*, meaning controlled isomorphisms of stratified spaces on the one hand, and fibered diffeomorphisms of fibered corners manifolds on the other, but in general, pinning down a suitable class of morphisms on the stratified spaces side is difficult. In addition to the requisite condition that a map $f : \mathcal{X} \rightarrow \mathcal{Y}$ be *stratified*, meaning that it sends strata to strata and restricts to a smooth map of manifolds thereon (which corresponds to the fibered condition of Definition 4.3), the typical conditions discussed in the literature are inadequate for our purposes. Indeed, the weakest typical condition is that the map f is ‘weakly controlled,’ meaning it maps tubes to tubes and commutes with the retractions. This is on the one hand too strong, since it amounts to extending the boundary fibrations to open neighborhoods of the boundary hypersurfaces and demanding an extension of the fiber bundle maps (4.3) to such neighborhoods, and it

does not seem that every fibered corners morphism satisfies such a condition. On the other hand, it is also too weak, since it does not impose the kind of algebraic behavior with respect to boundary defining functions that a general b-map satisfies. From our point of view, the right class of morphisms are those maps of smoothly stratified spaces which are descended from fibered corners morphisms as defined here, but it seems difficult to characterize these directly using only the data of stratified spaces.

There is little discussion of products of stratified spaces in the literature, but the definition of a smoothly stratified structure on the Cartesian product $\mathcal{X} \times \mathcal{Y}$ is obvious enough. It has strata $s_G \times s_H$ for $(G, H) \in \mathcal{M}_{1,0}(X) \times \mathcal{M}_{1,0}(Y)$, with control data $(\mathcal{T}_G \times \mathcal{T}_H, r_G \times r_H, \rho_G + \rho_H)$ (it seems any homogeneous combination of ρ_G and ρ_H would also suffice, such as $(\rho_G^p + \rho_H^p)^{1/p}$ for $p \geq 1$). This satisfies the universal property of the product for most reasonable notions of morphism of stratified spaces.

If we knew that the projections $\mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{X}$ and $\mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Y}$ lifted to fibered corners morphisms upon resolution, as well as the map into $\mathcal{X} \times \mathcal{Y}$ induced by a good map $\mathcal{W} \rightarrow \mathcal{X}$ and $\mathcal{W} \rightarrow \mathcal{Y}$, then an easy application of the universal property of the product in both categories would give a quick proof of the following result. However, since this is unavailable at present, we settle for a coordinate based proof.

Theorem 5.1 *Let X and Y be interior maximal manifolds with fibered corners, with associated smoothly stratified spaces $\mathcal{X}$ and $\mathcal{Y}$. Then, the ordered product $X \times Y$ is equivalent to the resolution of the product $\mathcal{X} \times \mathcal{Y}$ of stratified spaces.*

Proof Let

$$(y_1, x_1, y_2, x_2, \dots, y_n, x_n, y_{n+1}) \quad (5.1)$$

denote standard form coordinates on a (maximal) fibered corners manifold X . Iteratively collapsing the boundary fibers leads to singular coordinates

$$(y_1, r_1, r_1 y_2, r_2, r_2 y_3, \dots, r_n, r_n y_{n+1}), \quad r_i = r_{i-1} x_i = x_1 \cdots x_i$$

on the associated stratified space $\mathcal{X}$; conversely, the resolution procedure is implemented in coordinates by iteratively dividing the coordinates to the right of each r_i by r_i starting with r_1 .

Likewise, if Y is another fibered corners manifold with standard form coordinates

$$(y'_1, x'_1, y'_2, x'_2, \dots, y'_n, x'_n, y'_{n+1}) \quad (5.2)$$

with associated singular coordinates

$$(y'_1, r'_1, r'_1 y'_2, r'_2, r'_2 y'_3, \dots, r'_m, r'_m y'_{m+1})$$

on $\mathcal{Y}$, then singular coordinates on $\mathcal{X} \times \mathcal{Y}$ are given by

$$(y_1, y'_1, r_1, r_1 y_2, r'_1, r'_1 y'_2, \dots, r_n, r_n y_{n+1})$$

assuming without loss of generality that $m \leq n$. We will focus attention on the totally ordered chain of strata $(1, 1) < (1, 2) < (2, 3) < \cdots < (m-1, m) < (m, 0) < \cdots < (n, 0)$; others are similar.

The resolution procedure consists of first replacing r_1 and r'_1 by $r_1 + r'_1$ and $s_1 = r_1 / (r_1 + r'_1)$ (or $s'_1 = 1 - s_1 = r'_1 / (r_1 + r'_1)$), dividing all coordinates to the right by $r_1 + r'_1$. Then, near the lift of (say) $s'_1 = 0$ and $r_2 = 0$, the next step is to replace s'_1 and $s_2 = r_2 / (r_1 + r'_1)$ by $s'_1 + s_2$ and $s_2 / (s'_1 + s_2) = r_2 / (r'_1 + r_2)$, divide all coordinates to the right by $s'_1 + s_2$, and so

on. However, as is frequently the case with blow-up computations, it is more convenient here to use equivalent rational coordinates, in which (r_1, r'_1) is replaced instead by $(r_1, r'_1/r_1)$ and all coordinates to the right are divided by r_1 ; then $(r'_1/r_1, r_2/r_1)$ is replaced by $(r'_1/r_1, r_2/r'_1)$ and all coordinates to the right are divided by r'_1/r_1 , and so on. The end of this process results in standard form coordinates

$$(y_1, y'_1, x_1, y_2, \frac{x'_1}{x_1}, y'_2, \frac{x_1 x'_2}{x'_1}, y_3, \frac{x'_1 x'_2}{x_1 x'_2}, y'_3, \dots, y'_{m+1}, \frac{x_1 \dots x_{m+1}}{x'_1 \dots x'_m}, y_{m+1}, \dots, x_n, y_{n+1})$$

for the resolution of $\mathcal{X} \times \mathcal{Y}$ which are identical to the rational coordinates near $L(G_1, H_1) \cap L(G_1, H_2) \cdots \cap L(G_n, Y)$ for the ordered product $X \tilde{\times} Y$ based on (5.1) and (5.2). Other coordinate charts based on other totally ordered chains of strata involve similar computations. $\square$

If X is a manifold with fibered corners, then the natural fibered corners structure on the cone $C_{\max}(X)$ according to Definition 4.11 consists of fibrations

$$X \times 0 \rightarrow *, \quad G \times [0, \infty) \rightarrow B_G \times [0, \infty), \quad X \times [0, \infty) \rightarrow X \times [0, \infty).$$

It is clear that the stratified space associated with $C_{\max}(X)$ is the cone

$$\mathcal{C}(\mathcal{X}) = (\mathcal{X} \times [0, \infty)) / (\mathcal{X} \times \{0\})$$

with link $\mathcal{X}$. From the well-known fact that the product of cones is a cone over the join of the links, we obtain the following.

Corollary 5.2 *For fibered corners manifolds X and Y , the stratified space associated with $C_{\max}(X) \tilde{\times} C_{\max}(Y)$ is the cone*

$$\mathcal{C}(\mathcal{X}) \times \mathcal{C}(\mathcal{Y}) \cong \mathcal{C}(\mathcal{X} \star \mathcal{Y})$$

and in particular, the stratified space associated with the maximal join $X \tilde{\star}_{\max} Y$ is the topological join

$$\mathcal{X} \star \mathcal{Y} = (\mathcal{X} \times \mathcal{Y} \times [0, 1]) / \sim,$$

where $(x, y, 0) \sim (x', y, 0)$ for all $x, x' \in \mathcal{X}$ and $y \in \mathcal{Y}$ and $(x, y, 1) \sim (x, y', 1)$ for all $x \in \mathcal{X}, y, y' \in \mathcal{Y}$.

5.2 Many-body spaces

Many-body compactifications of vector spaces go back at least to [31], and have been discussed more recently in [6, 21]. Using notation from [21], recall that given a finite-dimensional vector space V and a linear system S_V , meaning a finite set of subspaces of V which is closed under intersection and contains $\{0\}$ and V , the many-body compactification $M(V)$ of V is the manifold with corners

$$M(V) = [\overline{V}; \{\partial \overline{S} : S \in S_V\}],$$

given by the blow-up in the radial compactification $\overline{V}$ of the boundaries of the subspaces in S_V , taken in order of size. The boundary hypersurfaces of $M(V)$ are indexed by the subspaces in S_V , ordered by containment; in fact $\mathcal{M}_{1,0}(M(V)) \cong S_V$ as ordered sets, with the interior indexed by $\{0\}$. In addition, $M(V)$ has a natural interior minimal fibered corners structure, with fibrations consisting of products

$$\phi_S : H_S = B_S \times F_S \rightarrow B_S, \quad \text{where } F_S = M(V/S), \quad B_S = \left[\partial \bar{S}; \left\{ \partial \bar{S}' : S' < S \in \mathcal{S}_V \right\} \right] \quad (5.3)$$

for each $S \in \mathcal{S}_V$, where V/S is equipped with the linear system $\mathcal{S}_{V/S} = \{S'/(S' \cap S) : S' \in \mathcal{S}_V\}$, and $B_S = H_S \subset M(S)$ is the maximal boundary face of the many-body compactification of S itself, with respect to the system $\mathcal{S}_S = \{S' \in \mathcal{S}_V : S' \subset S\}$. The boundary hypersurfaces of B_S decompose as

$$\partial_{S'} B_S = B_{S'} \times B_{S/S'}, \quad S' < S. \quad (5.4)$$

The fc-equivalence class of boundary defining functions on $M(V)$ is well defined by lifting rational combinations of radial functions on $\bar{V}$ which define $\partial \bar{S}$ for $S \in \mathcal{S}_V$.

As shown in [21], the many-body compactification is functorial with respect to *admissible* linear maps, meaning $f : V \rightarrow W$ satisfies $f^{-1}(\mathcal{S}_W) \subset \mathcal{S}_V$, with such f extending to b-maps $\hat{f} : M(V) \rightarrow M(W)$. If in addition $f(\mathcal{S}_V) = \mathcal{S}_W$ — what is called in [21] an *admissible quotient* — then $\hat{f}$ is a b-fibration and satisfies the conditions to be a morphism in the category of interior minimal fibered corners. Indeed, the restriction of the lifted map splits as a product with respect to (5.3) and (5.4).

It is clear that the stratified space associated with $M(V)$ by collapsing the fibers of all boundary fibrations (we do not collapse the fibers over the interior) is the radial compactification $\bar{V}$, with principal stratum V , and (closed) singular strata consisting of the boundaries $\partial \bar{S}$ of the subspaces in $\mathcal{S}_V$, the links of which are the normal quotients $\bar{V}/S$ with strata $\partial \bar{S}'/S$ for $S' > S$. In this category, a natural alternative to the usual product of $\bar{V}$ and $\bar{W}$ as stratified spaces (which does not resolve to a many-body compactification of a vector space) is the space $\bar{V} \times \bar{W}$ with closed singular strata $\partial(\bar{S} \times \bar{T})$ for $(S, T) \in \mathcal{S}_V \times \mathcal{S}_W$, the resolution of which is the many-body compactification $M(V \times W)$.

Note that, as spheres, the closed singular strata $\partial(\bar{S} \times \bar{T})$ are homeomorphic to the topological joins $\partial \bar{S} \star \partial \bar{T}$ of the singular strata of the factors, a property which generalizes to the interior minimal fibered corners setting since $B_G \tilde{\star}_{\min} B_H$ has associated stratified space the join $B_G \star B_H$ of the stratified spaces associated with B_G and B_H as a consequence of the results of §7.

Theorem 5.3 *Let $M(V)$ and $M(W)$ be many-body compactifications of $(V, \mathcal{S}_V)$ and $(W, \mathcal{S}_W)$, respectively. Then, the ordered product $M(V) \tilde{\times} M(W)$ as interior minimal manifolds with fibered corners is canonically isomorphic to the many-body compactification $M(V \times W)$ of $(V \times W, \mathcal{S}_V \times \mathcal{S}_W)$.*

Proof The lifted projections $M(V \times W) \rightarrow M(V)$ and $M(V \times W) \rightarrow M(W)$ are interior minimal morphisms, so by the universal property there is an associated map $h : M(V \times W) \rightarrow M(V) \tilde{\times} M(W)$, with induced map $h_{\sharp} : \mathcal{M}_{1,0}(M(V \times W)) \rightarrow \mathcal{M}_{1,0}(M(V) \tilde{\times} M(W))$ an isomorphism of ordered sets identified with $\mathcal{S}_{V \times W} \cong \mathcal{S}_V \times \mathcal{S}_W$. Both $M(V \times W)$ and $M(V) \tilde{\times} M(W)$ have interior identified with $V \times W$, on which h restricts to the identity, so it remains to show that h extends to an isomorphism over each boundary hypersurface. We proceed by induction on the total depth of (V, W) , meaning the sum $|\mathcal{S}_V| + |\mathcal{S}_W|$, which is equivalent to the depth of $M(V \times W)$ or $M(V) \tilde{\times} M(W)$ as fibered corners manifolds, with depth zero coinciding with the trivial case $V = W = \{0\}$.

The restriction to a hypersurface $h : H_{S \times T} \subset M(V \times W) \rightarrow L(H_S, H_T) \subset M(V) \tilde{\times} M(W)$ is a fibered map

$$\begin{array}{ccc} F_{S \times T} \times B_{S \times T} & \xrightarrow{h} & (F_S \tilde{\times} F_T) \times (B_S \tilde{\star} B_T) \\ \downarrow & & \downarrow \\ B_{S \times T} & \xrightarrow{h_{S \times T}} & B_S \tilde{\star} B_T \end{array} \quad (5.5)$$

where $\tilde{\star} = \tilde{\star}_{\min}$, with $B_S \tilde{\star} B_T$ replaced simply by B_S or B_T in case $T = 0$ or $S = 0$, respectively. Restricting over a point in the base and recalling that $F_S = M(V/S)$, $F_T = M(W/T)$, and $F_{S \times T} = M((V \times W)/(S \times T)) = M(V/S \times W/T)$, it is clear that the map on fibers is the universal map $M(V/S \times W/T) \rightarrow M(V/S) \tilde{\times} M(W/T)$, in particular independent of the point in the base, and hence the top row of (5.5) splits as a product. Since $(S, T) \neq (0, 0)$ here (as we are considering a proper boundary hypersurface), the pair $(V/S, W/T)$ has strictly lower total depth, and so the first factor $F_{S \times T} \rightarrow F_S \tilde{\times} F_T$ is an isomorphism by the inductive hypothesis.

Next we recall that the lower map $h_{S \times T}$ is induced by a thickened map $C_{\min}(B_{S \times T}) \rightarrow C_{\min}(B_S) \tilde{\times} C_{\min}(B_T)$ afforded by Lemma 4.12; however in this many-body setting this has a natural reinterpretation. Indeed, here we may generally identify the cone $C_{\min}(B_S)$ with the normal model $N_+ B_S$ for the maximal hypersurface $B_S \subset M(S)$ in the many-body compactification for S itself, and the thickened horizontal map on cones in part (b) of Lemma 4.12 then coincides with the normal derivative of the induced map between the many-body compactifications of the relevant subspaces. In particular the map $B_{S \times T} \rightarrow B_S \tilde{\star} B_T$ coincides with the restriction to the maximal boundary hypersurface of the map $M(S \times T) \rightarrow M(S) \tilde{\times} M(T)$, and again by induction it follows that the latter is an isomorphism, *except* in the case that $(S, T) = (V, W)$ itself.

To take care of this final case, we verify by hand that the map $M(V \times W) \rightarrow M(V) \tilde{\times} M(W)$ has full rank over the interior of the maximal boundary hypersurface $H_{V \times W} = B_{V \times W}$. Representing points in $V \times W$ by ‘product radial’ coordinates

$$(v, w) = (Rr\nu, Rs\omega), \quad \text{where } R = \sqrt{|v|^2 + |w|^2}, \quad r = \frac{|v|}{R}, \quad s = \frac{|w|}{R} = \sqrt{1 - r^2}$$

where ν and ω are respective unit vectors for some norms on V and W , leads to coordinates (x, r, ν, ω) near the boundary of the radial compactification $\bar{V} \times \bar{W}$, with $x = R^{-1}$. Assuming now that $r \in (0, 1)$ and ν and ω are limited to open sets disjoint from any of the proper subspaces in $\mathcal{S}_V$ or $\mathcal{S}_W$, these remain valid coordinates for the interior of the maximal boundary face $B_{V \times W} \subset M(V \times W)$. Likewise, (x', ν) , where $x' = |v|^{-1}$ and (x'', ω) , where $x'' = |w|^{-1}$ form coordinates for $\bar{V}$ and $\bar{W}$ lifting to $M(V)$ and $M(W)$, and the lifted projections $M(V \times W) \rightarrow M(V)$ and $M(V \times W) \rightarrow M(W)$ are given respectively by

$$(x, r, \nu, \omega) \mapsto \left(\frac{x}{r}, \nu\right) \quad \text{and} \quad (x, r, \nu, \omega) \mapsto \left(\frac{x}{\sqrt{1-r^2}}, \omega\right).$$

It follows that coordinates near the maximal boundary hypersurface of $M(V) \tilde{\times} M(W)$ are given by $(x', \frac{x'}{x''}, \nu, \omega)$, in which the lifted map $M(V \times W) \rightarrow M(V) \tilde{\times} M(W)$ takes the form

$$(x, r, \nu, \omega) \mapsto \left(\frac{x}{r}, \frac{r}{\sqrt{1-r^2}}, \nu, \omega\right)$$

which has full rank down to $\{x = 0\}$ for $r \in (0, 1)$. $\square$

6 Geometric structures

In this section we consider certain classes of metrics on interior minimal and maximal manifolds with fibered corners. A convenient and equivalent way to discuss these classes of metrics is in terms of their associated *geometric structures*, as encoded by associated *rescaled tangent bundles*, on which a Riemannian metric of the given type on the interior extends to a uniformly bounded inner product up to all boundary faces. Both structures to be considered begin with the following.

Definition 6.1 The algebra of *edge vector fields* is defined as

$$\mathcal{V}_e(X) = \{V \in \mathcal{V}_b(X) : V|_G(\phi_G^*(C^\infty(B_G))) = 0\}$$

or those vector fields whose restriction to every $G \in \mathcal{M}_1(X)$ is tangent to the fiber of ϕ_G . This definition has appeared in various forms in [2–4] and others sometimes under the name ‘iterated edge’ or ‘ie,’ and extends the definition due to [26] in the depth 1 case. This is a Lie subalgebra of $\mathcal{V}_b(X)$ and forms a locally free sheaf of constant rank, defining the *edge tangent bundle* by

$${}^eTX \rightarrow X, \quad \text{where } \mathcal{V}_e(X) = C^\infty(X; {}^eTX). \quad (6.1)$$

In local standard form coordinates, a frame for eTX is furnished by the vector fields

$$v_1 \frac{\partial}{\partial y_1}, v_1 \frac{\partial}{\partial x_1}, v_2 \frac{\partial}{\partial y_2}, v_2 \frac{\partial}{\partial x_2}, \dots, v_n \frac{\partial}{\partial y_n}, v_n \frac{\partial}{\partial x_n}, \frac{\partial}{\partial z}, \quad v_k = x_k \cdots x_n, \quad (6.2)$$

where $\frac{\partial}{\partial y_j}$ is shorthand for various partial derivatives $\frac{\partial}{\partial y_j^I}$ associated with the components of $y_j \in \mathbb{R}^{k_j}$ and we use $\frac{\partial}{\partial z}$ in place of $\frac{\partial}{\partial y_{n+1}}$. Since these vector fields will be of interest in both the interior maximal and minimal cases, and since the interior fibration will not play a role, we enumerate boundary hypersurfaces exclusively with positive integers in all cases throughout this section.

The vector fields of particular interest in the setting of interior maximal manifolds are the following.

Definition 6.2 A *wedge vector field* (called an ‘iterated incomplete edge’ vector field in [3, 4], with the name ‘wedge’ coming from [14] in the depth 1 case) is one of the form $\frac{1}{\rho_X}V$, where $V \in \mathcal{V}_e(X)$ and $\rho_X = \prod_{G \in \mathcal{M}_1(X)} \rho_G$ is a total boundary defining function. Such a vector field is singular at the boundary faces of X , and the set $\mathcal{V}_w(X)$ of wedge vector fields does not form a Lie algebra. Instead the *wedge tangent bundle* may be defined as a rescaling (see [30] for a general discussion of this procedure) of eTX by the requirement

$$\rho_X C^\infty(X; {}^wTX) = C^\infty(X; {}^eTX).$$

(See [2] for an alternate definition via the cotangent space ${}^wT^*X$.) Note that $\mathcal{V}_w(X)$ and wTX are independent of the choice of ρ_X . A local frame for wTX is given in standard form coordinates by

$$\frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_1}, w_1 \frac{\partial}{\partial y_2}, w_1 \frac{\partial}{\partial x_2}, \dots, w_{n-1} \frac{\partial}{\partial y_n}, w_{n-1} \frac{\partial}{\partial x_n}, w_n \frac{\partial}{\partial z}, \quad w_k = \frac{1}{x_1 \cdots x_k}. \quad (6.3)$$

As for bTX , both eTX and wTX bundles are canonically isomorphic to the usual tangent bundle over the interior of X , and a *wedge metric* on X is then by definition the Riemannian metric on X° induced by uniformly bounded inner product on wTX which is smooth on the interior and conormal up to the boundary (i.e., all b -derivatives are also uniformly bounded; this is in accord with the usual geometric analysis definition). Wedge metrics are incomplete on X (hence the term ‘incomplete iterated edge metric’ of [3, 4]), and contain the iterated conic metrics studied by Cheeger in [9] as special cases. While a general wedge metric is defined by an arbitrary inner product on wTX , a model wedge metric in standard form coordinates which shows off the iterated conic aspect of the geometry (and an example of what is called a ‘rigid wedge metric’ in [3]) is one of the form

$$\begin{aligned} g_X &= dy_1^2 + dx_1^2 + x_1^2(dy_2^2 + dx_2^2 + \cdots + x_{n-1}^2(dy_n^2 + dx_n^2 + x_n^2 dz^2) \cdots) \\ &= g_{B_1} + d\rho_1^2 + \rho_1^2 \kappa_{F_1} \end{aligned}$$

where dy_i^2 is shorthand for an inner product in the variables $y_i \in \mathbb{R}^{k_i}$, g_{B_1} denotes a Riemannian metric on B_1 and κ_{F_1} denotes a family of wedge metrics on F_1 .

In contrast, the following structure is of interest particularly in the setting of interior minimal manifolds.

Definition 6.3 The algebra of *quasi fibered boundary (QFB)*, or simply Φ vector fields is the set

$$\mathcal{V}_\Phi(X) = \{V \in \mathcal{V}_e(X) : V\rho_X \in \rho_X^2 C^\infty(X)\},$$

consisting of those edge vector fields which annihilate a total boundary defining function ρ_X to second order [10]. While $\mathcal{V}_e(X)$ and $\mathcal{V}_w(X)$ only depend on the fibered corners structure, $\mathcal{V}_\Phi(X)$ also depends on the choice of total boundary defining function ρ_X up to equivalence, where two total boundary defining functions ρ_X and ρ'_X are equivalent if their ratio is basic [10]. In particular, $\mathcal{V}_\Phi(X)$ is well defined by a choice of fc-equivalence classes of boundary defining functions on X .

As with edge vector fields above, $\mathcal{V}_\Phi(X)$ is a Lie subalgebra of $\mathcal{V}_b(X)$ and forms a locally free sheaf of constant rank, defining the Φ *tangent bundle* (which is again canonically isomorphic to TX in the interior) by

$${}^\Phi TX \rightarrow X, \quad \text{where } \mathcal{V}_\Phi(X) = C^\infty(X; {}^\Phi TX). \quad (6.4)$$

In local standard form coordinates a local frame for ${}^\Phi TX$ is given by

$$\begin{aligned} &v_1 \frac{\partial}{\partial y_1}, x_1 v_1 \frac{\partial}{\partial x_1}, v_2 \frac{\partial}{\partial y_2}, v_2 \left(x_2 \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_1} \right), \\ &\quad \dots, v_n \frac{\partial}{\partial y_n}, v_n \left(x_n \frac{\partial}{\partial x_n} - x_{n-1} \frac{\partial}{\partial x_{n-1}} \right), \frac{\partial}{\partial z}, \\ &v_k = x_k \cdots x_n. \end{aligned} \quad (6.5)$$

Note that the basis element $v_1 x_1 \frac{\partial}{\partial x_1}$ can be replaced by $v_1 x_k \frac{\partial}{\partial x_k}$ for any k by taking appropriate linear combinations.

A Φ metric on X is by definition a (complete) Riemannian metric on X° induced by a uniformly bounded inner product on ${}^\Phi TX$. Note that such a metric need not be smooth up to the boundary. When the fibers of the maximal hypersurfaces are trivial, a Φ metric is

known as a ‘quasi-asymptotically conical’ (QAC) metric [10, 12]. A particularly nice model Φ metric in local standard form coordinates is one of the form

$$\begin{aligned} \frac{dv_1^2}{v_1^4} + \frac{1}{v_1^2} (dy_1^2 + dx_1^2 + x_1^2(dy_2^2 + dx_2^2 + \cdots + x_{n-1}^2(dy_n^2 + x_n^2 dz^2) \cdots)) \\ = \frac{dv_1^2}{v_1^4} + \sum_{i=1}^{n-1} \frac{dx_i^2}{v_i^2} + \sum_{i=1}^n \frac{dy_i^2}{v_i^2} + dz^2. \end{aligned}$$

where again dy_i^2 is shorthand for an inner product in the variables $y_i \in \mathbb{R}^{k_i}$. Note that such a metric can support additional terms of the form $\frac{dv_i^2}{v_i^4}$, and in particular restricts to a Φ metric on any interior fiber of a boundary hypersurface.

6.1 Products of Φ structures

We first consider the product of Φ structures on interior minimal manifolds X and Y with fibered corners, proving an isomorphism between ${}^\Phi T X \oplus {}^\Phi T Y$ and ${}^\Phi T(X \times Y)$. In fact we give two proofs, one based on direct analysis of Φ -tangent bundles using Proposition 6.5, which decomposes ${}^\Phi T X$ with respect to the bundle structure of the compressed projection $\tilde{\phi}_H : N_+ H \rightarrow C_X(B_H)$ of Lemma 4.12. The second proof is based on an analysis of Φ metrics taken to be in a convenient local form.

The Φ structure on $X \times Y$ is well defined by the fc-equivalence class of boundary defining functions given by local rational combinations of representative boundary defining functions on X and Y , as discussed in Sect. 4.1. However, the following characterization of total boundary defining functions on $X \times Y$ is conceptually satisfying.

Proposition 6.4 *Let ρ_X and ρ_Y be representative total boundary defining functions on X and Y . Then, the reciprocal p -sum*

$$\rho_{X,Y,p} = (\rho_X^{-p} + \rho_Y^{-p})^{-1/p} = \frac{\rho_X \rho_Y}{(\rho_X^p + \rho_Y^p)^{1/p}}$$

is a representative total boundary defining function on $X \times Y$ for any $1 \leq p \in \mathbb{R}$.

Proof Consider a neighborhood in $X \times Y$ of maximal depth, meeting a maximal totally ordered chain of boundary hypersurfaces, which without loss of generality (exchanging the role of X and Y if necessary) contains hypersurfaces of the form $L(G, Y)$ but none of the form $L(X, H)$. Denote by ρ_G and ρ'_H representative boundary defining functions for $G \in \mathcal{M}_1(X)$ and $H \in \mathcal{M}_1(Y)$, respectively, and denote by $g = \rho_G \partial_{\rho_G}$ and $h = \rho'_H \partial_{\rho'_H}$ the generators of the associated monoids.

We claim that in such a neighborhood, $\rho_X = \prod_{G \in \mathcal{M}_1(X)} \rho_G$ is a representative total boundary defining function on $X \times Y$, while $\rho_X / \rho_Y = \prod_G \rho_G \prod_H \rho'^{-1}_H$ is basic. Indeed, both are rational combinations, and $\text{bd} \rho_X$ pairs to 1 with every generator $g + h$ or $g + 0$ of the associated monoid characterizing this neighborhood of $X \times Y$, so it is locally a total boundary defining function. Likewise, $\text{bd}(\rho_X / \rho_Y)$ pairs to 1 with every generator of the form $g + 0$ and to 0 with every generator of the form $g + h$, so it is locally the product of boundary defining functions for faces of the form $L(G, Y)$. On any such face ρ_X / ρ_Y vanishes identically, while at any face of the form $L(G', H)$ with $Y < H$, the boundary defining functions for $L(G, Y)$ are all basic since $L(G, Y) < L(G', H)$, thus ρ_X / ρ_Y is basic.

The result now follows from the observation that

$$\frac{\rho_X}{\rho_{X,Y,p}} = \frac{(\rho_X^p + \rho_Y^p)^{1/p}}{\rho_Y} = \left(\left(\frac{\rho_X}{\rho_Y} \right)^p + 1 \right)^{1/p}$$

is a smooth function of ρ_X/ρ_Y , hence basic, so ρ_X and $\rho_{X,Y,p}$ are locally equivalent for any p . $\square$

Proposition 6.5 *Let $G \in \mathcal{M}_1(X)$ where X is an interior minimal manifold with fibered corners, with normal model N_+G . Then, the differential of the compressed projection $\tilde{\phi}_G : N_+G \rightarrow C_X(B_G) = C_{\min}(B_G)$ of Lemma 4.12 extends by continuity from the interior to a short exact sequence*

$$0 \rightarrow {}^\Phi T F_G \rightarrow {}^\Phi T N_+G \rightarrow {}^\Phi T C_{\min}(B_G) \rightarrow 0 \quad (6.6)$$

of vector bundles over $G \times 0$.

Proof We verify this in standard form local coordinates assuming $G = G_k$, in which case $\tilde{\phi}_k$ takes the form

$$(y_1, x_1, y_2, x_2, \dots, y_k, x_k, \dots, y_n, x_n, z) \mapsto (y_1, x_1, \dots, y_k, t) \quad t = x_k \cdots x_n.$$

By direct computation, the basis (6.5) transforms as follows:

$$\begin{aligned} x_1 \cdots x_n x_1 \partial_{x_1} &\mapsto x_1 \cdots x_{k-1} t x_1 \partial_{x_1}, \\ x_i \cdots x_n \partial_{y_i} &\mapsto x_i \cdots x_{k-1} t \partial_{y_i}, \quad i \leq k \\ x_i \cdots x_n (x_i \partial_{x_i} - x_{i-1} \partial_{x_{i-1}}) &\mapsto x_i \cdots x_{k-1} t (x_i \partial_{x_i} - x_{i-1} \partial_{x_{i-1}}), \quad i < k, \\ x_k \cdots x_n (x_k \partial_{x_k} - x_{k-1} \partial_{x_{k-1}}) &\mapsto t(t \partial_t - x_{k-1} \partial_{x_{k-1}}), \\ x_{k+1} \cdots x_n (x_{k+1} \partial_{x_{k+1}} - x_k \partial_{x_k}) &\mapsto 0, \\ x_j \cdots x_n \partial_{y_j} &\mapsto 0, \quad j \geq k+1, \\ x_j \cdots x_n (x_j \partial_{x_j} - x_{j-1} \partial_{x_{j-1}}) &\mapsto 0, \quad j \geq k+2, \\ \partial_z &\mapsto 0. \end{aligned}$$

The first four lines of the above constitute a basis for ${}^\Phi T C_{\min}(B_k)$, with coordinates $(y_1, x_1, \dots, x_{k-1}, y_k, t)$, while the rest, spanning the kernel, constitute a basis for ${}^\Phi T F_k$, with coordinates $(y_{k+1}, \dots, y_n, x_n, z)$. $\square$

Theorem 6.6 *If X and Y are interior minimal fibered corners, then the lifted projection maps $\pi_X : X \times Y \rightarrow X$ and $\pi_Y : X \times Y \rightarrow Y$ induce an isomorphism*

$${}^\Phi T(X \times Y) \cong \pi_X^* {}^\Phi T X \oplus \pi_Y^* {}^\Phi T Y.$$

In particular, if g_X and g_Y are Φ -metrics on X and Y , then $\pi_X^ g_X \oplus \pi_Y^* g_Y$ is a Φ -metric on $X \times Y$.*

Proof The isomorphism is unique if it exists, since it extends the canonical isomorphism $T(X^\circ \times Y^\circ) \cong \pi_X^* T X^\circ \oplus \pi_Y^* T Y^\circ$ over the interior. To show existence, we proceed by induction on depth of $X \times Y$, starting with the isomorphism $T(X \times Y) \cong T X \oplus T Y$ in depth 0.

In general, a point of maximal depth in $X \times Y$ lies in an intersection of boundary hypersurfaces which must include a minimal hypersurface, without loss of generality of the form $L(G, Y)$, where G is a minimal boundary hypersurface of X ; in particular B_G has depth 0. The projection $X \times Y \rightarrow X$ induces via Lemma 4.12 a diagram

$$\begin{array}{ccc} N_+L(G, Y) & \longrightarrow & N_+G \\ \downarrow & & \downarrow \\ C(B_G) & \longrightarrow & C(B_G) \end{array}$$

where $C = C_{\min}$, which we can arrange to be the identity along the bottom row since we may take $\pi_X^* \rho_{\geq G} = \rho_{\geq L(G, Y)}$, and the restriction to the fibers over the cone end of which is the projection $F_G \tilde{\times} Y \rightarrow F_G$. By induction the projections induce an isomorphism $\Phi T(F_G \tilde{\times} Y) \cong \Phi T F_G \oplus \Phi T Y$ (since $F_G \tilde{\times} Y$ has strictly lower depth), and in light of Proposition 6.5 we obtain a diagram of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Phi T(F_G \tilde{\times} Y) & \longrightarrow & \Phi T N_+ L(G, Y) & \longrightarrow & \Phi T C(B_G) \longrightarrow 0 \\ & & \downarrow \cong & & & & \parallel \\ 0 & \longrightarrow & \Phi T F_G \oplus \Phi T Y & \longrightarrow & \Phi T N_+ G \oplus \Phi T Y & \longrightarrow & \Phi T C(B_G) \longrightarrow 0 \end{array}$$

in which the left and right maps are isomorphisms, inducing an isomorphism in the middle, leading to the desired isomorphism upon identifying the normal models $N_+ L(G, Y)$ and $N_+ G$ with neighborhoods in $X \tilde{\times} Y$ and X , respectively. $\square$

Alternate proof of Theorem 6.6 For the second proof, we use the characterization of $\Phi T X$ as the unique vector bundle (up to isomorphism) which extends $T X^\circ$ and to which any Φ metric g_X on X° extends to be non-degenerate and non-singular. We take advantage of the freedom to choose g_X to be in a particularly convenient form.

When X or Y is of depth 0, the result is trivial, and proceeding by induction on the depth of $X \tilde{\times} Y$, we may assume that X and Y are of positive depth. Letting G and H be minimal boundary hypersurfaces of X and Y , it suffices by induction to prove the result near the lift of $G \times H$. Working locally, we may assume ϕ_G and ϕ_H are trivial, and choose the defining Φ metrics g_X and g_Y to have the simple form

$$\begin{aligned} g_X &= \frac{d\rho_X^2}{\rho_X^4} + \frac{g_{B_G}}{\rho_X^2} + \kappa_G, \quad \text{in the region } \rho_X < \rho_{>G}, \\ g_Y &= \frac{d\rho_Y^2}{\rho_Y^4} + \frac{g_{B_H}}{\rho_Y^2} + \kappa_H, \quad \text{in the region } \rho_Y < \rho_{>H}, \end{aligned} \quad (6.7)$$

where g_{B_G} and g_{B_H} are smooth metrics on B_G and B_H , while κ_G and κ_H are Φ metrics associated with F_G and F_H with total boundary defining functions $\rho_{>G} = \prod_{G' > G} \rho_{G'}$ and $\rho_{>H}$. Since $\rho_G = \frac{\rho_X}{\rho_{>G}}$ is a boundary defining function for G , the condition $\rho_X < \rho_{>G}$ is equivalent to $\rho_G < 1$. With $u_X = \rho_X^{-1}$ and $u_Y = \rho_Y^{-1}$, this suggests using the polar coordinates $r = \sqrt{u_X^2 + u_Y^2}$, $\theta = \arctan \frac{u_X}{u_Y} = \arctan \frac{\rho_Y}{\rho_X}$, leading to the total boundary defining function $\rho := \rho_{X,Y,2} = r^{-1} = (\rho_X^{-2} + \rho_Y^{-2})^{-1/2}$ on $X \tilde{\times} Y$, with respect to which the Cartesian product g of g_X and g_Y takes the form

$$\begin{aligned} g &= dr^2 + r^2 (d\theta^2 + \sin^2 \theta g_{B_G} + \cos^2 \theta g_{B_H}) + \kappa_G + \kappa_H \\ &= \frac{d\rho^2}{\rho^4} + \frac{1}{\rho^2} (d\theta^2 + \sin^2 \theta g_{B_G} + \cos^2 \theta g_{B_H}) + \kappa_G + \kappa_H \\ &\quad \text{in the region } \rho < (\rho_{>G}^{-2} + \rho_{>H}^{-2})^{-1/2} \end{aligned} \quad (6.8)$$

on $\mathcal{U}_{L(G,H)} \cong (B_G \tilde{\times} B_H) \times (F_G \tilde{\times} F_H) \times [0, 1)_\rho$. By induction, the Cartesian product $\kappa_G + \kappa_H$ is a Φ metric associated with the manifold with fibered corners $F_G \tilde{\times} F_H$ with total boundary defining function $(\rho_{>G}^{-2} + \rho_{>H}^{-2})^{-1/2}$, while $d\theta^2 + \sin^2 \theta g_{B_G} + \cos^2 \theta g_{B_H}$ is manifestly a wedge metric on $B_G \tilde{\times} B_H = B_G \times B_H \times [0, \pi/2]_\theta$. It follows that (6.8) is locally a Φ metric on $X \tilde{\times} Y$ near $L(G, H)$ with respect to the total boundary defining function ρ , and hence by induction, is a Φ metric globally on $X \tilde{\times} Y$. $\square$

6.2 Products of wedge structures

We now turn to the product statement for wedge structures on interior maximal manifolds, in the form of Theorem 6.8. An initial attempt along the lines of the previous section would begin by observing that the map $\phi_G : N_+G \rightarrow B_G$ induces a short exact sequence

$$0 \rightarrow \rho_{<G}^{-1} {}^wTC(F_G) \rightarrow {}^wTN_+G \rightarrow {}^wTB_G \rightarrow 0$$

which, while true, turns out not to be immediately useful since it is not obvious that the lifted projections off of $X \tilde{\times} Y$ implement the correct rescalings necessary to identify the rescaled bundles $\rho_{<L(G,H)}^{-1} {}^wT(C(F_G) \tilde{\times} C(F_H))$ with $\rho_{<G}^{-1} {}^wTC(F_G)$ and $\rho_{<H}^{-1} {}^wTC(F_H)$.

Instead, we implement a dual approach involving a somewhat unnatural map from N_+G to $C(F_G)$, assuming $G = F_G \times B_G$ is a product. Since G is a product locally, and since we need not be concerned with uniqueness—the desired isomorphism ${}^wT(X \tilde{\times} Y) \cong {}^wTX \oplus {}^wTY$ being the unique extension of $T(X^\circ \times Y^\circ) \cong \pi_X^*TX^\circ \times \pi_Y^*TY^\circ$ as long as it exists—this unnaturality is tolerable.

Proposition 6.7 *Let $G = F_G \times B_G$ be a product-type hypersurface of an interior maximal manifold X with fibered corners. Then, the compressed projection*

$$N_+G \rightarrow C_{\max}(F_G), \quad v \mapsto (\pi_{F_G}(v), \rho_{\leq G}(v)), \quad \rho_{\leq G} = \prod_{G' \leq G} \rho_{G'}$$

induces a short exact sequence

$$0 \rightarrow {}^wTB_G \rightarrow {}^wTN_+(F_G \times B_G) \rightarrow {}^wTC_{\max}(F_G) \rightarrow 0$$

over the 0-section $G \subset N_+G$.

Proof We compute in standard form local coordinates supposing $G = G_k$, in which the map is given by

$$(y_1, x_1, \dots, y_k, x_k, y_{k+1}, \dots, y_n, x_n, z) \mapsto (t, y_{k+1}, x_{k+1}, \dots, y_n, x_n, z) \quad t = x_1 \cdots x_k.$$

The basis (6.3) is transformed as follows:

$$\begin{aligned} \frac{1}{x_1 \cdots x_{j-1}} \partial_{x_j} &\mapsto x_{j+1} \cdots x_n \partial_t, \quad j \leq k-1, \\ \frac{1}{x_1 \cdots x_{j-1}} \partial_{y_j} &\mapsto 0, \quad j \leq k, \\ \frac{1}{x_1 \cdots x_{k-1}} \partial_{x_k} &\mapsto \partial_t, \\ \frac{1}{x_1 \cdots x_{j-1}} \partial_{x_j} &\mapsto \frac{1}{tx_{k+1} \cdots x_{j-1}} \partial_{x_j}, \quad j \geq k+1, \\ \frac{1}{x_1 \cdots x_{j-1}} \partial_{y_j} &\mapsto \frac{1}{tx_{k+1} \cdots x_{j-1}} \partial_{y_j}, \quad j \geq k+1, \\ \frac{1}{x_1 \cdots x_n} \partial_z &\mapsto \frac{1}{tx_{k+1} \cdots x_n} \partial_z. \end{aligned}$$

The right-hand sides of the first two lines, constituting the kernel of the map over $G_k = \{x_k = 0\}$, form a basis for wTB_k , while the right-hand side of the remaining lines form a basis for ${}^wTC_{\max}(F_G)$. $\square$

Theorem 6.8 *For X and Y interior maximal manifolds with fibered corners, the lifted projections induce an isomorphism*

$${}^wT(X \tilde{\times} Y) \cong \pi_X^* {}^wTX \oplus \pi_Y^* {}^wTY, \quad (6.9)$$

In particular, if g_X and g_Y are wedge metrics on X and Y , respectively, then $\pi_X^* g_X \oplus \pi_Y^* g_Y$ is a wedge metric on $X \times Y$.

Proof Such an isomorphism is unique if it exists, being the continuation of the standard isomorphism $T(X^\circ \times Y^\circ) \cong TX^\circ \oplus TY^\circ$ over the interior, which also constitutes the result when X and Y both have depth 0. Proceeding then by induction on the depth of $X \times Y$, it suffices to consider a point in $X \times Y$ of maximal depth, lying in a maximal boundary hypersurface, which without loss of generality we may assume has the form $L(G, Y)$ where $G \in \mathcal{M}_1(X)$ is a maximal boundary hypersurface of X .

Restricting attention to a sufficiently small neighborhood, we may assume $G = F_G \times B_G$ and $L(G, Y) = (F_G) \times (B_G \times Y)$ are products, and consider via Proposition 6.7 the compressed projection diagram

$$\begin{array}{ccc} N_+ L(G, Y) & \longrightarrow & N_+ G \\ \downarrow & & \downarrow \\ C(F_G) & \longrightarrow & C(F_G) \end{array}$$

where $C = C_{\max}$, and which we can assume is the identity along the bottom row since we may take $\rho_{\leq L(G, Y)} = \pi_X^* \rho_{\leq G}$ by maximality. Over the cone end the diagram is fibered, given by the projection $B_G \times Y \rightarrow B_G$ on fibers. Since $B_G \times Y$ has lower depth, Proposition 6.7 and induction give a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & {}^w T(B_G \times Y) & \longrightarrow & {}^w T N_+ L(G, Y) & \longrightarrow & {}^w T C(F_G) \longrightarrow 0 \\ & & \downarrow \cong & & & & \parallel \\ 0 & \longrightarrow & {}^w T B_G \oplus {}^w T Y & \longrightarrow & {}^w T N_+ G \oplus {}^w T Y & \longrightarrow & {}^w T C(F_G) \longrightarrow 0 \end{array}$$

of short exact sequences, in which the left- and right-hand vertical maps are isomorphisms, assembling to the desired isomorphism in the middle upon identifying $N_+ G$ and $N_+ L(G, Y)$ with suitable neighborhoods in X and $X \times Y$. $\square$

Alternate proof of Theorem 6.8 For the second proof, we use the characterization of ${}^w T X$ as the unique vector bundle (up to isomorphism) which extends TX° and to which any wedge metric g_X on X° extends to be non-degenerate and non-singular.

The result is trivial when X or Y is of depth 0, and we then proceed by induction on the depth of $X \times Y$. Let G and H be minimal hypersurfaces of X and Y respectively; by induction it suffices to prove the result near the lift of $G \times H$. In particular, B_G and B_H are closed manifolds of depth 0, and, by localizing as necessary, we may assume that ϕ_G and ϕ_H are trivial, so $G = B_G \times F_G$ and $H = B_H \times F_H$.

In order to prove the isomorphism (6.9) we may freely choose the defining wedge metrics g_X and g_Y . Hence on neighborhoods $B_G \times F_G \times [0, 1)$ and $B_H \times F_H \times [0, 1)$ we may assume that g_X and g_Y have the form

$$g_X = d\rho_G^2 + g_{B_G} + \rho_G^2 \kappa_G \quad \text{and} \quad g_Y = d\rho_H^2 + g_{B_H} + \rho_H^2 \kappa_H \quad (6.10)$$

where κ_G and κ_H are wedge metrics on F_G and F_H , respectively, and where g_{B_G} and g_{B_H} are smooth metrics on B_G and B_H , respectively. Using the polar coordinates $r = \sqrt{\rho_G^2 + \rho_H^2}$, $\theta = \arctan \frac{\rho_G}{\rho_H}$, the Cartesian product $g = \pi_X^* g_X + \pi_Y^* g_Y$ takes the form

$$g = dr^2 + g_{B_G} + g_{B_H} + r^2 (d\theta^2 + \sin^2 \theta \kappa_G + \cos^2 \theta \kappa_H).$$

Clearly, this is a wedge metric on the neighborhood

$$L(G, H) \times [0, 1] \cong B_G \times B_H \times (F_G \tilde{*} F_H) \times [0, 1]$$

provided we know that the geometric join $d\theta^2 + \sin^2 \theta \kappa_G + \cos^2 \theta \kappa_H$ lifts to a wedge metric on $F_G \tilde{*} F_H$. However, by our inductive assumption on the depth, g is known to be a wedge metric away from $r = 0$, which in particular forces this geometric join to be a wedge metric on $F_G \tilde{*} F_H$. $\square$

7 Equivalence of minimal and maximal joins

As noted previously, the ordered products $X_{\max} \tilde{\times} Y_{\max}$ and $X_{\min} \tilde{\times} Y_{\min}$ are generally not diffeomorphic. By contrast, in this section we prove a rather remarkable diffeomorphism between the minimal and maximal joins $X \tilde{*}_{\max} Y$ and $X \tilde{*}_{\min} Y$. Note that since $X \tilde{*}_{\max} Y$ is interior minimal, while $X \tilde{*}_{\min} Y$ is interior maximal, we do not expect (nor do we obtain) a natural fibered corners isomorphism from one to the other; however it happens that the diffeomorphism does intertwine all the boundary fibrations (just not the interior fibration); in particular it follows that these two extremal versions of the join product have the same underlying stratified space, namely the topological join of the two associated stratified spaces.

Theorem 7.1 *Let X and Y be manifolds with fibered corners with total boundary defining functions $\rho_X = \prod_{G \in \mathcal{M}_1(X)} \rho_G$ and $\rho_Y = \prod_{H \in \mathcal{M}_1(Y)} \rho_H$. Then, the b-map*

$$\begin{aligned} (X \times \mathbb{R}_+) \times (Y \times \mathbb{R}_+) &\rightarrow (X \times \mathbb{R}_+) \times (Y \times \mathbb{R}_+), \\ ((x, t), (y, s)) &\mapsto ((x, s\rho_Y(y)), (y, t\rho_X(x))) \end{aligned} \quad (7.1)$$

lifts to a b-map (not a fibered corners morphism)

$$C_{\max}(X) \tilde{\times} C_{\max}(Y) \rightarrow C_{\min}(X) \tilde{\times} C_{\min}(Y)$$

the restriction of which to the principal boundary hypersurface is a diffeomorphism

$$X \tilde{*}_{\max} Y \cong X \tilde{*}_{\min} Y \quad (7.2)$$

In addition, (7.2) is fibered over boundary hypersurfaces, and constitutes a fibered corners isomorphism $X \tilde{}_{\max} Y \cong (X \tilde{*}_{\min} Y)_{\min}$.*

In particular, it follows from Corollary 5.2 that the stratified space associated with $X \tilde{}_{\min} Y$ is also the topological join $\mathcal{X} \star \mathcal{Y}$ of the stratified spaces associated with X and Y .*

Remark Note that the fibered corners structure on $C_{\min}(X)$ or $C_{\max}(X)$ does not make use of the interior fibration on X itself. Viewing the joins as blow-ups of $X \times Y \times [0, 1]$, the isomorphism (7.2) identifies boundary hypersurfaces in a manner which is somewhat surprising, namely:

- the lift of $X \times Y \times \{0\}$ is exchanged with the lift of $X \times Y \times \{1\}$,
- the lift of $G \times H \times [0, 1]$ for $G \in \mathcal{M}_1(X)$ and $H \in \mathcal{M}_1(Y)$ is sent to itself; however,
- the lift of $G \times Y \times \{0\}$ is exchanged with the lift of $G \times Y \times [0, 1]$, and
- the lift of $X \times H \times \{1\}$ is exchanged with the lift of $X \times H \times [0, 1]$.

Proof Enumerate the hypersurfaces of X and Y by $G_1, \dots, G_m$ and $H_1, \dots, H_n$, respectively, and suppose that they are ordered in the sense that $G_i < G_j$ implies $i < j$ and similarly for

H_i . On $C_{\max/\min}(X)$ (and its product with $C_{\max/\min}(Y)$) denote by g_i the monoid generator associated with the boundary hypersurface $G_i \times \mathbb{R}_+$ and by ξ the generator associated with the boundary hypersurface $X \times 0$. Likewise, on $C_{\max/\min}(Y)$ (and its product with $C_{\max/\min}(X)$) denote by h_i the monoid generator associated with the boundary hypersurface $H_i \times \mathbb{R}_+$ and by η the generator associated with the boundary hypersurface $Y \times 0$. On the unresolved product $C_{\max}(X) \times C_{\max}(Y)$, the map (7.1) is associated with monoid homomorphisms generated by

$$\xi \mapsto \eta, \quad \eta \mapsto \xi, \quad g_i \mapsto g_i + \eta, \quad h_j \mapsto \xi + h_j. \quad (7.3)$$

We proceed to show that the map lifts to the ordered products. The ordered product of the domain, $C_{\max}(X) \tilde{\times} C_{\max}(Y)$ is characterized as a blow-up via Theorem 2.1 by monoids generated by maximal ordered chains of sums $a + b$ where

$$\begin{aligned} a &\in \{\xi < g_1 < \cdots < g_m < 0\}, \\ b &\in \{\eta < h_1 < \cdots < h_n < 0\}. \end{aligned}$$

Such a chain must begin with $\xi + \eta$, and may have terms of the form $g_i + \eta$ or $\xi + h_j$ but not both, and also of the form $g_i + 0$ or $0 + h_j$ but not both. Though it is not really a generator, it is useful to imagine $0 + 0$ at the end of the chain. On the other hand, the ordered product of the range, $C_{\min}(X) \tilde{\times} C_{\min}(Y)$ is characterized by monoids generated by maximal ordered chains of sums $a + b$ where

$$\begin{aligned} a &\in \{0 < g_1 < \cdots < g_m < \xi\} \\ b &\in \{0 < h_1 < \cdots < h_n < \eta\}. \end{aligned}$$

Such an ordered chain must end with $\xi + \eta$, and may have terms of the form $g_i + \eta$ or $\xi + h_j$ but not both, and also of the form $g_i + 0$ or $0 + h_j$ but not both. Again, though it is not a generator, it is useful to imagine $0 + 0$ at the beginning of the chain.

To each maximal ordered chain of the source $C_{\max}(X) \tilde{\times} C_{\max}(Y)$, we associate a maximal ordered chain of the target $C_{\min}(X) \tilde{\times} C_{\min}(Y)$ by exchanging ξ with 0 and η with 0 in the list of generators (in particular $\xi + \eta$ is exchanged with $0 + 0$ at the start or end), and we claim that the monoid homomorphism generated by (7.3) induces a homomorphism from the monoid generated by this source chain into (but not onto) the monoid generated by the target chain. Indeed, the generators map as

$$\begin{array}{ll} \xi + \eta \mapsto \xi + \eta & g_i + h_j \mapsto g_i + h_j \\ \xi + 0 \mapsto 0 + \eta & 0 + \eta \mapsto \xi + 0 \\ g_i + 0 \mapsto g_i + \eta & 0 + h_j \mapsto \xi + h_j \\ g_i + \eta \mapsto (g_i + 0) + (\xi + \eta) & \xi + h_i \mapsto (0 + h_i) + (\xi + \eta) \end{array}$$

It follows from Theorem 2.1 that (7.1) lifts to a b-map $C_{\max}(X) \tilde{\times} C_{\max}(Y) \rightarrow C_{\min}(X) \tilde{\times} C_{\min}(Y)$. Note that the monoid homomorphism is not an isomorphism (it is injective but not surjective) owing to the last line; in particular the lifted map is not b-normal.

To understand what the map looks like when restricted to $X \tilde{\star}_{\max} Y \subset C_{\max}(X) \tilde{\times} C_{\max}(Y)$, recall that the b-normal monoids of a boundary hypersurface are the quotient of those from the ambient space by the generator of the said hypersurface, which in this case is $\xi + \eta$. Thus, the monoids of $X \tilde{\star}_{\max} Y$ are generated by ordered chains as described above with the omission of $\xi + \eta$ at the start, while the monoids of $X \tilde{\star}_{\min} Y$ are generated by ordered chains as for $C_{\min}(X) \tilde{\times} C_{\min}(Y)$ with the omission of $\xi + \eta$ at the end. Moreover, the induced map

on the quotient monoids is generated as follows:

$$\begin{aligned} [g_i + \eta] &\mapsto [(g_i + \eta) + \xi] = [g_i + 0], \\ [\xi + h_j] &\mapsto [\eta + (\xi + h_j)] = [0 + h_j], \\ [0 + \eta] &\mapsto [\xi + 0], \\ [\xi + 0] &\mapsto [0 + \eta]. \end{aligned} \quad (7.4)$$

Thus, each monoid of $X \tilde{\star}_{\max} Y$ is mapped isomorphically to a related monoid of $X \tilde{\star}_{\min} Y$ obtained by exchanging ξ with 0 and η with 0. In particular it follows that $X \tilde{\star}_{\max} Y \rightarrow X \tilde{\star}_{\min} Y$ is simple, b-normal and ordered if the domain is made interior minimal (or the target maximal).

To see that it is also fibered (and to check that the diffeomorphism on the interior extends up to the boundary faces, which does not follow automatically from the above), we resort to using coordinates. Before recording the general result, it is illustrative to consider some explicit examples setting $n = m = 2$ for simplicity. Beginning with standard form coordinates $(t, y_1, x_1, y_2, x_2, y_3)$ on $C_{\max}(X)$ and $(s, y'_1, x'_1, y'_2, x'_2, y'_3)$ on $C_{\max}(Y)$ taken so that the x_i and x'_i agree with the boundary defining functions used to construct the map, standard form coordinates on $C_{\max}(X) \tilde{\times} C_{\max}(Y)$ associated with the chain

$$\xi + \eta < \xi + h_1 < g_1 + h_1 < g_1 + h_2 < g_2 + h_2 < g_2 + 0 < (0 + 0)$$

(with notation as above) are given by

$$\left(s, y'_1, \frac{t}{s}, y_1, \frac{sx'_1}{t}, y'_2, \frac{tx_1}{sx'_1}, y_2, \frac{sx'_1x'_2}{tx_1}, y'_3, \frac{tx_1x_2}{sx'_1x'_2}, y_3 \right). \quad (7.5)$$

(Recall that the exponents of the rational boundary defining functions of the blow-up are the coefficients of the basis vectors which are dual to the basis defined by the chain above.) In a similar manner, with coordinates $(y_1, x_1, y_2, x_2, y_3, x_3, y_4, \tau)$ on $C_{\min}(X)$ and $(y'_1, x'_1, y'_2, x'_2, y'_3, x'_3, y'_4, \sigma)$ on $C_{\min}(Y)$, standard form coordinates on $C_{\min}(X) \tilde{\times} C_{\min}(Y)$ associated with the chain

$$(0 + 0) < 0 + h_1 < g_1 + h_1 < g_1 + h_2 < g_2 + h_2 < g_2 + \eta < \xi + \eta$$

are given by

$$\left(y'_1, \frac{x'_1x'_2\sigma}{x_1x_2\tau}, y_1, \frac{x_1x_2\tau}{x'_2\sigma}, y'_2, \frac{x'_2\sigma}{x_2\tau}, y_2, \frac{x_2\tau}{\sigma}, y'_3, \frac{\sigma}{\tau}, y_3, \sigma \right). \quad (7.6)$$

The maximal join is given by $s = 0$ in (7.5) and the minimal join is given by $\sigma = 0$ in (7.6), and the map (7.1), determined in coordinates by setting $\tau = sx'_1x'_2$ and $\sigma = tx_1x_2$, perfectly identifies $s = 0$ in (7.5) with $\sigma = 0$ in (7.6), in particular identifying the boundary fibrations which are given by projections since the coordinates are in standard form.

For another example, standard form coordinates on $C_{\max}(X) \tilde{\times} C_{\max}(Y)$ associated with the chain

$$\xi + \eta < \xi + h_1 < \xi + h_2 < \xi + 0 < g_1 + 0 < g_2 + 0 < (0 + 0)$$

are given by

$$\left(s, y'_1, x'_1, y'_2, x'_2, y'_3, \frac{t}{sx'_1x'_2}, y_1, x_1, y_2, x_2, y_3 \right), \quad (7.7)$$

whereas coordinates on $C_{\min}(X) \tilde{\times} C_{\min}(Y)$ associated with the chain

$$(0 + 0) < 0 + h_1 < 0 + h_2 < 0 + \eta < g_1 + \eta < g_2 + \eta < \xi + \eta$$

are given by

$$\left(y'_1, x'_1, y'_2, x'_2, y'_3, \frac{\tau x_1 x_2}{\sigma}, y_1, x_1, y_2, x_2, y_3, \sigma \right) \quad (7.8)$$

and again $\{s = 0\}$ in (7.7) is identified with $\{\sigma = 0\}$ in (7.8) by $\tau = s x'_1 x'_2$ and $\sigma = t x_1 x_2$.

In the general setting, standard form coordinates on $C_{\max}(X) \tilde{\times} C_{\max}(Y)$ associated with a given maximal ordered ‘source chain’ have the form

$$(\sigma_0, \zeta_1, \sigma_1, \zeta_2, \dots, \zeta_{m+n}, \sigma_{m+n}, \zeta_{m+n+1}) \quad (7.9)$$

where $(\sigma_0, \dots, \sigma_{m+n})$ are rational boundary defining coordinates made up from the $(s, x_1, \dots, x_m)$ and $(t, x'_1, \dots, x'_n)$ with exponents given by the coefficients of the basis which is dual to the source chain (with $\{\sigma_0 = 0\}$ defining principal boundary hypersurface $X \tilde{\times}_{\max} Y$), and ζ_i stands for those interior coordinates (y_j or y'_j for some j) for the base of the boundary fibration on $\{\sigma_i = 0\}$ which belong to the fiber of the boundary fibration on $\{\sigma_{i-1} = 0\}$. Likewise, standard form coordinates on $C_{\min}(X) \tilde{\times} C_{\min}(Y)$ associated with the ordered chain given by exchanging $\xi \leftrightarrow 0$ and $\eta \leftrightarrow 0$ in the source chain have a similar form

$$(\zeta_1, \sigma'_1, \zeta_2, \sigma'_2, \dots, \zeta_{m+n}, \sigma'_{m+n}, \zeta_{m+n+1}, \sigma'_{m+n+1}) \quad (7.10)$$

where $(\sigma'_1, \dots, \sigma'_{m+n+1})$ are rational boundary defining coordinates made up from $(x_1, \dots, x_m, \sigma)$ and $(x'_1, \dots, x'_n, \tau)$, with $\{\sigma'_{m+n+1} = 0\}$ defining the principal hypersurface $X \tilde{\times}_{\min} Y$, and with the same ordered sequence of interior coordinates ζ_j . The form of the boundary defining coordinates is entirely algebraic, and since (7.4) is an isomorphism, it follows that $\tau = s x_1 \cdots x_m$ and $\sigma = t x'_1 \cdots x'_n$ lifts to an identification $\sigma_i = \sigma'_i$ for $1 \leq i \leq m+n$. The interior coordinates ζ_j are identical between the source and target, and it follows that the induced map from $\{\sigma_0 = 0\}$ in (7.9) to $\{\sigma'_{m+n+1} = 0\}$ in (7.10) is a diffeomorphism which intertwines the boundary fibrations. $\square$

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Appendix A: Tube systems and fibrations

When working with manifolds with corners, it is often convenient to make use of tubular neighborhoods of boundary hypersurfaces.

Definition A.1 Let X be a manifold with corners. A *tube* for $G \in \mathcal{M}_1(X)$ is an open neighborhood $\mathcal{U}_G$ of G with one of the following data, which are equivalent up to possibly replacing $\mathcal{U}_G$ by a smaller neighborhood:

- (a) A map $\mathcal{U}_G \rightarrow G \times [0, \infty)$ which is a diffeomorphism onto its image and which restricts to the identity $1 : G \subset \mathcal{U}_G \rightarrow G \times \{0\}$.
- (b) A retraction $r_G : \mathcal{U}_G \rightarrow G$ and a non-degenerate local boundary defining function $\rho_G : \mathcal{U}_G \rightarrow [0, \infty)$.
- (c) A vector field ξ_G on $\mathcal{U}_G$ which is inward pointing at G .

The equivalence of (a) and (b) is given by equating the diffeomorphism with the product $r_G \times \rho_G : \mathcal{U}_G \rightarrow G \times [0, \infty)$, and the equivalence of (a) and (c) follows from the flow-out

of G by ξ_G in one direction (shrinking $\mathcal{U}_G$ if necessary) and the pullback of ∂_t on $[0, \infty)$ in the other direction.

Definition A.2 Let X be a manifold with fibered corners. A *tube system* on X consists of a tube $(\mathcal{U}_G, \xi_G)$ for each $G \in \mathcal{M}_1(X)$ as well as tubes $(\mathcal{U}_{G,G'}, \xi_{G,G'})$ for $\partial_G B_{G'} \subset B_{G'}$ for each $G < G'$ such that

- (a) $\mathcal{U}_G \cap \mathcal{U}_{G'} = \emptyset$ if $G \cap G' = \emptyset$; otherwise $[\xi_G, \xi_{G'}] = 0$ on $\mathcal{U}_G \cap \mathcal{U}_{G'}$,
- (b) if $G < G'$, then $\xi_{G'}$ is tangent to the fibers of ϕ_G at $G \cap \mathcal{U}_{G'}$,
- (c) if $G < G'$, then $\phi_{G'}(\mathcal{U}_G \cap G') \subset \mathcal{U}_{G,G'}$ and $\xi_G|_{\mathcal{U}_G \cap G'}$ is $\phi_{G'}$ -related to $\xi_{G,G'}$ on $\mathcal{U}_{G,G'} \subset B_{G'}$, and
- (d) $\{(\mathcal{U}_{G,G'}, \xi_{G,G'}) : G < G'\}$ forms a tube system on $B_{G'}$.

Note that the retractions and boundary defining functions for a tube system on X correspond to the tubes of the associated stratified space $\mathcal{X}$ as discussed in Sect. 5.1. The existence of tube systems is proved in [5, Prop. 3.7] and [11, Lem. 1.4], on which the proof of Proposition A.4 is based. In it we establish the existence of tube systems relative to a given interior fibration $\phi_X : X \rightarrow B_X$, though since we will also characterize interior fibrations in the process, proving an Ehresmann lemma for fibered corners, we make the following (temporary) definition:

Definition A.3 Let X be a manifold with fibered corners. (The fibration ϕ_X need not be defined.) An *interior fibration* is a surjective submersion $f : X \rightarrow Y$ onto an interior maximal manifold with fibered corners $Y = Y_{\max}$ which is ordered and fibered, and which has the property that whenever $f_{\#}(G) = H \in \mathcal{M}_1(Y)$, the bottom row of the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ \phi_G \downarrow & & \downarrow \phi_H \\ B_G & \xrightarrow[f_H]{\cong} & B_H \end{array}$$

is a diffeomorphism identifying $B_G \cong B_H$.

Proposition A.4 Let X be a manifold with fibered corners with representative boundary defining functions $\{\rho_G\}$ for the given fc-equivalence classes and $f : X \rightarrow Y$ an interior fibration in the sense of Definition A.3. Then, there exists a tube system on X with the following properties:

- (a) ξ_G is tangent to the fibers of f for every $G \in \mathcal{M}_1(X)$ such that $f_{\#}(G) = Y \in \mathcal{M}_0(Y)$.
- (b) $\xi_G(\rho_G) = 1$ at all boundary hypersurfaces of X for each $G \in \mathcal{M}_1(X)$. In particular, the boundary defining functions determined by the tubes $\mathcal{U}_G \cong G \times [0, \varepsilon_G)$ are fc-equivalent to the $\{\rho_G\}$.

Remark Observe that $X \rightarrow *$ and $1 : X \rightarrow X$ are always interior fibrations according to Definition A.3 and neither imposes any additional conditions on a tube system. Thus, the result with either of these interior fibrations furnishes the existence of tube systems in general.

Proof The proof is by induction on the depth of X ; if X has depth 0 then the statement is vacuous. Thus, assume the result holds for all spaces of depth strictly less than that of X ; in particular each B_G is equipped with a tube system satisfying properties (a) and (b)

with respect to representative boundary defining functions on B_G and the interior fibration $B_G \rightarrow Y$ in case $f_{\sharp}(G) = Y$ or $B_G \xrightarrow{\cong} B_H$ in case $f_{\sharp}(G) = H \in \mathcal{M}_1(Y)$.

We proceed to construct ξ_G for each $G \in \mathcal{M}_1(X)$ by another induction on $\mathcal{M}_1(X)$ in reverse order. For G maximal, there are two cases to consider: If $f_{\sharp}(G) = Y$, then $NG \subset \ker df$ and we may choose ξ_G near G to be transversal to G and tangent to the fibers of f . The condition that $\xi_G(\rho_G) = 1$ at G is consistent with tangency to the fibers of f since ρ_G and f are transversal. Note that for each $G' < G$, either $f_{\sharp}(G') = Y$, in which case $G' \rightarrow Y$ factors as the composition of $\phi_{G'} : G' \rightarrow B_{G'}$ with $f_{G'} : B_{G'} \rightarrow Y$, and taking ξ_G tangent to the fibers of $\phi_{G'}$ on G' is consistent with ξ_G being tangent to the fibers of f ; otherwise $f_{\sharp}(G') = H \in \mathcal{M}_1(Y)$ and in this case $\phi_{G'} : G' \rightarrow B_{G'}$ factors as the composition of $f : G' \rightarrow H$ with $\phi_H : H \rightarrow B_H \cong B_{G'}$, so taking ξ_G tangent to the fibers of f makes it automatically tangent to the fibers of $\phi_{G'}$ at G' . Either case is consistent with the condition that $\xi_G(\rho_G) = 1$ over G' since ρ_G is transverse to $\phi_{G'}$ for $G' < G$ as well as to f as noted above. Note that the flow of ξ_G just constructed preserves the distribution $\ker df$ by integrability of the latter. If on the other hand $f_{\sharp}(G) = H \in \mathcal{M}_1(Y)$, then we may simply take ξ_G to be transversal to G , tangent to the fibers of $\phi_{G'}$ for each $G' < G$, as well as f -related to some vector normal field to H ; this again ensures that the flow of ξ_G preserves $\ker f$ by the identity $f_*([\xi_G, \eta]) = [f_*\xi_G, f_*\eta] = 0$ for η tangent to the fibers of f . Again the condition that $\xi_G(\rho_G) = 1$ at every $G' \in \mathcal{M}_1(X)$ can be imposed since ρ_G is transverse to $\phi_{G'}$ for $G' < G$.

Proceeding now by induction, suppose that $\xi_{G'}$ has been chosen for all $G' > G$ to satisfy the conditions of Definition A.2 as well as conditions (a) and (b). We first define $\xi_G|_{G'}$ for each $G' > G$ to be a lift by $\phi_{G'}$ of the normal vector field $\xi_{G,G'}$ for $\partial G B_{G'} \subset B_{G'}$; such a lift is automatically tangent to the fibers of f in case $f_{\sharp}(G) = Y$ since $f : G' \rightarrow Y$ factors through $\phi_{G'}$ in this case and $\xi_{G,G'}$ is already tangent to the fibers of $B_{G'} \rightarrow Y$. Likewise, since $\rho_G|_{G'}$ is the pullback of a boundary defining function $\rho_{G,G'}$ for $\partial G B_{G'} \subset B_{G'}$ and we may assume that $\xi_{G,G'}(\rho_{G,G'}) = 1$, the condition that $\xi_G(\rho_G) = 1$ is satisfied here. Note that such lifts may be chosen consistently over $G' \cap G''$ by Definition A.2.(c) applied to $\xi_{G,G'}$ and $\xi_{G,G''}$. We then extend ξ_G over the tubular neighborhoods $\mathcal{U}_{G'}$ of each $G' > G$ by flowing out along $\xi_{G'}$; by Definition A.2.(a) these flow-outs are consistent on $\mathcal{U}_{G'} \cap \mathcal{U}_{G''}$. Since the $\xi_{G'}$ preserve $\ker f$, the extensions of ξ_G remain tangent to the fibers of f if applicable. Finally, away from the neighborhoods $\mathcal{U}_{G'}$ for $G' > G$ we proceed as in the base case, taking ξ_G to be normal to G , tangent to the fibers of f if $f_{\sharp}(G) = Y$ (and f -related to a normal vector field for $f_{\sharp}(G) \in \mathcal{M}_1(Y)$ otherwise), tangent to the fibers of $\phi_{G'}$ for $G' < G$, and such that $\xi_G(\rho_G) = 1$ over $G' \leq G$; then we use a partition of unity to assemble ξ_G globally to satisfy Definition A.2.(a)–(c) and with respect to the vector fields already constructed, as well as conditions (a) and (b). This completes the inner induction on $G \in \mathcal{M}_1(X)$, at the end of which X is equipped with a tube system having the required properties, completing the outer induction. $\square$

Proposition A.5 *Let $f : X \rightarrow Y$ be an interior fibration of manifolds with fibered corners in the sense of Definition A.3. Then, there exists a connection on $f : X \rightarrow Y$ with respect to which the lift $\tilde{\eta}$ of any vector field $\eta \in \mathcal{V}(Y)$ satisfies the following properties with respect to each $G \in f_{\sharp}^{-1}(Y) \subset \mathcal{M}_1(X)$:*

- $\tilde{\eta}|_G$ is ϕ_G -related to a lift $\tilde{\eta}_G$ of η to B_G by a connection on $f_G : B_G \rightarrow Y$ satisfying the same properties, and
- $\tilde{\eta}(\rho_G) = 0$ on a sufficiently small neighborhood of G for some boundary defining function ρ_G (coming from a tube system) in the fc -equivalence class on X .

Proof We fix a tube system on X satisfying the conditions of Proposition A.4 and use the associated boundary defining functions. The proof is by induction on the depth of the fiber of $f : X \rightarrow Y$; if this fiber has depth 0 then the two conditions are vacuous and any connection suffices. Thus, we may assume by induction that such a connection exists for every interior fibration with fiber depth less than that of f ; in particular we may assume each $f_G : B_G \rightarrow Y$ is equipped with a suitable connection for each $G \in f_{\sharp}^{-1}(Y)$, satisfying (b) with respect to the boundary defining functions of the induced tube system on B_G .

Consider now a maximal boundary hypersurface G such that $f_{\sharp}(G) = Y$ and consider the tubular neighborhood $\mathcal{U}_G \cong G \times [0, \varepsilon)$. The fibration $\phi_G \circ \text{pr}_1 : G \times [0, \varepsilon) \rightarrow B_G$ is transverse to $\rho_G = \text{pr}_2 : G \times [0, \varepsilon) \rightarrow [0, \varepsilon)$, so we may construct a connection on $\phi_G \circ \text{pr}_1$ whose horizontal distribution annihilates ρ_G in a possibly smaller neighborhood of G . Composing this connection with the connection on $f_G : B_G \rightarrow Y$ gives a connection with the required properties near G , since the lift $\tilde{\eta}_G$ of $\eta \in \mathcal{V}(Y)$ to B_G annihilates the boundary defining function $\rho_{G',G}$ for $\partial_{G'} B_G$ for each $G' < G \in f_{\sharp}^{-1}(Y)$, and by properties of the tube system $\rho_{G'}$ on $\mathcal{U}_G \cap \mathcal{U}_{G'}$ is a lift of $\rho_{G',G}$ on $\mathcal{U}_{G'} \cap \mathcal{U}_G$.

Having defined the connection on a neighborhood of G , we now consider the double, $\tilde{X}_G$, of X across G ; this consists of two copies of X glued along G with opposite orientations, and $f : X \rightarrow Y$ extends to a smooth map $\tilde{f} : \tilde{X}_G \rightarrow Y$ (provided the gluing is performed with respect to a choice of normal direction along $\ker df|_G$) which is again an interior fibration according to Definition A.3. The complement of $\mathcal{U}_G \subset X$ has a neighborhood which may be identified with an open set in $\tilde{X}_G$, and since $\tilde{f} : \tilde{X}_G \rightarrow Y$ has strictly lower depth fibers, the connection on $\mathcal{U}_G$ may be combined with a suitable connection on this neighborhood by the inductive assumption. $\square$

Corollary A.6 *An interior fibration in the sense of Definition A.3 is a locally trivial fiber bundle of manifolds with fibered corners. More precisely, for every $p \in Y$ there exists a neighborhood $\mathcal{O} \ni p$ and diffeomorphism $f^{-1}(\mathcal{O}) \cong f^{-1}(p) \times \mathcal{O}$ such that for each $G \in f_{\sharp}^{-1}(Y) \cap \mathcal{M}_1(X)$ the diagrams*

$$\begin{array}{ccc}
 f^{-1}(\mathcal{O}) & \xrightarrow{\cong} & f^{-1}(p) \times \mathcal{O} \\
 & \searrow f & \downarrow \text{pr}_2 \\
 & & \mathcal{O}
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 f^{-1}(\mathcal{O}) \cap G & \xrightarrow{\cong} & (f^{-1}(p) \cap G) \times \mathcal{O} \\
 \phi_G \downarrow & & \downarrow \phi_G \times 1 \\
 f_G^{-1}(\mathcal{O}) & \xrightarrow{\cong} & f_G^{-1}(p) \times \mathcal{O} \\
 & \searrow f_G & \downarrow \text{pr}_2 \\
 & & \mathcal{O}
 \end{array}$$

commute and the diffeomorphism $\mathcal{O} \times f^{-1}(p) \cong f^{-1}(\mathcal{O})$ can be arranged to pull back ρ_G to be independent of $\mathcal{O}$ in a sufficiently small neighborhood of $f^{-1}(p) \cap G$ for some representative boundary defining function ρ_G .

In particular, if $f : X \rightarrow Y$ is an interior fibration, then X can be equipped with the fibered corners structure in which $\phi_X = f$.

Proof Fix a neighborhood $\mathcal{O}'$ of an arbitrary point $p \in Y$ with coordinates $(u_1, \dots, u_n) \in \mathbb{R}_+^k \times \mathbb{R}^{n-k}$ and associated coordinate vector fields $\partial_{u_1}, \dots, \partial_{u_n}$. Let $\tilde{\eta}_1, \dots, \tilde{\eta}_n$ be lifts of the ∂_{u_i} on $f^{-1}(\mathcal{O}')$ with respect to a connection afforded by Proposition A.5. (These need not commute.) Then, the ordered flow

$$((t_1, \dots, t_n), q) \mapsto \exp(t_1 \tilde{\eta}_1) \cdots \exp(t_n \tilde{\eta}_n) q$$

induces a diffeomorphism $\mathcal{O} \times f^{-1}(p) \cong f^{-1}(\mathcal{O})$ where $\mathcal{O} \subset \mathcal{O}'$ is a suitably small neighborhood of p . By the properties of the vector fields $\tilde{\eta}_i$ the local diffeomorphism respects the

fibered corners structure on fibers in that it trivializes the boundary fibrations and representative boundary defining functions for $G \in f_{\sharp}^{-1}(Y)$ as required. $\square$

Corollary A.6 is used in [23, 24] to provide local trivializations for bundles of QFB manifolds, a construction that is also useful to see that the results of [7] automatically hold for QFB metrics.

Corollary A.7 *Let X be a manifold with fibered corners with interior fibration $\phi_X : X \rightarrow B_X$. Then standard form coordinates exist near any point of X .*

Proof Fix a tube system on X as per Proposition A.4. The boundary defining functions $\rho_G : \mathcal{U}_G \cong G \times [0, \infty) \rightarrow [0, \infty)$ associated with the tubes furnish local boundary defining coordinates $(x_{-n'}, \dots, x_{-1}, x_1, \dots, x_n)$ satisfying the requisite conditions, and by local triviality and the iterated condition of the boundary fibrations, interior coordinates $(y_{-n'}, \dots, y_0, y_1, \dots, y_{n+1})$ can be chosen so that the boundary fibrations correspond to projections. $\square$

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