

Running head(s):

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This article offers the construct *unitizing predicates* to name mental actions important for students' reasoning about logic. To unitize a predicate is to conceptualize (possibly complex or multipart) conditions as a single property that every example either has or does not have, thereby partitioning a universal set into examples and nonexamples. This explicates the cognitive work that supports students to unify various statements with the same logical form, which is conventionally represented by replacing parts of statements by logical variables p or $P(x)$. Using data from a constructivist teaching experiment with two undergraduate students, we document barriers to unitizing predicates and demonstrate how this activity influences students' ability to render mathematical statements and proofs as having the same logical structure.

Keywords: Reasoning about logic; Unitizing predicates; Conditional statements; Proof; Contrapositive

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Consider the following statements that we presented to students during teaching experiments on logic:

Theorem 1 For every integer x , if x is a multiple of 6, then x is a multiple of 3.

Theorem 2 For any integer x , if x is a multiple of 2 and a multiple of 7, then x is a multiple of 14.

Theorem 4¹ For any quadrilateral $\blacksquare ABCD$, if $\blacksquare ABCD$ is a rhombus, then the diagonal $\overline{AC}$ forms two congruent, isosceles triangles $\triangle ABC$ and $\triangle CDA$.

Theorem 9 Given any functions f, g that are continuous on the domain $[a, b]$, if $f(a) = g(b)$ and $f(b) = g(a)$, then there exists some c in $[a, b]$ such that $f(c) = g(c)$.

In what ways would we expect students to see these four theorem statements as being the same? How would that sameness influence students' reasoning about these theorems and their proofs? These questions illustrate a contextual way of considering the role of logic in students' mathematical reasoning. Logic concerns what unifies these statements—not their semantic content but the form of the statements: “For all $x \in S$, if $P(x)$, then $Q(x)$.” This generalized notation captures three aspects of each theorem: (a) each theorem talks about all objects in some universal set (integers, quadrilaterals, or pairs of continuous functions), (b) they are in conditional form (if . . . then . . .), and (c) they contain two predicates, meaning conditions true or false of each example, represented in function notation as $P(x)$ and $Q(x)$. A truth function maps each input object x to a truth value: T if the object makes that condition true and F if it makes that condition false.

This article explores this psychological question of how undergraduate students construct a sense of shared structure for statements of this type and how those conceptualizations influence any operative sense of logical structure among these statements and their proofs. Stated another way, we wanted to know how logic might arise in student activity and how they might learn certain logical principles. We propose the construct *unitizing predicates* to describe how students conceptualize the various conditions as entailing properties that each example either has or does not have. The predicate thus induces a partition of the set of examples (into a truth set and a falsity set), which affords a structural analogy between each statement and the corresponding set of objects to which it refers. This shared structure can unify these statements across the mathematical contexts (number theory, geometry, and calculus). We offer evidence for how unitizing predicates can be consequential for how students perceive logical structure within each context. Unitizing predicates influences how students reason about *truth conditions*—meaning

¹ We refer to these theorems by their numbering in our teaching experiment, so they are nonconsecutive.

conditions for when any universally quantified conditional is true or false—and types of proof (e.g., direct or contrapositive).

We explore how students connect theorems to help us understand the necessary cognitive work involved for students to construct logical form (i.e., shared syntax and reference structure) and reason about the logic of proofs. Imagining this work as translation from mathematical language to some formal language—such as a logical calculus or a diagrammatic representational system—can be helpful. Many previous studies document how students’ reasoning about mathematical language (and everyday language) differs from the intended structure of the formal language (Epp, 2003; Roh & Lee, 2011; Selden & Selden, 1995; Sellers et al., 2021; Stylianides et al., 2004). This gap is widely acknowledged as a barrier to students’ fruitful participation in proof-based mathematics at the university level.

How might students bridge this gap? Most studies of logic learning and transition to proof textbooks (e.g., Hammack, 2013) make the seemingly natural move of teaching logical structure in a formal language (ps , qs , truth tables, etc.) so that students might be able to accomplish this translation in context (Dawkins et al., 2022). In this approach, students might construct analogies among the statements given at the outset of this article by replacing the text after “if” and “then” with logical placeholders p and q that merely have truth values T or F. This entails a view of logic as removing meaning from statements and suggests that students would abstract logical relationships by ignoring aspects of the statements that are not shared (such as whether the statement is about integers, quadrilaterals, or functions). In other words, abstraction occurs in this view by removing information instead of constructing some new unifying structure. However, what is not shared between these statements is almost everything of interest about number theory, geometry, or calculus. This translation therefore involves a huge loss of information. As a consequence, this treatment of logic is not able to capture how the “if” and “then” parts of each statement are related through meaning (Dawkins & Norton, 2022; Piaget & Garcia, 1991).

We conducted constructivist teaching experiments (Steffe & Thompson, 2000) to investigate an alternative approach to how students might bridge this gap. Our approach does not intend for students to remove meaning from the statements they read, but for them to structure their interpretation of each statement (and their proofs) in a manner that affords structural analogies and repeated reasoning. In their survey of research on the teaching and learning of

logic, Durand-Guerrier et al. (2012) argued that “it is important to view logic as dealing with both the syntactic and semantic aspects of the organisation of mathematical discourse” (p. 385). They defined *syntax* as “the rules of integration of signs in a given system” (p. 378; i.e., the system of formal language) and *semantics* as “the relationship between signs and objects” (p. 378; i.e., how properties in the statements refer to objects). In the four statements from the beginning of this article, the shared “if . . . then . . .” grammar constitutes part of the syntax whereas the meanings of the number theoretic and geometric terms constitute part of the semantics. For logic to attend to both, it must capture how the theorems refer to mathematical objects (see Dawkins, 2019; Dawkins & Norton, 2022), rather than merely how to remove meaning from the statements to render them the same. We employed an approach to logic in which students construct a shared syntax by restructuring their reasoning about meaning. The present article contributes to answering the following research question central to our overall research agenda: How can students abstract logical structure and relationships in a manner that integrates with their reasoning about various mathematical topics? We propose the construct *unitizing predicates* to characterize how students can abstract the semantic structure of their mathematical reasoning about particular statements to afford the construction of logical structure that generalizes across contexts. We think of this mental action as an important cognitive activity that supports students to make productive use of formal logic in their reasoning in proof-based mathematics.

Goals for the Teaching Experiments and Tasks Used Therein

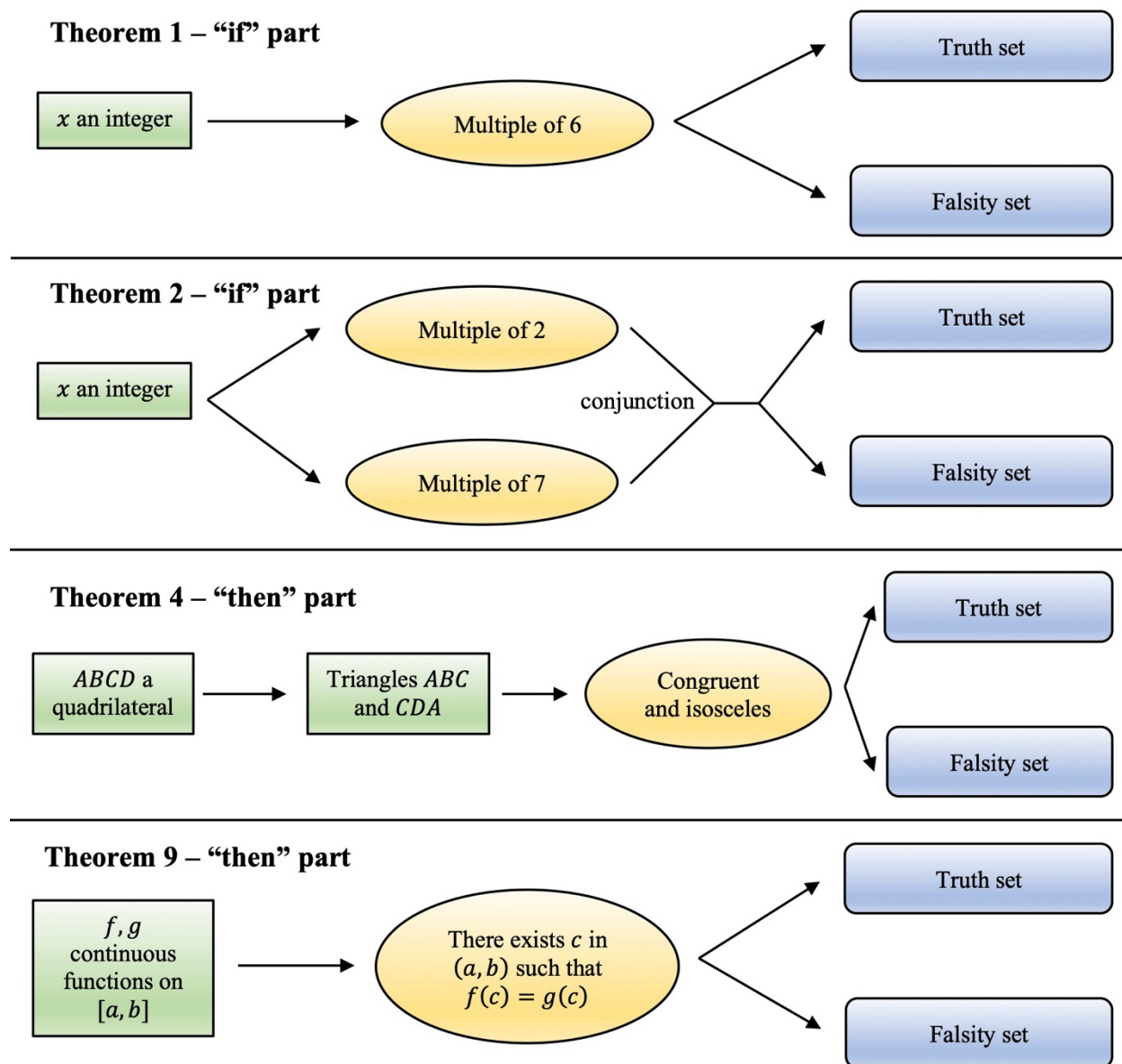
Although our broad goal was for students to construct logical structures that could unify different theorems and proofs, we needed specific learning goals to design learning activities. In this section, we briefly describe the learning goals of the teaching experiment featured in this article and the tasks used to pursue those goals. This will provide context for the literature we review in the next section. The data presented come from a teaching experiment with two undergraduate computer science majors whom we call Moria and April. We have conducted multiple experiments in this sequence. In the early experiments, we presented students with lists of statements of the same logical form, which they had to declare true or false and later negate (Dawkins, 2017, 2019; Dawkins & Cook, 2017; Hub & Dawkins, 2018). In the study with Moria and April, we presented the participants with universally quantified conditional theorems (including Theorems 1, 2, 4, and 9), each alongside two to four related proofs (to be described in

greater detail later). The students' tasks in each case were to determine (a) whether each proof proved or did not prove the given theorem and (b) if it did not prove the theorem, what other statement it proved or disproved.

We focused on students' ability to see various theorems as having the same form, so as to support reasoning about proof. To research this ability, we intentionally chose theorems featuring various relationships between objects and properties. If we conceptualize a predicate as a truth function that maps each object to a truth value (T if the object has the property or F if the object does not), then we can see how the predicates in the statements vary (Figure 1). In the diagram, rectangles represent objects that serve as inputs for the predicates, ovals represent properties the objects may or may not have, and rounded rectangles represent the sets of objects as partitioned by the truth-value output of the predicate. Both conditions in Theorem 1 are familiar such that students often perceive that these predicates correspond to a single category of numbers. The antecedent (the "if" part) in Theorem 2 is a conjunction, meaning students must somehow combine the properties to yield a new single predicate. The consequent (the "then" part) in Theorem 4 involves auxiliary objects (the triangles). Finally, Theorem 9 is the most complex in that the input of the predicates is pairs of functions, the antecedent is a conjunction, and the parts of the statement deal with different inputs (of the functions f, g), one of which is existentially quantified.

Figure 1

The Variety of Predicate Structures in Theorems 1, 2, 4, and 9



Our operative hypothesis guiding the experiment with April and Moria was that by reading and comparing proofs and their syntax, we could guide them to reason about principles of logical structures of mathematical proof regarding a theorem of the form “For all x in S , if $P(x)$, then $Q(x)$.” In particular, we wanted them to construct the principle of *contrapositive equivalence*, which entails that contrapositive proofs, which are those that begin “let not $Q(x)$ ” and end “thus, not $P(x)$,” prove the given theorem. The formulation of a logical principle like this assumes two things: (a) the efficacy of a proof depends on the syntax of the proof relative to the form of the conditional statement to be proven, and (b) this syntactic relationship between proof

and theorem does not depend on the particular choice of predicates P and Q . Because of the second assumption, we hypothesize that students may need to construct a unifying structure that helps them see these claims as relevantly similar. We claim unitizing predicates is a key part of constructing such a shared structure, specifically to stand in the place of semantic reference. Because of the first assumption, students may need to also construct some general account of how the proofs justify a theorem. This account must afford both the significance of the direction of the theorem/proof (from “if” to “then,” or from first line to last line) and the influence of negation.

Literature on Conditional Statements and Logic Learning

In this section, we review notable findings from psychological and mathematics education literature for background about the challenges and opportunities in student learning of logic. The psychological literature provides some insights about how people usually interpret conditional statements in everyday and abstract contexts. These findings inform us about why seeing various conditional statements as the same is challenging. Despite the variation in reasoning about conditionals depending on context, some recurrent patterns are worth noting, especially because they often conflict with formal logic. Specifically, unitizing predicates conflicts with some everyday modes of interpreting statements, which means that construing statements as we described earlier must be taught and learned.

The mathematics education literature offers further evidence that students need opportunities to learn how to interpret mathematical conditionals in ways compatible with formal logic. For students to consciously reason about the logic of mathematical conditionals, they need access to unifying ways of interpreting all such conditionals. Our focus on predicates and truth/falsity sets is only one such approach, so we will explain our choice to focus on it in our teaching experiments.

Cognitive Psychology’s Insights About How People Reason About Conditional Statements

Beginning with Wason’s (1966) famous studies, cognitive psychologists have extensively studied how people reason about statements in conditional form. Three major findings are worth noting (see Evans & Over, 2004, for a helpful summary):

1. How people interpret such statements varies depending on the context of the statement.

2. People have a tendency to reason about the statement “if P , then Q ” by focusing on the cases in which P and Q are true.
3. (Modifying the second phenomenon) the presence of negations greatly affects the patterns of interpretation such that people often interpret “if P , then not Q ” by focusing on cases in which P and Q are true.

Though many theories are available for explaining the various phenomena, probably the most well-evidenced theory is called the *suppositional account* that is based on the *Ramsey test* (Evans & Over, 2004; Oaksford & Chater, 2020). This theory posits that people affirm a conditional statement “If P , then Q ” whenever the conditional probability of Q is high given that P is true, which is the Ramsey test. This model is called the suppositional account because it assumes that people reason about the conditional by supposing the antecedent (P) is true. A key consequence of the probabilistic nature of this model is that people will affirm a conditional even in the presence of counterexamples (cases in which the antecedent is true and the consequent is false), which is an important empirical finding that this theory explains better than others.

How does the suppositional account compare with the interpretation of conditional statements in formal logic? First, if people simply imagine cases in which the antecedent is true, then the antecedent does not operate as a predicate (meaning it has both a truth set and a falsity set). People simply suppose it is true. Formal logic posits that a conditional is true whenever the antecedent is false, but most people believe that conditionals are simply irrelevant in such cases (Hoyles & Küchemann, 2002; Wason, 1966). This poses a challenge to contrapositive equivalence. If conditionals are only about the cases in which the antecedent is true, then often a conditional and its contrapositive will be about completely disjoint sets of objects. This makes it hard to argue how they could be related, much less equivalent.

Second, under this model the consequent also does not operate as a predicate on the whole universal set. Rather, people consider only whether the consequent is likely among the cases in which the antecedent is true.

Third, formal logic posits that a conditional is false whenever any counterexample exists. This is important for contrapositive equivalence because the conditions for being a counterexample to “if P , then Q ” and “if not Q , then not P ” are identical (P and not Q). Thus, contrapositive statements either both have or both lack counterexamples. However, the relative frequency of those counterexamples in the two statements can be quite different, meaning

contrapositive statements are not equivalent. Thus, the suppositional model helps explain why formal logic is a weak descriptive model for how people reason in the everyday. These findings suggest why instruction is needed to help students interpret mathematical statements in a manner more compatible with formal logic, such as interpreting parts of conditional statements as predicates.

Reasoning About Conditional Statements in Mathematics

Formal logic is often taught in undergraduate mathematics as part of the transition to proof-based mathematics (David & Zazkis, 2020; Dawkins et al., 2022). Logic must support students in comprehending some unifying structure among theorems in different mathematical contexts. It also should support them in understanding and justifying logical principles such as contrapositive equivalence. Although teaching approaches vary (see Durand-Guerrier & Dawkins, 2020), some key tools for teaching the logic of conditionals are available in the literature. First, we consider the traditional approach to teaching logic using symbols such as logical variables and truth tables, which appears in most transition-to-proof textbooks (Dawkins et al., 2022). Related to this, we first consider some findings about logic learning in the context of direct logic instruction. Next, we review the suppositional account when adapted to mathematical proving. Third, we consider the eliminating-counterexamples approach developed by Yopp (2017). The final section presents the set-based approach we use, which builds on our prior investigations.

Truth Tables and Abstract Syntax

Because logic is viewed as remaining the same when we vary the content or context of statements, it is often taught using abstract symbols that remove context or meaning (e.g., logical variables such as P and Q and truth tables). What are the results of such instruction? Stylianides et al. (2004) investigated students' understanding of contrapositive equivalence (in both everyday and mathematical contexts) after students had been taught the principle directly. Normative application of contrapositive equivalence varied greatly by context, and students applied it in mathematical settings only about 30% of the time.

Hawthorne and Rasmussen (2015) studied students' conceptual understanding of logical syntax and truth tables as taught in a discrete mathematics course. They found that logical notation posed barriers to students' meaning-making. They observed that students often struggled to interpret logical symbolisms as expressing a unified meaning for the truth and

falsehood of conditional statements. In framing the alternative, unified interpretation, Hawthorne and Rasmussen (2015) drew on Steffe's (1983, 1992) notion of unitizing numbers and Sfard and Linchevski's (1994) notion of reifying an algebraic expression. We find the analogy with algebra quite helpful inasmuch as teachers may try to teach—and students try to learn—algebra as manipulation of symbols that point to nothing beyond themselves (that is, they are purely syntactic). This approach to algebra learning has widely been discouraged among mathematics educators in favor of the more meaning-based approach in which algebraic symbol use is grounded in a semantically rich way of thinking that builds on cognitive construction of quantities (Hackenberg, 2013; Hackenberg & Lee, 2015). A similar commitment to meaning-making in logic guides the current study.

The Suppositional Account and Warranted Implications

Durand-Guerrier et al. (2012) criticized abstract instruction on logic because it divorces syntax from semantics. Some evidence even suggests that these formalisms do not adequately reflect the ways mathematicians reason (Rav, 1999). When Inglis (2006) studied how mathematicians reasoned about conditional statements in mathematics, he found that the suppositional account with one modification best captured their reasoning. Mathematicians largely reasoned about the conjectures by imagining the case in which the antecedent was true, but they did not affirm a conditional in the presence of counterexamples. They wanted to confirm that the consequent was certain given the antecedent, not just highly likely. Whereas other forms of reasoning gave mathematicians high levels of certainty, the mathematicians held that absolute certainty came from mathematical proof. Similarly, Weber and Alcock (2005) found that (in conflict with the truth-table definition) mathematicians wanted to affirm a conditional only when they believed the consequent could be proven from the antecedent, which the authors called *warranted implications*. We use the phrase “warranted implications” to refer to the findings described in both studies. That the meaning of conditional statements in mathematics is deeply connected to proving (both as warrants and as statements to be proven) is an essential insight. For this reason, our teaching experiment uses proof texts as a context for students to reason about logical structure.

However, we doubt that warranted implications alone will be sufficient to support novices in logical abstraction for two reasons. First, this criterion may not help students to see various theorems as the same because the actual proofs used to prove different statements vary

greatly. The concept of warranted implications captures the semantic aspect of logic—the meanings of the terms and relationships between terms—but it does not provide sufficient tools for reasoning about syntax and how it is shared across contexts. Second, we do not see clear ways that warranted implications can help students discover and justify logical principles such as contrapositive equivalence. As a result, we sense the need for an alternative truth condition for mathematical conditionals (meaning a generalized criteria for when any mathematical conditional is true and when it is false).

Eliminating Counterexamples

Yopp (2017) developed an instructional sequence intended to support middle school students in mathematical argumentation. His approach encouraged students to prove general conditional claims—“For all $x \in S$, if $P(x)$, then $Q(x)$ ”—by *eliminating counterexamples*, which means to argue why the negation—“There exists some $x \in S$ such that $P(x)$ and not $Q(x)$ ”—must be false. Part of the power of this approach is that shifting to the negation allows students to focus on specific examples rather than arbitrary examples. The instructional sequence invited students to list examples that make $P(x)$ true and others that make $Q(x)$ false before arguing why no examples can do both. This focus on examples seems particularly useful for middle school students. Yopp (2017) offered evidence that, after direct instruction on contrapositives using eliminating counterexamples, at least some students understand quite well why contrapositive statements are equivalent.

Eliminating counterexamples is thus another possible truth condition for teaching logic. It bridges syntax and semantics because it shows the repeated structure of different statements without trying to remove meaning. We do not focus on this truth condition largely because it does not express what it means for a theorem to be true; it rather gives a falsehood condition which can be shown to fail. We instead opt for a closely related meaning rooted in truth sets.

The Subset Meaning

Our investigations of student learning of logic began with teaching experiments in which we presented university students (who had not yet been taught logic) with lists of mathematical statements of the same logical form (e.g., disjunctions or conditionals). In a series of articles (Dawkins, 2017; Dawkins & Cook, 2017; Hub & Dawkins, 2018), we reported our observation that the students who most powerfully abstracted a shared logical structure for the statements were those who tended to associate each part of the statement with the set of objects that made it

true. Dawkins (2017) referred to this propensity as *reasoning with predicates*. This form of set-based reasoning allowed students to see the various statements as having the same logical form as well as to develop generalizable truth conditions. Specifically, a disjunction—“Given any $x \in S$, $P(x)$ or $Q(x)$ ”—was true precisely when the union of the truth sets of P and Q covered all of S . The set-based truth condition for “given any $x \in S$, if $P(x)$, then $Q(x)$ ” is that the statement is true whenever the truth set of P is a subset of the truth set of Q , which we call the *subset meaning*. The falsehood condition matches that in truth tables and eliminating counterexamples, because failing the subset meaning means an object exists that makes P true and Q false.

The subset meaning is compatible with eliminating counterexamples. Indeed, Hub and Dawkins (2018) reported that one student affirmed a conditional using the subset meaning—everything that makes P true also makes Q true—and then affirmed its contrapositive using the *empty-intersection meaning*—the set that makes Q false is disjoint from the set that makes P true. This final truth condition is quite similar to eliminating counterexamples, though it focuses on sets of objects instead of examples.

We argue that the subset meaning bridges syntax and semantics. We also conjecture that it can help students formulate and justify logical relationships such as contrapositive equivalence. Because of the power we observed that such set-based reasoning had for student construction of logic, we tried to guide students to formulate the subset meaning in all our subsequent experiments.

In summary, the literature reports on four truth conditions for conditional statements that might be used to teach logic in mathematics: truth tables, warranted implications, eliminating counterexamples, and the subset meaning. Our experiment seeks to draw on the positive potential of the last three, though for the reasons described here, we most directly try to foster the subset meaning for the truth of a conditional statement.

Theoretical Framework

Assimilation and Accommodation

We frame our account of students’ logical abstraction using tools from radical constructivism (von Glasersfeld, 1995), which is a particular interpretation of Piaget’s (1970) genetic epistemology. In particular, given our interest in the preconditions under which students sense that various statements are the same, we draw on the constructs of assimilation and accommodation. As von Glasersfeld (1995) explained, “Assimilation must instead be understood

as treating new material as *an instance of something known*. . . . Cognitive assimilation comes about when a cognizing organism fits an experience into a conceptual structure it already has” (p. 62). Reasoners assimilate experiences to a scheme, which entails actions (physical or mental) operating within some goal structure. In other words, acting on an assimilation induces some sort of expectation on the outcome of the action (which is why schemes are often called “anticipatory,” Hackenberg, 2010, p. 383). Further, von Glasersfeld (1995) framed assimilation as largely passive (preconscious) in the sense of the cognizing agent simply recognizing a new experience as known, which entails ignoring some aspects that distinguish the current experience from prior experiences. Steffe (2010) provided a useful four-part model of a scheme comprising goal structure, situation, activity, and result. If we consider a mathematical example, someone may have the *goal* of counting a collection organized into equal size subgroups. Perceiving the *situation* of equal groups, they may assimilate to their multiplication scheme and carry out the *action* of multiplying, producing the *result* of a total count. Steffe (2010) pointed out that someone may assimilate an experience to this scheme from any of these parts. For instance, if some amount is called a “product,” then it may be assimilated as the result of multiplication. Schemes represent streams of goal-directed physical or mental activity (such as computation, transformation, or even argumentation) that can be evoked and drawn on in a variety of ways.

Inasmuch as assimilation to a scheme also assimilates to an expectation, the situation or result of the action may conflict with the expectation. This induces “perturbation,” which may lead a reasoner to search through other aspects of experience that they might have previously ignored. This search may lead the reasoner to modify the scheme in some way (possibly constituting a new scheme in the process), which “would be an act of learning and we would speak of an ‘accommodation’” (von Glasersfeld, 1995, p. 66). Relevant to our study, von Glasersfeld pointed out:

The child’s experiential world also comes to contain other people, and the almost constant interaction with them is an even richer source of perturbation and consequent accommodations. . . . The most frequent cause of accommodation is the interaction, and especially linguistic interaction, with others. (p. 66)

Piaget (1977/2001) distinguished two primary forms of abstraction—empirical and reflexive—that differ depending on whether they operate on physical objects and experiences (empirical) or the reasoner’s schemes and operations (reflexive). Given our interest in students’

abstraction of their interpretation of mathematical statements and proof texts, we intend for students to draw reflective abstractions. Whereas some schemes are helpful to achieve goals like meeting an organism's physical needs, von Glasersfeld (1995) pointed out that many schemes operating on a reasoner's own mental activity achieve different kinds of goals:

Operative schemes are instrumental in helping organisms achieve a relatively coherent conceptual network of structures. . . . The viability of concepts on this higher, more comprehensive level of abstraction is not measured by their practical value, but by their non-contradictory fit into the largest possible conceptual network. (p. 68)

Indeed, noncontradiction among concepts held by the reasoner is the primary criterion by which Piaget characterized viability within the system of meanings known as formal logic (Beth & Piaget, 1966).

Unitizing Numbers and the Analogy to Logic

The construct we propose—unitizing predicates—is inspired by the work of Steffe (1983, 1992), Hackenberg (2010, 2013), and others on students' whole number and fraction schemes. Although those scholars' findings do not directly inform our claims about student reasoning about predicates in logic, we see certain parallels in the ways mental actions need to be combined to construct fractional quantities and logical structures (units).

Steffe (1983, 1992), operating within his interpretation of Piaget's genetic epistemology, proposed the notion of levels of units in children's reasoning about whole numbers. Constructing place value relies on children thinking of groups of 10 as both a single unit (a 10) that contains units (10 ones). Fraction understanding similarly requires children to recognize that splitting a unit into five equal parts creates units that are fifths. The disembedding operation describes when a child can reason about a fifth apart from the original unit without destroying the unit (Hackenberg, 2010). Doing so requires the simultaneous awareness of fifth as itself a unit in relation to another unit (a second-level unit). A key idea here is that mental actions like equipartitioning and iterating units create new units (closure) that can be operated on in relation to the original whole. This is the parallel we draw when we claim that students need to be able to recognize that actions such as negation, conjunction, and composition of predicates all create new predicates (closure). We thus refer to this as "unitizing a predicate."

We should clarify the parallel we are making between units of quantification and units of logic. Just as lengths are the units in the activity of quantifying length, so predicates are units in

the activity of classification. We might say that claims and their truth values are the units of more traditional logic, but classes are more fitting for our set-based approach. Each predicate classifies objects into examples and nonexamples. A conditional statement expresses an entailment between two classifications: Being in class A entails being in class B. Just as reasoning about $7/5$ requires quantification relative to two units (fifths and ones), most mathematical statements require reasoning about multiple classifications. Students must coordinate these classifications for the same object or for whole groups of objects.

To see the parallel in greater detail, consider a student reasoning about the predicate “is a kite and is a parallelogram.” Students who understand the two definitions will have a scheme by which they can reason through whether any given shape is a kite and (separately) is a parallelogram. Ostensibly, this would allow them to sequentially reason through why a square has both properties. This sequence of mental activity is analogous to equipartitioning and iterating. The relevant question is whether students actively anticipate that shapes exist that are kite-parallelograms, thus anticipating forming the truth set, without carrying out the whole sequence of activity. Like unitizing of numbers, students who can reason with this second-order unit can also unpack the unit into its constituent parts (such as seeing the set of kite-parallelograms as the intersection of two sets or those things that pass both tests). We also consider the need to unitize the negation of a predicate, such as “is not a multiple of 3,” because we have observed that students often interpret this as saying, “not in the set of multiples,” rather than, “in the set of nonmultiples.” “Is not a multiple of 3” must be able to act as a predicate in its own right to construct normative logical structure.

Another important distinction that Steffe (1987, 1992) and Hackenberg (2010, 2013) made, drawing on Piaget’s notions of abstraction, is between what students can do only in activity versus what they can hold out for reflection and reorganization. Some children can perform the actions of equipartitioning a unit into five parts and then iterating one of those parts seven times, but they cannot coordinate this new amount as $7/5$ because the units being iterated do not maintain the relationship with the original unit. In our studies of logic, students are quite often able to reason powerfully about the truth and falsehood of the statements we give them in an activity. Whether they can productively hold out for reflection their interpretation of a statement or their justification of that statement to be able to see how it relates to other statements and other justifications is another matter. This need to hold out a logical structure for

reflection is precisely what we intend for the subset meaning to facilitate. Furthermore, we shall argue that students' ability to unitize complex predicates can support them in constructing a unifying structure that would afford logical abstraction.

Methods

Participant Selection

The data presented stem from a constructivist teaching experiment (Steffe & Thompson, 2000) conducted with two undergraduate computer science majors at a medium-sized public university in the United States. These students were recruited from a calculus 3 class and received modest monetary compensation for their time. Our criteria for recruitment were that students were in calculus 3, being thus ready to move into proof-based courses, and had not yet taken any proof-based courses. Furthermore, we investigated their prior knowledge of logic using an online screening survey (Roh & Lee, 2018). We recruited only participants who displayed logical inconsistency, which was identified as reading an example proof and either (a) declaring that it validly proved a theorem true while still declaring the theorem false or (b) declaring that it validly proved the theorem false while still declaring the theorem true.² We thus had evidence that the study participants did not interpret mathematical proofs in normative ways before participating in the experiment.

Tasks and Instructional Sequence

As portrayed in Figure 2, we met with April and Moria for six sessions, each of which lasted between 1 and 1.5 hr. In each session, students were presented with sheets of paper containing a theorem and two to four proof-texts related to that theorem. We also provided any relevant definitions and prior theorems used in the proof-texts. We told them that the proofs contained no errors and their task was to determine which proofs proved the given theorem. If they judged a proof did not prove the theorem, they were to determine what it proved or disproved, explaining their reasoning in every case.

Figure 2

The Theorems and Proofs Presented to April and Moria by Session

² Roh and Lee (2018) reported that 21% of their sample showed logical inconsistency after completing a transition to proof course. Unpublished data show that students recruited from calculus and later courses do so at higher rates.

Session 1	Theorem 1: For every integer x , if x is a multiple of 6, then x is a multiple of 3.	•Proof 1.1 – Direct Proof •Proof 1.2 – Disproof of Converse •Proof 1.3 – Proof by Contrapositive
	Theorem 2: For any integer x , if x is a multiple of 2 and a multiple of 7, then x is a multiple of 14.	•Proof 2.1 – Direct Proof •Proof 2.2 – Proof of Converse
Session 2	Theorem 2': For any integer x , if x is a multiple of 4 and a multiple of 6, then x is a multiple of 24. (false)	•Proof 2.1' – Direct Proof (invalid) •Proof 2.2' – Proof of Converse (valid)
Session 3	Theorem 6: For any line segment $\overline{AB}$, if a point X is on the perpendicular bisector of $\overline{AB}$, then $AX = BX$.	•Proof 6.1 - Proof of Converse •Proof 6.2 - Direct Proof
	Theorem 5: For all quadrilaterals $\square ABCD$, if $\square ABCD$ is a rhombus, then it is a parallelogram.	•Proof 5.1 – Direct Proof •Proof 5.2 – Disproof of Converse •Proof 5.3 – Disproof of Inverse
Session 4	Theorem 4: For any quadrilateral $\square ABCD$, if $\square ABCD$ is a rhombus, then the diagonal $\overline{AC}$ forms two congruent, isosceles triangles $\triangle ABC$ and $\triangle CDA$.	•Proof 4.1 - Direct Proof •Proof 4.2 - Disproof of Converse •Proof 4.3 - Proof by Contrapositive
Session 5	Theorem 3: For any integer x , if x is not a multiple of 3, then $x^2 - 1$ is a multiple of 3.	•Proof 3.1 - Direct Proof •Proof 3.2 - Proof by Contrapositive •Proof 3.3 - Proof of Converse •Proof 3.4 - Proof of Inverse
Session 6	Theorem 9: Given any functions f, g that are continuous on the domain $[a, b]$, if $f(a) = g(b)$ and $f(b) = g(a)$, then there exists some c in $[a, b]$ such that $f(c) = g(c)$.	•Proof 9.1 - Direct Proof •Proof 9.2 - Disproof of Converse •Proof 9.3 - Proof by Contrapositive

The intent of the task sequence was for students to recognize how the theorems and proofs had parallel structures, which might allow them to abstract logical structure and proof techniques. Having them read multiple such pairs allowed us to observe when students saw theorems or proofs as the same and how those comparisons influenced their decisions about

which proofs proved the given theorem (or what other statement each proof proved). According to the screening survey, we knew the students did not give normative answers about which proofs proved a given theorem. On the basis of prior research, we had reason to believe students would not find contrapositive equivalence intuitively obvious (e.g., Stylianides et al., 2004). These served as our learning goals against which we could frame students' interpretations and toward which we could try to prompt their reasoning.

On the basis of our prior findings, we wanted to encourage April and Moria in reasoning about whole sets of objects and the subset truth condition. To prompt learning, the teacher/researcher (Cobb & Steffe, 1983) often questioned April and Moria about whether the proof proved the claim for the whole set of objects and invited them to attend to the set of objects that made a condition true. He also invited them to compare across the proofs either of the same theorem or of different theorems (prompting perturbation and accommodation). Further, at particular points the teacher/researcher asked students to adapt their thinking about previous proofs to later proofs. Most specifically, Moria and April introduced Euler diagrams to portray when one set of objects was a proper subset of another or alternatively the same set. Because this was consistent with our goals to promote the subset meaning, the interviewer consistently asked them to construct such diagrams thereafter.

Iterative and Retrospective Analysis

Consistent with teaching experiment methodology (Steffe & Thompson, 2000), the first author served as the teacher/researcher and the second author served as an outside observer, keeping field notes throughout teaching sessions. Both sought to form and test models of student reasoning throughout the teaching sessions and met between sessions to discuss student reasoning, form hypotheses about student understanding, and plan the tasks for the subsequent session accordingly. Planning for subsequent sessions largely involved selecting the theorem/proof pairs for the following session and forming hypotheses about how the students would interpret the tasks (this process led us to have students read the theorems out of the numerical order we initially assigned them). After the experiment, we continued a similar process of modeling student reasoning by forming and testing hypotheses about student meanings through retrospective analysis.

In accordance with our goals for students to abstract logical structure rooted in their reasoning about meaningful semantic content, our modeling attended to a few different aspects

of student understanding. First, we analyzed students' meanings for the mathematical concepts at hand and how those afforded lines of inference that justified the theorem or reflected the argument in the proof (semantic reasoning). Second, building on the framework in Hub and Dawkins (2018), we analyzed how students made comparisons across the texts and their activity. This meant that students might connect or distinguish—

1. a given proof and its associated theorem,
2. two proofs of the same theorem,
3. proofs of different theorems,
4. their decisions about which proofs prove the given theorem, or
5. their reasoning/explanation about a given proof or how it proves a theorem.

Naturally, these comparisons—which may entail assimilations, accommodations, or student explanation or why two things are not the same and thus cannot be assimilated—all depend on the students' evoked meanings for the mathematical concepts. Specifically, we attended to whether their reasoning was rooted in domain-specific meanings (about number theory or geometry) or rooted in domain-generalizable meanings (grammatical or set structure).

We searched for evidence of students not assimilating new theorem/proofs to prior lines of reasoning or explanation, even though we as experts saw an opportunity for such assimilation on the basis of shared logical structure. We then focused our analysis on student resolution of perturbations that arose either when the interviewer (to prompt perturbation) asked the student to connect to or repeat prior lines of reasoning or when the other student adopted a conflicting interpretation. In some cases, the student elaborated their understanding of the proof in a manner that afforded later accommodation. We formulated the construct unitizing predicates to explain some such cases. By analyzing when April and Moria assimilated across tasks and their processes of accommodation, we refined our models of their logical actions that either afforded or constrained their construction of the normative logic of these proofs, specifically contrapositive equivalence.

Results

To develop the construct of unitizing predicates, we share two storylines. The first story we share traces April and Moria's reasoning about the proofs by contraposition (Proofs 1.3, 4.3, and 9.3), though we address other tasks for context. Contrapositive proofs are of particular interest because they relate to one of our key learning goals. This story is particularly useful for

portraying the influence of unitizing because both students initially denied that the latter two contrapositive proofs proved their respective theorems, but then April changed her mind after unitizing the predicates in the theorems. Furthermore, unitizing predicates supported her in adapting her argument for why Proof 1.3 did prove Theorem 1 to the latter two proofs by contraposition.

The second story portrays some of the limitations students faced when they did not unitize a predicate, meaning they did not construe a complex condition as a single property that every example either had or did not have. In particular, Moria reasoned about a conjunctive condition (“is a kite and is a parallelogram”) in a two-part sequence in a manner that functionally changed the logical structure of a statement so that for her it did not have the same form as those they had previously discussed.

Unitizing Predicates and Contrapositive Equivalence

Proof 1.3: Constructing an Explanation of Contrapositive Proof

Proof 1.3 (Figure 3) was the first proof by contrapositive that April and Moria read. For this proof, April produced her first argument for why a proof of this form did prove, to which we hoped she might assimilate later proofs by contraposition. According to April’s reading of Proofs 1.1 and 1.2, she clearly understood that all multiples of 6 are multiples of 3, but that some multiples of 3 were not multiples of 6. This awareness was important for her interpretation of Proof 1.3 (by contrapositive). Using the suppositional account (which posits that the truth of Theorem 1 corresponds to the conditional probability of x being a multiple of 3 given that x is a multiple of 6), we might expect her to see Proof 1.3 as irrelevant to the theorem because it was about a different set of numbers (multiples of 6 versus nonmultiples of 3). Upon reading Proof 1.3, April responded, “I think this is proving the theorem because it’s saying if it’s not a multiple of 3, then it can’t be a multiple of 6.” She added, “Because I already know that you have to have 3 times 2 times something to be a multiple of 6 . . . Without being a multiple of 3, it’s never going to equal a multiple of 6.” She then elaborated this argument relative to the proof:

Everything that you throw into it is going to give a remainder, how it’s set up . . . you’ve already eliminated the fact that there’s ever going to be a 3 in this, so it just doesn’t formulate. It’s like making a cake without the flour or the sugar.

Figure 3

Theorem 1 and Its Contrapositive Proof

Definition 1: We say the integer n is a **multiple of d** whenever there exists some integer k such that $\frac{n}{d} = k$. This may also be written $n = k * d$. This can also be stated as “ n is divisible by d ” or that “ d divides n .”

Theorem to be proven 1: For every integer x , if x is a multiple of 6, then x is a multiple of 3.

Proof 1.3: Let x be any integer that is not a multiple of 3.

That means when we divide x by 3, we get a remainder of 1 or 2.

Then there exists some integer k such that $x = k * 3 + 1$ or $x = k * 3 + 2$.

If k is even, then there exists some integer s such that $k = s * 2$.

Substituting into the equations for x , we see:

$$\begin{aligned}x &= (s * 2) * 3 + 1 \\&= s * 6 + 1\end{aligned}$$

or

$$\begin{aligned}x &= (s * 2) * 3 + 2 \\&= s * 6 + 2.\end{aligned}$$

This means x is not a multiple of 6, because x is 1 or 2 greater than a multiple of 6.

If k is odd, then there exists some integer t such that $k = t * 2 + 1$.

Substituting into the equations for x , we see

$$\begin{aligned}x &= (t * 2 + 1) * 3 + 1 \\&= t * 6 + 4\end{aligned}$$

or

$$\begin{aligned}x &= (t * 2 + 1) * 3 + 2 \\&= t * 6 + 5\end{aligned}$$

This means x is not a multiple of 6, because it is 4 or 5 greater than a multiple of 6.

We see a few critical aspects of April’s reasoning that are worth describing in some detail. First, her meaning for “multiple of d ” was to factor out a d in an algebraic expression. Hence, she recognized that being a multiple of 6 entailed being “3 times 2 times something.” Thus, her meaning for multiple allowed her to construct an argument that matched the argument in Proof 1.1 (Direct proof) and afforded her a sense of conviction about this entailment.

Second, she gave meaning to the negation of “multiple of 3” in terms of “give a remainder,” which she was able to connect to the various equations in Proof 1.3. In our prior studies we have found that students want to substitute a positive description for a negative predicate, which greatly facilitates their reasoning (Dawkins, 2017). April unitized the negation by giving it a positive meaning in terms of remainders.

Third, April explains why not being a multiple of 3 justifies the theorem using the analogy to ingredients. Because the factored out 6 is composed of a 2 and 3, then to try to make a 6 without a 3 is “like making a cake without the flour or the sugar.” This argument implicitly

draws on the empty intersection meaning in that it shows no number can both be a nonmultiple of 3 and be a multiple of 6. April did not elaborate this argument to say, “therefore, all multiples of 6 are multiples of 3,” possibly because they had already justified this claim in the prior discussion. What is significant is that her sense that “multiple of 6” entails “multiple of 3” allowed her to justify why this proof about nonmultiples of 3 proved the theorem. She thus coordinated negating the predicate “is a multiple of 3” with the previously constructed relationship between “multiples of 6” and “multiples of 3,” which is a marker of unitizing the predicates.

Proof 4.3, Part 1: A Different Line of Reasoning in Geometry

Proof 4.3 (Figure 4) was the second proof by contraposition that April and Moria read. It differed from Proof 1.3 both because it was in a different mathematical context and because it contained a more complex predicate in the conclusion. It provided our first opportunity to see whether April would assimilate this proof to her prior argument affirming proof by contraposition. She did not do so initially, as evidenced by her rejecting that the proof proved the claim. However, she later accommodated this theorem to her previous argument once she had unitized predicates relevant to the theorem and developed a strong sense that being a rhombus entailed the theorem’s conclusion.

Figure 4

Theorem 4 and Its Contrapositive Proof

We shall denote the distance between the points A and B as “ AB .”

Definitions: Two segments $\overline{AB}$ and $\overline{CD}$ are **congruent** if the distance between their endpoints is the same, $AB = CD$.

A quadrilateral is a **rhombus** if all of its sides are congruent.

A triangle is **isosceles** if at least two of its sides are congruent.

Theorem to be proven 4: For any quadrilateral $\blacksquare ABCD$, if $\blacksquare ABCD$ is a rhombus, then the diagonal $\overline{AC}$ forms two congruent, isosceles triangles $\triangle ABC$ and $\triangle CDA$.

Proof 4.3: Let $\blacksquare ABCD$ be a quadrilateral such that when we form the diagonal $\overline{AC}$, the triangles $\triangle ABC$ and $\triangle CDA$ are not both isosceles and congruent.

This means either the triangles are not isosceles, not congruent, or both non-isosceles and non-congruent.

If the triangles are not isosceles, that means that none of their sides are congruent.

This means $AB \neq BC$, which means $\blacksquare ABCD$ is not a rhombus.

If the triangles are not congruent, that means at least one pair of corresponding sides are not congruent.

Clearly, $AC = CA$, so it must be the case that $AB \neq CD$ or $AB \neq DA$.

In both cases, $\blacksquare ABCD$ is not a rhombus.

Therefore, $\blacksquare ABCD$ is not a rhombus.

By the time April and Moria read Theorem 4 in Session 4, they had shown strong evidence of interpreting both number theory and geometry proofs by reasoning about sets of objects. For instance, they had discussed Theorem 5 (“For all quadrilaterals $\blacksquare ABCD$, if $\blacksquare ABCD$ is a rhombus, then it is a parallelogram.”) at the end of Session 3. April explained that Proof 5.2 (disproof of the converse) was “proving that the converse is not true.” The interviewer asked her to state the converse and she replied, “‘if it is a parallelogram, then it is a rhombus,’ which is not true.” April then offered a revised statement of what the proof proved (as opposed to disproved): “if it is a parallelogram, it is not necessarily a rhombus.” This second statement showed her attention to the whole set of parallelograms, only some of which were rhombi. Moria explained that Proof 5.2 “is doing the converse, which is an issue in this scenario because rhombus is a special case of parallelogram.” She later elaborated:

Moria: I just think about it in set notation. . . . I drew you a beautiful diagram [Figure 5]. I have this beautiful equilateral in here, 'cause it's a part of the quadrilaterals, but it's not exhaustive.

Interviewer: When you say “equilateral” you mean, you’re calling rhombuses “equilateral?”

M: Yeah, 'cause all their sides are equal it's gonna be equilateral. Fancy words.

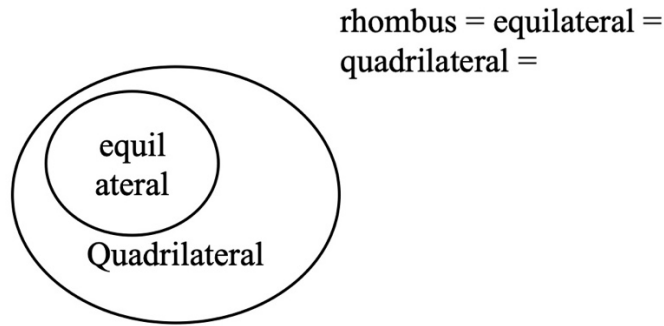
I: So that's what you mean by those circles? One inside the other.

M: Yeah, it's like nested, but not exhaustive.

Notice that Moria uses various phrases such as “special case,” “part of,” and “nested, but not exhaustive” to describe a (proper) subset relationship, which she then represented using an Euler diagram (Figure 5). However, Moria had shifted to relating rhombi to all quadrilaterals rather than to parallelograms.

Figure 5

Reproduction of Moria's Euler Diagram for Theorem 5



Theorem 4 was the first task we introduced with an unfamiliar property. After reading Proof 4.1 (direct proof), April summarized it saying, “It proves why a rhombus is an isosceles.” April thus used the term “isosceles” to stand for the antecedent condition. Reading Proof 4.2 (disproof of converse), she explained:

- April:* Why does 4.2 matter at all? Sure, it’s not always the case that a quadrilateral is a rhombus, but we don’t care. We’re saying “if it’s a rhombus, then this,” but if it’s not a rhombus it doesn’t seem relevant.
- Interviewer:* So you think Proof 4.2 is irrelevant to Theorem 4?
- A:* Yeah
- Moria:* Same page.
- A:* Yeah, ’cause you’re not even trying to prove the theorem. You’re talking about a completely different case from.
- M:* This is a different theorem I’d say.
- I:* Well, can you tell me what it does prove or disprove?
- A:* It proves that a quadrilateral does not mean rhombus.
- M:* We were just talking about this yesterweek, last week we were talking about this that just because it’s a quadrilateral doesn’t mean that it’s a rhombus because we’ve got these like nested conditions.

We interpret April’s argument that Proof 4.2 is irrelevant using the suppositional account (Evans & Over, 2004) in which one interprets a conditional by imagining the case in which the antecedent is true. In this case, this meant talking about rhombi. Because Proof 4.2 did not restrict itself to rhombi, it was irrelevant. From our perspective, this argument conflicted with

April's ability to see Proof 1.3 as proving Theorem 1, because the proof begins with cases outside the truth set of the antecedent (nonrhombi).

Proof 4.3, Part 2: Unitizing the Predicate "Fancy"

When the interviewer asked April and Moria what Proof 4.2 did prove or disprove, they assimilated this argument to their set-based model of Proof 5.2 from the previous session (Figure 5). As with that model, they did not construct a three-set model of antecedent (rhombi), consequent (parallelograms/made of two congruent, isosceles triangles), and universal set (quadrilaterals). Rather, they reasoned only about rhombi—what the theorem was about—and any other quadrilateral—irrelevant to the theorem. The interviewer thus began investigating how and whether they could think of the “then” part of Theorem 4 as having a truth set:

Interviewer: OK, so do you think this theorem is about any quadrilateral? Is what this theorem is about?

Moria: Theorem 4?

I: Uh, sorry no, not theorem. Proof 4.2. You're saying this is starting talking about any quadrilateral and it is not the case that any quadrilateral is a rhombus.

April: I mean it does limit the quadrilaterals with the first of the sentence, so it is still a subset of quadrilaterals but like rhombuses would be yet another subset within that subset. [. . .] 'Cause it's just saying that the diagonal will form two congruent isosceles triangles. So, a rhombus is like if all sides are congruent. [. . .] [*draws the diagram in Figure 6*] There is all quadrilaterals, and then there's whatever this is, and then there's rhombuses.

I: April, [*points to Figure 6*] what do you want that middle circle to mean? I see you have all quadrilaterals and rhombi, what's that middle circle?

A: Those are quadrilaterals such that the diagonal AC forms two congruent, isosceles triangles.

I: Mmhmm. You're bothered that you don't have a name for that?

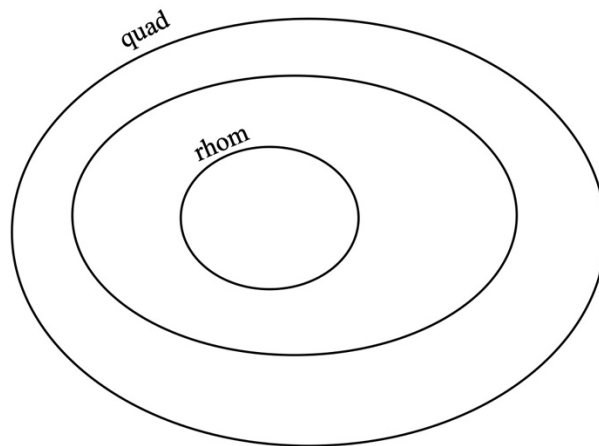
A: Yes. 'Cause I feel like there's a name for it.

I: Ah, well no. Well, I mean, the name for it could be quadrilaterals for which the diagonal AC forms two congruent, isosceles triangles.

- A: That is a lot of words to write in that circle.
- I: I agree. So, the question is, do you think that forms a group of triangles, or a group of quadrilaterals?
- A: Yes.
- I: And do you agree the way you have shown it in the picture it is not all quadrilaterals?
- A: Yeah, it's not all quadrilaterals.

Figure 6

Recreation of April's Euler Diagram for Theorem 4



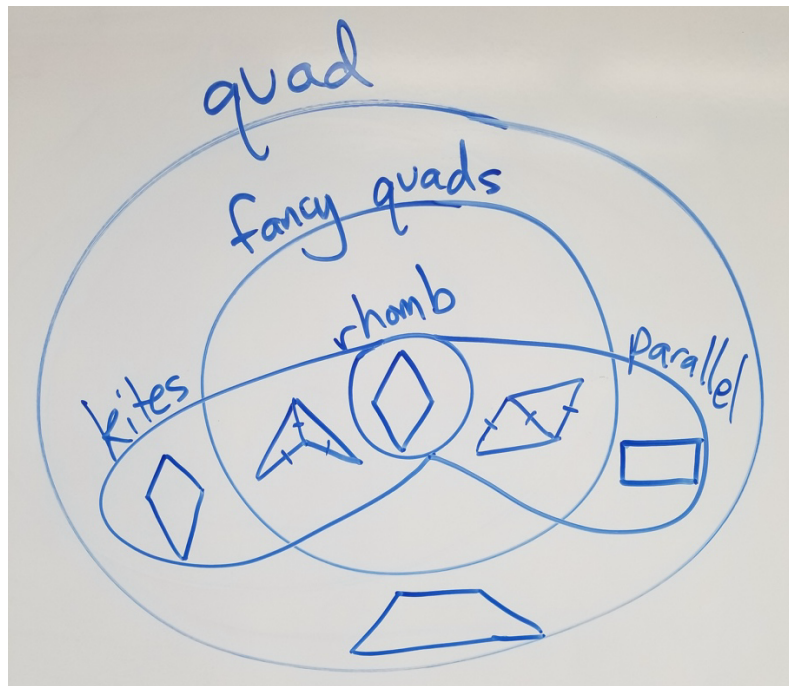
Pressed by the interviewer, April began to accommodate her model of the sets relevant to Theorem 4 to include both the truth set of the antecedent and the truth set of the consequent. She did not have particular trouble convincing herself that this condition constituted a distinct set nested within quadrilaterals that contained all rhombi, but it was a novel construction prompted by the interviewer question. April wanted a (brief) name for the set.

The interviewer invited April and Moria to identify examples outside that group, after which the pair placed example shapes in all three regions of the Euler diagram (a rhombus in the center, a chevron in the middle region, and a trapezoid outermost). The interviewer then invited Moria and April to name this middle set, for which April chose the name “fancy.” Moria wondered how this category related to parallelograms and concluded that they are “more conditional than what we are going for in this.” April drew in a region for parallelograms that

included rhombi but also stretched outside the fancy region. They again populated the new regions with example shapes (see Figure 7).

Figure 7

Reproduction of Moria and April's Euler Diagram for Theorem 4



When they then read Proof 4.3 (proof by contrapositive), April used the diagram to interpret the proof. She explained, “We’re in quadrilaterals [pointing to the outer region] and we can’t go into fancy quadrilaterals and therefore we cannot go into rhombuses.” Both students introduced analogies of dependence to explain their reasoning. Moria’s analogy was, “You didn’t pass algebra, so you can’t pass trig[onometry], so you can’t pass calc[ulus].” However, neither student at this point judged that Proof 4.3 proved Theorem 4. When the interviewer asked them to explain what Proof 4.3 proved, neither one explicitly connected the first line of that proof to “not fancy,” despite April’s association of the hypothesis with the outer ring of their diagram. Indeed, they seemed to struggle to articulate the hypotheses for Proof 4.3 (possibly because it involved negating the conjunction “congruent and isosceles”). Moria explained how if one of the two triangles was scalene, the unequal sides or the triangles having unequal area meant you did

not have a rhombus. Thus, although they found multiple ways to conceptualize how Proof 4.3's hypothesis entailed not being a rhombus, they did not judge this as relevant to Theorem 4.

This was an instance of students failing to assimilate this proof to the line of reasoning April exhibited for Proof 1.3, though we see their arguments as having the same structure. Rather, they may have assimilated Proof 4.3 to their suppositional reasoning that because Theorem 4 was about rhombi, any argument about other sets of quadrilaterals was irrelevant.

Proof 4.3, Part 3: Coordinating Negation With Property Entailment

Toward the end of Session 4, the interviewer wrote abbreviated forms of Theorem 4, its converse, its inverse, and its contrapositive on the board (using the phrase “not fancy” in the last two) with a reproduction of their diagram with three nested regions (like Figure 6). He asked Moria and April to interpret all four statements using the diagram. They were very comfortable explaining that Theorem 4 corresponded to the subset relation between rhombi and fancy quadrilaterals and that the converse was false because the relationship was “not exhaustive” and “you are going to have some fancy quadrilaterals that are not rhombuses.”

Both students agreed that the contrapositive statement (abbreviated as “If not fancy, then not a rhombus”) must be true. April explained that nonfancy quadrilaterals were in the outer region (covering it with both hands) and you cannot find a rhombus anywhere except the inner circle (the empty intersection meaning). The interviewer then returned to the question of whether Proof 4.3 proved Theorem 4. April now shifted to claiming that it can prove the theorem, explaining, “You need to have it so that it forms the isosceles and you need to have it that the sides are congruent. You need to have those properties to have a rhombus.”

What allowed April to shift from denying that Proof 4.3 proved Theorem 4 to affirming that it did prove it? First, she desired a way to conceptualize the “fancy” condition in Theorem 4 as a predicate of quadrilaterals. Initially, April referred to it as “isosceles,” which might mean she was thinking of it as a predicate of triangles and not of quadrilaterals, though she may have simply meant this as an elliptical reference to the long phrase. By naming the category “fancy,” she constituted it as a predicate of quadrilaterals. April and Moria then worked to arrange this new category within their existing categories of quadrilaterals, as portrayed in Figure 7. We thus see that unitizing predicates, in this case, entailed naming the condition, conceptualizing it, connecting it to other familiar categories, and negating it, as facilitated by the interviewer.

Regarding Proof 4.3, despite other relevant connections April made, she did not initially assimilate the hypothesis condition (“the triangles $\triangle ABC$ and $\triangle CDA$ are not both isosceles and congruent”) as equivalent to “not fancy.” Once the interviewer wrote down the contrapositive using the phrase “not fancy,” April and Moria assimilated Proof 4.3 to that statement and related it to their understanding that “rhombus” entailed “fancy.” April’s empty intersection argument for Proof 4.3—that no rhombi are among the nonfancy quadrilaterals—closely matched her argument for Proof 1.3. In the same way that having a factor of 2 and 3 was necessary to have a factor of 6, forming isosceles, congruent triangles was necessary to be a rhombus. Thus, April’s abstraction of her argument to the new mathematical context depended on her ability to unitize the predicate “fancy,” to unitize its negation in Proof 4.3, and to coordinate the entailments among the properties.

Proof 9.3

Proof 9.3 (Figure 8) was another proof by contraposition in yet another mathematical context. We observed a similar pattern to Proof 4.3 in which Moria and April initially did not assimilate it to the previous logical structure, but later did so once they unitized the predicates and constructed a sense of entailment between hypothesis and conclusion.

Figure 8

Theorem 9 and Proof 9.3 (Contraposition)

Intermediate Value Theorem: Given any function f that is continuous on the domain $[a, b]$, for any number N that is between $f(a)$ and $f(b)$, there exists some c in (a, b) such that $f(c) = N$.

Theorem to be proven 9: Given any functions f, g that are continuous on the domain $[a, b]$, if $f(a) = g(b)$ and $f(b) = g(a)$, then there exists some c in $[a, b]$ such that $f(c) = g(c)$.

Proof 9.3: Let f, g be continuous functions on the domain $[a, b]$ such that for all c in $[a, b]$, $f(c) \neq g(c)$.

Define the function $h(x) = f(x) - g(x)$ for all x in $[a, b]$.

h is continuous because f and g are.

Notice that since $f(c) \neq g(c)$ that means $f(c) - g(c) = h(c) \neq 0$ for all c in $[a, b]$.

This means h cannot have both positive and negative outputs on $[a, b]$ or else it would have a zero output by the Intermediate Value Theorem.

This means h is either always positive or always negative on $[a, b]$.

If $h(x)$ is positive for all x in the interval $[a, b]$, then $f(x) > g(x)$ on that interval.

From this we can infer that if $f(a) = g(b)$, then $g(a) < f(a) = g(b) < f(b)$; so $g(a) \neq f(b)$.

Or if $f(b) = g(a)$, then $g(b) < f(b) = g(a) < f(a)$; so $g(b) \neq f(a)$.

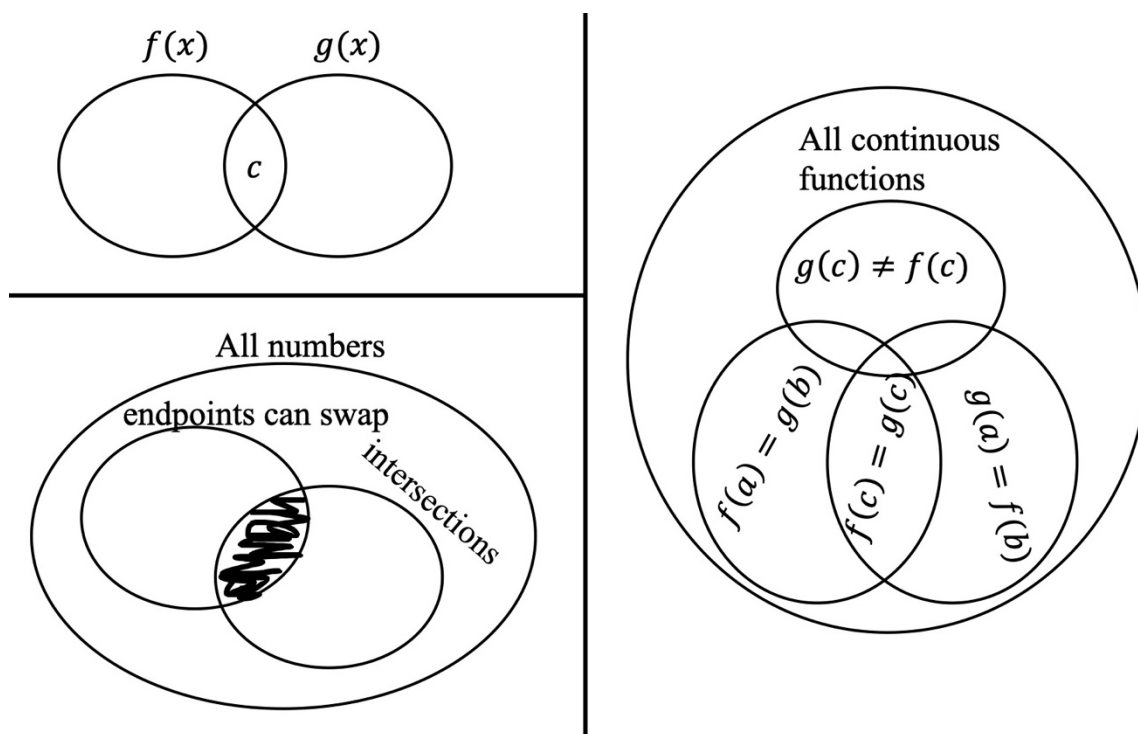
If $h(x)$ is negative for all x in $[a, b]$, then $f(x) < g(x)$ on that interval.

By a similar argument, it cannot be the case that both $f(a) = g(b)$ and $f(b) = g(a)$.

In the final session, Moria and April interpreted the proofs associated with Theorem 9 and tried to produce Euler diagrams for what the given proofs proved. Theorem 9 is more complex both because it is quantified over pairs of functions and because of the existential quantifier in the conclusion. Proof 9.3 (by contrapositive) begins “Let f, g be continuous functions on the domain $[a, b]$ such that for all c in $[a, b]$, $f(c) \neq g(c)$.” Figure 9 juxtaposes April’s initial diagram, her final diagram, and Moria’s final diagram. The first image portrays how April originally did not perceive this theorem as having the same structure as the previous theorems. She did not think of the regions as sets of (pairs of) functions, but rather associated each function to a set and the point of intersection to the overlap in the diagrams. Although previous conditional theorems conveyed a “nested condition,” she could not construe Theorem 9 as doing so. This again was an opportunity to see whether and how she could resolve the perturbation caused by the interviewer asking the students to construct a diagram like the ones they had previously produced.

Figure 9

Reproduction of Three Euler Diagrams for Theorem 9: April’s Initial (top left), April’s Final (bottom left), and Moria’s Final (right)



As the pair read Proof 9.2 (disproof of converse), they had trouble conceptualizing the conditions in Theorem 9. April originally described that the functions had to intersect at three points (which would be true if the antecedent stated $f[a] = g[a]$ and $f[b] = g[b]$). The interviewer helped them to construct a specific set of two functions that had the properties listed in Theorem 9. April then shifted from describing this as three intersections to saying that “endpoints can swap.”

When April and Moria initially read Proof 9.3, neither affirmed that it proved the theorem, though April assimilated this to the previous pattern of trying to prove, “If we don’t have the end, we can’t have the beginning.” One challenge in this proof by contraposition is properly negating the existentially quantified conclusion of the theorem. The interviewer asked April whether the hypothesis to Proof 9.3 was actually the negation of the end of Theorem 9. April confirmed this by explaining Theorem 9 “is expecting only one $[c]$ ” whereas Proof 9.3 said, “There is absolutely no cs , I didn’t choose the wrong c , there is no c here that could be an intersection.” Later in the interview, when April produced the middle diagram in Figure 9, April had unitized the hypothesis of the theorem to say “endpoints can swap” and the conclusion as saying the functions have “intersections.” She used the shading to portray that all functions for

which endpoints swap will intersect. Although her diagram was not the same image as the previous diagrams, April assimilated this to their idea of nested conditions. By the end of their discussion, April assimilated Proof 9.3 to her previous line of argument. She summarized, “If there is no intersection, there is definitely no endpoint swapping.”

We see strong parallels between April’s pattern of accommodation for Theorem 4 and Theorem 9. April needed to unitize the predicates in Theorem 9 to constitute properties that every pair of functions either had or did not have (like fancy quadrilaterals). Before doing so, she did not assimilate Theorem 9 to the set-based structure the students had used previously. As she found ways to unitize and name the conditions in the theorem, she was able to reconstitute the relationship as one of entailment as before. Once she construed Theorem 9 as one property entailing another, she was able to reiterate her argument regarding the contrapositive, namely that failing to have the consequent property precludes having the antecedent property.

Impediments to Unitizing Predicates

We noted in the previous episodes how, to reiterate her argument by contraposition, April had to (a) unitize the predicates, possibly by assigning a meaning such as endpoints swap, intersection, or naming the predicate; (b) unitize the negation through a meaning such as “give a remainder” or by negating a quantifier; and (c) construct relationships of entailment between predicates. This often took some time and discussion before she assimilated each new theorem and context to her subset meaning and her prior arguments. In this section, to further aid in characterizing the construct, we share an episode in which Moria struggled to unitize predicates in productive ways. This episode further demonstrates our argument that unitizing predicates serves as a precondition to students construing theorems and proofs as having the structures assumed within mathematical logic.

In one case, using the diagram in Figure 7, April conjectured that rhombi were the same as kite-parallelograms. The interviewer spontaneously asked the students to write proofs of Statement *: “Given any quadrilateral, if it is a kite and a parallelogram, then it is a rhombus.” They discussed how to begin:

April: So, we already have a quadrilateral, so we are already in this, so we just need to prove that it is a kite and a parallelogram.

- Moria:* So, we talk about the kite definition a little bit and the parallelogram definition a little bit and we extract what we need from both of them. Should we write down what we are thinking for proofs on both of those?
- A:* OK, so, it already, it has all the qualities of a kite and a parallelogram [. . .]
- M:* [Writes “For any quad $\blacksquare ABCD$, $\blacksquare ABCD$ is a rhombus if”] I am just trying to find a place for this definition. It says because . . .
- A:* It just doesn’t read like the proofs we have done in the past.
- M:* I realize I am doing more of a theorem statement than a proof [erases]. I feel better having written it out though.
- A:* Let something be something. Let our quadrilateral, or let $ABCD$ be a quadrilateral such that it has two pairs of adjacent sides that are congruent
- M:* OK how do we want to structure our proof? Are we gonna go, “because a quad, because a parallelogram has this identity and a kite has this identity, when you put them together you are going to get this” or should we start with the product and trickle our way out?
- A:* Either way. ’Cause what I was thinking is we can just define a quadrilateral that has all of the properties of the kite and all the properties of a parallelogram and then prove why.
- M:* OK, so you are thinking start with the product and then trickle through?
- A:* Yeah. Because that is what we are proving in the theorem is “if it is a kite and a parallelogram.” So, if it has the properties of a kite and the properties of a parallelogram, then it has the properties of a rhombus. Or is that not really proving anything, it’s just like picking information that works instead of proving?

The key disagreement that arose in this dialogue is between April’s idea that the quadrilateral “already, it has all the qualities of a kite and a parallelogram” whereas Moria wanted to consider the two conditions sequentially before you “put them together.” At this point, Moria drew a nonkite parallelogram and a nonparallelogram kite before drawing a rhombus with all sides marked congruent (see Figure 10). April instead drew two congruent sides verbally citing the kite property and then a third equal side while verbally citing the parallelogram property. She explained her difficulty, “I am trying to keep them separate but to have both of those properties,

everything has to be [congruent].” April was apparently aware that she could not draw a diagram of a kite parallelogram without drawing a rhombus, which may explain her question of whether this was “really proving anything.”

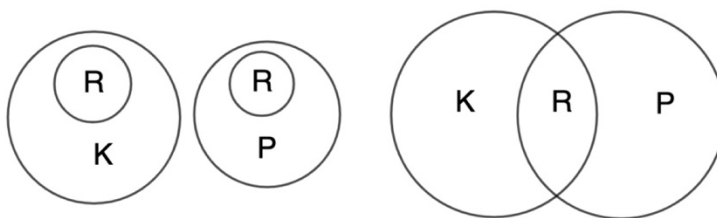
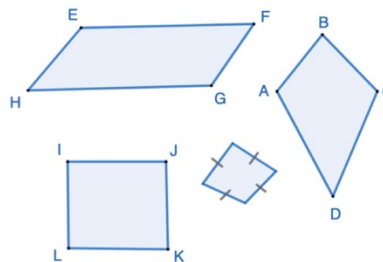
Figure 10

*Reproduction of Moria’s Proof and Diagrams for Statement **

Pf: By definition a kite has two pairs of adjacent sides that are congruent such that $AB = BC$ and $CD = DA$ but $AB \neq CD$.

And by definition a parallelogram has two pairs of opposite sides that are congruent such that $EH = FG$ and $HG = EF$ but $EH \neq EF$.

If the quadrilateral is a kite and a parallelogram all sides would be congruent, $IJ = KL$, $IL = KJ$ and $KJ = IJ$, therefore it would be a rhombus.



Moria proceeded to construct a proof following her idea of drawing separately from the two definitions (see Figure 10). She specifically gave the shapes distinct names and diagrams. Midway through Moria’s proof production, April asked for permission to write the proof “how I would word it.” April constructed her proof to mirror the structure of prior direct proofs, beginning with “Let $\blacksquare ABCD$ be a quadrilateral that is a kite and a parallelogram” and ending by claiming that it is a rhombus. The interviewer invited the students to compare and contrast their proofs. They both affirmed that the proofs “are saying the same thing.” Moria portrayed her thinking by drawing two separate Euler diagrams showing that rhombi were a “special case” of kite and also a subset of parallelograms (Figure 10). Moria represented April’s proof by drawing two overlapping circles, which she labeled kite (“K”) and parallelogram (“P”), naming the overlap as rhombus (“R”).

We conjecture that Moria had trouble unitizing conjunctive predicates as constituting a single predicate, such as “kite and parallelogram.” As further evidence of this, Moria never

connected this theorem to their subset meaning or to set equivalence. Specifically, her diagram for April's proof simply identified rhombi with the intersection of kites and parallelograms and she could not distinguish the two subset claims between these sets ($R \subseteq K \cap P$ and $K \cap P \subseteq R$). We explain this inasmuch as Moria saw subset and equivalence relationships as involving two sets, and Moria conceived Statement * as involving three sets: kites, parallelograms, and rhombi. Hence, she could not accommodate Statement * to the previous set relationships. We see the same pattern in her diagram in Figure 9 (regarding Theorem 9), namely that she represented a conjunctive condition using a pair of circles rather than one. Moria needed to construct the conjunctions in sequence by imagining a parallelogram and then making it a kite instead of simply constituting the compound predicate simultaneously, as April did. The ability to render a conjunction as a single predicate is another instance of unitizing predicates, which has consequences for abstracting logical structure across theorems and proofs.

Discussion

The goal of this article was to define and illustrate the construct unitizing predicates as well as to demonstrate some of the implications it can have for students' reasoning about logic. To unitize predicates is to recognize that, regardless of the types of objects discussed or the specific properties that define the categories, the conditions in universally quantified mathematical conditionals create categories of objects. This separation of objects into those satisfying a predicate and those that do not forms a structure that recurs across various theorems, which shows what these different statements share by virtue of their logical structure. Indeed, our findings demonstrate how unitizing predicates allowed April to construct a recurrent structure even in the presence of negations, conjunctions, and auxiliary objects in the theorems she read. More significantly, unitizing supported April in reiterating her argument for why a contrapositive proof proved a theorem. In other words, unitizing predicates was an important precondition both for her to construct a shared logical form for the various statements/proofs and to understand and justify logical relationships.

In the examples of both Proof 4.3 and Proof 9.3, April initially denied that the contrapositive proof proved the given theorem (she judged them irrelevant because they talked about a distinct class of objects, as predicted by the suppositional account). Once she unitized the predicates in the theorem, confirmed that the antecedent entailed the consequent, and unitized the negations of the predicates in the proof, she assimilated each new contrapositive proof to her

original argument as to why Proof 1.3 proved Theorem 1. In this sense, our set-based approach blends the shared syntax across the statements with the specific semantic features of each context. The syntax comprised the conditional grammar of the statements, the first-line to the last-line structure of the proofs, the fixed transformation from conditional to contrapositive, the recurring set structure, and the subset meaning for the truth of a conditional. The semantic features of each theorem included the predicates that she unitized and the entailments between the predicates as expressed in each proof.

April completed significant cognitive work to conceptualize each predicate in a way that allowed her to see it as a single condition that each example had or lacked. In addition, the interviewer played a crucial role in supporting this construal. For instance, when April seemed to search for a word for (what she later called) fancy quadrilaterals, the interviewer asked “do you think that forms . . . a set of quadrilaterals?” This was one of a few key moves that seemed to help foster unitization so that April could hold the logical structure out for reflection. These moves also included (a) reasoning about truth sets, (b) naming complex conditions, and (c) representing the sets/predicates using spatial diagrams. In this experiment, we stopped short of inviting April or Moria to adopt symbols that might be further used in abstraction and reflection, such as logical variables P and Q . Although we have fostered such symbol use in subsequent experiments, this experiment with April and Moria was vital in helping us understand the cognitive work that might render those symbols meaningful. We anticipate that further abilities to symbolize the logical structure may support rich abstraction, but not without the facilitating activity that students learn to unitize increasingly complex types of predicates, such as those that appear in later undergraduate proof-based courses.

Our second episode featuring Moria and April’s proof productions helps develop the construct unitizing predicates by showing what it looks like for a student not to have done so. For Moria, this occurred because of the conjunction of two familiar properties. Students who have not unitized a predicate may still be able to determine whether a given condition is true or false in a case, but this will require a sequence of activity. They will not see the condition as a property that objects have or do not have. In the case of negative conditions, students may not see “not a rectangle” as a property because it points to a lack. What is significant here is that such reasoning induces disanalogies between statements and proofs, which may preclude treating statements as logically the same.

This study of two students' reasoning has limitations with respect to the generalizability of the findings, which is why this article's goal was to define a construct that could be further explored. This study alone cannot speak to a number of related questions, such as how frequently students experience difficulty unitizing predicates, in what parts of the proof-based curriculum this activity is particularly important, how not unitizing predicates influences students' reasoning about logic, or how students taught using more conventional logic symbols and truth tables come to reason about the logical structure of mathematical statements and proofs in later courses. The current study took an interventionist approach, trying to foster learning along a particular trajectory. Although we encourage and are conducting more studies to understand this trajectory for teaching logic, we see much room for studying logic learning as it is currently occurring in classrooms. Mathematics education research on the learning of logic is still quite young, and future studies could do much more to investigate what students learn from symbolic logic instruction and how they draw on those ideas in subsequent proof-oriented courses. More broadly, when and how do students reason about logical structure as part of their proof-oriented mathematical work, what aspects of logical structure are important for this reasoning, and what mental activities and representation systems support their reasoning?

Future studies could also extend our understanding of unitizing predicates. We have identified a few challenges to students' ability to unitize predicates: negatively defined predicates, compound predicates (including "and," "or," and quantifiers), and predicates involving auxiliary objects. Therefore, researchers might explore ways to identify tasks in which students may benefit from engaging in unitizing predicates. In proof-based mathematics, we see two such tasks: reasoning about implications and constructing sets.

In addition, we saw how April's unitizing predicates helped her construct a sense of entailment between conditions. Many theorems have a hypothesis containing multiple conditions that students may need to unitize. Researchers therefore have an opportunity to explore whether students reason about these compound conditions as a single property that defines a set of objects and how that influences their reasoning about such theorems or the proofs of those theorems.

Furthermore, mathematicians use set-builder notation to construct sets defined by a given property. Research could explore how students reason about sets with complex objects and conditions (such as the set of all homomorphisms from one group to another) to learn more about whether and how students can reason about the defining conditions as a single property that

forms a category of objects. Scholars could explore whether and when the distinction between reasoning about properties in activity versus being able to hold out the category for reflection helps explain aspects of students' proof-oriented reasoning, as we see it did for April's reasoning about logic.

Conclusions

The primary contribution of this article is to set forth the construct unitizing predicates to describe a mental activity that may be important in students' abstraction of logical structure within mathematical statements and proofs. Unitizing predicates describes students' ability to construe a (possibly complex and multipart) mathematical condition as a predicate, which means a property that every example either has or does not have. We showed how the ability to unitize predicates helped one student to adapt her argument for why a contrapositive proof proved a conditional theorem to new theorem–proof pairs. In contrast, we showed an example of how not unitizing a compound predicate inhibited another student from interpreting a new theorem and its proof as having the same structure as the others she had read and discussed.

Unitizing predicates provides a useful research tool and learning goal for logic instruction. Consistent with the mathematics education commitment to help students to develop abstractions from within their own mathematical activity and to develop conceptual understanding of these abstractions, we have sought for students to construct logical structure using set-based reasoning. We argue that this construct holds promise for conceptual understanding of mathematical logic. We hope that this article contributes to ongoing design efforts to understand what logic learning entails and how it can help foster powerful ways of reasoning essential to proof-based mathematical activity.

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