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RESEARCH

A threefold violating a local-to-global principle for rationality

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Abstract

In this note we construct an example of a smooth projective threefold that is irrational over $\mathbb{Q}$ but is rational at all places. Our example is a complete intersection of two quadrics in $\mathbb{P}^5$, and we show it has the desired rationality behavior by constructing an explicit element of order 4 in the Tate–Shafarevich group of the Jacobian of an associated genus 2 curve.

Keywords: Rationality, Local-to-global principle

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1 Introduction

Let X be a smooth projective variety over a global field k, and suppose that for every place v of k, the variety $X \times_k k_v$ is k_v-rational (i.e. birational to projective space over k_v). Is X k-rational?

In dimension 1, k_v-rationality at all places implies k-rationality, since conics satisfy the Hasse principle, and existence of a point characterizes rationality for conics. In dimension 2, this local-to-global principle for rationality again holds if X is a del Pezzo surface of degree at least 5 [1, Section IV]. In smaller degree, however, there are examples of del Pezzo surfaces that are $\mathbb{Q}$-unirational and rational over all completions of $\mathbb{Q}$, but have a Brauer group obstruction to $\mathbb{Q}$-rationality [2, Example 3.3 and the following remark].

We study the case of threefolds, constructing a threefold complete intersection of two quadrics $X \subset \mathbb{P}^5$ over $\mathbb{Q}$ that is rational at all places but *irrational* over $\mathbb{Q}$. To our knowledge, ours is the first explicit such example of dimension ≥ 3 in the literature. Note that $\operatorname{Br} X = \operatorname{Br} \mathbb{Q}$ for such X [3], so the rationality obstruction used in the surface example of [2] vanishes; furthermore, $\operatorname{Pic}(X_{\overline{\mathbb{Q}}}) = \mathbb{Z}$ has trivial Galois action, so its Galois module structure does not obstruct rationality over $\mathbb{Q}$.

Theorem 1 *Let $\mathcal{X} = Q_1 \cap Q_2 \subset \mathbb{P}^5_{\mathbb{Z},[u:v:w:x:y:z]}$, where the quadrics Q_1, Q_2 are defined by*

$$Q_1 = uv + uw - 4vw + 2vz + 2wz + x^2 - 2xz + y^2 - z^2,$$
$$Q_2 = uv - uw + uy - 2v^2 + 2vx - 2wy + 2wz + 2xz.$$

Then $X := \mathcal{X} \times_{\mathbb{Z}} \mathbb{Q}$ is a smooth projective threefold that is $\mathbb{Q}$-unirational, $\mathbb{R}$-rational, and $\mathbb{Q}_p$-rational for all primes p, but is irrational over $\mathbb{Q}$.

Furthermore, the reduction of $\mathcal{X}$ modulo p is $\mathbb{F}_p$-rational for all primes $p \neq 2$.

The intermediate Jacobian of X is the Jacobian of the curve $C : z^2 = -t^6 - 3t^5 + 2t^4 + 3t^3 - 3t^2 - 3t - 2$, which appeared in [4]. Bruin–Stoll, when combined with recent work of Fisher–Yan [5], show that $\mathbf{Pic}^1_C$ is a nontrivial element of $Ш(\mathbf{Pic}^0_C)$. The recent rationality criterion of Hassett–Tschinkel over $\mathbb{R}$ [6] and Benoist–Wittenberg over arbitrary fields [7] (see also [8] for $k \subset \mathbb{C}$) shows that the existence of a k-point on a certain 2-cover of $\mathbf{Pic}^1_C$ determines rationality of X (see Sect. 2); thus, we show Theorem 1 by constructing an explicit example where this 2-cover violates the Hasse principle (and, more precisely, has order 4 in the Tate–Shafarevich group).

In [9] the authors, together with Sankar, Viray, and Vogt, construct examples of conic bundle threefolds over $\mathbb{Q}$ that are irrational over $\mathbb{R}$ (and hence irrational over $\mathbb{Q}$), rational over $\mathbb{C}$, and become rational modulo all primes of good reduction (for the discriminant double cover). Earlier, examples of threefold intersections of two quadrics that are irrational over $\mathbb{R}$ and rational modulo all primes of good reduction appeared implicitly in Hassett–Tschinkel's work [6, Construction 1, Theorem 36]. The difficulty in constructing an intersection of quadrics X as in Theorem 1 lies in distinguishing between rationality over $\mathbb{Q}$ and $\mathbb{R}$, and in determining the behavior at the bad primes. Note that X cannot have everywhere good reduction, since otherwise its intermediate Jacobian would be a nontrivial abelian variety with everywhere good reduction, which is impossible by [10, 11].

The Hasse principle for smooth intersections of two quadrics in $\mathbb{P}^5$ is an open question. Wittenberg showed it holds under the assumptions of Schinzel's hypothesis and finiteness of Ш for elliptic curves [12]. Recently, Iyer–Parimala showed that if X contains a k_v-line for all places v of k and if $\mathrm{disc}(Q_1) = 1$, then X contains a k-point [13, Theorem 0.2]. The condition $\mathrm{disc}(Q_1) = 1$ can be replaced with assuming that $\mathrm{Div}^1(C)(k) \neq \emptyset$[1], where C is the genus 2 curve associated to the discriminant of the pencil spanned by Q_1 and Q_2 [13, Corollary 10.2] (see also [14, Corollaire 8.7]). In their paper, Iyer–Parimala also study the period-index conjecture for C. A threefold intersection of quadrics defines a Brauer class on the associated genus 2 curve C by taking the Azuyama algebra of the even Clifford algebra associated to the pencil of quadrics (see Sect. 2.1). Iyer–Parimala conjecture that, for elements in $Ш(\mathrm{Br}\, k(C)) := \ker\left(H^2(k(C), \mathbb{G}_m) \to \prod_v H^2(k_v(C), \mathbb{G}_m)\right)$, the period is equal to the index. They prove their conjecture for the 2-torsion of $Ш(\mathrm{Br}\, C)$ under the assumption that $\mathrm{Div}^1(C)(k) \neq \emptyset$ [13, Corollary 5.4]. As a consequence of Theorem 1, we obtain an example of a Brauer class supporting their conjecture in the case when $\mathrm{Div}^1(C)(k) = \emptyset$:

Corollary 1 *Let Q_1 and Q_2 be as in Theorem 1, and let C be the genus 2 curve over $\mathbb{Q}$ defined by $z^2 = -t^6 - 3t^5 + 2t^4 + 3t^3 - 3t^2 - 3t - 2$ (which has $\mathbf{Pic}^1_C(\mathbb{Q}) = \emptyset$). Then the pencil of quadric fourfolds spanned by Q_1 and Q_2 defines a class $\beta \in Ш(\mathrm{Br}\, C)$ with period and index equal to 2.*

[1] $\mathrm{Div}^i(C)(k)$ denotes the set of k-rational divisors of degree i. $\mathbf{Pic}^i_C$ denotes the degree i component of the Picard scheme, and its k-points are the k-rational divisor classes of degree i.

From Q_1 and Q_2, β can be written down explicitly [15, Lemma 10], which may be of independent interest in the study of Brauer classes on curves.

After the first version of this article appeared on the arXiv, Kunyavskii [16] has used different methods to construct new examples of varieties violating the local-to-global principle for rationality and where the Brauer obstruction vanishes. He constructs examples of algebraic tori over any global field k, which are irrational over k because $\mathrm{Pic}(X_{\overline{k}})$ is not stably permutation, and surface examples in characteristic $\neq 2$, which are irrational over k by results of Iskovskikh on conic bundle surfaces.

Finally, Theorem 1 raises the following question:

Question 1 *Is there a smooth projective threefold over $\mathbb{Q}$ that is rational over $\mathbb{R}$, has $\mathbb{F}_p$-rational reduction mod p for all primes p, has a $\mathbb{Q}$-point, and is irrational over $\mathbb{Q}$?*

2 Intersections of two quadrics in $\mathbb{P}^5$ and genus 2 curves

In this section we recall results about the geometry of a pencil of two quadrics. The geometry of the variety of maximal linear spaces in an intersection of two quadrics is a rich theory that has been widely studied. We will only address the case of quadrics in $\mathbb{P}^5$ over fields of characteristic not 2, because this is the generality that we will require; we refer the reader to [3, 17] for other dimensions and to [7, Section 4] for arbitrary characteristic.

Let k be a field of characteristic not equal to 2, and let $X = Q_1 \cap Q_2 \subset \mathbb{P}^5_k$ be a complete intersection of two quadrics. Let M_1 and M_2 denote the Gram matrices of Q_1 and Q_2 with respect to the same basis, and define the polynomial

$$f(t) = -\det(M_1 - tM_2).$$

Then X is a smooth threefold if and only if the polynomial $f(t)$ is not identically zero and has 6 distinct roots [17, Proposition 2.1].

If X is smooth, then its intermediate Jacobian is the Jacobian of the genus 2 curve C defined by $z^2 = f(t)$, and its Fano variety of lines $F_1(X)$ is a smooth surface [17, Theorem 2.6]. It is classically known that X is k-rational if $F_1(X)(k) \neq \emptyset$, since projection from a line on X gives a rationality construction [3, Proposition 2.2]. Wang showed that $F_1(X)$ is a torsor under $\mathbf{Pic}^0_C$, and that $2[F_1(X)] = [\mathbf{Pic}^1_C]$ as torsors under $\mathbf{Pic}^0_C$ over k [18, Theorem 1.1]. By extending the Clemens–Griffiths intermediate Jacobian rationality obstruction to a torsor condition over non-closed fields and using Wang's work, Hassett–Tschinkel (over $\mathbb{R}$) and Benoist–Wittenberg (over arbitrary fields) characterized rationality of X over any field:

Theorem 2 [6, Theorem 36] [7, Theorem A] [8, Theorem 24] *Let $X \subset \mathbb{P}^5_k$ be a smooth complete intersection of two quadrics. Then X is k-rational if and only if $F_1(X)(k) \neq \emptyset$.*

In particular, X is rational at all places but irrational over $\mathbb{Q}$ if and only if the surface $F_1(X)$ violates the Hasse principle; this failure of the Hasse principle should be explained by a Brauer–Manin obstruction on $F_1(X)$ [19, Theorem 6.2.3].

As explained above, a pencil of quadrics naturally gives rise to a genus 2 curve. In the other direction, given a genus 2 curve C over k, Bhargava–Gross–Wang characterize when C can arise from an intersection of two quadrics in the above manner [20, Theorem 24]. Furthermore, in this case, they give conditions for the existence of lines on X.

Theorem 3 [20, Theorem 29] *Let $C : z^2 = f(t)$ be a smooth projective curve of genus 2 defined over a field k of characteristic $\neq 2$. The following two conditions are equivalent:*

1. *There exist quadrics $Q_1, Q_2 \subset \mathbb{P}^5$ over k such that $f(t) = -\det(M_1 - tM_2)$ and the intersection $Q_1 \cap Q_2$ contains a line over k.*
2. $\mathrm{Div}^1(C)(k) \neq \emptyset$.

The above result does *not* imply that every C over a global field k with local points everywhere is obtained from an intersection of quadrics $X \subset \mathbb{P}^5$ such that X has k_v-lines everywhere: it may be the case that no single choice of quadrics over k simultaneously works for all places.

Indeed, the existence of such an X implies the $\mathbf{Pic}^0_C$-torsor $F_1(X)$ is locally soluble and satisfies $2[F_1(X)] = [\mathbf{Pic}^1_C]$; hence, $F_1(X)$ is a locally soluble 2-cover of $\mathbf{Pic}^1_C$ (see [20, page 2] for definitions), and in particular the class of $\mathbf{Pic}^1_C$ is divisible by 2 in the Tate–Shafarevich group of $\mathbf{Pic}^0_C$. Bhargava–Gross–Wang showed that this necessary condition is sufficient:

Theorem 4 [20, Theorem 31] *Let k be a global field of characteristic $\neq 2$, and let $C : z^2 = f(t)$ be a smooth projective curve of genus 2 defined over k such that $\mathrm{Div}^1(C)(k_v) \neq \emptyset$ for all places v of k. The following two conditions are equivalent:*

1. *There exist quadrics $Q_1, Q_2 \subset \mathbb{P}^5$ defined over k such that $f(t) = -\det(M_1 - tM_2)$ and the intersection $Q_1 \cap Q_2$ contains a line over k_v for all places v.*
2. *$\mathbf{Pic}^1_C$ admits a locally soluble 2-cover over k.*

Combining Theorems 2, 3, and 4, together with the fact that $2[F_1(X)] = [\mathbf{Pic}^1_C]$ as $\mathbf{Pic}^0_C$-torsors, gives the following sufficient condition for the existence of an irrational X that is everywhere locally rational.

Corollary 2 *Let k be a global field of characteristic $\neq 2$. Let $C : z^2 = f(t)$ be a smooth projective curve of genus 2 defined over k. Assume the following conditions hold:*

1. $\mathrm{Div}^1(C)(k_v) \neq \emptyset$ *for all places v of k,*
2. *$\mathbf{Pic}^1_C(k) = \emptyset$, and*
3. *$\mathbf{Pic}^1_C$ admits a locally soluble 2-cover over k.*

Then there exists a smooth complete intersection of two quadrics $X \subset \mathbb{P}^5_k$ such that the intermediate Jacobian of X is $\mathbf{Pic}^0_C$, and X is k_v-rational for all places v of k but irrational over k.

In particular, conditions (1)–(3) imply that $\mathrm{Div}^1(C)$ violates the Hasse principle, and that $Ш(\mathbf{Pic}^0_C)$ contains an element of order 4.

Note that even if C satisfies the conditions in Corollary 2, it need not be the case that *every* complete intersection of quadrics with associated genus 2 curve C is rational at all places. (In the language of [20], the existence of locally soluble orbits does not imply that every orbit is locally soluble; see [20, Proof of Theorem 31].)

Remark 1 Condition (2) in Corollary 2 is not necessary for the existence of a threefold complete intersection of two quadrics violating the local-to-global principle for rationality. Indeed, over a global field k of characteristic $\neq 2$, if $\mathrm{Div}^1(C)(k) \neq \emptyset$, then any element of the 2-Selmer group of $\mathbf{Pic}^0_C$ corresponds to an intersection of quadrics X with locally soluble $F_1(X)$ [21, Theorem 11] [20, Theorem 31]. However, given an arbitrary complete intersection of quadrics X associated to such a C, it could be the case that $F_1(X)(k) \neq \emptyset$. If $\mathbf{Pic}^1_C(k) \neq \emptyset$, then we do not know a way to determine explicit equations for X ensuring that the torsor $F_1(X)$ is nontrivial.

2.1 A Brauer class arising from the intersection of two quadrics

The complete intersection X defines a natural Brauer class on the genus 2 curve C as follows. Let $\mathcal{Q} \to \mathbb{P}^1$ be the pencil of quadric fourfolds spanned by Q_1 and Q_2. The even Clifford algebra of this pencil defines an Azumaya algebra on the discriminant double cover $C \to \mathbb{P}^1$ and hence a Brauer class $\beta \in \operatorname{Br} C$. The proof of [22, Lemma 5.2.3] shows:

Theorem 5 (c.f. [22, Lemma 5.2.3]) *Let k be a field of characteristic $\neq 2$, and let $X \subset \mathbb{P}^5_k$ be a smooth complete intersection of two quadrics with $X(k) \neq \emptyset$. Then $F_1(X)(k) \neq \emptyset$ if and only if $\beta \in \operatorname{Br} C$ is trivial.*

While this Brauer class naturally arises as a Severi–Brauer variety of relative dimension 3 over C, the process of quadric reduction on the pencil of quadrics spanned by Q_1 and Q_2 gives rise to a Severi–Brauer variety of relative dimension 1 over C representing β [22, Theorem 1.8.7]. Hence, whenever the characteristic of k is not 2, if $X(k) \neq \emptyset$ and X contains no lines, then X defines a class in $\operatorname{Br} C$ of period and index equal to 2.

2.2 Genus 2 curves with $\mathbf{Pic}^1_C$ violating the Hasse principle

We construct our example by searching the literature for genus 2 curves satisfying the conditions in Corollary 2, performing the procedure described in [20, Section 2] (for suitable choices of s and α) to recover a complete intersection of quadrics X whose discriminant is associated to this curve, and then verifying that $F_1(X)$ has local points everywhere.

Any genus 2 curve C over a local field k_v necessarily has $\mathbf{Pic}^1_C(k_v) \neq \emptyset$ [23, Corollary 4, Footnote 10], but the existence of a k_v-rational divisor *class* of degree 1 does not necessarily imply that $\operatorname{Div}^1(C)(k_v) \neq \emptyset$. Moreover, many genus 2 curves C that have been previously shown to have interesting arithmetic do not satisfy the criteria in [20, Theorem 24] for $\mathbf{Pic}^0_C$ to arise globally as the intermediate Jacobian of a complete intersection of two quadrics. For example, a smooth genus 2 curve C over $\mathbb{Q}$ with no $\mathbb{R}$-points and with $\operatorname{Div}^1(C)(\mathbb{Q}_p) \neq \emptyset$ for all primes p has $\mathbf{Pic}^1_C(\mathbb{Q}) = \emptyset$ by [23, Theorem 11] (see [24, Section 3] for a description of the Brauer–Manin obstruction, and [23, $t > 0$ case of Proposition 26] for an example of such a curve). However, a genus 2 curve with $C(\mathbb{R}) = \emptyset$ will never arise from an intersection of quadrics over $\mathbb{R}$ [20, Section 7.2].

The example constructed in [23, Proposition 28] with both C and $\mathbf{Pic}^1_C$ violating the Hasse principle does arise over $\mathbb{Q}$ from an intersection of quadrics and satisfies condition (2) in Corollary 2; however, Poonen–Stoll show conditionally on the Birch–Swinnerton-Dyer conjecture that $\text{Ш}(\mathbf{Pic}^0_C) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, so condition (3) does not hold. See also [25] for additional examples of genus 2 curves where C and $\mathbf{Pic}^1_C$ violate the Hasse principle.

To prove Theorem 1, we use a genus 2 curve C from [4]. Bruin–Stoll and Fisher–Yan [5] exhibit 38 examples of curves C satisfying the conditions of Corollary 2. More precisely, Bruin–Stoll construct 42 curves C such that:

(a) C is locally soluble,
(b) (conditionally) $\mathbf{Pic}^1_C(\mathbb{Q}) = \emptyset$, and
(c) $\mathbf{Pic}^1_C$ has a locally soluble 2-cover

For (b), Bruin–Stoll reduce showing $\mathbf{Pic}^1_C(\mathbb{Q}) = \emptyset$ to computing the rank of the Jacobian of C. Then, they assume either the Generalized Riemann Hypothesis or the Birch–Swinnerton-Dyer conjecture for C in order to conditionally compute the rank for each of these 42 curves [4, Sections 2 and 3.4]. Recently, Fisher–Yan [5] developed a new method for computing the Cassels–Tate pairing on the 2-Selmer group of a genus 2 Jacobian, which allows them show that the Jacobian has expected rank for all but 4 of these curves. (The 4 unresolved cases are #2, #21, #23, and #33 [26].) In particular, they show that (b) holds unconditionally for these 38 genus 2 curves [5, Section 4.3].

3 Construction of the example

Let Q_1, Q_2 be the quadratic forms in Theorem 1, and let M_i be the symmetric matrix associated to Q_i. Then

$$-\det(M_1 - tM_2) = -t^6 - 3t^5 + 2t^4 + 3t^3 - 3t^2 - 3t - 2.$$

The genus 2 curve $z^2 = -\det(M_1 - tM_2)$ is curve #22 in the `BSD-data.txt` file [27] accompanying [4]. This and all of the computational claims in the proof below can be verified with `Magma` code provided in our `Github` repository [28].

3.1 Equations for $F_1(X)$

To show that X has $\mathbb{Q}_p$-lines for all primes p, we show that the Fano variety of lines $F_1(X)$ has smooth $\mathbb{F}_p$-points for all primes p and then apply Hensel's lemma. Such points are automatic for primes of good reduction by Lang's theorem, since in the smooth case $F_1(X)$ is a $\mathbf{Pic}^0_C$-torsor. For the primes of bad reduction, we must work with the explicit equations for $F_1(X)$. To do this, we work in a standard affine patch of $\mathrm{Gr}(2, 6)$; see for example [29, Example 6.6]. In the accompanying `Magma` code, we use the affine patch $\mathbb{A}^8_{(t_j)}$ of $\mathrm{Gr}(2, 6)$ parametrizing lines given by the row space of the matrix

$$\begin{pmatrix} t_1 & 1 & 0 & t_3 & t_5 & t_7 \\ t_2 & 0 & 1 & t_4 & t_6 & t_8 \end{pmatrix}.$$

Explicitly, on this chart of $\mathrm{Gr}(2, 6)$ we consider lines $\mathbb{P}^1 \hookrightarrow \mathbb{P}^5$ of the form $[r : s] \mapsto [t_1 r + t_2 s : r : s : t_3 r + t_4 s : t_5 r + t_6 s : t_7 r + t_8 s]$.[2] Requiring a line to be contained in the quadric Q_i defines conditions on the t_j (eqnsi in the `Magma` code); together these equations for Q_1 and for Q_2 define the Fano surface of lines on this affine patch. In the `Magma` code, this chart is called Fanopatch.

[2] We work with this particular affine patch because, on many other patches, $F_1(X)$ has no smooth $\mathbb{F}_2$-points. In practice, the most difficult condition to verify is the existence of $\mathbb{Q}_2$-points on $F_1(X)$.

3.2 Proof of Theorem 1 and Corollary 1

Starting from C, we found the equations for Q_1 and Q_2 by applying the algorithm described in [20, Section 2] for $s = \frac{1}{2}$ and $\alpha^{-1} = (-\theta^4-2\theta^3+4\theta^2-4)(1+2\theta+\theta^2-4\theta^3+2\theta^4+\theta^5)^5$ (in the notation of their paper), and then performing a simplifying coordinate change. Note that $[1:0:0:0:0:0] \in \mathbb{P}^5_{\mathbb{Q}}$ exhibits a $\mathbb{Q}$-point on X, so X is unirational over $\mathbb{Q}$ [3, Proposition 2.3].

Next, we show that X becomes rational at all places. To do this, we will show that $F_1(X)$ has a $\mathbb{Q}_v$-point for all places of $\mathbb{Q}$ (including the real place), so that X has a $\mathbb{Q}_v$-line and is thus rational over $\mathbb{Q}_v$ by [3, Proposition 2.2]. First, C has two real Weierstrass points, branched over points in $\mathbb{P}^1$ whose coordinates are approximately $[-3.26599 : 1]$ and $[-1.13643 : 1]$. Then [20, Section 7.2] shows that $F_1(X)$ necessarily has a real point.

Now fix a prime p. If p does not divide the discriminant of C, then the reduction of X mod p is smooth [17, Proposition 2.1], so $F_1(X)$ has a $\mathbb{Q}_p$-point (see Sect. 3.1 above).

Then it remains to check for the primes p dividing the discriminant of C that the reduction of $F_1(X)$ modulo p contains a smooth $\mathbb{F}_p$-point, which we do by checking directly on the affine open chart described in Sect. 3.1. The two primes dividing the discriminant are 2 and 149743897.

- For $p = 2$, a smooth $\mathbb{F}_2$-point is given by $(1, 1, 0, 0, 1, 1, 0, 0) \in \mathbb{A}^8_{\mathbb{F}_2}$ on the affine patch discussed in Sect. 3.1.
- For $p = 149743897$, on the same affine patch, a smooth $\mathbb{F}_{149743897}$-point is given by

$$(10276, 859210, 113976451, 113430900, 122036333, 94785567, 35411179, 25838500) \in \mathbb{A}^8_{\mathbb{F}_{149743897}}.$$

In each case, Hensel's lemma yields a $\mathbb{Q}_p$-point on $F_1(X)$ and hence an $\mathbb{Q}_p$-line on X. Thus, we have checked that X is rational at all places of $\mathbb{Q}$.

We next consider $\mathcal{X}$ over $\mathbb{Z}$ as given in Theorem 1, and show that the mod p reduction $X_p := \mathcal{X} \times_{\mathbb{Z}} \mathbb{F}_p$ is $\mathbb{F}_p$-rational for all primes $p \neq 2$. For $p \notin \{2, 149743897\}$ the reduction X_p is smooth, so this follows from [6, paragraph after Construction 1]. For $p = 149743897$, since we have shown that $X_{149743897}$ contains a line, [3, Proposition 2.2] implies that $X_{149743897}$ is $\mathbb{F}_{149743897}$-rational if it is not a cone. Checking that $X_{149743897}$ is non-conical is equivalent, by [3, Lemma 1.12], to checking that the Jacobian matrix of (Q_1, Q_2) does not vanish identically at any point. Indeed, the singular locus consists of a single point $[10925789 : 85737939 : 85378598 : 93099029 : 51694582 : 1] \in \mathbb{P}^5_{\mathbb{F}_{149743897}}$, and the Jacobian matrix has rank 1 at this point. Thus, we have verified $\mathbb{F}_p$-rationality of X_p for all $p \neq 2$. (The mod 2 reduction X_2 is reducible and non-reduced.)

It remains to show the irrationality of X over $\mathbb{Q}$. To show this, we claim that $F_1(X)$ has order 4 in $Ш(\mathbf{Pic}^0_C)$; by Theorem 2 this implies X is irrational over $\mathbb{Q}$. Since $2[F_1(X)] = [\mathbf{Pic}^1_C]$ in the Weil–Châtelet group of $\mathbf{Pic}^0_C$ [18], it suffices to show that $\mathbf{Pic}^1_C$ has no $\mathbb{Q}$-points. For this, Bruin–Stoll show that if the Jacobian of C has expected rank (which is 0 in this case), then $\mathbf{Pic}^1_C(\mathbb{Q}) = \emptyset$ [4, Section 2]. They then conditionally compute the rank, assuming the Birch–Swinnerton-Dyer conjecture for C. Recent work of Fisher–Yan *unconditionally* computes the rank [5, Section 4.3]; details are provided in the Magma code `g2ctp_examples.m` in [26], where this curve is called Bruin Stoll #22. Thus, $\mathbf{Pic}^1_C$ has no $\mathbb{Q}$-points, and the claim follows.

Finally, Corollary 1 follows from Theorem 1 (using Theorem 2 and the discussion in Section 2.1). Note that $\mathrm{Div}^1(C)(\mathbb{Q}) = \emptyset$ because $\mathbf{Pic}^1_C(\mathbb{Q}) = \emptyset$. □

Remark 2 In Theorem 1, local solubility of $F_1(X)$ at $\mathbb{R}$ is automatic because the curve C has two real Weierstrass points [20, Section 7.2]. In general, if C has any number of real Weierstrass points, Krasnov has characterized when X contains a real line, by giving an isotopy classification using the signatures of the matrices $t_0M_1 - t_1M_2$ as (t_0, t_1) varies over the unit circle in $\mathbb{R}^2$ [30].

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Code availability The `Magma` code accompanying this paper is available in our `Github` repository https://github.com/lena-ji/local-global-2quadrics.

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