J. reine angew. Math., Ahead of Print
DOI 10.1515/crelle-2024-0077

Journal für die reine und angewandte Mathematik

Symmetries of Fano varieties

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Abstract. Prokhorov and Shramov proved that the BAB conjecture, which Birkar later proved, implies the uniform Jordan property for automorphism groups of complex Fano varieties of fixed dimension. This property in particular gives an upper bound on the size of finite semi-simple groups (i.e., those with no nontrivial normal abelian subgroups) acting faithfully on n-dimensional complex Fano varieties, and this bound only depends on n. We investigate the geometric consequences of an action by a certain semi-simple group: the symmetric group. We give an effective upper bound for the maximal symmetric group action on an n-dimensional Fano variety. For certain classes of varieties – toric varieties and Fano weighted complete intersections – we obtain optimal upper bounds. Finally, we draw a connection between large symmetric actions and boundedness of varieties, by showing that the maximally symmetric Fano fourfolds form a bounded family. Along the way, we also show analogues of some of our results for Calabi–Yau varieties and log terminal singularities.

1. Introduction

In this paper, we study automorphisms of Fano varieties. The automorphism group of a Fano variety satisfies the so-called *Jordan property*. This property states that any finite subgroup of the automorphism group contains a normal abelian subgroup of bounded index. Moreover, for n-dimensional Fano varieties, there is a uniform upper bound for this index that only depends on n ([48, Theorem 1.8] and [6, Corollary 1.5]). In particular, if a finite semi-simple group acts on an n-dimensional Fano variety, its order is bounded above in terms of n. The symmetric groups S_n, for $n \geq 5$, are very natural examples of semi-simple groups.

In dimension 1, any symmetric group acts on some curve of general type; however, the symmetric actions on elliptic and rational curves are much more limited. For instance, S_4 is the largest symmetric group acting on a rational curve. In higher dimensions, the Jordan property gives a (non-explicit) upper bound for the order of symmetric groups acting on n-dimensional Fano varieties. Similar behavior is expected in the case of Calabi–Yau varieties (see, e.g., [42, Conjecture 4.47]). However, in neither of these cases do we understand how to

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During the preparation of this article, L. Esser was partially supported by NSF grant DMS-2054553, and L. Ji was partially supported by NSF MSPRF grant DMS-2202444.

control the size of the symmetric group in terms of the dimension of the variety endowed with the action.

As a first naïve example, one can consider symmetric actions on projective spaces $\mathbb{P}^n$. Although S_4 acts on $\mathbb{P}^1$ and S_6 acts on $\mathbb{P}^3$, in most dimensions, the largest symmetric group that acts faithfully on $\mathbb{P}^n$ is S_{n+2}, via the standard representation of that group in $\mathrm{GL}_{n+1}(\mathbb{C})$. In fact, S_{n+2} is the largest symmetric group inside $\mathrm{Aut}(\mathbb{P}^n) = \mathbb{P}\mathrm{GL}_{n+1}(\mathbb{C})$ for $n = 2$ and $n \geq 4$ (cf. Table 2). However, this example is not optimal, even among rational varieties: in each dimension n, there exists a smooth rational Fano variety that admits an S_{n+3}-action (Example 8.7). Moreover, there exist (conjecturally irrational) Fano varieties with even larger symmetric actions (see Section 8). On the other hand, based on work by J. Xu [62] on actions of p-groups for $p > \dim X + 1$, we give a first asymptotic upper bound for the order of symmetric groups acting on n-dimensional Fano varieties (Fano varieties in this paper have klt singularities, by definition).

Theorem 1. *Let $S_{m(n)}$ be the largest symmetric group acting faithfully on an n-dimensional Fano variety. Then we have that*

$$\lim_{n\to\infty} \frac{m(n)}{(n+1)^2} \leq 1.$$

By means of global-to-local techniques, we show that the previous statement admits an analogue for klt singularities.

Theorem 2. *Let $S_{\ell(n)}$ be the largest symmetric group acting faithfully on an n-dimensional klt singularity. Then we have that*

$$\lim_{n\to\infty} \frac{\ell(n)}{n^2} \leq 1.$$

Subsequent work of Kollár–Zhuang studies actions of p-groups for small primes to improve Theorem 1 and Theorem 2 to linear bounds [35, Corollary 20]. We emphasize that Theorem 1 is also expected to hold for Calabi–Yau varieties. However, Theorem 2 does not hold for log canonical singularities. Indeed, every symmetric group acts on some 3-dimensional log canonical singularity (see, e.g., [20, Theorem 6]).

1.1. Weighted complete intersections and toric varieties. For more restrictive classes of varieties, we can prove sharp bounds on symmetric actions in every dimension. First, we find the largest symmetric action on a simplicial toric variety (not necessarily Fano) in every dimension.

Theorem 3 (cf. Theorem 4.1). *Let X be a complete simplicial toric variety of dimension n. Suppose that the symmetric group S_k acts faithfully on X. If $n = 1$, 2, or 3, then $k \leq n + 3$; if $n \geq 4$, then $k \leq n + 2$.*

These bounds are sharp for each n. If equality is achieved and $n \neq 2, 4$, then $X \cong \mathbb{P}^n$. If $n = 2$, then $k = 5$ if and only if $X \cong \mathbb{P}^1 \times \mathbb{P}^1$. If $n = 4$, then $k = 6$ if and only if $X \cong \mathbb{P}^4$ or $X \cong \mathbb{P}^2 \times \mathbb{P}^2$.

For $n \geq 3$, however, toric varieties do not have the largest symmetric actions among all Fano varieties. Rather, we expect the optimal examples to be (quasismooth) weighted complete intersections. In Sections 5 and 6, we prove sharp bounds on symmetric actions on these varieties in every dimension. In particular, we prove the following result.

Theorem 4 (cf. Theorem 5.1). *Let $S_{m(n)}$ be the largest symmetric group acting faithfully on an n-dimensional quasismooth Fano weighted complete intersection. Then we have that*

$$\lim_{n\to\infty} \frac{m(n)}{n+1} = 1.$$

Though the asymptotics are the same as in the toric case, $m(n)$ is on the order of $n + \sqrt{2n}$ here. We obtain a precise formula for this $m(n)$ in Theorem 5.1, where we also prove an analogous statement for Calabi–Yau weighted complete intersections. We expect a similar statement to Theorem 4 in the case of weighted complete intersection klt singularities (see Question 8.16).

An n-dimensional Fano variety is said to be *maximally symmetric* if it admits the largest symmetric action among n-dimensional Fano varieties. We define *maximally symmetric* (quasismooth) Fano weighted complete intersections similarly. Our next aim is to describe maximally symmetric Fano weighted complete intersections.

The following theorem gives a characterization of maximally symmetric Fano complete intersections. In the theorem below, we say that a complete intersection in $\mathbb{P}^N$ is *totally symmetric* if its defining ideal is contained in the ring of symmetric polynomials in the variables $x_0, \ldots, x_N$.

Theorem 5. *Let X be a maximally symmetric quasismooth Fano weighted complete intersection of dimension $n \neq 2$, where the maximal action is by S_k. Then there is a finite cover $X \to Y$, where Y is a totally symmetric complete intersection in $\mathbb{P}^{k-1}$.*

In the theorem below, the *index* of $-K_X$ refers to the largest positive integer r such that $-K_X$ is divisible by r in the class group $\operatorname{Cl} X$. A *Fano–Fermat variety* is a Fano complete intersection in the projective space $\mathbb{P}^N$ that is cut out by Fermat hypersurfaces.

Theorem 6 (cf. Theorem 6.2). *Let X be a maximally symmetric quasismooth Fano weighted complete intersection of dimension n with largest possible index of $-K_X$, where the maximal action is by S_k. Then X is S_k-equivariantly isomorphic to a Fano–Fermat variety.*

More precisely, we will show that a maximally symmetric Fano weighted complete intersection with maximal index is isomorphic to the Fano–Fermat variety in Example 8.1.

1.2. Symmetries and boundedness. Boundedness of Fano varieties is an important topic in birational geometry. Kollár, Miyaoka, and Mori proved the boundedness of n-dimensional smooth Fano varieties [32]. Birkar proved the boundedness of n-dimensional Fano varieties with minimal log discrepancy bounded away from zero [6]. Other constraints on invariants are also known to give boundedness of n-dimensional Fano varieties, such as bounding the degree and alpha-invariant away from zero [31]. In these cases, the invariant that defines a bounded family of Fano varieties is a measure of singularities. We prove a boundedness result in a novel direction – we show that Fano 4-folds with large symmetric actions form bounded families.

Theorem 7 (cf. Theorem 7.8). *The class of S_8-equivariant klt Fano 4-folds is bounded.*

In contrast, the S_7-equivariant klt Fano 4-folds are unbounded (see Example 8.10).

It is worth mentioning that we are not aware of moduli in the bounded family from Theorem 7. This means that all the examples that we know are isolated (see Example 8.9 and Question 8.14). Note that, for $n \leq 3$, the classification shows that there are only finitely many maximally symmetric Fanos of dimension n (see [15, 47]); in fact, for $n = 3$, there is only one up to conjugation.

The proof of Theorem 7 uses several results in geometry: finite actions on spheres [64], finite actions on rationally connected varieties [8], dual complexes of log Calabi–Yau pairs [34], and boundedness of Fano varieties [6]. We expect that maximally symmetric n-dimensional Fano varieties form bounded families (see Question 8.11). Let us emphasize that the behavior described in Theorem 7 is not expected for actions by other finite groups. For instance, every n-dimensional toric Fano variety admits the action of $(\mathbb{Z}/m)^n$ for m arbitrarily large. However, in the case of finite abelian actions, we have some structural theorems instead. In [39, Theorem 2], it is proved that n-dimensional Fano varieties with $(\mathbb{Z}/m)^n$-actions for m large are compactifications of $\mathbb{G}_m^n$.

In a similar vein, we prove a local statement for 5-dimensional klt singularities with faithful S_8-actions. In this case, as it is usual in the domain of singularities, we only get a bounded family up to degeneration (see Definition 2.10).

Theorem 8. *Let $\epsilon > 0$. The class of S_8-equivariant 5-dimensional klt singularities with minimal log discrepancy at least ϵ forms a family which is bounded up to degeneration.*

We summarize the largest known maximal symmetric actions on various types of varieties and topological spaces in Table 1.

Dimension	Fano	Calabi–Yau	Rational	Sphere
1	4*	3*	4*	3*
2	5*	6*	5*	4*
3	7*	7	6*	5*
4	8	8	7	6*
5	9	10	8	7
$n \gg 0$	$n + \sqrt{2n} + O(1)$	$n + \sqrt{2n} + O(1)^{\dagger}$	$n + 3$	$n + 2$

Table 1. Each table entry shows the maximal k for which S_k is known to act faithfully on an object of the indicated class and dimension, to our knowledge. In the case of the n-sphere, we consider topological actions. Entries with * are known to be optimal. The expressions for Fano and Calabi–Yau n-folds are approximate; precise formulas appear in Section 5. In the asymptotic Calabi–Yau case (†), examples with these asymptotics are only known for infinitely many values of n, rather than all n. See Remark 8.2.

The fact that S_4 is the largest symmetric group acting on $\mathbb{P}^1$ is a classical result (see, e.g., [2]). For the action of S_5 on Fano surfaces, we refer to the work of Dolgachev and Iskovskikh [15], which shows that this action is realized on the quadric surface

$$\left\{\sum x_i = \sum x_i^2 = 0\right\} \subset \mathbb{P}^4,$$

the Clebsch diagonal cubic surface

$$\left\{\sum x_i = \sum x_i^3 = 0\right\} \subset \mathbb{P}^4,$$

and the degree 5 del Pezzo surface $\overline{M}_{0,5}$. The fact that S_3 is the largest symmetric group acting on curve of genus 1 is classical (see, e.g., [56]). The fact that S_6 is the largest symmetric group acting on Calabi–Yau surfaces follows from the work of Mukai and Fujiki [25, 43]. For the actions of symmetric groups on rationally connected varieties of dimension at most 3, we refer the reader to the work of Blanc, Cheltsov, Duncan, and Prokhorov [8, 46, 47], which shows that the maximal action is realized (uniquely up to conjugation) on the symmetric sextic Fano threefold

$$\left\{\sum x_i = \sum x_i^2 = \sum x_i^3 = 0\right\} \subset \mathbb{P}^6.$$

For smooth actions of symmetric groups on spheres of dimension at most 4, we refer the reader to the work of Mecchia and Zimmerman [37, 64]. The symmetric group S_{n+2} acts on the n-sphere for any $n \geq 1$ as the symmetries of the (boundary of the) regular $(n+1)$-simplex. The examples for the remaining table entries appear in Section 8.

1.3. Outline. We begin with preliminary results in Section 2. In Section 3, we prove the quadratic bounds for Fano varieties and klt singularities in Theorems 1 and 2. Next, we study toric varieties and weighted complete intersections. We prove Theorem 3 on toric varieties in Section 4. In Sections 5 and 6, we study weighted complete intersections. In Section 5, we consider Fano and Calabi–Yau weighted complete intersections: we show the implications of a large symmetric action on the defining equations of such a weighted complete intersection, and we prove Theorem 4. In Section 6, we study the maximally symmetric Fano case and prove Theorems 5 and 6. Next, in Section 7, we prove the boundedness results of Theorems 7 and 8. Finally, in Section 8, we end the article with several examples and questions.

Notation. We work over the field of complex numbers $\mathbb{C}$. Throughout the article, $\mathbb{Z}/m$ denotes the cyclic group with m elements. Further, S_k and A_k denote the symmetric and alternating groups, respectively, on a set of order k. For a finite set W, we also use S_W to denote the symmetric group on W.

Let X be a variety. Its automorphism group (regarded with the reduced scheme structure) will be denoted $\mathrm{Aut}(X)$. For a subscheme $Z \subset X$, we let $\mathrm{Aut}(X, Z) \leq \mathrm{Aut}(X)$ denote the subgroup of automorphisms that fix Z (not necessarily pointwise). The Weil divisor class group of a normal variety X is denoted $\mathrm{Cl}\, X$.

2. Preliminaries

In this section, we recall some preliminaries regarding representation theory of symmetric groups, singularities of the MMP, Fano and Calabi–Yau varieties, and boundedness of varieties.

2.1. Representation theory of symmetric and alternating groups. In this section, we review the linear and projective representation theory of alternating and symmetric groups that are required for our proofs. For the projective representation theory of A_k and S_k, we refer to [57, 58].

Definition 2.1 ([59, Section 6.9]). A *central extension* of a group G is an extension $1 \to K \to H \to G \to 1$ such that K is in the center of H. Then H is called a *universal central extension* of G if, for every central extension $1 \to K' \to H' \to G \to 1$, there is a unique homomorphism from H to H' over G,

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d] & H \arrow[r] \arrow[d, "\exists!"] & G \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & K' \arrow[r] & H' \arrow[r] & G \arrow[r] & 1. \end{tikzcd}$$

If a universal central extension of G exists, then it is unique up to isomorphism over G. A group G has a universal central extension if and only if it is perfect [59, Theorem 6.9.5].

Central extensions of finite groups G are important for classifying their projective representations, i.e., embeddings $G \hookrightarrow \mathrm{Aut}(\mathbb{P}^n) = \mathbb{P}\mathrm{GL}_{n+1}(\mathbb{C})$. Indeed, any projective representation of G in $\mathbb{P}\mathrm{GL}_{n+1}(\mathbb{C})$ gives rise to a linear representation of a central extension in $\mathrm{GL}_{n+1}(\mathbb{C})$, whose projectivization is the original representation. For the alternating and symmetric groups, one can use a single central extension to classify all projective representations. This classification was first achieved by Schur [55].

Example 2.2 ([59, Example 6.9.10]). The standard representation $A_k \to \mathrm{SO}_{k-1}$ of the alternating group gives rise to a central extension $1 \to \mathbb{Z}/2 \to 2 \cdot A_k \to A_k \to 1$ by restricting the central extension $1 \to \mathbb{Z}/2 \to \mathrm{Spin}_{k-1}(\mathbb{R}) \to \mathrm{SO}_{k-1} \to 1$. For $k \geq 5$, the group A_k is perfect, and the universal central extension is the *Schur covering group*, which we denote $\widetilde{A}_k$. The Schur multiplier is

$$H^2(A_k, \mathbb{C}^*) = \begin{cases} 0, & k \leq 3, \\ \mathbb{Z}/2, & k \in \{4,5\} \cup \mathbb{Z}_{\geq 8}, \\ \mathbb{Z}/6, & k \in \{6,7\}. \end{cases}$$

For $k = 5$ and $k \geq 8$, $\widetilde{A}_k$ is the double cover $2 \cdot A_k$.

For $k = 6, 7$, there are additional covers $3 \cdot A_k$ and $6 \cdot A_k$ (which are central extensions of A_k by $\mathbb{Z}/3$ and $\mathbb{Z}/6$, respectively), and we have $\widetilde{A}_k = 6 \cdot A_k$. See [60, Sections 2.7.3 and 2.7.4] for the constructions of the triple covers $3 \cdot A_k$.

Example 2.3. The Schur multiplier of S_k is given by

$$H^2(S_k, \mathbb{C}^*) = \begin{cases} 0, & k \leq 3, \\ \mathbb{Z}/2, & k \geq 4. \end{cases}$$

Unlike in the case of A_k, S_k is not a perfect group and there is no universal central extension. In fact, the standard representation $S_k \to \mathrm{O}_{k-1}$ gives rise to two possible central extensions

k	S_k	$\widetilde{S}_k$
4	3	2
5	4	4
6	5	4
≥ 7	$k-1$	$2^{\lfloor (k-1)/2 \rfloor}$

Table 2. The table above summarizes the representation theory of S_k and $\widetilde{S}_k$ for $k \geq 4$. The second column shows the degree of the smallest faithful representation of S_k, and the third of $\widetilde{S}_k$.

$2 \cdot S_k^{\pm}$ of order 2 when $k \geq 4$. These are the restrictions of $\mathrm{Pin}^{\pm}(\mathbb{R}) \to \mathrm{O}_{k-1}$, where $\mathrm{Pin}^{\pm}(\mathbb{R})$ is one of the two pin groups. Both extensions are maximal, but they are only isomorphic when $k = 6$.

However, the representation theory of S_k^+ is essentially the same as that of S_k^-. In particular, the dimensions of their irreducible representations are the same [57, page 93]. From now on, we will denote by $\widetilde{S}_k$ a Schur cover of S_k, and not make a distinction between the two possible choices for $k \geq 4$.

The automorphism groups of varieties appearing in this paper are often the quotients of linear algebraic groups by central subgroups. For example,

$$\mathrm{Aut}(\mathbb{P}^n) \cong \mathbb{P}\mathrm{GL}_{n+1}(\mathbb{C}) = \mathrm{GL}_{n+1}(\mathbb{C})/\mathbb{C}^*,$$

where $\mathbb{C}^*$ is the group of scalar matrices. To identify embeddings of S_k or A_k in these automorphism groups, we therefore need to understand the representation theory of their central extensions. A faithful linear representation of S_k or $\widetilde{S}_k$ of dimension $n+1$, for instance, gives a projective representation of S_k in dimension n. Table 2 lists the smallest degrees of faithful representations of S_k and $\widetilde{S}_k$ for all $k \geq 4$. The smallest value in each row determines the largest symmetric group acting on $\mathbb{P}^n$, namely S_4 if $n = 1$, S_6 if $n = 3$, and S_{n+2} for all other values of n.

In Section 5, we will require the following further lemma about symmetric group representations. It is expressed in terms of the function

$$c_{\mathrm{Fano}}(n) := n + \left\lceil \frac{1 + \sqrt{8n+9}}{2} \right\rceil,$$

which will be important in Section 5.

Lemma 2.4. *Let $n \geq 4$ and $k \geq c_{\mathrm{Fano}}(n)$. Let S_k be the symmetric group of order k, and let $\widetilde{S}_k$ be a representation group of S_k. Unless $n = 4$ and $k = c_{\mathrm{Fano}}(4) = 8$, the only irreducible representations of $\widetilde{S}_k$ with dimension at most $2n+2$ are: the trivial representation, the sign representation (of dimension 1), the standard representation of S_k (of dimension $k-1$), and the tensor product of the standard and sign representations (of dimension $k-1$).*

In the special case of $n = 4$, $k = 8$, there is also a faithful representation of $\widetilde{S}_8$ of dimension 8. Any polynomial in 8 variables which is $\widetilde{S}_8$-invariant up to sign for this representation is contained in the invariant ring $\mathbb{C}[z_1, \ldots, z_8]^{\widetilde{A}_8}$ given by restriction of the basic spin representation to the subgroup $\widetilde{A}_8$. This ring has lowest degree generators h_1, h_2, h_3 of degrees $2, 8$, and 8, respectively.

k	A_k	$2 \cdot A_k$	$3 \cdot A_k$	$\widetilde{A}_k$
4	3	2	N/A	2
5	3	2	N/A	2
6	5	4	3	6
7	6	4	6	6
≥ 8	$k-1$	$2^{\lfloor (k-2)/2 \rfloor}$	N/A	$2^{\lfloor (k-2)/2 \rfloor}$

Table 3. The table above summarizes the smallest degrees of the faithful representations of A_k and its central extensions. The Schur covering group $\widetilde{A}_k$ is $2 \cdot A_k$ for $k = 4, 5$, $k \geq 8$ and $6 \cdot A_k$ for $k = 6, 7$.

Proof. The smallest faithful representation of $\widetilde{S}_k$ is the *basic spin representation*, which has dimension $2^{(k-2)/2}$ if k is even and dimension $2^{(k-1)/2}$ if k is odd [57, Section 3]. These dimensions are greater than $2k$ and hence greater than $2n+2$ when $k \geq 11$. The only remaining cases are $n = 4$ and $n = 5$, where $k \geq c_{\text{Fano}}(4) = 8$ or $k \geq c_{\text{Fano}}(5) = 9$. Omitting the special case of $n = 4$, $k = 8$, the basic spin representations of $\widetilde{S}_k$ have dimension at least 16 in both cases, which is larger than $2n+2$ for either value of n.

If a representation of $\widetilde{S}_k$ is not faithful, then it factors through S_k. Therefore, it remains to bound the sizes of representations of S_k. As above, the assumptions of the lemma imply $k \geq 8$. A result of Rasala [52, Result 2] gives that the first three dimensions of irreducible representations of S_k are 1, $k-1$, and $\frac{1}{2}k(k-3)$ for $k \geq 9$. The irreducible representations of dimension 1 and $k-1$ are precisely those stated in the lemma. The third value, $\frac{1}{2}k(k-3)$, grows quadratically in k and is greater than $2n+2$ when $k \geq 9$. In the $k = 8$ case, the next largest representation of S_8 has dimension 14, which is greater than $10 = 2 \cdot 4 + 2$.

Now we return to the case $n = 4$, $k = 8$. The smallest dimensions of representations of $\widetilde{S}_8$ are $1, 7, 8$ (all others have dimension greater than $2n + 2 = 10$). The 1- and 7-dimensional representations are the ones already listed. The representations of dimension 8 are the basic spin representations. A polynomial which is invariant up to sign under the $\widetilde{S}_8$-action is in particular an $\widetilde{A}_8$-invariant polynomial, where $\widetilde{A}_8$ is the (unique) Schur double cover of A_8.

The dimensions of the graded pieces of the invariant ring $\mathbb{C}[z_1, \ldots, z_8]^{\widetilde{A}_8}$ are readily computable using Molien's formula, for example using `gap`. This computation yields that the first few generators have degrees 2, 8, and 8. □

Next, we collect results on the minimal degree characters of the alternating group and its Schur cover. We will use these results in Section 4.

Lemma 2.5. *For $k \geq 4$, let $\widetilde{A}_k$ be the Schur covering group of A_k. The minimal degree faithful representations of A_k and its central extensions are summarized in Table* 3. *If $k = 8$ or $k \geq 10$, then the smallest degree of a nontrivial irreducible representation of $\widetilde{A}_k$ is $k-1$, and it factors through the standard representation of A_k.*

Proof. The faithful representation of $2 \cdot A_k$ is the basic spin representation, which has degree $2^{\lfloor (k-2)/2 \rfloor}$ by [58]. If a representation of $\widetilde{A}_k$ is not faithful and if $k \neq 6, 7$, then it factors through A_k. Every irreducible character of A_k is obtained from the restriction of an irreducible character of S_k (see [30, Statement 20.13 (3)]), so Lemma 2.4 implies that A_k has exactly one nontrivial irreducible character of minimal degree $k-1$, which is the standard representation. For $k \geq 8$, we have $k-1 \geq 2^{\lfloor (k-2)/2 \rfloor}$. Strict inequality holds for $k=8$ and $k \geq 10$, so the unique smallest degree representation of $\widetilde{A}_k$ comes from the standard representation of A_k.

If $k = 6$ or 7, then $2 \cdot A_k$ is no longer the Schur cover, and we need to also consider the minimal degree faithful representations of $3 \cdot A_k$ and $6 \cdot A_k$. These can be computed directly using `gap`. □

2.2. Singularities and positivity of pairs. In this subsection, we briefly recall some terminology related to singularities of pairs. We refer the reader to [33].

Definition 2.6. A *log pair* (X, B) is a couple consisting of a normal quasi-projective variety X and an effective divisor B for which $K_X + B$ is a $\mathbb{Q}$-Cartier divisor. Let $\pi \colon Y \to X$ be a projective morphism from a normal variety. Let $E \subset Y$ be a prime divisor. The *log discrepancy* of (X, B) at E is the rational number $1 - \mathrm{coeff}_E(B_Y)$, where B_Y is defined by the formula

$$K_Y + B_Y = \pi^*(K_X + B).$$

We say that (X, B) is *Kawamata log terminal* or *klt* for short if all the log discrepancies of (X, B) are positive. We say that (X, B) is *log canonical* or *lc* for short if all the log discrepancies are nonnegative. We say that X is *klt* (resp. *lc*) if the pair $(X, 0)$ is klt (resp. lc).

In Section 7, we will consider group actions on log pairs.

Definition 2.7. Let (X, B) be a log pair. We write $G \leq \mathrm{Aut}(X, B)$ if G is a group acting on X and $g^* B = B$ for every $g \in G$. In particular, every element of G maps components of B to components of B with the same coefficient.

Let X be an algebraic variety, $G \leq \mathrm{Aut}(X)$ a finite subgroup, and $\pi \colon X \to Y := X/G$ the quotient. We say that π is *quasi-étale* if it is étale over an open subset whose complement has codimension at least 2.

The main objects of study of this article are Fano and Calabi–Yau varieties.

Definition 2.8. We define a *Fano pair* to be a log pair (X, B) with klt singularities for which $-(K_X + B)$ is ample. If $B = 0$, then we simply say that X is a *Fano variety*. A *Calabi–Yau variety* is a variety X with klt singularities for which $K_X \sim_{\mathbb{Q}} 0$. A *log Calabi–Yau pair* is a log pair (X, B) with log canonical singularities for which $K_X + B \sim_{\mathbb{Q}} 0$.

Note that we allow log Calabi–Yau pairs to have log canonical singularities. This is a natural assumption to make when considering boundaries on Fano varieties that induce a log Calabi–Yau structure.

2.3. Boundedness of varieties and singularities. In this subsection, we recall some concepts about boundedness of varieties and singularities. In Section 7, we will prove some results regarding boundedness of Fano varieties and klt singularities admitting large symmetric actions.

Definition 2.9. Let $\mathcal{C}$ be a class of log pairs. We say that the class $\mathcal{C}$ is *log bounded* if the following condition holds. There exist a finite type morphism $\mathcal{X} \to T$ and a boundary $\mathcal{B}$ on $\mathcal{X}$ such that every element $(X, B) \in \mathcal{C}$ is isomorphic to $(\mathcal{X}_t, \mathcal{B}_t)$ for some closed point $t \in T$. If we consider a class of varieties instead of pairs, then we simply say that the class of varieties is *bounded*.

Many classes of varieties or log pairs satisfy a boundedness condition when certain invariants are fixed. However, this is not the case for singularities. Even if we fix many invariants for a class of singularities, it is likely that the resulting class is not bounded. This happens because, unlike projective varieties, the versal deformation space of singularities tends to be infinite-dimensional and many singularities in the versal deformation space will share the same invariants as the central fiber. In order to fix this issue, we use the following definition.

Definition 2.10. Let $\mathcal{C}$ be a class of singularities. We say that $\mathcal{C}$ is *bounded up to degeneration* if the following condition is satisfied. There exists a bounded class $\mathcal{B}$ of singularities such that, for every element $(X; x) \in \mathcal{C}$, there exists a flat family $\mathcal{X} \to C \ni \{0\}$ of singularities for which $(\mathcal{X}_c; x_c) \simeq (X; x)$ for some $c \in C$ and $(\mathcal{X}_0; x_0) \in \mathcal{B}$.

In other words, we say that a class of singularities is bounded up to degeneration if the elements of this class are deformations of singularities in a bounded class.

2.4. A smoothness lemma. We conclude the preliminaries with a smoothness lemma that will be used to construct examples in Section 8. In particular, certain complete intersections of Fermat hypersurfaces are smooth. The notation $p_k = p_k(x_0, \dots, x_N) := \sum_{i=0}^{N} x_i^k$ denotes the k-th power sum equation in $N + 1$ variables.

Lemma 2.11 ([53]). *For any positive integers $m \leq N - 1$, the intersection of Fermat hypersurfaces*

$$X := \{p_1(x_0, \dots, x_N) = p_2(x_0, \dots, x_N) = \cdots = p_m(x_0, \dots, x_N)\} \subset \mathbb{P}^N$$

is smooth and irreducible of dimension $N - m$.

Proof. This follows directly from results in [53]. Indeed, the affine cone C_X over the variety X is the subvariety in $\mathbb{A}^{N+1}$ cut out by the same equations. For $m \leq N - 1$, [53, Lemma 9.4] shows that C_X is irreducible of dimension $N - m + 1$; hence X is irreducible of the indicated dimension. Then [53, Lemma 9.3] shows that $C_X \setminus \{0\}$ is smooth, so X is smooth as well. $\square$

3. Bounds for symmetric actions

In this section, we study upper bounds for symmetric actions on Fano varieties and klt singularities. First, we show an explicit quadratic upper bound for k where S_k is a symmetric group acting faithfully on an n-dimensional Fano variety. As mentioned in the introduction, recent results of Kollár–Zhuang improve the bound in Theorem 1 to a linear bound, namely $m(n) \leq 4n + 1$ (see [35, Corollary 20]).

Proposition 3.1. *For any integer $n \geq 1$, let p_n be the smallest prime greater than $n+1$. There exists an integer $m(n) < p_n(n+1)$ such that, for any n-dimensional rationally connected variety X over a field of characteristic 0 and embedding $S_k \hookrightarrow \operatorname{Bir} X$, we have $k \leq m(n)$. In particular, for $n \gg 0$,*

$$m(n) < \Big(1 + \frac{1}{5000 \ln^2(n+1)}\Big)(n+1)^2.$$

Following a suggestion of Serge Cantat and Yuri Prokhorov, our argument uses a result of J. Xu [62].

Proof. For a prime p and integer $i \geq 1$, let $W_p(i)$ denote the isomorphism class of Sylow p-subgroups of the symmetric group S_{p^i}. Note that $W_p(1) \cong \mathbb{Z}/p$ and that $W_p(i)$ is non-abelian for $i \geq 2$ (see [54, Theorem 7.27]).

Let p_n be the smallest prime greater than $n+1$. If $k \geq p_n(n+1)$, then the Sylow p_n-subgroups of S_k contain either $(\mathbb{Z}/p_n)^{\oplus(n+1)}$ or $W_{p_n}(i)$ for some $i \geq 2$ as a direct factor [54, page 176]. Then S_k contains a p_n-group that either has rank greater than n or is non-abelian, so by [62, Main Theorem], there does not exist an embedding $S_k \hookrightarrow \operatorname{Bir} X$. Thus, we have $k < p_n(n+1)$. For $n \geq 468991632$, we have

$$p_n \leq \Big(1 + \frac{1}{5000 \ln^2(n+1)}\Big)(n+1)$$

by [16, Corollary 5.5], so we conclude that

$$m(n) < \Big(1 + \frac{1}{5000 \ln^2(n+1)}\Big)(n+1)^2. \qquad \square$$

Then the proof of Theorem 1 follows.

Proof of Theorem 1. This follows by taking the limit in Proposition 3.1. □

Now, we turn to symmetric actions on klt singularities. We provide an upper bound for k, where S_k is a symmetric group acting faithfully on an n-dimensional klt singularity. First, we prove the following lemma about finite actions on normal varieties, which we will also apply later in Section 7.

Lemma 3.2. *Let X be a normal variety and E a prime divisor on X. If $G \leq \operatorname{Aut}(X)$ is a finite subgroup that fixes E pointwise, then G is a normal cyclic subgroup of* $\operatorname{Aut}(X, E)$.

Proof. By [9, Corollary 2.13], G is cyclic. For normality, let $h \in \operatorname{Aut}(X, E)$, $g \in G$, and $x \in E$. Then $h^{-1}(x) \in E$, so $(hgh^{-1})(x) = (hg)(h^{-1}(x)) = h(h^{-1}(x)) = x$. □

Proof of Theorem 2. Let $(X; x)$ be an n-dimensional klt singularity and S_k a symmetric group acting on $(X; x)$. Let $\pi \colon X \to Y$ be the quotient of X by S_k. Let B_Y be the divisor with standard coefficients, i.e., coefficients in the set $\{1 - \frac{1}{n} \mid n \in \mathbb{Z}_{>0} \cup \{\infty\}\}$, for which $\pi^*(K_Y + B_Y) = K_X$. Then the pair $(Y, B_Y; y)$ is klt, where $y = \pi(x)$. Let $\varphi_Y \colon Y' \to Y$ be a projective birational morphism satisfying the following conditions:

- φ_Y extracts a unique prime divisor E' over y,

- the pair $(Y', E' + \varphi_{Y*}^{-1} B_Y)$ has plt singularities, and
- the divisor $-(K_{Y'} + E' + \varphi_{Y*}^{-1} B_Y)$ is ample over Y.

This projective birational morphism exists by [61, Lemma 1]. Let $\varphi_X \colon X' \to X$ be the projective birational morphism obtained by base change and $\pi' \colon X' \to Y'$ the corresponding quotient map. Let F be the reduced preimage of E' on X'. Then the pair (X', F) is plt and $-(K_{X'} + F)$ is ample over X. By the connectedness of log canonical centers, we can conclude that F is prime. Indeed, by contradiction, let $F = \sum_{i=1}^{k} F_i$ and assume that $k \geq 2$. By [24, Connectedness Principle], we conclude that F is connected over X, so there are two components F_i and F_j that intersect. As an intersection of log canonical centers is a union of log canonical centers (see [1, Theorem 1.1 (ii)]), we are led to a contradiction of the fact that (X', F) is plt. Thus, F is prime. By construction, the projective birational morphism $X' \to X$ is S_k-equivariant. Hence, S_k fixes F. By Lemma 3.2, we conclude that S_k acts faithfully on F. Note that F is a Fano type variety, so it is a rationally connected variety. We conclude that S_k acts on a rationally connected variety of dimension at most $n-1$. Hence, the statement follows from Theorem 1 by taking the limit. □

4. Symmetries of toric varieties

In this section, we give an upper bound for symmetric actions on complete simplicial toric varieties.

Theorem 4.1. *Let X be a complete simplicial toric variety of dimension n. Suppose that the symmetric group S_k acts faithfully on X. If $n = 1$, 2, or 3, then $k \leq n+3$; if $n \geq 4$, then $k \leq n+2$.*

These bounds are sharp for each n. If equality is achieved and $n \neq 2, 4$, then $X \cong \mathbb{P}^n$. If $n = 2$, then $k = 5$ if and only if $X \cong \mathbb{P}^1 \times \mathbb{P}^1$. If $n = 4$, then $k = 6$ if and only if $X \cong \mathbb{P}^4$ or $X \cong \mathbb{P}^2 \times \mathbb{P}^2$.

The idea of the proof of Theorem 4.1 is to use the structure of the automorphism group of a toric variety developed in [11]. Briefly, the automorphism group of a toric variety X admits a two-step filtration, the associated graded pieces of which roughly correspond to symmetries of the Cox ring of X preserving the grading, and symmetries of the fan of X, respectively. The S_{n+2}-action on $\mathbb{P}^n$ exhibits an example of symmetries "coming from" graded automorphisms of the Cox ring (Example 8.6). In contrast, the S_n-action on $\prod_{i=1}^{n} \mathbb{P}^1$ by permuting the factors comes from the symmetries of the fan. For $X = \mathbb{P}^1 \times \mathbb{P}^1$, the S_5-action is obtained from an A_5-action on $\mathbb{P}^1$ and a $\mathbb{Z}/2$-action exchanging the factors. We first recall the results from [11] that we will need about automorphisms of a toric variety. For a general reference on toric varieties, see [26].

Throughout this section, let X be a complete simplicial toric variety of dimension n, defined by a fan Δ in $N = \mathbb{Z}^n$. Let $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. Let $T := N \otimes_{\mathbb{Z}} \mathbb{C}^*$ be the torus acting on X. We will use $\Delta(1)$ to denote the set of one-dimensional cones (rays) of Δ and $d = |\Delta(1)|$ for the total number of rays. The free abelian group $\mathbb{Z}^{\Delta(1)}$ of T-invariant Weil divisors on X fits into an exact sequence

$$1 \to M \to \mathbb{Z}^{\Delta(1)} \to \mathrm{Cl}\, X \to 1, \qquad (4.1)$$

where $M \to \mathbb{Z}^{\Delta(1)}$ is defined by $m \mapsto \sum_{\rho\in\Delta(1)}\langle m, n_\rho\rangle D_\rho$. In particular, $\mathrm{Cl}\,X$ is a finitely generated abelian group with $\mathrm{rank}((\mathrm{Cl}\,X)_{\mathbb{Q}}) = d - n$. The *degree* of an element of $\mathbb{Z}^{\Delta(1)}$ is defined to be its class in $\mathrm{Cl}\,X$.

The toric variety X may be constructed as a geometric quotient $(\mathbb{C}^{\Delta(1)} \setminus Z)/G$, where G is the algebraic group defined as $G := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}\,X, \mathbb{C}^*)$ and Z is the exceptional set defined by the vanishing of a certain monomial ideal. The action of the group G on $\mathbb{C}^{\Delta(1)}$ is induced by the quotient morphism $\mathbb{Z}^{\Delta(1)} \to \mathrm{Cl}\,X$.

The coordinate ring $R := \mathbb{C}[x_\rho \mid \rho \in \Delta(1)]$ of the space $\mathbb{C}^{\Delta(1)}$ acquires a grading from this action by $\mathrm{Cl}\,X$. The resulting graded ring, known as the *Cox ring*, plays a major role in the study of toric varieties. In particular, its structure is closely related to that of the automorphism group of X.

Before stating this connection, we will introduce some more notation related to the Cox ring and that set of rays $\Delta(1)$ of the fan of X. For each $\alpha \in \mathrm{Cl}\,X$, let R_α be the graded piece of R of elements of degree α; then $R = \bigoplus_{\alpha_i} R_{\alpha_i}$.

We will pay particular attention to the graded pieces containing variables x_ρ for $\rho \in \Delta(1)$. Indeed, partition $\Delta(1)$ into disjoint subsets $\Delta(1) = \Delta_1 \sqcup \cdots \sqcup \Delta_s$, where each Δ_i corresponds to a set of variables with the same degree α_i. For each α_i, one may write $R_{\alpha_i} = R'_{\alpha_i} \oplus R''_{\alpha_i}$, where R'_{α_i} is spanned by the monomials x_ρ for $\rho \in \Delta_i$.

The dimension n of X constrains the possible values for the sizes of the Δ_i in the above partition.

Lemma 4.2. *Write $d_i = |\Delta_i|$ and $d = |\Delta(1)| = \sum_{i=1}^{s} d_i$. The following hold.*

(1) *$\sum_{i=1}^{s}(d_i - 1) \le n$. In particular, $d_i \le n + 1$ for each i.*

(2) *If $\sum_{i=1}^{s}(d_i - 1) = n$, then $X \cong \mathbb{P}^{d_1-1} \times \cdots \times \mathbb{P}^{d_s-1}$.*

Proof. We have that $s \ge \mathrm{rank}((\mathrm{Cl}\,X)_{\mathbb{Q}}) = d - n$, so $d - s \le n$, and part (1) follows from this inequality. For part (2), label the d rays on X as $\rho_1, \ldots, \rho_d$. For each ray ρ_j, let D_j be the corresponding torus-invariant divisor. For each fixed Δ_i with $d_i \ge 2$, consider the differences $\{D_j - D_k \mid \rho_j \ne \rho_k \in \Delta_i\}$. Each such difference is in the kernel of the map to $\mathrm{Cl}\,X$ in (4.1), so it is in the image of an element of M. For each Δ_i, there are $d_i - 1$ independent such differences, and the set Σ of differences across all Δ_i extends to a basis of $\mathbb{Z}^{\Delta(1)}$. By the assumption, the $\mathbb{Z}$-span of Σ has rank $n = \mathrm{rank}\,M$; thus, it is equal to the image of $M \to \mathbb{Z}^{\Delta(1)}$.

We claim that this implies $d_i \ge 2$ for all i. Indeed, any $m \in M$ has image

$$\sum_{j=1}^{d}\langle m, n_{\rho_j}\rangle D_j$$

which must be in the span of Σ by the conclusion of the last paragraph. Hence, if $d_i = 1$ for some i and $\Delta_i = \{\rho_j\}$, then D_j does not belong to such a difference, so the ray ρ_j satisfies $\langle m, n_{\rho_j}\rangle = 0$ for all $m \in M$. This would imply that the ray is 0, which is impossible.

For each Δ_i, the sublattice of $\mathbb{Z}^{\Delta(1)}$ generated by $\{\rho \mid \rho \in \Delta_i\}$ intersects the image of M in the $d_1 - 1$ rank sublattice of the image of M generated by the ray differences. The preimages M_i of these sublattices in M decompose M as a direct sum, each component of which evaluates to zero identically on any rays not in the corresponding Δ_i. We also get a dual decomposition $N = \bigoplus_{i=1}^{s} N_i$. The fan generated by the $d_i - 1$ rays of Δ_i in N_i is clearly that of $\mathbb{P}^{d_i-1}$, so $X \cong \mathbb{P}^{d_1-1} \times \cdots \times \mathbb{P}^{d_s-1}$. □

To prove Theorem 4.1, we will use the following results of Cox on automorphisms of simplicial toric varieties [11]. These results realize the automorphism group of X as a quotient of a group of automorphisms of the affine variety $\mathbb{C}^{\Delta(1)} \setminus Z$. In particular, let $\widetilde{\mathrm{Aut}}^0(X)$ be the centralizer of the group G in the automorphism group of $\mathbb{C}^{\Delta(1)} \setminus Z$, and let $\widetilde{\mathrm{Aut}}(X)$ be the normalizer.

Theorem 4.3 ([11]). *Let X be a complete simplicial toric variety, and let R be the Cox ring of X. We denote by* $\mathrm{Aut}_g(R)$ *the group of automorphisms of this ring preserving the grading. Let* $G = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}\, X, \mathbb{C}^*)$*, and let* $\mathrm{Aut}(N, \Delta)$ *be the group of lattice isomorphisms of N that preserve the fan Δ.*

(1) *There is a natural isomorphism* $\widetilde{\mathrm{Aut}}^0(X) \cong \mathrm{Aut}_g(R)$. *In particular, G is in the center of* $\mathrm{Aut}_g(R)$.

(2) $\mathrm{Aut}_g(R)$ *is isomorphic to the semidirect product $U \rtimes G_s$, where U is the unipotent radical and* $G_s = \prod_{i=1}^{s} \mathrm{GL}(R'_{\alpha_i})$. *In particular, any finite subgroup of* $\mathrm{Aut}_g(R)$ *is conjugate to a subgroup of* $\prod_{i=1}^{s} \mathrm{GL}(R'_{\alpha_i})$.

(3) *The connected component of the identity in* $\mathrm{Aut}(X)$ *is* $\mathrm{Aut}^0(X) \cong \mathrm{Aut}_g(R)/G$.

(4) $\mathrm{Aut}(N, \Delta) \hookrightarrow S_{\Delta(1)}$ *and*

$$\widetilde{\mathrm{Aut}}(X)/\widetilde{\mathrm{Aut}}^0(X) \cong \mathrm{Aut}(X)/\mathrm{Aut}^0(X) \cong \mathrm{Aut}(N, \Delta)\Big/\prod_{i=1}^{s} S_{\Delta_i}.$$

Proof. Most of the statements are taken directly from [11]. The assertions of (1) are [11, Theorem 4.2 (iii)]. Part (2) is contained in [11, Proposition 4.3 (iv)] (see also [12]), except the "in particular" in (2), which follows from the structure theory of Lie groups (see, e.g., [28, Proposition VIII.4.2]). Part (3) is [11, Corollary 4.7 (iii)], and finally, part (4) follows from [11, Corollary 4.7 (v) and the proof of Theorem 4.2 (ii)]. □

Now we begin the proof of Theorem 4.1. Theorem 4.3 (4) shows that an action on X decomposes into a part in $\mathrm{Aut}^0(X)$ and an action on the fan. We will consider these two situations separately. First, we consider the case where A_k or S_k is a subgroup of $\mathrm{Aut}^0(X)$.

Lemma 4.4. *Let X be a complete simplicial toric variety of dimension n, and let $\Delta_1 \sqcup \cdots \sqcup \Delta_s$ be the partition of $\Delta(1)$ by degrees defined before Lemma 4.2. Let $k \geq 5$ be an integer, and let Γ be the alternating group A_k or the symmetric group S_k. If $\Gamma \leq \mathrm{Aut}^0(X)$, then*

$$n \geq \begin{cases} 1 & \text{if } k = 5, \\ 2 & \text{if } k = 6, \\ 3 & \text{if } k = 7, \\ k-2 & \text{if } k \geq 8 \end{cases} \quad \text{if } \Gamma = A_k, \qquad n \geq \begin{cases} 3 & \text{if } k = 5, 6, \\ k-2 & \text{if } k \geq 7 \end{cases} \quad \text{if } \Gamma = S_k.$$

Furthermore, if equality holds, then $X \cong \mathbb{P}^n$.

Proof. By Theorem 4.3 (1) and (3), G is contained in the center of $\mathrm{Aut}_g(R)$, and we have an isomorphism $\mathrm{Aut}^0(X) \cong \mathrm{Aut}_g(R)/G$. This induces a central extension

$$1 \to K \to H \to \Gamma \to 1$$

with $K \le G$ and $H \le \mathrm{Aut}_g(R)$. If $\Gamma = A_k$, then by Example 2.2, we can assume

$$H \cong \begin{cases} \widetilde{A}_k \text{ or } A_k & \text{if } k = 5 \text{ or } k \ge 8, \\ \widetilde{A}_k,\ 3 \cdot A_k,\ 2 \cdot A_k, \text{ or } A_k & \text{if } k = 6, 7. \end{cases}$$

If $\Gamma = S_k$, we can assume by Example 2.3 that $H \cong S_k$ or $H \cong \widetilde{S}_k$.

By Theorem 4.3 (2), we may assume H is contained in $\prod_{i=1}^s \mathrm{GL}(R'_{\alpha_i})$. Projection onto each factor induces a representation $H \to \mathrm{GL}(R'_{\alpha_i}) = \mathrm{GL}_{d_i}(\mathbb{C})$. The composition

$$H \hookrightarrow \mathrm{Aut}_g(R) \to \mathrm{Aut}^0(X)$$

surjects onto Γ, so using Table 3 and Table 2, we conclude that some $1 \le i \le s$ satisfies

$$d_i \ge \begin{cases} 2 & \text{if } k = 5, \\ 3 & \text{if } k = 6, \\ 4 & \text{if } k = 7, \\ k-1 & \text{if } k \ge 8, \end{cases} \quad \text{if } \Gamma = A_k, \qquad d_i \ge \begin{cases} 4 & \text{if } k = 5, 6, \\ k-1 & \text{if } k \ge 7, \end{cases} \quad \text{if } \Gamma = S_k.$$

By Lemma 4.2, we have $n \ge d_i - 1$, and if equality holds, then $X \cong \mathbb{P}^n$. This shows the lemma. □

Now we need to deal with the case $S_k \hookrightarrow \mathrm{Aut}(N, \Delta)/\prod_{i=1}^s S_{\Delta_i}$.

Lemma 4.5. *Let $n \ge 2$ and k be integers, and let X be a complete simplicial toric variety of dimension n. If $S_k \le \mathrm{Aut}(N, \Delta)/\prod_{i=1}^s S_{\Delta_i}$, then $k \le n + 1$.*

Proof. For each positive integer m, define $I_m \subset \{1, \dots, s\}$ to be the set of indices i for which $|\Delta_i| = m$. Then Lemma 4.2 (1) implies that $\{1, \dots, s\} = I_1 \sqcup \cdots \sqcup I_{n+1}$, since all higher I_m must be empty. We first claim that the map $S_k \hookrightarrow \mathrm{Aut}(N, \Delta)/\prod_{i=1}^s S_{\Delta_i}$ induces a natural embedding $S_k \hookrightarrow S_{I_1} \times \cdots \times S_{I_{n+1}}$, where each S_{I_m} is the symmetric group on partition pieces Δ_i of $\Delta(1)$ of size m.

Indeed, an element of $\mathrm{Aut}(N, \Delta)$ is an automorphism of the lattice N preserving the fan Δ, so in particular, it preserves the linear equivalence of rays in Δ (see [11, page 26] for more details). Hence, every member of a collection Δ_i of linearly equivalent rays is sent to a member of a single collection Δ_j; furthermore, we have $|\Delta_i| = |\Delta_j|$. Therefore, mapping $\varphi \in \mathrm{Aut}(N, \Delta)$ to the assignments $i \mapsto j$ defines a group homomorphism

$$\mathrm{Aut}(N, \Delta) \to S_{I_1} \times \cdots \times S_{I_{n+1}}.$$

The kernel of this homomorphism is precisely the subgroup $\prod_{i=1}^s S_{\Delta_i}$, so it descends to

$$\mathrm{Aut}(N, \Delta)/\prod_{i=1}^s S_{\Delta_i} \hookrightarrow S_{I_1} \times \cdots \times S_{I_{n+1}}.$$

Composing with the inclusion of the subgroup S_k gives the desired embedding above.

Therefore, S_k acts on each set $I_1, \dots, I_{n+1}$ of collections of linearly independent rays of a given size. We assume by way of contradiction that $k \geq n+2$. Each set I_m for $m \geq 2$ has size at most n by Lemma 4.2 (1), so S_k cannot act faithfully on any of these sets. Therefore, the composite homomorphism $S_k \hookrightarrow S_{I_1} \times \cdots \times S_{I_{n+1}} \to S_{I_1}$ with the projection onto the first factor must be an injection, that is, S_k acts faithfully on size 1 linear equivalence classes of rays. We will use this fact to find a faithful S_k representation of small dimension.

Let $V \subset N_{\mathbb{Q}}$ be the $\mathbb{Q}$-vector space spanned by $\{\rho \in \Delta_i \mid i \in I_1\}$. There is a restriction homomorphism

$$\operatorname{Aut}(N, \Delta)/\prod_{i=1}^{s} S_{\Delta_i} \to \operatorname{GL}(V).$$

Indeed, for $\varphi \in \operatorname{Aut}(N, \Delta)$, let $\varphi_{\mathbb{Q}} \in \operatorname{GL}(N_{\mathbb{Q}})$ denote the extension of φ by scalars to the vector space $N_{\mathbb{Q}}$. Then $\varphi_{\mathbb{Q}}(V) = V$ because φ preserves the collection of rays with linear equivalence class of size 1. It follows that we have a natural restriction map $\operatorname{Aut}(N, \Delta) \to \operatorname{GL}(V)$. Moreover, any element of $\prod_{i=1}^{s} S_{\Delta_i} \leq \operatorname{Aut}(N, \Delta)$ is sent to the identity transformation under this restriction, since it must fix every ray in a spanning set of V.

Thus, we have a homomorphism $S_k \to \operatorname{GL}(V)$. Since we have already shown that the subgroup $S_k \leq \operatorname{Aut}(N, \Delta)/\prod_{i=1}^{s} S_{\Delta_i}$ acts faithfully on the collection of rays ρ with index in I_1, the composite homomorphism $S_k \to \operatorname{GL}(V)$ must be injective. This proves that V is a faithful representation of S_k. Since $\dim_{\mathbb{Q}} V \leq \operatorname{rank} N = n$, we must have that $k \leq n+1$ by Table 2. This contradicts the assumed bound on k. □

Finally, we will consider the situation where a subgroup $S_k \leq \operatorname{Aut}(X)$ has the property $S_k \cap \operatorname{Aut}^0(X) = A_k$.

Lemma 4.6. *Let $n \geq 2$ and $k \geq 5$ be integers, and let X be a complete simplicial toric variety of dimension n. Suppose that $S_k \leq \operatorname{Aut}(X)$ is a subgroup of automorphisms with the property that $S_k \cap \operatorname{Aut}^0(X) = A_k$, and S_k is not a subgroup of $\operatorname{Aut}^0(X)$. Then there must exist at least two distinct indices i such that $d_i = |\Delta_i|$ satisfies*

$$d_i \geq \begin{cases} 2 & \text{if } k = 5, \\ 3 & \text{if } k = 6, \\ 4 & \text{if } k = 7, \\ k-1 & \text{if } k \geq 8. \end{cases}$$

Proof. Since $S_k \cap \operatorname{Aut}^0(X) = A_k \leq \operatorname{Aut}^0(X) = \operatorname{Aut}_g(R)/G$, as in Lemma 4.4, we have representations of $\widetilde{A}_k$ on the factors $\operatorname{GL}(R'_{\alpha_i})$ whose product is the reductive subgroup G_s of Theorem 4.3 (2). At least one of these must be faithful. Therefore, it follows that the dimension $d_i = |\Delta_i| = \dim(R'_{\alpha_i})$ must satisfy the inequalities in the lemma for some i.

We will assume that there is exactly one index satisfying the inequalities of the lemma, and then derive a contradiction. We may assume this index is 1. Then the representation of $\widetilde{A}_k$ on each $\operatorname{GL}(R'_{\alpha_i})$ is trivial for $i \geq 2$. Next, consider the preimage H of the entire $S_k \leq \operatorname{Aut}(X)$ inside $\widetilde{\operatorname{Aut}}(X)$, so that $S_k \cong H/G$. The group G is of multiplicative type, hence reductive, so H, being an extension of S_k by G, is also reductive.

We saw in Theorem 4.3 (2) that $\widetilde{\operatorname{Aut}^0}(X)$, the connected component of the identity in $\widetilde{\operatorname{Aut}}(X)$, contains the reductive subgroup $G_s = \prod_{i=1}^{s} \operatorname{GL}(R'_{\alpha_i})$. One can find an analogous reductive subgroup of $\widetilde{\mathrm{Aut}}(X)$ as follows. It was shown in [11, page 27] that $\widetilde{\mathrm{Aut}}(X)$ is generated by $\widetilde{\mathrm{Aut}}^0(X)$ and elements of the form P_φ, where $\varphi \in \mathrm{Aut}(N, \Delta)$ is an automorphism of the fan. The automorphism P_φ is constructed on the level of $\mathbb{C}^{\Delta(1)}$ as the corresponding permutation matrix on rays; this automorphism then descends to the quotient $X = (\mathbb{C}^{\Delta(1)} \setminus Z)/G$ (see [11, page 26]). The subgroup generated by G_s and the P_φ is a reductive subgroup G_s' of $\widetilde{\mathrm{Aut}}(X)$ with the property that $UG_s' = \widetilde{\mathrm{Aut}}(X)$. (Here, U is the unipotent radical of $\widetilde{\mathrm{Aut}}^0(X)$ from Theorem 4.3 (2); it is also the unipotent radical of $\widetilde{\mathrm{Aut}}(X)$.) Thus, by [28, Proposition VIII.4.2], the group H is conjugate to a subgroup of G_s'. We may therefore assume $H \leq G_s'$.

Now pick a transposition τ of order 2 in $S_k \leq \mathrm{Aut}(X)$, so that τ and A_k generate the subgroup S_k. For a lift $\widetilde{\tau} \in \widetilde{\mathrm{Aut}}(X)$ of τ, we have by assumption that $\widetilde{\tau} \in G_s'$. Since G_s is normal in G_s', we may write $\widetilde{\tau}$ as a composition $P_\varphi \circ h$, where $\varphi \in \mathrm{Aut}(N, \Delta)$ and $h \in G_s$. By assumption, Δ_1 is the unique largest piece of the partition, so the permutation on partition pieces that φ induces must fix the piece Δ_1. After changing φ by an element σ of $\prod_{i=1}^s S_{\Delta_i}$ (the corresponding P_σ is in G_s), we may even assume P_φ induces the identity permutation on Δ_1. Both $\widetilde{\tau}$ and h therefore act by the same linear transformation when restricted to the space R'_{α_1}; we shall denote by $\widetilde{\tau}'$ the automorphism in $G_s \leq \widetilde{\mathrm{Aut}}^0(X)$ that acts by this linear transformation in R'_{α_1} and is constant on all other x_ρ, $\rho \in \Delta(1)$.

Let $\tau' \in \mathrm{Aut}(X)$ be the image of $\widetilde{\tau}'$. The point is now to show that τ' and A_k generate a copy of S_k just as τ and A_k do, but this time inside of $\mathrm{Aut}^0(X)$. Indeed, we have that $\widetilde{\tau}^{-1}\widetilde{\tau}'$ is trivial on R'_{α_1}, so its image $\tau^{-1}\tau'$ in $\mathrm{Aut}(X)$ commutes with any $g \in A_k$. This implies $(\tau')^{-1} g \tau' = \tau^{-1} g \tau \in A_k$ for any such g. Therefore, the group Γ generated by τ' and A_k has order $2 \cdot |A_k| = k!$ and the action by τ' on the normal subgroup A_k by conjugation is the same as that of τ. This shows Γ has the same semidirect product structure as S_k does, so $\Gamma \cong S_k$. This contradicts the assumption that there is no embedding $S_k \hookrightarrow \mathrm{Aut}^0(X)$, completing the proof. □

Putting the above results together, we can now prove Theorem 4.1.

Proof of Theorem 4.1. For each dimension n, we may assume that k is at least the upper bound given in the statement of Theorem 4.1 (if not, the conclusion holds automatically). Under this assumption, we show that k must in fact equal this bound and characterize the optimal examples. First, we deal with $n = 1$. A one-dimensional normal complete toric variety is isomorphic to $\mathbb{P}^1$, so $X = \mathbb{P}^1$ and $S_4 \leq \mathbb{P}\mathrm{GL}_2(\mathbb{C})$ is the largest symmetric action.

From now on, we consider $n \geq 2$ so that we may assume $k \geq 5$. Therefore, A_k is simple. The cokernel of $S_k \cap \mathrm{Aut}^0(X) \to S_k \leq \mathrm{Aut}(X)$ is either trivial, $\mathbb{Z}/2$, or S_k.

If the cokernel is trivial, then we have an embedding $S_k \hookrightarrow \mathrm{Aut}^0(X)$. Using Lemma 4.4, we may get a bound on n. For $n = 2$, the lemma implies $k < 5$, contradicting the maximality assumption $k \geq 5$. Therefore, no maximal symmetric actions on toric surfaces occur in the case of trivial cokernel. For $n = 3$, any embedding $S_k \hookrightarrow \mathrm{Aut}^0(X)$ satisfies $k \leq 6$, and for $n \geq 4$, we must have $k \leq n + 2$. So, for all $n \geq 2$, our assumption that k is maximal means that the inequalities are equalities and $X \cong \mathbb{P}^n$, again by Lemma 4.4. In particular, $\mathbb{P}^n$ achieves the optimal bound in dimensions $n \geq 3$.

If the cokernel of $S_k \cap \mathrm{Aut}^0(X) \to S_k \leq \mathrm{Aut}(X)$ is all of S_k, then we have

$$S_k \hookrightarrow \mathrm{Aut}(X)/\mathrm{Aut}^0(X) \cong \mathrm{Aut}(N, \Delta)/\prod_{i=1}^{s} S_{\Delta_i}.$$

We claim that this case produces no maximally symmetric examples. Indeed, Lemma 4.5 shows that, for each n, $k \leq n+1$. Therefore, k falls short of the maximum possible value laid out in Theorem 4.1.

Finally, we consider the case where the cokernel of $S_k \cap \mathrm{Aut}^0(X) \to S_k \leq \mathrm{Aut}(X)$ is $\mathbb{Z}/2$. We can suppose without loss of generality that there is no embedding $S_k \hookrightarrow \mathrm{Aut}^0(X)$, or else we would be back in the trivial cokernel case. This situation is characterized by Lemma 4.6.

Begin with the $n = 2$ case. If $k \geq 6$, we would have by Lemma 4.6 that $d_1, d_2 \geq 3$, contradicting Lemma 4.2 (1). This leaves only $k = 5$ to consider. Lemma 4.6 gives $d_1, d_2 \geq 2$, so in fact, $d_1 = d_2 = 2$, or else we would again contradict Lemma 4.2 (1). Thus, $X \cong \mathbb{P}^1 \times \mathbb{P}^1$ by Lemma 4.2 (2). On the other hand, we know that S_5 acts faithfully on the toric variety $\mathbb{P}^1 \times \mathbb{P}^1$ (see Example 8.1 for $n = 2$). Therefore, it follows that $\mathbb{P}^1 \times \mathbb{P}^1$ is the unique optimal example for $n = 2$.

Now consider $n = 3$. If $k \geq 6$, then Lemma 4.6 shows that we would have (without loss of generality) $d_1, d_2 \geq 3$, contradicting Lemma 4.2 (1). Therefore, we get no new maximal examples.

For $n = 4$, $k \geq 7$, we would have $d_1, d_2 \geq 4$, once again a contradiction. The remaining possibility is $k = 6$, where we need $d_1 = d_2 = 3$. This implies that $X \cong \mathbb{P}^2 \times \mathbb{P}^2$ by Lemma 4.2 (2). Conversely, we claim that $\mathrm{Aut}(\mathbb{P}^2 \times \mathbb{P}^2) \cong \mathbb{P}\mathrm{GL}_3 \wr \mathbb{Z}/2$ (see [36, Theorem 1]) contains a copy of S_6. This is because $A_6 \leq \mathbb{P}\mathrm{GL}_3(\mathbb{C})$ and S_6 is a semidirect product of A_6 and $\mathbb{Z}/2$. This semidirect product is a subgroup in the wreath product generated by a twisted diagonal embedding of A_6 and the transposition of factors. Therefore, $\mathbb{P}^2 \times \mathbb{P}^2$ is another optimal example for $n = 4$.

For $n = 5$, the assumption $k \geq 7$ means $d_1, d_2 \geq 4$, a contradiction. Finally, when $n \geq 6$, we can assume $k \geq n+2$, so we would have $d_1, d_2 \geq n+1$, so $\sum_{i=1}^{s}(d_i - 1) \geq 2n > n$. In summary, no maximal examples can occur in this case for $n \geq 5$. □

5. Symmetries of weighted complete intersections

In this section, we find the largest symmetric group which can act on a Fano or Calabi–Yau variety which is a quasismooth weighted complete intersection of dimension n. We will first review a few key definitions.

We say that a weighted projective space $\mathbb{P} := \mathbb{P}(a_0, \ldots, a_N)$ is *well-formed* if

$$\gcd(a_0, \ldots, \widehat{a}_i, \ldots, a_N) = 1 \quad \text{for all } 1 \leq i \leq N.$$

A subvariety X of $\mathbb{P}$ is *well-formed* if $\mathbb{P}$ is well-formed and

$$\dim X - \dim(X \cap \mathrm{Sing}(\mathbb{P})) \geq 2,$$

where by convention the empty set has dimension -1. The subvariety X is *quasismooth* if its preimage in $\mathbb{A}^{N+1} \setminus \{0\}$ is smooth. We will always work with quasismooth weighted complete intersections throughout this paper. For a thorough introduction to weighted complete intersections, see [29].

The main theorem of this section is as follows.

Theorem 5.1. *Let X be a quasismooth weighted complete intersection of dimension n. Suppose that the symmetric group S_k acts faithfully on X. The following hold.*

(1) *If X is Fano, then*

$$k \leq n + \left\lceil \frac{1+\sqrt{8n+9}}{2} \right\rceil.$$

This bound is sharp for every n.

(2) *If X is Calabi–Yau, then*

$$k \leq n + \left\lfloor \frac{1+\sqrt{8n+9}}{2} \right\rfloor + 1.$$

Example 8.1 shows that (1) is sharp in every dimension. In the Calabi–Yau case, the bound $k \leq 4$ given by (2) for $n = 1$ is not sharp because S_3 is the largest symmetric action on a smooth elliptic curve, by the proof of Proposition 5.4 below. It is unclear whether (2) is always sharp in higher dimensions (see Remark 8.2).

For succinctness, we will use the following abbreviations for the functions above throughout the section:

$$c_{\mathrm{Fano}}(n) := n + \left\lceil \frac{1+\sqrt{8n+9}}{2} \right\rceil, \quad c_{\mathrm{CY}}(n) := n + \left\lfloor \frac{1+\sqrt{8n+9}}{2} \right\rfloor + 1.$$

Remark 5.2. Notice that these two functions satisfy $c_{\mathrm{CY}}(n) \geq c_{\mathrm{Fano}}(n)$, they never differ by more than 1, and they are equal unless the fractional expression is an integer. It is also true that $c_{\mathrm{Fano}}(n-1)$ and $c_{\mathrm{CY}}(n-1)$ are both strictly smaller than $c_{\mathrm{Fano}}(n)$ for all n. Since we expect Example 8.1 to be a maximally symmetric Fano for each n, this suggests that the hypothesis of Theorem 7.9 is likely to hold. It also provides some evidence that the proof of Theorem 7 in Section 7 should extend to higher dimensions, because we expect that the maximal S_k which can act on a Fano variety of dimension n cannot act faithfully on either a Calabi–Yau or a Fano variety of dimension $n-1$. This in turn is one of the key inductive steps to generalizing Theorem 7 (see the remarks before Question 8.11).

Throughout Sections 5 and 6, we will use the following notation.

Notation 5.3. Let $X := X_{d_1,\ldots,d_m} \subset \mathbb{P} := \mathbb{P}(a_0, \ldots, a_N)$ be a quasismooth weighted complete intersection defined by m weighted homogeneous equations $f_1, \ldots, f_m$, of degrees $d_1, \ldots, d_m$, respectively. The dimension of X is $n := N - m$. Assume the symmetric group S_k acts faithfully on X.

We will first deal with some low-dimensional cases that are known via other means, so that we may exclude them later.

Proposition 5.4. *Let X be a quasismooth weighted complete intersection which is*

(a) *Fano of dimension $n \leq 3$, or*

(b) *Calabi–Yau of dimension $n \leq 2$.*

Suppose X has a faithful action of S_k. Then the upper bounds in Theorem 5.1 *hold.*

Proof. We may always replace X with a well-formed quasismooth complete intersection which is isomorphic [49, Lemma 2.3], so assume X is well-formed. We will consider the statements for Fano and Calabi–Yau varieties separately. Begin with case (1), where X is Fano.

If the dimension of X is 1, then $X \cong \mathbb{P}^1$ since it is a klt Fano variety, and it is well known that S_4 is the largest symmetric group that embeds in $\mathbb{P}\mathrm{GL}_2(\mathbb{C})$ (see, e.g., [2]). Since $c_{\mathrm{Fano}}(1) = 4$, this proves the theorem in this case.

When $n = 2$, X is rational, so S_k embeds in the Cremona group Cr(2). The finite subgroups of Cr(2) have been classified (see [15]); the largest symmetric group action that appears is by S_5, which again agrees with $c_{\mathrm{Fano}}(2) = 5$.

Finally, when $n = 3$, a resolution of singularities of X is a rationally connected variety, so $S_k \leq \mathrm{Bir}(V)$ for V some rationally connected threefold. A result of Prokhorov shows that, for $k \geq 8$, S_k does not admit an embedding into $\mathrm{Bir}(V)$ for V any rationally connected threefold [47, Proposition 1.1]. Since $c_{\mathrm{Fano}}(2) = 7$, this proves the bound for $n = 3$.

We next turn to the Calabi–Yau case. If $\dim X = 1$, then X is a smooth genus 1 curve. Its automorphism group is a semidirect product of the automorphism group of an elliptic curve with the (abelian) group on translations of X. Since the automorphism group of a complex elliptic curve is cyclic of order 2, 4, or 6, the largest possible symmetric group action on X is by S_3. This S_3 is in fact achieved, for instance, by the permutation of variables on the cubic curve $X = \{x^3 + y^3 + z^3 = 0\} \subset \mathbb{P}^2$. We have $3 < c_{\mathrm{CY}}(1) = 4$.

Let $n = 2$. We have that $c_{\mathrm{CY}}(2) = 6$. By the adjunction formula, $K_X \cong \mathcal{O}_X$. Since K_X is Cartier, X has canonical singularities. Suppose that $\widetilde{X} \to X$ is a minimal resolution of singularities; then $\widetilde{X}$ is an abelian surface or a smooth K3 surface and the action of S_k lifts to $\widetilde{X}$ (see [18, Proposition 2.2]). If $\widetilde{X}$ is an abelian surface, then $\mathrm{Aut}(\widetilde{X})$ is a semidirect product of the (abelian) group of translations with the subgroup $\mathrm{Aut}(\widetilde{X}, 0)$ which preserves the identity point. When $k > c_{\mathrm{CY}}(2) = 6$, $A_k \leq S_k$ is simple, so it must embed in $\mathrm{Aut}(\widetilde{X}, 0)$. This is impossible by the classification of automorphism groups of complex tori of dimension 2 (see [25]).

If instead $\widetilde{X}$ is a K3 surface, then the action of any finite subgroup $H \leq \mathrm{Aut}(\widetilde{X})$ on the one-dimensional vector space $H^0(\widetilde{X}, K_{\widetilde{X}}) \cong \mathbb{C}$ gives an exact sequence

$$1 \to H_{\mathrm{symp}} \to H \to \mathbb{Z}/m \to 1,$$

where m is a positive integer and H_{symp} is the kernel of the representation, which acts by symplectic automorphisms [44]. As above, if $S_k \leq \mathrm{Aut}(\widetilde{X})$ for $k > 6$, we would have a symplectic group of automorphisms on a K3 surface isomorphic to A_k. This is impossible by the classification of finite symplectic actions on such surfaces [43], proving the required inequality on k. $\square$

Before proving Theorem 5.1 in higher dimensions, we show some lemmata that we will need in the proof. The first reduction step is to show that we may assume that no defining equation f_i of X contains a linear term, i.e., x_j is not a monomial in f_i for any j, i.

Lemma 5.5. *Let $X_{d_1,\ldots,d_m} \subset \mathbb{P}(a_0, \ldots, a_N)$ be a quasismooth weighted complete intersection of dimension at least* 3. *Then there is a quasismooth, well-formed weighted complete intersection $X' \subset \mathbb{P}(a'_0, \ldots, a'_{N'})$ that is isomorphic to X and such that none of the equations defining X' contains a linear term.*

Proof. The argument is the same as in [49, Proposition 2.9]. Indeed, suppose without loss of generality that x_N is a monomial in f_m, so that $f_m = x_N - g$ for some polynomial g depending only on the other variables $x_0, \ldots, x_{N-1}$. After a change of variables $x_N - g \mapsto x_N$, we may assume that $f_m = x_N$, and that no other f_i contain the variable x_N.

Indeed, if another f_i does contain x_N, we may modify f_i by subtracting a multiple of $f_m = x_N$ to eliminate that term. This changes the defining equations, but neither the degrees, nor the ideal defining the complete intersection. Since the equation $x_N = 0$ cuts out $\mathbb{P}(a_0, \dots, a_{N-1}) \subset \mathbb{P}$, X is a codimension $m-1$ weighted complete intersection X' in this smaller weighted projective space. It follows from the quasismoothness of X that X' is also quasismooth. Though it may happen that $\mathbb{P}(a_0, \dots, a_{N-1})$ is not well-formed, X' will be isomorphic to another quasismooth complete intersection in a well-formed weighted projective space $\mathbb{P}(a'_0, \dots, a'_{N-1})$ by [49, Lemma 2.3]. If the resulting equations contain a linear term, we may repeat this process until the assumptions are satisfied. □

Using this lemma, we can assume that every X we consider in the proof of Theorem 5.1 will be quasismooth and well-formed. This in particular means that the adjunction formula holds for X, i.e., $K_X \cong \mathcal{O}_X(d_1 + \dots + d_m - a_0 - \dots - a_N)$ (see [14, Theorem 3.3.4]).

A quasismooth weighted complete intersection with no linear terms must also satisfy certain conditions on degrees [29, Lemma 18.14 (i)].

Lemma 5.6. *Let $X_{d_1,\dots,d_m} \subset \mathbb{P}(a_0, \dots, a_N)$ be a well-formed quasismooth complete intersection such that none of the equations defining X contains a linear term. Rearrange degrees and weights such that $d_1 \leq \dots \leq d_m$ and $a_0 \leq \dots \leq a_N$. Then the following inequalities hold.*

(1) $d_{m-j} > a_{N-j}$ *for all* $0 \leq j \leq m-1$.

(2) *If* $m \geq \dim X + 1$, *then* $d_{m-j} - a_{N-j} \geq a_{\dim X - j}$ *for all* $0 \leq j \leq \dim X$.

Proof. Part (1) is [29, Lemma 18.14 (i)]. The statement of [29, Lemma 18.14 (i)] assumes that the complete intersection X is not the intersection of a linear cone with other hypersurfaces, i.e., that $d_i \neq a_j$ for any i and j. However, their proof only requires that no linear term appears in any of the equations $f_1, \dots, f_m$ defining X, which is precisely what we assumed.

Part (2) is [10, Proposition 3.1 (2)]. (Once again, the statement of [10, Proposition 3.1 (2)] assumes that the complete intersection X is not the intersection of a linear cone with other hypersurfaces, i.e., that $d_i \neq a_j$ for all i and j, but the same comment made in part (1) shows that the proof extends to our situation.) □

Next, we bound the codimension of Fano and Calabi–Yau weighted complete intersections satisfying the conditions above on linear terms.

Lemma 5.7. *Suppose that $X_{d_1,\dots,d_m} \subset \mathbb{P}(a_0, \dots, a_N)$ is a well-formed quasismooth complete intersection such that none of the equations defining X contains a linear term.*

(1) *If X is Fano, then the codimension m satisfies $m < (N+1)/2$. Equivalently, the dimension n of X satisfies $2n + 2 > N + 1$.*

(2) *If X is Calabi–Yau, then the codimension m satisfies $m \leq (N+1)/2$. Equivalently, the dimension n of X satisfies $2n + 2 \geq N + 1$.*

Proof. After reordering, we may assume $d_1 \leq \dots \leq d_m$ and $a_0 \leq \dots \leq a_N$. Suppose that $m \geq (N+1)/2$ so that $m \geq N - m + 1 = \dim X + 1$. If this does not hold, then the conclusion of either part of the lemma is already true, and there is nothing to prove. In the case $m \geq (N+1)/2$, both parts of Lemma 5.6 apply.

Since $d_{m-j} \geq a_{N-j} + a_{\dim X - j}$ for $j = 0, \dots, \dim X$ by Lemma 5.6 (2), we have that

$$d_m + d_{m-1} + \dots + d_{m-\dim X} \geq a_N + \dots + a_{N-\dim X} + a_{\dim X} + \dots + a_0. \tag{5.1}$$

In the case that we have a strict inequality $m > \dim X + 1$, there are more degrees d_i that have not appeared in the inequality above. We may now apply Lemma 5.6 (1) to the remaining degrees (i.e., for indices $j = \dim X + 1, \dots, m-1$). Summing these inequalities gives that

$$d_{m-\dim X - 1} + \dots + d_1 > a_{N-\dim X-1} + \dots + a_{N-m+1}. \tag{5.2}$$

Because $N - m + 1 = \dim X + 1$, all the weights of $\mathbb{P}$ appear on the right-hand side of either (5.1) or (5.2). Adding these two inequalities thus yields $d_1 + \dots + d_m \geq a_0 + \dots + a_N$. Furthermore, this inequality is strict unless $m = \dim X + 1$ exactly.

By the adjunction formula [14, Theorem 3.3.4],

$$K_X = \mathcal{O}_X(d_1 + \dots + d_m - a_0 - \dots - a_N).$$

Under the assumption $m \geq (N+1)/2$, we have therefore shown that

$$d_1 + \dots + d_m - a_0 - \dots - a_N$$

is nonnegative, and hence X is not Fano. This completes the proof of part (1) of the lemma.

The complete intersection X is Calabi–Yau if and only if $d_1 + \dots + d_m = a_0 + \dots + a_N$. Again under the assumption $m \geq (N+1)/2$, we have shown that this can only occur if we have equality $m = (N+1)/2$. This proves (2) of the lemma. □

Having finished the preliminaries, we now begin the main part of the proof of Theorem 5.1. We will next prove some general properties of higher-dimensional weighted complete intersections with large symmetric actions, which will be key to finishing the proof of Theorem 5.1. Indeed, the following lemma assumes that the S_k-action on X of dimension n satisfies $k \geq c_{\text{Fano}}(n)$ in the Fano case or $k > c_{\text{CY}}(n)$ in the Calabi–Yau case. These assumptions put big constraints on X, and we will show later that strict inequality will lead to a contradiction. For Fano weighted complete intersections, we include the case of equality $k = c_{\text{Fano}}(n)$ because it will be useful for the classification of maximal examples in Section 6.

Lemma 5.8. *Let S_k act faithfully on a well-formed quasismooth weighted complete intersection X of dimension n such that no equation of X contains a linear term. Suppose that either*

(a) *X is Fano, $n \geq 4$, and $k \geq c_{\text{Fano}}(n)$, or*

(b) *X is Calabi–Yau, $n \geq 3$, and $k > c_{\text{CY}}(n)$.*

Then, after an appropriate change of variables, the following properties hold.

(1) *The subgroup $A_k \leq S_k$ acts by the standard representation in the first $k-1$ variables $x_0, \dots, x_{k-2}$, which all have the same weight b, and acts trivially on the remaining variables $x_{k-1}, \dots, x_N$.*

(2) *The equations* $f_1, \dots, f_m$ *are contained in the ideal*

$$\langle \sigma_2, \dots, \sigma_k, V, x_{k-1}, \dots, x_N \rangle,$$

where $\sigma_2, \dots, \sigma_k$*, are the elementary symmetric polynomials in*

$$x_0, \dots, x_{k-2}, y := -x_0 - \cdots - x_{k-2},$$

and V *is the Vandermonde polynomial in these variables.*

(3) *Any collection of* α *equations among* $\{f_1, \dots, f_m\}$*, for any* $1 \leq \alpha \leq m$*, have total degree at least* $(\frac{1}{2}(\alpha+1)(\alpha+2)-1)b$.

Proof. Since $n \geq 3$, the subgroup $S_k \leq \mathrm{Aut}(X)$ lifts to a subgroup $S_k \leq \mathrm{Aut}(\mathbb{P})$ by [50, Theorem 1.3] (here we use that S_k is a reductive group in characteristic zero).

By Theorem 4.3 the automorphism group $\mathrm{Aut}(\mathbb{P})$ of weighted projective space is described by an exact sequence $1 \to \mathbb{C}^* \to \mathrm{Aut}(S) \to \mathrm{Aut}(\mathbb{P}) \to 1$, where $S = \mathbb{C}[x_0, \dots, x_N]$ is the polynomial ring with each variable x_i of weight a_i, and $\mathrm{Aut}(S)$ is the group of graded automorphisms of this ring. The subgroup $\mathbb{C}^*$ is the group of "scalar transformations" which, for each i, map $x_i \mapsto t^{a_i} x_i$ for some $t \in \mathbb{C}^*$. Since the Schur multiplier $H^2(S_k, \mathbb{C}^*)$ is $\mathbb{Z}/2$ for $k \geq 4$, the map $S_k \to \mathrm{Aut}(\mathbb{P})$ lifts to $\widetilde{S}_k \to \mathrm{Aut}(S)$, where $\widetilde{S}_k$ is one of the two representation groups of S_k, a central extension of order 2.

Theorem 4.3 (2) (see also the proof of [17, Lemma 3.5]) shows that any finite subgroup of $\mathrm{Aut}(S)$ is conjugate to one inside the reductive subgroup $\prod_\ell \mathrm{GL}_{N_\ell}(\mathbb{C}) \leq \mathrm{Aut}(S)$ given by the group of automorphisms that do not "mix" variables with weights of different sizes. Here, $\sum_\ell N_\ell = N+1$ is the total number of weights. (For example, when $\mathbb{P} = \mathbb{P}(5,5,2,2,2)$, we have a $\mathrm{GL}_2(\mathbb{C})$ acting on the first two variables, and a $\mathrm{GL}_3(\mathbb{C})$ on the last three.) Projection to each factor $\mathrm{GL}_{N_\ell}(\mathbb{C})$ gives a linear representation of $\widetilde{S}_k$. Since the original map $S_k \to \mathrm{Aut}(\mathbb{P})$ was injective, at least one of these representations must be a faithful linear representation of S_k or $\widetilde{S}_k$.

Let $I \subset \mathbb{C}[x_0, \dots, x_N]$ be the weighted homogeneous prime ideal defining the weighted complete intersection X. Then I is invariant under the $\widetilde{S}_k$-action. By Nakayama's lemma, $I/\mathfrak{m}I$ is a $\mathbb{C}$-vector space with dimension the minimal number of generators of I, where $\mathfrak{m}$ is the irrelevant ideal of the graded polynomial ring $\mathbb{C}[x_0, \dots, x_N]$. But the minimal number of generators of I is m, the codimension of X. This is at most $n+1$ by Lemma 5.7 and hence less than $k-1$, so the action of $\widetilde{S}_k$ on $I/\mathfrak{m}I$ is trivial up to sign by the classification of $\widetilde{S}_k$ representations (Table 2). Thus, we may choose a set of weighted homogeneous generators $f_1, \dots, f_m$ for I such that each f_i is $\widetilde{S}_k$-invariant up to sign.

We saw above that, after an appropriate change of variables in $\mathbb{P}$, the S_k-action on X lifts to an $\widetilde{S}_k$-action on $\mathbb{C}[x_0, \dots, x_N]$ which acts linearly on each vector space $\mathbb{C}^{N_\ell}$ of variables of each given weight. By Lemma 5.7, X is a complete intersection in $\mathbb{P}(a_0, \dots, a_N)$ where $N+1 < 2n+2$ (in the Fano case) or $N+1 \leq 2n+2$ (in the Calabi–Yau case).

Claim. *In this setting, we have an irreducible* $(k-1)$*-dimensional linear representation of* $\widetilde{S}_k$ *inside a space* $\mathbb{C}^{N_\ell}$ *of variables of the same weight in* $\mathbb{C}[x_0, \dots, x_N]$.

To show the claim, we consider several different cases. The total number $N+1$ of weights is at most $2n+2$. Thus, Lemma 2.4 implies that if $n \geq 4$ and we are not in the special case $n = 4$ and $k = 8$, the only irreducible representations which are of small enough dimension to comprise the $\widetilde{S}_k$-action on some $\mathbb{C}^{N_\ell}$ are of dimension 1 and $k-1$. They cannot all be dimension 1, or else S_k would not act faithfully on X. We conclude that there is a $(k-1)$-dimensional representation on the weights.

It remains to consider the exceptional cases

(1) $n = 3$ and X Calabi–Yau, and

(2) $(n, k) = (4, 8)$.

First, consider when X is Calabi–Yau and $n = 3$. The assumptions of the lemma mean

$$k \geq 8 = c_{\mathrm{CY}}(3) + 1.$$

If $k > 8$, all representations of $\widetilde{S}_k$ other than those of dimension 1 and $k-1$ have dimension larger than $2n + 2 \geq N + 1$ as above. When $k = 8$, we could conceivably have that $N + 1 = 8$ and that $\widetilde{S}_8$ acts faithfully by the basic spin representation of dimension 8. This would mean that all 8 weights $a_0, \ldots, a_7$ are equal, so actually, $X \subset \mathbb{P}^7$ is a smooth complete intersection of codimension 4. The only way this is possible (since there are no linear equations) is if X is a $(2,2,2,2)$-complete intersection. But up to scaling, there is only one polynomial of degree 2 which is $\widetilde{S}_8$-invariant up to sign (see Lemma 2.4), a contradiction.

When $n = 4$ and X is Calabi–Yau, then the assumption implies $k > 8$. So the remaining exceptional case is when X is Fano and $n = 4$. Lemma 5.7 guarantees that the number of weights is less than $2n + 2 = 10$. If there are 8 weights and we want to fit a basic spin representation of $\widetilde{S}_8$, we have that $X \subset \mathbb{P}^7$ again, and the invariant polynomials do not have low enough degree as above. If there are 9 weights, and the first 8 are part of the faithful spin representation of $\widetilde{S}_8$, then $X \subset \mathbb{P}(1^{(8)}, a)$ has codimension 4. At least one equation includes the variable corresponding to a, and all four must involve invariant (up to sign) polynomials in the first 8 variables. The total degree is therefore more than $a + 2 + 8 + 8$, so X could not be Fano. This shows the claim.

Therefore, we have a $(k-1)$-dimensional linear representation inside a space $\mathbb{C}^{N_\ell}$ of variables of the same weight in $\mathbb{C}[x_0, \ldots, x_N]$. Since $2(k-1) = 2k - 2 > 2n + 2 \geq N + 1$, there is exactly one ℓ with the above property. After reordering the variables, we can conclude the following: $\widetilde{S}_k$ acts by the standard representation of S_k (or its tensor product with the sign representation) on the variables $x_0, \ldots, x_{k-2}$, which all must be of the same weight b. In addition, it acts trivially, or by the sign representation, on all other variables $x_{k-1}, \ldots, x_N$, which could all be of different weights. Since all the representations that appear are actually S_k representations rather than just $\widetilde{S}_k$ representations, we will only work with S_k from now on. In particular, we now know that $A_k \leq S_k$ acts by the standard representation in the first $k-1$ variables and acts trivially on the remaining ones. This completes the proof of (1).

We saw above that each of $f_1, \ldots, f_m$ must be S_k-invariant up to sign. In particular, all these equations are A_k-invariant, so $f_1, \ldots, f_m \in \langle \sigma_2, \ldots, \sigma_k, V, x_{k-1}, \ldots, x_N \rangle$, where $\sigma_2, \ldots, \sigma_k$ are the elementary symmetric polynomials in $x_0, \ldots, x_{k-2}, y := -x_0 - \cdots - x_{k-2}$, and V is the Vandermonde polynomial in these variables. Indeed, for the usual permutation representation of A_k on $\mathbb{C}^k$, the invariant ring would be generated by the first k elementary symmetric polynomials and the Vandermonde polynomial. The standard representation is the subspace of the permutation representation $\mathbb{C}^k$ where the variables add to zero, so the invariants are given as above, with σ_1 omitted, and $x_0 + x_1 + \cdots + x_{k-2} + y = 0$. This shows (2).

In order to prove (3), first observe as above that the assumption

$$k \geq c_{\mathrm{Fano}}(n) \quad \text{or} \quad k > c_{\mathrm{CY}}(n)$$

implies $k - 1 > \frac{N+1}{2}$, so more than half the total variables belong to the set permuted by S_k. Also, $k > n$, so the codimension satisfies $m = N - n \geq N - k$, that is, there are more equations than there are variables not belonging to the permutation action. From here, we make a few additional simple observations.

Each of the polynomials $f_1, \ldots, f_m$ must involve some variable $x_0, \ldots, x_{k-2}$, or else X would fail to be quasismooth. This follows from the same type of arguments as the proof of Lemma 5.7. Indeed, suppose one of the equations, say f_1, does not include any of $x_0, \ldots, x_{k-2}$. Then let Π be the $(k-1)$-plane in $\mathbb{A}^{N+1}$ given by $x_{k-1} = \cdots = x_N = 0$. The intersection

$$Z := \{f_2 = \cdots = f_m = 0\} \cap \Pi$$

has positive dimension because $k > m$, so choose a point $p \in Z \setminus \{0\} \subset \mathbb{A}^{N+1}$. Then f_1 and all its derivatives are identically zero at p, so the affine cone over X is singular at p by the Jacobian criterion.

Hence, we conclude that each equation f_i involves at least one of the elementary symmetric polynomials or V. Next, note that, for any $1 \leq \alpha \leq m$, it is impossible for a subcollection of α of the equations, say $\{f_1, \ldots, f_\alpha\}$, to be contained in an ideal generated by $\alpha - 1$ or fewer elements of the set $\{\sigma_2, \ldots, \sigma_k, V\}$. Otherwise, the locus where these $\alpha - 1$ elements and $f_{\alpha+1}, \ldots, f_m$ are zero would be a subvariety of $\mathbb{P}$ of codimension at most $m - 1$ contained in X, but this contradicts the fact that X is codimension m.

We will use the above observation to show by induction on α that, for any subset of α equations from $\{f_1, \ldots, f_\alpha\}$, with $1 \leq \alpha \leq m$, we have

$$\deg(f_1) + \cdots + \deg(f_\alpha) \geq \left(\frac{(\alpha+1)(\alpha+2)}{2} - 1\right) b.$$

Here, b is the weight from part (1) of the lemma. In the base case, we have already shown that a single equation f_1 must include some polynomial from $\{\sigma_2, \ldots, \sigma_k, V\}$, so it has degree at least $\deg(\sigma_2) = 2b$, since σ_2 has the smallest degree. By the inductive hypothesis, suppose that any subset of $\alpha - 1$ polynomials from $T = \{f_1, \ldots, f_\alpha\}$ satisfies the corresponding inequality on degree. Some equation from T must be of degree at least $\deg(\sigma_{\alpha+1})$, or else only $\sigma_2, \ldots, \sigma_\alpha$ would appear in equations in T, contradicting the previous paragraph. Since the sum of degrees of the other $\alpha - 1$ equations is at least $\deg(\sigma_2) + \cdots + \deg(\sigma_\alpha)$, this completes the induction. We note that the Vandermonde polynomial has degree $\binom{k}{2} b$, which is larger than the degree of any of the elementary symmetric polynomials; thus, its degree did not feature in the bounds just proved. □

The following lemma will nearly finish the proof.

Lemma 5.9. *Suppose that the same assumptions from Lemma* 5.8 *hold. Then the total degree $d = d_1 + \cdots + d_m$ of X satisfies*

$$d \geq a_{k-1} + \cdots + a_N + \left(\frac{(k-n-1)(k-n)}{2} - 1\right) b,$$

where b is the weight in Lemma 5.8 (1). *If equality holds, then $N = k - 2$, so there are no additional weights on the right-hand side, and $\mathbb{P}(a_0, \ldots, a_N) \cong \mathbb{P}^{k-2}$.*

Proof. Assume that $f_1, \dots, f_m$ are ordered by increasing degree. By Lemma 5.8 (3), the first $k-n-2$ equations satisfy

$$\begin{aligned}\deg(f_1) + \cdots + \deg(f_{k-n-2}) &\geq \deg(\sigma_2) + \cdots + \deg(\sigma_{k-n-1}) \\ &= \Big(\frac{(k-n-1)(k-n)}{2} - 1\Big)b.\end{aligned}$$

If $m > k-n-2$, then we may apply Lemma 5.6 (1) for $0 \leq j \leq N-k+1$ to obtain

$$\deg(f_{k-n-1}) + \cdots + \deg(f_m) > a_{k-1} + \cdots + a_N. \tag{5.3}$$

(Contrary to the notation of that lemma, the weights $a_{k-1}, \dots, a_N$ might not be the largest of the a_i, but the same inequality will certainly also hold for a different subset of weights with smaller total.) Here, we note that $m \geq N-k+2$ because $k \geq n+2 = N-m+2$.

Adding the two inequalities together yields the inequality in the statement of the lemma, and we see that equality can only occur when there is no contribution from (5.3). This only occurs when $m = k-n-2$, so that $N = k-2$ and all the weights are the same. Since our weighted projective space is well-formed, this implies $b = 1$ and $\mathbb{P}(a_0, \dots, a_N) \cong \mathbb{P}^{k-2}$. □

We can now conclude the proof of Theorem 5.1.

Proof of Theorem 5.1. In light of Proposition 5.4, we may assume that $n \geq 4$ for X Fano, and $n \geq 3$ for X Calabi–Yau. By Lemma 5.5, we may exclusively consider well-formed X with the property that no defining equation has a linear term. Finally, we may also assume that the S_k-action satisfies $k \geq c_{\mathrm{Fano}}(n)$ in the Fano case, or $k > c_{\mathrm{CY}}(n)$ in the Calabi–Yau case. Indeed, if these inequalities on k are not satisfied, then the conclusion of Theorem 5.1 automatically holds. In summary, we have reduced to the setting where the conditions of Lemma 5.8 and Lemma 5.9 are satisfied, so we may apply the conclusions of these lemmata.

For X to be Fano (resp. Calabi–Yau), we must have

$$d < a_0 + \cdots + a_N = (k-1)b + a_{k-1} + \cdots + a_N \quad \text{(resp. } \leq \text{)}.$$

This inequality together with Lemma 5.9 implies

$$\frac{(k-n-1)(k-n)}{2} < k \quad \text{(resp. } \leq \text{)}. \tag{5.4}$$

For a fixed n, (5.4) with a strict inequality holds for an integer k if and only if

$$n + 1 + \frac{1 - \sqrt{8n+9}}{2} < k < n + 1 + \frac{1 + \sqrt{8n+9}}{2}.$$

Therefore,

$$k \leq n + \Big\lceil \frac{1 + \sqrt{8n+9}}{2} \Big\rceil = c_{\mathrm{Fano}}(n).$$

Example 8.1 shows that the bound for Fano X is sharp for all $n \geq 1$.

Similarly, in the Calabi–Yau case, (5.4) with a non-strict inequality holds for an integer k if and only if

$$n + 1 + \frac{1 - \sqrt{8n+9}}{2} \leq k \leq n + 1 + \frac{1 + \sqrt{8n+9}}{2}.$$

Therefore,

$$k \leq n + \left\lfloor \frac{1 + \sqrt{8n+9}}{2} \right\rfloor + 1 = c_{\mathrm{CY}}(n).$$

This completes the proof. $\square$

Proof of Theorem 4. It follows from Theorem 5.1 that the largest symmetric group action on a weighted complete intersection is by

$$c_{\mathrm{Fano}}(n) = n + \left\lceil \frac{1 + \sqrt{8n+9}}{2} \right\rceil.$$

Taking the limit of $c_{\mathrm{Fano}}(n)/(n+1)$ as $n \to \infty$ gives the required result. $\square$

6. Maximally symmetric varieties

In this section, we study maximally symmetric Fano weighted complete intersections. These are the Fano weighted complete intersections of dimension n which have a faithful action by S_k, where $k = c_{\mathrm{Fano}}(n)$ is the largest possible.

Using the setup of Section 5, we can further limit the possible behavior of maximally symmetric Fano weighted complete intersections.

Proposition 6.1. *Suppose that X is a maximally symmetric Fano weighted complete intersection of dimension $n \geq 4$ with action by S_k, i.e., $k = c_{\mathrm{Fano}}(n)$. Suppose further that X is quasismooth and well-formed and no defining equation contains a linear term. Then X is embedded in either $\mathbb{P}^{k-2}$ or $\mathbb{P}^{k-1}(1^{(k-1)}, a)$.*

As before, X will denote a quasismooth weighted complete intersection

$$X_{d_1,\dots,d_m} \subset \mathbb{P}(a_0, \dots, a_N)$$

defined by equations $f_1, \dots, f_m$ which are weighted homogeneous of degrees $d_1, \dots, d_m$, respectively.

Proof. The conditions of Lemma 5.8 are met, so the three properties listed there hold for X. We retain the notation from that lemma.

In Lemma 5.9, we applied Lemma 5.8 (3) with $\alpha = k - n - 2$ and Lemma 5.6 (1) to obtain a lower bound on the total degree $d = d_1 + \cdots + d_m$ of the complete intersection X. Now, we will do nearly the same thing with a different value of α to obtain another useful bound on d. From now on, order the equations $f_1, \dots, f_m$ by increasing degree.

If the dimension of the ambient weighted projective space $\mathbb{P}$ is $N = k - 2$, then we have $\mathbb{P} \cong \mathbb{P}^{k-2}$ by Lemma 5.8 (1). Otherwise, there is at least one weight *not* contained in the faithful S_k-representation, so that $N \geq k - 1$. Since $N = n + m$, this implies $k - n - 1 \leq m$, so we may apply Lemma 5.8 (3) to the first $\alpha = k - n - 1$ equations $f_1, \dots, f_{k-n-1}$ to obtain

$$d_1 + \cdots + d_{k-n-1} \geq \Big(\frac{(k-n)(k+1-n)}{2} - 1\Big) b.$$

Since X is Fano, the total degree is less than the sum of the weights. That is,

$$\begin{aligned} a_0 + \cdots + a_N = (k-1)b + a_{k-1} + \cdots + a_N &> d_1 + \cdots + d_m \\ &\geq \Big(\frac{(k-n)(k-n+1)}{2} - 1\Big)b + d_{k-n} + \cdots + d_m. \end{aligned}$$

Rearranging this expression gives that

$$\Big(k - \frac{(k-n)(k-n+1)}{2}\Big)b > (d_{k-n} + \cdots + d_m) - (a_{k-1} + \cdots + a_N).$$

The key point is that $k = c_{\mathrm{Fano}}(n)$ is the largest integer k which satisfies $k - \frac{(k-n-1)(k-n)}{2} > 0$ (see (5.4)). On the left-hand side, we have replaced k by $k+1$ in the fraction, so the left-hand side must now be nonpositive. Hence, the right-hand side is actually negative, i.e.,

$$a_{k-1} + \cdots + a_N > d_{k-n} + \cdots + d_m. \tag{6.1}$$

Now we will assume that $N > k-1$, i.e., there are at least *two* weights not contained in the faithful S_k-representation, and derive a contradiction. In inequality (6.1), there are $N-k+2$ weights on the left and $m-(k-n)+1 = N-k+1$ degrees on the right; in particular, the assumption that $N > k-1$ means there is a nonzero number of terms on the right-hand side.

Reorder $a_{k-1}, \ldots, a_N$ by increasing size. Recall that $k > n+1$, so $m-1 > m-k+n$. Then we have $d_m > a_N, d_{m-1} > a_{N-1}, \ldots, d_{k-n} > a_k$ by Lemma 5.6 (1). We claim that we can improve the first inequality to $d_m \geq a_N + a_{k-1}$. Indeed, some equation must involve x_N, or else the image of the coordinate point $p := (0, \ldots, 0, 1) \in \mathbb{A}^{N+1}$ of x_N is in X and all partial derivatives of all equations vanish there, contradicting quasismoothness. If x_N ever appears with an exponent of at least 2, $d_m \geq 2a_N \geq a_N + a_{k-1}$ and we are done. If not, since there are no linear terms, x_N always appears multiplied by other variables and hence $p \in X$. We must then have a monomial of the form $x_j x_N$ with $j \neq N$ in some equation, or else once again all equations would have all partial derivatives vanishing at p. But this j cannot be from $0, \ldots, k-2$, because those variables only appear as part of the polynomials $\sigma_2, \ldots, \sigma_k, V$, which all have degree at least 2. We conclude that $j \in \{k-1, \ldots, N-1\}$, so the largest degree d_m is at least $a_N + a_{k-1}$.

In summary, $d_m \geq a_N + a_{k-1}, d_{m-1} > a_{N-1}, \ldots, d_{k-n} > a_k$. Adding these together contradicts inequality (6.1). We have thus shown that $k-2 \leq N \leq k-1$. That is, the ambient weighted projective space $\mathbb{P}$ is of the form either $\mathbb{P}^{k-2}$ or $\mathbb{P}(1^{(k-1)}, a)$, where in the second case, we note that $\mathbb{P}(b^{(k-1)}, a)$ is not well-formed unless $b = 1$. □

Theorem 5 states that maximally symmetric Fano weighted complete intersections are finite covers of complete intersections in $\mathbb{P}^N$ cut out by symmetric polynomials. This will now follow quickly from Proposition 6.1. We omit the case of dimension $n = 2$ in Theorem 5 because the largest symmetric group inside Cr(2) is S_5, and there is a copy of S_5 contained in Cr(2) acting regularly on the degree 5 del Pezzo surface, for which it is not clear whether the required cover exists.

Proof of Theorem 5. As usual, we first deal with low-dimensional cases. When $n = 1$, $\mathbb{P}^1$ is the only quasismooth Fano weighted complete intersection, so the theorem is trivial. For $n = 3$, [47, Proposition 1.1 (ii)] shows that any three-dimensional S_7-Mori fiber space over a rationally connected base is equivariantly isomorphic to the complete intersection of Fermat hypersurfaces of degrees $1, 2$, and 3 in $\mathbb{P}^6$. It follows in particular that this is the only maximally symmetric quasismooth Fano weighted complete intersection of dimension 3.

For $n \geq 4$, we can apply Proposition 6.1. We reduce as before to the case where X has no linear terms and is well-formed using Lemma 5.5. This shows that X is isomorphic to a weighted complete intersection in $\mathbb{P}^{k-2}$ or $\mathbb{P}^{k-1}(1^{(k-1)}, a)$. In either case, Lemma 5.8 (2) already showed that the variables $x_0, \ldots, x_{k-1}$ only appear in the equations of X in the form of elementary symmetric polynomials in $x_0, \ldots, x_{k-1}, y$, plus the Vandermonde polynomial. However, no equation f_i may involve the Vandermonde polynomial V, or else the degree would be too high to be Fano. Hence, X is defined by equations which are all symmetric in $x_0, \ldots, x_{k-1}, y$.

We may now "add back on" an additional weight equal to 1 to make the standard representation into the permutation representation; indeed, we saw that the equations for X are combinations of elementary symmetric polynomials of degrees $2, \ldots, k$ in

$$x_0, \ldots, x_{k-2}, y = -x_0 - \cdots - x_{k-2}.$$

Add the variable y of weight 1 and the extra linear relation $x_0 + \cdots + x_{k-2} + y = 0$ to see the same X as living inside $\mathbb{P}^{k-1}$ or $\mathbb{P}^k(1^{(k)}, a)$, this time defined by invariants of the permutation representation in the first k variables.

If $X \subset \mathbb{P}^{k-1}$, we can take the finite cover $X \to X$ to be the identity and we are done. If $X \subset \mathbb{P}^k(1^{(k)}, a)$, consider the restriction to X of the rational map $\pi\colon \mathbb{P}^k(1^{(k)}, a) \dashrightarrow \mathbb{P}^{k-1}$ forgetting the last weight. For X to be quasismooth in $\mathbb{P}^k(1^{(k)}, a)$, there must be a monomial of the form z^r appearing in some f_i, where z is the variable of weight a. Otherwise, the coordinate point of z would be contained in X, and all partial derivatives of all equations would vanish there, since the other variables always appear as part of symmetric polynomials of degree at least 2.

It follows that the restriction $\pi|_X\colon X \to \operatorname{im}(X)$ is a morphism because the only basepoint of π is the coordinate point of the last variable, which we saw cannot be contained in X. The image $Y := \operatorname{im}(X)$ is clearly defined by symmetric polynomials in $\mathbb{P}^{k-1}$, and the map has finite fibers, hence is finite. □

Nontrivial finite covers do appear in maximally symmetric examples; see Example 8.8. We can say something more precise in the case that a maximally symmetric Fano weighted complete intersection in addition has the largest possible index of $-K_X$. Theorem 6 is a direct consequence of the following statement.

Theorem 6.2. *Let X be a quasismooth Fano weighted complete intersection of dimension $n \geq 2$ with faithful S_k-action, where $k = c_{\mathrm{Fano}}(n)$ is the upper bound of Theorem* 5.1. *Then the index i_X of $-K_X$ satisfies*

$$i_X \leq k - \frac{(k-n)(k-n-1)}{2}.$$

When equality holds, X is equivariantly isomorphic to the intersection of Fermat hypersurfaces of degrees $1, \ldots, k-n-1$ *in* $\mathbb{P}^{k-1}$.

Proof. As usual, we will first deal with low dimensions. By [15], the possible actions of S_5 on del Pezzo surfaces are on $\mathbb{P}^1 \times \mathbb{P}^1$, the Clebsch diagonal cubic surface, and the degree 5 del Pezzo. Only the first case has the maximal index of $-K_X$ equal to 2, and this $\mathbb{P}^1 \times \mathbb{P}^1$ is the quadric which is a Fano–Fermat complete intersection in $\mathbb{P}^4$. In dimension 3, we know that, up to equivariant isomorphism, the unique S_7-action on a Fano quasismooth weighted complete intersection is on the $(1, 2, 3)$-Fano–Fermat complete intersection in $\mathbb{P}^6$, of index 1.

Suppose $n \geq 4$. Assume that X is well-formed and has no linear terms in its defining equations. We may assume this without loss of generality by Lemma 5.5, which allows us to replace the original $X \subset \mathbb{P}$ with an isomorphic $X' \subset \mathbb{P}'$ with the desired properties. By taking the S_k-action on X' to be the one induced by this isomorphism, we can ensure $X \cong X'$ is equivariant.

By [50, Theorem 2.15], the class group of a quasismooth well-formed weighted complete intersection of dimension at least 3 is isomorphic to $\mathbb{Z}$ with generator $\mathcal{O}_X(1)$. Therefore, the index of $-K_X$ for a Fano weighted complete intersection equals $a_0 + \cdots + a_N - d$, where $d = d_1 + \cdots + d_m$ is the total degree of X. We know from Lemma 5.9 that the total degree satisfies

$$d \geq a_{k-1} + \cdots + a_N + \Big(\frac{(k-n-1)(k-n)}{2} - 1\Big)b,$$

while the sum of the weights is $a_0 + \cdots + a_N = (k-1)b + a_{k-1} + \cdots + a_N$. Therefore,

$$i_X \leq \Big(k - \frac{(k-n-1)(k-n)}{2}\Big)b.$$

We learned in Proposition 6.1 that $b = 1$ in all maximal examples, so this gives the desired index inequality.

Suppose now that equality holds. Lemma 5.9 also showed that this inequality can only be an equality if X is actually a complete intersection in $\mathbb{P}^{k-2}$ defined by invariants of the standard representation. The codimension of X is therefore $k-n-2$. The minimum total degree of $k-n-2$ equations is precisely $\frac{(k-n-1)(k-n)}{2} - 1$, and this can only occur when the defining ideal is $\langle \sigma_2, \ldots, \sigma_{k-n-1} \rangle$. Add back on the extra weight as above (this operation is an S_k-equivariant isomorphism on X) and note that the elementary symmetric polynomials $\sigma_1, \ldots, \sigma_\alpha$ generate the same ideal as the Fermat polynomials $p_1, \ldots, p_\alpha$ (over a field of characteristic zero). Therefore, X is S_k-equivariantly isomorphic to the Fano–Fermat complete intersection $\{p_1 = \cdots = p_{k-n-1} = 0\} \subset \mathbb{P}^{k-1}$ of Example 8.1. Note that this X is smooth by Lemma 2.11. □

Proof of Theorem 6. In each dimension $n \geq 2$, Theorem 6 directly follows from Theorem 6.2. The latter theorem omits the case of $n = 1$ because the index of $-K_{\mathbb{P}^1}$ is 2 rather than 1, as the formula predicts. Nevertheless, $\mathbb{P}^1$ is still a Fano–Fermat variety

$$\{x_0 + x_1 + x_2 + x_3 = x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0\} \subset \mathbb{P}^3$$

with the S_4-action by permutation, so Theorem 6 holds in all dimensions. □

7. Symmetries and boundedness

In this section, we prove statements about the boundedness of Fano 4-folds and 5-dimensional klt singularities admitting S_8-actions. First, we recall the concepts of dual complexes and coregularity.

Definition 7.1. Let E be a simple normal crossing divisor on a smooth variety X. The *dual complex* $\mathcal{D}(E)$ is the CW complex whose vertices correspond to the components of E and whose k-cells correspond to the irreducible components of the intersection of $k+1$ components of E.

Let (X, Γ) be a log Calabi–Yau pair. Let $\pi: Y \to X$ be a log resolution of (X, Γ). Write $\pi^*(K_X + \Gamma) = K_Y + \Gamma_Y$. Let S_Y be the sum of all the components of Γ_Y that appear with coefficient 1. The *dual complex* $\mathcal{D}(Y, \Gamma_Y)$ of (Y, Γ_Y) is the CW complex $\mathcal{D}(S_Y)$.

In [13, Theorem 3], the authors show that the homotopy class of $\mathcal{D}(S_Y)$ is independent of the chosen log resolution. More precisely, given two log resolutions $Y \to X$ and $Y' \to X$ of (X, Γ), the dual complexes $\mathcal{D}(S_Y)$ and $\mathcal{D}(S_{Y'})$ are simple homotopy equivalent to each other. Thus, we have a well-defined *dual complex* of $\mathcal{D}(X, \Gamma)$. If G is a finite group acting on (X, Γ), we may consider a G-equivariant log resolution of the pair. Hence, G acts on $\mathcal{D}(X, \Gamma)$. The dual complex of a log Calabi–Yau pair is in general a pseudo-manifold [24, Theorem 1.6]; however, in dimension at most 4, we know that they are orbifolds [34, Proposition 5].

Definition 7.2. Let (X, Γ) be a log Calabi–Yau pair. The *coregularity* of (X, Γ), written $\operatorname{coreg}(X, \Gamma)$, is defined to be $\dim X - \dim \mathcal{D}(X, \Gamma) - 1$. Let (X, B) be a log Fano pair. The *coregularity* of (X, B) is the minimum among the coregularities of (X, Γ) where $\Gamma \geq B$ and (X, Γ) is log Calabi–Yau. The coregularity of an n-dimensional log Fano pair is contained in the set $\{0, \dots, n\}$.

The concept of coregularity has recently been connected with log canonical thresholds, indices of Calabi–Yau pairs, and complements of Fano varieties (see [19, 21, 22]). For log Calabi–Yau pairs, the coregularity is independent of the chosen crepant model.

Lemma 7.3 ([22, Proposition 3.11]). *Let (X, Γ) be a log Calabi–Yau pair. Let (X', Γ') be a crepant model of X, i.e., a birational log Calabi–Yau pair for which there exists a common resolution $p: Y \to X$ and $q: Y \to X'$ with $p^*(K_X + \Gamma) = q^*(K_{X'} + \Gamma')$. Then*

$$\operatorname{coreg}(X, \Gamma) = \operatorname{coreg}(X', \Gamma').$$

The following lemma states that the dimension of dual complexes of log Calabi–Yau pairs is preserved under finite quotients.

Lemma 7.4. *Let (X, Γ) be a log Calabi–Yau pair. Let $G \leq \operatorname{Aut}(X, \Gamma)$ be a finite group. Let $Y := X/G$, let $p: X \to Y$ be the quotient morphism, and let Γ_Y be the boundary divisor for which $p^*(K_Y + \Gamma_Y) = K_X + \Gamma$. Then we have that* $\dim \mathcal{D}(Y, \Gamma_Y) = \dim \mathcal{D}(X, \Gamma)$.

Proof. We proceed by induction on the dimension of X. The case of dimension 1 is clear. By passing to a G-equivariant dlt modification, we may assume that (X, Γ) is dlt. By [24, Theorem 1.6], the dual complex $\mathcal{D}(X, \Gamma)$ is an equidimensional pseudo-manifold. If (X, Γ) is klt, then the statement is clear, since in this case, both (X, Γ) and (Y, Γ_Y) are klt, so their dual complexes have dimension -1. Thus, we may assume that $\lfloor \Gamma \rfloor$ is non-empty. Let $S \subset \lfloor \Gamma \rfloor$ be an irreducible component and S_Y the image of S on Y. Let (S, Γ_S) be the log Calabi–Yau pair obtained by adjunction of (X, Γ) to S, and let (S_Y, Γ_{S_Y}) be the log Calabi–Yau pair obtained by adjunction of (Y, Γ_Y) to S_Y. Note that $\dim \mathcal{D}(X, \Gamma) = \dim(S, \Gamma_S) + 1$

and $\dim \mathcal{D}(Y, \Gamma_Y) = \dim(S_Y, \Gamma_{S_Y}) + 1$. Indeed, the dual complex $\mathcal{D}(S, \Gamma_S)$ is the link of the vertex v_S corresponding to S in $\mathcal{D}(X, \Gamma)$, and the analogous statement holds for (S_Y, Γ_{Y_S}) and (Y, Γ_Y). Let $G_S \leq G$ be the subgroup fixing S. Then G_S acts on (S, Γ_S). By construction, we have that $p_S \colon S \to S_Y$ is the quotient morphism by G_S and $p_s^*(K_{S_Y} + \Gamma_{S_Y}) = K_S + \Gamma_S$. By induction on the dimension, we have that $\dim \mathcal{D}(S, \Gamma_S) = \dim \mathcal{D}(S_Y, \Gamma_{S_Y})$. This finishes the proof. □

Now, we prove the main global statement of this section. To do so, we first prove lemmata regarding alternating group actions on Calabi–Yau surfaces and Calabi–Yau 3-folds, and subgroups of the special orthogonal groups.

Lemma 7.5. *Let H be a finite group and $H \to A_8$ a surjective group homomorphism. Let (X, Γ) be a 2-dimensional log Calabi–Yau pair. Then (X, Γ) does not admit a faithful H-action.*

Proof. We proceed by contradiction. Let (X, Γ) be a log canonical Calabi–Yau surface that admits a faithful action by H. By passing to an H-equivariant dlt modification, we may assume that (X, Γ) is dlt.

First, assume that $\Gamma \neq 0$. Then we may run an H-equivariant K_X-MMP that terminates with a Mori fiber space. Let $X \to X_1 \to \cdots \to X_k$ be the steps of this MMP and let $X_k \to Z$ be the Mori fiber space. Let Γ_k be the pushforward of Γ on X_k. First, assume that Z is a point. So X_k is a Fano surface. Let N be the kernel of the homomorphism $H \to A_8$. The quotient $Y := X_k/N$ is a Fano type surface that admits an A_8-action. In particular, an A_8-equivariant resolution of Y is a smooth rational surface with a faithful A_8-action. This is impossible, as the plane Cremona group does not admit a subgroup isomorphic to A_8 (see [15]). Now, assume that Z is a curve. We have a short exact sequence $1 \to G_F \to H \to G_Z \to 1$, where G_F acts on the general fiber of $X_k \to Z$ and G_Z acts on Z. By the canonical bundle formula, the curve Z has genus either 0 or 1. Note that either G_F or G_Z admits a surjective homomorphism to A_8. Thus, we get a faithful action on a curve by a group G that surjects onto A_8. By taking the quotient by the kernel of $G \to A_8$, we obtain a curve of genus at most 1 that admits a faithful A_8-action. This is impossible due to the classification of finite subgroups of $\mathbb{P}\mathrm{GL}_2(\mathbb{C})$ and the classification of finite actions on genus 1 curves; indeed, the automorphism group of a genus 1 curve C is a semidirect product of an abelian translation group with a cyclic group of order 2, 4, or 6 (see also the proof of Proposition 5.4).

Now, assume that $\Gamma = 0$. Since we have assumed (X, Γ) is dlt, this means that X is a klt Calabi–Yau surface. Let $Y \to X$ be an H-equivariant resolution and $\varphi^*(K_X) = K_Y + D_Y$. If $D_Y \neq 0$, then we proceed as in the previous paragraph. Thus, we may assume that X has canonical singularities and Y is a smooth surface with $K_Y \sim_{\mathbb{Q}} 0$. By the Enriques–Kodaira classification of surfaces, Y is a K3 surface, an Enriques surface, an abelian surface, or a hyperelliptic surface. We will show that each of these four cases leads to a contradiction.

If Y is a K3 surface, then as in the proof of Proposition 5.4, we have an exact sequence

$$1 \to H^{\mathrm{symp}} \to H \to \mathbb{Z}/m \to 1,$$

where H^{symp} is a finite group acting by symplectic automorphisms on the K3 surface Y. We conclude that H^{symp} surjects onto A_8. In particular, $|H^{\mathrm{symp}}| \geq 8!/2$. This leads to a contradiction by the classification of finite groups acting symplectically on K3 surfaces (see [43]).

If Y is an Enriques surface, let $\widetilde{Y} \to Y$ be the universal cover. Then $\widetilde{Y}$ is a K3 surface, and there is a finite group $\widetilde{H}$ acting on $\widetilde{Y}$ that surjects onto A_8, which is a contradiction by the previous case.

If Y is an abelian surface, let $T_Y \leq \mathrm{Aut}(Y)$ be the group of translations. Then we have an exact sequence

$$1 \to H_Y \cap T_Y \to H_Y \to G_Y \to 1.$$

Since T_Y is abelian, we conclude that $H_Y \cap T_Y$ does not surject onto A_8. So G_Y must surject onto A_8. Observe that G_Y is a group of automorphisms of the abelian surface that fixes the identity and surjects onto A_8. This contradicts the classification of finite groups acting on abelian surfaces (see [25]).

Finally, if Y is a hyperelliptic surface, then [4] gives a contradiction. This completes the proof. □

Lemma 7.6. *Let H be a finite group and $H \to A_8$ a surjective group homomorphism. Let (X, Γ) be a 3-dimensional log Calabi–Yau pair with $\Gamma \neq 0$. Then (X, Γ) does not admit a faithful H-action.*

Proof. By means of contradiction, assume that a Calabi–Yau 3-fold (X, Γ) with a faithful A_8-action exists. We run an H-equivariant K_X-MMP. Since $\Gamma \neq 0$, this MMP must terminate with a Mori fiber space $X_k \to Z$. By replacing X with X_k and Γ with its pushforward on X_k, we may assume that X itself admits a Mori fiber space $X \to Z$. If Z is a point, then X is a Fano variety. But A_8 does not act faithfully on a rationally connected 3-fold [8], so we get a contradiction. Assume that Z is positive-dimensional. We have a short exact sequence

$$1 \to G_F \to H \to G_Z \to 1,$$

where G_F acts on the general fiber of $X_k \to Z$ and G_Z acts on the base Z. By an equivariant version of the canonical bundle formula (see [40, Lemma 2.32]), we obtain a G_Z-equivariant boundary B_Z such that (Z, B_Z) is Calabi–Yau and log canonical. Note that either G_F or G_Z admits a surjective homomorphism onto A_8. In either case, we get a group G surjecting onto A_8 and acting on a log Calabi–Yau pair of dimension at most 2. This contradicts Proposition 5.4 and Lemma 7.5. □

Lemma 7.7. *Let G be a finite subgroup of* $\mathrm{O}(k)$ *for* $k \leq 4$. *Then G does not admit a surjective homomorphism to A_8.*

Proof. It is enough to consider finite subgroups of $\mathrm{SO}(k)$ for $k \leq 4$. The statement is clear for $k \leq 3$. Indeed, a finite subgroup of $\mathrm{SO}(k)$ with $k \leq 3$ is cyclic, dihedral, icosahedral, tetrahedral, or octahedral. For $k = 4$, recall that we have a short exact sequence

$$1 \to \mathbb{Z}/2 \to \mathrm{SO}(4) \to \mathrm{SO}(3) \times \mathrm{SO}(3) \to 1.$$

Thus, if there is a finite subgroup of $\mathrm{O}(4)$ that surjects onto A_8, then there is a finite subgroup of $\mathrm{SO}(3)$ that surjects onto A_8. This leads to a contradiction. □

Now, we are ready to prove the boundedness of S_8-equivariant Fano 4-folds. In what follows, we show a version of Theorem 7 for log pairs. This version for log pairs will be used to prove Theorem 8.

Theorem 7.8. *Let $\mathcal{B} \subset [0,1]$ be a set satisfying the DCC and $\overline{\mathcal{B}} \subset \mathbb{Q}$. Let $\mathcal{F}_{4,8,\mathcal{B}}$ be the class of 4-dimensional S_8-equivariant klt pairs (X, B) for which $-(K_X + B)$ is ample and* $\operatorname{coeff}(B) \subset \mathcal{B}$. *Then the class $\mathcal{F}_{4,8,\mathcal{B}}$ is log bounded.*

Proof of Theorem 7.8. We will show that the class of S_8-equivariant klt log Fano 4-dimensional pairs is log bounded.

Let (X, B) be a klt S_8-equivariant log Fano 4-dimensional pair with $\operatorname{coeff}(B) \subset \mathcal{B}$, and let $\pi\colon X \to Y$ be the quotient. By Riemann–Hurwitz, we can write $\pi^*(K_Y + B_Y) = K_X + B$, where B_Y is an effective divisor. By [23, Lemma 5.2], there exists a set $\mathcal{C} \subset [0,1]$ satisfying the DCC and $\overline{\mathcal{C}} \subset \mathbb{Q}$ such that $\operatorname{coeff}(B_Y) \in \mathcal{C}$. The set $\mathcal{C}$ only depends on $\mathcal{B}$; hence, it is independent of the chosen pair (X, B). So (Y, B_Y) is a klt log Fano pair. Indeed, a pair is klt if and only if a finite pullback of it is klt; see [39, Proposition 2.11]. We proceed in three cases, depending on $\operatorname{coreg}(Y, B_Y)$.

Case 1: *In this case, we assume that the coregularity of the pair* (Y, B_Y) *is* 4. In this case, every log Calabi–Yau structure (Y, Γ_Y) with $\Gamma_Y \geq B_Y$ satisfies that (Y, Γ_Y) is klt. Hence, (Y, B_Y) is an exceptional Fano pair [5, Section 2.15]. By Birkar's boundedness of exceptional Fano pairs [5, Theorem 1.11], we conclude that (Y, B_Y) is log bounded. That is, there exist constants k_0, k_1 such that, for any (Y, B_Y) log Fano klt pair of dimension 4 and coregularity 4, there is a very ample line bundle $\mathcal{A}_Y$ on Y with $\mathcal{A}_Y^4 \leq k_0$ and $\mathcal{A}_Y^3 \cdot B_Y \leq k_1$. Choose $A_Y \in |\mathcal{A}_Y|$ with no components in the branch locus of π. Then $A_X := \pi^* A_Y$ satisfies $A_X^4 = (8!)^4 A_Y^4 \leq (8!)^4 k_0$, so X is bounded. In particular, we have $\mathcal{A}_X^3 \cdot -K_X \leq k_2$ for some constant k_2 independent of X. On the other hand, note that $\mathcal{A}_X^3 \cdot (K_X + B) \leq 0$, so

$$\mathcal{A}_X^3 \cdot B \leq \mathcal{A}_X^3 \cdot -K_X \leq k_2.$$

Since the coefficients of B are bounded below, we conclude that every component of B has degree bounded above with respect to $\mathcal{A}_X$. Thus, we conclude that the pairs (X, B) are log bounded.

Case 2: *In this case, we assume that the coregularity of the pair* (Y, B_Y) *is* 3. By [41, Lemma 2.18] and [23, Theorem 1.2], there exists a constant N such that the following holds: for any klt log Fano pair (Y, B_Y) with coregularity 3, there exists $\Gamma_Y \geq B_Y$ such that

- (Y, Γ_Y) is log canonical,
- $\mathcal{D}(Y, \Gamma_Y)$ is zero-dimensional, and
- $N(K_Y + \Gamma_Y) \sim 0$.

Let (X, Γ) be the log Calabi–Yau pair defined by

$$K_X + \Gamma = \pi^*(K_Y + \Gamma_Y).$$

Then the following hold:

- $S_8 \leq \operatorname{Aut}(X, \Gamma)$,
- $\mathcal{D}(X, \Gamma)$ is zero-dimensional (by Lemma 7.4), and
- $N(K_X + \Gamma) \sim 0$.

By [24, Theorem 1.6], $\mathcal{D}(X, \Gamma)$ is either one point or two points. First, assume $\mathcal{D}(X, \Gamma)$ is two points. Let $(X', \Gamma') \to (X, \Gamma)$ be an S_8-equivariant dlt modification and $E_0, E_1 \subset \lfloor \Gamma' \rfloor$

the two components. Note that E_0 and E_1 are each A_8-invariant. So we may run an A_8-equivariant $(K_{X'} + \Gamma' - E_0 - E_1)$-MMP, which terminates with a Mori fiber space, because $K_{X'} + \Gamma' - E_0 - E_1$ is not pseudo-effective. Let

$$\begin{array}{ccccccccc} X' & \dashrightarrow & X_1 & \dashrightarrow & X_2 & \dashrightarrow & \cdots & \dashrightarrow & X_k \\ &&&&&&&& \downarrow \\ &&&&&&&& Z \end{array}$$

be the steps of the MMP and $X_k \to Z$ the A_8-equivariant Mori fiber space. Let $E_{0,k}$ and $E_{1,k}$ be the pushforwards of E_0 and E_1, respectively, on X_k. Since $E_{0,k} + E_{1,k}$ is ample over Z, either $E_{0,k}$ or $E_{1,k}$ is horizontal over Z. Furthermore, they have trivial intersection by the assumption on $\mathcal{D}(X, \Gamma)$. Thus, both divisors $E_{0,k}$ and $E_{1,k}$ must be horizontal over the base; otherwise, they would have nontrivial intersection. Since $\rho^{A_8}(X_k/Z) = 1$ and both $E_{0,k}$ and $E_{1,k}$ are A_8-invariant, we conclude that $E_{0,k}$ and $E_{1,k}$ are each ample over Z. Then a general fiber F of $X_k \to Z$ has dimension 1. Indeed, if a general fiber F of $X_k \to Z$ has dimension at least 2, then $E_{0,k}|_F$ and $E_{1,k}|_F$ intersect nontrivially, leading to a contradiction. Hence, $\dim Z = 3$ and Z is rationally connected (as it is the image of the rationally connected variety X_k). Moreover, letting Γ_k denote the pushforward of Γ', the general fiber of $(X_k, \Gamma_k) \to Z$ is isomorphic to $(\mathbb{P}^1, \{0\} + \{\infty\})$. So we have an exact sequence

$$1 \to G_F \to A_8 \to G_Z \to 1,$$

where G_F acts on the log general fiber $(\mathbb{P}^1, \{0\} + \{\infty\})$ and G_Z acts on Z. As A_8 is simple, we have that G_Z is either trivial or A_8. The latter case does not happen by [8]. In the former case, we must have that $G_F \cong A_8$; however, $\mathrm{Aut}(\mathbb{P}^1, \{0\} + \{\infty\})$ is an extension of $\mathbb{G}_m$ and $\mathbb{Z}/2$, which does not admit an embedding of A_8. Thus, we obtain a contradiction.

Now $\mathcal{D}(X, \Gamma)$ is a single point. Let $\pi\colon (X', \Gamma') \to (X, \Gamma)$ be an S_8-equivariant dlt modification. The divisor $E := \lfloor \Gamma' \rfloor$ is fixed by S_8. We proceed in two cases, depending on whether or not $E = \Gamma'$.

If $E \neq \Gamma'$, write $\Gamma' = E + F$ with $F > 0$, and run an S_8-equivariant $(K_{X'} + \Gamma' - E)$-MMP. Call the steps

$$\begin{array}{ccccccccc} X' & \dashrightarrow & X_1 & \dashrightarrow & X_2 & \dashrightarrow & \cdots & \dashrightarrow & X_k \\ &&&&&&&& \downarrow{\scriptstyle \pi} \\ &&&&&&&& Z \end{array}$$

and denote by E_i, Γ_i, F_i the pushforwards. Here, $\pi\colon X_k \to Z$ is the equivariant Mori fiber space. If $\dim Z = 1$, then we get a contradiction by analyzing the action on the general fiber, which is a Fano 3-fold, and the base, which is a rational curve. Thus, we assume that $\dim Z \geq 2$ or $\dim Z = 0$. In this case, since E_k is ample over Z, we conclude that E_k intersects every irreducible divisor on X_k. Indeed, if the irreducible divisor is vertical over Z, then E_k intersects positively every curve contained in such divisor. On the other hand, if the irreducible divisor is horizontal over Z, then it intersects E_k on the general fiber. However, note that every divisor that is contracted by this MMP must intersect the strict transform of E positively. Indeed, every curve that is contracted on this MMP is $(-E)$-negative. Thus, if the strict transform of F on X_k is trivial, then F_j and E_j intersect for some $j < k$. On the other hand, if the strict transform of F on X_k is nontrivial, then F_k and E_k intersect. Thus, for some $1 \leq j \leq k$, we have that E_j intersects F_j. Let (E_j, Γ_{E_j}) be the pair obtained by adjunction of (X_j, Γ_j) to E_j. As $F_j \cap E_j \neq \emptyset$, we have that $\Gamma_{E_j} \neq 0$. The kernel of the action of S_8 on E_j is normal and cyclic by Lemma 3.2, hence trivial. So S_8 acts faithfully on the 3-dimensional log Calabi–Yau pair (E_j, Γ_{E_j}) with $\Gamma_{E_j} \neq 0$. This contradicts Lemma 7.6.

It remains to show the case $\Gamma' = E$. Note that $\Gamma \neq 0$. If the dlt modification π is nontrivial, then $\Gamma' = E + \pi_*^{-1}\Gamma$, which is a contradiction. So π is trivial, and we have $\Gamma = E$ and the pair (X, Γ) is dlt. In particular, (X, Γ) is plt and, by construction, $N(K_X + \Gamma) \sim 0$. If $a_F(X) < \frac{1}{N}$ for some $F \neq \Gamma$ over X, then $a_F(X, \Gamma) = 0$, which contradicts that (X, Γ) is plt. We conclude that X is $\frac{1}{N}$-lc and Fano, and hence bounded by [6, Theorem 1.1]. Then the boundedness of the pair (X, B) follows as in the first step.

Case 3: *In this case, we assume that the coregularity of the pair* (Y, B_Y) *is* ≤ 2. We will show that this case does not happen. In this case, we know there exists $\Gamma_Y > B_Y \geq 0$ with $K_Y + \Gamma_Y \sim_{\mathbb{Q}} 0$ and $3 \geq \dim \mathcal{D}(Y, \Gamma_Y) \geq 1$. Let (X, Γ) be the log pullback of (Y, Γ_Y) to X. We are in the setting of Lemma 7.4, so $\dim \mathcal{D}(X, \Gamma) \in \{1, 2, 3\}$ and $K_X + \Gamma \sim_{\mathbb{Q}} 0$.

Let $(X', \Gamma') \to (X, \Gamma)$ be an S_8-equivariant dlt modification. The profinite completion $\widehat{\pi}_1(\mathcal{D}(X', \Gamma'))$ corresponds to a quasi-étale cover $(Z', \Gamma_{Z'}) \to (X', \Gamma')$ such that $\mathcal{D}(Z', \Gamma_{Z'})$ is PL-homeomorphic to either a sphere $\mathbb{S}^k$ or a disk $\mathbb{D}^k$ with $k \leq 3$ (see [34, Theorem 2 and Paragraph 33]). Since $(Z', \Gamma_{Z'}) \to (X', \Gamma')$ is associated to the universal cover of $\mathcal{D}(X', \Gamma')$, there is a finite group G surjecting onto S_8 and acting on $\mathcal{D}(Z', \Gamma_{Z'})$. If $\mathcal{D}(Z', \Gamma_{Z'})$ is a disk, then G acts on $\partial\mathcal{D}(Z', \Gamma_{Z'}) \cong_{\mathrm{PL}} \mathbb{S}^{k-1}$ with $k \leq 3$. So, in either case, G acts on a triangulation of a sphere $\mathbb{S}^k$ with $k \leq 3$. In particular, G acts continuously on $\mathbb{S}^k$ with $k \leq 3$. By [45, Theorem 1.1], there is a smooth faithful action of G on $\mathbb{S}^k$ with $k \leq 3$. Every finite smooth action on a sphere of dimension at most 3 is conjugate to an orthogonal action (see, e.g., [64, page 1]). Hence, we have a homomorphism $G \to \mathrm{O}(k)$ with $k \leq 4$. Let H denote the kernel. By Lemma 7.7, we conclude that H surjects onto A_8. So H acts trivially on either $\mathcal{D}(Z', \Gamma_{Z'})$ or its boundary, so in particular, the H-action on $\mathcal{D}(Z', \Gamma_{Z'})$ has a fixed vertex $v \in \mathcal{D}(Z', \Gamma_{Z'})$. Let E_v be the corresponding divisor on $\lfloor \Gamma_{Z'} \rfloor$. Then E_v is fixed by every element of H. By Lemma 3.2, the subgroup of H that fixes E_v pointwise is normal and cyclic. This subgroup must have trivial image in A_8, so the quotient H' of H by this subgroup still surjects onto A_8 and acts faithfully on E_v. Let (E_v, Γ_v) be the pair obtained by adjunction of $(Z', \Gamma_{Z'})$ to E_v. Since $\dim \mathcal{D}(Z', \Gamma_{Z'}) \geq 1$, we have $\Gamma_v \neq 0$. So H' acts faithfully on a 3-dimensional log Calabi–Yau pair (E_v, Γ_v) with $\Gamma_v \neq 0$. This contradicts Lemma 7.6. □

Now, we turn to give a proof of the boundedness up to degeneration of 5-dimensional S_8-equivariant klt singularities. The global-to-local argument used in the proof of Theorem 8 is very similar to that of Theorem 2.

Proof of Theorem 8. Let $\mathcal{K}_{5,8,\epsilon}$ be the class of 5-dimensional S_8-equivariant klt singularities $(X; x)$ with $\mathrm{mld}(X; x) > \epsilon$. We show that the class $\mathcal{K}_{5,8,\epsilon}$ is bounded up to degeneration. Let $(X; x)$ be an element of $\mathcal{K}_{5,8,\epsilon}$. Let $\pi\colon (X; x) \to (Y; y)$ be the quotient of $(X; x)$ by the S_8-action. Then there is a boundary B_Y with standard coefficients for which $(Y, B_Y; y)$ is klt and $\pi^*(K_Y + B_Y) = K_X$. By [61, Lemma 1], there exists a blow-up $\varphi_Y\colon Y' \to Y$ that extracts a unique prime divisor E' that maps to $y \in Y$ and satisfies the following:

- the pair $(Y', E' + \varphi_{Y*}^{-1} B_Y)$ has plt singularities, and
- the divisor $-(K_{Y'} + E' + \varphi_{Y*}^{-1} B_Y)$ is ample over Y.

Let $X' \to X$ be the projective birational morphism obtained by fiber product. Then X' admits an S_8-action and its quotient is Y'. Let $\pi'\colon X' \to Y'$ be the corresponding quotient map. Let $K_{X'} + F = \pi'^{*}(K_{Y'} + E' + \varphi_{Y*}^{-1} B_Y)$. Since (X', F) is the finite pullback of a plt pair, we conclude that it is itself plt. By connectedness of log canonical centers, we conclude that F is prime. Thus, F is the unique prime divisor that maps to $x \in X$. Note that $-(K_{X'} + F)$ is ample over X. On the other hand, the pair (X', F) admits a faithful S_8-action. By Lemma 3.2, we conclude that F admits a faithful S_8-action. Let (F, B_F) be the log pair obtained by adjunction of (X', F) to F. By construction, the following conditions hold:

- F is 4-dimensional,
- (F, B_F) is klt,
- $-(K_F + B_F)$ is ample, and
- B_F has standard coefficients.

By Theorem 7.8, we conclude that (F, B_F) belongs to the log bounded class $\mathcal{F}_{4,8}$. Then the class $\mathcal{K}_{5,8,\epsilon}$ is bounded up to degeneration by [27, Theorem 1.1]. We give more details for the benefit of the reader: we may degenerate the singularity $(X; x)$ to the orbifold cone of F with respect to the $\mathbb{Q}$-polarization $-F|_F$. The degree of this $\mathbb{Q}$-polarization is bounded above if the mld of $(X; x)$ is bounded below. The central fiber of this degeneration is a cone singularity that belongs to a bounded family by [38, Theorem 1]. □

We finish this section by proving a *birational* boundedness statement for maximally symmetric Fano varieties. The following theorem states that maximally symmetric Fano varieties are birationally bounded, provided some hypothesis that is supported by Theorem 5.1. Observe that birational boundedness is much weaker than boundedness. For example, there are countably many toric Fano varieties of dimension n for $n \geq 2$; however, the class of toric Fano varieties of dimension n is birationally bounded, as each of these varieties is birational to $\mathbb{P}^n$.

Theorem 7.9. *Let $m(n)$ be the maximum integer for which $S_{m(n)}$ acts faithfully on an n-dimensional Fano variety. Let $\ell(d)$ be the maximum integer for which $A_{\ell(d)}$ acts faithfully on a d-dimensional Fano variety. Assume that $m(n) > \ell(d)$ for every $d \leq n-1$. Then the class of maximally symmetric n-dimensional Fano varieties is birationally bounded.*

Proof. Let X be a maximally symmetric n-dimensional Fano variety. Let S_m be the symmetric group acting on X. Let $X' \to X$ be an equivariant resolution of singularities. The Fano variety X is rationally connected, so X' is rationally connected. We run an S_m-equivariant minimal model program $X' \dashrightarrow X'_1 \dashrightarrow X'_2 \dashrightarrow \cdots \dashrightarrow X'_k$ for $K_{X'}$. Since X' is rationally connected and smooth, then $K_{X'}$ is not pseudo-effective, so we have an equivariant Mori fiber space $X'_k \to Z$. To show the result, it suffices to show that $\dim Z = 0$, since then X'_k is a terminal n-dimensional Fano variety of Picard rank one, so it belongs to a bounded family by [6, Theorem 1.1].

To show that $\dim Z = 0$, assume by contradiction that $\dim Z \geq 1$. By the assumption $m(n) > \ell(d)$, we conclude that A_m does not act on the general fiber of $X'_k \to Z$, so it must act on Z. Note that Z is rationally connected, being the image of a rationally connected variety. We take an A_m-equivariant resolution of singularities $Z' \to Z$. The variety Z' is rationally connected and smooth, so $K_{Z'}$ is not pseudo-effective. We run an A_m-equivariant MMP for $K_{Z'}$.

Proceeding inductively, we obtain a d-dimensional Fano variety that admits an A_m-action. This contradicts the fact that $m(n) > \ell(d)$ for $d \leq n-1$. □

8. Examples and questions

In this section, we consider several examples and questions related to the results of the article.

Example 8.1. Given a dimension n, let

$$m := c_{\mathrm{Fano}}(n) - n - 1 = \left\lceil \frac{1+\sqrt{8n+9}}{2} \right\rceil - 1.$$

Let X be the following complete intersection in $\mathbb{P}^{n+m}$:

$$X := \left\{ \sum_{i=0}^{n+m} x_i = \sum_{i=0}^{n+m} x_i^2 = \cdots = \sum_{i=0}^{n+m} x_i^m = 0 \right\} \subset \mathbb{P}^{n+m}.$$

Then X is a smooth Fano complete intersection of dimension n by Lemma 2.11. The symmetric group S_{n+m+1} acts on X by permutation of the variables and it is clear that this action is faithful.

This example is a maximally symmetric Fano weighted complete intersection for each dimension n by Theorem 5.1 and has the largest possible index among such maximal examples by Theorem 6.2. We expect this to be a maximally symmetric Fano variety in every dimension.

Remark 8.2. Example 8.1 gives examples in any dimension n showing that the bound for Fanos in Theorem 5.1 is sharp. One may ask whether the same can be done for Calabi–Yau complete intersections for $n \geq 3$ (recall that, by Proposition 5.4, the bound $c_{\mathrm{CY}}(n)$ is sharp for $n = 2$ but not for $n = 1$). One obstacle to finding Calabi–Yau examples is that Lemma 2.11 no longer holds if the degrees are not $(1, 2, \dots, m)$.

For $n = 3$, the $(1, 2, 4)$-Fermat complete intersection in $\mathbb{P}^6$ is smooth and therefore is a maximally symmetric Calabi–Yau weighted complete intersection. However, for $n = 4$, the degree $(1, 2, 5)$-Fermat complete intersection in $\mathbb{P}^7$ is singular (and it even has non-isolated singularities). For $n = 4$, it turns out that the degree $(1, 3, 4)$-Fermat complete intersection is smooth and thus exhibits a smooth maximally symmetric example, i.e., it achieves $c_{\mathrm{CY}}(4) = 8$. In general, the numerics to ensure smoothness seem complicated.

Nevertheless, the upper bound $c_{\mathrm{CY}}(n)$ is achieved for infinitely many values of n, namely when there happens to exist an m such that the complete intersection with degrees $(1, 2, \dots, m)$ in $\mathbb{P}^{n+m}$ is Calabi–Yau.

Question 8.3. Is the bound in Theorem 5.1 for quasismooth Calabi–Yau weighted complete intersections sharp for all $n \geq 2$?

Example 8.4. In any dimension n, there exist maximally symmetric Fano–Fermat complete intersections of index 1. Indeed, this happens if X has degrees $(d_1, \dots, d_m)$ with

$$m = c_{\mathrm{Fano}}(n) - n - 1 \quad \text{and} \quad n + m + 1 - \sum_{i=1}^{m} d_i = 1.$$

(For a concrete example, take $d_i = i$ for all $1 \le i \le m-1$ and $d_m = n + m - \frac{(m-1)m}{2}$.) However, as with Remark 8.2, the numerics to ensure smoothness seem complicated. For example, for $n = 5$, the degree $(1,3,4)$-Fermat–Fano in $\mathbb{P}^8$ is smooth, but the degree $(1,2,5)$-Fermat–Fano is singular.

If X is smooth and if $n \ge 210$, then X is birationally superrigid and in particular irrational by [63, Theorem 1.2]. Moreover, for $n \ge 4$, any smooth such X is conjecturally birationally rigid and hence irrational [51, Conjecture 5.1].

For $n = 3$, the X in Example 8.4 is known as the symmetric sextic Fano threefold; it is a smooth Fano threefold with an intermediate Jacobian obstruction to rationality [3]. (Moreover, any embedding of S_7 into the birational automorphism group of a rationally connected threefold is conjugate to this action [47, Proposition 1.1 (ii)].) For general n, however, it is not clear how to guarantee smoothness in Example 8.4.

Question 8.5. Do there exist index 1 Fano–Fermat complete intersections as in Example 8.4 that are smooth for all n? In particular, do there exist irrational examples of maximally symmetric Fano–Fermat varieties for $n \gg 0$?

For rational varieties, S_{n+1} always acts on $\mathbb{P}^n$ by permutation of coordinates, so we have an embedding $S_{n+1} \le \mathbb{P}\mathrm{GL}_{n+1}(\mathbb{C}) \le \mathrm{Cr}(n)$. In fact, one can get $S_{n+2} \le \mathbb{P}\mathrm{GL}_{n+1}(\mathbb{C})$.

Example 8.6. For $n \ge 1$, the projective representation $S_{n+2} \to \mathbb{P}\mathrm{GL}_{n+1}(\mathbb{C})$ of degree $n+1$ defines a faithful action of S_{n+2} on $\mathbb{P}^n$. Theorem 4.1 shows that this is the best one can do among toric varieties (apart from the $n = 2$ case).

There are also easy examples of rational n-dimensional Fanos with S_{n+3}-actions.

Example 8.7. Let $n \ge 1$ and define

$$X := \left\{ \sum_{i=0}^{n+2} x_i = \sum_{i=0}^{n+2} x_i^2 = 0 \right\} \subset \mathbb{P}^{n+2}.$$

Here, X is smooth by Lemma 2.11 and it is isomorphic to a quadric; thus, it is a smooth rational n-dimensional Fano with a faithful S_{n+3}-action.

Not all maximally symmetric Fano weighted complete intersections are complete intersections in projective space, that is, nontrivial finite covers can arise in Theorem 5.

Example 8.8. Let $X \subset \mathbb{P}^9(1^{(9)}, 2)$ be the following smooth weighted complete intersection, where the variables of the weighted projective space are $x_0, \ldots, x_8, y$:

$$X := \left\{ \sum_{i=0}^{8} x_i = \sum_{i=0}^{8} x_i^2 = \sum_{i=0}^{8} x_i^3 = y^2 - \sum_{i=0}^{8} x_i^4 = 0 \right\}.$$

Then X is a Fano fivefold since $K_X \cong \mathcal{O}_X(-1)$ and it carries a faithful S_9-action by permutation of the x_i. By Theorem 5.1, X is a maximally symmetric weighted complete intersection of dimension 5. It is a double cover of the Fermat $(1,2,3)$-complete intersection in $\mathbb{P}^8$, which is the highest index maximally symmetric example by Theorem 6.2.

In Theorem 7, we showed that the class of S_8-equivariant klt Fano 4-folds is bounded. In fact, we are only aware of the following members in this class.

Example 8.9 (Examples of Fano 4-folds with S_8-actions). Define the following Fano–Fermat complete intersections in $\mathbb{P}^7$:

(1) X_{123} of degrees $(1,2,3)$, and

(2) X_{124} of degrees $(1,2,4)$.

Then X_{123} and X_{124} are smooth S_8-equivariant Fano 4-folds with $\rho = 1$.

In contrast, the class of S_7-equivariant klt Fano 4-folds is *unbounded*.

Example 8.10. Let X be the symmetric sextic Fano 3-fold (Example 8.4 for $n = 3$). Then X is a smooth Fano variety of Picard rank one and admits faithful S_7-action. The divisor class $-K_X$ is invariant under the action of S_7, and the divisor $D \in |-4K_X|$ defined by $\sum_{i=0}^{6} x_i^4$ is invariant under the S_7-action. For $m \gg 0$, define

$$Y_m := \mathbb{P}(\mathcal{O}_X \oplus \mathcal{O}_X(-mD)) \xrightarrow{\pi_m} X.$$

Note that Y_m is endowed with an S_7-action. Let $\mathcal{O}_{Y_m}(1)$ be the associated relative ample bundle. Let E_m be the section corresponding to the exact sequence

$$1 \to \mathcal{O}_X \to \mathcal{O}_X \oplus \mathcal{O}_X(-mD) \to \mathcal{O}_X(-mD) \to 1$$

and let F_m be the section corresponding the exact sequence

$$1 \to \mathcal{O}_X(-mD) \to \mathcal{O}_X \oplus \mathcal{O}_X(-mD) \to \mathcal{O}_X \to 1.$$

The cokernel of $\pi_m^*(\mathcal{O}_X) \to \mathcal{O}_{Y_m}(1)$ is $\mathcal{O}_{E_m}(1)$, so the normal bundle of E_m is

$$\mathcal{O}_{E_m}^{\vee} \otimes \mathcal{O}_{E_m}(-m\pi_m^* D) \simeq \mathcal{O}_{E_m}(4mK_{E_m}).$$

Analogously, the normal bundle of F_m is $\mathcal{O}_{F_m}(-4mK_{F_m})$. Note that

$$K_{Y_m} + E_m + F_m \sim \pi_m^*(K_X) \tag{8.1}$$

and the divisor $\pi_m^*(-K_X)$ is nef. We claim that (Y_m, E_m) is log Fano. Indeed, if C is not contained in F_m, then $(K_{Y_m} + E_m) \cdot C < 0$ by the linear equivalence (8.1). On the other hand, if C is contained in F_m, then $-(K_{Y_m} + E_m) \cdot C = F_m \cdot C + \pi_m^*(-K_X) \cdot C > 0$ by the normal bundle computation. Therefore, for $\epsilon > 0$ small enough, the pair $(Y_m, (1-\epsilon)E_m)$ is a klt Fano pair. We conclude that Y_m is a Mori dream space by [7, Corollary 1.3.2]. Thus, we may run an S_7-equivariant MMP for any S_7-invariant divisor on Y_m. By the normal bundle computation, E_m is covered by E_m-negative curves. Since E_m has Picard rank 1, the S_7-equivariant E_m-MMP has a single step and contracts E_m to a point. Let $\varphi_m\colon Y_m \to X_m$ be the S_7-equivariant contraction of E_m to a point. We obtain a variety X_m of Picard rank one. Note that X_m is endowed with the action of S_7. Next, we compute the mld of X_m. To do this, let $C \subset E_m$ be a curve. Then the following sequence of equalities hold:

$$\Big(K_{Y_m} + E_m - \frac{1}{4m} E_m\Big) \cdot C = K_{E_m} \cdot C - \frac{1}{4m} E_m|_{E_m} \cdot C$$
$$= K_{E_m} \cdot C - \frac{1}{4m} 4m K_{E_m} \cdot C = 0.$$

By the contraction theorem, we have that

$$\varphi_m^*(K_{X_m}) = K_{Y_m} + \Big(1 - \frac{1}{4m}\Big) E_m.$$

We conclude that X_m is an S_7-equivariant klt Fano variety with $\operatorname{mld}(X_m) = \frac{1}{4m}$. Indeed, the pair (Y_m, E_m) is a log resolution of X_m. The minimal log discrepancies of a bounded set of projective varieties can only take finitely many values. We conclude that the varieties X_m form a sequence of unbounded S_7-equivariant klt Fano 4-folds.

In any dimension n, the construction in Example 8.10 shows that, given a smooth S_k-equivariant $(n-1)$-dimensional Fano variety of Picard rank one, one can construct an unbounded family of n-dimensional S_k-equivariant klt Fano varieties.

In the proof of Theorem 7, to prove that S_8-equivariant Fano 4-folds form a bounded family, we use the following facts:

(1) the group S_8 acts neither on Fano varieties of dimension at most 3 nor Calabi–Yau varieties of dimension at most 2 (see, e.g., [8]),

(2) the group S_8 does not act smoothly on spheres of dimension at most 3 (see [64]),

(3) the dual complex $\mathcal{D}(X, \Gamma)$ of a log Calabi–Yau pair of dimension at most 4 is a quotient of a sphere of dimension at most 3 (see [34, Proposition 5]), and

(4) the boundedness of Fano 4-folds with log discrepancies bounded away from zero (see [6, Theorem 1.1]).

Statement (3) is expected to hold in any dimension (see [34, Question 4]). On the other hand, if $m(n)$ is the largest integer for which S_m acts faithfully on an n-dimensional Fano variety, then we expect that S_m does not act on an ℓ-dimensional Fano variety with $\ell \leq n-1$. Similarly, we expect that S_m does not act on an ℓ-dimensional Calabi–Yau variety with $\ell \leq n-1$. Thus, we expect (1) and (2) to have analogous statements in higher dimensions. Finally, (4) is known to hold in any dimension. This leads us to the following question.

Question 8.11. For $n \geq 4$, is the family of maximally symmetric n-dimensional Fano varieties bounded?

Although we do not know whether Fano 4-folds with S_8-actions are maximally symmetric, Theorem 7 implies that Fano 4-folds endowed with an action of S_k, with $k \geq 8$, are bounded. Hence, the previous question has a positive answer in dimension 4. We do not have enough evidence for a positive answer of Question 8.11 in higher dimensions. A better understanding of symmetric actions on Calabi–Yau varieties is needed to tackle this question. The following question is very related to the boundedness one.

Question 8.12. Are maximally symmetric n-dimensional Fano varieties equivariantly exceptional? That is, are the quotients exceptional Fano varieties?

This holds in all the examples that we consider (Example 8.9). However, our tools do not allow to prove this statement in dimension 4. It would be interesting to find, if possible, singular examples of S_8-equivariant Fano 4-folds. We do not know the existence of these.

Question 8.13. Are there examples of singular maximally symmetric Fano varieties? What about singular S_8-equivariant Fano 4-folds?

In a similar vein, all the examples of maximally symmetric Fano varieties that we know are isolated. This motivates the following question.

Question 8.14. Do maximally symmetric Fano varieties have nontrivial moduli?

For $n \geq 4$, the largest symmetric group that can embed into the Cremona group

$$\mathrm{Cr}(n) = \mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$$

of rank n is not known. Proposition 3.1 gives a quadratic bound, which we do not expect to be sharp. Example 8.7 shows that S_{n+3} always embeds into $\mathrm{Cr}(n)$. We also note that, for $n \geq 3$, the integer $c_{\mathrm{Fano}}(n)$ defined in Theorem 5.1 (1) is always strictly greater than $n + 3$.

Question 8.15. For $n \geq 4$, is S_{n+3} the largest symmetric group that admits an embedding into $\mathrm{Cr}(n)$? In particular, are all maximally symmetric Fano varieties irrational for $n \geq 3$?

Finally, we expect that Theorem 2 can be improved by replacing n^2 with n. However, this problem seems very challenging. For instance, one would need to prove the analogous projective statement for Fano varieties. However, we expect that weighted complete intersection singularities should be easier to deal with. We propose the following question.

Question 8.16. Let $S_{m(n)}$ be the largest symmetric group acting on an n-dimensional weighted complete intersection klt singularity. Do we have that $\lim_{n\to\infty} m(n)/n = 1$?

We expect that ideas similar to those of the proof of Theorem 5 lead to a positive answer for Question 8.16. However, working in the local setting introduces extra difficulties, such as not having a well-defined degree of the equations that cut out the singularity.

Acknowledgement. The authors would like to thank Caucher Birkar, Serge Cantat, Igor Dolgachev, Mattia Mecchia, Yusuke Nakamura, Yuri Prokhorov, Zinovy Reichstein, and Burt Totaro for many useful comments. The authors also thank the anonymous referee for their careful reading and helpful comments. Part of this work was carried out during a visit of L. Ji to the University of California, Los Angeles and a visit of J. Moraga to the University of Michigan. The authors would like to thank these institutions, as well as Burt Totaro and Mircea Mustaţă, for their hospitality and nice working environment.

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Eingegangen 29. September 2023, in revidierter Fassung 5. September 2024