

PARTIAL DATA INVERSE PROBLEMS FOR NONLINEAR MAGNETIC SCHRÖDINGER EQUATIONS

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ABSTRACT. We prove that the knowledge of the Dirichlet-to-Neumann map, measured on a part of the boundary of a bounded domain in \mathbb{R}^n , $n \geq 2$, can uniquely determine, in a nonlinear magnetic Schrödinger equation, the vector-valued magnetic potential and the scalar electric potential, both being nonlinear in the solution.

1. Introduction

We investigate an inverse boundary value problem for the nonlinear magnetic Schrödinger equations. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be an open connected bounded domain with smooth boundary $\partial\Omega$, we consider the boundary value problem

$$(1.1) \quad (D + A(x, u))^2 u + q(x, u) = 0 \quad \text{in } \Omega,$$

with the boundary condition $u = f$ on $\partial\Omega$. Here the vector-valued function $A = (A_1, \dots, A_n)$ is the nonlinear magnetic potential, modeling the effect of an external magnetic field, the scalar function q represents the nonlinear electric potential and D denotes $-i\nabla_x$. The Dirichlet-to-Neumann (DN) map for the equation is defined by

$$(1.2) \quad \Lambda_{A,q} : W^{2-1/p,p}(\partial\Omega) \rightarrow W^{1-1/p,p}(\partial\Omega), \quad f \mapsto \nu \cdot (\nabla u + iA(x, u)u) |_{\partial\Omega},$$

where ν is the unit outer normal to $\partial\Omega$.

The type of inverse boundary value problem was first formulated by Calderón [2] for the linear conductivity equation $\nabla \cdot \gamma(x)\nabla u = 0$ when he sought to determine the electrical conductivity $\gamma(x)$ of a medium by making boundary measurements of electric voltage and current. The unique determination was proved in [40] in dimension $n \geq 3$ by solving the problem of determining an electric potential $q(x)$ in a Schrödinger operator $-\Delta + q$ from the boundary Dirichlet and Neumann data. Since then, the inverse problem has been extensively studied in various generalized cases. The inverse boundary value problem for the linear magnetic Schrödinger equation, where $A(x, z) = A(x)$ and $q(x, z) = q(x)$, has been considered in [3, 6, 8, 11, 13, 14, 16, 23, 27, 36, 37] and the references therein. Specifically, due to a gauge invariance, one can only expect to recover uniquely the magnetic field $\text{curl } A$ and q from the boundary DN-map.

In dealing with the inverse problems for nonlinear PDEs, a standard approach based on the first order linearization of the DN-map was introduced to identify the linear reaction from the medium, then the full nonlinear medium for certain cases. See for instance [15, 17, 18, 20, 21, 39] for the demonstration of the approach in solving the inverse problems for certain semilinear, quasilinear elliptic equations and parabolic

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equations. Recently the higher order linearization of the DN-map has been applied in determining the full nonlinearity of the medium for several different equations. The method was successfully applied to solve inverse problems for nonlinear hyperbolic equations on the spacetime [26], where in contrast the underlying problems for linear hyperbolic equations are still open, see also [5, 33] and the references therein. In particular, the second order linearization of the nonlinear boundary map was studied in [4, 22, 38, 39] for nonlinear elliptic equations. Moreover, this higher order linearization technique was also applied to study elliptic equations with power-type nonlinearities, see [10, 24, 25, 28, 31, 30, 34, 35]. A demonstration of the method can be found in [1] on nonlinear Maxwell's equations, in [29] on nonlinear kinetic equations, and in [32] on semilinear wave equations.

Given a semilinear elliptic PDE whose leading term is the Laplacian operator, we apply the higher order linearization of the DN-map with respect to the small perturbation around the zero solution. The knowledge of the DN-map, measured partially or completely on the boundary, determines an integral of the product of the m -th order term of the nonlinear parameter and $m + 1$ harmonic functions or their derivatives. A density argument of the products of harmonic functions or their derivatives is crucial in proving the uniqueness of the m -th order term. For the inverse problem of the linear equation with DN-map measured only on part of the boundary, the density of the product of harmonic functions, which vanish on a closed proper subset of the boundary, was first shown in [7]. More specifically, in [7], this density argument relies on a Runge type approximation result and an idea of propagating exponential decay estimates for FBI transforms by the use of maximum principle as in the Kashiwara's watermelon theorem. In [25, 30], this density argument was directly used, along with unique continuation and the maximum principle, to show unique determination of a potential function $q(x)$ in a model equation $-\Delta u + q(x)u^2 = 0$ or in a more general equation of the form $-\Delta u + V(x, u) = 0$, assuming partial data. The argument in [7] was then generalized in [24] where the authors of [24] proved the density of the products of the gradients of two harmonic functions, which vanish on part of the boundary, and then use it to show the unique determination of a nonlinear potential $q(x)$ in the equation $-\Delta u + q(x)(\nabla u \cdot \nabla u) = 0$.

1.1. Problem setup and strategy. In this paper, the main objective is to determine the nonlinear vector potential $A(x, z)$ and the scalar potential $q(x, z)$ in (1.1) from the boundary DN-map (1.2). We briefly state our strategy using the higher order linearization technique as follows.

Suppose that two sets of potentials (A_1, q_1) and (A_2, q_2) satisfy

$$(1.3) \quad \text{the map } \mathbb{C} \ni z \mapsto A_j(\cdot, z) \text{ is holomorphic with values in } W^{1,\infty}(\Omega; \mathbb{C}^n),$$

$$(1.4) \quad \text{the map } \mathbb{C} \ni z \mapsto q_j(\cdot, z) \text{ is holomorphic with values in } L^\infty(\Omega; \mathbb{C}),$$

and

$$(1.5) \quad A_j(x, 0) = 0, \quad q_j(x, 0) = \partial_z q_j(x, 0) = 0$$

for $j = 1, 2$. We have that the potentials admit the following expansions:

$$A_j(x, z) = \sum_{k=0}^{\infty} \partial_z^k A(x, 0) \frac{z^k}{k!}, \quad q_j(x, z) = \sum_{k=0}^{\infty} \partial_z^k q_j(x, 0) \frac{z^k}{k!}.$$

Fixing a positive integer $m \geq 1$, let ε_k be small positive numbers and $f_k \in W^{2-1/p, p}(\partial\Omega)$ for $k = 1, \dots, m$. We denote $\varepsilon := (\varepsilon_1, \dots, \varepsilon_m)$ and

$$u_j := u_j(x; \varepsilon), \quad j = 1, 2$$

to be the unique small solution of the Dirichlet problem

$$(1.6) \quad \begin{cases} (D + A_j(x, u_j))^2 u_j + q_j(x, u_j) = 0 & \text{in } \Omega, \\ u_j = \varepsilon_1 f_1 + \dots + \varepsilon_m f_m & \text{on } \partial\Omega. \end{cases}$$

We establish the well-posedness for this Dirichlet problem with small data in the Appendix and the DN-map is thus well-defined. Moreover, by Theorem 2 we know that the finite difference u_j/ε_k is bounded in $W^{2,p}(\Omega)$ (the bound is independent of ε), hence u_j is differentiable in ε_k and the derivatives satisfy the linearized Laplace equation. (See for example, [19, 38] for a more detailed exposition.) By expanding u_j in the small perturbation parameter ε_k and noting that $u_j(x; 0) \equiv 0$ due to the well-posedness, we have that the first order term

$$v_{j,k} := \partial_{\varepsilon_k} u_j|_{\varepsilon=0}$$

is indeed a harmonic function in Ω satisfying $v_{j,k}|_{\partial\Omega} = f_k$ for $k = 1, \dots, m$, $j = 1, 2$.

- Remark 1.** (1) We point out that in this setup, we have harmonic functions $v_{1,k} = v_{2,k}$ for $k = 1, \dots, m$ in Ω since they agree on the whole boundary.
- (2) In other cases, the domain Ω might have unknown geometrical features. For example, if there is an unknown inclusion or obstacle embedded in Ω , or if the part of the boundary where we cannot measure the DN-map has an unknown geometry, then we would have u_j to be the solutions to the magnetic Schrödinger equation in Ω_j associated with A_j and q_j for $j = 1, 2$. This implies that $v_{1,k}$ and $v_{2,k}$ are harmonic functions in potentially different domains Ω_j . These scenarios are discussed in [30] for elliptic equations, where one can show under certain assumptions, using unique continuation, that the domains in above examples are indeed identical.
- (3) In this paper, we focus on the case where the domain is known to be Ω . For the partial data inverse problems, we assume that the subsets of the boundary: Γ_1 and Γ_2 , be where Dirichlet data and Neumann data are measured respectively. By the definition of the partial DN-map in (1.11), we have the harmonic function $v_{j,k} = \partial_{\varepsilon_k} u_j|_{\varepsilon=0}$, $j = 1, 2$ satisfying the boundary condition

$$v_{1,k}|_{\partial\Omega} = v_{2,k}|_{\partial\Omega} = f_k \quad \text{with } \text{supp}(f_k) \subset \Gamma_1.$$

Therefore, in the partial data setting, we still have $v_{1,k} = v_{2,k}$ in Ω , hence we simply denote

$$v_k := v_{j,k} \quad \text{for } k = 1, \dots, m$$

from this point on.

To reconstruct $A(x, z)$ and $q(x, z)$, it is sufficient to consider the unique determination of $\partial_z A(x, 0), \partial_z^2 A(x, 0), \dots$ and $\partial_z^2 q(x, 0), \partial_z^3 q(x, 0), \dots$ in Ω due to (1.5). The proof is based on induction steps and is sketched as follows. We start with the second order linearization. Let $m = 2$ and denote

$$w_j := \partial_{\varepsilon_1} \partial_{\varepsilon_2} u_j|_{\varepsilon=0}.$$

Then w_j is the solution to the problem

$$(1.7) \quad -\Delta w_j + Q_j^{(2)}(v_1, v_2) = 0 \quad \text{in } \Omega, \quad w_j|_{\partial\Omega} = 0,$$

where $v_k = v_{j,k}$ for $k = 1, 2$, as discussed in Remark 1, and

$$(1.8) \quad Q_j^{(2)}(v_1, v_2) := 3\partial_z A_j(x, 0) \cdot (v_1 Dv_2 + v_2 Dv_1) + 2D_x \cdot \partial_z A_j(x, 0)v_1 v_2 + \partial_z^2 q_j(x, 0)v_1 v_2,$$

defined in (2.4). Assume that the DN-maps associated to (A_1, q_1) and (A_2, q_2) are identical. We will obtain the integral identity

$$\int_{\Omega} \left(Q_1^{(2)}(v_1, v_2) - Q_2^{(2)}(v_1, v_2) \right) v_3 \, dx = 0,$$

where v_3 is a third harmonic function in Ω with certain boundary condition. One can see that the integral involves several complicated terms of products of harmonic functions and their gradients, as well as mixtures of the vector and scalar potentials, unlike the cases studied in [24, 30].

In the spirit of [7, 24], one can potentially use the corrected harmonic exponentials

$$v(x, \zeta) = e^{\frac{-ix \cdot \zeta}{h}} + w(x, \zeta), \quad \zeta \in \mathbb{C}^n, \quad \zeta \cdot \zeta = 0$$

that vanishes on a closed proper subset of the boundary, the idea of propagating exponential decay estimates for FBI transforms and a proper version of Runge-type approximation, to prove an improved density result. However, the exponential decay propagation is difficult to derive for the associated FBI type transform of the vector-valued potential (multiplied by the complex phase). Another major difficulty comes from the entanglement of A and q in the mixture of terms.

Instead, we combine the previously established density result in [7] and the corrected harmonic exponentials together to obtain the local uniqueness of the potentials. Then we conduct the local-to-global step, as in the previous work, using the H^1 Runge-type approximation. Our key step relies on a transport equation for the harmonic functions, which helps decouple the potentials.

The argument can be easily generalized to the case $m > 2$ by induction.

1.2. Main result. Let us present our main result where we show that partial data on the boundary is sufficient to uniquely determine the nonlinear potentials in the magnetic Schrödinger equation. Meanwhile, for the completeness of the paper, we also provide a separate proof for the situation with full data in the Appendix.

Let $\Omega \subset \mathbb{R}^n$ be an open connected bounded domain with smooth boundary $\partial\Omega$. Let u be the solution to the boundary value problem for the magnetic Schrödinger equation with nonlinearity:

$$(1.9) \quad \begin{cases} (D + A(x, u))^2 u + q(x, u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases}$$

where $A(x, z) \in W^{1,\infty}(\Omega \times \mathbb{C}; \mathbb{C}^n)$ and $q(x, z) \in L^\infty(\Omega \times \mathbb{C}; \mathbb{C})$ are both C^∞ in z , and $D := -i\nabla$. Assume that A and q satisfy (1.3)-(1.5). We will show that the Dirichlet problem (1.9) has a unique solution $u \in W^{2,p}(\Omega)$ for sufficiently small boundary condition $f \in W^{2-1/p,p}(\partial\Omega)$ where $p > n$. It is clear that the equation (1.9) with $f = 0$ admits the zero solution $u = 0$. Then the full boundary DN-map for such small functions is defined by

$$(1.10) \quad \Lambda_{A,q} : W^{2-1/p,p}(\partial\Omega) \rightarrow W^{1-1/p,p}(\partial\Omega), \quad f \mapsto \nu \cdot (\nabla u + iA(x, u)u) |_{\partial\Omega},$$

where ν is the unit outer normal to $\partial\Omega$. We also define the partial boundary DN-map as follows. Let $\Gamma_1, \Gamma_2 \subset \partial\Omega$ be two arbitrary, nonempty open subsets. Then the partial boundary DN-map is defined by

$$(1.11) \quad \Lambda_{A,q}^{\Gamma_1, \Gamma_2}(f) = \Lambda_{A,q}(f)|_{\Gamma_2} \quad \text{for all } f \in W^{2-1/p,p}(\partial\Omega) \text{ with } \text{supp}(f) \subset \Gamma_1.$$

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be an open connected bounded domain with C^∞ boundary $\partial\Omega$ and let $\Gamma_1, \Gamma_2 \subset \partial\Omega$ be arbitrary nonempty open subsets of $\partial\Omega$. Suppose that two sets of coefficients (A_1, q_1) and (A_2, q_2) satisfy (1.3)-(1.5) and*

$$\nu \cdot \partial_z^k A_1(x, 0) = \nu \cdot \partial_z^k A_2(x, 0) \quad \text{on } \Gamma_1 \cap \Gamma_2 \quad \text{for } k \geq 1.$$

Let $\Lambda_{A_j, q_j}^{\Gamma_1, \Gamma_2}$ be the above partial boundary DN-map associated to (A_j, q_j) for $j = 1, 2$. Suppose that $\Lambda_{A_1, q_1}^{\Gamma_1, \Gamma_2}(f) = \Lambda_{A_2, q_2}^{\Gamma_1, \Gamma_2}(f)$ for any $f \in W^{2-1/p,p}(\partial\Omega)$, $n < p < \infty$ with $\text{supp}(f) \subset \Gamma_1$ and $\|f\|_{W^{2-1/p,p}(\partial\Omega)} < \delta$, where $\delta > 0$ is a sufficiently small constant. Then

$$A_1 = A_2 \quad \text{and} \quad q_1 = q_2 \quad \text{in } \Omega.$$

Remark 2. *When $\Gamma_1 = \Gamma_2 = \partial\Omega$, this is the uniqueness result for the inverse problem with full boundary data. In particular, it can be shown by a separate and direct method as seen in the Appendix.*

We comment here due to the assumption (1.5), the first order linearization of the DN-map provides boundary measurements of the harmonic functions in Ω . As commented in Remark 1, we could adopt the argument in [30] to show the unique determination of obstacles embedded in Ω or the unknown geometry of the inaccessible part of the boundary.

Another important observation is that our result shows that there is no gauge invariance for this problem.

The paper is organized as follows. The higher order linearization technique is detailed in Section 2 and the crucial integral identity is also derived there. Then the proof of Theorem 1 is given in Section 3. The well-posedness for the boundary value problem of the nonlinear magnetic Schrödinger equation is established in Appendix A. Finally, an alternative proof of the uniqueness of potentials with full boundary measurements is provided in Appendix B.

2. The higher order linearization

In this section, we use the higher order linearization approach to derive a key integral identity encoding the information of the discrepancy of the potentials A and q , as stated in Proposition 2. We start by considering the $m = 2$ case and then extend it to the higher order terms by induction steps.

For $m \geq 2$, let $\varepsilon := (\varepsilon_1, \dots, \varepsilon_m)$ with $\varepsilon_k > 0$ and let $f_k \in W^{2-1/p,p}(\partial\Omega)$ with $\text{supp}(f_k) \subset \Gamma_1$, $k = 1, \dots, m$. Under the assumptions of Theorem 1, the boundary value problem

$$(2.1) \quad \begin{cases} (D + A_j(x, u))^2 u_j + q_j(x, u_j) = 0 & \text{in } \Omega, \\ u_j = \varepsilon_1 f_1 + \dots + \varepsilon_m f_m & \text{on } \partial\Omega, \end{cases}$$

admits a unique solution $u_j = u_j(x; \varepsilon)$ for $|\varepsilon|$ small enough.

2.1. For $m = 2$ case. We recall the condition (1.5). Given the boundary condition $f = \varepsilon_1 f_1 + \varepsilon_2 f_2$ with $\text{supp}(f_k) \subset \Gamma_1$ for small enough $|\varepsilon|$, following the steps described in the introduction, the first order linearization of (2.1) around the zero solution $u_j(x; 0) = 0$ gives that $v_{j,k} := \partial_{\varepsilon_k} u_j(x; \varepsilon)|_{\varepsilon=0}$, $k = 1, 2$, is harmonic function satisfying

$$(2.2) \quad -\Delta v_{j,k} = 0 \quad \text{in } \Omega, \quad v_{j,k}|_{\partial\Omega} = f_k.$$

This indeed implies that

$$v_k := v_{1,k} = v_{2,k} \quad \text{in } \Omega.$$

Next we perform the second order linearization, then it gives that the function

$$w_j := \partial_{\varepsilon_1} \partial_{\varepsilon_2} u_j(x; \varepsilon)|_{\varepsilon=0}$$

is the solution to

$$(2.3) \quad -\Delta w_j + Q^{(2)}(v_1, v_2) = 0 \quad \text{in } \Omega, \quad w_j|_{\partial\Omega} = 0,$$

where

$$(2.4) \quad Q^{(2)}(v_1, v_2) := 3\partial_z A_j(x, 0) \cdot (v_1 Dv_2 + v_2 Dv_1) + 2D_x \cdot \partial_z A_j(x, 0)v_1 v_2 + \partial_z^2 q_j(x, 0)v_1 v_2,$$

with the partial D_x meaning the derivative with respect to the first variable of $A_j(x, u)$. Then the $O(\varepsilon_1 \varepsilon_2)$ term in the expansion of the DN-map is

$$(2.5) \quad \partial_{\varepsilon_1} \partial_{\varepsilon_2} |_{\varepsilon=0} [\Lambda_{A_j, q_j}^{\Gamma_1, \Gamma_2}(\varepsilon_1 f_1 + \varepsilon_2 f_2)] = (\partial_\nu w_j + 2i\nu \cdot \partial_z A_j(x, 0)f_1 f_2)|_{\Gamma_2}.$$

Remark 3. Since $v_k \in W^{2,p}(\Omega)$, Sobolev embedding theorem implies that $v_k \in C^{1,\alpha}(\bar{\Omega})$, $0 < \alpha < 1$. Note that $C^{1,\alpha}(\bar{\Omega})$ is an algebra under pointwise multiplication. Therefore, $f_1 f_2 = v_1 v_2|_{\partial\Omega} \in C^{1,\alpha}(\partial\Omega)$. Combining it with $\partial_\nu w_j \in C^\alpha(\partial\Omega)$, the right-hand side of (2.5) is at least in $L^\infty(\partial\Omega)$.

We then have the integral identity in the $m = 2$ case.

Proposition 1. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be an open connected bounded domain with C^∞ boundary $\partial\Omega$ and let $\Gamma_1, \Gamma_2 \subset \partial\Omega$ be arbitrary nonempty open subsets of $\partial\Omega$. Given two sets of potentials (A_1, q_1) and (A_2, q_2) that satisfy the conditions (1.3)-(1.5) and

$$\nu \cdot \partial_z A_1(x, 0) = \nu \cdot \partial_z A_2(x, 0) \quad \text{on } \Gamma_1 \cap \Gamma_2,$$

we then have that if $\Lambda_{A_1, q_1}^{\Gamma_1, \Gamma_2} = \Lambda_{A_2, q_2}^{\Gamma_1, \Gamma_2}$ (for small boundary data), then for any harmonic functions v_1, v_2, v_3 with

$$\text{supp}(v_1|_{\partial\Omega}), \text{supp}(v_2|_{\partial\Omega}) \subset \Gamma_1 \quad \text{and} \quad \text{supp}(v_3|_{\partial\Omega}) \subset \Gamma_2,$$

we have

$$(2.6) \quad \int_{\Omega} \left(Q_1^{(2)}(v_1, v_2) - Q_2^{(2)}(v_1, v_2) \right) v_3 \, dx = 0,$$

where $Q_j^{(2)}(v_1, v_2)$ is given by (2.4) with A, q replaced by A_j, q_j for $j = 1, 2$.

Proof. Let v_1 and v_2 be harmonic functions with boundary conditions $f_k := v_k|_{\partial\Omega}$ and $\text{supp}(f_k) \subset \Gamma_1$ for $k = 1, 2$. From the fact that $\Lambda_{A_1, q_1}^{\Gamma_1, \Gamma_2}(\varepsilon_1 f_1 + \varepsilon_2 f_2) = \Lambda_{A_2, q_2}^{\Gamma_1, \Gamma_2}(\varepsilon_1 f_1 + \varepsilon_2 f_2)$ for small $\varepsilon = (\varepsilon_1, \varepsilon_2)$, we have that

$$\partial_\nu w_1 + 2i\nu \cdot \partial_z A_1(x, 0) f_1 f_2 = \partial_\nu w_2 + 2i\nu \cdot \partial_z A_2(x, 0) f_1 f_2 \quad \text{on } \Gamma_2,$$

where w_1, w_2 are solutions to

$$(2.7) \quad -\Delta w_j + Q_j^{(2)}(v_1, v_2) = 0 \quad \text{in } \Omega, \quad w_j|_{\partial\Omega} = 0.$$

Since $\text{supp}(f_1), \text{supp}(f_2) \subset \Gamma_1$ and $\nu \cdot \partial_z A_1(x, 0) = \nu \cdot \partial_z A_2(x, 0)$ on $\Gamma_1 \cap \Gamma_2$, one has

$$\nu \cdot \partial_z A_1(x, 0) f_1 f_2 = \nu \cdot \partial_z A_2(x, 0) f_1 f_2 \quad \text{on } \Gamma_2,$$

which leads to

$$\partial_\nu w_1|_{\Gamma_2} = \partial_\nu w_2|_{\Gamma_2}.$$

Multiplying (2.7) by any harmonic function v_3 in Ω with $\text{supp}(v_3|_{\partial\Omega}) \subset \Gamma_2$ and applying Green's formula, we then derive that

$$\int_{\Omega} \left(Q_1^{(2)}(v_1, v_2) - Q_2^{(2)}(v_1, v_2) \right) v_3 \, dx = \int_{\partial\Omega \setminus \Gamma_2} (\partial_\nu w_1 - \partial_\nu w_2) v_3 \, dS = 0.$$

Thus, the proof is complete. \square

2.2. Induction steps in $m \geq 2$. Let $m \geq 2$ and suppose that

$$\partial_z^k A_1(x, 0) = \partial_z^k A_2(x, 0) \quad \text{for } k = 1, \dots, m-2,$$

$$\partial_z^k q_1(x, 0) = \partial_z^k q_2(x, 0) \quad \text{for } k = 2, \dots, m-1.$$

Combining the base case (1.5), we have that

$$(2.8) \quad \partial_z^k A_1(x, 0) = \partial_z^k A_2(x, 0) \quad \text{for } k = 0, \dots, m-2,$$

and

$$(2.9) \quad \partial_z^k q_1(x, 0) = \partial_z^k q_2(x, 0) \quad \text{for } k = 0, \dots, m-1.$$

Let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)$ with small enough $\varepsilon_k > 0$ and let $f_k \in W^{2-1/p, p}(\partial\Omega)$ with $\text{supp}(f_k) \subset \Gamma_1$ for $k = 1, \dots, m$. Again, from Theorem 2, there exists a unique small solution $u_j = u_j(x; \varepsilon)$ to the problem

$$(2.10) \quad \begin{cases} (D + A_j(x, u_j))^2 u_j + q_j(x, u_j) = 0 & \text{in } \Omega, \\ u_j = \varepsilon_1 f_1 + \dots + \varepsilon_m f_m & \text{on } \partial\Omega, \end{cases}$$

for $j = 1, 2$.

Generally, for any positive integer $m \geq 2$, we define the function $Q_j^m(v_1, \dots, v_m)$ by

$$(2.11) \quad \begin{aligned} Q_j^{(m)}(v_1, \dots, v_m) &:= (m+1) \partial_z^{m-1} A_j(x, 0) \cdot D(v_1 \dots v_m) \\ &\quad + m(D_x \cdot \partial_z^{m-1} A_j(x, 0)) v_1 \dots v_m + \partial_z^m q_j(x, 0) v_1 \dots v_m. \end{aligned}$$

The general integral identity is summarized in the following proposition:

Proposition 2. *Let (A_1, q_1) and (A_2, q_2) satisfy the conditions (1.3)-(1.5). Moreover, suppose for $m \geq 2$, (2.8) and (2.9) are satisfied and*

$$(2.12) \quad \nu \cdot \partial_z^{m-1} A_1(x, 0) = \nu \cdot \partial_z^{m-1} A_2(x, 0) \quad \text{on } \Gamma_1 \cap \Gamma_2$$

holds. If $\Lambda_{A_1, q_1}^{\Gamma_1, \Gamma_2} = \Lambda_{A_2, q_2}^{\Gamma_1, \Gamma_2}$ for small data, then for any harmonic functions v_1, \dots, v_{m+1} satisfying

$$\text{supp}(v_1|_{\partial\Omega}), \dots, \text{supp}(v_m|_{\partial\Omega}) \subset \Gamma_1 \quad \text{and} \quad \text{supp}(v_{m+1}|_{\partial\Omega}) \subset \Gamma_2,$$

we have

$$(2.13) \quad \int_{\Omega} \left(Q_1^{(m)}(v_1, \dots, v_m) - Q_2^{(m)}(v_1, \dots, v_m) \right) v_{m+1} dx = 0.$$

Before proving Proposition 2, we first need to look more closely at the derivative of u_j with respect to ε , which is stated in Lemma 1.

We start with defining the notation E_k for $1 \leq k \leq m$ to be a product of k distinct operators of the form ∂_{ε_i} and setting $\varepsilon = 0$. For example, for distinct numbers $1 \leq \ell_1, \ell_2, \dots, \ell_k \leq m$, we have that $\partial_{\varepsilon_{\ell_1}} \partial_{\varepsilon_{\ell_2}} \dots \partial_{\varepsilon_{\ell_k}} u|_{\varepsilon=0}$ is a representative of $E_k u$.

Lemma 1. *Suppose that (2.8) and (2.9) hold. Let u_j be the solution to (2.10). For any $1 \leq k < m$, then we have*

$$E_k u_1 = E_k u_2.$$

Proof. To demonstrate this for $k = 1$, we apply ∂_{ε_i} , $1 \leq i \leq m$, to the equation (2.10) for u_j and set $\varepsilon = 0$. As before we find $\partial_{\varepsilon_i} u_j|_{\varepsilon=0}$ satisfies the linear equation

$$\Delta \partial_{\varepsilon_i} u_j|_{\varepsilon=0} = 0,$$

with boundary condition $(\partial_{\varepsilon_i} u_j|_{\varepsilon=0})|_{\partial\Omega} = f_i$. Since this holds for $j = 1, 2$, we conclude that $E_1 u_1 = E_1 u_2$.

Now we proceed by the induction argument. Suppose that

$$(2.14) \quad E_i u_1 = E_i u_2$$

holds for any E_i with $1 \leq i < k$, and we want to show $E_k u_1 = E_k u_2$. Without loss of generality, we consider the operator

$$E_k = \partial_{\varepsilon_1} \dots \partial_{\varepsilon_k}|_{\varepsilon=0}.$$

By applying E_k to (2.10), we have

$$0 = -\Delta E_k u_j + \Psi_k(u_j, A_j, q_j),$$

where the term Ψ_k is defined by

$$\begin{aligned} \Psi_k(u_j, A_j, q_j) \\ := \Psi_k(E_1 u_j, \dots, E_{k-1} u_j, \partial_z^1 A_j(x, 0), \dots, \partial_z^{k-1} A_j(x, 0), \partial_z^2 q_j(x, 0), \dots, \partial_z^k q_j(x, 0)) \end{aligned}$$

and contains derivatives of order at most $k-1$ in u_j , derivatives of order at most $k-1$ in A_j , and derivatives of order at most k in q_j with respect to the variable z . Since $k < m$, we have $k \leq m-1$ and $k-1 \leq m-2$. Therefore combining (2.8), (2.9), and (2.14), we have

$$\Psi_k(u_1, A_1, q_1) = \Psi_k(u_2, A_2, q_2).$$

Therefore the conclusion is that

$$-\Delta(E_k u_1) = -\Delta(E_k u_2).$$

Moreover, $E_k u_1$ and $E_k u_2$ share the same boundary condition, then one can conclude

$$E_k u_1 = E_k u_2$$

as desired. \square

Proof of Proposition 2. For the given harmonic functions v_1, \dots, v_m , whose boundary traces are supported on Γ_1 , we consider the solution u_j to (2.10) with $f_k = v_k|_{\partial\Omega}$ ($k = 1, \dots, m$). It is not hard to see by Lemma 1 that $\partial_{\varepsilon_k} u_1|_{\varepsilon=0} = \partial_{\varepsilon_k} u_2|_{\varepsilon=0} = v_k$.

Applying the operator $\partial_{\varepsilon_1} \dots \partial_{\varepsilon_m}|_{\varepsilon=0}$ to (2.10), we get

$$\begin{aligned} 0 = & -\Delta(\partial_{\varepsilon_1} \dots \partial_{\varepsilon_m} u_j|_{\varepsilon=0}) \\ & + (m+1)\partial_z^{m-1} A_j(x, 0) \cdot D(v_1 \dots v_m) + m(D_x \cdot \partial_z^{m-1} A_j(x, 0))v_1 \dots v_m \\ & + \partial_z^m q_j(x, 0)v_1 \dots v_m + R_m(u_j, A_j, q_j). \end{aligned}$$

Here the remaining term R_m contains derivatives of order at most $m-1$ in u_j , at most $m-2$ in $A_j(x, 0)$, and at most $m-1$ in $q_j(x, 0)$ with respect to z variable. For $j = 1, 2$, if we write

$$\phi_j := \partial_{\varepsilon_1} \dots \partial_{\varepsilon_m} u_j|_{\varepsilon=0}$$

and use the notation introduced in (2.11), then we can write this as

$$\Delta\phi_j = Q_j^{(m)}(v_1, \dots, v_m) + R_m(u_j, A_j, q_j) \quad \text{in } \Omega, \quad \phi_j|_{\partial\Omega} = 0.$$

From Lemma 1 ($E_\ell u_1 = E_\ell u_2$, $1 \leq \ell < m$) and the assumptions on A_j and q_j , we see that

$$R_m(u_1, A_1, q_1) = R_m(u_2, A_2, q_2).$$

Therefore if we subtract the equation for $j = 2$ from the equation for $j = 1$, we get

$$\Delta(\phi_1 - \phi_2) = Q_1^{(m)}(v_1, \dots, v_m) - Q_2^{(m)}(v_1, \dots, v_m).$$

From the equality of the DN-maps $\Lambda_{A_1, q_1}^{\Gamma_1, \Gamma_2} = \Lambda_{A_2, q_2}^{\Gamma_1, \Gamma_2}$, Lemma 1 ($E_k u_1 = E_k u_2$, $1 \leq k < m$), and the condition (2.12), we can easily derive

$$(2.15) \quad \partial_\nu \phi_1|_{\Gamma_2} = \partial_\nu \phi_2|_{\Gamma_2}.$$

Now let v_{m+1} be harmonic and $\text{supp}(v_{m+1}|_{\partial\Omega}) \subset \Gamma_2$, and consider the integral

$$\int_{\Omega} \Delta(\phi_1 - \phi_2) v_{m+1} \, dx.$$

Similar to the case $m = 2$ discussed in the proof of Proposition 1, by performing the integration by parts, we get

$$\int_{\Omega} \Delta(\phi_1 - \phi_2) v_{m+1} \, dx = \int_{\Omega} (\phi_1 - \phi_2) \Delta v_{m+1} \, dx = 0,$$

with no boundary terms, thanks to the equality (2.15), $\phi_1|_{\partial\Omega} = \phi_2|_{\partial\Omega} = 0$ and $\text{supp}(v_{m+1}|_{\partial\Omega}) \subset \Gamma_2$. This gives us

$$\int_{\Omega} \left(Q_1^{(m)}(v_1, \dots, v_m) - Q_2^{(m)}(v_1, \dots, v_m) \right) v_{m+1} \, dx = 0.$$

This finishes the proof. \square

From Proposition 1 and Proposition 2, we have proved that the integral identity (2.13) holds for $m \geq 2$. In the next section, we will focus on extracting the information about potentials A and q from this identity.

3. Proof of Theorem 1

3.1. A key lemma. We will see below that by using the integral identity (2.13) in Proposition 2 and the density result in Theorem 1.1 in [7], we can derive a much simpler identity that will be the key component to show the desired uniqueness result.

To this end, we first simplify the notations by denoting the discrepancy in $\partial_z^{m-1}A_1$ and $\partial_z^{m-1}A_2$ as

$$\tilde{A}_{m-1}(x) := \partial_z^{m-1}A_2(x, 0) - \partial_z^{m-1}A_1(x, 0),$$

and also denoting the discrepancy in $\partial_z^m q_1$ and $\partial_z^m q_2$ as

$$\tilde{q}_m(x) := \partial_z^m q_2(x, 0) - \partial_z^m q_1(x, 0).$$

By applying the integration by parts and the boundary condition $\nu \cdot \partial_z^{m-1}A_1(x, 0) = \nu \cdot \partial_z^{m-1}A_2(x, 0)$, the identity in Proposition 2 now becomes

$$\begin{aligned} 0 &= \int_{\Omega} \left(Q_2^{(m)}(v_1, \dots, v_m) - Q_1^{(m)}(v_1, \dots, v_m) \right) v_{m+1} \, dx \\ &= \int_{\Omega} \left(-(D \cdot \tilde{A}_{m-1})v_{m+1} - (m+1)(\tilde{A}_{m-1} \cdot Dv_{m+1}) + \tilde{q}_m v_{m+1} \right) v_1 \dots v_m \, dx \end{aligned}$$

for harmonic functions v_1, \dots, v_{m+1} such that

$$\text{supp}(v_1|_{\partial\Omega}), \dots, \text{supp}(v_m|_{\partial\Omega}) \subset \Gamma_1 \quad \text{and} \quad \text{supp}(v_{m+1}|_{\partial\Omega}) \subset \Gamma_2.$$

We then have the following result by using the density of the product of harmonic functions in L^1 space.

Lemma 2. *Suppose that the conditions in Proposition 2 hold and harmonic functions v_3, \dots, v_m have nontrivial boundary data. Then the following identity holds:*

$$(3.1) \quad -(D \cdot \tilde{A}_{m-1})v_{m+1} - (m+1)(\tilde{A}_{m-1} \cdot Dv_{m+1}) + \tilde{q}_m v_{m+1} = 0$$

almost everywhere (a.e.) in Ω for $m \geq 2$.

Proof. When $m = 2$, we apply Theorem 1.1 in [7], stating that the set of products $v_1 v_2$ of harmonic functions in $C^\infty(\bar{\Omega})$ that vanish on a closed proper subset $\tilde{\Gamma}$ of $\partial\Omega$ is dense in $L^1(\Omega)$. We immediately obtain

$$-(D \cdot \tilde{A}_1)v_3 - 3(\tilde{A}_1 \cdot Dv_3) + \tilde{q}_2 v_3 = 0$$

a.e. in Ω .

For $m > 2$, we also apply Theorem 1.1 in [7] to obtain

$$\left(-(D \cdot \tilde{A}_{m-1})v_{m+1} - (m+1)(\tilde{A}_{m-1} \cdot Dv_{m+1}) + \tilde{q}_m v_{m+1} \right) v_3 \dots v_m = 0$$

a.e. in Ω . By the fact that $\bigcup_{j=3}^m v_j^{-1}(0)$ has measure zero, we then have (3.1) for any $m > 2$. \square

For simplicity of notations, for $m \geq 2$, we recast (3.1) as the following transport equation,

$$(3.2) \quad F(x) \cdot Dv_{m+1} + g(x)v_{m+1} = 0,$$

where

$$(3.3) \quad F(x) := -(m+1)\tilde{A}_{m-1}(x), \quad g(x) := -(D \cdot \tilde{A}_{m-1})(x) + \tilde{q}_m(x).$$

3.2. Uniqueness result. The proof of Theorem 1 relies heavily on the following result, that is, Proposition 3 below: if the equation

$$F(x) \cdot Dv_{m+1} + g(x)v_{m+1} = 0 \quad \text{in } \Omega$$

holds for all harmonic functions $v_{m+1} \in C^\infty(\bar{\Omega})$ with $\text{supp}(v_{m+1}|_{\partial\Omega}) \subset \Gamma_2 \subset \partial\Omega$, then $F = 0$ and $g = 0$ in Ω . It is clear that the identity $F = 0 = g$ implies $\tilde{A}_{m-1} = 0 = \tilde{q}_m$. Thus, the uniqueness of the potentials follows immediately.

The key strategy is to first show that F and g vanish locally, that is, $F(x)$ and $g(x)$ vanish a.e. in a neighborhood of a point $x_0 \in \Gamma_2$. Next we extend this local result to the global one. The detailed argument is stated in the proof of Proposition 3.

To begin, we first construct the harmonic function v_{m+1} as in [7]. Without loss of generality, we let $x_0 = 0$, the tangent plane to $\partial\Omega$ at x_0 be given by $x_1 = 0$ and

$$\Omega \subset \{x \in \mathbb{R}^n : |x + \mathbf{e}_1| < 1\}, \quad \tilde{\Gamma}_2 := \partial\Omega \setminus \Gamma_2 = \{x \in \partial\Omega : x_1 \leq -2c\}$$

for some constant $c > 0$. Here

$$\mathbf{e}_j = (0, \dots, 0, 1, 0, \dots, 0)$$

with the j^{th} component equals to 1.

Remark 4. We comment here that one can apply a transformation to achieve above conditions for the domain (or by the conformal mapping in [7] for non-convex domains). More specifically, given the transformation $T : \tilde{\Omega} \rightarrow \Omega$, $x = T(y)$, the transport equation

$$F(x) \cdot Dv + g(x)v = 0 \quad \text{in } \Omega$$

becomes

$$\tilde{F}(y) \cdot \left(\frac{\partial y}{\partial x} \right)^T D_y \tilde{v} + \tilde{g}(y)\tilde{v} = 0 \quad \text{in } \tilde{\Omega},$$

where

$$\tilde{v}(y) = v \circ T(y), \quad \tilde{F}(y) = F \circ T(y), \quad \tilde{g}(y) = g \circ T(y).$$

It is not hard to see that with the transformation satisfying $\det \left(\frac{\partial y}{\partial x} \right) \neq 0$ a.e., we have that $\left(\frac{\partial y}{\partial x} \right)^T \tilde{F}(y) = 0$ and $\tilde{g}(y) = 0$ in $\tilde{\Omega}$ implies that $F(x) = 0$ and $g(x) = 0$ in Ω .

For $\zeta \in \mathbb{C}^n$ such that $\zeta \cdot \zeta = 0$ and a cut-off function $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $\chi = 1$ on $\tilde{\Gamma}_2$ and $\text{supp}(\chi) \subset \{x \in \mathbb{R}^n : x_1 \leq -c\}$, we consider the harmonic function

$$(3.4) \quad v_{m+1}(x, \zeta) = e^{-ix \cdot \zeta/h} + \tilde{w}(x, \zeta), \quad h > 0,$$

with $v_{m+1}|_{\tilde{\Gamma}_2} = 0$, where \tilde{w} is the solution to the Dirichlet problem

$$\begin{cases} \Delta \tilde{w} = 0 & \text{in } \Omega, \\ \tilde{w}|_{\partial\Omega} = -\left(e^{-ix \cdot \zeta/h} \chi\right)|_{\partial\Omega}. \end{cases}$$

Then it is clear to see

$$\|\tilde{w}\|_{H^1(\Omega)} \leq C \|e^{-ix \cdot \zeta/h} \chi\|_{H^{1/2}(\partial\Omega)} \leq C(1 + h^{-1}|\zeta|)^{1/2} e^{\frac{1}{h} H_K(\text{Im } \zeta)},$$

where H_K is the supporting function of the compact subset $K = \text{supp}(\chi) \cap \partial\Omega$ of the boundary and is defined by

$$H_K(\vec{d}) = \sup_{x \in K} x \cdot \vec{d}, \quad \text{for } \vec{d} \in \mathbb{R}^n.$$

In particular, from the property of χ , one can further derive that when $\text{Im } \zeta_1 \geq 0$,

$$\|\tilde{w}\|_{H^1(\Omega)} \leq C(1 + h^{-1}|\zeta|)^{1/2} e^{-\frac{\epsilon}{h} \text{Im } \zeta_1} e^{\frac{1}{h} |\text{Im } \zeta'|},$$

where $\zeta = (\zeta_1, \zeta')$ with ζ' being the $(n-1)$ dimensional coordinate vector.

Similarly, we can also derive that for any $\alpha \in \mathbb{Z}_+^n$, the remainder function \tilde{w} satisfies

$$\begin{aligned} \|\partial^\alpha \tilde{w}\|_{H^1(\Omega)} &\leq C \|\partial^\alpha (e^{-ix \cdot \zeta/h} \chi)\|_{H^{1/2}(\partial\Omega)} \\ &\leq C(1 + h^{-1}|\zeta|)^{(1+|\alpha|)/2} e^{-\frac{\epsilon}{h} \text{Im } \zeta_1} e^{\frac{1}{h} |\text{Im } \zeta'|}, \end{aligned}$$

which gives the upper bound of $\tilde{w} \in H^{|\alpha|+1}(\Omega)$. By the Sobolev embedding theorem [9], when $|\alpha| - [\frac{n}{2}] - 1 \geq 0$, one has

$$(3.5) \quad \|\tilde{w}\|_{C^1(\bar{\Omega})} \leq C(1 + h^{-1}|\zeta|)^{(1+|\alpha|)/2} e^{-\frac{\epsilon}{h} \text{Im } \zeta_1} e^{\frac{1}{h} |\text{Im } \zeta'|}.$$

Now we are ready to show the following proposition.

Proposition 3. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be an open bounded connected set with smooth boundary and Γ_2 be an open nonempty proper subset of $\partial\Omega$. Denote $\tilde{\Gamma}_2 = \partial\Omega \setminus \Gamma_2$. Let $F \in L^\infty(\Omega; \mathbb{C}^n)$ and $g \in L^\infty(\Omega; \mathbb{C})$. Suppose for all harmonic functions $v_{m+1} \in C^\infty(\bar{\Omega})$ with $\text{supp}(v_{m+1}|_{\partial\Omega}) \subset \Gamma_2$ such that*

$$(3.6) \quad F(x) \cdot Dv_{m+1}(x) + g(x)v_{m+1}(x) = 0 \quad \text{a.e. in } \Omega.$$

Then we have $F = 0$ and $g = 0$ a.e. in Ω .

Remark 5. *We remark here that in the full data setting ($\Gamma_2 = \partial\Omega$), an individual proof of this proposition can be found in Appendix B.*

Proof. Step 1: Local result. As discussed above, we take $x_0 = 0 \in \Gamma_2$ without loss of generality. Substituting v_{m+1} defined in (3.4) with nontrivial boundary data $v_{m+1}|_{\partial\Omega} \neq 0$ into (3.6), we obtain

$$(3.7) \quad F(x) \cdot \zeta = hF(x) \cdot D\tilde{w}e^{ix \cdot \zeta/h} + hg(x) + hg(x)\tilde{w}e^{ix \cdot \zeta/h}.$$

Let $\zeta = (i, 1, 0, \dots, 0)^T$. Then for all $x \in \bar{\Omega}$ such that $x_1 > -c$, we have from the above estimates (3.5) for \tilde{w} that

$$|\tilde{w}(x)e^{ix \cdot \zeta/h}|, \quad |D\tilde{w}(x)e^{ix \cdot \zeta/h}| \leq Ce^{-\frac{\pi}{h} \text{Im } \zeta_1} (1 + h^{-1}|\zeta|)^{(1+|\alpha|)/2} e^{-\frac{\epsilon}{h} \text{Im } \zeta_1} e^{\frac{1}{h} |\text{Im } \zeta'|}.$$

When $h \rightarrow 0$, this implies that the right-hand side of (3.7)

$$hF(x) \cdot D\tilde{w}e^{ix \cdot \zeta/h} + hg(x) + hg(x)\tilde{w}e^{ix \cdot \zeta/h}$$

vanishes. Hence $F(x) \cdot \zeta = 0$, at every point $x \in \overline{\Omega}$ with $x_1 > -c$, i.e., in a neighborhood of $x_0 = 0$. Similarly, by choosing $\zeta' = (i, -1, 0, \dots, 0)^T$ instead, we can derive that $F(x) \cdot \zeta' = 0$. These two identities $F(x) \cdot \zeta = 0$ and $F(x) \cdot \zeta' = 0$ indicate the first two components of $F(x)$ indeed vanish.

Furthermore, by choosing other

$$\zeta = i\mathbf{e}_1 + \mathbf{e}_j \quad \text{for } j = 3, \dots, n,$$

one can show that the other components of $F(x)$ vanish too, which implies that $F = 0$. Thus, we can also obtain that $g = 0$ from the equation (3.6) and the fact that $v_{m+1}^{-1}(0)$ has measure zero. Finally we have derived that $F = 0$ and $g = 0$ in a neighborhood of every point $x_0 \in \Gamma_2$ provided that (3.6) hold for all harmonic functions v_{m+1} with boundary data that is supported in Γ_2 .

Step 2: Global result. To extend the local result to any point x_1 of Ω , we take a point $x_0 \in \Gamma_2$ and let $\theta : [0, 1] \rightarrow \overline{\Omega}$ be a C^1 curve joining x_0 and x_1 such that $\theta(0) = x_0$ and $\theta'(0)$ is the inner normal to $\partial\Omega$ at x_0 , and $\theta(t) \in \Omega$ for $t \in (0, 1]$. We set

$$\Theta_\varepsilon(t) = \{x \in \overline{\Omega} : d(x, \theta([0, t])) \leq \varepsilon\},$$

a closed neighborhood of the curve $\theta(s)$, $s \in [0, t]$. Let

$$I = \{t \in [0, 1] : F = 0, \quad g = 0 \quad \text{a.e. on } \Theta_\varepsilon(t) \cap \Omega\}.$$

The above local result indicates that $0 \in I$ if $\varepsilon > 0$ is small enough. Moreover, it is clear that I is a closed subset of $[0, 1]$. If we can further show that I is also open, then we can get $I = [0, 1]$, which further implies that $x_1 \notin \text{supp}(F) \cup \text{supp}(g)$. Since x_1 is an arbitrary point in Ω , we then have $F = 0$ and $g = 0$ in Ω . This will complete the proof of the global result.

To show that I is open in $[0, 1]$, we take $t \in I$ and $\varepsilon > 0$ small enough so that $\partial\Theta_\varepsilon(t) \cap \partial\Omega \subset \Gamma_2$. It is easy to see that the set $\Omega \setminus \Theta_\varepsilon(t)$ can be smoothed out into an open subset Ω_1 of Ω with smooth boundary so that

$$\Omega_1 \supset \Omega \setminus \Theta_\varepsilon(t), \quad \partial\Omega \cap \partial\Omega_1 \supset \tilde{\Gamma}_2.$$

We further augment the set Ω by smoothing out the set $\Omega \cup B(x_0, \varepsilon')$ with $0 < \varepsilon' \ll \varepsilon$ sufficiently small, into an open set Ω_2 so that

$$\partial\Omega_2 \cap \partial\Omega \supset \partial\Omega \cap \partial\Omega_1 = \partial\Omega_1 \cap \partial\Omega_2 \supset \tilde{\Gamma}_2.$$

Now we let G_2 be the Green kernel associated to the open set Ω_2 and

$$-\Delta_y G_2(x, y) = \delta(x - y) \quad \text{in } \Omega_2, \quad G_2(x, y)|_{\partial\Omega_2} = 0.$$

We consider the function

$$\Phi(x; y) := F(y) \cdot D_y G_2(x, y) + g(y) G_2(x, y), \quad y \in \Omega_1, \quad x \in \Omega_2 \setminus \overline{\Omega_1}.$$

It is clear that $\Phi(x; y)$ is harmonic in x on $\Omega_2 \setminus \overline{\Omega_1}$ for a fixed $y \in \Omega_1$. Since $F(y) = 0$ and $g(y) = 0$ for $y \in \Theta_\varepsilon(t) \cap \Omega$, we can extend $\Phi(x; y)$ by zero to $y \in \Omega$. When $x \in \Omega_2 \setminus \overline{\Omega}$, the Green function $G_2(x, y)$ is a harmonic function in y on Ω with $G_2(x, \cdot)|_{\tilde{\Gamma}_2} = 0$. By (3.6), we have

$$\Phi(x; y) = 0, \quad \text{for a.e. } y \in \Omega, \quad x \in \Omega_2 \setminus \overline{\Omega}.$$

Since $\Phi(x; y)$ is harmonic in x on $\Omega_2 \setminus \overline{\Omega_1}$ and the set $\Omega_2 \setminus \overline{\Omega_1}$ is connected, by the unique continuation, we then have

$$\Phi(x; y) = 0, \quad \text{for a.e. } y \in \Omega_1, \quad x \in \Omega_2 \setminus \overline{\Omega_1}.$$

By Lemma 2.2 of [24] (H^1 -density), we have that for any $v \in C^\infty(\overline{\Omega_1})$ harmonic with $v|_{\partial\Omega_1 \cap \partial\Omega_2} = 0$ and arbitrary small $\epsilon > 0$, there exists $a \in C^\infty(\overline{\Omega_2})$ with $\text{supp}(a) \subset \Omega_2 \setminus \overline{\Omega_1}$ such that

$$\left\| v(y) - \int_{\Omega_2} G_2(x, y) a(x) dx \right\|_{H^1(\Omega_1)} < \epsilon.$$

We multiply $\Phi(x; y)$ by $a(x)$ and then integrate it with respect to x on Ω_2 . We obtain

$$F(y) \cdot D_y \int_{\Omega_2} G_2(x, y) a(x) dx + g(y) \int_{\Omega_2} G_2(x, y) a(x) dx = 0, \quad \text{a.e. } y \in \Omega_1,$$

and, moreover, we can derive that

$$\begin{aligned} & \|F \cdot Dv + gv\|_{L^2(\Omega_1)} \\ & \leq \left\| F \cdot D \int_{\Omega_2} G_2(x, \cdot) a(x) dx + g \int_{\Omega_2} G_2(x, \cdot) a(x) dx \right\|_{L^2(\Omega_1)} + C\epsilon = C\epsilon \end{aligned}$$

for arbitrary small $\epsilon > 0$. This implies that

$$F \cdot Dv + gv = 0 \quad \text{a.e. in } \Omega_1$$

for $v \in C^\infty(\overline{\Omega_1})$ harmonic with $v|_{\partial\Omega_1 \cap \partial\Omega_2} = 0$. By the above local result in Step 1, we then have $F = 0$ and $g = 0$ in an open neighborhood of $\partial\Omega_1 \setminus (\partial\Omega_1 \cap \partial\Omega_2)$ and this implies that F and g vanish on a slightly larger neighborhood $\Theta_\varepsilon(t')$, $t' > t$ of the curve. This proves that I is open, hence completes the proof. \square

Proof of Theorem 1. From Proposition 2, we have the integral identity holds for $m = 2$. By applying Lemma 2, Proposition 3 and (3.3), we have $F = 0$, $g = 0$, which implies that

$$\partial_z A_1(x, 0) = \partial_z A_2(x, 0), \quad \partial_z^2 q_1(x, 0) = \partial_z^2 q_2(x, 0).$$

Given any integer $m > 2$, by induction argument, suppose that for $k = 2, \dots, m-1$, the following are true:

$$\partial_z^{k-1} A_1(x, 0) = \partial_z^{k-1} A_2(x, 0), \quad \partial_z^k q_2(x, 0) = \partial_z^k q_1(x, 0).$$

We want to show that $\partial_z^{m-1} A_1(x, 0) = \partial_z^{m-1} A_2(x, 0)$ and $\partial_z^m q_1(x, 0) = \partial_z^m q_2(x, 0)$ also hold.

From above, we have known that A_j and q_j satisfy the conditions (2.8) and (2.9) and thus we can apply Proposition 2 to get the integral (2.13) for such $m > 2$. Applying Lemma 2 and Proposition 3 again, we then derive that $F = 0$, $g = 0$, which gives

$$0 = \partial_z^{m-1} A_2(x, 0) - \partial_z^{m-1} A_1(x, 0)$$

and

$$0 = \partial_z^m q_2(x, 0) - \partial_z^m q_1(x, 0).$$

Therefore, we complete the proof of Theorem 1. \square

Appendix A. Well-posedness of the nonlinear magnetic Schrödinger equation

In this section, we prove that the boundary value problem (1.9) is well-posed if the small boundary data is given. The analysis is based on the contraction mapping principle.

Theorem 2 (Well-posedness). *Let $A(x, z)$ and $q(x, z)$ satisfy (1.3)-(1.4). Moreover, suppose that $q(x, 0) = 0$ and 0 is not a Dirichlet eigenvalue of the linear operator*

$$\mathcal{L}_0 := (D + A(x, 0))^2 + \partial_z q(x, 0).$$

Then there exists a small constant $\varepsilon > 0$ such that for any $\|f\|_{W^{2-1/p,p}(\partial\Omega)} \leq \varepsilon$, the boundary value problem

$$(A.1) \quad \begin{cases} (D + A(x, u))^2 u + q(x, u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases}$$

admits a unique solution $u \in W^{2,p}(\Omega)$. Moreover, there exists a constant $C > 0$ independent of f such that

$$(A.2) \quad \|u\|_{W^{2,p}(\Omega)} \leq C \|f\|_{W^{2-1/p,p}(\partial\Omega)}.$$

Proof. We will use contraction mapping principle to show the existence of solution to (1.9).

Step 1: Linearization. First, for $A(x, z)$ and $q(x, z)$ satisfying (1.3)-(1.4), we use the Taylor formulas

$$\begin{aligned} A(x, z) &= A(x, 0) + A_r(x, z)z, \\ q(x, z) &= \partial_z q(x, 0)z + q_r(x, z)z^2, \end{aligned}$$

where we denote

$$A_r(x, z) := \int_0^1 \partial_z A(x, tz) dt, \quad q_r(x, z) := \int_0^1 \partial_z^2 q(x, tz)(1-t) dt.$$

Given $f \in W^{2-1/p,p}(\partial\Omega)$ for $p \in (n, +\infty)$, by Theorem 9.15 of [12], there exists a unique solution $u_0 \in W^{2,p}(\Omega)$ to the Dirichlet problem

$$(A.3) \quad \begin{cases} \mathcal{L}_0 u_0 := (D + A(x, 0))^2 u_0 + \partial_z q(x, 0)u_0 = 0 & \text{in } \Omega, \\ u_0 = f & \text{on } \partial\Omega. \end{cases}$$

Moreover, we have

$$\|u_0\|_{W^{2,p}(\Omega)} \leq C \|f\|_{W^{2-1/p,p}(\partial\Omega)}.$$

(This can be obtained by extending f to a $W^{2,p}(\Omega)$ function and apply Lemma 9.17 of [12] to the equation for the difference of the solution and the extended function.)

Thus, if u is a solution to (1.9) we have the remainder function $v := u - u_0$ satisfying the following problem

$$(A.4) \quad \mathcal{L}_0 v = \mathcal{F}(v), \quad v|_{\partial\Omega} = 0,$$

where

$$\begin{aligned}\mathcal{F}(v) := & -(D + A(x, 0)) \cdot [A_r(x, u_0 + v)(u_0 + v)^2] \\ & - A_r(x, u_0 + v)(u_0 + v) \cdot (D + A(x, 0))(u_0 + v) \\ & - A_r(x, u_0 + v)^2(u_0 + v)^3 - q_r(x, u_0 + v)(u_0 + v)^2.\end{aligned}$$

By Theorem 9.15 in [12] again, for $F \in L^p(\Omega)$, there exists a unique solution $\tilde{u} \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ to the equation $\mathcal{L}_0 \tilde{u} = F \in L^p(\Omega)$ in Ω with trivial boundary data. We denote the solution operator by

$$\mathcal{L}_0^{-1} : L^p(\Omega) \rightarrow W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega),$$

which is the continuous operator $F \mapsto \tilde{u}$ and thus $\mathcal{L}_0^{-1}(F)$ is the solution to $\mathcal{L}_0 \tilde{u} = F \in L^p(\Omega)$ in Ω with trivial boundary condition. Therefore, we are looking for the unique fixed point v of $\mathcal{L}_0^{-1} \circ \mathcal{F}$.

Step 2: A contraction map. In what follows, we will show that $\mathcal{L}_0^{-1} \circ \mathcal{F}$ is indeed a contraction map on a suitable subset X_δ of $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Here we denote the set X_δ for $1 > \delta > 0$ by

$$X_\delta := \{v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \mid \|v\|_{W^{2,p}(\Omega)} \leq \delta\}.$$

We first show that $(\mathcal{L}_0^{-1} \circ \mathcal{F})(X_\delta) \subset X_\delta$. Recalling that, by the Sobolev embedding theorem, we have $W^{2,p}(\Omega) \hookrightarrow C^1(\Omega)$ if $p > n$. For $v \in X_\delta$, we have $v, u_0 \in C^1(\Omega)$ since $v, u_0 \in W^{2,p}(\Omega)$. Thus we have that $A_r(x, u_0(x) + v(x))$ and $q_r(x, u_0(x) + v(x))$ are both bounded in Ω . Moreover, since

$$\begin{aligned}(A.5) \quad & D \cdot [A_r(x, u_0(x) + v(x))] \\ &= \int_0^1 D_x \cdot \partial_z A(x, t(u_0 + v)) \, dt + \int_0^1 t \partial_z^2 A(x, t(u_0 + v)) \, dt \cdot D(u_0 + v),\end{aligned}$$

one can derive that

$$\begin{aligned}\|\mathcal{F}(v)\|_{L^p(\Omega)} &\leq C\|u_0 + v\|_{C^1(\Omega)}\|u_0 + v\|_{L^p(\Omega)} \\ &\leq C\|u_0 + v\|_{W^{2,p}(\Omega)}^2 \leq C(\|u_0\|_{W^{2,p}(\Omega)}^2 + \|v\|_{W^{2,p}(\Omega)}^2).\end{aligned}$$

This implies that, for $\|f\|_{W^{2-1/p,p}(\partial\Omega)} < \varepsilon$ and $p > n$, one has

$$\begin{aligned}(A.6) \quad & \|\mathcal{L}_0^{-1}(\mathcal{F}(v))\|_{W^{2,p}(\Omega)} \leq C\|\mathcal{F}(v)\|_{L^p(\Omega)} \\ & \leq C(\|f\|_{W^{2-1/p,p}(\partial\Omega)}^2 + \|v\|_{W^{2,p}(\Omega)}^2) \leq C(\varepsilon^2 + \delta^2).\end{aligned}$$

Therefore, for ε and δ small enough, the operator $\mathcal{L}_0^{-1} \circ \mathcal{F}$ maps X_δ into itself.

Next we show that $\mathcal{L}_0^{-1} \circ \mathcal{F}$ is a contraction on X_δ . To this end, we take $v_1, v_2 \in X_\delta$ and consider

$$\begin{aligned}\|\mathcal{L}_0^{-1} \circ \mathcal{F}(v_1) - \mathcal{L}_0^{-1} \circ \mathcal{F}(v_2)\|_{W^{2,p}(\Omega)} &= \|\mathcal{L}_0^{-1}(\mathcal{F}(v_1) - \mathcal{F}(v_2))\|_{W^{2,p}(\Omega)} \\ &\leq C\|\mathcal{F}(v_1) - \mathcal{F}(v_2)\|_{L^p(\Omega)}.\end{aligned}$$

In addition, we rewrite

$$\begin{aligned}-\mathcal{F}(v) = & D \cdot (A_r(x, u_0 + v))(u_0 + v)^2 + 3(A_r(x, u_0 + v) \cdot D(u_0 + v))(u_0 + v) \\ & + 2A(x, 0) \cdot A_r(x, u_0 + v)(u_0 + v)^2 + A_r(x, u_0 + v)^2(u_0 + v)^3 \\ & + q_r(x, u_0 + v)(u_0 + v)^2.\end{aligned}$$

Thus, $\mathcal{F}(v_2) - \mathcal{F}(v_1)$ is the sum of the following two terms

$$\begin{aligned} \text{I} &= D \cdot (A_r(x, u_0 + v_1))[(u_0 + v_1)^2 - (u_0 + v_2)^2] \\ &\quad + 3A_r(x, u_0 + v_1) \cdot [D(u_0 + v_1)(u_0 + v_1) - D(u_0 + v_2)(u_0 + v_2)] \\ &\quad + 2A(x, 0) \cdot A_r(x, u_0 + v_1)[(u_0 + v_1)^2 - (u_0 + v_2)^2] \\ &\quad + A_r(x, u_0 + v_1)^2[(u_0 + v_1)^3 - (u_0 + v_2)^3] \\ &\quad + q_r(x, u_0 + v_1)[(u_0 + v_1)^2 - (u_0 + v_2)^2], \\ \text{II} &= [D \cdot (A_r(x, u_0 + v_1)) - D \cdot (A_r(x, u_0 + v_2))](u_0 + v_2)^2 \\ &\quad + 3(A_r(x, u_0 + v_1) - A_r(x, u_0 + v_2)) \cdot D(u_0 + v_2)(u_0 + v_2) \\ &\quad + 2A(x, 0) \cdot [A_r(x, u_0 + v_1) - A_r(x, u_0 + v_2)](u_0 + v_2)^2 \\ &\quad + [A_r(x, u_0 + v_1)^2 - A_r(x, u_0 + v_2)^2](u_0 + v_2)^3 \\ &\quad + (q_r(x, u_0 + v_1) - q_r(x, u_0 + v_2))(u_0 + v_2)^2. \end{aligned}$$

For the first term, we obtain

$$\begin{aligned} \|\text{I}\|_{L^p(\Omega)} &\leq C\{(\|u_0\|_{C^1(\Omega)} + \|v_1\|_{C^1(\Omega)} + \|v_2\|_{C^1(\Omega)})\|v_1 - v_2\|_{L^p(\Omega)} \\ &\quad + (\|u_0\|_{L^p(\Omega)} + \|v_1\|_{L^p(\Omega)} + \|v_2\|_{L^p(\Omega)})\|v_1 - v_2\|_{C^1(\Omega)}\} \\ &\leq C(\|u_0\|_{W^{2,p}(\Omega)} + \|v_1\|_{W^{2,p}(\Omega)} + \|v_2\|_{W^{2,p}(\Omega)})\|v_1 - v_2\|_{W^{2,p}(\Omega)} \\ &\leq C(\varepsilon + \delta)\|v_1 - v_2\|_{W^{2,p}(\Omega)}. \end{aligned}$$

For II, we have

$$\begin{aligned} \|\text{II}\|_{L^p(\Omega)} &\leq C\|u_0 + v_2\|_{C^1(\Omega)}^2 \{ \|D \cdot [A_r(x, u_0 + v_1)] - D \cdot [A_r(x, u_0 + v_2)]\|_{L^p(\Omega)} \\ &\quad + \|A_r(x, u_0 + v_1) - A_r(x, u_0 + v_2)\|_{L^p(\Omega)} \\ &\quad + \|q_r(x, u_0 + v_1) - q_r(x, u_0 + v_2)\|_{L^p(\Omega)} \}. \end{aligned}$$

By (A.5) and that $D \cdot \partial_z A(x, z)$, $\partial_z^2 A(x, z)$, $\partial_z A(x, z)$ and $\partial_z^2 q(x, z)$ are all Lipschitz in z (where the Lipschitz constants are independent of x by the boundedness of $\partial_z^k A$ and $\partial_z^k q$), we obtain

$$\|\text{II}\|_{L^p(\Omega)} \leq C\|u_0 + v_2\|_{C^1(\Omega)}^2 \|v_1 - v_2\|_{W^{1,p}(\Omega)} \leq C(\varepsilon^2 + \delta^2)\|v_1 - v_2\|_{W^{2,p}(\Omega)}.$$

Combining above estimates together, we obtain

$$\|\mathcal{F}(v_1) - \mathcal{F}(v_2)\|_{L^p(\Omega)} \leq C(\delta + \varepsilon + \delta^2 + \varepsilon^2)\|v_1 - v_2\|_{W^{2,p}(\Omega)}.$$

Therefore, $\mathcal{L}_0^{-1} \circ \mathcal{F}$ is a contraction on X_δ for ε and δ small enough. Using the contraction mapping theorem, there exists a unique fixed point $v \in X_\delta$ of $\mathcal{L}_0^{-1} \circ \mathcal{F}$, namely,

$$(\mathcal{L}_0^{-1} \circ \mathcal{F})(v) = v,$$

and hence v solves (A.4). Substituting the fixed point v into the second inequality of (A.6), we then have

$$\|v\|_{W^{2,p}(\Omega)} \leq C(\varepsilon\|f\|_{W^{2-1/p,p}(\partial\Omega)} + \delta\|v\|_{W^{2,p}(\Omega)}).$$

For δ small enough, this gives

$$\|v\|_{W^{2,p}(\Omega)} \leq C\|f\|_{W^{2-1/p,p}(\partial\Omega)}.$$

Finally, we obtain $u = u_0 + v \in W^{2,p}(\Omega)$ which solves (1.9) and satisfies

$$\|u\|_{W^{2,p}(\Omega)} \leq C\|f\|_{W^{2-1/p,p}(\partial\Omega)}.$$

□

Appendix B. An alternative proof of the full boundary data result

In this section, we provide a separate proof to show that the nonlinear potentials can be uniquely recovered when the boundary data are given on the whole boundary.

Proof of Theorem 1 (when $\Gamma_1 = \Gamma_2 = \partial\Omega$). We will begin by reproving Proposition 3 here when $\Gamma_2 = \partial\Omega$. From identity (3.2), we substitute harmonic function

$$v_{m+1} = e^{\zeta \cdot x},$$

into (3.2), where $\zeta \in \mathbb{C}^n$ satisfy $\zeta \cdot \zeta = 0$. Then we have

$$(B.1) \quad F(x) \cdot \zeta + g(x) = 0.$$

Since ζ is arbitrary with $\zeta \cdot \zeta = 0$, we can take

$$\zeta = h\mathbf{e}_1 + i h\mathbf{e}_j$$

for $j = 2, \dots, n$ and $h \in \mathbb{R}$. We then obtain from (B.1) that

$$(B.2) \quad F(x) \cdot (\mathbf{e}_1 + i\mathbf{e}_j) = 0$$

as $h \rightarrow \infty$. Similarly, we can take

$$\zeta' = h\mathbf{e}_1 - i h\mathbf{e}_j,$$

then we have

$$(B.3) \quad F(x) \cdot (\mathbf{e}_1 - i\mathbf{e}_j) = 0.$$

Adding these two equations (B.2) and (B.3) together, we get

$$F(x) \cdot \mathbf{e}_1 = 0,$$

which implies the first component of F vanishes. Following similar argument as above, we can then conclude $F = 0$ in Ω . Thus, from (3.2), we can also derive $g = 0$ if we have known $F = 0$.

Finally, by following a similar argument as in the Proof of Theorem 1 for the partial data setting in Section 3, we obtain the uniqueness result with complete data. □

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