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Noncommutative branch-cut quantum gravity with a self-coupling inflaton scalar field: The wave function of the Universe

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Abstract

This article focuses on the implications of a noncommutative formulation of branch-cut quantum gravity. Based on a mini-superspace structure that obeys the noncommutative Poisson algebra, combined with the Wheeler–DeWitt equation and Hořava–Lifshitz quantum gravity, we explore the impact of a scalar field of the inflaton-type in the evolution of the Universe's wave function. Taking as a starting point the Hořava–Lifshitz action, which depends on the scalar curvature of the branched Universe and its derivatives, the corresponding wave equations are derived and solved. The noncommutative quantum gravity approach adopted preserves the diffeomorphism property of General Relativity, maintaining compatibility with the Arnowitt–Deser–Misner Formalism. In this work we delve deeper into a mini-superspace of noncommutative variables, incorporating scalar inflaton fields and exploring inflationary models, particularly chaotic and nonchaotic scenarios. We obtained solutions to the wave equations without resorting to numerical approximations. The results indicate that the noncommutative algebraic space captures low and high spacetime scales, driving the exponential acceleration of the Universe.

KEYWORDS

branch cut cosmology, inflation, scalar field

1 | INTRODUCTION

Branch cut gravity (BCG), a theoretical alternative to the inflation model, based on the mathematical augmentation technique of closure and existential completeness (Manders 1989), represents an analytically continuous extension of general relativity (Einstein 1916, 1917) to the complex plane (Bodmann et al. 2022; Bodmann et al. 2023a,b; de Freitas Pacheco et al. 2022; Hess et al. 2022; Weber et al. 2024a,b; Zen Vasconcellos et al. 2019; Zen Vasconcellos et al. 2021a,b; Zen Vasconcellos et al. 2022). Such mathematical procedures have proven extremely useful both in quantum mechanics (Dirac 1937) and in pseudocomplex general relativity (pc-GR) (Hess & Greiner 2009) with direct physical (Aharonov & Bohm 1959; Wu et al. 2021) and cosmological (Hess 2017; Hess et al. 2016; Hess & Boller 2020) manifestations.

The branch-cut formulation corresponds to the complexification of the Friedman-Lemaître-Robertson-Walker (FLRW) metric (Friedman 1922; Lemaître 1927; Robertson 1935; Walker 1937), resulting in a sum of field equations associated to continuously distributed single-poles with infinitesimal residues, arranged along a line in the complex plane (for details, see ref. Bodmann et al. 2022; Bodmann et al. 2023a,b; de Freitas Pacheco et al. 2022; Hess et al. 2022; Weber et al. 2024a,b; Zen Vasconcellos et al. 2019, 2022; Zen Vasconcellos et al. 2021a,b). Through a Riemann integration process, this complexification procedure gives rise to a new scale factor, denoted as $\ln^{-1}[\beta(t)]$, which characterizes a topological foliated spacetime structure.

In this work, based on a recently developed commutative and noncommutative formulation of the Poisson algebra, which combines the branch-cut cosmology, the Wheeler–DeWitt equation and the Hořava–Lifshitz quantum gravity (Bodmann et al. 2023a,b; Weber et al. 2024a,b). We will show that the introduction of a noncommutative structure, which is equivalent to introducing a minimal length will lead to sensible changes in the development of the universe.

Considering a mini-superspace framework, we study the implications of an inflaton-type scalar field and the corresponding potential in the acceleration of the Universe. Given a set of elements $a, b, c \ldots$, usually over the field of real or complex numbers, equipped with a bilinear mapping, a Poisson algebra A corresponds to an associative algebra together with a Lie algebra structure $\{,\}$, that satisfies Leibniz's law:

$$\{a, bc\} = \{a, b\}c + b\{a, c\}, \tag{1}$$

with $a, b, c \ldots \in A$. The element $\{a, b\}$ is called the Poisson bracket of a and b. The Poisson algebra is generally associated with a associative commutative algebra over a commutative ring R whose structure on A is defined by an R-bilinear skew-symmetric mapping $\{.,.\} : A \times A \to A$, such that $(A, \{.,.\})$ is a Lie algebra over R. The Poisson algebra however can be extended to the noncommutative environment, as addressed in this work.

2 | NONCOMMUTATIVE BRANCH-CUT QUANTUM GRAVITY

The starting point of this study is a recently developed noncommutative formulation of branch-cut cosmology based on the Wheeler–DeWitt–Hořava–Lifshitz quantum gravity (Bodmann et al. 2023a,b; Weber et al. 2024a,b), a renormalizable theory that obeys Lorentz invariance at low energy while breaking this symmetry at high energy is a consequence of the implicit presence of a minimal length (Bertolami & Zarro 2011; Hořava 2009).

In the BCG formulation, the corresponding quantum action $\mathcal{S}_{BCG}$ depends on the scalar curvature of the branched Universe, $\mathcal{R}$, and on its derivatives, in different orders (Abreu et al. 2019; Bertolami & Zarro 2011; Bodmann et al. 2023a,b; Cordero et al. 2019; García-Compe & Mata-Pacheco 2022; Hess et al. 2022; Hořava 2009; Vieira et al. 2020). Based on this formulation, we investigate the effects of noncommutativity in a mini-superspace of variables obeying Poisson's algebra on the accelerating behavior of the wave function of the Universe (Bodmann et al. 2023a,b).

The development of a formalism based on a noncommutative algebra, follows the paths shown in (Abreu et al. 2019): (a) The insertion, in the Hořava–Lifshitz formalism, of the action of a perfect fluid, characterized by a dimensionless number ω, associated with the variable $v(t)$, a quantum variable that spans, with $u(t)$, dual reciprocal spaces, and whose canonically conjugated momentum is represented by p_v. (b) The former commutative variables $\{u, p_u, v, p_v\}$ satisfy now a noncommutative algebra, defined as:

$$\{u, v\} = \sigma \ ; \quad \{p_u, p_v\} = \alpha \ ; \quad \{u, p_v\} = \gamma;$$
$$\{v, p_u\} = \chi \ ; \quad \{u, p_u\} = \{v, p_v\} = 1, \tag{2}$$

where p_u and p_v represent the canonically conjugated momenta associated to u and v.

The final step in building the formalism is (c) to carry out a linear transformation of the original noncommutative phase space configuration into a commutative representation. This transformation allows the incorporation of the noncommutative algebra, through the insertion of the new set of variables $\{\tilde{u}, \tilde{p}_u, \tilde{v}, \tilde{p}_v\}$ and the consequent modification of the quantum gravity phase space, into the intrinsic structure of the cosmic quantum dynamics. In this new set, the momentum variables can be substituted by derivatives in terms of the coordinates, as done in the standard quantization procedure.

We introduce a mapping which relates the commutative $\{\tilde{u}, \tilde{p}_u, \tilde{v}, \tilde{p}_v\}$ and the noncommutative phase space set of variables $\{u, p_u, v, p_v\}$ and we adopt a canonical quantization procedure for the variables $u(t)$ and $v(t)$ along with their corresponding conjugate momenta p_{u} and p_{v}, in order to be treated as quantum operators (for details, please consult Bodmann et al. (2023a)).

For convenience, we choose $\sigma = 0$, which results in $\tilde{u} = u$ and $\tilde{v} = v$. Furthermore, from now on, we skip the tilde symbol of the commuting variables. We adopt the canonical quantization procedure, arriving at a canonical super-Hamiltonian equation for the wave function of the Universe on the basis of a characteristic equation (Bodmann et al. 2023a),

$$\mathcal{H}(\xi,\eta)\Psi(\xi,\eta) = \frac{1}{2}\frac{N}{\eta}\Bigg[-p^2_{\eta,\gamma,\alpha} + g_r - g_m\eta - g_k\eta^2 - g_q\eta^3 + g_\Lambda\eta^4 + \frac{g_s}{\eta^2} + \frac{\alpha}{\eta^{3\alpha-2}} - \frac{\alpha\xi}{\eta^{3\alpha-1}} + \frac{1}{\eta^{3\alpha-1}}p_\xi\Bigg]\Psi(\xi,\eta) = 0, \quad (3)$$

where $-p^2_{\eta,\gamma,\alpha}$ is defined as

$$-p^2_{\eta,\gamma,\alpha} \equiv \frac{\partial^2}{\partial\eta^2} + \frac{\gamma}{\eta^{3\alpha-1}}\frac{\partial}{\partial\eta} = -\left(-i\frac{\partial}{\partial\eta}\right)\left(-i\frac{\partial}{\partial\eta}\right) + \frac{i|\gamma|}{\eta^{3\alpha-1}}\frac{\partial}{\partial\eta} \equiv -p^2_\eta - p_{\eta,\gamma,\alpha}, \quad (4)$$

with

$$p_{\eta,\gamma,\alpha} \equiv -\frac{i|\gamma|}{\eta^{3\alpha-1}}\frac{\partial}{\partial\eta} \mapsto \frac{\gamma}{\eta^{3\alpha-1}}\left(-i\frac{\partial}{\partial\eta}\right) = \frac{\gamma}{\eta^{3\alpha-1}}p_\eta. \quad (5)$$

In expression (3) the following change of the variables u and v in terms of new independent variables ξ and η was introduced:

$$\xi = \varphi(u,v), \quad \text{and} \quad \eta = \eta(u,v), \quad (6)$$

where $\eta(u,v)$ is a differentiable function that satisfies the nondegeneracy condition of the Jacobian $D(\xi,\eta)/D(u,v)$ in the given domain and $\varphi(u,v)$ represents the general integral.

Still with respect to expression (3), g_i denote running coupling constants: g_k, g_Λ, g_r, and g_s represent respectively the curvature, cosmological constant, radiation, and stiff matter running coupling constants (Bertolami & Zarro 2011; Maeda et al. 2010), $g_m u$ describes the contribution of baryon matter combined with dark matter, and $g_q u^3$ denotes a quintessence contribution. The g_r, and g_s running coupling constants can be positive or negative, without affecting the stability of the solutions. Because $\sigma = 0$ we can set $\eta = u$ and $\xi = v$.

Stiff matter contribution in turn is determined by the $p = \omega\rho$ condition in the corresponding equation of state.

The parameterization of curvature, cosmological constant, radiation, stiff matter, baryon matter combined with dark matter, and quintessence running coupling constants are in tune with the Wilkinson Microwave Anisotropy Probe (WMAP) observations (Hinshaw et al. 2013).

The solutions of the WdW equation, represented by a geometric functional of compact manifolds and matter fields, describe the evolution of the quantum wave function of the Universe (Hartle & Hawking 1983; Hawking 1982), $\Psi(\eta,\xi)$.

The corresponding expression for the wave function of the Universe is defined here in terms of the variables $\eta(t)$ that characterize the scale factor of the branch-cut cosmology and its dual complementary counterpart, represented by $\xi(t)$. The meaning of this solution in the BCG allows, under certain conditions, the separation of variables in such a way that the final solution $\Psi(\eta,\xi)$ can be written in the form of a product $\Psi(\eta)\Psi(\xi)$ so that the corresponding component solutions represent, in the same way as the variables η and ξ, complementary dual spaces solutions. Said simply, its meaning can be summarized as: the wave function describes different states of the branch-cut Universe which corresponds to different stages of its evolution. Another possible interpretation: the wave function is a probability amplitude for the Universe to have some space geometry, or to be found in some point of the Wheeler mini superspace, evolving as a function of the variables η and ξ.

3 | MINI-SUPERSPACE OF VARIABLES

We consider in the following a mini-superspace of variables $(\eta(t), \xi(t), \phi(t))$, where $\phi(t)$ represents the scalar inflaton quantum field. We adopt for the action of the scalar field the following expression (Kiritsis & Kofinas 2009; Tavakoli et al. 2021)

$$S_\phi = \int_{\mathcal{M}} d^3x\, dt\, N\sqrt{g}\left[\frac{1}{N^2}\left(\dot{\phi} - N^i\partial_i\phi\right)^2 - \mathcal{V}(\partial_i\phi,\phi)\right]. \quad (7)$$

Assuming homogeneous and isotropic cosmological settings we have $N_i = 0$ (Kiritsis & Kofinas 2009; Tavakoli et al. 2021), and the action of the scalar field $\phi(t)$, given by expression (7), may be written as

$$S_\phi = \int_{\mathcal{M}} d^3x\, dt\, N\sqrt{g}\left(\frac{1}{2}\frac{1}{N^2}F(\phi)\dot{\phi}^2 - V(\phi)\right), \tag{8}$$

with $V(\phi)$ denoting the inflation potential and where $F(\phi)$ represents a coupling function. In the following, from the total action determined by adding the Hořava–Lifshitz and the scalar field actions, the Hamiltonian associated with the mini-superspace of variables may be obtained.

3.1 | Noncommutative Hamiltonian

As noted above, the η variable can be identified with the scale parameter of the universe, u, and ξ, its complementary dual counterpart, with v. We are familiar with the fact that, in quantum mechanics, energy-momentum and space-time correspond to dual and complementary vector spaces. In noncommutative branch-cut cosmology, $\eta - \xi$ scan, in the same way as in standard quantum mechanics, complementary dual vector spaces, bringing to light the complexity of an algebraic structure that captures small and large spatial-temporal scales. The question that arises, and which we do not intend to discuss in this work, in view of the comparison with standard quantum mechanics, back to Snyder's vision for a minimum space-time scale (Snyder 1947), is whether these dual variables are coupled through a minimum uncertainty-like relation.

The momenta conjugate to the dynamical variables $(\eta(t), \xi(t), \phi(t))$ can be obtained by definition as $p_q = \partial L/\partial \dot{q}$, where L defines the total Lagrangian of the system, resulting in

$$p_\eta = -\frac{1}{N}\eta\dot{\eta}\ ; p_\xi = -\frac{1}{N}\xi\dot{\xi}\ ; \quad \text{and} \quad p_\phi = \frac{1}{N}F(\phi)u^3\dot{\phi}. \tag{9}$$

The total Hamiltonian then reads

$$\begin{aligned}\mathcal{H} = &\frac{1}{2}\frac{N}{\eta}\left[-p^2_{\eta,\gamma,\alpha} + g_r - g_m\eta - g_k\eta^2 - g_q\eta^3 + g_\Lambda\eta^4 + \frac{g_s}{\eta^2}\right]\\ &+ \frac{1}{2}\frac{N}{\eta}\left[\frac{\alpha}{\eta^{3\alpha-2}} - \frac{\alpha\xi}{\eta^{3\alpha-1}} + \frac{1}{\eta^{3\alpha-1}}p_\xi\right]\\ &+ \frac{1}{2}\frac{N}{\eta}\left[\frac{1}{u^{3\omega-1}F(\phi)}p^2_\phi + 2V(\phi)\right].\end{aligned} \tag{10}$$

In the following we promote the canonical conjugate momenta p_η, p_ξ and p_ϕ into operators

$$p_\eta \to -i\frac{\partial}{\partial\eta}; \quad p_\xi \to -i\frac{\partial}{\partial\xi}; \quad \text{and} \quad p_\phi \to -i\frac{\partial}{\partial\phi}. \tag{11}$$

For $\omega = \alpha = 1/3$, the condition $H\Psi(\eta, \xi, \phi) = 0$ implies the following separable equations

$$\left(\frac{\partial^2}{\partial\eta^2} + \gamma\frac{\partial}{\partial\eta} + \tilde{g}_r - \tilde{g}_m\eta - g_k\eta^2 - g_q\eta^3 + g_\Lambda\eta^4 + \frac{g_s}{\eta^2}\right)\Psi(\eta) = 0; \tag{12}$$

with $\tilde{g}_r \equiv g_r - C$ and $\tilde{g}_m \equiv g_m - 1/3$,

$$\left(i\frac{\partial}{\partial\xi} - \frac{1}{3}\xi + C\right)\Psi(\xi) = 0; \tag{13}$$

and

$$\left(\frac{1}{F(\phi)}p^2_\phi + 2V(\phi) - C\right)\Psi(\phi) = 0. \tag{14}$$

In these expressions, $\Psi(\eta, \xi, \phi) = \Psi(\eta)\Psi(\xi)\Psi(\phi)$ represents the wave function of the Universe in a separable form. Following the interpretation given previously for the wave function $\Psi(\eta, \xi)$, here η, ξ, and ϕ scan a complementary dual space of variables and its solutions also scan, from the point of view of quantum gravitation, a dual and complementary space of solutions for the wave function of the branch-cut Universe.

In Figure 1, we present the solutions $\Psi(\eta)$ of Equation (12), for different initial conditions and a collection of parameters.[1] In the left panel, the wave function increases continuously in the negative sector of the scale η $(= u)$ of the universe. The right figure shows the positive sector of η. A strong increase of the wave function indicates an accelerating behavior of the universe. Note, the significant change provoked by a change of sign of the γ parameter. In Figure 2 the function $\Psi(\xi)$ is plotted, for $\alpha = 1/3$ and $C = 1$, for different ranges, reflecting again a significant change in the properties of the wave function.

Here, we have to make a comment on the interpretation of the wave function, which is not solved yet definitively. The result, presented here, gives only a rough idea of what is happening. In a future contribution (in preparation) we intend to discuss this matter in further detail.

3.2 | Modeling chaotic and nonchaotic inflation

In what follows, we assume a polynomial coupling function for the scalar field, $F(\phi) = \lambda\phi^m$, and we model chaotic and nonchaotic inflation. We model chaotic inflation by

[1] In order to solve the differential equations, we used MATHEMATICA PRO PREMIUM (Wolfram Research, Inc., 2023), which enabled us to obtain the solutions without recurring to approximations.

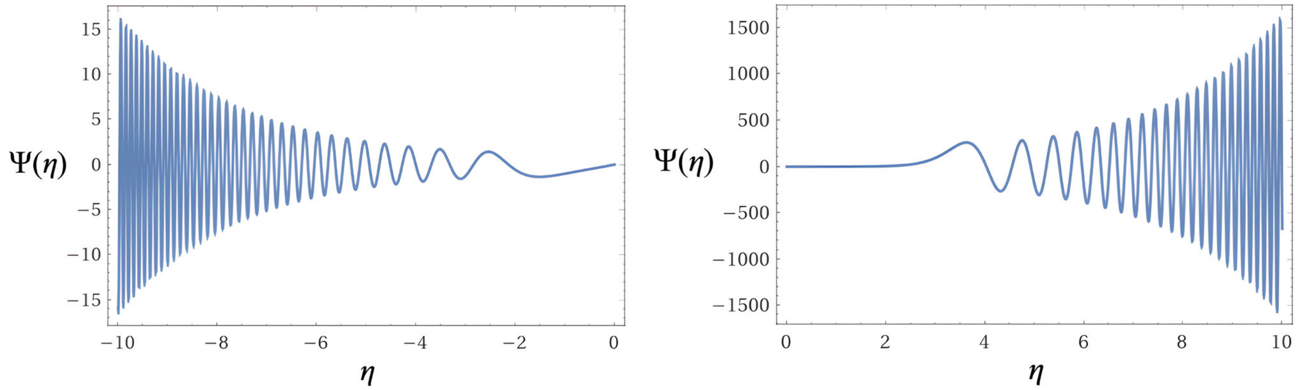

FIGURE 1 Typical individual solutions of the $\eta(t)$-component ($\Psi(\eta)$) of the wave function of the Universe, $\Psi(\eta, \xi, \phi)$, using the noncommutative approach given by Equation (12) and the boundary conditions $\Psi(-1) = -1$, on the left figure, and $\Psi(1) = 1$ on the right figure. Concerning the α, χ, and γ set of parameter of the noncommutative Poisson algebra, $\alpha = 1/3$, χ is implicitly considered in the structure of the scale factor η and its dual counterpart ξ as a result of the variable transformation and imposition of a canonical solution. With respect to the γ parameter, $\gamma = -1$ in both figures. In the calculation of Equation (12), the parameter values are: $\tilde{g}_r = 0.4$; $\tilde{g}_m = 0.6138$; $g_k = 1$; $g_q = 0.7$; $g_\Lambda = 0.333$; $g_s = 0.03$.

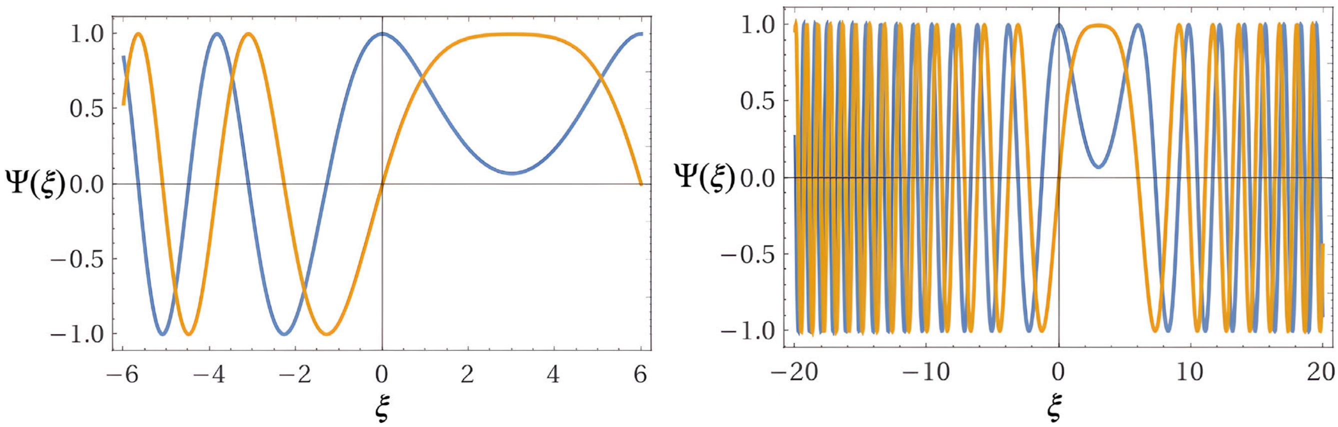

FIGURE 2 Typical individual solutions of the ξ-component ($\Psi(\xi)$) of the wave function of the Universe, $\Psi(\eta, \xi, \phi)$, for the noncommutative approach according to Equation (13), and different ranges.

using the potential $V(\phi) = \frac{1}{2} g_\phi^2 \phi^2$, so Equation (14) then reads

$$\left[-\frac{\partial^2}{\partial \phi^2} + \lambda \phi^m \left(g_\phi^2 \phi^2 - C\right)\right] \Psi(\phi) = 0. \tag{15}$$

For the nonchaotic inflation we adopt in the following the Fubini potential to simulate inflation

$$V(\phi) = \frac{\beta}{4}(\phi - \phi_c)^4 - \frac{1}{2} g_\phi^2 (\phi - \phi_c)^2. \tag{16}$$

Combining this equation with expression (16), we obtain

$$\left[\frac{\partial^2}{\partial \phi^2} - \lambda \phi^m \left(\frac{\beta}{2}(\phi - \phi_c)^4 - g_\phi^2 (\phi - \phi_c)^2 - C\right)\right] \Psi(\phi) = 0. \tag{17}$$

In Figure 3 two types of potentials are plotted, the left one for the chaotic inflation and the right one for the original nonchaotic inflationary model. For reasons of completeness, Figure 4 show typical individual solutions for $\Psi(\phi)$, the Φ-component of the wave function of the Universe, $\Psi(u, \phi)$. The figure on the left corresponds to chaotic inflation according to Equation (15), with the initial condition $\Psi(1) = 1$ and $\lambda g_\phi^2 < 0$. In turn, the figure on the right corresponds to nonchaotic inflation, according to Equation (17), with the initial condition $\Psi(1) = 1$, $\lambda\beta < 0$, $\lambda g_\phi^2 < 0$, and $\lambda C < 0$. Although the solutions exhibit some similarity to Figure 1, it is important to highlight that the presence of the inflaton field anticipates the effects of the acceleration of the Universe towards the boundary region of separation between the two evolutionary cosmic phases: the current phase and its mirror counterpart. This is in comparison with results that do not include the presence of this scalar field. Also here, the wave equations are solved without recurring to approximations. The left

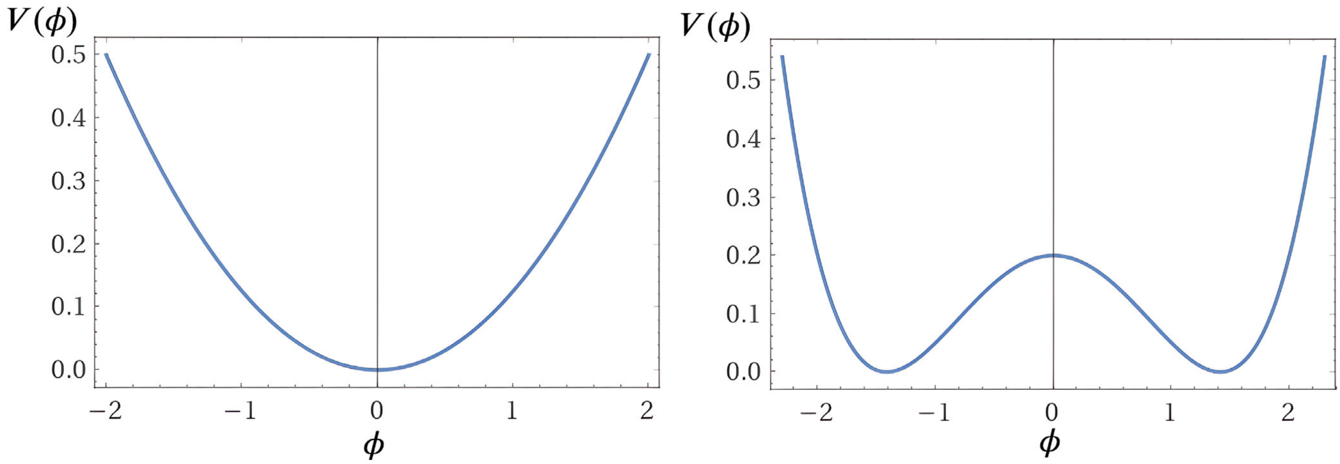

FIGURE 3 On the left, generic form of the potential for the chaotic inflationary scenario. On the right, a typical form of the potential for the original nonchaotic inflationary model, based on the Fubini proposal (de Alfaro et al. 1976).

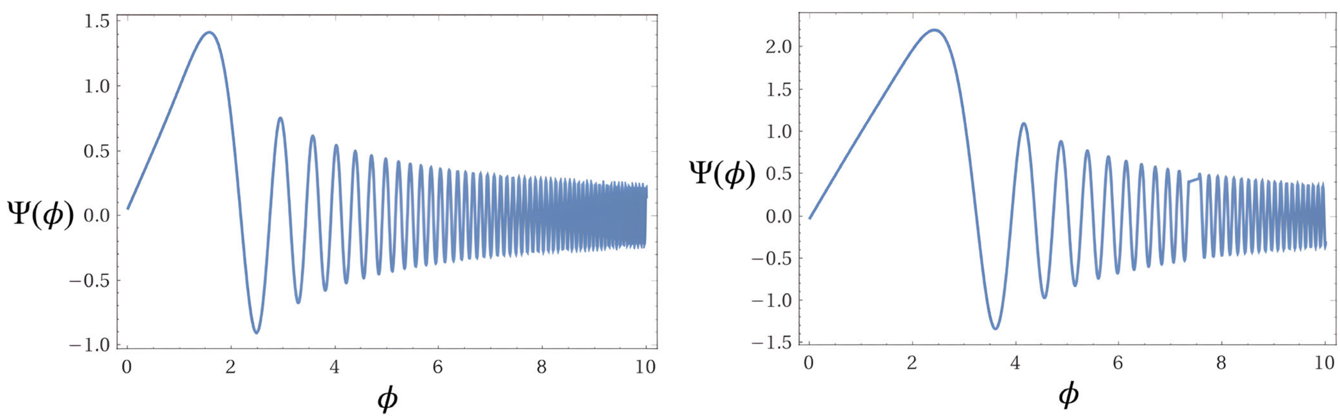

FIGURE 4 For reasons of completeness, the figures show typical individual solutions for $\Psi(\phi)$, the Φ-component of the wave function of the Universe, $\Psi(u, \phi)$. The figure on the left corresponds to chaotic inflation according to Equation (15), with the initial condition $\Psi(1) = 1$ and $\lambda g_\phi^2 < 0$. The figure on the right corresponds to nonchaotic inflation, according to Equation (17), with the initial condition $\Psi(1) = 1$, $\lambda\beta < 0$, $\lambda g_\phi^2 < 0$, and $\lambda C < 0$. Although the solutions exhibit some similarity to Figure 1, it is important to highlight that the presence of the inflaton field anticipates the effects of the acceleration of the Universe towards the boundary region of separation between the two evolutionary cosmic phases: the current phase and its mirror counterpart. This is in comparison with results that do not include the presence of this scalar field.

figure is for a negative λg_ϕ^2 and the right one for a positive λg_ϕ^2. The behavior of $\Psi(\phi)$ is quite distinct for the two cases. While for a negative λg_ϕ^2 the wave function shows an oscillatory behavior, for a positive λg_ϕ^2 it is exploding for positive ϕ. The oscillatory behavior is a remnant of the phase contribution, that is, $\Psi(\phi) = |\Psi(\phi)|e^{i\mu}$ with μ as the phase. The real part is then $|\Psi(\phi)|\cos(\mu)$. For the interpretation of the wave function as a probability distribution, only the absolute value of the wave function matters.

A crucial point in our results: these solutions are similar to the commutative formulation considering that the inflaton field was not inserted in the original algebraic structure, being introduced into the formalism in an ad hoc manner. This raises a relevant question to be considered in a future contribution to this volume. There is still the question of the correct interpretation of the wave function. This is a still unresolved issue and we plan to investigate it in a future contribution.

4 | SUMMARY AND FINAL REMARKS

One main motivation was to study the effects of a noncommutative Poisson algebra of the coordinates and momenta on the evolution of the universe. We demonstrated that the effects are important and change the outcome, compared to the commutative structure.

We have studied the properties of the wave function of the universe, taking into account the effects of a noncommutative Poisson algebra for the coordinates and momenta. This corresponds to effectively introducing a minimal length scale. A transformation was used to obtain a new set of coordinates and momenta, satisfying the standard algebra. The Hamiltonian in terms of the new variables was constructed and the corresponding Wheeler–DeWitt equations derived. These equations were solved without recurring to approximations. Different scenarios were studied, varying not only the parameters involved but also the initial conditions.

Which of the set of parameters finally describes our universe and which initial condition has to be applied, is still out of our possibilities to determine. Nevertheless, we hope that the results presented can help to understand better the BCG but also the effects of a minimal length on the evolution of the universe, leading at the end to testable predictions.

AUTHOR CONTRIBUTIONS

Conceptualization: C.A.Z.V.; *Methodology*: C.A.Z.V., B.A.L.B., P.O.H., J.A.deF.P., D.H., F.W., and M.M.; *Software*: C.A.Z.V., B.A.L.B., M.R., and M.M.; *Validation*: C.A.Z.V., B.A.L.B., D.H., P.O.H., J.A.deF.P., and F.W.; *Formal analysis*: C.A.Z.V., B.A.L.B., P.O.H., J.A.deF.P., D.H., and F.W.; *Investigation*: C.A.Z.V., B.A.L.B., P.O.H., J.A.deF.P., M.R., G.A.D., M.M., and F.W.; *Resources*: C.A.Z.V.; *Data curation*: C.A.Z.V. and B.A.L.B.; *Writing–original draft preparation*: C.A.Z.V.; *Writing—review and editing*: C.A.Z.V., B.A.L.B., P.O.H., J.A.deF.P., D.H., G.A.D., M.R., M.M., and F.W.; *Visualization*: C.A.Z.V. and B.A.L.B.; *Supervision*: C.A.Z.V.; *Project administration*: C.A.Z.V. All authors have read and agreed to the published version of the article.

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