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Uniqueness in an inverse problem of fractional elasticity

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We study a nonlinear inverse problem for fractional elasticity. In analogy to the classical problem of linear elasticity, we consider the unique recovery of the Lamé parameters associated with a linear, isotropic fractional elasticity operator from fractional Dirichlet-to-Neumann data. In our analysis, we make use of a fractional matrix Schrödinger equation via a generalization of the so-called Liouville reduction to the case of fractional elasticity. We conclude that unique recovery is possible if the Lamé parameters agree and are constant in the exterior, and their Poisson ratios agree everywhere. Our study is motivated by the significant recent activity in the field of nonlocal elasticity.

1. Introduction

(a) The inverse problem for isotropic fractional elasticity

Inspired by the recent special issue on nonlocal elasticity of the Philosophical Transactions of the Royal Society [1], in this paper, we consider an inverse problem for a fractional elasticity operator of the form

$$\mathbf{E}^s u := (\nabla\cdot)^s(\mathcal{C}(x,y)\nabla^s u)$$

for a fixed $s\in(0,1)$. The fractional gradient ∇^s and the fractional divergence $(\nabla\cdot)^s$ appearing in the definition of $\mathbf{E}^s$ go back to [2] and they will be defined in detail in §2. They can be thought of as nonlocal counterparts of the classical gradient and divergence operators, with the expected properties $(\nabla^s)^* = (\nabla\cdot)^s$ and $(\nabla\cdot)^s\nabla^s = (-\Delta)^s$.

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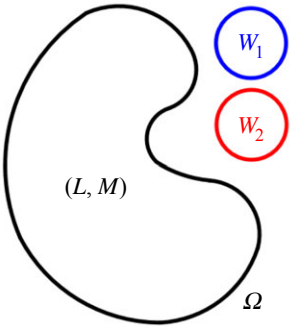
The tensor $\mathcal{C}(x,y)$ is itself a nonlocal counterpart of the classical stiffness tensor C. Section 3 is dedicated to the detailed definition of $\mathbf{E}^s$.



We make the assumption that C is isotropic, that is, its properties are completely described by two Lamé parameters L, M (see §2e). The scalar functions L, M are assumed to be constant outside a domain of interest Ω, and unknown within it. The goal of the inverse problem we wish to study is to recover L, M (and thus C) within Ω from measurements performed on its exterior $\Omega_e := \mathbb{R}^n \setminus \overline{\Omega}$. To this end, we start by considering the direct problem

$$\mathbf{E}^s u = 0 \quad \text{in } \Omega$$

and

$$u = f \quad \text{in } \Omega_e,$$

for which we prove well-posedness in a weak sense for any sufficiently regular exterior datum f. This means that given any f defined on Ω_e (e.g. we might have $f \in C_c^\infty(\Omega_e)$) there is one and only one solution $u_f \in H^s(\mathbb{R}^n)$ to the direct problem. Building on this, we define the exterior measurements as a nonlocal Dirichlet-to-Neumann (DN) map $\Lambda_{L,M}$, associating each exterior datum f to the corresponding nonlocal Neumann data $\mathbf{E}^s u_f|_{\Omega_e}$. It is clear that $\Lambda_{L,M}$ carries information about the Lamé parameters L, M, as these are involved in the definition of the unique solution u_f. More specifically, we would like to recover L, M within Ω from partial data, that is, from the knowledge of $\Lambda_{L,M} f|_{W_2}$ for all $f \in C_c^\infty(W_1)$, where W_1, W_2 are non-empty, open and disjoint subsets of Ω_e. This means that the exterior data f will be supported in W_1, while the measurements will be performed on W_2 only. These restrictions represent the physical situation in which not all of the exterior is accessible for measurement. We thus ask the following inverse problem:

Q: *Does* $\Lambda_{L_1,M_1} f|_{W_2} = \Lambda_{L_2,M_2} f|_{W_2}$ *for all* $f \in C_c^\infty(W_1)$ *imply that* $L_1 = L_2$ *and* $M_1 = M_2$ within Ω?

Note that this inverse problem is strongly nonlinear, because the measurements depend in a nonlinear way on the coefficients of the equation.

(b) Related Calderón problems

Before entering in the details of the discussion of the problem at hand, in this subsection, we briefly describe the inverse problems of classical and fractional conductivity, as well as that of classical elasticity. These are related to the problem we consider and share interesting common features. We leave an accurate literature discussion of fractional gradient elasticity to the coming §2d.

The objective of the inverse problem for the classical conductivity equation, which is also known as the *Calderón problem*, consists of finding an unknown conductivity γ within a bounded

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domain Ω using measurements performed on the boundary $\partial\Omega$. Here, the potential u is a scalar function solving the direct problem

$$\nabla \cdot (\gamma \nabla u) = 0 \quad \text{in } \Omega$$

and

$$u = f \quad \text{on } \partial\Omega,$$

and the measurements take the form of the (local) DN map

$$\Lambda_\gamma : H^{1/2}(\partial\Omega) \ni u|_{\partial\Omega} \mapsto \gamma \nabla u|_{\partial\Omega} \in H^{-1/2}(\partial\Omega).$$

One can consider this problem in both the isotropic and anisotropic cases, which correspond to scalar and matrix valued conductivities. We refer to the survey [3] and to the references therein for a detailed discussion of the main results and techniques involved in the study of the classical Calderón problem.

The *fractional Calderón problem* has been introduced in the seminal paper [4, preprint published in 2016] as the inverse problem related to the fractional Schrödinger operator $(-\Delta)^s + q$. It was later shown [5] that one can define a fractional conductivity operator related to the fractional Schrödinger one via a nonlocal Liouville reduction. The fractional conductivity equation is

$$\mathbf{C}^s_\gamma u := (\nabla \cdot)^s(\Theta \cdot \nabla^s u) = 0 \quad \text{in } \Omega$$

and

$$u = f \quad \text{in } \Omega_e,$$

where $(\nabla \cdot)^s, \nabla^s$ are the fractional divergence and gradient (see §2), and the potential u is a scalar function. When the conductivity matrix verifies $\Theta(x,y) = \gamma^{1/2}(x)\gamma^{1/2}(y)Id$, i.e. it is a separable, symmetric function and a multiple of the identity matrix, we say that the problem is isotropic. The anisotropic problem results from considering a matrix Θ of general form. The inverse problem in this case consists in recovering Θ from the (nonlocal) DN-map

$$\Lambda^s_\gamma : H^s(W) \ni u|_W \mapsto \mathbf{C}^s_\gamma u|_W \in H^{-s}(W),$$

where $W \subset \Omega_e$ is open, bounded and such that $\overline{\Omega} \cap \overline{W} = \emptyset$. In light of the cited fractional Liouville reduction, the results available for the inverse problem related to the fractional conductivity operator are strongly linked to the ones for the fractional Calderón problem. We refer to the survey paper [6] and §2d for many references on the topic.

Finally, the aim of the inverse problem for classical elasticity is to determine the stiffness tensor C within a bounded domain Ω from boundary measurements of stresses and deformations. In the special case of isotropic elasticity, the stiffness tensor is completely determined by two scalar functions λ, μ called Lamé parameters, which are thus to be recovered from measurements. The direct problem takes the form

$$\nabla \cdot (C : \nabla u) = 0 \quad \text{in } \Omega$$

and

$$u = f \quad \text{on } \partial\Omega,$$

while the DN-map is defined as

$$\Lambda_C : H^{1/2}(\partial\Omega) \ni u|_{\partial\Omega} \mapsto (C : \nabla u)|_{\partial\Omega} \in H^{-1/2}(\partial\Omega).$$

We observe that in this case the function u represents a deformation, and is thus of vectorial nature. Accordingly, the vector equation above can rather be seen as a system of scalar equations. For a concise exposition of the main concepts of the theory of classical elasticity which will be needed in our arguments, we refer to §2e. Section 2d contains many references on the inverse problem for classical isotropic elasticity.

After this brief discussion, it is clear that the inverse problem we are interested in is at once a fractional counterpart of the inverse problem of classical elasticity (in much the same way as the

fractional Calderón problem is related to the classical one), and a generalization of the fractional Calderón problem to a specific vectorial case (as for the local inverse problems for conductivity and elasticity).

(c) Main results and techniques

We build on the work of [4]. We will first use the assumption on the DN maps in order to obtain an integral identity, the so-called Alessandrini identity, relating the difference of the DN maps to the differences of the Lamé parameters via some special solutions to the direct problem. Then, we shall test the Alessandrini identity with aptly chosen solutions in order to deduce the desired result. Such solutions will be produced by means of a Runge approximation property, which we will prove for our equation. The proof of the Runge approximation property will itself rely on a unique continuation property (UCP), which is the key point of the technique.

However, the approach described above does not directly lead to the desired result. It is instead necessary to first reduce the given problem to a more manageable one of Schrödinger type. We do so by means of the so-called *fractional Liouville reduction*, which is reminiscent of the techniques used for both the classical and fractional conductivity equations (see [5] for the latter), but has so far not been used in the context of elasticity. One of the principal contributions of this work is precisely the introduction of a new, sophisticated vectorial Liouville reduction adapted to the problem of fractional elasticity. After the reduction, the direct problem reads

$$(-\Delta)^{s-1} Dw - w \cdot Q = 0 \quad \text{in } \Omega$$

and

$$w = g \quad \text{in } \Omega_e,$$

where D is a fixed differential operator of second order, Q is a new potential containing information relative to the Lamé parameters L, M, the new exterior datum g is computed from f, and the new solution w is computed from u (see §5 for the details).

The reduction from the fractional elasticity equation to matrix Schrödinger equation is possible for arbitrary Lamé coefficients. However, in order to make a reduction on the level of exterior measurements and to derive a suitable integral identity, we need to assume that the Poisson ratio is a fixed (unknown) function:

Definition 1.1. If $(L_1, M_1), (L_2, M_2)$ are two couples of Lamé parameters, we write $(L_1, M_1) \sim (L_2, M_2)$ if and only if they coincide on Ω_e and $\nu_1 = \nu_2$ on $\mathbb{R}^n$, where the Poisson ratio ν relative to a couple of Lamé parameters (L, M) is $\nu := L/((n-1)L + 2M)$ (see §2e for the precise definition).

As it turns out, if the Poisson ratio is a fixed function, then the technique described above will work for the transformed problem and will allow us to determine the matrix potential Q from the nonlocal DN map. We can then recover the Lamé parameters (L, M) from Q and prove the following uniqueness result.

Theorem 1.2. *Let $\Omega, W_1, W_2 \subset \mathbb{R}^n$ be bounded open sets such that $W_1, W_2 \subset \Omega_e$, and assume $s \in (0, 1)$. Let (L_1, M_1) and (L_2, M_2) be two couples of Lamé parameters satisfying assumptions (A 1)–(A 3). If $(L_1, M_1) \sim (L_2, M_2)$ and*

$$\Lambda_{L_1, M_1} f|_{W_2} = \Lambda_{L_2, M_2} f|_{W_2} \quad \text{for all } f \in C_c^\infty(W_1),$$

then $(L_1, M_1) = (L_2, M_2)$.

Here assumptions (A 1)–(A 3) state that L, M are $(2s + \varepsilon)$-Hölder regular, constant outside of Ω and enjoy a certain positivity condition (see §3).

(d) Background in fractional gradient elasticity

We consider nonlocal elasticity, that is, space-nonlocality, when nonlocal stress is defined as the Riesz fractional integral of the strain field in space. There exist several complex phenomena,

occurring possibly in damage zones [7], that cannot be addressed by classical local continua. For a recent overview, we refer to [8]. While the notion of nonlocal elasticity dates back to the work of Mindlin [9,10] (who considered a variational formulation resulting in stable and well-posed solutions of boundary value problems, enabling the removal of singularities from dislocations and cracks), we focus on a fractional generalization of the classical Eringen integral model [11,12] of nonlocality. That is, we consider nonlocal elasticity based on the so-called fractional (or generalized) linear 'gradient'-elasticity model [13,14]

$$\boldsymbol{\sigma} = C : [\boldsymbol{\epsilon} + a_s(-\Delta)^s \boldsymbol{\epsilon}], \quad a_s = \ell_s^2,$$

where $(-\Delta)^s$ is a fractional Laplacian in the Riesz form, $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, ℓ_s is a material parameter signifying an internal length scale, $\boldsymbol{\sigma}$ is the stress tensor, and $\boldsymbol{\epsilon}$ is the strain tensor. Assuming that the stiffness tensor C is isotropic and has constant coefficients results in the following fractional elasticity operator $\mathbf{T}^s$ from [14]:

$$\mathbf{T}^s u := \nabla \cdot (C(1 + a_s(-\Delta)^s)\nabla u) = \nabla \cdot (C\nabla u) + a_s \nabla \cdot (C(-\Delta)^s \nabla u).$$

The operator $\mathbf{E}^s$ which we study in the present paper generalizes $\mathbf{T}^s$ to the case of variable coefficients (see §2c for the precise definitions of the fractional operators used here). In order to see this, compute

$$\mathbf{E}^{s+1} u := (\nabla \cdot)^{s+1}(C^{1/2}(x) : C^{1/2}(y)\nabla^{s+1} u) = \nabla \cdot \{(\nabla \cdot)^s C^{1/2}(x) : C^{1/2}(y)\nabla^s \nabla u\},$$

where the relations $\nabla^{s+1} = \nabla^s \nabla$ and the corresponding one for the fractional divergence are definitions, in accordance to [15]. If the coefficients are assumed to be constant, a straightforward computation shows that

$$\mathbf{E}^{s+1} u = \nabla \cdot ((\nabla \cdot)^s C \nabla^s \nabla u) = \nabla \cdot (C(\nabla \cdot)^s \nabla^s \nabla u) = \nabla \cdot (C(-\Delta)^s \nabla u),$$

and thus

$$\mathbf{T}^s u = a_s \mathbf{E}^{s+1} u + \nabla \cdot (C\nabla u).$$

In an alternative introduction of a model for fractional elasticity directly through a kernel, one develops a fractional Taylor series using the Caputo fractional derivative (involving a left-sided Riemann–Liouville fractional integral) of its Fourier transform [16,17]. The physical basis of such an introduction is an assumption pertaining to fractional spatial dispersion for a nonlocal elastic continuum.

Zorica & Oparnica [18] presented time-fractional wave equations that model hereditary viscoelastic behaviour and space-fractional wave equations associated with certain nonlocal elasticity models. For a number of fractional wave equations, the authors provided mathematical evidence of energy dissipation and conservation.

An inverse problem with nonlocal elasticity was considered by Askes & Aifantis [19]. For an important overview of general fractional derivative equations, containing fractional time and space derivatives, and inverse problems examining ill-posedness in space dimension 1, we refer to Jin & Rundell [20].

The theory of linear elasticity has given rise to classical inverse problems which are related to our question, and can rather be considered the main inspiration of our study. We briefly present the main concepts of the classical theory of linear elasticity in §2e (see also [21]). The operator of classical elasticity is

$$\mathbf{E} u := \nabla \cdot (C\nabla u),$$

where C is the stiffness tensor. Since $\mathbf{E}$ is a local operator, the measurements involved in the inverse problems are performed on the boundary $\partial\Omega$ of the domain Ω rather than in its exterior Ω_e. Most of the results concern the isotropic case where C depends on two scalar Lamé parameters L and M. The main uniqueness results for this inverse problem in three dimensions are in [22–25]. They state that if M is sufficiently close to a constant, or if either L or M is a fixed (unknown) function, then the Lamé parameters are uniquely determined by the boundary measurements. In [25], one also finds the classical analogue of theorem 1.2 stating that the Lamé

parameters are determined if the Poisson ratio is a fixed function. The two-dimensional case is considered in [26]. Many other different aspects of this inverse problem have been considered, such as inclusion detection [27,28], identification of the elastic moduli in beams, plates and other geometric configurations [24,29–31], the linearized problem [32–34], uniqueness [22,23,35,36] and identification of residual stresses [37–40], among others. We refer to the survey [41] for many more results.

The inverse problem we study is related to the space-fractional Calderón problem, which was introduced in the seminal paper [4] as a nonlocal counterpart to the classical Calderón problem arising in electric impedance tomography. Uniqueness was achieved in the case of bounded potentials and fractional exponent $s \in (0,1)$ [4] also with a single measurement [42], and later extended to rough potentials [43] and all positive fractional exponents $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ [15]. Perturbed versions of the same problem were studied e.g. in [44] for $s \in (1/2,1)$ and first order perturbations and [45] for $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and general high order local perturbations. Nonlocal perturbations were studied in some specific cases in [46–48]. Uniqueness has also been studied in numerous other settings, including the fractional magnetic Schrödinger equation [15,49–52] and the fractional heat equation [53,54]. A very recent fractional elasticity equation with constant principal coefficients [55] and the fractional conductivity equation [5] bear a strong connection with the present study. Moreover, the fractional Schrödinger equation was studied in the semilinear setting [50–52,56]. We refer to the surveys [6,57] for more information about the fractional Calderón problem.

(e) Organization of the rest of the article

The remaining part of the paper is organized as follows. Section 2 contains preliminaries from functional analysis, classical elasticity theory and nonlocal vector calculus, as well as the definitions of the notations used in the article. Section 3 defines and describes the main object of study of the paper, the fractional elasticity operator $\mathbf{E}^s$. The Dirichlet problem related to such operator is studied in §4, and in §5, it is shown to be equivalent to a Dirichlet problem for the fractional Schrödinger equation. An integral identity relating coefficients and measured data, the so-called Alessandrini identity, is shown to hold in §6. In §7, we prove the Runge approximation property and eventually the main theorem 1.2.

2. Preliminaries

In this section, we recall the definitions of relevant function spaces, define some concepts from mathematical physics related to the problem of elasticity, and establish some useful notations.

(a) Tensor products and contractions

We will make wide use of the concepts of tensor product and contraction, mainly with respect to vectors. This will let us write our equations in a more understandable way. We let $\mathbb{N}$ be the set of *strictly* positive integers.

Definition 2.1 (Tensor product). Let $m,n \in \mathbb{N}$, and assume a,b are multi-indices belonging to $\mathbb{N}^m, \mathbb{N}^n$, respectively. Consider two tensors $A_{\alpha_1,\dots,\alpha_m}$ and $B_{\beta_1,\dots,\beta_n}$, with $\alpha_i \in \{1,\dots,a_i\}$ for $i=1,\dots,m$ and $\beta_j \in \{1,\dots,b_j\}$ for $j=1,\dots,n$, respectively. The *tensor product* $A \otimes B$ is the new tensor of elements

$$(A \otimes B)_{\alpha_1,\dots,\alpha_m,\beta_1,\dots,\beta_n} = A_{\alpha_1,\dots,\alpha_m} B_{\beta_1,\dots,\beta_n}.$$

In particular, if $m=n=1$ (and thus $A \in \mathbb{R}^a, B \in \mathbb{R}^b$ are vectors) the tensor product $A \otimes B$ is just the $a \times b$ matrix of elements

$$(A \otimes B)_{\alpha,\beta} = A_\alpha B_\beta, \quad \text{for } \alpha = 1,\dots,a \text{ and } \beta = 1,\dots,b.$$

Definition 2.2 (Tensor contraction of order k). Let m,n,a,b,A,B be as in definition 2.1. Assume that $k \in \mathbb{N}$ is such that $k \le \min\{m,n\}$, with $a_{m+\ell-k} = b_\ell$ for all $\ell \in \{1,\dots,k\}$. We define the *k-th*

contraction of tensors A and B as the new tensor given by

$$(A \cdot {}_k B)_{\alpha_1,\ldots,\alpha_{m-k},\beta_{k+1},\ldots,\beta_n} = A_{\alpha_1,\ldots,\alpha_{m-k},\gamma_1,\ldots,\gamma_k} B_{\gamma_1,\ldots,\gamma_k,\beta_{k+1},\ldots,\beta_n},$$

where we assume the Einstein summation convention on repeated indices. The ranges of the α, β indices appearing in the above formula are the same as in the definitions of A, B, while the γ_ℓ index has range in $\{1,\ldots,b_\ell\}$ for $\ell = 1,\ldots,k$.

In the present work, we apply the above definition only with $k=1$ and $k=2$, so we use the standard symbols $\cdot$ and : in place of the more general $\cdot_1$ and $\cdot_2$. It is useful to observe that when A, B are vectors or matrices the above contraction operator $\cdot$ coincides with the usual scalar product and matrix multiplication.

The definitions are immediately extended to functions. If A, B are tensor-valued functions defined on some set Ω, then we let

$$A \otimes B : x \mapsto A(x) \otimes B(x), \quad \text{for all } x \in \Omega$$

and

$$A \cdot {}_k B : x \mapsto A(x) \cdot {}_k B(x), \quad \text{for all } x \in \Omega.$$

Accordingly, if V, W are sets of tensor-valued functions defined on some set Ω, then we let $V \otimes W$ be the new set of functions on Ω given by

$$V \otimes W := \{v \otimes w, v \in V, w \in W\}.$$

Remark 2.3. This should not be confused with the familiar Cartesian product $V \times W$. For example, if $a, b \in \mathbb{N}$ and V, W are sets of vector-valued functions $v : \Omega \mapsto \mathbb{R}^a$ and $w : \Omega \mapsto \mathbb{R}^b$, respectively, then the elements of $V \times W$ map Ω to $\mathbb{R}^{a+b}$, while the ones of $V \otimes W$ map Ω to $\mathbb{R}^{a \times b}$. In the present work, we shall often have $a = 2$ and $V = \{v\}$ for some fixed function $v : \Omega \to \mathbb{R}^2$.

The following Lemma collects some elementary properties of tensor products and contractions, which can be easily proved using the index notation:

Lemma 2.4. *Let $m, n, p \in \mathbb{N}$, and assume that a, b, c are multi-indices belonging to $\mathbb{N}^m, \mathbb{N}^n, \mathbb{N}^p$, respectively. Consider three tensors*

$$A_{\alpha_1,\ldots,\alpha_m}, \quad \text{with } \alpha_i \in \{1,\ldots,a_i\} \text{ for } i = 1,\ldots,m,$$

$$B_{\beta_1,\ldots,\beta_n}, \quad \text{with } \beta_j \in \{1,\ldots,b_j\} \text{ for } j = 1,\ldots,n$$

and

$$C_{\gamma_1,\ldots,\gamma_p}, \quad \text{with } \gamma_k \in \{1,\ldots,c_k\} \text{ for } k = 1,\ldots,p.$$

The following equalities hold whenever the tensor contractions are well-defined:

(*i*) $(A \otimes B) \cdot {}_n C = (A \otimes C) \cdot {}_n B$ *for* $n = p$,
(*ii*) $A \cdot {}_k (B \otimes C) = (A \cdot {}_k B) \otimes C$ *for all* $k \in \mathbb{N}$,
(*iii*) $(A \cdot {}_k B) \cdot {}_q C = A \cdot {}_k (B \cdot {}_q C)$ *for all* $k, q \in \mathbb{N}$ *with* $n \geq k + q$,
(*iv*) $A \cdot {}_{n+p} (B \otimes C) = (A \cdot {}_p C) \cdot {}_n B$.

If A, B are tensor-valued functions of $x \in \Omega$, then the equalities

(*v*) $\nabla(A \otimes B) = \nabla A \otimes B + A \otimes \nabla B$ *for* $m = a = 1$,
(*vi*) $\nabla(A \cdot {}_m B) = \nabla A \cdot {}_m B + \nabla B \cdot {}_m A$ *for* $m = n$,
(*vii*) $\nabla \cdot (A \otimes B) = A \otimes \nabla \cdot B + B^T \cdot \nabla A$ *for* $m = a = 1, n = 2$

also hold whenever the tensor contractions are well-defined.

Remark 2.5. For all vectors v, we use the convention $(\nabla v)_{ij} := \partial_i v_j$.

(b) Fractional Sobolev spaces

Let $r \in \mathbb{R}$, and assume that $\Omega, F \subset \mathbb{R}^n$, respectively, are an open and a closed set. We indicate by $H^r(\mathbb{R}^n)$ the usual L^2-based Bessel potential spaces. These are endowed with the norm

$$||u||_{H^r} := ||(1+|\xi|^2)^{r/2}\hat{u}(\xi)||_{L^2(\mathbb{R}^n)}.$$

Here by $\hat{u}(\xi) = \mathcal{F}u(\xi) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\,dx$ we indicate the Fourier transform. Using the same notation as in [58] and [4], we also define the following fractional Sobolev spaces:

$$H^r(\Omega) := \{u|_\Omega, u \in H^r(\mathbb{R}^n)\},$$
$$\widetilde{H}^r(\Omega) := \text{closure of } C_c^\infty(\Omega) \text{ in } H^r(\mathbb{R}^n)$$
and
$$H^r_F(\mathbb{R}^n) := \{u \in H^r(\mathbb{R}^n) : \text{ supp}(u) \subset F\}.$$

In particular, $H^r(\Omega)$ is endowed with the quotient norm $||u||_{H^r(\Omega)} := \inf\{||w||_{H^r} : w \in H^r(\mathbb{R}^n), w|_\Omega = u\}$. One sees that the following inclusions and identities

$$\widetilde{H}^r(\Omega) \subseteq H^r(\Omega), \quad \widetilde{H}^r(\Omega) \subseteq H^r_{\overline{\Omega}} \quad \text{and} \quad H^r_F \subseteq H^r,$$

and

$$(\widetilde{H}^r(\Omega))^* = H^{-r}(\Omega) \quad \text{and} \quad (H^r(\Omega))^* = \widetilde{H}^{-r}(\Omega)$$

hold. If in addition Ω is known to be Lipschitz, then $\widetilde{H}^r(\Omega) = H^r_{\overline{\Omega}}$ for all $r \in \mathbb{R}$, as shown in [45,58].

We shall also use products of Sobolev spaces. If $r \in \mathbb{R}$, the space $H^r(\mathbb{R}^n) \times H^r(\mathbb{R}^n)$ consists of all the vectors $u = (u_1, u_2)$ with $u_1, u_2 \in H^r(\mathbb{R}^n)$, endowed with the norm

$$||u||^2_{H^r(\mathbb{R}^n)\times H^r(\mathbb{R}^n)} := ||u_1||^2_{H^r(\mathbb{R}^n)} + ||u_2||^2_{H^r(\mathbb{R}^n)}.$$

(c) The fractional Laplacian and other nonlocal operators

Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $u \in \mathcal{S}$, the set of Schwartz functions. The fractional Laplacian of u can be defined as

$$(-\Delta)^s u := \mathcal{F}^{-1}(|\xi|^{2s}\hat{u}(\xi)),$$

which makes $(-\Delta)^s$ a continuous map from $\mathcal{S}$ to L^∞ (see [4]). The definition above can be uniquely extended in such a way that $(-\Delta)^s$ acts as a continuous operator $(-\Delta)^s : H^r(\mathbb{R}^n) \to H^{r-2s}(\mathbb{R}^n)$ for all $r \in \mathbb{R}$. Further extensions are possible to Sobolev spaces of negative exponent and to L^p-based Sobolev spaces [4,15]. It is possible to give many other definitions of the fractional Laplacian, which at least in the case $s \in (0,1)$ can be shown to be equivalent to ours [59]. In particular, the fractional Laplacian can be defined as a singular integral by

$$(-\Delta)^s u(x) := C_{n,s}\, PV \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy,$$

where $C_{n,s} := 4^s\Gamma(n/2+s)/\pi^{n/2}|\Gamma(-s)|$.

The fractional Laplacian presents a nonlocal behaviour, as made evident by the following property it enjoys:

Theorem 2.6 (UCP for the fractional Laplacian). *Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, and assume $u \in H^r(\mathbb{R}^n)$ for some $r \in \mathbb{R}$. If $u = (-\Delta)^s u = 0$ in a non-empty open set V, then $u \equiv 0$ in $\mathbb{R}^n$.*

The proof of this theorem can be found in [15], where the authors explore also the case of negative exponent s. The main case $s \in (0,1)$ upon which the proof is based was proved in [4]. One more property of the fractional Laplacian we shall use is the following fractional Poincaré inequality.

Theorem 2.7. *Let $n \geq 1$, $s \geq t \geq 0$, $K \subset \mathbb{R}^n$ a compact set and $u \in H^s_K(\mathbb{R}^n)$. There exists a constant $\widetilde{c} = \widetilde{c}(n,K,s) > 0$ such that*

$$\|(-\Delta)^{t/2}u\|_{L^2(\mathbb{R}^n)} \leq \widetilde{c}\|(-\Delta)^{s/2}u\|_{L^2(\mathbb{R}^n)}.$$

Many variously flavoured proofs of the above statement can be found in [15].

Next, we shall define nonlocal counterparts to the gradient and divergence operators as in [5,15]. These are special instances of the general nonlocal vector calculus operators introduced in [2,60]. For $u \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ and $s \in (0,1)$, the fractional gradient is

$$\nabla^s u(x,y) := (u(y) - u(x)) \otimes \zeta(x,y), \quad \text{where } \zeta(x,y) := \frac{C_{n,s}^{1/2}}{\sqrt{2}} \frac{x-y}{|x-y|^{n/2+s+1}}$$

and since one sees that $||\nabla^s u||_{L^2(\mathbb{R}^{2n})} \leq ||u||_{H^s(\mathbb{R}^n)}$, the definition is extended to act as $H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n})$ by density. Many more properties of the fractional gradient can be found in [49]. The fractional divergence $(\nabla \cdot)^s$ is defined as the adjoint of ∇^s, that is

$$\langle (\nabla \cdot)^s v, u \rangle := \langle v, \nabla^s u \rangle$$

for all $v \in L^2(\mathbb{R}^{2n}), u \in H^s(\mathbb{R}^n)$. Thus $(\nabla \cdot)^s : L^2(\mathbb{R}^{2n}) \to H^{-s}(\mathbb{R}^n)$. It is useful to keep in mind that for more regular functions one has

$$(\nabla \cdot)^s v(x) = \frac{C_{n,s}^{1/2}}{\sqrt{2}} \int_{\mathbb{R}^n} \frac{v(x,y) + v(y,x)}{|x-y|^{n/2+s+1}} \cdot (x-y)\, \mathrm{d}y.$$

More generally, we define the fractional gradient of order $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ as $\nabla^s := \nabla^{s-\lfloor s \rfloor} \nabla^{\lfloor s \rfloor}$, and a corresponding relation holds by definition for the fractional divergence. Most importantly, one sees that the property

$$(\nabla \cdot)^s \nabla^s u = (-\Delta)^s u$$

holds in $H^s(\mathbb{R}^n)$. Finally, we define the j-th fractional derivative as

$$\partial_j^s u(x,y) := \nabla^s u(x,y) \cdot e_j = \frac{C_{n,s}^{1/2}}{\sqrt{2}} \frac{u(y) - u(x)}{|x-y|^{n/2+s+1}} (x_j - y_j).$$

In particular, this definition implies that $\partial_j^s u \cdot \partial_i^s v = \partial_i^s u \cdot \partial_j^s v$ holds for all vectors u, v and all $i, j \in \{1, \ldots, n\}$. This property, which of course does not hold in the classical case $s = 1$, will be fundamental for our arguments.

(d) Hölder spaces

Let $r \in (0,1)$. Following [61], we define the Hölder space $C^r(\mathbb{R}^n)$ as the set of bounded functions u such that

$$|u(x) - u(y)| \leq C|x-y|^r$$

for all $x, y \in \mathbb{R}^n$. If $k \in \mathbb{N}$, we let $C^k(\mathbb{R}^n)$ be the set of all bounded continuous functions u such that $D^\beta u$ is bounded and continuous for all multi-indexes $|\beta| \leq k$. Finally, if $r \in \mathbb{R}^+ \setminus \mathbb{Z}^+$ we define $C^r(\mathbb{R}^n)$ to be the set of all functions $u \in C^{\lfloor r \rfloor}(\mathbb{R}^n)$ with $D^\beta u \in C^{r-\lfloor r \rfloor}(\mathbb{R}^n)$ for all multi-indexes $|\beta| = \lfloor r \rfloor$.

Next, we shall list some properties of Hölder spaces which will be useful in our arguments. It follows immediately from the definition that for all $r, r' \in \mathbb{R}^+$ one has the embedding

$$r \leq r' \Rightarrow C^{r'}(\mathbb{R}^n) \subseteq C^r(\mathbb{R}^n). \tag{2.1}$$

The Hölder space $C^r(\mathbb{R}^n)$ is clearly closed under composition with smooth functions, that is

$$u \in C^r(\mathbb{R}^n),\ F \in C^\infty(\mathbb{R}) \Rightarrow F(u) \in C^r(\mathbb{R}^n). \tag{2.2}$$

It is also known that $C^r(\mathbb{R}^n)$ is an algebra for all $r \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, which means

$$u, v \in C^r(\mathbb{R}^n) \Rightarrow uv \in C^r(\mathbb{R}^n). \tag{2.3}$$

Let $m \in \mathbb{R}$ and assume $\Psi \in OPS_{1,0}^m$, i.e. Ψ is a pseudodifferential operator with symbol in the Hörmander class $S_{1,0}^m$. If both $r, r - m \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, then

$$\Psi : C^r(\mathbb{R}^n) \to C^{r-m}(\mathbb{R}^n). \tag{2.4}$$

For later purposes, we also show that the operator $(-\Delta)^{s-1}\partial_i\partial_j$ maps $C^r(\mathbb{R}^n)$ to $C^{r-2s}(\mathbb{R}^n)$. We write the symbol as

$$|\xi|^{2s-2}\xi_i\xi_j = \psi(\xi)|\xi|^{2s-2}\xi_i\xi_j + (1-\psi(\xi))|\xi|^{2s-2}\xi_i\xi_j$$

where $\psi \in C_c^\infty(\mathbb{R}^n)$ satisfies $\psi = 1$ near 0, and note that the second symbol has the right mapping properties by (2.4). The first symbol gives rise to a convolution operator $u \mapsto k * u$, where

$$k = \mathcal{F}^{-1}(\psi) * \mathcal{F}^{-1}(|\xi|^{2s-2}\xi_i\xi_j).$$

Now the first function in the convolution is Schwartz and the second one is homogeneous of order $-n-2s$ and smooth outside of 0 [62], which proves that $k \in L^1(\mathbb{R}^n)$. Since $\psi(\xi)$ is compactly supported, the Fourier characterization of Hölder spaces [63] implies that $u \mapsto k * u$ is bounded between any two Hölder spaces. This proves that whenever $r, r-2s \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, one has

$$(-\Delta)^{s-1}\partial_i\partial_j : C^r(\mathbb{R}^n) \to C^{r-2s}(\mathbb{R}^n). \tag{2.5}$$

(e) Fundamentals of the classical theory of linear elasticity

This section is a brief introduction to some of the fundamental concepts in the theory of linear elasticity. Our main reference in this respect is Landau's book [21]; however, we do not necessarily restrict our discussion to the case $n = 3$.

The theory of elasticity studies the mechanics of the deformations of continuous media. Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set representing a physical body. When forces are applied to Ω, the body answers to that by changing shape and volume, i.e. each point $x \in \Omega$ is transferred to a new location $x' \in \mathbb{R}^n$. Thus it is possible to define a *displacement vector field* u such that $u(x) = x' - x$ for all $x \in \Omega$.

By computing the change in distance between two points which were originally close to each other, one sees that the new infinitesimal distance $\mathrm{d}l'$ is related to the original one $\mathrm{d}l$ by

$$\mathrm{d}l'^2 - \mathrm{d}l^2 = 2u_{ik}\,\mathrm{d}x_i\,\mathrm{d}x_k,$$

where

$$u_{ik} := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k}\right)$$

is called the *strain tensor*. It is customary in the case of small deformations to neglect the second order terms of the strain tensor and use instead what is known as *Cauchy's strain tensor*, *linear strain tensor* or *small strain tensor*:

$$\varepsilon_{ik} := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) \approx u_{ik}.$$

One immediately sees that the tensor ε is symmetric.

When deformed, the body leaves its equilibrium state and some *internal forces* (or *stresses*) F are generated, which attempt to return the body to its original undeformed state. The resultant of such forces acting on a region $\Omega' \subset \Omega$ can be computed as $\int_{\Omega'} F\,\mathrm{d}V$. Assuming that such forces do not act on a distance, one should be able to express their resultant as an integral over $\partial\Omega'$. This suggests that F should be of the form $\nabla \cdot \sigma$, where the new tensor σ is called *stress tensor*. Further investigation of the total moment of the forces acting on Ω' reveals that the stress tensor β should also be symmetric.

Because we are interested in a theory of *linear* elasticity, we shall assume a linear relationship between the stress tensor σ, representing the forces acting on Ω, and the strain tensor ε, which represents the resulting deformation of the body. This gives rise to the following *Hooke's Law* (or *constitutive equation of linear elasticity*)

$$\sigma_{ij} = C_{ijlm}\varepsilon_{lm},$$

where the new fourth-order tensor C, which completely describes the elastic behaviour of the body, is called *elasticity* or *stiffness tensor*. Using *Newton's second law*, we can eventually define the

operator of classical elasticity **E** as

$$\mathbf{E}u := \nabla \cdot (C(\nabla u + \nabla u^T)). \tag{2.6}$$

One can also associate a *potential energy* U to **E**

$$U(u) := \frac{1}{2}\int_{\mathbb{R}^n} C_{ijlm}(x)\varepsilon_{ij}(x)\varepsilon_{lm}(x)\,\mathrm{d}x.$$

This quantity is often assumed to be non-negative, with $U(u) = 0$ holding if and only if $u = 0$.

Let us now assume that the material is *isotropic*, i.e. it is completely characterized by properties which are independent of direction. In this case, the elasticity tensor can be expressed as

$$C_{ijlm} = \lambda\delta_{ij}\delta_{lm} + \mu(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}),$$

where λ, μ are called *Lamé parameters*. Consequently, Hooke's Law becomes

$$\sigma_{ij} = \lambda\,\mathrm{tr}(\varepsilon)\delta_{ij} + 2\mu\varepsilon_{ij} = \left(\lambda + \frac{2\mu}{n}\right)\mathrm{tr}(\varepsilon)\delta_{ij} + 2\mu\left(\varepsilon_{ij} - \frac{\mathrm{tr}(\varepsilon)}{n}\delta_{ij}\right).$$

The quantities $k := \lambda + 2\mu/n$ and μ (this last one sometimes indicated by G) are, respectively, called *bulk* and *shear moduli*. Thermodynamic considerations ensure the positivity of both k and μ.

The corresponding tensors $tr(\varepsilon)\mathrm{Id}$ and $\varepsilon - (\mathrm{tr}(\varepsilon)/n)\mathrm{Id}$ are called *dilational* and *deviatoric strain tensors*. The first one of them represents the *hydrostatic compression* of the body, i.e. a deformation in scale but not in shape, and it is independent of the coordinate system. On the other hand, the second tensor represents a *pure shear*, i.e. a deformation in which the volume is unchanged, and it is trace-free.

It is also possible to compute the strain tensor given the stress tensor. By Hooke's Law, we have

$$\mathrm{tr}(\sigma) = (n\lambda + 2\mu)\,\mathrm{tr}(\varepsilon) = nk\,\mathrm{tr}(\varepsilon),$$

and since $k > 0$, we can write $\sigma_{ij} = (\lambda\,\mathrm{tr}(\sigma)/nK)\delta_{ij} + 2\mu\varepsilon_{ij}$. This eventually gives

$$\varepsilon_{ij} = \frac{1}{2\mu}\left(\sigma_{ij} - \frac{\lambda\,\mathrm{tr}(\sigma)}{nk}\delta_{ij}\right) = s_{ijlm}\sigma_{lm},$$

where the new tensor $s_{ijlm} := (\delta_{il}\delta_{jm}/2\mu) - (\lambda\delta_{ij}\delta_{lm}/2n\mu k)$ is the *isotropic compliance tensor*.

The Poisson effect indicates the physical phenomenon observed in the study of elastic materials in which a body reacts to a compression (resp. extension) along one axis with an extension (resp. compression) in the perpendicular directions. The *Poisson ratio* ν is defined as the amount of transverse extension divided by the amount of axial compression. It is easily computed in the case of a *homogeneous deformation* of a thin rod along its axis. If such axis is oriented in the e_n direction and the applied pressure is p, then $\sigma_{ii} = p\delta_{in}$ for all $i \in \{1, \ldots, n\}$. Therefore,

$$\varepsilon_{ii} = \frac{1}{2\mu}\left(\sigma_{ii} - \frac{\lambda\,\mathrm{tr}(\sigma)}{nk}\right) = \frac{p}{2\mu}\left(\delta_{in} - \frac{\lambda}{nk}\right), \quad \text{for all } i \in \{1, \ldots, n\},$$

which gives

$$\nu := -\frac{\varepsilon_{11}}{\varepsilon_{nn}} = -\frac{\delta_{1n} - (\lambda/nk)}{\delta_{nn} - (\lambda/nk)} = \frac{\lambda}{nk - \lambda} = \frac{k - (2\mu/n)}{(n-1)k + (2\mu/n)} = \frac{(k/\mu) - (2/n)}{(n-1)(k/\mu) + (2/n)}.$$

Given the positivity of k and μ, it is always true that $\nu \in (-1, 1/(n-1))$. It is also clear that ν depends only on the ratio k/μ rather than on the two moduli taken separately.

3. The isotropic fractional elasticity operator

In this section, we introduce a model for linear fractional elasticity derived from the classical one, which is related to the model proposed by Tarasov and Aifantis in [14]. To this end, we use the fractional divergence and gradient, and the result will be a new self-adjoint operator $\mathbf{E}^s$. We also assume throughout the paper the stiffness tensor C to be isotropic and such that the associated Lamé parameters L, M satisfy

(A 1) there exists $\varepsilon > 0$ such that $L, M \in C^{2s+\varepsilon}(\mathbb{R}^n)$,
(A 2) there exist $L_0, M_0 \in \mathbb{R}$ such that $L - L_0 = M - M_0 = 0$ in Ω_e and
(A 3) the functions M and $K := L + (2M/n)$ are positive and bounded away from 0.

Given that our nonlocal operators act on two-point functions, we need a preparatory lemma.

Lemma 3.1 (Square root of stiffness tensor). *Let the Lamé parameters L, M of the isotropic stiffness tensor C be as in assumptions (A 1)–(A 3). There exists a unique pair of real valued functions λ, μ verifying (A 1)–(A 3) and*

$$C^{1/2} : C^{1/2} = C,$$

where the new tensor $C^{1/2}$ is defined as

$$C^{1/2}_{ijlm} := \lambda\delta_{ij}\delta_{lm} + \mu(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}).$$

Moreover, we have

$$\begin{aligned} C^{1/2}_{ij\alpha\beta}(x)C^{1/2}_{\alpha\beta lm}(y) = (n\lambda(x)\lambda(y) + 2\lambda(x)\mu(y) + 2\lambda(y)\mu(x))\delta_{ij}\delta_{lm} \\ + 2\mu(x)\mu(y)\delta_{il}\delta_{jm} + 2\mu(x)\mu(y)\delta_{im}\delta_{jl}, \end{aligned} \tag{3.1}$$

and therefore

$$C^{1/2}_{ij\alpha\beta}(x)C^{1/2}_{\alpha\beta lm}(y) = C^{1/2}_{lm\alpha\beta}(y)C^{1/2}_{\alpha\beta ij}(x).$$

Proof. Note that

$$\begin{aligned} C^{1/2}_{ij\alpha\beta}C^{1/2}_{\alpha\beta lm} &= (\lambda\delta_{ij}\delta_{\alpha\beta} + \mu\delta_{i\alpha}\delta_{j\beta} + \mu\delta_{i\beta}\delta_{j\alpha})(\lambda\delta_{\alpha\beta}\delta_{lm} + \mu\delta_{\alpha l}\delta_{\beta m} + \mu\delta_{\alpha m}\delta_{\beta l}) \\ &= (n\lambda^2 + 4\lambda\mu)\delta_{ij}\delta_{lm} + 2\mu^2\delta_{il}\delta_{jm} + 2\mu^2\delta_{im}\delta_{jl} \end{aligned}$$

and

$$C_{ijlm} = L\delta_{ij}\delta_{lm} + M\delta_{il}\delta_{jm} + M\delta_{im}\delta_{jl}.$$

We look for λ, μ such that $C^{1/2}_{ij\alpha\beta}C^{1/2}_{\alpha\beta lm} = C_{ijlm}$. Since μ must be positive, we have $\mu := \sqrt{M/2}$, which also ensures that μ is bounded away from 0. This gives two possible choices for λ, namely $\lambda_\pm = 1/n\left(\pm\sqrt{2M + nL} - \sqrt{2M}\right)$. However, the required positivity of the coefficient $k := \lambda + 2\mu/n$ ensures that $\lambda = 1/n\left(\sqrt{2M + nL} - \sqrt{2M}\right)$. Now $k = \sqrt{K/n}$, and thus it is bounded away from 0. This proves that λ, μ satisfy condition (A 3). Given that the square root is a smooth function when considered far from 0, by formula (2.2), we deduce $\mu, k \in C^{2s+\varepsilon}(\mathbb{R}^n)$, which in turn implies $\lambda \in C^{2s+\varepsilon}(\mathbb{R}^n)$ as well and proves (A 1). Since L, M are constant outside of Ω, so must be λ, μ too, which proves (A 2). The last equalities in the statement of the lemma follow easily from the computations above. ∎

Similarly to what was done in [5], we can define the new fractional elasticity operator

$$\mathbf{E}^s u := (\nabla\cdot)^s(C^{1/2}(x) : C^{1/2}(y)(\nabla^s u + \nabla^s u^T)(x, y)). \tag{3.2}$$

Observe that this corresponds to taking as fractional Cauchy's strain tensor

$$\varepsilon^s_{ik} := \frac{1}{2}(\partial^s_k u_i + \partial^s_i u_k), \quad \text{i.e. } \varepsilon^s := \frac{1}{2}(\nabla^s u + \nabla^s u^T), \tag{3.3}$$

and then as a fractional, 'symmetrized' version of Hooke's Law

$$\sigma^s_{ij}(x, y) = C^{1/2}_{ij\alpha\beta}(x)C^{1/2}_{\alpha\beta lm}(y)\varepsilon^s_{lm}(x, y).$$

We next prove the following lemma about $\mathbf{E}^s$, which motivates definition (3.2):

Lemma 3.2 (Properties of $\mathbf{E}^s$). *Let L, M satisfy assumptions (A 1), (A 2), and let $s \in (0, 1)$. The operator $\mathbf{E}^s$ is self-adjoint, and it maps $H^s(\mathbb{R}^n)$ to $H^{-s}(\mathbb{R}^n)$.*

Proof. We start from the proof of the mapping property. By lemma 3.1 both λ and μ belong to $C^{2s+\varepsilon}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$, which implies that the two-point functions $\lambda(x)\lambda(y)$, $\mu(x)\mu(y)$ and $\lambda(x)\mu(y)$ all belong to $L^\infty(\mathbb{R}^{2n})$. The result then follows by the mapping properties of ∇^s and $(\nabla\cdot)^s$ from §2c and equation (3.1).

In order to see the self-adjointness, recall the minor and major symmetries of the stiffness tensor $C^{1/2}_{ij\alpha\beta} = C^{1/2}_{ji\alpha\beta}$, $C^{1/2}_{ij\alpha\beta} = C^{1/2}_{\alpha\beta ij}$, as well as the last equality from lemma 3.1. Then we compute for $\phi \in H^s(\mathbb{R}^n)$

$$\begin{aligned}
\langle \mathbf{E}^s u, \phi\rangle &= \langle (\nabla\cdot)^s (C^{1/2}(x) : C^{1/2}(y)(\nabla^s u + \nabla^s u^T)), \phi\rangle \\
&= \langle C^{1/2}(x) : C^{1/2}(y)(\nabla^s u + \nabla^s u^T), \nabla^s \phi\rangle \\
&= \int_{\mathbb{R}^{2n}} C^{1/2}_{ij\alpha\beta}(x) C^{1/2}_{\alpha\beta lm}(y)(\partial^s_m u_l + \partial^s_l u_m)\partial^s_j \phi_i \,\mathrm{d}x\,\mathrm{d}y \\
&= 2\int_{\mathbb{R}^{2n}} C^{1/2}_{ij\alpha\beta}(x) C^{1/2}_{\alpha\beta lm}(y)\partial^s_m u_l \partial^s_j \phi_i \,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\mathbb{R}^{2n}} \partial^s_m u_l \, C^{1/2}_{ij\alpha\beta}(x) C^{1/2}_{\alpha\beta lm}(y)(\partial^s_j \phi_i + \partial^s_i \phi_j)\,\mathrm{d}x\,\mathrm{d}y \\
&= \langle \nabla^s u, C^{1/2}(x) : C^{1/2}(y)(\nabla^s \phi + \nabla^s \phi^T)\rangle \\
&= \langle u, (\nabla\cdot)^s (C^{1/2}(x) : C^{1/2}(y)(\nabla^s \phi + \nabla^s \phi^T))\rangle = \langle u, \mathbf{E}^s \phi\rangle.
\end{aligned}$$

■

Remark 3.3. For $n = 1$, $\mathbf{E}^s$ reduces to the fractional conductivity operator. In fact, in that case C is a positive scalar function $\gamma : \mathbb{R} \to \mathbb{R}$, and $\nabla^s u$ can be written as

$$\nabla^s u(x,y) = \nabla^s u^T(x,y) = \frac{\mathcal{C}^{1/2}_{1,s}}{\sqrt{2}} \frac{u(y) - u(x)}{|x-y|^{1/2+s}} \operatorname{sgn}(x-y).$$

We also define the fractional potential energy U^s as

$$U^s(u) := \frac{1}{2}\int_{\mathbb{R}^{2n}} C^{1/2}_{ij\alpha\beta}(x) C^{1/2}_{\alpha\beta lm}(y)\varepsilon^s_{ij}(x,y)\varepsilon^s_{lm}(x,y)\,\mathrm{d}y\,\mathrm{d}x,$$

and prove the following lemma:

Lemma 3.4 (Positive definiteness of U^s). *Let $K \subset \mathbb{R}^n$ be a compact set. There exist two constants $c, C > 0$ such that the inequality $c||u||_{H^s} \le U^s(u) \le C||u||^2_{H^s}$ holds for all $u \in H^s_K(\mathbb{R}^n)$. As a consequence, $U^s(u) = 0$ if and only if $u = 0$.*

Proof. We first claim that

$$C^{1/2}_{ij\alpha\beta}(x) C^{1/2}_{\alpha\beta lm}(y) v_i w_j v_l w_m \gtrsim v_i w_j v_i w_j = |v|^2|w|^2 \tag{3.4}$$

holds for all vectors $v = v(x,y)$, $w = w(x,y)$ in $\mathbb{R}^n$. In fact, if (3.4) holds, then the definition of the fractional strain tensor (3.3), the symmetries of $C^{1/2}$ and the definition of $\nabla^s u$ imply that

$$\begin{aligned}
U^s(u) &= 2\int_{\mathbb{R}^{2n}} C^{1/2}_{ij\alpha\beta}(x) C^{1/2}_{\alpha\beta lm}(y)(\nabla^s u)_{ij}(\nabla^s u)_{lm}\,\mathrm{d}x\,\mathrm{d}y \\
&\gtrsim \langle \nabla^s u, \nabla^s u\rangle_{L^2(\mathbb{R}^{2n})} = \langle (-\Delta)^s u, u\rangle_{L^2(\mathbb{R}^n)} = ||(-\Delta)^{s/2} u||^2_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Using the fractional Poincaré inequality from theorem 2.7 for $u \in H^s_K(\mathbb{R}^n)$, we finally get

$$||u||^2_{H^s(\mathbb{R}^n)} \le ||u||^2_{L^2(\mathbb{R}^n)} + ||(-\Delta)^{s/2} u||^2_{L^2(\mathbb{R}^n)} \lesssim ||(-\Delta)^{s/2} u||^2_{L^2(\mathbb{R}^n)} \lesssim U^s(u).$$

Thus for the lower bound it suffices to show (3.4). Using lemma 3.1, we see that the left-hand side of (3.4) can be written as

$$\left(nk(x)k(y) + \frac{2(n-2)}{n}\mu(x)\mu(y)\right)(v\cdot w)^2 + 2\mu(x)\mu(y)|v|^2|w|^2,$$

where we recall our previous definition $k := \lambda + 2\mu/n$. Since $k(x), \mu(x)$ are known to be larger than a positive constant for all $x \in \mathbb{R}^n$ and also $n \geq 2$, the lower bound is proved. The reverse inequality $U^s(u) \lesssim ||u||^2_{H^s}$ follows immediately from the boundedness of the coefficients L, M (see also formula (4.2)). This completes the proof of the lemma. ∎

While the definition of the fractional elasticity operator $\mathbf{E}^s$ is very useful in order to prove its self-adjointness, when it comes to studying other properties it is more convenient to rewrite it in a different way.

Lemma 3.5 (Reduction lemma). *Let $s \in (0,1)$ and $u \in H^s(\mathbb{R}^n)$. Then in weak sense, we have*

$$\begin{aligned}(n/2+s)\boldsymbol{E}^s u = {} & (2n+4s+n')\mu(-\Delta)^s(u\mu) + nk(-\Delta)^s(uk) \\ & - 2n's\mu(-\Delta)^{s-1}\nabla\nabla\cdot(u\mu) - 2nsk(-\Delta)^{s-1}\nabla\nabla\cdot(uk) \\ & + 2su\cdot\{n'\mu(-\Delta)^{s-1}\nabla^2\mu + nk(-\Delta)^{s-1}\nabla^2 k\} \\ & - u\{(2n+4s+n')\mu(-\Delta)^s\mu + nk(-\Delta)^s k\}, \end{aligned} \tag{3.5}$$

where $n' := 2(n-2)/n$.

Proof. **Step 1.** By the computations in lemmas 3.2 and 3.1, we see that for all test functions $\phi \in H^s(\mathbb{R}^n)$

$$\begin{aligned}\frac{1}{2}\langle \mathbf{E}^s u, \phi\rangle &= \langle C^{1/2}_{ij\alpha\beta}(x) C^{1/2}_{\alpha\beta lm}(y)\partial_l^s u_m, \partial_i^s \phi_j\rangle \\ &= \langle (n\lambda(x)\lambda(y) + 2\lambda(x)\mu(y) + 2\lambda(y)\mu(x))\partial_j^s u_j, \partial_i^s\phi_i\rangle \\ &\quad + 2\langle \mu(x)\mu(y)\partial_j^s u_i, \partial_j^s\phi_i\rangle + 2\langle \mu(x)\mu(y)\partial_i^s u_j, \partial_j^s\phi_i\rangle \\ &= \langle (n\lambda(x)\lambda(y) + 2\lambda(x)\mu(y) + 2\lambda(y)\mu(x) + 2\mu(x)\mu(y))\partial_i^s u_j, \partial_j^s\phi_i\rangle \\ &\quad + 2\langle \mu(x)\mu(y)\partial_j^s u_i, \partial_j^s\phi_i\rangle. \end{aligned}$$

For the last equality, we used the relation $\partial_j^s u_j \partial_i^s \phi_i = \partial_i^s u_j \partial_j^s \phi_i$, which follows directly from the definition of the fractional gradient. Using the definitions $k := \lambda + 2\mu/n$ and $n' := 2(n-2)/n$, we can write

$$\begin{aligned}\langle \mathbf{E}^s u, \phi\rangle &= \langle 2(nk(x)k(y) + n'\mu(x)\mu(y))(\nabla^s u)^T + 4\mu(x)\mu(y)\nabla^s u, \nabla^s\phi\rangle \\ &=: \langle a(x,y)\nabla^s u + b(x,y)(\nabla^s u)^T, \nabla^s\phi\rangle. \end{aligned}$$

By the definition of the fractional gradient, we then have

$$\begin{aligned}\langle \mathbf{E}^s u, \phi\rangle &= \langle a(x,y)(u(y)-u(x))\otimes\zeta + b(x,y)\zeta\otimes(u(y)-u(x)), (\phi(y)-\phi(x))\otimes\zeta\rangle \\ &= \langle a(x,y)|\zeta|^2(u(y)-u(x)) + b(x,y)(\zeta\otimes(u(y)-u(x)))\cdot\zeta, \phi(y)-\phi(x)\rangle \\ &= \langle a(x,y)|\zeta|^2(u(y)-u(x)), \phi(y)-\phi(x)\rangle \\ &\quad + \langle b(x,y)(\zeta\otimes\zeta)\cdot(u(y)-u(x)), \phi(y)-\phi(x)\rangle, \end{aligned}$$

where at the second and third steps, we used lemma 2.4. Since

$$\frac{x-y}{|x-y|^{m+2}} = \frac{\nabla_y(|x-y|^{-m})}{m} = -\frac{\nabla_x(|x-y|^{-m})}{m}$$

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holds for all $m \neq 0$, and also the identity $\nabla\psi \otimes v = \nabla(\psi v) - \psi\nabla v$ holds for scalar ψ and vector v (see again lemma 2.4), we have

$$\begin{aligned}\frac{(x-y)\otimes(x-y)}{|x-y|^{n+2s+2}} &= \frac{\nabla_y(|x-y|^{-(n+2s)})}{n+2s}\otimes(x-y)\\ &= \frac{1}{n+2s}\left(\frac{Id}{|x-y|^{n+2s}} + \nabla_y\left(\frac{x-y}{|x-y|^{n+2s}}\right)\right)\\ &= \frac{1}{n+2s}\left(\frac{Id}{|x-y|^{n+2s}} - \frac{\nabla_y\nabla_x(|x-y|^{-(n+2s-2)})}{n+2s-2}\right),\end{aligned}$$

which implies

$$\begin{aligned}(\zeta\otimes\zeta)\cdot(u(y)-u(x)) = &\frac{|\zeta|^2(u(y)-u(x))}{n+2s}\\ &- \frac{C_{n,s}\nabla_y\nabla_x(|x-y|^{-(n+2s-2)})}{2(n+2s)(n+2s-2)}\cdot(u(y)-u(x)).\end{aligned}$$

Using $C_{n,s} = 2s(n+2s-2)C_{n,s-1}$, we get

$$\begin{aligned}\langle \mathbf{E}^s u, \phi\rangle = &\left\langle\left(a(x,y) + \frac{b(x,y)}{n+2s}\right)|\zeta|^2(u(y)-u(x)), \phi(y)-\phi(x)\right\rangle\\ &- \frac{sC_{n,s-1}}{n+2s}\langle b(x,y)\nabla_y\nabla_x(|x-y|^{-(n+2s-2)})\cdot(u(y)-u(x)), \phi(y)-\phi(x)\rangle\\ =: &\frac{I_1 - sI_2}{n/2+s}.\end{aligned}$$

Step 2. For the first term I_1, we compute

$$(n+2s)a(x,y) + b(x,y) = 2(2n+4s+n')\mu(x)\mu(y) + 2nk(x)k(y),$$

and then obtain

$$\begin{aligned}I_1 &= (2n+4s+n')\langle\mu(x)\mu(y)\nabla^s u, \nabla^s\phi\rangle + n\langle k(x)k(y)\nabla^s u, \nabla^s\phi\rangle\\ &= (2n+4s+n')\langle\mathbf{C}^s_{\mu^2}u, \phi\rangle + n\langle\mathbf{C}^s_{k^2}u, \phi\rangle,\end{aligned}$$

where $\mathbf{C}^s_{\mu^2}, \mathbf{C}^s_{k^2}$ are fractional conductivity operators, as studied in [5]. Since by assumptions (A 1)–(A 3) μ^2, k^2 are conductivities in the sense of [5], theorem 3.1 from this paper can be applied. This leads to

$$\begin{aligned}I_1 = &(2n+4s+n')\langle\mu(-\Delta)^s(u\mu) - \mu u(-\Delta)^s(\mu-\mu_0), \phi\rangle\\ &+ n\langle k(-\Delta)^s(uk) - ku(-\Delta)^s(k-k_0), \phi\rangle.\end{aligned} \tag{3.6}$$

For the second term I_2, we want to integrate by parts twice. For the sake of simplicity, we will show our computations only for the term with k coming from b (the term with μ is treated in the

same way). For $w := ku$, we have the integral

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}} \nabla_y \nabla_x (|x-y|^{-(n+2s-2)}) : [(\phi(y)-\phi(x)) \otimes (k(x)w(y) - k(y)w(x))]\, \mathrm{d}y\, \mathrm{d}x \\
&= -\int_{\mathbb{R}^{2n}} \nabla_x (|x-y|^{-(n+2s-2)}) \cdot \{(\nabla \cdot \phi)(y)(k(x)w(y) - k(y)w(x)) \\
&\quad + (\phi(y)-\phi(x)) \cdot (k(x)\nabla w(y) - \nabla k(y) \otimes w(x))\}\, \mathrm{d}y\, \mathrm{d}x \\
&= \int_{\mathbb{R}^{2n}} \frac{1}{|x-y|^{n+2s-2}} (\nabla \cdot)_x \{(\nabla \cdot \phi)(y)(k(x)w(y) - k(y)w(x)) \\
&\quad + (\phi(y)-\phi(x)) \cdot (k(x)\nabla w(y) - \nabla k(y) \otimes w(x))\}\, \mathrm{d}y\, \mathrm{d}x \\
&= \int_{\mathbb{R}^{2n}} \frac{1}{|x-y|^{n+2s-2}} \{(\nabla k(x) \cdot w(y) - k(y)(\nabla \cdot w)(x))(\nabla \cdot \phi)(y) \\
&\quad + (w(x) \otimes \nabla k(y) - k(x)(\nabla w(y))^T) : \nabla \phi(x) \\
&\quad + (\nabla w(y) \cdot \nabla k(x) - \nabla k(y)(\nabla \cdot w)(x)) \cdot (\phi(y) - \phi(x))\}\, \mathrm{d}y\, \mathrm{d}x \\
&= \int_{\mathbb{R}^{2n}} \frac{1}{|x-y|^{n+2s-2}} \{(\nabla k(x) \cdot w(y) - k(y)(\nabla \cdot w)(x))(\nabla \cdot \phi)(y) \\
&\quad + (w(y) \otimes \nabla k(x) - k(y)(\nabla w(x))^T) : \nabla \phi(y) \\
&\quad + [\nabla w(y) \cdot \nabla k(x) - \nabla k(y)(\nabla \cdot w)(x) \\
&\quad - \nabla w(x) \cdot \nabla k(y) + \nabla k(x)(\nabla \cdot w)(y)] \cdot \phi(y)\}\, \mathrm{d}y\, \mathrm{d}x,
\end{aligned}$$

where at the last step we exchanged the x, y variables in the last two terms. Observe that the $n + 2s - 2 < n$ exponent ensures that all the above integrals are well-defined. If we define the new operator $R := (-\Delta)^{s-1}/C_{n,s-1}$, we can rewrite the last line as

$$\begin{aligned}
&\int_{\mathbb{R}^n} \{R\nabla k \cdot w - kR(\nabla \cdot w)\}(\nabla \cdot \phi) + \{w \otimes R\nabla k - kR(\nabla w)^T\} : \nabla \phi \\
&\quad + \{\nabla w \cdot R\nabla k - \nabla kR(\nabla \cdot w) - R\nabla w \cdot \nabla k + R\nabla k(\nabla \cdot w)\} \cdot \phi\, \mathrm{d}y.
\end{aligned}$$

We integrate by parts in the first two terms one more time. Since R commutes with the derivatives, we get many cancellations, and eventually

$$\begin{aligned}
&\int_{\mathbb{R}^n} -\nabla\{R\nabla k \cdot w - kR(\nabla \cdot w)\} \cdot \phi - \nabla \cdot \{w \otimes R\nabla k - kR(\nabla w)^T\} \cdot \phi \\
&\qquad + \{\nabla w \cdot R\nabla k - \nabla kR(\nabla \cdot w) - R\nabla w \cdot \nabla k + R\nabla k(\nabla \cdot w)\} \cdot \phi\, \mathrm{d}y \\
&= 2\langle kR\nabla\nabla \cdot w - w \cdot R\nabla^2 k, \phi\rangle \\
&= 2\langle kR\nabla\nabla \cdot (ku) - ku \cdot R\nabla^2 k, \phi\rangle.
\end{aligned}$$

Coming back to I_2, we have obtained

$$\begin{aligned}
I_2 &= 2n\langle k(-\Delta)^{s-1}\nabla\nabla \cdot (ku) - ku \cdot (-\Delta)^{s-1}\nabla^2 k, \phi\rangle \\
&\quad + 2n'\langle \mu(-\Delta)^{s-1}\nabla\nabla \cdot (\mu u) - \mu u \cdot (-\Delta)^{s-1}\nabla^2 \mu, \phi\rangle,
\end{aligned} \tag{3.7}$$

and the wanted formula finally follows as a combination of (3.6) and (3.7); however, we still need to make sure that all the involved terms make sense in $H^{-s}(\mathbb{R}^n)$. Recall that by assumptions (A 1), (A 2) there exist $\mu_0, k_0 > 0$ such that $\tilde{\mu} := \mu - \mu_0 \in C_c^{2s+\varepsilon}(\Omega)$ and $\tilde{k} := k - k_0 \in C_c^{2s+\varepsilon}(\Omega)$. For each term, we can use a decomposition of the kind

$$\mu(-\Delta)^s(u\mu) = \tilde{\mu}(-\Delta)^s(u\tilde{\mu}) + \mu_0\tilde{\mu}(-\Delta)^s u + \mu_0(-\Delta)^s(u\tilde{\mu}) + \mu_0^2(-\Delta)^s u,$$

and thus it suffices to study the terms in (3.6) and (3.7) with μ, k substituted by $\tilde{\mu}, \tilde{k}$.

Given that $\tilde{\mu}, \tilde{k} \in C^{2s+\epsilon}(\mathbb{R}^n)$, they both map $H^s(\mathbb{R}^n)$ to itself by [63], §3.3.2. Since the space of multipliers on $H^s(\mathbb{R}^n)$ coincides with that of the multipliers on $H^{-s}(\mathbb{R}^n)$ (see [45,64]), we have $\mu(-\Delta)^s(u\mu) \in H^{-s}(\mathbb{R}^n)$. This same reasoning shows that the first two terms on the right-hand sides of (3.6) and (3.7) all make sense in $H^{-s}(\mathbb{R}^n)$.

For the remaining parts of (3.6) and (3.7), we can proceed as follows, taking as an example the term $u \cdot \tilde{\mu}(-\Delta)^{s-1}\nabla^2\tilde{\mu}$. It suffices to show $\tilde{\mu}(-\Delta)^{s-1}\nabla^2\tilde{\mu} \in L^\infty(\mathbb{R}^n)$, since this is a set of multipliers on $L^2(\mathbb{R}^n)$. Moreover, given that $\tilde{\mu} \in C_c^{2s+\varepsilon}(\Omega) \subset L^\infty(\mathbb{R}^n)$, it is enough to show that the operator $(-\Delta)^{s-1}\nabla^2$ maps $C_c^{2s+\varepsilon}(\Omega)$ to $L^\infty(\mathbb{R}^n)$. However, by formula (2.5), we have $(-\Delta)^{s-1}\nabla^2 : C^{2s+\varepsilon}(\mathbb{R}^n) \to C^\varepsilon(\mathbb{R}^n)$, so that the wanted mapping property follows from $C^\varepsilon(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$. This concludes the proof. ∎

Remark 3.6. The above reduction can similarly be performed if $u \in H^s(\mathbb{R}^n)$ is matrix-valued instead than vector-valued. In this case, the fractional elasticity operator $\mathbf{E}^s$ is weakly defined as

$$\langle \mathbf{E}^s u, \phi \rangle = 2\langle C^{1/2}_{ij\alpha\beta}(x) C^{1/2}_{\alpha\beta lm}(y) \partial_l^s u_{mp}, \partial_i^s \phi_{jp} \rangle,$$

for all matrix-valued test functions $\phi \in H^s(\mathbb{R}^n)$. The first step of the proof, which only deals with the vector ζ, and formula (3.6) are unchanged in the matrix case, apart from the additional component indicated by the index p. For the term I_2 in this case, we compute the integral

$$\begin{aligned}&\int_{\mathbb{R}^{2n}} \partial_{y,i}\partial_{x,j}(|x-y|^{-(n+2s-2)})[(\phi_{il}(y)-\phi_{il}(x))(k(x)w_{jl}(y)-k(y)w_{jl}(x))]\,\mathrm{d}y\,\mathrm{d}x\\ &\quad = 2\int_{\mathbb{R}^n} \phi : (kR\nabla\nabla \cdot w - R\nabla^2 k \cdot w)\,\mathrm{d}y\end{aligned}$$

following the same integration by parts technique shown in the second step of the proof of lemma 3.5. If in particular there exists a scalar function r such that $u = rId$, and thus w commutes with all matrices, we obtain that (3.5) holds. This observation will be used in the last steps of the proof of the main theorem.

4. Well-posedness and the Dirichlet-to-Neumann map

We begin this section by defining our problem of interest. Let $s \in (0,1)$, and assume $\Omega \subset \mathbb{R}^n$ is a bounded open set. In the *direct problem for the isotropic fractional elasticity equation*, we are given an exterior value $f \in H^s(\mathbb{R}^n)$, and we want to find a weak solution u to

$$\begin{cases} \mathbf{E}^s u = 0 & \text{in } \Omega \\ u = f & \text{in } \Omega_e. \end{cases} \tag{4.1}$$

Here, the condition $u = f$ in Ω_e should be intended in the sense that $u - f \in \widetilde{H}^s(\Omega)$. In order to define what we mean by a weak solution, we introduce the following bilinear form. Using the definition of the operator, we write for $u, v \in C_c^\infty(\mathbb{R}^n)$

$$B_{L,M}(u,v) := \langle C^{1/2}(x) : C^{1/2}(y)\nabla^s u, \nabla^s v \rangle.$$

It is immediately seen that $B_{L,M}$ is symmetric. Boundedness in $H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$ follows easily thanks to the assumption $L, M \in L^\infty(\mathbb{R}^n)$, which implies $\lambda, \mu \in L^\infty(\mathbb{R}^n)$ as well:

$$\begin{aligned}|B_{L,M}(u,v)| &\le ||C^{1/2}(x) : C^{1/2}(y)\nabla^s u||_{L^2(\mathbb{R}^{2n})} ||\nabla^s v||_{L^2(\mathbb{R}^{2n})}\\ &\lesssim ||\nabla^s u||_{L^2(\mathbb{R}^{2n})} ||\nabla^s v||_{L^2(\mathbb{R}^{2n})} \le ||u||_{H^s(\mathbb{R}^n)} ||v||_{H^s(\mathbb{R}^n)}.\end{aligned} \tag{4.2}$$

With this, we can extend the definition of $B_{L,M}$ to act on $H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$ by density. We can now say that $u \in H^s(\mathbb{R}^n)$ is a weak solution to (4.1) if and only if $B_{L,M}(u,\phi) = 0$ for all $\phi \in \widetilde{H}^s(\Omega)$, and $u - f \in \widetilde{H}^s(\Omega)$. More generally, we say that $u \in H^s(\mathbb{R}^n)$ is a weak solution to the inhomogeneous

problem

$$\begin{cases} \mathbf{E}^s u = F & \text{in } \Omega \\ u = f & \text{in } \Omega_e, \end{cases} \tag{4.3}$$

where $F \in H^{-s}(\Omega)$, if and only if $B_{L,M}(u,\phi) = F(\phi)$ holds for all $\phi \in \widetilde{H}^s(\Omega)$, and $u - f \in \widetilde{H}^s(\Omega)$.

For our problem (4.3), we have the following well-posedness result:

Proposition 4.1 (Well-posedness). *Let $s \in (0,1)$ and assume $\Omega \subset \mathbb{R}^n$ is a bounded open set. For any $f \in H^s(\mathbb{R}^n)$ and $F \in H^{-s}(\Omega)$, there exists a unique $u \in H^s(\mathbb{R}^n)$ such that $u - f \in \widetilde{H}^s(\Omega)$ and*

$$B_{L,M}(u,\phi) = F(\phi) \quad \textit{for all } \phi \in \widetilde{H}^s(\Omega).$$

Moreover, the following estimate holds:

$$||u||_{H^s(\mathbb{R}^n)} \leq C(||f||_{H^s(\mathbb{R}^n)} + ||F||_{H^{-s}(\Omega)}).$$

Proof. By letting $\tilde{u} := u - f$, we can reduce the above problem to the one of finding a unique $\tilde{u} \in \widetilde{H}^s(\Omega)$ such that $B_{L,M}(\tilde{u},\phi) = \tilde{F}(\phi)$, where $\tilde{F} := F - B_{L,M}(f,\,\cdot\,)$ belongs to $(\widetilde{H}^s(\Omega))^*$ because of the boundedness estimate (4.2) for the bilinear form:

$$|\tilde{F}(\phi)| \leq |F(\phi)| + |B_{L,M}(f,\phi)| \leq (\|F\|_{H^{-s}(\Omega)} + c||f||_{H^s(\mathbb{R}^n)})||\phi||_{H^s(\mathbb{R}^n)}.$$

Observe that $B_{L,M}(\,\cdot\,,\,\cdot\,)$ gives an equivalent inner product on $\widetilde{H}^s(\Omega)$, because by lemma 3.4 the fractional potential energy $U^s(v)$ always verifies $||v||^2_{H^s} \lesssim U^s(v)$, and also it vanishes if and only if $v = 0$. In fact,

$$\begin{aligned} B_{L,M}(v,v) &= \langle \nabla^s v, C^{1/2}(x) : C^{1/2}(y) \nabla^s v \rangle \\ &= \int_{\mathbb{R}^{2n}} C^{1/2}_{ij\alpha\beta}(x) C^{1/2}_{\alpha\beta lm}(y) \varepsilon^s_{ij}(x,y) \varepsilon^s_{lm}(x,y)\, \mathrm{d}y\, \mathrm{d}x = 2U^s(v). \end{aligned}$$

The Riesz representation theorem now ensures the existence of a bounded linear operator $G : H^{-s}(\Omega) \to \widetilde{H}^s(\Omega)$ associating each functional in $H^{-s}(\Omega)$ to its unique representative in the inner product given by $B_{L,M}(\,\cdot\,,\,\cdot\,)$ on $\widetilde{H}^s(\Omega)$. Thus the wanted (unique) solution $\tilde{u} \in \widetilde{H}^s(\Omega)$ can be defined as $\tilde{u} := G\tilde{F}$, and it verifies

$$B_{L,M}(\tilde{u},\phi) = \tilde{F}(\phi) \quad \text{for all } \phi \in \widetilde{H}^s(\Omega).$$

The boundedness of G and the definition of $\tilde{u}$ eventually give the estimate

$$\begin{aligned} ||u||_{H^s(\mathbb{R}^n)} &\leq ||f||_{H^s(\mathbb{R}^n)} + ||\tilde{u}||_{H^s(\Omega)} = ||f||_{H^s(\mathbb{R}^n)} + ||G\tilde{F}||_{H^s(\Omega)} \\ &\leq ||f||_{H^s(\mathbb{R}^n)} + C||\tilde{F}||_{H^{-s}(\Omega)} \leq C(||f||_{H^s(\mathbb{R}^n)} + ||F||_{H^{-s}(\Omega)}). \end{aligned}$$

■

In light of proposition 4.1, we can define a Poisson operator $P_{L,M}$ of $H^s(\mathbb{R}^n)$ into itself: if $f \in H^s(\mathbb{R}^n)$ is any exterior datum, then $P_{L,M}f$ is by definition the unique solution to the homogeneous problem (4.1). It of course follows from proposition 4.1 that $P_{L,M}$ is a bounded operator.

Remark 4.2. Let f, g be exterior values, let $u_f := P_{L,M}f$ and $u_g := P_{L,M}g$ be the unique solutions corresponding to them, and let e_f, e_g be any extensions. Then

$$\begin{aligned} B_{L,M}(u_f, e_g) &= \langle C^{1/2}(x) : C^{1/2}(y) \nabla^s u_f, \nabla^s e_g \rangle = \langle \mathbf{E}^s u_f, e_g \rangle = \langle \mathbf{E}^s u_f, e_g \rangle_{\Omega_e} \\ &= \langle \mathbf{E}^s u_f, u_g \rangle_{\Omega_e} = \langle \mathbf{E}^s u_f, u_g \rangle = \langle u_f, \mathbf{E}^s u_g \rangle, \end{aligned}$$

where we used the properties of $(\nabla \cdot)^s$, the fact that $\mathbf{E}^s u_f = 0$ in Ω and the self-adjointness of $\mathbf{E}^s$. Now following the same computations backwards gives $B_{L,M}(u_f, e_g) = B_{L,M}(u_g, e_f)$.

With the well-posedness of the direct problem, we can now define the DN map. Consider first the abstract trace space $X := H^s(\mathbb{R}^n)/\widetilde{H}^s(\Omega)$. It is such that two functions $f_1, f_2 \in H^s(\mathbb{R}^n)$ belong to the same equivalence class if and only if they agree in Ω_e, in the sense that $f_1 - f_2 \in \widetilde{H}^s(\Omega)$. If Ω happens to be Lipschitz, then it has been proved in [4, p. 463] that $X = H^s(\Omega_e)$.

Lemma 4.3 (DN map). *Let $s \in (0,1)$, and assume that $\Omega \subset \mathbb{R}^n$ is a bounded open set. There exists a continuous, self-adjoint linear map*

$$\Lambda_{L,M} : X \to X^* \quad \text{defined by } \langle \Lambda_{L,M}[f], [g] \rangle := B_{L,M}(P_{L,M}f, g),$$

where $f, g \in H^s(\mathbb{R}^n)$.

Proof. The proof is quite standard, and it follows the arguments presented in [4,45,47]. The DN map $\Lambda_{L,M}$ is well-defined because of the well-posedness of the direct problem: in fact, since $[f] = f + \widetilde{H}^s(\Omega)$, we have $P_{L,M}f' = P_{L,M}f$ for all $f' \in [f]$. Moreover, $B_{L,M}(P_{L,M}f, g') = B_{L,M}(P_{L,M}f, g)$ for all $g' \in [g]$ by the definition of the Poisson operator. The boundedness of the bilinear form and the well-posedness estimate give the continuity of $\Lambda_{L,M}$. Finally, the self-adjointness of $\Lambda_{L,M}$ follows from remark 4.2:

$$\langle \Lambda_{L,M}[f], [g] \rangle = B_{L,M}(P_{L,M}f, e_g) = B_{L,M}(P_{L,M}g, e_f) = \langle \Lambda_{L,M}[g], [f] \rangle.$$

■

5. The fractional Liouville reduction

In this section, we show an equivalence between our original problem (4.1) and a Schrödinger-like problem in which the nonlocal part does not depend on the coefficients. In analogy to the classical transformation from the conductivity equation to Schrödinger's, we call this procedure *fractional Liouville reduction* (see also [5]).

Recall that every vector valued $u \in H^s(\mathbb{R}^n)$ admits a *Helmholtz decomposition*, i.e. it can be written as $u = \nabla\phi + F$, where ϕ is the Newtonian potential of $\nabla \cdot u$ and $\nabla \cdot F = 0$. See e.g. [65] and references therein.

This allows us to define an operator $(\cdot)' : u \mapsto u'$ for all $u \in H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$ such that, if the Helmholtz decomposition of u is

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \nabla\phi_1 + F_1 \\ \nabla\phi_2 + F_2 \end{pmatrix}, \quad \text{with } \nabla \cdot F_1 = \nabla \cdot F_2 = 0,$$

then $u' \in H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$ is

$$u' := \begin{pmatrix} (2n + 4s + n' + 2n's)\nabla\phi_1 + (2n + 4s + n')F_1 \\ n(1 + 2s)\nabla\phi_2 + nF_2 \end{pmatrix}.$$

Recall that $n' = 2(n-2)/n$. The operator $(\cdot)'$ is bounded, with

$$||u'||_{H^s \times H^s} \lesssim \sum_{j=1}^{2} (||\nabla\phi_j||_{H^s} + ||F_j||_{H^s}) \leq ||u||_{H^s \times H^s}. \tag{5.1}$$

Moreover, if we define the differential operator D acting as

$$D \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} := \begin{pmatrix} D_1 u_1 \\ D_2 u_2 \end{pmatrix} := - \begin{pmatrix} d_1 \Delta u_1 + d_2 \nabla\nabla \cdot u_1 \\ d_3 \Delta u_2 + d_4 \nabla\nabla \cdot u_2 \end{pmatrix}$$
$$:= - \begin{pmatrix} (2n + 4s + n')\Delta u_1 + 2n's\nabla\nabla \cdot u_1 \\ n\Delta u_2 + 2ns\nabla\nabla \cdot u_2 \end{pmatrix},$$

we can compute

$$-Du = \begin{pmatrix} ((2n + 4s + n')\Delta + 2n's\nabla\nabla\cdot)(\nabla\phi_1 + F_1) \\ (n\Delta + 2ns\nabla\nabla\cdot)(\nabla\phi_2 + F_2) \end{pmatrix}$$

$$= \begin{pmatrix} (2n+4s+n'+2n's)\Delta\nabla\phi_1 + (2n+4s+n')\Delta F_1 \\ n(1+2s)\Delta\nabla\phi_2 + n\Delta F_2 \end{pmatrix}$$
$$= \Delta \begin{pmatrix} (2n+4s+n'+2n's)\nabla\phi_1 + (2n+4s+n')F_1 \\ n(1+2s)\nabla\phi_2 + nF_2 \end{pmatrix} = \Delta u',$$

which shows that $Du = -\Delta u'$. In particular, $(-\Delta)^{s-1}Du = (-\Delta)^s u'$ whenever $u \in H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$.

Proposition 5.1 (Fractional Liouville reduction). *Let $L, M \in C^{2s+\varepsilon}(\mathbb{R}^n)$ satisfy assumptions (A 1)–(A 3), and assume $f \in C_c^\infty(W)$ with $W \subset \Omega_e$ open and bounded. Define*

$$\Gamma(x) := (\mu(x), k(x)) \quad \text{and} \quad Q(x) := \frac{\Gamma(x)}{|\Gamma(x)|^2} \cdot (-\Delta)^{s-1} D(\Gamma(x) \otimes Id).$$

If $u \in H^s(\mathbb{R}^n)$ solves the original problem

$$\begin{cases} \boldsymbol{E}^s u = F & \text{in } \Omega \\ u = f & \text{in } \Omega_e \end{cases} \tag{5.2}$$

in the weak sense for some $F \in L^2(\Omega)$, then $w := \Gamma \otimes u$ solves the transformed problem

$$\begin{cases} (-\Delta)^{s-1} Dw - w \cdot Q = G & \text{in } \Omega \\ w = \Gamma \otimes f & \text{in } \Omega_e \end{cases} \tag{5.3}$$

in the weak sense in $\Gamma \otimes H^s$, where $G := \Gamma \otimes (n/2+s)F/|\Gamma|^2$. Conversely, if $w \in \Gamma \otimes H^s(\mathbb{R}^n)$ solves (5.3) in the weak sense in $\Gamma \otimes H^s$ for some $G \in \Gamma \otimes L^2(\Omega)$, then $u := \Gamma \cdot w/|\Gamma|^2$ solves (5.2) in the weak sense, where $F = \Gamma \cdot G/(n/2+s)$.

Observe that saying that $w \in \Gamma \otimes H^s(\mathbb{R}^n)$ is a solution in the weak sense in $\Gamma \otimes H^s$ means that

$$B_Q(w, \Gamma \otimes \phi) = \langle G, \Gamma \otimes \phi \rangle, \quad \text{for all } \phi \in \widetilde{H}^s(\Omega).$$

This can be equivalently written as

$$\Gamma \cdot ((-\Delta)^{s-1} Dw - w \cdot Q) = \Gamma \cdot G \text{ in } \Omega.$$

Proof. Because of the assumptions on L, M we have $\Gamma \in C^{2s+\varepsilon}(\mathbb{R}^n) \times C^{2s+\varepsilon}(\mathbb{R}^n)$, with $\Gamma(x) = (\mu_0, k_0) =: \gamma$ for all $x \in \Omega_e$. Therefore,

$$Q(x) = \frac{\Gamma(x)}{|\Gamma(x)|^2} \cdot (-\Delta)^{s-1} D((\Gamma(x) - \gamma) \otimes Id),$$

where $\Gamma(x) - \gamma =: \tilde{\Gamma} \in C_c^{2s+\varepsilon}(\Omega) \times C_c^{2s+\varepsilon}(\Omega)$. By formula (2.5), we have $(-\Delta)^{s-1} D((\Gamma(x) - \gamma) \otimes Id) \in C^\varepsilon(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$. Since $\Gamma(x)/|\Gamma(x)|^2$ is also in $L^\infty(\mathbb{R}^n)$ by property (2.2), we conclude that $Q \in L^\infty(\mathbb{R}^n)$.

Let now $u \in H^s(\mathbb{R}^n)$. In terms of the new symbols introduced in the statement of the proposition, after a straightforward computation the equality from lemma 3.5 can equivalently be written as

$$\Gamma \cdot ((-\Delta)^{s-1} D(\Gamma \otimes u) - (\Gamma \otimes u) \cdot Q) = \left(\frac{n}{2} + s\right) \mathbf{E}^s u. \tag{5.4}$$

Observe that problem (5.3) is well-posed in the Hilbert space $\Gamma \otimes H^s(\mathbb{R}^n)$. In fact, in $(\Gamma \otimes H^s(\mathbb{R}^n)) \times (\Gamma \otimes H^s(\mathbb{R}^n))$ the bilinear form

$$B_Q(v_1, v_2) := \langle (-\Delta)^{s/2} v_1', (-\Delta)^{s/2} v_2 \rangle - \langle v_1 \cdot Q, v_2 \rangle$$

is clearly bounded, and it is also coercive by equation (5.4)

$$\begin{aligned} B_Q(\Gamma\otimes u,\Gamma\otimes u) &= \langle (-\Delta)^{s-1}D(\Gamma\otimes u)-(\Gamma\otimes u)\cdot Q,\Gamma\otimes u\rangle \\ &= \langle \Gamma\cdot((-\Delta)^{s-1}D(\Gamma\otimes u)-(\Gamma\otimes u)\cdot Q),u\rangle \\ &= (n/2+s)\langle \mathbf{E}^s u,u\rangle \\ &= (2n+4s)U^s(u)\geq 0. \end{aligned}$$

Thus (5.3) has a unique solution of the kind $w=\Gamma\otimes v$ for some $v\in H^s(\mathbb{R}^n)$, provided that $G\in L^2(\Omega)$. Of course, one must have $v=f$ in Ω_e.

If now $u\in H^s(\mathbb{R}^n)$ solves (5.2) and $G:=\Gamma\otimes((n/2+s)F/|\Gamma|^2)$, then by (5.4) $w:=\Gamma\otimes u$ solves (5.3). Conversely, if $w=\Gamma\otimes v$ solves (5.3) in $\Gamma\otimes H^s$ and $F=\Gamma\cdot G/(n/2+s)$, then by (5.4) v solves (5.2). Therefore, $u=\Gamma\cdot w/|\Gamma|^2=\Gamma\cdot(\Gamma\otimes v)/|\Gamma|^2=v$ must also solve (5.2). ∎

We also define the adjoint bilinear form

$$B_Q^*(u,v):=\langle(-\Delta)^{s/2}u,(-\Delta)^{s/2}v'\rangle-\langle v\cdot Q,u\rangle,$$

which of course shares the same boundedness inequality as B_Q and can similarly be extended to act on $(\Gamma\otimes\widetilde{H}^s(\Omega))\times(\Gamma\otimes\widetilde{H}^s(\Omega))$. It is also clear that we have $B_Q(u,v)=B_Q^*(v,u)$. Given the well-posedness in $\Gamma\otimes H^s$ of problem (5.3), we can define the Poisson operator P_Q associating to the exterior datum $\Gamma\otimes f$ the unique solution $w=\Gamma\otimes v$ to the problem (5.3) with $G=0$. We can also define the DN map Λ_Q in a similar fashion as in our lemma 4.3:

$$\begin{aligned} &\langle\Lambda_Q[\Gamma\otimes f_1],[\Gamma\otimes f_2]\rangle \\ &\quad:=B_Q(P_Q(\Gamma\otimes f_1),\Gamma\otimes f_2) \\ &\quad=\langle(-\Delta)^{s/2}(P_Q(\Gamma\otimes f_1))',(-\Delta)^{s/2}(\Gamma\otimes f_2)\rangle-\langle P_Q(\Gamma\otimes f_1)\cdot Q,\Gamma\otimes f_2\rangle \\ &\quad=\langle\Gamma\cdot((-\Delta)^{s-1}DP_Q(\Gamma\otimes f_1)-P_Q(\Gamma\otimes f_1)\cdot Q),f_2\rangle, \end{aligned}$$

and similarly for Λ_Q^*. With the usual computation (see e.g. the analogous result in [45]), we get $\langle\Lambda_Q[g_1],[g_2]\rangle=\langle[g_1],\Lambda_Q^*[g_2]\rangle$, which motivates the choice of symbols.

6. The Alessandrini identity

The most important instruments needed for proving our main theorem are the so-called Alessandrini identity and Runge approximation property, which we study in this and the next section. Let us start from a simple lemma relating the DN maps of the original and transformed problems:

Lemma 6.1 (Relation between the DN maps). *Let $f_j\in C_c^\infty(W_j)$ for $j=1,2$, where $W_1,W_2\subset\Omega_e$ are open, bounded and disjoint. Then the following equation holds:*

$$\left(\frac{n}{2}+s\right)\langle\Lambda_{L,M}[f_1],[f_2]\rangle=\langle\Lambda_Q[\Gamma\otimes f_1],[\Gamma\otimes f_2]\rangle. \tag{6.1}$$

Proof. Let u_1 be the unique solution to problem (5.2) corresponding to the exterior value f_1, and let w_1 be the unique solution to (5.3) corresponding to u_1 via the fractional Liouville reduction. In light of formula (5.4), we can compute

$$\begin{aligned} \left(\frac{n}{2}+s\right)\langle\Lambda_{L,M}[f_1],[f_2]\rangle=\left(\frac{n}{2}+s\right)\langle\mathbf{E}^s u_1,f_2\rangle &= \langle\Gamma\cdot((-\Delta)^{s-1}Dw_1-w_1\cdot Q),f_2\rangle \\ &= \langle(-\Delta)^{s-1}Dw_1-w_1\cdot Q,\Gamma\otimes f_2\rangle=\langle\Lambda_Q[\Gamma\otimes f_1],[\Gamma\otimes f_2]\rangle. \end{aligned}$$

Therefore, complete knowledge of the DN map $\Lambda_{L,M}$ is equivalent to knowledge of the DN map Λ_Q on functions g_j of the kind $\Gamma\otimes f_j$. ∎

Next, we state and prove the Alessandrini identity:

Lemma 6.2 (Alessandrini identity). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $s \in (0,1)$. Let L_j, M_j for $j = 1,2$ be two sets of Lamé parameters satisfying assumptions (A 1)–(A 3), corresponding to Γ_j, Q_j for $j = 1,2$ through the fractional Liouville reduction. Assume $\Gamma_1(x) = \Gamma_2(x) =: \gamma$ for all $x \in \Omega_e$ and that the relative Poisson ratios of (L_1, M_1) and (L_2, M_2) agree in $\mathbb{R}^n$, i.e. $(L_1, M_1) \sim (L_2, M_2)$. Then the following integral identity holds for all $f_1, f_2 \in C_c^\infty(\Omega_e)$*

$$\left(\frac{n}{2} + s\right) \langle (\Lambda_{L_1,M_1} - \Lambda_{L_2,M_2})[f_1], [f_2] \rangle = \langle u_1 \cdot (Q_1 - Q_2), u_2^* \rangle,$$

where $u_1 := P_{Q_1}(\gamma \otimes f_1)$ and $u_2^* := P_{Q_2}^*(\gamma \otimes f_2)$.

Remark 6.3. Let ν be the Poisson ratio corresponding to the Lamé parameters M, K. By the definition of ν and lemma 3.1, we see that there is a one-to-one correspondence between ν and the ratio $r := \mu/k$ of the Lamé parameters μ, k of the square root of the stiffness tensor. Thus $(L_1, M_1) \sim (L_2, M_2)$ if and only if $\mu_1/k_1 = \mu_2/k_2$, that is, if and only if $\Gamma_1 = \rho\Gamma_2$ for some fixed function ρ. We say that in this case Γ_1 and Γ_2 are themselves in gauge, and we indicate this by $\Gamma_1 \sim \Gamma_2$.

Proof of lemma 6.2. The proof is a computation following from lemma 6.1:

$$\begin{aligned}
&\left(\frac{n}{2} + s\right) \langle (\Lambda_{L_1,M_1} - \Lambda_{L_2,M_2})[f_1], [f_2] \rangle \\
&= (n/2 + s)(\langle \Lambda_{L_1,M_1}[f_1], [f_2] \rangle - \langle \Lambda_{L_2,M_2}[f_1], [f_2] \rangle) \\
&= \langle \Lambda_{Q_1}[\Gamma_1 \otimes f_1], [\Gamma_1 \otimes f_2] \rangle - \langle \Lambda_{Q_2}[\Gamma_2 \otimes f_1], [\Gamma_2 \otimes f_2] \rangle \\
&= \langle \Lambda_{Q_1}[\gamma \otimes f_1], [\gamma \otimes f_2] \rangle - \langle [\gamma \otimes f_1], \Lambda_{Q_2}^*[\gamma \otimes f_2] \rangle \\
&= B_{Q_1}(P_{Q_1}(\gamma \otimes f_1), \Gamma_2 \otimes f_2) - B_{Q_2}^*(P_{Q_2}^*(\gamma \otimes f_2), \Gamma_1 \otimes f_1) \\
&= B_{Q_1}(u_1, u_2^*) - B_{Q_2}^*(u_2^*, u_1) \\
&= \langle u_1 \cdot (Q_1 - Q_2), u_2^* \rangle.
\end{aligned}$$

Here, we have used the fact that $\Gamma_1 \sim \Gamma_2$ in order to deduce that $B_{Q_1}(u_1, \Gamma_2 \otimes f_2) = B_{Q_1}(u_1, u_2^*)$, and similarly for the other term. In fact, this will be true as soon as

$$B_{Q_1}(u_1, \Gamma_2 \otimes v_2) = 0$$

for all $v_2 \in \widetilde{H}^s(\Omega)$, which is granted by the fact that u_1 is a weak solution in $\Gamma_1 \otimes H^s$ and $\Gamma_2 = \rho\Gamma_1$. ∎

Given that the right-hand side of the Alessandrini identity from lemma 6.2 only contains the difference of the transformed potentials Q_1 and Q_2, we can at most hope to recover Q from the complete knowledge of the DN map $\Lambda_{L,M}$. This suggests that we may encounter a gauge invariance for our inverse problem: if many different couples of Lamé parameters (L, M) give rise to the same transformed potential Q, they will remain indistinguishable. Thus we are now left with two tasks: to find appropriate solutions to use in the Alessandrini identity which will let us recover information about Q, and to study the relative gauge. These problems will be considered in the coming sections.

7. Runge approximation property and proof of the main theorem

Because of the particular exterior values associated with the solutions appearing in our Alessandrini identity, we do not need to prove a full Runge approximation property in the sense of [4] or [45]. We rather need only the following result:

Lemma 7.1 (Runge approximation property). *Let $\Omega, W \subset \mathbb{R}^n$ be bounded open sets such that $W \subset \Omega_e$, and assume $s \in (0,1)$. Define*

$$\mathcal{R} := \left\{ \frac{\Gamma \cdot P_Q^*(\Gamma \otimes f)}{|\Gamma|^2} - f : f \in C_c^\infty(W) \right\} \subset \widetilde{H}^s(\Omega).$$

Then the set $\{w|_\Omega : w \in \mathcal{R}\}$ is dense in $L^2(\Omega)$. The same result holds when we substitute P_Q to P_Q^* in the definition of $\mathcal{R}$.

Proof. By the Hahn–Banach theorem, it is enough to show that any $F \in L^2(\Omega)$ such that $\langle F, v\rangle = 0$ for all $v \in \mathcal{R}$ must vanish identically. Fix any $F \in L^2(\Omega)$ with such property, and consider the problem

$$(-\Delta)^{s-1} Dw - w \cdot Q = -\frac{\Gamma \otimes F}{|\Gamma|^2} \quad \text{in } \Omega$$

and

$$w = 0 \quad \text{in } \Omega_e.$$

It has a unique weak solution in $\Gamma \otimes \widetilde{H}^s(\Omega)$ of the form $w = \Gamma \otimes \phi$ by proposition 5.1. Then for any $f \in C_c^\infty(W)$, we have

$$\begin{aligned} 0 &= \left\langle F, \frac{\Gamma \cdot P_Q^*(\Gamma \otimes f)}{|\Gamma|^2} - f \right\rangle \\ &= \left\langle F, \Gamma \cdot \frac{P_Q^*(\Gamma \otimes f) - \Gamma \otimes f}{|\Gamma|^2} \right\rangle = \left\langle \frac{\Gamma \otimes F}{|\Gamma|^2}, P_Q^*(\Gamma \otimes f) - \Gamma \otimes f \right\rangle \\ &= \langle (-\Delta)^{s-1} Dw - w \cdot Q, \Gamma \otimes f\rangle - \langle (-\Delta)^{s-1} Dw - w \cdot Q, P_Q^*(\Gamma \otimes f)\rangle. \end{aligned}$$

The second term on the right-hand side is

$$B_Q(w, P_Q^*(\Gamma \otimes f)) = B_Q^*(P_Q^*(\Gamma \otimes f), w) = 0,$$

because of the fact that $w \in \Gamma \otimes \widetilde{H}^s(\Omega)$ and the definition of weak solution to the adjoint problem. Thus we are left with

$$\begin{aligned} 0 &= \langle (-\Delta)^{s-1} Dw - w \cdot Q, \Gamma \otimes f\rangle \\ &= \langle (-\Delta)^{s-1} Dw - w \cdot Q, \gamma \otimes f\rangle = \langle (-\Delta)^{s-1} Dw, \gamma \otimes f\rangle \end{aligned}$$

because of the assumption that the supports of f, w are disjoint. Eventually,

$$0 = \langle (-\Delta)^{s-1} Dw, \gamma \otimes f\rangle = \langle (-\Delta)^s w', \gamma \otimes f\rangle = \langle (-\Delta)^s (\gamma \cdot w'), f\rangle,$$

which by the arbitrariety of f implies $(-\Delta)^s(\gamma \cdot w') = 0$ in W. The UCP for the fractional Laplacian and the exterior datum of w now imply $\gamma \cdot w' \equiv 0$. Thus in Ω by lemma 2.4

$$\begin{aligned} \frac{(\gamma \cdot \Gamma)F}{|\Gamma|^2} = \gamma \cdot \frac{\Gamma \otimes F}{|\Gamma|^2} &= \gamma \cdot (-(-\Delta)^{s-1} Dw + w \cdot Q) \\ &= -(-\Delta)^s(\gamma \cdot w') + \gamma \cdot (w \cdot Q) = \gamma \cdot (w \cdot Q) \\ &= \gamma \cdot ((\Gamma \otimes \phi) \cdot Q) = (\gamma \cdot \Gamma)(\phi \cdot Q), \end{aligned}$$

and by the positivity of $\gamma \cdot \Gamma$, we get $\phi \cdot Q = F/|\Gamma|^2$ in Ω. Therefore, in Ω

$$(-\Delta)^s w' = w \cdot Q - \frac{\Gamma \otimes F}{|\Gamma|^2} = \Gamma \otimes \left(\phi \cdot Q - \frac{F}{|\Gamma|^2}\right) = 0,$$

which means that w' solves

$$(-\Delta)^s w' = 0 \quad \text{in } \Omega$$

and

$$w' = 0 \quad \text{in } \Omega_e.$$

By the well-posedness of the direct problem for the fractional Laplacian, we deduce $w' \equiv 0$, which entails $w \equiv 0$ and eventually $F \equiv 0$. ∎

With this result at hand, we can prove our main theorem.

Proof of theorem 1.2. **Step 1.** Given that the known data can always be restricted, we can without loss of generality assume that the sets W_1 and W_2 are disjoint. Let $f_j \in C_c^\infty(W_j)$ for $j = 1, 2$, and define $u_1 := P_{Q_1}(\gamma \otimes f_1)$, $u_2 := P^*_{Q_2}(\gamma \otimes f_2)$. By the definition of the Poisson operators, we have $u_1 = \Gamma_1 \otimes v_1$, $u_2^* = \Gamma_2 \otimes v_2$ for some $v_1, v_2 \in H^s(\mathbb{R}^n)$. Thus by the Alessandrini identity from lemma 6.2 and lemma 2.4 it holds that

$$\begin{aligned} 0 &= \left(\frac{n}{2} + s\right) \langle (\Lambda_{L_1,M_1} - \Lambda_{L_2,M_2})[f_1], [f_2] \rangle \\ &= \langle u_1 \cdot (Q_1 - Q_2), u_2^* \rangle \\ &= \langle (\Gamma_1 \otimes v_1) \cdot (Q_1 - Q_2), \Gamma_2 \otimes v_2 \rangle \\ &= \langle v_1 \cdot (Q_1 - Q_2), (\Gamma_1 \cdot \Gamma_2) v_2 \rangle. \end{aligned} \tag{7.1}$$

Let now g_1, g_2 be any functions belonging to $C_c^\infty(\Omega)$. Using the Runge approximation property from lemma 7.1, we can find two sequences $\{f_{j,i}\}_i \subset C_c^\infty(W_j)$, $j = 1, 2$, such that

$$v_{1,i} := \frac{\Gamma_1 \cdot P_{Q_1}(\gamma \otimes f_{1,i})}{|\Gamma_1|^2} = f_{1,i} + g_1 + r_{1,i}, \quad \text{with } ||r_{1,i}||_{L^2(\Omega)} \leq \frac{1}{i}$$

and

$$v_{2,i} := \frac{\Gamma_2 \cdot P^*_{Q_2}(\gamma \otimes f_{2,i})}{|\Gamma_2|^2} = f_{2,i} + g_2 + r_{2,i}, \quad \text{with } ||r_{2,i}||_{L^2(\Omega)} \leq \frac{1}{i}.$$

Substituting $v_{1,i}, v_{2,i}$ into (7.1) gives

$$\begin{aligned} 0 &= \langle v_1 \cdot (Q_1 - Q_2), (\Gamma_1 \cdot \Gamma_2) v_2 \rangle \\ &= \langle (f_{1,i} + g_1 + r_{1,i}) \cdot (Q_1 - Q_2), (\Gamma_1 \cdot \Gamma_2)(f_{2,i} + g_2 + r_{2,i}) \rangle \\ &= \langle (g_1 + r_{1,i}) \cdot (Q_1 - Q_2), (\Gamma_1 \cdot \Gamma_2)(g_2 + r_{2,i}) \rangle \end{aligned}$$

by the support assumptions. Moreover, the terms containing the errors $r_{j,i}$ vanish as $i \to \infty$, since for example

$$|\langle r_{1,i} \cdot (Q_1 - Q_2), (\Gamma_1 \cdot \Gamma_2) r_{2,i} \rangle| \leq ||r_{1,i} \cdot (Q_1 - Q_2)||_{L^2(\Omega)} ||(\Gamma_1 \cdot \Gamma_2) r_{2,i}||_{L^2(\Omega)} \leq \frac{C}{i^2},$$

and similarly for the other ones. Therefore, we are left with

$$0 = \langle g_1 \cdot (Q_1 - Q_2), (\Gamma_1 \cdot \Gamma_2) g_2 \rangle.$$

Since $\Gamma_1 \cdot \Gamma_2 = \mu_1\mu_2 + k_1k_2 > 0$, by the arbitrariety of $g_1, g_2 \in C_c^\infty(\Omega)$, we obtain $Q_1 = Q_2$ in Ω.

Step 2. We are left with the task of proving that $Q_1 = Q_2$ in Ω implies $\Gamma_1 = \Gamma_2$. By remark 6.3, the assumption $(L_1, M_1) \sim (L_2, M_2)$ already implies that $\Gamma_2 = r\Gamma_1$ for some fixed but unknown function r. Thus Γ_2 solves

$$\Gamma_2 \cdot (-\Delta)^{s-1} D(\Gamma_2 \otimes Id) = |\Gamma_2|^2 Q_1 \quad \text{in } \Omega$$

and

$$\Gamma_2 = \gamma \quad \text{in } \Omega_e,$$

which implies that r solves

$$\Gamma_1 \cdot ((-\Delta)^{s-1} D(\Gamma_1 \otimes (rId)) - (\Gamma_1 \otimes (rId)) \cdot Q_1) = 0 \quad \text{in } \Omega$$

and

$$r = 1 \quad \text{in } \Omega_e.$$

By remark 3.6, equation (3.5) holds when $u = rId$. Observe that the right-hand side of equation (3.5) can be rewritten as the left-hand side of the above equation in Ω (see proposition 5.1). This lets us deduce that $\mathbf{E}^s_{L_1,M_1}(rId) = 0$ holds in Ω. Because the direct problem for the fractional elasticity equation can be showed to be well-posed in the matrix case as well by the same strategy used in proposition 4.1, we conclude that it must necessarily be $r \equiv 1$. ∎

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