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RESOLVENT ESTIMATES FOR VISCOELASTIC SYSTEMS OF EXTENDED MAXWELL TYPE AND THEIR APPLICATIONS*

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Abstract. In the theory of viscoelasticity, an important class of models admits a representation in terms of springs and dashpots. Widely used members of this class are the Maxwell model and its extended version. This paper concerns resolvent estimates for the system of equations for the anisotropic, extended Maxwell model (EMM) and its marginal model; special attention is paid to the introduction of augmented variables. This leads to the augmented system that will also be referred to as the "original" system. A reduced system is then formed which encodes essentially the EMM; it is a closed system with respect to the particle velocity and the difference between the elastic and viscous strains. Based on resolvent estimates, it is shown that the original and reduced systems generate C_0-groups and the reduced system generates a C_0-semigroup of contraction. Naturally, the EMM can be written in an integro-differential form with a relaxation tensor. However, there is a difference between the original and integro-differential systems, in general, with consequences for whether their solutions generate semigroups or not. Finally, an energy estimate is obtained for the reduced system, and it is proven that its solutions decay exponentially as time tends to infinity. The limiting amplitude principle follows readily from these two results.

Key words. viscoelasticity, anisotropy, resolvent estimates, limiting amplitude principle

MSC codes. 35B37, 35B40, 35L55, 74D05

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1. Introduction.

Let $\Omega \subset \mathbb{R}^d$ with $d = 2, 3$ be a bounded domain on which a spring-dashpot model of a viscoelastic medium is defined. We assume that its boundary $\partial\Omega$ is connected and Lipschitz smooth. We divide $\partial\Omega$ into $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$, where $\Gamma_D, \Gamma_N \subset \partial\Omega$ are connected open sets, and we assume that $\Gamma_D \neq \emptyset$, $\Gamma_D \cap \Gamma_N = \emptyset$ and if $d = 3$, then their boundaries $\partial\Gamma_D$, $\partial\Gamma_N$ are Lipschitz smooth. We emphasize that the setup with $\partial\Omega$, Γ_D, Γ_N underpins the consideration of the so-called mixed type boundary condition. Our analysis extends to the case where $\partial\Omega$ consists of several connected components and Γ_D, Γ_N are unions of these components.

Let $x \in \Omega$ be a point in space and $t \in \mathbb{R}$ be time. For each $1 \leq j \leq n$ with a fixed $n \in \mathbb{N}$, let $C_j = C_j(x)$ be a stiffness tensor and $\phi_j = \phi_j(x,t)$ be a tensor describing

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the effect of viscosity; these are rank 4 and rank 2 tensors, respectively. Further, let $\rho = \rho(x)$ be the density defined on Ω and each $\eta_j = \eta_j(x)$, $1 \le j \le n$, be the viscosity of the jth dashpot, respectively.

Throughout this paper, the assumptions for C_j, η_j, and ρ are as follows.

Assumption 1.1.

(i) $C_j, \eta_j, \rho \in L^\infty(\Omega)$.

(ii) (full symmetry) $(C_j)_{klrs} = (C_j)_{rskl} = (C_j)_{klsr}$, $j \le k, l, r, s \le d$ a.e. in Ω.

(iii) (strong convexity) There exists a constant $\alpha_0 > 0$ such that for any $d \times d$ real symmetric matrix $w = (w_{kl})$

$$(C_j w)w \ge \alpha_0 |w|^2 \ \text{ a.e. in } \Omega, \tag{1.1}$$

where $|w| := \sqrt{\sum_{k,l=1}^d w_{kl}^2}$, and $C_j w$, $(C_j w)w$ are defined as follows:

$$\begin{cases} (k,l)\text{-component } (C_j w)_{kl} \text{ of the } d \times d \text{ matrix } C_j w \text{ is given as} \\ (C_j w)_{kl} := \sum_{r,s=1}^d (C_j)_{klrs} w_{rs}, \\ (C_j w)w \text{ is defined as } (C_j w)w = \sum_{k,l=1}^d \left(\sum_{r,s=1}^d (C_j)_{klrs} w_{rs} \right) w_{kl}. \end{cases}$$

(iv) There exist $\beta_0 > 0$ and $\gamma_0 > 0$ such that

$$\eta_j \ge \beta_0,\ 1 \le j \le n \text{ and } \rho \ge \gamma_0 \qquad \text{a.e. in } \Omega.$$

Historically, viscoelasticity is introduced through relaxation leading to systems of integro-differential equations, which we refer to as VID systems [1, 9]. In the case of special, parametric models, representable by springs and dashpots [9], augmented variables can be introduced to cast the systems of integro-differential equations into systems of differential equations. This alternative mathematical description affects the meaning of initial values. This also affects whether the solutions form a semigroup. Here, we will present the analysis for and clarify the properties of the two different types of systems associated with the so-called extended Maxwell model (EMM); see Figure 1. We will also consider its related extended standard linear solid model (ESLSM); see Figure 2 for an illustration of one unit. In the case of the EMM, the solutions of the integro-differential (ID) system do not generate a semigroup, but the solutions of the augmented system of differential equations (AD) generate not only a semigroup but also a group. The ID system generates exponentially decaying solutions; exponentially decaying solutions are generated by the AD only upon a reduction of the system eliminating quasi-static modes. In fact, the proofs make use of distinct energy functions. We will present the proof for the ID system in a companion paper. In various remarks, we will indicate which results apply to the ESLSM.

As a direct application of the result on exponential decay in time, we show that the solutions of the reduced system satisfy the limiting amplitude principle. That is, if a time-harmonic vibration is given on a part of the boundary, this principle implies how fast solutions of this system converge to time-harmonic solutions. We note that the time-harmonic vibration given on the boundary has a transitional period of time before becoming a time-harmonic vibration. If this convergence is fast, one can quickly generate many time-harmonic solutions by switching the frequency of the time-harmonic vibration given on a part of the boundary. This principle has been accepted without mathematical proof in applications. In this paper, we give a mathematical justification of this principle. For the Kelvin–Voigt model, which can be directly described as a system of differential equations, the exponential decay

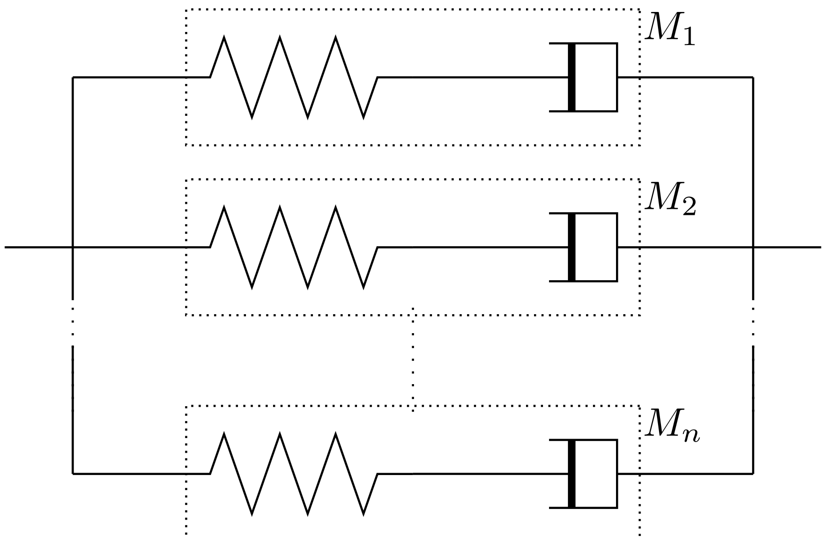

FIG. 1. *EMM and its one unit called the Maxwell model, where the zigzag and piston describe a spring and a dashpot, respectively. Here, M_j denotes the jth Maxwell constituent.*

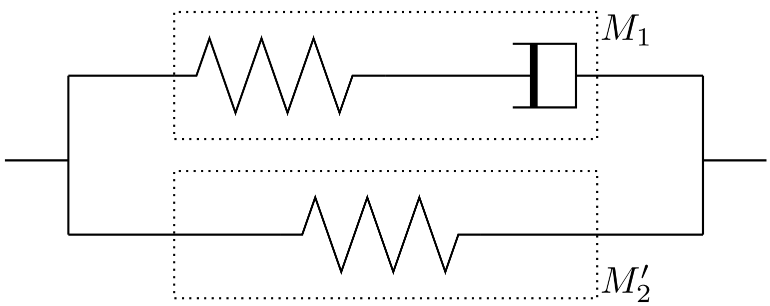

FIG. 2. *Standard linear solid model.*

of solutions was proven before. By using the inner product introduced in [15], it follows that this system generates a holomorphic semigroup and has the mentioned decay of solutions. Based on these results the limiting amplitude was proven when the medium is isotropic in [5] and [7] for the one space dimensional case and the three space dimensional case, respectively. Applications of the limiting amplitude principle appear, for example, in magnetic resonance elastography (MRE) [12] and in exploration seismology with vibroseis. In MRE, one uses a special pulse-echo sequence of MRI and measures the time-harmonic quasi-shear wave inside the tissue of an organ generated by a time-harmonic vibration at its surface; this was analyzed by Papazoglou et al. [13] using an ESLSM.

The remainder of the paper is organized as follows. In section 2, we introduce the AD system and the ID system. We also analyze the relation between the ID system and the AD system and give what causes their difference. In section 3, after giving some notations convenient for this paper, we write the AD system to the first order system with respect to the time derivative. Then, we state the semigroup property of this system which will be proved in the next section. Section 4 is the core part of this paper, devoted to giving resolvent estimates for the AD system, which leads to giving not only its semigroup property but also its group property. In section 5, besides these properties of the AD system, we discuss the abstract Cauchy problem for the AD system. To show the other subject of this paper, the limiting amplitude principle for the AD system, we need to have at least the following two conditions to be satisfied. They are that the resolvent set contains the imaginary axis of the complex plane and that solutions of the AD system have a uniform decaying property, for example, polynomial or exponential as $t \to \infty$. However, these conditions do not hold for the AD system due to the existence of a stationary solution. Hence, we introduce a reduced system (RD), and then prove that it generates a C_0-group of contraction in section 6 and that it has the exponentially decaying property of solutions in section 7; thus this system satisfies the mentioned two conditions. We conclude by establishing the limiting amplitude principle for the reduced system. The last section is devoted to giving some conclusions and pointing out the new idea deriving the resolvent estimate for the AD system.

2. AD system and ID system. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with $d = 2, 3$ and Lipschitz smooth boundary $\partial\Omega$. We consider Ω as a reference domain on which we consider a small viscoelastic deformation. The EMM is a spring-dashpot model connecting n-number of Maxwell models in parallel (see Figure 1).

Corresponding to each of these models, we denote its pair of strain and stress as (e_i, σ_i) in accordance with the labeling number $i = 1, \dots, n$. Also, we denote by (e, σ) the pair of strain and stress of the EMM. Then, the strain and stress relation for each model and the equation of motion for the EMM are given as follows:

$$\tag{2.1} \begin{aligned} &M_i \ : \ \begin{cases} \sigma_i^s = C_i e_i^s, \quad \eta_i \partial_t e_i^d = \sigma_i^d \\ e_i = e_i^s + e_i^d, \quad \sigma_i = \sigma_i^s = \sigma_i^d \end{cases} \qquad (i = 1, \dots, n), \\ &\text{EMM} \ : \ \begin{cases} e = e[u] = \frac{1}{2}(\nabla u + (\nabla u)^{\mathbf{t}}), \\ \rho \partial_t^2 u = \operatorname{div} \sigma, \\ \sigma = \sum_{i=1}^n \sigma_i, \quad e = e_1 = \cdots = e_n, \end{cases} \end{aligned}$$

where "$\mathbf{t}$" denotes the transpose. Also, hereby, for each $i = 1, \dots, n$, (e_i^s, σ_i^s) and (e_i^d, σ_i^d) are the pairs of strain and stress for the spring and dashpot, respectively. Also, C_i and η_i for each $i = 1, \dots, n$ are the elasticity tensor of the spring and viscosity of the dashpot, respectively.

Now, we introduce the AD system for the EMM. In the absence of an exterior force, the vibration with small deformation of a viscoelastic medium on Ω, modeled as the AD system, is expressed in terms of the elastic displacement $u = u(x,t)$ and viscous strains $\phi = \phi(x,t) = (\phi_1(x,t), \dots, \phi_n(x,t))$ as follows:

$$\tag{2.2} \begin{cases} \rho \partial_t^2 u - \operatorname{div} \sigma[u, \phi] = 0, \quad \sigma[u,\phi] = \sum_{j=1}^n \sigma_j[u, \phi_j], \\ \eta_j \partial_t \phi_j - \sigma_j[u, \phi_j] = 0, \quad j = 1, \dots, n, \\ u = 0 \quad \text{on} \quad \Gamma_D, \ |\Gamma_D| > 0, \\ \sigma[u,\phi]\nu = 0 \quad \text{on} \quad \Gamma_N, \ |\Gamma_N| > 0, \\ (u, \partial_t u, \phi)|_{t=0} = (u^0, v^0, \phi^0) \quad \text{on} \quad \Omega, \end{cases}$$

requiring compatibility between the initial and boundary values, where ν is the outward unit normal of $\partial\Omega$, $|\Gamma_D|$, $|\Gamma_N|$ are the respective measures of Γ_D, Γ_N, and

$$\tag{2.3} \sigma_j[u, \phi_j] = C_j(e[u] - \phi_j), \, \phi_j := e_j^d.$$

Here, using (2.1), (2.3) is derived as follows:

$$\begin{aligned}\sigma_j[u,\phi_j] &= \sigma_j = \sigma_j^s = C_j e_j^s \\ &= C_j(e_j - e_j^d) = C_j(e - e_j^d) = C_j(e[u] - \phi_j).\end{aligned}$$

Remark 2.1. We interpret the system for the ESLSM as a special case of the system for the EMM in the sense that $\eta_j^{-1}C_j = 0$ for some of the values of j, and omitting the ϕ_j for such values.

The AD system (2.2) and its energy dissipation structure were studied in [17] and in [10], based on a preceding work [8], where a simplified AD system without the inertia term and its energy dissipation structure were considered.

The first two equations of (2.2) are considered on $\Omega \times \mathbb{R}$ unless otherwise specified in the further analysis. The function spaces for the solution and the initial data will be specified later, in sections 3, 4, and 5, while addressing the unique solvability of (2.2); this will follow from the existence of a C_0-(semi)group for (2.2).

Next, we discuss the ID system associated with the EMM. Essentially, this system follows directly from (2.2) upon setting $\phi^0 = 0$. With this initial condition, one can integrate

$$\begin{cases}\eta_j \partial_t \phi_j = \sigma_j[u,\phi_j] = C_j(e[u] - \phi_j), \\ \phi_j(0) = 0,\end{cases} \tag{2.4}$$

to yield

$$\phi_j(t) = \int_0^t e^{-(t-s)\eta_j^{-1}C_j}\,\eta_j^{-1}C_j\, e[u(s)]ds, \tag{2.5}$$

where the x-dependence of the different functions and tensors is suppressed. With the first equation of (2.2), we directly obtain

$$\sigma[u] = G(x,0)e[u](t) + \int_0^t (\partial_t G)(x,t-s)e[u(s)]\,ds \tag{2.6}$$

with ϕ being eliminated and signifying a description in terms of relaxation tensor

$$G(x,t) := \sum_{j=1}^n e^{-t\eta_j^{-1}C_j} C_j. \tag{2.7}$$

The ID system for the EMM is then given by

$$\begin{cases}\rho\,\partial_t^2 u - \operatorname{div}\sigma[u] = 0, \\ u = 0 \quad \text{on} \quad \Gamma_D, \\ \sigma[u]\nu = 0 \quad \text{on} \quad \Gamma_N, \\ (u,\partial_t u)|_{t=0} = (u^0, v^0) \quad \text{on} \quad \Omega.\end{cases} \tag{2.8}$$

Here we can include the ESLSM by setting $\eta_j C_j = 0$ for some of the values of j.

We remark that this system is equivalent to (2.2) subject to the restriction $\phi^0 = 0$. Also, this initial condition avoids the occurrence of stationary solutions. The mentioned reduced system, which will be discussed later, is a (closed) system given in terms of other dependent variables; the reduced system does not have stationary solutions either. However, it generates such solutions upon transforming the variables back to the original ones. We remark that this restriction has profound consequences such as preventing the generation of a semigroup. The well-posedness of this system in appropriate function spaces and the exponential decay of solutions in these function spaces are presented in a companion paper [3]. We note that the exponential of the relaxation tensor is evaluated through an expansion and contractions of rank 4 tensor C_j with itself.

Ahead of the further analysis of properties of solutions, we note that obtaining exponential decay is a nontrivial matter. In the case that EMM includes an ESLS, neither the ϕ_j's nor the solution u of the ID system exhibits this decay. In the case of a pure EMM, the exponential decay of each ϕ_j depends on the relation between the lower bound of positive symmetric matrix $\eta_j^{-1}C_j$ and the exponential decay rate of solutions of the ID system.

3. AD system and its semigroup. We rewrite the AD system by introducing the following notations. We let $\breve{C} := \operatorname{block\,diag}(C_1, \dots, C_n)$, $\breve{\eta} := \operatorname{block\,diag}(\eta_1 I_d, \dots, \eta_n I_d)$, where I_d stands for the $d \times d$ identity matrix and $\breve{I} := \operatorname{block\,diag}(I_d, \dots, I_d)$. To be more precise, for example, $\breve{C}$ is defined for $C_1, \dots, C_n$ as

$$\breve{C} = \begin{pmatrix} C_1 & O & \cdots & 0 \\ O & C_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ O & O & \cdots & C_n \end{pmatrix}.$$

Then, by using these notations, we can write $\sigma[u, \phi]$ in the forms

$$\begin{aligned} \sigma[u,\phi] &= \sum_{j=1}^{n} \sigma_j[u,\phi_j] = \sum_{j=1}^{n} C_j(e[u]-\phi_j) \\ &= \operatorname{trace}\left[\operatorname{diag}(C_1(e[u]-\phi_1), C_2(e[u]-\phi_2), \dots, C_n(e[u]-\phi_n))\right] \\ &= \mathfrak{T}\{\breve{C}(e[u]\breve{I}-\phi)\}, \end{aligned} \tag{3.1}$$

where $\mathfrak{T}$ denotes the trace for the diagonal blocks. Also, we understand the multiplications $\breve{C}\phi$, $e[u]\breve{I}$ and $\breve{C}(e[u]\breve{I})$ as follows:

$$\begin{cases} \breve{C}\phi := \operatorname{block\,diag}(C_1\phi_1, \dots, C_n\phi_n) \\ \text{with } C_j\phi_i := ((C_j\phi_i)_{kl}), (C_j\phi_i)_{kl} = \sum_{r,s=1}^{n} (C_j)_{klrs}(\phi_i)_{rs}, \\ e[u]\breve{I} := \operatorname{block\,diag}(e[u], \dots, e[u]), \\ \breve{C}(e[u]\breve{I}) := \operatorname{block\,diag}(C_1 e[u], \dots, C_n e[u]). \end{cases}$$

In order to write the equations in (2.2) as a first order system with respect to the time derivative, we let $u_1 = u$, $u_2 = \partial_t u$, and $U := (u_1, u_2, \phi)^{\mathbf{t}}$. Then, using (2.2) and (3.1), we get

$$\begin{cases} \partial_t u_1 = u_2, \\ \partial_t u_2 = \rho^{-1} \operatorname{div} \sigma[u_1,\phi] = \rho^{-1} \operatorname{div} \mathfrak{T}\{\breve{C}(e[u_1]\breve{I}-\phi)\}, \\ \partial_t \phi = \breve{\eta}^{-1}\{\breve{C}(e[u_1]\breve{I}-\phi)\}, \end{cases} \tag{3.2}$$

which takes the form $\partial_t U = AU$ with

$$A = \begin{pmatrix} O & I_d & O \\ \rho^{-1}\operatorname{div}\mathfrak{T}\{\breve{C}(e[\cdot]\breve{I})\} & O & -\rho^{-1}\operatorname{div}\mathfrak{T}\{\breve{C}\cdot\} \\ \breve{\eta}^{-1}\{\breve{C}(e[\cdot]\breve{I})\} & O & -\breve{\eta}^{-1}\{\breve{C}\cdot\} \end{pmatrix}, \tag{3.3}$$

where O denotes the zero matrix. Since we need to prepare several notations to define the domain $D(A)$ of A, we will give it later when we introduce another operator A_Z.

In the later resolvent analysis which we will give in section 4 for $\tilde{A}$ pertaining to A, we need to consider the resolvent equation for the (u_1, u_2) components separately. For that, we have to give an appropriate regularity for ϕ so that it will not affect the boundary condition over Γ_N. Hence, we have to modify the blockwise component $-\rho^{-1}\mathrm{div}\mathfrak{T}\{\breve{C}\cdot\}$ of A not to decrease the regularity when it is applied to ϕ. More precisely, we want to have the term $-\rho^{-1}\mathrm{div}\,\mathfrak{T}\{(\lambda I+\breve{\eta}^{-1}\breve{C})^{-1}\breve{C}(\mathfrak{z}\omega)\}\in L^2_\rho(\Omega)$ which is the term of a component of the inhomogeneous term of (4.4) equivalent to λ-equation (4.1) given later in section 4. Therefore, we introduce A_Z by modifying A as follows:

$$A_Z := \mathfrak{Z}^{-1}A\mathfrak{Z} = \begin{pmatrix} O & I_d & O \\ \rho^{-1}\mathrm{div}\,[\mathfrak{T}\{\breve{C}(e[\cdot]\breve{I})\}] & O & -\rho^{-1}\mathrm{div}\,[\mathfrak{T}\{(\breve{C}\mathfrak{z})\cdot\}] \\ \mathfrak{z}^{-1}\breve{\eta}^{-1}\{\breve{C}(e[\cdot]\breve{I})\} & O & -\mathfrak{z}^{-1}\breve{\eta}^{-1}[(\breve{C}\mathfrak{z})\cdot] \end{pmatrix}, \tag{3.4}$$

where

$$\mathfrak{Z} := \begin{pmatrix} I_d & O & O \\ O & I_d & O \\ O & O & \mathfrak{z} \end{pmatrix}, \tag{3.5}$$

$\mathfrak{z} = \zeta\breve{I}$ with

$$\zeta := (-\Delta)^{-1/2}:\ L^2(\Omega) \xrightarrow{\sim} H^1_0(\Omega).$$

Here Δ is the Laplace operator on Ω supplemented with the Dirichlet boundary condition on $\partial\Omega$ and $H^1_0(\Omega) := \{\phi_z \in H^1(\Omega) : \phi\big|_{\partial\Omega} = 0\}$.

Our immediate challenge is to show that A_Z generates a group on the Hilbert space $W_Z = S_\infty \times L^2(\Omega)$ given as the direct product of the Hilbert spaces $S_\infty := K(\Omega)\times L^2_\rho(\Omega)$ and $L^2(\Omega)$. Here, $K(\Omega) := \{v_1 \in H^1(\Omega) : v_1 = 0 \text{ in } \Omega\}$, $L^2_\rho(\Omega) := L^2(\Omega)$ and the inner products are as follows:

$$\begin{cases} (\ ,\ )_{W_Z} := (\ ,\ )_{S_\infty} + (\ ,\ ), \\ (V, V')_{S_\infty} := (C\nabla v_1, \nabla v_1') + (v_2, v_2')_\rho \text{ for } V = (v_1, v_2)^{\mathbf{t}},\ V' = (v_1', v_2')^{\mathbf{t}} \in S_\infty \end{cases} \tag{3.6}$$

with $C := \sum_{j=1}^n C_j$, $L^2(\Omega)$-inner product $(\ ,\ )$, and $L^2_\rho(\Omega)$-inner product $(v_2, v_2')_\rho = \int_\Omega v_2 \bar{v_2'} \rho\, dx$ for $v_2,\ v_2' \in L^2_\rho(\Omega)$. Then, domain $D(A_Z)$ of A_Z is given as

$$D(A_Z) := \{V \in W_Z : A_Z V \in \mathcal{L}^2(\Omega)\} \tag{3.7}$$

with $\mathcal{L}^2(\Omega) := L^2_\rho(\Omega)\times L^2(\Omega)\times L^2(\Omega)$. Also, the postponed definition of the domain $D(A)$ is given as

$$D(A) := \{U \in W :\ AU \in \mathcal{L}^2(\Omega)\}, \tag{3.8}$$

where $W := S_\infty \times H^1_0(\Omega)$ equipped with the inner product $(\ ,\ )_W := (\ ,\ )_{S_\infty} + (\ ,\ )_{H^1_0(\Omega)}$, and $(\ ,\ )_{H^1_0(\Omega)}$ denotes the $H^1_0(\Omega)$-inner product.

To see the mentioned generation of a group based on the Hille–Yoshida theorem [11], [14], we need to prove that A_Z is a densely defined closed operator on W_Z with the estimate

$$\|(\lambda I - A_Z)^{-1}\| \le (|\lambda| - \beta)^{-1}, \quad |\lambda| > \beta,$$

for the resolvent $(\lambda I - A_Z)^{-1}$ with an appropriate constant $\beta > 0$. Here $\|\cdot\|$ is the operator norm for the bounded linear operator $(\lambda I - A_Z)^{-1}$ on W_Z, and I is the identity operator. Here, we have used the following convention for the notation I. That is, even though the I here is the multiplication operator by the $(n+2)d \times (n+2)d$ identity matrix, we just write this operator as I. This convention will be used for the resolvent and the associated λ-equation (see (4.1) given just below) in the rest of this paper.

4. Resolvent estimate. To analyze the resolvent $(\lambda I - A_Z)^{-1}$, consider the λ-equation

$$(\lambda I - A_Z)V = F, \tag{4.1}$$

where $V = (v_1, v_2, \phi_z)^{\mathbf{t}} \in D(A_Z),\ F = (f_1, f_2, \omega)^{\mathbf{t}} \in \mathcal{L}^2(\Omega)$.

The componentwise expression of (4.1) is given as

$$\begin{cases} \lambda v_1 - v_2 = f_1, \\ \lambda v_2 - \rho^{-1}\mathrm{div}\,[\mathfrak{T}\{\breve{C}(e[v_1]\breve{I})\}] + \rho^{-1}\mathrm{div}\,[\mathfrak{T}\{(\breve{C}\mathfrak{z})\phi_z\}] = f_2, \\ \lambda\phi_z - \mathfrak{z}^{-1}\breve{\eta}^{-1}\{\breve{C}(e[v_1]\breve{I})\} + \mathfrak{z}^{-1}\breve{\eta}^{-1}[(\breve{C}\mathfrak{z})\phi_z] = \omega. \end{cases} \tag{4.2}$$

Substituting $(v_1, v_2, \phi_z)^{\mathbf{t}} = (u_1, u_2, \mathfrak{z}^{-1}\phi)^{\mathbf{t}}$, that is, $U = (u_1, u_2, \phi)^{\mathbf{t}} = \mathfrak{Z}V$, a computation yields

$$\begin{cases} \lambda u_1 = u_2 + f_1, \\ \lambda u_2 = \lambda\rho^{-1}\mathrm{div}\,\mathfrak{T}\{(\lambda I + \breve{\eta}^{-1}\breve{C})^{-1}\breve{C}\,(e[u_1]\breve{I})\} \\ \qquad\quad -\rho^{-1}\mathrm{div}\,\mathfrak{T}\{(\lambda I + \breve{\eta}^{-1}\breve{C})^{-1}\breve{C}(\mathfrak{z}\omega)\} + f_2, \\ \lambda\phi = \lambda(\lambda I + \breve{\eta}^{-1}\breve{C})^{-1}[\breve{\eta}^{-1}\{\breve{C}(e[u_1]\breve{I})\} + \mathfrak{z}\omega], \end{cases} \tag{4.3}$$

which can be written in the matrix form

$$(\lambda I - \tilde{A})U = (f_1, f_2 - \rho^{-1}\mathrm{div}\,\mathfrak{T}\{(\lambda I + \breve{\eta}^{-1}\breve{C})^{-1}\breve{C}(\mathfrak{z}\omega)\}, \lambda(\lambda I + \breve{\eta}^{-1}\breve{C})^{-1}(\mathfrak{z}\omega))^{\mathbf{t}}, \tag{4.4}$$

with

$$\tilde{A} := \begin{pmatrix} O & I_d & O \\ \lambda\rho^{-1}\mathrm{div}\,\mathfrak{T}\{(\lambda I + \breve{\eta}^{-1}\breve{C})^{-1}(\breve{C}e[\cdot]\breve{I})\} & O & O \\ \lambda(\lambda I + \breve{\eta}^{-1}\breve{C})^{-1}\breve{\eta}^{-1}\{\breve{C}(e[\cdot]\breve{I})\} & O & O \end{pmatrix}. \tag{4.5}$$

By $U = \mathfrak{Z}V$, it is clear that (4.1) is equivalent to

$$(\lambda I - A)U = \mathfrak{Z}F. \tag{4.6}$$

Further, we will see later that (4.4) is equivalent to (4.6). Note that the boundary condition over Γ_N of (2.2) is always able to express using the $(2,1)$-blockwise component of A, A_Z, $\tilde{A}$.

To see (4.4) is equivalent to (4.6), observe that the componentwise description (4.6) is given as

$$\begin{cases} \lambda u_1 = u_2 + f_1, \\ \lambda u_2 = \rho^{-1}\mathrm{div}\,\mathfrak{T}\{\breve{C}(e[u_1]\breve{I} - \phi)\} + f_2, \\ \lambda\phi = \breve{\eta}^{-1}\{\breve{C}(e[u_1]\breve{I} - \phi)\} + \mathfrak{z}\omega. \end{cases} \tag{4.7}$$

Then, using the third equation of (4.7), we find that

$$\phi = (\lambda I + \breve{\eta}^{-1}\breve{C})^{-1}[\breve{\eta}^{-1}\{\breve{C}(e[u_1]\breve{I})\} + \mathfrak{z}\omega]. \tag{4.8}$$

Combining (4.7) and (4.8), we obtain

$$\begin{aligned} \lambda^2 u_1 &= \lambda u_2 + \lambda f_1 \\ &= \rho^{-1}\mathrm{div}\,\mathfrak{T}\{\breve{C}(e[u_1]\breve{I})\} - \rho^{-1}\mathrm{div}\,\mathfrak{T}\{\breve{C}\phi\} + f_2 + \lambda f_1 \\ &= \rho^{-1}\mathrm{div}\,\mathfrak{T}\{\breve{C}(e[u_1]\breve{I})\} + f_2 + \lambda f_1 \\ &\quad - \rho^{-1}\mathrm{div}\,\mathfrak{T}\{\breve{C}(\lambda I + \breve{\eta}^{-1}\breve{C})^{-1}[\breve{\eta}^{-1}\{\breve{C}(e[u_1]\breve{I})\} + \mathfrak{z}\omega]\}. \end{aligned} \tag{4.9}$$

Now, observe that

$$\begin{aligned} &\breve{C}(e[u_1]\breve{I}) - \breve{C}(\lambda I + \breve{\eta}^{-1}\,\breve{C})^{-1}\,\breve{\eta}^{-1}\,\breve{C}(e[u_1]\breve{I}) \\ &\quad = \breve{C}(e[u_1]\breve{I}) - \breve{\eta}^{-1}\,\breve{C}(\lambda I + \breve{\eta}^{-1}\,\breve{C})^{-1}\,\breve{C}(e[u_1]\breve{I}) \\ &\quad = [I - \breve{\eta}^{-1}\,\breve{C}(\lambda I + \breve{\eta}^{-1}\,\breve{C})^{-1}]\,\breve{C}\,(e[u_1]\breve{I}) \\ &\quad = \lambda(\lambda I + \breve{\eta}^{-1}\,\breve{C})^{-1}\,\breve{C}\,(e[u_1]\breve{I}). \end{aligned} \tag{4.10}$$

Combining (4.9) and (4.10), we have

$$\begin{aligned} \lambda u_2 &= \lambda\rho^{-1}\mathrm{div}\,\mathfrak{T}\{(\lambda I + \breve{\eta}^{-1}\,\breve{C})^{-1}\,\breve{C}\,(e[u_1]\breve{I})\} \\ &\quad - \rho^{-1}\mathrm{div}\,\mathfrak{T}\{\breve{C}(\lambda I + \breve{\eta}^{-1}\breve{C})^{-1}\mathfrak{z}\omega\} + f_2 \\ &= \lambda\rho^{-1}\mathrm{div}\,\mathfrak{T}\{(\lambda I + \breve{\eta}^{-1}\,\breve{C})^{-1}\,\breve{C}\,(e[u_1]\breve{I})\} \\ &\quad - \rho^{-1}\mathrm{div}\,\mathfrak{T}\{(\lambda I + \breve{\eta}^{-1}\breve{C})^{-1}\breve{C}\mathfrak{z}\omega\} + f_2. \end{aligned} \tag{4.11}$$

This immediately implies (4.4). Since this argument is reversible, we have the equivalence.

We note that the rightmost column of $\tilde{A}$ in the blockwise sense is zero. We will later appreciate that this form of $\tilde{A}$ implies that 0 is not in the resolvent set of A_Z.

The remainder of this section is devoted to proving that A_Z generates a C_0-semigroup for both $t \geq 0$ and $t \leq 0$. This is accomplished by proving the following estimate upon analyzing the relevant λ-equation.

PROPOSITION 4.1. *There exists a constant $\beta > 0$ such that the resolvent exists for λ satisfying $|\lambda| > \beta$, and it satisfies the estimate*

$$\|(\lambda I - A_Z)^{-1}\| \leq (|\lambda| - \beta)^{-1}, \quad |\lambda| > \beta, \tag{4.12}$$

where $\|\cdot\|$ is the operator norm on the space W_Z.

Proof. The proof is quite long and it is divided into two cases: $\lambda > 0$ and $\lambda < 0$. The proofs for both cases are similar. So, after introducing some notations, we will provide some brief orientation for the proof for the case $\lambda > 0$. Let $W_\lambda := S_\lambda \times H_0^1(\Omega)$ with $S_\lambda := K(\Omega) \times L^2_\rho(\Omega)$. S_λ is a Hilbert space equipped with the inner product $(\cdot,\cdot)_{S_\lambda}$ depending on λ and given as

$$(V, V')_{S_\lambda} := (B_\lambda \nabla v_1, \nabla v_1') + (v_2, v_2')_\rho, \quad V = (v_1, v_2)^{\mathbf{t}}, \quad V' = (v_1', v_2')^{\mathbf{t}}, \tag{4.13}$$

where

$$B_\lambda \nabla v_1 := \lambda \mathfrak{T}\{(\lambda I + \breve{\eta}^{-1}\,\breve{C})^{-1}(\breve{C}e[v_1]\breve{I})\}.$$

In the further analysis, we will consider the asymptotic limits, B_∞ and S_∞, of B_λ and S_λ, which are defined via $B_\infty = \lim_{\lambda\to\infty} B_\lambda$ and the induced inner product. They are

$$B_\infty \nabla v_1 = \mathfrak{T}\{(\breve{C}e[v_1]\breve{I})\} = C\nabla v_1 \tag{4.14}$$

and the space W which we defined before. Also, let $\tilde{A}_u$ be the upper left submatrix of $\tilde{A}$, which has the form

$$\tilde{A}_u := \begin{pmatrix} O & I_d \\ \lambda\rho^{-1}\mathrm{div}\,\mathfrak{T}\{(\lambda I + \breve{\eta}^{-1}\breve{C})^{-1}(\breve{C}e[\cdot]\breve{I})\} & O \end{pmatrix} = \begin{pmatrix} O & I_d \\ \rho^{-1}\mathrm{div}(B_\lambda \nabla[\cdot]) & O \end{pmatrix}. \tag{4.15}$$

Here, we remark that special inner product (4.13) handles the boundary condition on Γ_N for the operator $\tilde{A}_u$.

Now, for the case $\lambda > 0$, we briefly give an orientation to the subsequent arguments. Having the equivalence relationship between three equations (4.1), (4.4), (4.6), the arguments are given in the following three steps:

(i) Show the existence of the resolvent $(\lambda I - \tilde{A}_u)^{-1}$ and prove that it satisfies the Hille–Yoshida type estimate:

$$\|(\lambda I - \tilde{A}_u)^{-1}\| \le \lambda^{-1},\ \lambda > 0,$$

where the norm $\|\ \|$ is the operator norm on S_λ.

(ii) Lift this up to show the existence of the resolvent $(\lambda I - \tilde{A})^{-1}$ for $\lambda > 0$ and the estimate

$$\|U\|_{W_\lambda} \le (\lambda - m)^{-1}\|\mathfrak{Z}F\|_{W_\lambda},\ \lambda > m,$$

for some $m > 0$.

(iii) By showing the estimates between the norms $\|\ \|_{W_\lambda}$ and $\|\ \|_W$, the above estimate in (ii) holds with respect to the norm $\|\ \|_W$ for larger λ. Then, this immediately implies (4.12).

Returning to continue the proof, we consider the equation $\tilde{A}_u Y_u = G$ with $G = (\hat{f}_1, \hat{f}_2)^{\mathbf{t}}$ and $Y_u = (y_1, y_2)^{\mathbf{t}}$, which is equivalent to $y_2 = \hat{f}_1$, $\rho^{-1}\mathrm{div}(B_\lambda \nabla y_1) = \hat{f}_2$. The second equation is related to the Neumann boundary condition on Γ_N. In terms of the continuous sesquilinear form

$$\mathfrak{a}'(y_1, z) := \int_\Omega (B_\lambda \nabla y_1)\overline{\nabla z},\ y_1, z \in K(\Omega), \tag{4.16}$$

we consider the linear map $K(\Omega) \ni y_1 \to \rho^{-1}\mathrm{div}(B_\lambda \nabla y_1)$ as the bounded linear operator $\mathfrak{A}' : K(\Omega) \to K(\Omega)'$ defined by

$$\mathfrak{a}'(y_1, z) = (\mathfrak{A}' y_1, z)_\rho,\ y_1, z \in K(\Omega),$$

where $K(\Omega)'$ is the dual space of $K(\Omega)$ with respect to the inner product $(\cdot,\cdot)_\rho$. Furthermore, we let $\mathfrak{A}:\ S_\lambda \to S_\lambda'$ be the operator defined by replacing the left lower block of the operator matrix in (4.15) by $\mathfrak{A}'$, that is,

$$\mathfrak{A} = \begin{pmatrix} O & I_d \\ \mathfrak{A}' & O \end{pmatrix},$$

where S_λ' is the dual space of S_λ with respect to the $L^2(\Omega) \times L^2_\rho(\Omega)$-inner product. Then we identify $\tilde{A}_u$ with $\mathfrak{A}|_{D(\tilde{A}_u)}$, with

$$D(\tilde{A}_u) := \{Y_u = (y_1, y_2)^{\mathbf{t}} \in S_\lambda : \mathfrak{A} Y_u \in L^2(\Omega) \times L^2_\rho(\Omega)\}.$$

By the coercivity of the sesquilinear form (4.16) due to Korn's inequality [4], we have $D(\tilde{A}_u) = S_\lambda$ [16, Chapter 3, section 2].

An elementary calculation leads to

$$\begin{aligned}(\tilde{A}_u Y_u, Y_u)_{S_\lambda} &= (B_\lambda \nabla y_2, \nabla y_1) + (\operatorname{div}\{B_\lambda \nabla y_1\}, y_2)\\ &= -(y_2, \operatorname{div}\{B_\lambda \nabla y_1\}) + (\operatorname{div}\{B_\lambda \nabla y_1\}, y_2)\\ &= -\{(y_2, \operatorname{div}\{B_\lambda \nabla y_1\}) - (\operatorname{div}\{B_\lambda \nabla y_1\}, y_2)\}\\ &= -(Y_u, \tilde{A}_u Y_u)_{S_\lambda}\end{aligned} \tag{4.17}$$

for $Y_u \in D(\tilde{A}_u)$. Using this equality, we obtain for $\lambda > 0$ and $Y_u \in D(\tilde{A}_u)$ the estimate

$$\begin{aligned}\|(\lambda I - \tilde{A}_u)Y_u\|^2_{S_\lambda} &= ((\lambda I - \tilde{A}_u)Y_u, (\lambda I - \tilde{A}_u)Y_u)_{S_\lambda}\\ &= \lambda^2 \|Y_u\|^2_{S_\lambda} + \|\tilde{A}_u Y_u\|^2_{S_\lambda}\\ &\geq \lambda^2 \|Y_u\|^2_{S_\lambda}.\end{aligned} \tag{4.18}$$

The case $\lambda > 0$. For any $\lambda > 0$, the bijectivity of the map

$$\lambda I - \tilde{A}_u : D(\tilde{A}_u) \to S_\lambda$$

can be shown using the unique solvability of the aforementioned variational problem by recalling the Korn inequality. Hence, in terms of the operator norm, (4.18) implies

$$\|(\lambda I - \tilde{A}_u)^{-1}\| \leq \lambda^{-1}, \quad \lambda > 0. \tag{4.19}$$

Now, we lift $\tilde{A}_u$ to $W_\lambda := S_\lambda \times H^1_0(\Omega)$ to define an operator which is nothing but $\tilde{A}$, which clearly has $D(\tilde{A}) = W_\lambda$, where $D(\tilde{A})$ is the domain of $\tilde{A}$ simply given as

$$D(\tilde{A}) := D(\tilde{A}_u) \times H^1_0(\Omega).$$

For any $\lambda > 0$,

$$\lambda I - \tilde{A} : D(\tilde{A}) \to W_\lambda$$

is bijective.

Combining the relevant components of (4.3) and (4.19), we find that

$$\begin{aligned}\|(u_1, u_2)\|_{S_\lambda} &\leq \lambda^{-1} \|(f_1, f_2 - \rho^{-1} \operatorname{div} \mathfrak{T}\{(\lambda I + \breve{\eta}^{-1}\breve{C})^{-1}\breve{C}(\mathfrak{z}\omega)\})\|_{S_\lambda}\\ &\leq \lambda^{-1}\|(f_1, f_2)\|_{S_\lambda} + m_1 \lambda^{-2} \|\mathfrak{z}\omega\|_{H^1_0(\Omega)}\end{aligned} \tag{4.20}$$

for some positive constant m_1. With the third equation of (4.3) and (4.20), we get

$$\begin{aligned}\|\phi\|_{H^1_0(\Omega)} &= \|(\lambda I + \breve{\eta}^{-1}\breve{C})^{-1}[\breve{\eta}^{-1}\{\breve{C}(e[u_1]\breve{I})\} + \mathfrak{z}\omega]\|_{H^1_0(\Omega)}\\ &\leq m_2 \lambda^{-1} \|(u_1, u_2)\|_{S_\lambda} + \lambda^{-1}(1 + m_3 \lambda^{-1}) \|\mathfrak{z}\omega\|_{H^1_0(\Omega)}\\ &\leq m_4 \lambda^{-2} \|(f_1, f_2)\|_{S_\lambda} + \lambda^{-1}(1 + m_5 \lambda^{-1}) \|\mathfrak{z}\omega\|_{H^1_0(\Omega)},\end{aligned} \tag{4.21}$$

where m_2, m_3, m_4, m_5 are positive constants chosen appropriately.

We introduce m according to

$$m := m_1 + m_4 + m_5. \tag{4.22}$$

For $\lambda > m$, we have the elementary estimate

$$\begin{aligned}\lambda^{-1} + m\lambda^{-2} &= (\lambda - m)^{-1}(\lambda - m)\lambda^{-1}(1 + m\lambda^{-1}) \\ &= (\lambda - m)^{-1}(1 - m\lambda^{-1})(1 + m\lambda^{-1}) \\ &= (\lambda - m)^{-1}(1 - m^2\lambda^{-2}) \\ &\leq (\lambda - m)^{-1}.\end{aligned} \tag{4.23}$$

Combining (4.20), (4.21), and (4.23), we then obtain

$$\begin{aligned}\|U\|^2_{W_\lambda} &\leq \lambda^{-2}(1 + m_4\lambda^{-1})^2\|(f_1, f_2)\|^2_{S_\lambda} + \lambda^{-2}(1 + m_1\lambda^{-1} + m_5\lambda^{-1})^2\|\mathfrak{z}\omega\|^2_{H^1_0(\Omega)} \\ &\leq (\lambda^{-1} + m\lambda^{-2})^2\|(f_1, f_2)\|^2_{S_\lambda} + (\lambda^{-1} + m\lambda^{-2})^2\|\mathfrak{z}\omega\|^2_{H^1_0(\Omega)} \\ &\leq (\lambda^{-1} + m\lambda^{-2})^2\|\mathfrak{Z}F\|^2_{W_\lambda} \\ &\leq (\lambda - m)^{-2}\|\mathfrak{Z}F\|^2_{W_\lambda},\end{aligned} \tag{4.24}$$

where $\|\cdot\|^2_{W_\lambda} = \|\cdot\|^2_{S_\lambda} + \|\cdot\|^2_{H^1_0(\Omega)}$.

However, the norm of W_λ depends on λ. To proceed, we need an estimate similar to (4.24) with respect to a norm independent on λ. To this end, we consider the asymptotic behavior of the norm of W_λ as $|\lambda| \to \infty$.

We revisit (4.13) and note that the asymptotic behavior of B_λ determines how the desired estimate can be obtained. We have

$$\begin{aligned}(B_\lambda\nabla v_1, \nabla v_1) &= (\lambda\mathfrak{T}\{(\lambda I + \breve{\eta}^{-1}\breve{C})^{-1}(\breve{C}e[v_1]\breve{I})\}, \nabla v_1) \\ &= (\mathfrak{T}\{(I + \lambda^{-1}\breve{\eta}^{-1}\breve{C})^{-1}(\breve{C}e[v_1]\breve{I})\}, \nabla v_1) \\ &\leq (\mathfrak{T}\{\breve{C}e[v_1]\breve{I}\}, \nabla v_1) + l_1\lambda^{-1}(\mathfrak{T}\{\breve{C}e[v_1]\breve{I}\}, \nabla v_1) \\ &\leq (C\nabla v_1, \nabla v_1) + l_2\lambda^{-1}\|\nabla y_1\|^2_{L^2(\Omega)}\end{aligned} \tag{4.25}$$

with positive constants l_1, l_2. Then, by (3.6), (3.8), and (4.25), we obtain the estimates

$$\begin{aligned}\|Y\|^2_W &\leq (1 + k_1\lambda^{-1})\|Y\|^2_{W_\lambda}, \\ \|Y\|^2_{W_\lambda} &\leq (1 + k_2\lambda^{-1})\|Y\|^2_W,\end{aligned} \tag{4.26}$$

where k_1, k_2 are suitable positive constants. Combining (4.24) and (4.26), we have, for $\lambda > m \geq 1$, that

$$\begin{aligned}\|U\|^2_W &\leq (1 + k_1\lambda^{-1})\|U\|^2_{W_\lambda} \\ &\leq (1 + k_1\lambda^{-1})(\lambda - m)^{-2}\|\mathfrak{Z}F\|^2_{W_\lambda} \\ &\leq (1 + k_1\lambda^{-1})(\lambda - m)^{-2}(1 + k_2\lambda^{-1})\|\mathfrak{Z}F\|^2_W \\ &\leq (1 + k\lambda^{-1})(\lambda - m)^{-2}\|\mathfrak{Z}F\|^2_W,\end{aligned} \tag{4.27}$$

where k is given by

$$k := 1 + k_1 + k_2 + k_1k_2. \tag{4.28}$$

For $\lambda > m + k$, we have the elementary estimate,

$$\begin{aligned}(1 + k\lambda^{-1})(\lambda - m)^{-2} &\leq (1 + k(\lambda - m)^{-1})(\lambda - m)^{-2} \\ &= (\lambda - m - k)^{-2}(\lambda - m - k)^2(\lambda - m)^{-2}(1 + k(\lambda - m)^{-1}) \\ &= (\lambda - m - k)^{-2}(1 - k(\lambda - m)^{-1})^2(1 + k(\lambda - m)^{-1}) \\ &\leq (\lambda - m - k)^{-2}(1 - k(\lambda - m)^{-1})(1 + k(\lambda - m)^{-1}) \\ &\leq (\lambda - m - k)^{-2}.\end{aligned} \tag{4.29}$$

Combining (4.27) and (4.29), we have that for $\lambda > m+k$, the following estimate holds:

$$\|U\|_W^2 \le (\lambda - m - k)^{-2}\|\mathfrak{Z}F\|_W^2. \tag{4.30}$$

In terms of V, (4.30) is given as

$$\|V\|_{W_Z} \le (\lambda - m - k)^{-1}\|F\|_{W_Z},\ \lambda > m+k. \tag{4.31}$$

The case $\lambda < 0$. It is clear that if $\lambda \le -l_0$ for a large enough $l_0 > 0$, then $(V, V')_{S_\lambda}$ in (4.13) is a norm. We repeat the arguments from (4.13) to (4.18), and conclude having the estimate

$$\|(\lambda I - \tilde{A}_u)^{-1}\| \le (-\lambda)^{-1}, \quad \lambda \le -l_0, \tag{4.32}$$

in terms of the operator norm on S_λ. Combining (4.4) and (4.32), we have for $\lambda \le -l_1$, l_1 sufficiently large,

$$\begin{aligned}\|(u_1,u_2)\|_{S_\lambda} &\le (-\lambda)^{-1}\|(f_1, f_2 - \rho^{-1}\mathrm{div}\,\mathfrak{T}\{(\lambda I + \breve{\eta}^{-1}\breve{C})^{-1}\breve{C}(\mathfrak{z}\omega)\})\|_{S_\lambda}\\ &\le (-\lambda)^{-1}\|(f_1,f_2)\|_{S_\lambda} + m_1\lambda^{-2}\|\mathfrak{z}\omega\|_{H_0^1(\Omega)};\end{aligned} \tag{4.33}$$

here, $m_1 \ge l_1 > 0$ in (4.20) large so that they can be used here. With the third equation of (4.3) and (4.33), we obtain, for $\lambda \le -l_1$,

$$\begin{aligned}\|\phi\|_{H_0^1(\Omega)} &= \|(\lambda I + \breve{\eta}^{-1}\breve{C})^{-1}[\breve{\eta}^{-1}\{\breve{C}(e[u_1]\breve{I})\} + \mathfrak{z}\omega]\|_{H_0^1(\Omega)}\\ &\le m_2(-\lambda)^{-1}\|(u_1,u_2)\|_{S_\lambda} + (-\lambda)^{-1}(1 + m_3(-\lambda)^{-1})\|\mathfrak{z}\omega\|_{H_0^1(\Omega)}\\ &\le m_4\lambda^{-2}\|(f_1,f_2)\|_{S_\lambda} + (-\lambda)^{-1}(1 + m_5(-\lambda)^{-1})\|\mathfrak{z}\omega\|_{H_0^1(\Omega)}.\end{aligned} \tag{4.34}$$

We have taken m_2, m_3, m_4, m_5 in (4.21) large to be able to use a common notation. We then repeat the arguments from (4.23) to (4.31) and get

$$\|V\|_{W_Z} \le (-\lambda - m - k)^{-1}\|F\|_{W_Z},\ -\lambda > m+k. \tag{4.35}$$

Combining (4.31) and (4.35), we find that

$$\|V\|_{W_Z} \le (|\lambda| - m - k)^{-1}\|F\|_{W_Z},\ |\lambda| > m+k. \tag{4.36}$$

Therefore, we obtained (4.12) for A_Z with $\beta := m+k$. ∎

5. C_0-group for the AD system. In this section, based on the resolvent estimates in the previous section, we will discuss the generation of a C_0-group for the AD system. For that, we first recall the following standard theorem (see Corollary of Theorem 5.6 on page 296 of [11] or Theorem 6.3 on page 23 of [14]).

THEOREM 5.1. *Let $\mathcal{A}$ be a closed operator on a Banach space X having a dense domain of definition in X. If there exists $\beta \ge 0$ such that for $|\lambda| > \beta$, the resolvent $(\lambda I - \mathcal{A})^{-1}$ of $\mathcal{A}$ exists and satisfies*

$$\|(\lambda I - \mathcal{A})^{-1}\| \le (|\lambda| - \beta)^{-1},\ |\lambda| > \beta, \tag{5.1}$$

then $\mathcal{A}$ generates a C_0-group on X.

With Theorem 5.1, the resolvent estimate in the previous section implies the following.

THEOREM 5.2. *A_Z generates a C_0-group on W_Z and $\rho(A_Z) \supset \{\lambda \in \mathbb{R} : |\lambda| > \beta\}$, where $\rho(A_Z)$ denotes the resolvent set of A_Z and $\beta = m + k$ with m and k defined as* (4.22) *and* (4.28), *respectively.*

Abstract Cauchy problem. As an immediate consequence of this theorem, we obtain the following theorem.

THEOREM 5.3.

(i) *Let $F \in C^1([0,\infty); W_Z)$ and consider the following Cauchy problem:*

$$\begin{cases} \dfrac{d}{dt} V(t) = A_Z V(t) + F(t), \quad t > 0, \\ V(0) = V^0 \in D(A_Z). \end{cases} \tag{5.2}$$

Then, there exists a unique strong solution $V = V(t) \in C^0([0,\infty); W_Z) \cap C^1((0,\infty); W_Z)$ of (5.2). *Here, in addition to the usual conditions for the solution of* (5.2), *the strong solution $V(t)$ has to be differentiable almost everywhere in $(0,\infty)$ and $dV(t)/dt \in L^1((0,T); W_Z)$ for each $T > 0$.*

(ii) *Concerning the regularity of the solution of* (5.2), *let $F \in C^{m+1}([0,\infty); W_Z)$ for $m \in \mathbb{N}$ and assume that the condition $V^\ell \in D(A_Z)$, $1 \le l \le m$, referred to as the compatibility condition of order m, holds. Here, the V^ℓ's are defined as*

$$V^\ell := A_Z V^{l-1} + F^{(l-1)}(0) \text{ with } F^{(l-1)} := \frac{d^{l-1}F}{dt^{l-1}}.$$

Consider

$$V(t) := \sum_{l=0}^{m-1} \frac{t^l}{l!} V^l + \int_0^t \frac{(t-s)^{m-1}}{(m-1)!} \tilde{V}(s)\, ds, \tag{5.3}$$

where $\tilde{V} \in C^0([0,\infty); W_Z) \cap C^1((0,\infty); W_Z)$ is the unique strong solution to the Cauchy problem

$$\begin{cases} \dfrac{d}{dt} \tilde{V}(t) = A_Z \tilde{V}(t) + F^{(m)}(t), \quad t > 0, \\ \tilde{V}(0) = V^m. \end{cases} \tag{5.4}$$

Then, $V = V(t) \in C^m([0,\infty); W_Z) \cap C^{m+1}((0,\infty); W_Z)$ is the unique strong solution to (5.2) *in the space $C^0([0,\infty); W_Z) \cap C^1((0,\infty); W_Z)$.*

(iii) *Statements* (i) *and* (ii) *hold upon replacing $t > 0$, $[0,\infty)$, and $(0,\infty)$ by $t < 0$, $(-\infty, 0]$, and $(-\infty, 0)$, respectively. Furthermore, the two solutions in the above two different intervals match at $t = 0$ up to mth order derivatives if $F(t) \in C^{m+1}(\mathbb{R}; W_Z)$, and this $F(t)$ together with $V^0 \in D(A_Z)$ satisfies the compatibility condition of order m. Hence,* (ii) *implies the existence of a unique strong solution $V \in C^m(\mathbb{R}; W_Z)$ to* (5.2).

We refer to Theorem 5.6 of [11] and section 4.2 of [14] for the study of the abstract Cauchy problem and the generation of a C_0-semigroup; we mention section 1.6 [14], where the relation between the C_0-semigroup and the C_0-group is established. As for (5.3), there is a similar formula in [2, 6]. By quite a formal argument, except for verifying the commutativity of A_Z and $\int_0^t \cdot\, ds$, it follows that $V(t)$ given by (5.3) is the unique solution to (5.2) in the space $C^0([0,\infty); W_Z) \cap C^1((0,\infty); W_Z)$.

The verification of the mentioned commutativity can be shown by using that A_Z is a closed operator.

We conclude this section by showing that $0 \notin \rho(A_Z)$. This yields an obstruction to guaranteeing decaying properties of the solutions, which would require that $\{i\mu : \mu \in \mathbb{R}\} \subset \rho(A_Z)$. By relation (3.4) between A_Z and A via the isomorphism $\mathfrak{Z}$, we have $\rho(A_Z) = \rho(A)$. Hence, it is enough to show $0 \notin \rho(A)$. Let $U \in W$ satisfy $AU = 0$. Then, using (4.7), we have

$$\begin{cases} u_2 = 0, \\ \operatorname{div} \mathfrak{T}\{\breve{C}\,(e[u_1]\breve{I})\} - \operatorname{div} \mathfrak{T}\{\breve{C}\phi\} = 0, \\ \breve{C}(e[u_1]\breve{I}) - \breve{C}\phi = 0. \end{cases} \tag{5.5}$$

Now, let $0 \neq \phi \in L^2(\Omega)$ and search for a u_1 that satisfies $e[u_1]\breve{I} = \phi$. Then u_1 is given as the unique nonzero solution $u_1 \in K(\Omega)$ of the following boundary value problem:

$$\begin{cases} \operatorname{div} \mathfrak{T}\{\breve{C}(e[u_1]\breve{I}\} = \operatorname{div} \mathfrak{T}\{\breve{C}\phi\}, \\ u_1 = 0 \quad \text{on } \Gamma_D, \\ (\mathfrak{T}\{\breve{C}(e[u_1]\breve{I})\})\nu = (\mathfrak{T}\{\breve{C}\phi\})\nu \quad \text{on } \Gamma_N. \end{cases} \tag{5.6}$$

We note here that the above boundary condition on Γ_N comes from the boundary condition $\sigma[u_1, \phi]\nu = 0$ on Γ_N, which is consistent with the occurrence of the inhomogeneous term in the first equation of (5.6). Hence, there exists $0 \neq U = (u_1, 0, \phi)^{\mathbf{t}} \in D(A)$, $AU = 0$. As a consequence, we have $0 \notin \rho(A)$.

6. Reduced system, C_0-group, and abstract Cauchy problem.

To mitigate the fact that $0 \notin \rho(A)$, we introduce a reduction of the original system. We then establish that this system generates a C_0-group.

The reduced system. We let $v = \partial_t u$ and $\psi = e[u]\breve{I} - \phi$. We observe that (2.2) contains the following closed subsystem:

$$\begin{cases} \partial_t v = \rho^{-1} \operatorname{div} \mathfrak{T}\{\breve{C}\psi\}, \\ \partial_t \psi = -\breve{\eta}^{-1}\breve{C}\psi + e[v]\breve{I}, \\ v = 0 \quad \text{on} \quad \Gamma_D, \quad (\mathfrak{T}\{\breve{C}\psi\})\nu = 0 \quad \text{on} \quad \Gamma_N, \\ (v, \psi) = (v^0, e[u^0]\breve{I} - \phi^0) \quad \text{at } t = 0. \end{cases} \tag{6.1}$$

Remark 6.1. By assuming that the initial value for u in (6.1) is u^0, we obtain a solution (u, v, ϕ) of (2.2) from the relation $v = \partial_t u$, $\psi = e[u]\breve{I} - \phi$.

Next, we rewrite this initial boundary value problem (6.1) as an abstract Cauchy problem. To begin with, we let

$$L\cdot = \begin{pmatrix} 0 & \rho^{-1} \operatorname{div} \mathfrak{T}\{\breve{C}\cdot\} \\ e[\cdot]\breve{I} & -\breve{\eta}^{-1}\breve{C}\cdot \end{pmatrix} \tag{6.2}$$

and its domain $D(L)$ be given as

$$D(L) := \left\{ (v, \psi) \in K(\Omega) \times L^2(\Omega) : L \begin{pmatrix} v \\ \psi \end{pmatrix} \in L^2_\rho(\Omega) \times L^2(\Omega) \right\}.$$

We equip Hilbert space $H := L^2_\rho(\Omega) \times L^2(\Omega)$ with the inner product

$$(V, V')_H := (v, v')_\rho + (\psi, \psi')_{\breve{C}} \tag{6.3}$$

for $V = (v, \psi)^{\mathbf{t}}$, $V' = (v', \psi')^{\mathbf{t}}$, where $(\psi, \psi')_{\breve{C}} := (\breve{C}\psi, \psi')$ for ψ, $\psi' \in L^2(\Omega)$. This inner product is equivalent to the standard inner product of $L^2_\rho(\Omega) \times L^2(\Omega)$. It can be shown that $\overline{D(L)} = H$ and L is a closed operator in H in a way similar to how this was done for $\tilde{A}_u$. Then, the abstract Cauchy problem takes the form

$$\begin{cases} \partial_t \begin{pmatrix} v \\ \psi \end{pmatrix} = L \begin{pmatrix} v \\ \psi \end{pmatrix}, \\ (v, \psi) = (v^0, \psi^0) \quad \text{at } t = 0, \end{cases} \tag{6.4}$$

where $\psi^0 := e[u^0]\breve{I} - \phi^0$.

Next, we show that L generates a semigroup. To begin with, as we did in section 4, we consider the λ-equation associated with the first equation of (6.4):

$$(\lambda I - L) \begin{pmatrix} v \\ \psi \end{pmatrix} = \begin{pmatrix} f \\ \omega \end{pmatrix} \in H \text{ with } \lambda = \sigma + i\mu,\ \sigma,\ \mu \in \mathbb{R}. \tag{6.5}$$

This equation is equivalent to the system

$$\begin{cases} \lambda v = \rho^{-1} \mathrm{div} \mathfrak{T}\{\breve{C}\psi\} + f, \\ \lambda \psi = -\breve{\eta}^{-1} \breve{C}\psi + e[v]\breve{I} + \omega, \end{cases} \tag{6.6}$$

supplemented with the boundary condition given in (6.1). Since $\eta_j^{-1} C_j > 0$ for $1 \le j \le n$, there exists a $\delta > 0$ such that

$$\sigma I + \eta_j^{-1} C_j \ge \delta,\ 1 \le j \le n \text{ for } \sigma \ge -\delta_0 \text{ with a constant } \delta_0 > 0. \tag{6.7}$$

For given v, ψ can be obtained from the second equation of (6.6) based on (6.7). Substituting the result into the first equation of (6.6) gives

$$\lambda v = \rho^{-1} \mathrm{div} \mathfrak{T}\{\breve{C}(\lambda \breve{I} + \breve{\eta}^{-1}\breve{C})^{-1}(e[v]\breve{I})\} + \rho^{-1} \mathrm{div} \mathfrak{T}\{\breve{C}(\lambda \breve{I} + \breve{\eta}^{-1}\breve{C})^{-1}\omega\} + f. \tag{6.8}$$

The boundary condition, $\mathfrak{T}\{(\breve{C}\psi)\}\nu = 0$ on Γ_N, becomes

$$\{\mathfrak{T}(\{\breve{C}(\lambda \breve{I} + \breve{\eta}^{-1}\breve{C})^{-1}(e[v]\breve{I})\})\}\nu = -\mathfrak{T}\{\breve{C}(\lambda \breve{I} + \breve{\eta}^{-1}\breve{C})^{-1}\omega\}\nu \quad \text{on } \Gamma_N. \tag{6.9}$$

Here, we note the following identity: For any positive symmetric tensor K,

$$\begin{aligned} &(i\mu I + K)^{-1} = Q + iR, \\ &Q = K^{-1} - \mu^2 K^{-1}(\mu^2 I + K^2)^{-1},\ R = -\mu(\mu^2 I + K^2)^{-1}. \end{aligned} \tag{6.10}$$

Applying this identity to $(\lambda \breve{I} + \breve{\eta}^{-1}\breve{C})^{-1} = (i\mu \breve{I} + K)^{-1}$ with $K = \sigma \breve{I} + \breve{\eta}^{-1}\breve{C}$ leads to

$$\lambda v = \rho^{-1} \mathrm{div} \mathfrak{T}\{\mathcal{I}(e[v]\breve{I})\} + \rho^{-1} \mathrm{div} \mathfrak{T}\{\mathcal{I}\omega\} + f, \tag{6.11}$$

where $\mathcal{I}$ is given by

$$\mathcal{I} := (K - i\mu \breve{I})(\mu^2 \breve{I} + K^2)^{-1}\breve{C}. \tag{6.12}$$

Variational form. Associated to (6.11), we introduce a bilinear form, $B(v,w)$, on $K(\Omega)$ as follows:

$$B(v,w) := (\mathfrak{T}\{\mathcal{I}(e[v]\breve{I})\}, e[w]) + \lambda(v,w)_\rho,\ v,w \in K(\Omega). \tag{6.13}$$

Then, the variational problem which is equivalent to the boundary value problem for (6.6) with the aforementioned boundary conditions is given by

$$B(v,w) = (\mathrm{div}\mathfrak{T}\{\mathcal{I}\omega\}, w) + (f,w)_\rho \quad \text{for all } w \in K(\Omega). \tag{6.14}$$

Now, we split the bilinear form B into two parts upon splitting $\mathcal{I}$ into its real and imaginary parts,

$$\mathcal{I} = K(\mu^2\breve{I} + K^2)^{-1}\breve{C} + i\{-\mu(\mu^2\breve{I} + K^2)^{-1}\breve{C}\}. \tag{6.15}$$

That is,

$$\begin{aligned} &B = B_1 + iB_2,\\ &B_1(v,w) = (\mathfrak{T}\{K(\mu^2\breve{I} + K^2)^{-1}\breve{C}(e[v]\breve{I})\}, e[w]) + \sigma(v,w)_\rho,\\ &B_2(v,w) = -\mu(\mathfrak{T}\{(\mu^2\breve{I} + K^2)^{-1}\breve{C}(e[v]\breve{I}), e[w]) + \mu(v,w)_\rho. \end{aligned} \tag{6.16}$$

Here, B_1 and B_2 are both continuous symmetric bilinear forms on $K(\Omega)$.

Using that

$$\mathfrak{T}\{K(\mu^2\breve{I} + K^2)^{-1}\breve{C}(e[v]\breve{I})\} = \sum_{j=0}^{n} C_j K_j (\mu^2 I + K_j^2)^{-1} e[v], \tag{6.17}$$

where each K_j is given by $K_j = \sigma I + \eta_j^{-1} C_j$, and (6.7), it follows that B_1 is coercive for such σ due to the Korn inequality. Hence, the above variational problem is uniquely solvable by the usual argument [11, Chapter 3, section 9].

Generation of contractive C_0-group for L. We begin with the following.

LEMMA 6.2. *Let L be given by* (6.2). *The following holds true:*

$$\left\| (\lambda I - L)\begin{pmatrix} v \\ \psi \end{pmatrix} \right\|_H \geq \lambda \left\| \begin{pmatrix} v \\ \psi \end{pmatrix} \right\|_H, \quad \lambda > 0,\ \begin{pmatrix} v \\ \psi \end{pmatrix} \in D(L). \tag{6.18}$$

Proof. For $\left(\begin{smallmatrix} v \\ \psi \end{smallmatrix}\right) \in D(L)$, we have

$$\begin{aligned} \left(L\begin{pmatrix} v \\ \psi \end{pmatrix}, \begin{pmatrix} v \\ \psi \end{pmatrix} \right)_H &= (\rho^{-1}\mathrm{div}\mathfrak{T}\{\breve{C}\psi\}, v)_\rho + (-\breve{\eta}^{-1}\breve{C}\psi + e[v]\breve{I}, \psi)_{\breve{C}}\\ &= -(\psi, \breve{C}(e[v]\breve{I})) - (\breve{\eta}^{-1}\breve{C}\psi, \psi)_{\breve{C}} + (e[v]\breve{I}, \psi)_{\breve{C}}, \end{aligned} \tag{6.19}$$

$$\begin{aligned} \left(\begin{pmatrix} v \\ \psi \end{pmatrix}, L\begin{pmatrix} v \\ \psi \end{pmatrix} \right)_H &= (v, \rho^{-1}\mathrm{div}\mathfrak{T}\{\breve{C}\psi\})_\rho + (\psi, -\breve{\eta}^{-1}\breve{C}\psi + e[v]\breve{I})_{\breve{C}}\\ &= -(\breve{C}(e[v]\breve{I}), \psi) - (\breve{\eta}^{-1}\breve{C}\psi, \psi)_{\breve{C}} + (\psi, e[v]\breve{I})_{\breve{C}}. \end{aligned} \tag{6.20}$$

Combining (6.19) and (6.20), we find that

$$\left(L\begin{pmatrix} v \\ \psi \end{pmatrix}, \begin{pmatrix} v \\ \psi \end{pmatrix} \right)_H + \left(\begin{pmatrix} v \\ \psi \end{pmatrix}, L\begin{pmatrix} v \\ \psi \end{pmatrix} \right)_H = -2(\breve{\eta}^{-1}\breve{C}\psi, \psi)_{\breve{C}} \leq 0. \tag{6.21}$$

Hence, for $\lambda > 0$, $\binom{v}{\psi} \in D(L)$,

$$
\begin{aligned}
&\left\| (\lambda I - L) \begin{pmatrix} v \\ \psi \end{pmatrix} \right\|_H^2 \\
&= \lambda^2 \left\| \begin{pmatrix} v \\ \psi \end{pmatrix} \right\|_H^2 + \left\| L \begin{pmatrix} v \\ \psi \end{pmatrix} \right\|_H^2 - \lambda \left(L \begin{pmatrix} v \\ \psi \end{pmatrix}, \begin{pmatrix} v \\ \psi \end{pmatrix} \right)_H - \lambda \left(\begin{pmatrix} v \\ \psi \end{pmatrix}, L \begin{pmatrix} v \\ \psi \end{pmatrix} \right)_H \\
&\geq \lambda^2 \left\| \begin{pmatrix} v \\ \psi \end{pmatrix} \right\|_H^2 .
\end{aligned} \tag{6.22}
$$

Thus we have proved estimate (6.18). ☐

Using the aforementioned solvability of the variational problem, we get the bijectivity of the map $\lambda I - L : D(L) \to H$ for λ with $\mathrm{Re}\lambda > 0$ and, hence, the resolvent $(\lambda I - L)^{-1}$ satisfies

$$\|(\lambda I - L)^{-1}\| \leq \lambda^{-1}, \ \lambda > 0, \tag{6.23}$$

where $\|\cdot\|$ is the operator norm on H. Furthermore, for the resolvent set $\rho(L)$ of L, we have the inclusion

$$\{\sigma \geq -\delta_0\} \subset \rho(L) \text{ for some } \delta_0 > 0, \tag{6.24}$$

where δ_0 is the one given in (6.7). With these results and this observation, we formulate the following theorem.

THEOREM 6.3. *L generates a C_0-semigroup, e^{tL}, of contraction on H and the imaginary axis is in $\rho(L)$.*

Remark 6.4. It is possible to improve estimate (6.18). In fact, we have

$$\|(\lambda I - L)^{-1}\| \leq (\lambda^2 + \varepsilon_2)^{-1/2}, \quad \lambda > \beta, \tag{6.25}$$

for some $\varepsilon_2 > 0$ and $\beta < 0$.

Proof. Recalling (6.21), we modify estimate (6.22) as follows. We consider

$$L \begin{pmatrix} v \\ \psi \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \tag{6.26}$$

which is equivalent to

$$\begin{cases} \rho^{-1} \mathrm{div} \mathfrak{T}\{\breve{C}\psi\} = g_1, \\ -\breve{\eta}^{-1}\breve{C}\psi + e[v]\breve{I} = g_2, \end{cases} \tag{6.27}$$

where $g_1 \in L^2_\rho(\Omega)$ and $g_2 \in L^2(\Omega)$. Then, from (6.27), we have

$$(\rho g_1, v) = -(\mathfrak{T}\{\breve{C}\psi\}, e[v]) = (\mathfrak{T}(-\breve{\eta} e[v]\breve{I} + \breve{\eta} g_2), e[v]),$$

which is nothing but

$$-(|\eta| e[v], e[v]) = -(\mathfrak{T}(\breve{\eta} g_2), e[v]) + (\rho g_1, v).$$

Hence, by the positivity of $|\eta|$, we have

$$\varepsilon_1 (\|v\| + \|\nabla v\|) \leq \|g_1\|_\rho + \|g_2\| \tag{6.28}$$

for some $\varepsilon_1 > 0$, where $\|\cdot\|$ denotes the $L^2(\Omega)$-norm. From this estimate and the second equation of (6.27), we conclude that

$$2\varepsilon_2\|(v,\psi)^{\mathbf{t}}\|_H^2 \le \|(g_1,g_2)^{\mathbf{t}}\|_H^2 \tag{6.29}$$

for some $\varepsilon_2 > 0$. Hence, we have obtained

$$\left\|L\begin{pmatrix} v\\ \psi\end{pmatrix}\right\|_H^2 \ge 2\varepsilon_2\left\|\begin{pmatrix} v\\ \psi\end{pmatrix}\right\|_H^2. \tag{6.30}$$

Now, we choose a constant $\beta < 0$ such that $2\lambda(\breve\eta^{-1}\breve C\psi,\psi)+\varepsilon_2\|\psi\|^2 \ge 0$ for $\lambda > \beta$. Then, using (6.21), we have for $\lambda > \beta$ that

$$\begin{aligned}\left\|(\lambda I - L)\begin{pmatrix} v\\ \psi\end{pmatrix}\right\|_H^2 &\\ = \lambda^2\left\|\begin{pmatrix} v\\ \psi\end{pmatrix}\right\|_H^2 &+ \left\|L\begin{pmatrix} v\\ \psi\end{pmatrix}\right\|_H^2 - \lambda\left(L\begin{pmatrix} v\\ \psi\end{pmatrix},\begin{pmatrix} v\\ \psi\end{pmatrix}\right)_H - \lambda\left(\begin{pmatrix} v\\ \psi\end{pmatrix},L\begin{pmatrix} v\\ \psi\end{pmatrix}\right)_H\\ \ge (\lambda^2+\varepsilon_2)&\left\|\begin{pmatrix} v\\ \psi\end{pmatrix}\right\|_H^2.\end{aligned} \tag{6.31}$$

Thus, we have proven estimate (6.25). ∎

Next, we give an estimate of $(\lambda I - L)^{-1}$ for negative λ.

LEMMA 6.5. *There exists a positive constant l such that*

$$\left\|(\lambda I - L)\begin{pmatrix} v\\ \psi\end{pmatrix}\right\|_H \ge (-\lambda - l)\left\|\begin{pmatrix} v\\ \psi\end{pmatrix}\right\|_H, \quad -\lambda > l,\ \begin{pmatrix} v\\ \psi\end{pmatrix}\in D(L). \tag{6.32}$$

Proof. From (6.21) and (6.22), we have that if $\lambda \le -l_2$ for some positive constant l_2, then

$$\begin{aligned}&\left\|(\lambda I - L)\begin{pmatrix} v\\ \psi\end{pmatrix}\right\|_H^2\\ &\quad = \lambda^2\left\|\begin{pmatrix} v\\ \psi\end{pmatrix}\right\|_H^2 + \left\|L\begin{pmatrix} v\\ \psi\end{pmatrix}\right\|_H^2 + 2\lambda(\breve\eta^{-1}\breve C\psi,\psi)_{\breve C}\\ &\quad \ge (\lambda^2 - 2|\lambda|M)\left\|\begin{pmatrix} v\\ \psi\end{pmatrix}\right\|_H^2,\end{aligned} \tag{6.33}$$

where $M > l_2$ is a positive constant. Since

$$\begin{aligned}\lambda^2 - 2|\lambda|M &= \lambda^2 - 4|\lambda|M + 4M^2 + 2|\lambda|M - 4M^2\\ &= (|\lambda|-2M)^2 + 2M(|\lambda|-2M),\end{aligned} \tag{6.34}$$

we find that for $-\lambda > l$

$$\left\|(\lambda I - L)\begin{pmatrix} v\\ \psi\end{pmatrix}\right\|_H^2 \ge (|\lambda| - l)^2\left\|\begin{pmatrix} v\\ \psi\end{pmatrix}\right\|_H^2, \tag{6.35}$$

where $l = 2M$. ∎

Once again, from the aforementioned solvability of the variational problem, we have the following resolvent estimate:

$$\|(\lambda I - L)^{-1}\| \le (|\lambda| - l)^{-1},\ -\lambda > l, \tag{6.36}$$

where $\|\cdot\|$ is the operator norm on H.

Therefore, combining the two resolvent estimates (6.23) and (6.36), we arrive at the following.

THEOREM 6.6. *L generates a C_0-group on H.*

Abstract Cauchy problem. We obtain the unique solvability of the abstract Cauchy problem for the reduced system as we had obtained this for the full AD system. In Theorem 5.3, V becomes $(v,\psi)^{\mathbf{t}}$, and A_Z simply needs to be replaced by L and W_Z by H.

7. Decaying property and limiting amplitude principle for the reduced system. In this section, we first prove that any solution (v,ψ) of (6.1) whose initial data satisfies the compatibility condition of order 2 decays exponentially as $t\to\infty$. Then, combining this result with the fact that $\rho(L)$ contains the imaginary axis (see (6.24)), we prove the limiting amplitude principle for the AD system.

Exponential energy decay of solutions for the AD system. We start by introducing the energy

$$E(v,\psi):=\frac{1}{2}\|v\|_\rho^2+\frac{1}{2}\sum_{j=1}^n (C_j\psi_j,\psi_j), \tag{7.1}$$

where $(\ ,\ )$ denotes the $L^2(\Omega)$-inner product.

LEMMA 7.1. *The energy defined in* (7.1) *satisfies*

$$\frac{d}{dt}E(v,\psi)=-\sum_{j=1}^n \eta_j\|e[v]-\partial_t\psi_j\|^2, \tag{7.2}$$

where $\|\ \|$ denotes the $L^2(\Omega)$-norm.

Proof. The first equation of (6.1) implies that

$$(\rho\partial_t v,v)=\left(\operatorname{div}\sum_{j=1}^n C_j\psi_j,v\right)=-\sum_{j=1}^n (C_j\psi_j,e[v]). \tag{7.3}$$

The second equation of (6.1) gives

$$\eta_j(\partial_t\psi_j,\partial_t\psi_j)=(\eta_j e[v]-C_j\psi_j,\partial_t\psi_j)=(\eta_j e[v],\partial_t\psi_j)-(C_j\psi_j,\partial_t\psi_j). \tag{7.4}$$

A straightforward computation, using (7.3), (7.4), and the second equation of (6.1), yields

$$\begin{aligned}\frac{d}{dt}E(v,\psi)&=\sum_{j=1}^n (C_j\psi_j,\partial_t\psi_j)+(\rho\partial_t v,v)\\&=-\sum_{j=1}^n \eta_j\|\partial_t\psi_j\|^2+\sum_{j=1}^n(\eta_j e[v],\partial_t\psi_j)-\sum_{j=1}^n(C_j\psi_j,e[v])\\&=-\sum_{j=1}^n \eta_j\|\partial_t\psi_j\|^2+\sum_{j=1}^n 2\eta_j(e[v],\partial_t\psi_j)-\sum_{j=1}^n\eta_j\|e[v]\|^2\\&=-\sum_{j=1}^n\eta_j\|e[v]-\partial_t\psi_j\|^2,\end{aligned} \tag{7.5}$$

which is the statement of the lemma. ∎

We now differentiate (6.1) in t to obtain the following system for $(\partial_t v, \partial_t \psi)$:

(7.6)
$$\begin{cases} \partial_t^2 v = \rho^{-1}\mathrm{div}\mathfrak{T}\{\breve{C}\partial_t\psi\}, \\ \partial_t^2\psi = -\breve{\eta}^{-1}\breve{C}\partial_t\psi + e[\partial_t v]\breve{I}, \\ \partial_t v = 0 \quad \text{on} \quad \Gamma_D, \quad (\mathfrak{T}\{\breve{C}\partial_t\psi\})\nu = 0 \quad \text{on} \quad \Gamma_N, \\ (\partial_t v, \partial_t\psi)|_{t=0} \text{ is obtained by using the first and second equations of (6.1).} \end{cases}$$

The associated energy is

$$E(\partial_t v, \partial_t\psi) := \frac{1}{2}\|\partial_t v\|_\rho^2 + \frac{1}{2}\sum_{j=1}^n (C_j\partial_t\psi_j, \partial_t\psi_j),$$

which satisfies

$$\frac{d}{dt}E(\partial_t v, \partial_t\psi) = -\sum_{j=1}^n \eta_j\|e[\partial_t v] - \partial_t^2\psi_j\|^2 \tag{7.7}$$

in analogy to the statement in Lemma 7.1. We then define a higher energy, $\bar{E}(v,\psi)$, as

$$\bar{E}(v,\psi) = E(v,\psi) + E(\partial_t v, \partial_t\psi). \tag{7.8}$$

For simplicity of notations, we define

$$\|\tilde{v}\|_\rho^2 := \|v\|_\rho^2 + \|\partial_t v\|_\rho^2, \quad \|\tilde{\psi}\|^2 = \|\psi\|^2 + \|\partial_t\psi\|^2.$$

Then, from these definitions, we have

$$a_1(\|\tilde{v}\|_\rho^2 + \|\tilde{\psi}\|^2) \le \bar{E}(v,\psi) \le b_1(\|\tilde{v}\|_\rho^2 + \|\tilde{\psi}\|^2) \tag{7.9}$$

for some positive constants a_1, b_1 with $a_1 < 1 < b_1$. Further, using (7.2) and (7.7), we obtain from the second equation of (6.1),

$$\begin{aligned} \frac{d}{dt}\bar{E}(v,\psi) &= -\sum_{i=1}^n \eta_i\|e[v] - \partial_t\psi_i\|^2 - \sum_{i=1}^n \eta_i\|e[\partial_t v] - \partial_t^2\psi_i\|^2 \\ &\le -a_2\|\tilde{\psi}\|^2 \end{aligned} \tag{7.10}$$

for some positive constant $a_2 < 1$. Comparing (7.9) and (7.10), we need to amend $\bar{E}(v,\psi)$ through adding a function f_E so that $\frac{d}{dt}f_E$ has a contribution $-\|\partial_t v\|_\rho^2$. We define such an f_E by

$$f_E = \left(\sum_{j=1}^n C_j\psi_j, e[v]\right).$$

Using (6.1), a direct computation yields

$$\begin{aligned} \frac{d}{dt}f_E &= \left(\sum_{j=1}^n C_j\partial_t\psi_j, e[v]\right) + \left(\sum_{j=1}^n C_j\psi_j, e[\partial_t v]\right) \\ &= \left(\sum_{j=1}^n C_j\partial_t\psi_j, e[v]\right) - \left(\mathrm{div}\sum_{j=1}^n C_j\psi_j, \partial_t v\right) \\ &\le b_2\|\tilde{\psi}\|^2 - \|\partial_t v\|_\rho^2. \end{aligned} \tag{7.11}$$

Further, using the second equation of (6.1) and (7.9), we have

$$|f_E| \le b_3 \|\tilde{\psi}\|^2 \le \frac{b_3}{a_1} \bar{E}(v,\psi) \tag{7.12}$$

for some positive constant $b_3 > 1$. Based on these estimates, we define the amended energy as

$$\tilde{E}(v,\psi) := \bar{E}(v,\psi) + \frac{a_1 a_2}{2b_2 b_3} f_E. \tag{7.13}$$

Here, we note that $\{a_1, a_2\}$ and $\{b_1, b_2, b_3\}$ can be taken arbitrarily small and large, respectively. Hence, $\tilde{E}(v,\psi)$ is equivalent in the sense of norms to $\|\tilde{v}\|^2 + \|\tilde{\psi}\|^2$.

Combining (7.9) to (7.12) leads to the estimate for the time derivative,

$$\begin{aligned} \frac{d}{dt}\tilde{E}(v,\psi) &\le -a_2\|\tilde{\psi}\|^2 + \frac{a_1 a_2}{2b_3}\|\tilde{\psi}\|^2 - \frac{a_1 a_2}{2b_2 b_3}\|\partial_t v\|_\rho^2 \\ &\le -\frac{a_1 a_2}{2b_2 b_3}(\|\partial_t v\|_\rho^2 + \|\tilde{\psi}\|^2) \\ &\le -\frac{a_1 a_2}{2b_1 b_2 b_3}\bar{E}(v,\psi) \\ &\le -a_4 \tilde{E}(v,\psi), \end{aligned} \tag{7.14}$$

where $a_4 = (a_1 a_2)/(3b_1 b_2 b_3)$. Equation (7.14) implies the following exponential decay of solution (v,ψ) of (6.1):

$$a_1(\|\tilde{v}\|_\rho^2 + \|\tilde{\psi}\|^2) \le \bar{E}(v,\psi) \le 2\tilde{E}(v,\psi) \le 2\tilde{E}(v(0),\psi(0))e^{-a_4 t}. \tag{7.15}$$

This proves the next theorem.

THEOREM 7.2. *Let $(v(t),\psi(t)) \in C^2([0,\infty); H^1(\Omega)) \times C^2(([0,\infty); L^2(\Omega))$ be the solution of* (6.1) *satisying the compatibility condition of order* 2*. Then, there exists a constant $a_4 > 0$ independent of the initial values such that $(v(t),\psi(t))$ is exponentially decaying of order $O(e^{-a_4 t})$ as $t \to \infty$ with respect to the square root of the higher energy* (7.8).

Limiting amplitude principle. Next, we state the limiting amplitude principle for the AD system and give its proof. To begin with, we consider the following initial boundary value problem for the reduced system:

$$\begin{cases} \partial_t v = \rho^{-1}\mathrm{div}\mathfrak{T}\{\breve{C}\psi\}, \\ \partial_t \psi = -\breve{\eta}^{-1}\breve{C}\psi + e[v]\breve{I}, \\ v = e^{i\kappa t}\chi \tilde{f}\big|_{\Gamma_D} \quad \text{on} \quad \Gamma_D, \quad (\mathfrak{T}\{\breve{C}\psi\})\nu = 0 \quad \text{on} \quad \Gamma_N, \\ (v,\psi) = (0,0) \quad \text{at } t = 0, \end{cases} \tag{7.16}$$

where $\kappa > 0$ is a fixed angular frequency, $\tilde{f} \in H^1(\Omega)$, and $\chi = \chi(t) \in C^\infty([0,\infty))$ with the properties $\chi(t) = 0$ near $t = 0$, $\chi(t) = 1$, $t \ge t_0$ for a fixed $t_0 > 0$. The input over Γ_D is time harmonic after the time t_0. Here, note that the reason why we introduced χ is to make the boundary condition and the initial condition of (7.16) compatible.

In order to transform the boundary condition of (7.16) to a homogeneous one, we consider the following boundary value problem:

$$\begin{cases} i\kappa\tilde{v}_0 - \rho^{-1}\mathrm{div}\mathfrak{T}\{\breve{C}\tilde{\psi}_0\} = 0, \\ i\kappa\tilde{\psi}_0 + \breve{\eta}^{-1}\breve{C}\tilde{\psi}_0 - e[\tilde{v}_0]\breve{I} = 0, \\ \tilde{v}_0 = \tilde{f}\big|_{\Gamma_D} \quad \text{on } \Gamma_D, \quad (\mathfrak{T}\{\breve{C}\tilde{\psi}_0\})\nu = 0 \quad \text{on } \Gamma_N. \end{cases} \tag{7.17}$$

The unique solvability of this problem can be shown as follows. Due to the fact $\tilde{f} \in H^1(\Omega)$, it can be reduced to that of the boundary value problem for (6.6) with $\lambda = i\kappa$, that is, by transforming the inhomogeneous boundary condition on Γ_D to the homogeneous one. We use that (6.6) is uniquely solvable even in the case that f is in the dual space of $K(\Omega)$ (cf. (6.24)). Hence, there exists a unique solution $(\tilde{v}_0, \tilde{\psi}_0) \in H^1(\Omega) \times L^2(\Omega)$ of (7.17).

We define (v_0, ψ_0) by

$$(v_0, \psi_0) = \chi(t)(\tilde{v}_0, \tilde{\psi}_0) \tag{7.18}$$

and seek a solution (v, ψ) of (7.16) of the form

$$v = e^{i\kappa t} v_0 + \tilde{v}, \ \psi = e^{i\kappa t}\psi_0 + \tilde{\psi}. \tag{7.19}$$

Then $(\tilde{v}, \tilde{\psi})$ has to satisfy the following initial boundary value problem:

$$\begin{cases} \partial_t \tilde{v} - \rho^{-1}\mathrm{div}\mathfrak{T}\{\breve{C}\tilde{\psi}\} = -e^{i\kappa t}\dot{\chi}(t)\tilde{v}_0 =: \dot{\chi}(t)\tilde{F}_1(\kappa), \\ \partial_t \tilde{\psi} + \breve{\eta}^{-1}\breve{C}\tilde{\psi} - e[\tilde{v}]\breve{I} = -e^{i\kappa t}\dot{\chi}(t)\tilde{\psi}_0 =: \dot{\chi}(t)\tilde{F}_2(\kappa), \\ \tilde{v} = 0 \quad \text{on } \Gamma_D, \quad (\mathfrak{T}\{\breve{C}\tilde{\psi}\})\nu = 0 \quad \text{on } \Gamma_N, \\ (\tilde{v}, \tilde{\psi}) = 0 \quad \text{at } t = 0. \end{cases} \tag{7.20}$$

Here, $\dot{\chi}(t) := \frac{d\chi}{dt}(t)$. As

$$F(t) := (\dot{\chi}(t)\tilde{F}_1(\kappa), \dot{\chi}(t)\tilde{F}_2(\kappa))^{\mathbf{t}} \in C^\infty([0,\infty); H)$$

is 0 near $t = 0$, any order of the compatibility condition for (7.20) is satisfied. Using the semigroup, e^{tL}, solutions $\tilde{V} := (\tilde{v}, \tilde{\psi})^{\mathbf{t}}$ take the form

$$\tilde{V}(t) = \int_0^t e^{(t-s)L}\, e^{i\kappa s}\dot{\chi}(s)\, ds \tilde{F}(\kappa), \tag{7.21}$$

where $\tilde{F}(\kappa) = (\tilde{F}_1(\kappa), \tilde{F}_2(\kappa))^{\mathbf{t}}$. By a straightforward computation, we obtain

$$\tilde{V}(t) = e^{tL}\left(I + \int_0^{t_0} (i\kappa I - L)\, e^{(i\kappa I - L)s}(1 - \chi(s))\, ds\right)\tilde{F}(\kappa) = e^{tL}\left(I + O(1)\right)\tilde{F}(\kappa), \tag{7.22}$$

where $O(1)$ denotes a term which is uniformly bounded in time. Hence, by the exponential decay of solutions for the reduced system, we have

$$\|\tilde{V}(t)\| = O(e^{-a_4 t}) \text{ as } t \to \infty. \tag{7.23}$$

We have proved the following theorem.

THEOREM 7.3. *Let* $(v(t), \psi(t)) \in C^2([0,\infty); H^1(\Omega)) \times C^2(([0,\infty); L^2(\Omega))$ *be the solution of* (6.1) *satisfying the compatibility condition of order* 2. *Then, there exists a constant* $a_4 > 0$ *independent of the initial data such that* $(v(t), \psi(t))$ *converges to* $e^{i\kappa t}(\tilde{v}_0, \tilde{\psi}_0)$ *exponentially fast of order* $O(e^{-a_4 t})$ *as* $t \to \infty$ *with respect to the amended energy* (7.13). *Here* $(\tilde{v}_0, \tilde{\psi}_0) \in H^1(\Omega) \times L^2(\Omega)$ *is the unique solution of* (7.17).

8. Conclusion and discussion. We first summarize the results which we obtained in this paper. For typical spring-dashpot models, the EMM (extended Maxwell model) and its marginal model the ESLSM (extended standard linear solid model), which we simply refer to as the EMM, we showed that the solutions of the EMM generate C_0-groups. Among these solutions, those satisfying zero initial viscous strains satisfy the ID system (integro-differential system). However, due to the convolutional integral term, that is, the memory term with an integral kernel called the relaxation tensor, the solutions of the ID system do not generate any semigroup. Concerning the property of solutions, this is a big difference between the EBM and the ID system. Further, concerning the decay property of solutions as $t \to \infty$, the solutions of the EMM do not have the decaying property. To mitigate this fact, we introduced the reduced system, which is a subsystem of the EMM. Then, we proved that the solutions of this system not only generate a contractive C_0 group but also decay exponentially as $t \to \infty$. By combining this with the property of resolvent of the system, we proved the limiting amplitude property for the reduced system. We would like to emphasize that we analyzed the EMM for the case where the tensors are heterogeneous and anisotropic. Despite the importance of heterogeneity and anisotropy in the field of rheology, and earth and planetary science, there are not many mathematical studies on the spring-dashpot models for such a case. Except some results [8, 10, 17] on numerical analysis by the group of the second co-author of this paper, we haven't seen any other papers giving results similar to ours for the EMM.

Next, we point out a special mathematical method which we used to prove that the operator A_Z generates a C_0-group. Here, A_Z is the operator which describes the AD system after transforming it to the first order system with respect to the t-derivative. Since the size of A_Z is very large and it has a complicated form, it is very hard to directly estimate A_Z. The idea we used to show the existence of the resolvent of A_Z and its Hille–Yoshida type estimate was as follows. Instead of analyzing the λ-equation associated to A_Z, we look at an equivalent λ-equation described by another operator $\tilde{A}$ and analyzed $\tilde{A}_u$, the upper left block of $\tilde{A}$ (see (4.15)). We showed that its resolvent exists and it satisfies the Hille–Yoshida type estimate with respect to an operator norm depending on λ. Then, we lifted this up for $\tilde{A}$, and then for A_Z with respect to an operator norm depending on λ, which is the operator norm on the space W_λ depending on λ. By the asymptotic behavior of the norm of the space W_λ, we finally succeeded in proving the Hille–Yoshida type resolvent estimate for the operator A_Z with respect to the operator norm on W_Z which does not depend on λ. As far as we know, we haven't seen such a method in any other papers.

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