



Stable Recovery of Coefficients in an Inverse Fault Friction Problem

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Communicated by M. ORTIZ

Abstract

We consider the inverse fault friction problem of determining the friction coefficient in the Tresca friction model, which can be formulated as an inverse problem for differential inequalities. We show that the measurements of elastic waves during a rupture uniquely determine the friction coefficient at the rupture surface with explicit stability estimates.

1. Introduction

The study of earthquake physics remains highly challenging through its complex dynamics and multifaceted nature. Nearly all aspects of earthquake ruptures are controlled by the friction along a fault, where these commonly occur, that progressively increases with tectonic forcing. Indeed, in a recent Annual Review of Earth and Planetary Sciences, it was stated that “determining the friction during an earthquake is required to understand when and where earthquakes occur” (Brodsky et al. [2]). Some common approach has been developed retrieving the stress evolution at each point of the fault as dictated by the slip history obtained from the kinematic inverse rupture problem; we mention work by Ide and Takeo [13], who determined the spatiotemporal slip distribution on an assumed fault plane of the 1995 Kobe earthquake by “waveform inversion” and then numerically solved the elastodynamic equations to determine the stress distribution and constitutive relations on the fault plane. However, seismologists studying earthquake dynamics have reported that both stress and friction on a fault are still poorly known and difficult to constrain with observations (Causse, Dalguer and Mai [4]). Here, we address the question whether this is possible, in principle.

We study the recovery of a time- and space-dependent friction coefficient via the slip rate and normal and tangential stresses, using the Tresca model (see, e.g., the book of Sofonea and Matei [20]), at a pre-existing fault from “near-surface” elastic-wave, that is, seismic displacement data. This dynamic inverse friction problem

can be regarded as an inverse problem for differential inequalities, as the Tresca friction model can be formulated through variational inequalities as seen in many contact mechanics problems (e.g., [6, 20]). While inverse problems for differential equations have been widely studied, inverse problems for differential inequalities have not yet received much attention. Our approach is based on the quantitative unique continuation for the elastic wave equation established in our recent work [5], where we studied the kinematic inverse rupture problem of determining the friction force at the rupture surface from seismic displacement data. Itou and Kashiwabara [14] recently analyzed the Tresca model on a fault coupled to the elastic wave equation; we exploit their results in our study of the inverse problem. We also mention recent work by Hirano and Itou [11] on deriving an analytical solution to the slip rate distribution of self-similar rupture growth under a distance-weakening friction model. As a disclaimer, while we address the most fundamental question, we do ignore more complex physics such as thermo-mechanical effects.

We remark that in the past two decades, quite many studies have been devoted to the practical determination of fault frictional properties by analyzing slowly-evolving afterslip following large earthquakes. For very recent results, see Zhao and Yue [23]. Afterslip is the fault slip process in response to an instantaneous coseismic stress change, in which the slip velocity decrease corresponds to the stress releasing by itself. Its “self-driven” nature provides a framework to model the slip process with the fault frictional properties alone. For a review, see Yue et al. [22]. Afterslip is analyzed with quasi-static deformation, that is, with the elastostatic system of equations, typically using geodetic data.

Let $M \subset \mathbb{R}^3$ be a bounded connected open set with smooth boundary, modelling the solid Earth. Let $\overline{\Sigma_f} \subset M$ be a connected orientable embedded smooth surface with nonempty smooth boundary satisfying $\overline{\Sigma_f} \cap \partial M = \emptyset$, modelling the rupture surface. Consider the elastic wave equation

$$\rho \partial_t^2 \mathbf{u} - \mu \Delta \mathbf{u} - (\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) = 0 \quad \text{in } (M \setminus \overline{\Sigma_f}) \times (-T, T), \quad (1)$$

with the Tresca friction condition (e.g., [6, 14, 15]) on the rupture surface Σ_f :

$$\begin{cases} \sigma_n = F_n \text{ is given,} \\ [\sigma_n]_-^+ = 0, \quad [\sigma_\tau]_-^+ = 0, \\ |\sigma_\tau| < g \implies [\partial_t \mathbf{u}_\tau]_-^+ = 0, \\ |\sigma_\tau| = g \implies [\partial_t \mathbf{u}_\tau]_-^+ \cdot \boldsymbol{\sigma}_\tau = g \mid [\partial_t \mathbf{u}_\tau]_-^+ \mid. \end{cases} \quad (2)$$

Here $\sigma_n = (\boldsymbol{\sigma}(\mathbf{u})\mathbf{n}) \cdot \mathbf{n}$ and $\sigma_\tau = \boldsymbol{\sigma}(\mathbf{u})\mathbf{n} - \sigma_n \mathbf{n}$ are the normal and tangential components of the stress tensor $\boldsymbol{\sigma}(\mathbf{u})$, where $\mathbf{n}$ is the unit normal vector of the rupture surface Σ_f . The stress tensor $\boldsymbol{\sigma}(\mathbf{u})$ is defined as

$$\boldsymbol{\sigma}(\mathbf{u}) = 2\mu \boldsymbol{\varepsilon}(\mathbf{u}) + \lambda (\text{tr } \boldsymbol{\varepsilon}(\mathbf{u})) \mathbf{I}, \quad \boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T). \quad (3)$$

The notation $[\cdot]_-^+$ stands for the jump across the rupture surface; more precisely,

$$[\partial_t \mathbf{u}_\tau]_-^+ := \partial_t (\mathbf{u}_\tau^+ - \mathbf{u}_\tau^-), \quad \mathbf{u}_\tau^\pm := \lim_{h \rightarrow 0^\pm} \mathbf{u}_\tau(z + h\mathbf{n}, t), \quad z \in \Sigma_f, \quad (4)$$

where $\mathbf{u}_\tau$ is the tangential component of $\mathbf{u}$, with respect to the unit normal vector field $\mathbf{n}$ smoothly extended to a neighborhood of Σ_f . The friction force g at Σ_f is of the form

$$g = \mathcal{F} |F_n|, \quad (5)$$

where $\mathcal{F} > 0$ is the friction coefficient. Note that the friction force g and the friction coefficient $\mathcal{F}$ may depend on time.

Regarding the direct problem for the Tresca friction model above, the weak formulation is understood in the variational sense (see, e.g., [6, 14, 20]). Let $V_0 = \{\mathbf{v} \in H^1(M \setminus \overline{\Sigma_f}) : \mathbf{v} = 0 \text{ on } \partial M\}$. Recall that with the Dirichlet condition on the boundary ∂M , the problem of finding $\mathbf{u}$ satisfying the Tresca friction model (1-2) is formulated as finding $\mathbf{u}$ such that for all $\mathbf{v} \in H^1(V_0)$ and all $t \in (0, T)$, the following variational inequality holds:

$$\begin{aligned} & \rho \int_{M \setminus \overline{\Sigma_f}} (\mathbf{v} - \partial_t \mathbf{u}) \partial_t^2 \mathbf{u} \, dV + a(\mathbf{u}, \mathbf{v} - \partial_t \mathbf{u}) \\ & + \int_{\Sigma_f} g \cdot |[\mathbf{v}_\tau]_-^+| \, dA - \int_{\Sigma_f} g \cdot |[\partial_t \mathbf{u}_\tau]_-^+| \, dA + \int_{\Sigma_f} F_n \cdot [\mathbf{v}_n]_-^+ \, dA \geq 0. \end{aligned} \quad (6)$$

Here dV , dA stand for the volume and area element of M and Σ_f , respectively, and a is a bilinear symmetric form defined by $a(\mathbf{v}, \mathbf{w}) := \int_{M \setminus \overline{\Sigma_f}} \text{tr}(\boldsymbol{\sigma}(\mathbf{v})^T \boldsymbol{\varepsilon}(\mathbf{w})) \, dV$. If one assumes

$$\begin{aligned} \mathbf{u}_0 &:= \mathbf{u}(\cdot, 0) \in H^1(V_0), \quad \partial_t \mathbf{u}(\cdot, 0) \in H^1(V_0), \\ F_n &\in H^2(0, T; L^2(\Sigma_f)), \quad g \in H^2(0, T; L^2(\Sigma_f)), \end{aligned}$$

and the compatibility conditions at $t = 0$,

$$\begin{cases} \text{div } \boldsymbol{\sigma}(\mathbf{u}_0) \in L^2(M \setminus \overline{\Sigma_f}), \\ \sigma_n(\mathbf{u}_0^+) = \sigma_n(\mathbf{u}_0^-) = F_n(\cdot, 0) & \text{on } \Sigma_f, \\ \boldsymbol{\sigma}_\tau(\mathbf{u}_0^+) = \boldsymbol{\sigma}_\tau(\mathbf{u}_0^-) = 0, \quad \partial_t \mathbf{u}_\tau(\cdot, 0) = 0 & \text{on } \Sigma_f, \end{cases}$$

then the friction problem above has a unique solution [14],

$$\mathbf{u} \in W^{1,\infty}(0, T; H^1(V_0)) \cap W^{2,\infty}(0, T; L^2(M \setminus \overline{\Sigma_f})). \quad (7)$$

In this paper, we consider the inverse problem of determining the friction coefficient $\mathcal{F}$ in the Tresca friction model above. For the sake of presentation, we consider the elastic wave equation (1) on the time interval $(-T, T)$. Let $M \subset \mathbb{R}^3$ be a bounded connected open set with smooth boundary, modelling the solid Earth. Let $\overline{\Sigma_f} \subset M$ be a connected orientable embedded smooth surface with nonempty smooth boundary, modelling the rupture surface. Denote the *a priori* bound for the norm of the elastic wave $\mathbf{u}$ in the space (7) by

$$\|\mathbf{u}\|_{W^{1,\infty}(-T, T; H^1(M \setminus \overline{\Sigma_f}))} + \|\mathbf{u}\|_{W^{2,\infty}(-T, T; L^2(M \setminus \overline{\Sigma_f}))} \leq \Lambda_0. \quad (8)$$

We impose the following assumption on the regularity of the normal stress F_n and the friction coefficient $\mathcal{F}$. Assume that $F_n, \mathcal{F} \in C^{0,1}(\Sigma_f \times [-T, T])$, and

$$|F_n| \geq c_0 > 0, \quad \|F_n\|_{C^{0,1}(\Sigma_f \times [-T, T])} \leq C_0, \quad \|\mathcal{F}\|_{C^{0,1}(\Sigma_f \times [-T, T])} \leq C_0. \quad (9)$$

In addition, assume that the parameters ρ, λ, μ in the elastic wave equation (1) to be smooth and time-independent on $M \setminus \overline{\Sigma_f}$.

We prove the following result on the inverse fault friction problem during a rupture:

Theorem 1. *Let $M \subset \mathbb{R}^3$ be a bounded connected open set with smooth boundary. Let $\overline{\Sigma_f} \subset M$ be a connected orientable embedded smooth surface with nonempty smooth boundary, satisfying $\overline{\Sigma_f} \cap \partial M = \emptyset$. Consider the elastic wave equation (1) with smooth time-independent parameters and the Tresca friction condition (2). Assume the normal stress F_n and the friction coefficient $\mathcal{F}$ satisfy (9) on Σ_f . Assume $|\partial_t \mathbf{u}_\tau|^\pm| \geq c_1$ on $\Sigma_f \times [-T, T]$ for some constant $c_1 > 0$ and a priori norm (8) for the elastic wave $\mathbf{u}$. Suppose that we are given the elastic wave $\mathbf{u}$ on an interior open set U satisfying $\overline{U} \subset M \setminus \overline{\Sigma_f}$ up to sufficiently large time T . Then we have the following conclusions.*

- (1) *The friction coefficient $\mathcal{F}$ on $\Sigma_f \times [-\frac{T}{2}, \frac{T}{2}]$ is uniquely determined by $\mathbf{u}$ on $U \times [-T, T]$.*
- (2) *Suppose that we have two systems with friction coefficients $\mathcal{F}_1, \mathcal{F}_2$, and we are given the corresponding elastic waves $\mathbf{u}_1, \mathbf{u}_2$ on $U \times [-T, T]$. Then there exist constants $\widehat{\varepsilon}_0, C, c > 0$ such that for any $0 < \varepsilon_0 < \widehat{\varepsilon}_0$, if*

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_{H^2(U \times [-T, T])} \leq \varepsilon_0,$$

then the friction coefficients satisfy

$$\|\mathcal{F}_1 - \mathcal{F}_2\|_{L^2(\Sigma_f \times [-\frac{T}{2}, \frac{T}{2}])} \leq C(\log |\log \varepsilon_0|)^{-c},$$

where the constants $\widehat{\varepsilon}_0, C$ depend on $c_0, C_0, \Lambda_0, c_1, T$, parameters of the elastic wave equation and M, U, Σ_f , and c is an absolute constant.

We remark that one could also formulate the result assuming a lower bound for $|\partial_t \mathbf{u}_\tau|^\pm|$ on a subset of Σ_f and then recovers the friction coefficient $\mathcal{F}$ in the same subset. Theoretically speaking, our method can also work with measurements of the elastic waves on an open subset of the boundary ∂M if additional regularity of the elastic waves on ∂M can be assumed, see [5, Remark 1] or [3, Theorem 3]; however, this additional boundary regularity required is not provided by the regularity class (7).

The proof of Theorem 1 is divided into two parts: the measurements of $\mathbf{u}$ on U determine $\mathbf{u}$ near Σ_f , and the latter determines $\mathcal{F}$ under the Tresca friction condition. The first part, also known as the kinematic inverse rupture problem [10], has mostly been done in our recent work [5]. Our method was based on the quantitative unique continuation for the elasticity system, motivated by [7, 21]. However, regularity issues remain: the actual regularity of waves is not enough for the quantitative unique continuation arguments to work on the whole domain, which is addressed in Sect. 2. The second part is discussed in Sect. 3.

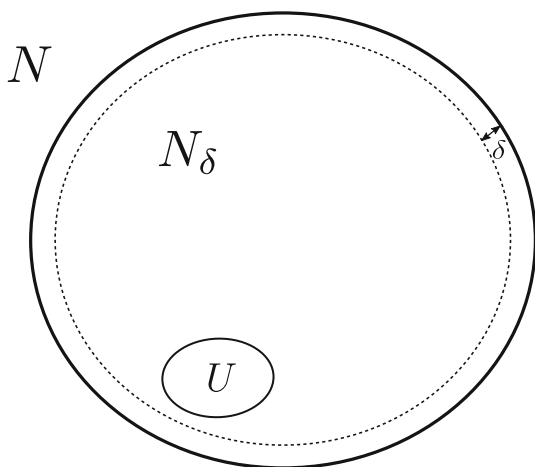


Fig. 1. Setting of Lemma 1 and Proposition 1

2. Interior Regularity

Let $N \subset \mathbb{R}^3$ be a bounded connected open set with smooth boundary. Let $\delta > 0$ be small, and denote the interior by

$$N_\delta := \{x \in N : d(x, \partial N) \geq \delta\}. \quad (10)$$

Let $\mathbf{u}$ be an elastic wave satisfying the elastic wave equation (1) on the whole set N (without the presence of a rupture surface). Recall [7, Lemma 5.1] that the elastic wave equation can be decomposed into a system of hyperbolic equations for $(\mathbf{u}, \operatorname{div} \mathbf{u}, \operatorname{curl} \mathbf{u})$. Let $U \subset N$ be a connected open subset, and we choose δ sufficiently small such that $\overline{U} \subset N_\delta$.

We apply Proposition 3.2 in [5] to $(\mathbf{u}, \operatorname{div} \mathbf{u}, \operatorname{curl} \mathbf{u})$ in N_δ : for sufficiently small h and sufficiently large T (specified in [5, Proposition 3.2]), if

$$\|\mathbf{u}\|_{H^2(U \times [-T, T])} \leq \varepsilon_0,$$

then

$$\begin{aligned} \|\mathbf{u}\|_{L^2(N_\delta \times [-\frac{T}{2}, \frac{T}{2}])} &\leq C \exp(h^{-cn}) \frac{\Lambda_\delta}{\left(\log\left(1 + h \frac{\Lambda_\delta}{\varepsilon_0}\right)\right)^{\frac{1}{2}}} \\ &\quad + C \|\mathbf{u}\|_{H^1(N_\delta \times [-T, T])} h^{\frac{1}{n+1}}, \end{aligned} \quad (11)$$

where $n = \dim(N) = 3$, and

$$\Lambda_\delta := \|(\mathbf{u}, \operatorname{div} \mathbf{u}, \operatorname{curl} \mathbf{u})\|_{H^1(N_\delta \times [-T, T])}. \quad (12)$$

Observe that Λ_δ essentially asks for H^2 regularity, while the solution of the original direct problem (7) is only in H^1 in space. One way to resolve the issue

is to use interior regularity estimate and then use Sobolev embedding to get an immediate estimate for the boundary layer.

We assume that the elastic wave $\mathbf{u}$ in N is in the following energy class:

$$\|\mathbf{u}\|_{W^{1,\infty}(-T,T;H^1(N))} + \|\mathbf{u}\|_{W^{2,\infty}(-T,T;L^2(N))} \leq \Lambda_0. \quad (13)$$

Lemma 1. *The following H^2 interior regularity estimate holds for the elastic wave equation (1) on $N \times [-T, T]$:*

$$\|\mathbf{u}\|_{H^2(N_{2\delta} \times [-T, T])} \leq C_T C_M \delta^{-4} \left(\|\mathbf{u}\|_{W^{1,\infty}(-T,T;H^1(N))} + \|\mathbf{u}\|_{W^{2,\infty}(-T,T;L^2(N))} \right).$$

That gives a bound

$$\Lambda_\delta \leq C_T C_N \delta^{-4} \Lambda_0, \quad (14)$$

where C_T depends only on T , and C_N depends only on geometric parameters of N .

Proof. Suppose that φ is a weak solution of the elliptic equation on N ,

$$\Delta \varphi = f, \quad \varphi \in H^1(N), \quad f \in L^2(N).$$

Let χ be a cut-off function satisfying $\chi|_{N_\delta} = 1$, $\chi|_{\partial N} = 0$ and $\|\chi\|_{C^2(N)} \leq C\delta^{-2}$. Then

$$\varphi_0 := \chi \varphi$$

satisfies

$$\begin{cases} \Delta \varphi_0 = \chi \Delta \varphi + [\Delta, \chi] \varphi =: \tilde{f}, \\ \varphi_0|_{\partial N} = 0, \end{cases}$$

where

$$\|\tilde{f}\|_{L^2(N)} \leq C\delta^{-2} \left(\|\varphi\|_{H^1(N)} + \|f\|_{L^2(N)} \right).$$

Hence, $\varphi_0 \in H^2(N)$ by boundary regularity for elliptic equations (e.g., [9, Theorem 4 in Chapter 6.3]). Then, it follows that $\varphi|_{N_\delta} \in H^2(N_\delta)$, and

$$\|\varphi\|_{H^2(N_\delta)} \leq \|\varphi_0\|_{H^2(N)} \leq C_N \delta^{-2} \left(\|\varphi\|_{H^1(N)} + \|f\|_{L^2(N)} \right). \quad (15)$$

The constant C_N depends on the first Dirichlet eigenvalue on N which is uniformly bounded below by diameter and curvature bounds (e.g., [16, Theorem 8]). The same argument is valid for $\varphi \in L^2(N)$ and $f \in H^{-1}(N)$ with constant $C_N \delta^{-2}$.

Now we switch to the notations in our first paper [5],

$$(\mathbf{u}, v, \mathbf{w}) := (\mathbf{u}, \operatorname{div} \mathbf{u}, \operatorname{curl} \mathbf{u}). \quad (16)$$

Recall that the elastic wave equation (1) can be decomposed into the system ([7, Lemma 5.1] or [5, Lemma A.1])

$$\begin{cases} \frac{\rho}{\mu} \partial_t^2 \mathbf{u} - \Delta \mathbf{u} + A_1(\mathbf{u}, v) = 0, \\ \frac{\rho}{2\mu + \lambda} \partial_t^2 v - \Delta v + A_2(\mathbf{u}, v, \mathbf{w}) = 0, \\ \frac{\rho}{\mu} \partial_t^2 \mathbf{w} - \Delta \mathbf{w} + A_3(\mathbf{u}, v, \mathbf{w}) = 0, \end{cases} \quad (17)$$

where A_i are first order and has no time derivative.

Then consider the second equation. Since $\mathbf{u} \in W^{1,\infty}(-T, T; H^1(N))$ from (13), then

$$v, \mathbf{w} \in W^{1,\infty}(-T, T; L^2(N)), \quad A_2(\mathbf{u}, v, \mathbf{w}) \in W^{1,\infty}(-T, T; H^{-1}(N)).$$

On the other hand, since $\mathbf{u} \in W^{2,\infty}(-T, T; L^2(N))$ from (13), then

$$v \in W^{2,\infty}(-T, T; H^{-1}(N)), \quad \partial_t^2 v \in L^\infty(-T, T; H^{-1}(N)).$$

Then using the second equation gives

$$\Delta v \in L^\infty(-T, T; H^{-1}(N)).$$

Thus from the interior regularity argument (15) (for L^2 - H^{-1}), we have

$$v|_{N_\delta} \in L^\infty(-T, T; H^1(N_\delta))$$

with constant $C_N \delta^{-2}$ in the regularity estimate.

Next consider the first equation on N_δ . Using the improved interior regularity for $v|_{N_\delta}$, we see that $A_1(\mathbf{u}, v)|_{N_\delta} \in L^\infty(-T, T; L^2(N_\delta))$. This gives

$$\Delta \mathbf{u}|_{N_\delta} \in L^\infty(-T, T; L^2(N_\delta)),$$

considering that $\partial_t^2 \mathbf{u} \in L^\infty(-T, T; L^2(N))$. Thus by shrinking the domain by another δ and using (15), we obtain

$$\mathbf{u}|_{N_{2\delta}} \in L^\infty(-T, T; H^2(N_{2\delta}))$$

with constant $C_N \delta^{-4}$ in the regularity estimate, i.e.,

$$\|\mathbf{u}\|_{L^\infty(-T, T; H^2(N_{2\delta}))} \leq C_N \delta^{-4} \left(\|\mathbf{u}\|_{W^{1,\infty}(-T, T; H^1(N))} + \|\mathbf{u}\|_{W^{2,\infty}(-T, T; L^2(N))} \right). \quad (18)$$

Thus, from (18) and (13), we obtain the estimate

$$\begin{aligned} \|\mathbf{u}\|_{H^2(N_{2\delta} \times [-T, T])} &\leq C_T \left(\|\mathbf{u}\|_{L^\infty(-T, T; H^2(N_{2\delta}))} + \|\mathbf{u}\|_{W^{1,\infty}(-T, T; H^1(N))} \right. \\ &\quad \left. + \|\mathbf{u}\|_{W^{2,\infty}(-T, T; L^2(N))} \right) \\ &\leq C_T C_N \delta^{-4} \left(\|\mathbf{u}\|_{W^{1,\infty}(-T, T; H^1(N))} + \|\mathbf{u}\|_{W^{2,\infty}(-T, T; L^2(N))} \right). \end{aligned}$$

□

Proposition 1. *Let $N \subset \mathbb{R}^3$ be a bounded connected open set with smooth boundary and $U \subset N$ be a connected open set. Let $\mathbf{u}$ be a solution of the elastic wave equation (1) in $N \times [-T, T]$ with a priori norm (13). Then, there exist constants $\widehat{\varepsilon}_0$, C , $c > 0$ such that for any $0 < \varepsilon_0 < \widehat{\varepsilon}_0$, if*

$$\|\mathbf{u}\|_{H^2(U \times [-T, T])} \leq \varepsilon_0,$$

then

$$\|\mathbf{u}\|_{L^2(N \times [-\frac{T}{2}, \frac{T}{2}])} \leq C(\log |\log \varepsilon_0|)^{-c},$$

where the constants $\widehat{\varepsilon}_0$, C depend on Λ_0 , T , parameters of the elastic wave equation and geometric parameters of N , and c is an absolute constant.

Proof. We have already estimated the L^2 -norm of $\mathbf{u}$ on $N_\delta \times [-T/2, T/2]$ from (11) and Lemma 1. The L^2 -norm on the remaining part follows from the Sobolev embedding theorem. Namely, consider

$$\mathcal{N}_\delta^T := \{x \in N : d(x, \partial N) \leq \delta\} \times [-\frac{T}{2}, \frac{T}{2}].$$

Apply the Sobolev embedding theorem (e.g., Theorem 4.12 in [1]) to the space $N \times [-T, T]$ of dimension $n + 1 = 4$, which satisfies the uniform cone condition (Definition 4.8 in [1]),

$$\|\mathbf{u}\|_{L^{\frac{2(n+1)}{n-1}}(N \times [-T, T])} \leq C\|\mathbf{u}\|_{H^1(N \times [-T, T])} \leq C\Lambda_0.$$

Then,

$$\|\mathbf{u}\|_{L^2(\mathcal{N}_\delta^T)} \leq \|\mathbf{u}\|_{L^{\frac{2(n+1)}{n-1}}(N \times [-T, T])} (\text{Vol}(\mathcal{N}_\delta^T))^{\frac{1}{n+1}} \leq C\Lambda_0 \delta^{\frac{1}{n+1}}.$$

Hence, we have

$$\begin{aligned} \|\mathbf{u}\|_{L^2(N \times [-\frac{T}{2}, \frac{T}{2}])} &\leq \|\mathbf{u}\|_{L^2(N_\delta \times [-\frac{T}{2}, \frac{T}{2}])} + \|\mathbf{u}\|_{L^2((N \setminus N_\delta) \times [-\frac{T}{2}, \frac{T}{2}])} \\ &\leq C \exp(h^{-cn}) \frac{\delta^{-4} \Lambda_0}{\left(\log\left(1 + h \frac{\delta^{-4} \Lambda_0}{\varepsilon_0}\right)\right)^{\frac{1}{2}}} + C\Lambda_0 h^{\frac{1}{n+1}} + C\Lambda_0 \delta^{\frac{1}{n+1}}. \end{aligned} \quad (19)$$

From here, we need to balance the parameters δ , h , ε_0 . Choose $\delta = h$ such that the three terms on the right-hand side of (19) are equal, and we get

$$\delta = h = C(\log |\log \varepsilon_0|)^{-c}, \quad (20)$$

for some constant c depending only on $n = 3$, and for some constant C independent of h . The condition $h < h_0$ gives the choice for $\widehat{\varepsilon}_0$:

$$\widehat{\varepsilon}_0 = \left(\exp \exp(C^{-1} h_0^{-1/c})\right)^{-1}. \quad (21)$$

Inserting the choice of δ , h back into (19) gives the desired form of the estimate. $\square$

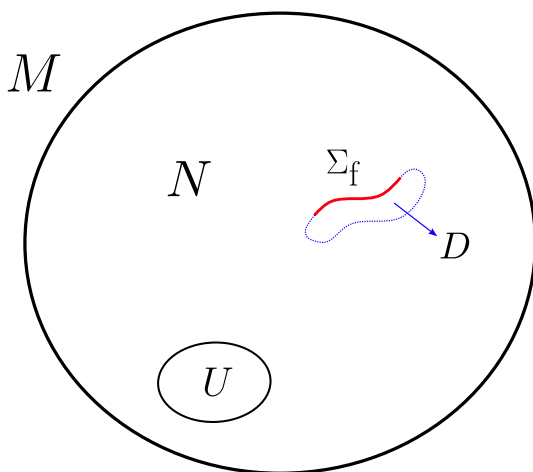


Fig. 2. Applying the quantitative unique continuation to the connected open set $N = M \setminus D$ with smooth boundary, where D is a compact smooth manifold whose boundary extends Σ_f

3. Inverse Friction Problem

Using Proposition 1, we consider the inverse problem of determining the friction coefficient $\mathcal{F}$ in the Tresca friction model. As Proposition 1 is formulated without the presence of the rupture surface Σ_f , we apply the result to a smooth manifold whose boundary extends Σ_f . Of course the choice of such a manifold can be arbitrary and in this paper we use a type as illustrated in Fig. 2.

Lemma 2. *Let $M \subset \mathbb{R}^3$ be a bounded connected open set with smooth boundary, $\overline{\Sigma_f}$ be a connected orientable smooth embedded compact rupture surface with nonempty smooth boundary satisfying $\overline{\Sigma_f} \cap \partial M = \emptyset$, and $U \subset M$ be a connected open set satisfying $\overline{U} \subset M \setminus \overline{\Sigma_f}$. Let $\mathbf{u}$ be a solution of the elastic wave equation (1) with a priori norm (8). If, for sufficiently small ε_0 ,*

$$\|\mathbf{u}\|_{H^2(U \times [-T, T])} \leq \varepsilon_0,$$

then for any $\alpha \in (0, \frac{1}{2})$, we have

$$\|\sigma(\mathbf{u})|_{\Sigma_f}\|_{H^{-\frac{1}{2}-\alpha}(\Sigma_f \times [-\frac{T}{2}, \frac{T}{2}])} \leq C \Lambda_0^{1-\alpha} \varepsilon_1^\alpha,$$

where $\varepsilon_1 := C(\log |\log \varepsilon_0|)^{-c}$. The same estimate also holds for the components σ_n, σ_τ . The constant C depends on Λ_0, T , parameters of the elastic wave equation and M, U, Σ_f , and c is an absolute constant.

Proof. First, we construct a connected open set $N \subset M$ with smooth boundary such that $U \subset N$ and $\Sigma_f \subset \partial N$. Let $\mathbf{n}$ be a smooth unit normal vector field on Σ_f , and consider the ray $z + r\mathbf{n}$, $r \geq 0$, from a point $z \in \Sigma_f$. Since $\overline{\Sigma_f}$ is smoothly embedded in $\mathbb{R}^3$, there exists sufficiently small $r_0 > 0$ such that the nearest point of

$z + r\mathbf{n}$ in Σ_f is z for all $r \in [0, r_0]$ and all $z \in \Sigma_f$. This gives a smooth embedding $f : \Sigma_f \times [0, r_0] \rightarrow \mathbb{R}^3$ such that $f(\Sigma_f \times \{0\}) = \Sigma_f$. Note that r_0 is chosen such that the closure of the image of f does not intersect with ∂M or $\bar{U}$. The boundary of the image $f(\Sigma_f \times [0, r_0])$ extends Σ_f , and can be smoothened over a neighborhood of $\partial \Sigma_f$ without intersecting ∂M or $\bar{U}$. This construct a compact smooth manifolds $D \subset M$ with smooth boundary satisfying $\Sigma_f \subset \partial D$, see Fig. 2. Note that the complement $M \setminus D$ is connected due to $\partial \Sigma_f \neq \emptyset$. Moreover, the manifold D can be constructed such that the curvature tensor of ∂D and its covariant derivatives are bounded depending on $r_0, d(\bar{\Sigma}_f, \partial M), d(\bar{\Sigma}_f, \bar{U})$ and geometric parameters of Σ_f . Hence, Proposition 1 is applicable to the connected open set $N := M \setminus D$ with smooth boundary $\partial N = \partial M \cup \partial D$.

Applying Proposition 1 to $N = M \setminus D$ gives $\|\mathbf{u}\|_{L^2(N \times [-\frac{T}{2}, \frac{T}{2}])} \leq \varepsilon_1$. We have a priori norm $\|\mathbf{u}\|_{H^1(N \times [-T, T])} \leq \Lambda_0$ by (8). By interpolation, we have

$$\|\mathbf{u}\|_{H^{1-\alpha}(N \times [-\frac{T}{2}, \frac{T}{2}])} \leq \Lambda_0^{1-\alpha} \varepsilon_1^\alpha, \quad \forall \alpha \in (0, 1). \quad (22)$$

Recall the system (17) for $(\mathbf{u}, v, \mathbf{w}) := (\mathbf{u}, \operatorname{div} \mathbf{u}, \operatorname{curl} \mathbf{u})$. Then

$$\|v\|_{H^{-\alpha}(N \times [-\frac{T}{2}, \frac{T}{2}])} + \|\mathbf{w}\|_{H^{-\alpha}(N \times [-\frac{T}{2}, \frac{T}{2}])} \leq 2\Lambda_0^{1-\alpha} \varepsilon_1^\alpha. \quad (23)$$

To proceed further, we recall the $H_{(k,s)}$ -norm in $\mathbb{R}^{n+1}$ (see [12, Definition B.1.10]) defined as

$$\|\mathbf{u}\|_{(k,s)}^2 = \int_{\mathbb{R}^{n+1}} |\widehat{\mathbf{u}}(\xi)|^2 (1 + |\xi|^2)^k (1 + |\xi'|^2)^s d\xi, \quad (24)$$

with respect to the coordinates $y = (x', x^n) \in \mathbb{R}^n \times \mathbb{R}$. Note that when $s = 0$, the $H_{(k,0)}$ -norm above is equivalent to the usual H^k -norm. The idea is using partial hypoellipticity (e.g., [12, Appendix B] or [8, Chapter 26.1]) to trade regularity between normal and tangential components.

Since the $H_{(k,s)}$ -norm was defined above for functions on $\mathbb{R}^{n+1}$, we apply the technique to local coordinates and patch up using partition of unity in the standard way. Let $\{B_k\}_{k=1}^N$ be a finite open cover of a small neighborhood of Σ_f in $\bar{N}$, and let $\{\chi_k\}_{k=1}^N$ be a partition of unity subordinate to the open cover. Setting the diameter of each open set B_k smaller than the injectivity radius of $\bar{N}$, one can work in the boundary normal coordinate $(y^1, y^2, y^3) \in \mathbb{R}_{y^1 \geq 0}^3$ of $\bar{N}$, where $y^1 = d(y, \partial N)$ is the coordinate normal to ∂N . Let $\Phi_k : B_k \rightarrow \mathbb{R}_{y^1 \geq 0}^3$ be the smooth (boundary normal) coordinate function on each B_k such that Φ_k maps $\Sigma_f \cap B_k$ to an open set of $\{(y^1, y^2, y^3) \in \mathbb{R}^3 : y^1 = 0\}$. Using the notation (24), we write $v, \mathbf{w} \in H_{(-\alpha,0)}$ from (23) and thus $P_2 v = -A_2(\mathbf{u}, v, \mathbf{w}) \in H_{(-1-\alpha,0)}$, where $P_2 = \frac{\rho}{2\mu+\lambda} \partial_t^2 - \Delta$ by the second equation in (17). We denote by $P_{2,k}$ the push-forward of the operator P_2 to $\Phi_k(B_k)$, that is, for any smooth function $\hat{f}$ on $\Phi_k(B_k)$,

$$P_{2,k} \hat{f} := (P_2(\hat{f} \circ \Phi_k)) \circ \Phi_k^{-1}. \quad (25)$$

In the coordinate $\mathbf{y} = (y^1, y^2, y^3)$ of $\Phi_k(B_k) \subset \mathbb{R}_{y^1 \geq 0}^3$, the operator $P_{2,k}$ has the standard form $P_{2,k} = \frac{\rho}{2\mu+\lambda} \partial_t^2 - \partial_{y^1}^2 - a(\mathbf{y}, \partial_{y^2}, \partial_{y^3}) + \text{first-order terms}$. Then, we

apply [12, Theorem B.2.9] to $(\chi_k v) \circ \Phi_k^{-1}$ (which is a function in the half-plane supported in $\Phi_k(B_k)$),

$$\begin{aligned} \|(\chi_k v) \circ \Phi_k^{-1}\|_{(1-\alpha, -1)} &\leq C \left(\|(\chi_k v) \circ \Phi_k^{-1}\|_{(-\alpha, 0)} + \|P_{2,k}((\chi_k v) \circ \Phi_k^{-1})\|_{(-1-\alpha, 0)} \right) \\ &\leq C \left(\|(\chi_k v) \circ \Phi_k^{-1}\|_{(-\alpha, 0)} + \|(\chi_k P_2 v) \circ \Phi_k^{-1}\|_{(-1-\alpha, 0)} \right. \\ &\quad \left. + \|[P_2, \chi_k]v \circ \Phi_k^{-1}\|_{(-1-\alpha, 0)} \right) \\ &\leq C \left(\|v\|_{(-\alpha, 0)} + \|A_2(\mathbf{u}, v, \mathbf{w})\|_{(-1-\alpha, 0)} \right) \\ &\leq C \Lambda_0^{1-\alpha} \varepsilon_1^\alpha, \end{aligned}$$

where the constants C depend on the C^2 -norms of Φ_k , Φ_k^{-1} , χ_k , which, in turn, depend on the geometry of Σ_f . Note that the commutator $[P_2, \chi_k]v$ in the above is first order in v .

Let $P_1 = \frac{\rho}{\mu} \partial_t^2 - \Delta$ in the first equation $P_1 \mathbf{u} = -A_1(\mathbf{u}, v)$ in (17), and denote by $P_{1,k}$ the push-forward of the operator P_1 to $\Phi_k(B_k)$. Now using $(\chi_k \mathbf{u}) \circ \Phi_k^{-1} \in H_{(1-\alpha, 0)} \subset H_{(1-\alpha, -1)}$ and the regularity $(\chi_k v) \circ \Phi_k^{-1} \in H_{(1-\alpha, -1)}$ just obtained above, we have

$$\begin{aligned} \|(\chi_k A_1(\mathbf{u}, v)) \circ \Phi_k^{-1}\|_{(-\alpha, -1)} &\leq C \left(\|\nabla((\chi_k \mathbf{u}) \circ \Phi_k^{-1})\|_{(-\alpha, -1)} + \|\nabla((\chi_k v) \circ \Phi_k^{-1})\|_{(-\alpha, -1)} \right. \\ &\quad \left. + \|(\chi_k \mathbf{u}) \circ \Phi_k^{-1}\|_{(-\alpha, -1)} + \|(\chi_k v) \circ \Phi_k^{-1}\|_{(-\alpha, -1)} \right) \\ &\leq C \left(\|(\chi_k \mathbf{u}) \circ \Phi_k^{-1}\|_{(1-\alpha, -1)} + \|(\chi_k v) \circ \Phi_k^{-1}\|_{(1-\alpha, -1)} \right) \\ &\leq C \left(\|\mathbf{u}\|_{(1-\alpha, 0)} + \|(\chi_k v) \circ \Phi_k^{-1}\|_{(1-\alpha, -1)} \right) \\ &\leq C \Lambda_0^{1-\alpha} \varepsilon_1^\alpha, \end{aligned}$$

where the constants C depend on the C^1 -norms of Φ_k , Φ_k^{-1} , χ_k . Note that we have used the fact that A_1 is a first-order linear operator in $\mathbf{u}, v$. Using [12, Theorem B.2.9] again to $(\chi_k \mathbf{u}) \circ \Phi_k^{-1}$, we have

$$\begin{aligned} \|(\chi_k \mathbf{u}) \circ \Phi_k^{-1}\|_{(2-\alpha, -1)} &\leq C \left(\|(\chi_k \mathbf{u}) \circ \Phi_k^{-1}\|_{(1-\alpha, 0)} + \|P_{1,k}((\chi_k \mathbf{u}) \circ \Phi_k^{-1})\|_{(-\alpha, -1)} \right) \\ &\leq C \left(\|\mathbf{u}\|_{(1-\alpha, 0)} + \|(\chi_k A_1(\mathbf{u}, v)) \circ \Phi_k^{-1}\|_{(-\alpha, -1)} \right. \\ &\quad \left. + \|[P_1, \chi_k]\mathbf{u} \circ \Phi_k^{-1}\|_{(-\alpha, -1)} \right) \\ &\leq C \left(\|\mathbf{u}\|_{(1-\alpha, 0)} + \|\mathbf{u}\|_{(1-\alpha, -1)} \right. \\ &\quad \left. + \|(\chi_k A_1(\mathbf{u}, v)) \circ \Phi_k^{-1}\|_{(-\alpha, -1)} \right) \\ &\leq C \Lambda_0^{1-\alpha} \varepsilon_1^\alpha. \end{aligned}$$

This gives $\nabla((\chi_k \mathbf{u}) \circ \Phi_k^{-1}) \in H_{(1-\alpha, -1)}$ and

$$\|\nabla((\chi_k \mathbf{u}) \circ \Phi_k^{-1})\|_{(1-\alpha, -1)} \leq C \Lambda_0^{1-\alpha} \varepsilon_1^\alpha.$$

Choosing $0 < \alpha < 1/2$, the trace of $\nabla((\chi_k \mathbf{u}) \circ \Phi_k^{-1})$ on $\Phi_k(B_k \cap \Sigma_f) \times [-T/2, T/2]$ is well-defined in $H^{-\frac{1}{2}-\alpha}$ by [12, Theorem B.2.7] with norm estimate,

$$\|\nabla((\chi_k \mathbf{u}) \circ \Phi_k^{-1})|_{\Phi_k(B_k \cap \Sigma_f)}\|_{H^{-\frac{1}{2}-\alpha}} \leq C_\alpha \|\nabla((\chi_k \mathbf{u}) \circ \Phi_k^{-1})\|_{(1-\alpha, -1)} \leq C \Lambda_0^{1-\alpha} \varepsilon_1^\alpha.$$

Changing back to the original Euclidean coordinate, we have $\nabla(\chi_k \mathbf{u})|_{B_k \cap \Sigma_f} \in H^{-\frac{1}{2}-\alpha}$ on $(B_k \cap \Sigma_f) \times [-T/2, T/2]$. Since $\mathbf{u} \in H^{1-\alpha}(N \times [-T/2, T/2])$, then $\mathbf{u}|_{\Sigma_f} \in L^2(\Sigma_f \times [-T/2, T/2])$ if $\alpha < 1/2$, and thus

$$\chi_k \nabla \mathbf{u}|_{B_k \cap \Sigma_f} \in H^{-\frac{1}{2}-\alpha}((B_k \cap \Sigma_f) \times [-\frac{T}{2}, \frac{T}{2}]),$$

with the same norm estimate as above. Hence,

$$\begin{aligned} \|\nabla \mathbf{u}|_{\Sigma_f}\|_{H^{-\frac{1}{2}-\alpha}(\Sigma_f \times [-\frac{T}{2}, \frac{T}{2}])} &= \left\| \sum_{k=1}^N \chi_k \nabla \mathbf{u}|_{\Sigma_f} \right\|_{H^{-\frac{1}{2}-\alpha}(\Sigma_f \times [-\frac{T}{2}, \frac{T}{2}])} \\ &\leq N \max_k \|\chi_k \nabla \mathbf{u}|_{B_k \cap \Sigma_f}\|_{H^{-\frac{1}{2}-\alpha}(\Sigma_f \times [-\frac{T}{2}, \frac{T}{2}])} \\ &\leq C N \Lambda_0^{1-\alpha} \varepsilon_1^\alpha. \end{aligned}$$

Then, the regularity for the stress tensor $\sigma(\mathbf{u})$ on Σ_f immediately follows from the definition (3). Contracting the stress tensor $\sigma(\mathbf{u})$ gives the same estimate for the components σ_n, σ_τ . $\square$

To prove our main result, we need the following lemma:

Lemma 3. *Let $c > 0$ be a constant and let $\Omega \subset \mathbb{R}^n$, $n > 2$, be a bounded open set. Suppose $\mathbf{f} \in H^1(\Omega; \mathbb{R}^m)$ with value in $\mathbb{R}^m$, $m \geq 1$. Let $\rho: \mathbb{R}^m \rightarrow \mathbb{R}^m$ be defined by*

$$\rho(\xi) = \frac{\xi}{\max(|\xi|, c)}, \quad \xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m. \quad (26)$$

Denote by $T_\rho \mathbf{f} = \rho \circ \mathbf{f}$ the composition operator. Then, T_ρ is a continuous operator from $H^1(\Omega; \mathbb{R}^m)$ to $H^1(\Omega; \mathbb{R}^m)$, and

$$\|T_\rho \mathbf{f}\|_{H^1(\Omega; \mathbb{R}^m)} \leq C(m, c, \Omega) \left(1 + \|\mathbf{f}\|_{H^1(\Omega; \mathbb{R}^m)}\right).$$

Proof. Let $j = 1, \dots, m$. To show $T_\rho \mathbf{f} \in H^1(\Omega; \mathbb{R}^m)$, it is enough to show that $\rho_j \circ \mathbf{f} \in H^1(\Omega)$ for all j , where the function $\rho_j: \mathbb{R}^m \rightarrow \mathbb{R}$ is the component of ρ , namely,

$$\rho_j(\xi) = \frac{\xi_j}{\max(|\xi|, c)}, \quad \xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m. \quad (27)$$

According to [18, Theorem 1], this follows after showing that ρ_j is locally Lipschitz and that there is a uniform constant $C > 0$ such that

$$|\partial_{\xi_k} \rho_j(\xi)| \leq C, \quad k = 1, \dots, m, \quad (28)$$

almost everywhere in $\mathbb{R}^m$, for any $j = 1, \dots, m$.

For $|\xi| \neq c$, the function ρ_j is smooth and it is straightforward to check that

$$|\partial_{\xi_k} \rho_j| \leq \frac{2}{c}, \quad j, k = 1, \dots, m.$$

Hence it suffices to verify that ρ_j is Lipschitz across $|\xi| = c$. Let $|\xi| \leq c$ and $|\eta| \geq c$. Then

$$\rho_j(\xi) - \rho_j(\eta) = \frac{\xi_j}{c} - \frac{\eta_j}{|\eta|},$$

and

$$\begin{aligned} |\rho_j(\xi) - \rho_j(\eta)| &\leq \left| \frac{\xi_j}{c} - \frac{\xi_j}{|\eta|} \right| + \left| \frac{\xi_j}{|\eta|} - \frac{\eta_j}{|\eta|} \right| \leq \frac{1}{c} |\eta| - c| + \frac{1}{c} |\xi_j - \eta_j| \\ &\leq \frac{1}{c} |\eta| - |\xi| + \frac{1}{c} |\xi_j - \eta_j| \leq \frac{2}{c} |\xi - \eta|. \end{aligned}$$

This shows that ρ_j is Lipschitz on $\mathbb{R}^m$ with uniform Lipschitz constant $2/c$, for all $j = 1, \dots, m$. Moreover, [18, Theorem 1] gives an estimate for the norm $\|\rho_j \circ \mathbf{f}\|_{H^1(\Omega)}$,

$$\|\rho_j \circ \mathbf{f}\|_{H^1(\Omega)} \leq C(m, c, \Omega) \left(1 + \|\mathbf{f}\|_{H^1(\Omega; \mathbb{R}^m)} \right), \quad j = 1, \dots, m.$$

For the continuity of T_ρ , it is necessary to use the specific form of the function ρ in (26), as the continuity does not hold for all composition operators in general, see, e.g., [19, Section 1]. In our case, for $\xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m$, we define

$$\mathbf{h}(\xi) := \rho(\xi) - \frac{\xi}{c} = \begin{cases} \frac{\xi}{|\xi|} - \frac{\xi}{c}, & |\xi| \geq c, \\ 0, & |\xi| < c. \end{cases} \quad (29)$$

It is clear that $\mathbf{h}$ is uniformly Lipschitz on $\mathbb{R}^m$ as ρ is uniformly Lipschitz on $\mathbb{R}^m$. We apply [19, Example 3.1] to the Lipschitz function $\mathbf{h}$ with $S = \{\xi \in \mathbb{R}^m : |\xi| \leq c\}$, the closed ball of radius c . Since $\mathbf{h}$ is identically 0 in S and $\mathbf{h}$ is smooth in the complement of S , it is enough to verify that, for every $\xi_0 \in \partial S$,

$$\lim_{\substack{\xi \rightarrow \xi_0 \\ \xi \notin S}} \nabla \mathbf{h}(\xi) \cdot \tau = 0, \quad \text{for all } \tau \in T_{\partial S}(\xi_0), \quad (30)$$

where $T_{\partial S}(\xi_0)$ denotes the tangent space of $\partial S = \{\xi \in \mathbb{R}^m : |\xi| = c\}$ at $\xi_0 \in \partial S$. For $\xi \in S \setminus \partial S$, we see $\nabla \mathbf{h}(\xi) = 0$ since $\mathbf{h}$ is identically 0 in S . On the complement $\mathbb{R}^m \setminus S$, since $\mathbf{h}|_{\mathbb{R}^m \setminus S}$ can be smoothly extended to a function $\tilde{\mathbf{h}}(\xi) := \xi/|\xi| - \xi/c$ on a neighborhood of ∂S , we have

$$\lim_{\substack{\xi \rightarrow \xi_0 \\ \xi \notin S}} \nabla \mathbf{h}(\xi) \cdot \tau = \nabla \tilde{\mathbf{h}}(\xi_0) \cdot \tau = 0, \quad \text{for all } \tau \in T_{\partial S}(\xi_0).$$

In the above, the fact that $\nabla \tilde{\mathbf{h}}(\xi_0) \cdot \tau = 0$ is due to the definition of tangent vector, namely, by differentiating $(\tilde{h}_j \circ \gamma)(s)$ at $s = 0$, where $\tilde{h}_j$, $j = 1, \dots, m$, are

the components of $\tilde{\mathbf{h}}$ and $\gamma(s)$ is a smooth curve in ∂S with the initial vector $\gamma'(0) = \tau \in T_{\partial S}(\xi_0)$. Then the claim immediately follows since $\tilde{\mathbf{h}}_j \circ \gamma = 0$ due to $\tilde{\mathbf{h}} = 0$ on ∂S . This verifies (30), and then [19, Example 3.1] gives the continuity of $T_{\mathbf{h}}$ and thus the continuity of T_{ρ} . $\square$

Now we prove Theorem 1.

Proof of Theorem 1. Since $|\llbracket \partial_t \mathbf{u}_\tau \rrbracket^\pm| \geq c_1 > 0$ on $\Sigma_f \times [-T, T]$, i.e., the fault is slipping everywhere, it follows from (9) that

$$g = |\sigma_\tau| = \mathcal{F}|F_n| \in C^{0,1}(\Sigma_f \times [-T, T]), \quad \|g\|_{C^{0,1}(\Sigma_f \times [-T, T])} \leq C_0^2. \quad (31)$$

On the other hand, the friction condition (2) implies that

$$\sigma_\tau = g \frac{\llbracket \partial_t \mathbf{u}_\tau \rrbracket_-^+}{|\llbracket \partial_t \mathbf{u}_\tau \rrbracket^\pm|}. \quad (32)$$

Denote

$$\mathbf{u}_+ := \mathbf{u}|_N, \quad \mathbf{u}_- := \mathbf{u}|_{\text{int}(D)}, \quad (33)$$

where N, D are the manifolds with smooth boundary constructed in the proof of Lemma 2, see Fig. 2. Let us denote

$$E(N) := W^{1,\infty}(-T, T; H^1(N)) \cap W^{2,\infty}(-T, T; L^2(N)). \quad (34)$$

As $\mathbf{u}_+ \in E(N)$ by (7), we can extend $\mathbf{u}_+$ to a small neighborhood $\tilde{N}$ of N in M so that the extension of $\mathbf{u}_+$ is in the same regularity class $E(\tilde{N})$. Similarly, we can extend $\mathbf{u}_-$ to a small neighborhood $\tilde{D}$ of D in M so that the extension of $\mathbf{u}_-$ is in the same regularity class $E(\tilde{D})$. Let V be a small open neighborhood of Σ_f in M . As $\Sigma_f \subset \partial D$, in this way, the functions $\mathbf{u}_+, \mathbf{u}_-$ are both extended to V and the extensions are in $E(V)$. Then $\partial_t \mathbf{u}_+, \partial_t \mathbf{u}_- \in H^1(V \times [-T, T])$, and thus the tangential component of the difference

$$\mathbf{f} := \partial_t(\mathbf{u}_+ - \mathbf{u}_-)_\tau \in H^1(V \times [-T, T]). \quad (35)$$

Recall the definition of tangential component near Σ_f in (4) where the unit normal vector field $\mathbf{n}$ is smoothly extended to a neighborhood of Σ_f . By Lemma 3, the function $\mathbf{f} / \max(|\mathbf{f}|, c_1) \in H^1(V \times [-T, T])$. Hence, as V is an open neighborhood of Σ_f in M , the trace of $\mathbf{f} / \max(|\mathbf{f}|, c_1)$ onto Σ_f is in $H^{1/2}(\Sigma_f \times [-T, T])$.

Denote the composition operator $T_\rho \mathbf{f} := \rho \circ \mathbf{f}$, where $\rho : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the Lipschitz function with uniform Lipschitz constant $4/c_1$ defined in (26), taking $c = c_1$. In the above, we have shown that $\text{tr}(T_\rho \mathbf{f}) \in H^{1/2}(\Sigma_f \times [-T, T])$, where tr stands for the trace operator onto Σ_f . Now we show that $\text{tr}(T_\rho \mathbf{f}) = T_\rho(\text{tr} \mathbf{f})$ in $L^2(\Sigma_f \times [-T, T])$, which would imply that $T_\rho(\text{tr} \mathbf{f}) \in H^{1/2}(\Sigma_f \times [-T, T])$. Let $\mathbf{f}_k \in C^\infty(\bar{V} \times [-T, T])$ be a sequence of smooth vector-valued functions such that $\mathbf{f}_k \rightarrow \mathbf{f}$ as $k \rightarrow \infty$ in $H^1(V \times [-T, T])$. Then, as $k \rightarrow \infty$,

$$\|T_\rho(\text{tr} \mathbf{f}) - T_\rho(\text{tr} \mathbf{f}_k)\|_{L^2(\Sigma_f \times [-T, T])} \leq \frac{4}{c_1} \|\text{tr} \mathbf{f} - \text{tr} \mathbf{f}_k\|_{L^2(\Sigma_f \times [-T, T])}$$

$$\leq \frac{4}{c_1} C \|f - f_k\|_{H^1(V \times [-T, T])} \rightarrow 0,$$

and by the continuity of the composition operator T_ρ from Lemma 3,

$$\|\text{tr}(T_\rho f) - \text{tr}(T_\rho f_k)\|_{L^2(\Sigma_f \times [-T, T])} \leq C \|T_\rho f - T_\rho f_k\|_{H^1(V \times [-T, T])} \rightarrow 0.$$

Since $T_\rho(\text{tr} f_k) = \text{tr}(T_\rho f_k)$ for smooth functions, we see that $T_\rho(\text{tr} f) = \text{tr}(T_\rho f)$ in $L^2(\Sigma_f \times [-T, T])$. As $\text{tr} f = [\partial_t \mathbf{u}_\tau]^\pm$ by definition, we conclude that $T_\rho([\partial_t \mathbf{u}_\tau]^\pm) \in H^{1/2}(\Sigma_f \times [-T, T])$. Thus, due to the condition $|\partial_t \mathbf{u}_\tau|^\pm| \geq c_1 > 0$ on $\Sigma_f \times [-T, T]$, we have

$$\frac{[\partial_t \mathbf{u}_\tau]^\pm}{|[\partial_t \mathbf{u}_\tau]^\pm|} \in H^{\frac{1}{2}}(\Sigma_f \times [-T, T]). \quad (36)$$

Using the equation (32), (31) and (36) imply that $\sigma_\tau \in H^{1/2}(\Sigma_f \times [-T, T])$. Namely, denote by A_g the multiplication operation by g . Since g is Lipschitz by (31), the operator $A_g : L^2(\Sigma_f \times [-T, T]) \rightarrow L^2(\Sigma_f \times [-T, T])$ and $A_g : H^1(\Sigma_f \times [-T, T]) \rightarrow H^1(\Sigma_f \times [-T, T])$ are bounded. Hence $A_g : H^s(\Sigma_f \times [-T, T]) \rightarrow H^s(\Sigma_f \times [-T, T])$ is a bounded operator for any $s \in (0, 1)$ by [17, Theorem 5.1]. Thus, by (32) and (36),

$$\sigma_\tau \in H^{\frac{1}{2}}(\Sigma_f \times [-T, T]), \quad (37)$$

with an estimate of the norm depending on Λ_0, C_0, c_1 and Σ_f .

Suppose that we have two systems with friction coefficients $\mathcal{F}_1, \mathcal{F}_2$ at Σ_f , and we are given elastic waves $\mathbf{u}_1, \mathbf{u}_2$ that are close in the sense that

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_{H^2(U \times [-T, T])} \leq \varepsilon_0.$$

Denote by $\sigma_n^{(j)}, \sigma_\tau^{(j)}$ ($j = 1, 2$) the components of the stress tensors corresponding to the two systems. Since the stress tensor $\sigma(\mathbf{u})$ is linear in $\mathbf{u}$, applying Lemma 2 to $\mathbf{u}_1 - \mathbf{u}_2$ gives

$$\begin{aligned} \|\sigma_\tau^{(1)} - \sigma_\tau^{(2)}\|_{H^{-\frac{1}{2}-\alpha}(\Sigma_f \times [-\frac{T}{2}, \frac{T}{2}])} &\leq C \Lambda_0^{1-\alpha} \varepsilon_1^\alpha, \\ \|\sigma_n^{(1)} - \sigma_n^{(2)}\|_{H^{-\frac{1}{2}-\alpha}(\Sigma_f \times [-\frac{T}{2}, \frac{T}{2}])} &\leq C \Lambda_0^{1-\alpha} \varepsilon_1^\alpha. \end{aligned}$$

Then picking $\alpha = 1/4$, by interpolation with $\sigma_\tau \in H^{1/2}$ by (37) and $\sigma_n = F_n \in C^{0,1} \subset H^1$ by (9), we have

$$\begin{aligned} \|\sigma_\tau^{(1)} - \sigma_\tau^{(2)}\|_{L^2(\Sigma_f \times [-\frac{T}{2}, \frac{T}{2}])} &\leq C(C_0, \Lambda_0, T, c_1) \varepsilon_1^{\frac{1}{10}}, \\ \|\sigma_n^{(1)} - \sigma_n^{(2)}\|_{L^2(\Sigma_f \times [-\frac{T}{2}, \frac{T}{2}])} &\leq C(C_0, \Lambda_0, T) \varepsilon_1^{\frac{1}{7}}. \end{aligned}$$

Hence,

$$\|\mathcal{F}_1 - \mathcal{F}_2\|_{L^2(\Sigma_f \times [-\frac{T}{2}, \frac{T}{2}])} = \left\| \frac{|\sigma_\tau^{(1)}|}{|\sigma_n^{(1)}|} - \frac{|\sigma_\tau^{(2)}|}{|\sigma_n^{(2)}|} \right\|_{L^2(\Sigma_f \times [-\frac{T}{2}, \frac{T}{2}])}$$

$$\begin{aligned}
&\leq c_0^{-2} \left\| \left(|\sigma_\tau^{(1)}| - |\sigma_\tau^{(2)}| \right) |\sigma_n^{(2)}| - \left(|\sigma_n^{(1)}| - |\sigma_n^{(2)}| \right) |\sigma_\tau^{(2)}| \right\|_{L^2} \\
&\leq c_0^{-2} C_0 \left\| |\sigma_\tau^{(1)}| - |\sigma_\tau^{(2)}| \right\|_{L^2} + c_0^{-2} C_0^2 \left\| |\sigma_n^{(1)}| - |\sigma_n^{(2)}| \right\|_{L^2} \\
&\leq c_0^{-2} C_0 \left\| \sigma_\tau^{(1)} - \sigma_\tau^{(2)} \right\|_{L^2(\Sigma_f \times [-\frac{T}{2}, \frac{T}{2}])} + c_0^{-2} C_0^2 \left\| \sigma_n^{(1)} \right. \\
&\quad \left. - \sigma_n^{(2)} \right\|_{L^2(\Sigma_f \times [-\frac{T}{2}, \frac{T}{2}])} \\
&\leq C(c_0, C_0, \Lambda_0, T, c_1) \varepsilon_1^{\frac{1}{10}},
\end{aligned}$$

which proves the stability part (2). The uniqueness part (1) is a consequence of the stability when $\varepsilon_0 \rightarrow 0$. $\square$

Acknowledgements. The authors thank Erik Burman for pointing out the reference on the regularity of the direct problem, and thank the anonymous referee for pointing out a mistake in an earlier version of the paper. MVdH was supported by the Simons Foundation under the MATH + X program, the National Science Foundation under grant DMS-2108175, and the corporate members of the Geo-Mathematical Imaging Group at Rice University. M.L. and J.L. were supported by the PDE-Inverse project of the European Research Council of the European Union and the Research Council of Finland, grants 273979 and 284715. L.O. was supported by the European Research Council of the European Union, grant 101086697 (Local), and the Research Council of Finland, grants 347715 and 353096. Views and opinions expressed are those of the authors only and do not necessarily reflect those of the European Union or the other funding organizations.

Funding Open Access funding provided by University of Helsinki (including Helsinki University Central Hospital).

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(Received October 27, 2023 / Accepted June 18, 2024)

Published online July 1, 2024

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