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A HILBERT–TYPE INEQUALITY FOR FOURIER COEFFICIENTS

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Abstract. There is extensive literature on Hilbert inequality and its many extensions. In this work we obtain inequalities involving Fourier coefficients of a Hölder continuous function. The results given here are valid without any assumption of monotonicity or signs of the Fourier coefficients.

1. Introduction

Hardy-Hilbert inequalities have a long history ([3]) with a substantial amount of literature. For sequences of real numbers $\{c_j\}$ and $\{d_j\}$ with $\sum c_j^2 < \infty$ and $\sum d_j^2 < \infty$, the basic inequalities are of the form

$$\left| \sum_{1 \leqslant j,k \leqslant \infty} c_j d_k h_{jk} \right| \leqslant \pi \left[\sum c_j^2\right]^{1/2} \left[\sum d_j^2\right]^{1/2},$$

where $h_{jk} = 1/(j+k)$, or $h_{jk} = 1/(j+k-1)$ or

$$h_{jk} = 1/(j-k) \text{ if } j \neq k,\ h_{jk} = 0 \text{ when } j = k.$$

Montgomery and Vaughan ([6]) extended the inequality whereby

$$h_{jk} = 1/(\lambda_j - \lambda_k) \quad \text{if} \quad j \neq k, \quad h_{jk} = 0 \quad \text{when} \quad j = k,$$

where $\{\lambda_j\}$ is an increasing sequence of real numbers with the constraint $\lambda_{j+1} - \lambda_j \geqslant f$, $j \geqslant 1$, for some positive constant f.

There are many extensions of discrete and integral versions of Hardy-Hilbert inequalities. A homogenous kernel $H(x,y)$, $x,y > 0$, is of order β if $H(tx,ty) = t^{\beta} H(x,y)$, $t > 0$. Extensive results are available for Hardy-Hilbert type inequalities for quadratic forms involving homogeneous kernels for discrete and continuos cases. A good account of these results can be found in the books [4] and [7], and in a recent survey article [2].

In this work we consider different kind of inequalities. Let h be a function on $[0,1]$ which is Hölder continuous with exponent θ, $0 < \theta < 1$, ie,

$$w(h) = \sup_{0 \leqslant x,y \leqslant 1} |h(x) - h(y)|/|x-y|^{\theta} < \infty \tag{1.1}$$

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Here we examine bounds for the quadratic forms $\sum_{j\neq k} c_j c_k h_{jk}$ and $\sum c_j c_k g_{jk}$, where

$$h_{jk} = (\rho_j \rho_k)^{\theta/2} \frac{\lambda_j - \lambda_k}{j-k}, \quad \text{with} \quad \rho_j = \pi(j-1/2),\ j \neq k, \tag{1.2}$$

$$g_{jk} = (\rho_j \rho_k)^{\theta/2} \frac{\lambda_j + \lambda_k}{j+k-1}, \tag{1.3}$$

$$\lambda_j = \lambda_j^C = \int_0^1 h(x)\cos(\rho_j x)dx \text{ or } \lambda_j = \lambda_j^S = \int_0^1 h(x)\sin(\rho_j x)dx.$$

Since λ_j^C and λ_j^S decay like $\rho_j^{-\theta}$, though not necessarily monotone (even in magnitude), we would expect h_{jk} and g_{jk} to behave like homogeneous kernels in ρ_j, ρ_k of order $\beta = -1$. We obtain the upper bound of the absolute values of $\sum_{j\neq k} c_j c_k h_{jk}$ and $\sum c_j c_k g_{jk}$ which involve the constant $w(h)$ given in (1.1) as well as the value of $h(1)$ or $h(0)$ depending on whether the quadratic form involves $\{\lambda_j^C\}$ or $\{\lambda_j^S\}$. We make no claim that the constants in the upper bounds are the best possible. We note in passing that $\{\phi_j = \sqrt{2}\cos(\rho_j x)\}$ is an orthonormal basis for $L_2 = L_2[0,1]$, the space of square integrable functions on $[0,1]$. Similarly, $\{\psi_j(x) = \sqrt{2}\sin(\rho_j x)\}$ is also an orthonormal basis for L_2. Thus $\{\lambda_j^C\}$ and $\{\lambda_j^S\}$ are proportional to the Fourier coefficients of h with respect to the bases $\{\phi_j\}$ and $\{\psi_j\}$ respectively.

We mention that, using the methods described in this work, it is possible to obtain similar bounds when λ_j is of the form $\int_0^1 h(x)\cos(\pi jx)dx$, or $\int_0^1 h(x)\cos(2\pi jx)dx$, or $\int_0^1 h(x)\sin(\pi jx)dx$, or $\int_0^1 h(x)\sin(2\pi jx)dx$. However, we do not present them here.

Section 2 lists the main results. Section 3 contains the proofs.

2. The main results

We begin this section with a well known result on Hilbert type inequalities involving homogeneous kernels of order $\beta = -1$ ([1], [5]).

THEOREM 1. *(a) Let H be a homogeneous kernel of order $\beta = -1$, $H(x,y) \geqslant 0$ for all $x, y > 0$. Assume that $H(1,y)y^{-1/2}$ and $H(y,1)y^{-1/2}$ are decreasing in y. If $H_1 = \int_0^\infty H(1,y)y^{-1/2}dy < \infty$, then for any sequences $\{c_j\}$ and $\{d_j\}$ with $\sum c_j^2 < \infty$ and $\sum d_j^2 < \infty$, the following holds*

$$\left|\sum c_j d_k H(j,k)\right| \leqslant H_1 \left\{\sum c_j^2\right\}^{1/2} \left\{\sum d_j^2\right\}^{1/2}.$$

(b) Let H be as in part (a) above. Additionally assume that $H(1,y)y^{-1/2}$ and $H(y,1)y^{-1/2}$ are convex in y. Denoting $j_1 = j - 1/2$, for any sequences $\{c_j\}$ and $\{d_j\}$, we have

$$\left|\sum c_j d_k H(j_1,k_1)\right| \leqslant H_1 \left\{\sum c_j^2\right\}^{1/2} \left\{\sum d_j^2\right\}^{1/2}.$$

We should point out that the double sums in Theorem 1 include the diagonal, ie, $\sum c_j d_j H(j,j)$ in part (a), and $\sum c_j d_j H(j_1, j_1)$ in part (b).

Before we state the main results, we present two simple lemmas. The first lemma is rather easy to verify and is stated without proof.

LEMMA 1. *If f and g are both non-negative, non-increasing (or non-decreasing), and convex, then their product fg is also non-negative, non-increasing (or non-decreasing), and convex.*

Let

$$\gamma_*^C = (2/\pi)w(h)S_\theta + (2/\pi)^{1-\theta}|h(1)|,$$
$$\gamma_*^S = (2/\pi)w(h)S_\theta + (2/\pi)^{1-\theta}|h(0)|, \tag{2.1}$$

where

$$S_\theta = \int_0^{\pi/2} x^\theta \sin(x)dx. \tag{2.2}$$

LEMMA 2. *Let λ_j^C and λ_j^S be as defined in the Introduction and let γ_*^C and γ_*^S be as in* (2.1). *Then*

$$\rho_j^\theta|\lambda_j^C| \leqslant \gamma_*^C, \;\; \rho_j^\theta|\lambda_j^S| \leqslant \gamma_*^S.$$

The proof of Lemma 2 will be given later. Let

$$A(\theta) = 2\pi[1+\tan(\pi\theta/2)].$$

We now state our main results. The proof of Theorem 2 involves carefully bounding the quadratic form $Q = \sum_{j\neq k} c_jc_kh_{jk}$ by sum of two appropriate quadratic forms. The first quadratic form uses Theorem 1 and the second quadratic form uses the well-known Hilbert inequality for $\{(j-k)^{-1},\ j\neq k\}$.

THEOREM 2. *Let h_{jk} be as in* (1.2). *Then for any sequence $\{c_j\}$ of real numbers with $\sum c_j^2 < \infty$, we have*

$$\left|\sum_{j\neq k} c_jc_kh_{jk}^C\right| \leqslant \gamma_*^C A(\theta)\sum c_j^2,$$

and

$$\left|\sum_{j\neq k} c_jc_kh_{jk}^S\right| \leqslant \gamma_*^S A(\theta)\sum c_j^2.$$

We now write down another result involving $\{g_{jk}\}$ defined in (1.3). It's proof is simple.

THEOREM 3. *Let g_{jk} be as in* (1.3). *Then for any sequence $\{c_j\}$ of real numbers with $\sum c_j^2 < \infty$, we have*

$$\left|\sum c_jc_kg_{jk}^C\right| \leqslant \gamma_*^C 2\pi\sec(\pi\theta/2)\sum c_j^2,$$

and

$$\left|\sum c_jc_kg_{jk}^S\right| \leqslant \gamma_*^S 2\pi\sec(\pi\theta/2)\sum c_j^2.$$

Here the sums include the diagonal elements.

REMARK 1. The constant $A(\theta)$ behaves well as long as θ stays away from 1. However, as θ approaches 1, $A(\theta) \to \infty$. It is not possible to remedy this when θ is near 1 as can be seen when $\theta = 1$ and $\lambda_j = \rho_j^{-1}$, and in that case $h_{jk} = -\pi(\rho_j\rho_k)^{-1/2}$. Take $c_j = \rho_j^{-1/2}$, $1 \leqslant j \leqslant n$ and $c_j = 0$ if $j > n$. The quadratic form with $\sum c_j^2 = 1$ is unbounded since

$$-\sum_{j,k} c_j c_k h_{jk} \Big/ \sum c_j^2 = \pi \sum_{j=1}^{n} \rho_j^{-1} \to \infty.$$

as $n \to \infty$.

REMARK 2. Focus of this paper is on the case $0 < \theta < 1$. However, when $\theta \to 0$, the upper bounds in Theorem 2 become simple. When $\theta \to 0$, $A(\theta) \to 2\pi$, and the upper bounds are

$$\gamma_*^C A(\theta) \to 4[w(h) + |h(1)|], \quad \gamma_*^S A(\theta) \to 4[w(h)| + |h(0)|].$$

REMARK 3. We are not aware of any nice simple formula for the integral S_θ given in (2.2) which appears in the expressions for γ_*^C and γ_*^S. However, the following reasoning is suggested by the referee. For each x, x^θ is convex in θ, then so is S_θ. Since $S_0 = S_1 = 1$, we have $S_\theta \leqslant 1$. Consequently,

$$\gamma_*^C \leqslant (2/\pi)w(h) + (2/\pi)^{1-\theta}|h(1)|, \quad \gamma_*^C \leqslant (2/\pi)w(h) + (2/\pi)^{1-\theta}|h(0)|.$$

REMARK 4. When $\theta \to 0$, the upper bounds given in Theorem 3 converge to the same limiting quantities listed in Remark 2 However, the bounds diverge to infinity as θ approaches 1. It is not possible to remedy this when θ is near 1 as can be seen when $\theta = 1$ and $\lambda_j = \rho_j^{-1}$, and in that case $g_{jk} = \pi(\rho_j\rho_k)^{-1/2}$. Take $c_j = \rho_j^{-1/2}$, $1 \leqslant j \leqslant n$ and $c_j = 0$ if $j > n$, then $\sum c_j c_k g_{jk} / \sum c_j^2$ is equal to $\pi \sum_{j=1}^{n} \rho_j^{-1}$ which diverges to infinity as $n \to \infty$.

REMARK 5. Note that $h_{jk} = h_{kj}$ and $g_{jk} = g_{kj}$. It then follows that we can obtain Hilbert type inequalities for $\sum_{j \neq k} c_j d_k h_{jk}$ and $\sum c_j d_k g_{jk}$ with the same upper bounds given in Theorems 2 and 3, where $\sum c_j^2 < \infty$ and $\sum d_j^2 < \infty$.

3. The proofs

Proof of Theorem 2. Denote $\rho_j^\theta \lambda_j$ by γ_j, and we know from Lemma 2 that

$$\sup_j |\gamma_j| \leqslant \gamma_*,$$

where γ_* has two different expressions for cosines and sines. The proof involves bounding the quadratic form $Q = \sum_{j \neq k} c_j c_k h_{jk}$ by the sum of two appropriate quadratic forms: the first quadratic form uses Theorem 1 and the second quadratic form uses the Hilbert inequality for $\{1/(j-k) : j \neq k\}$. Note that h_{jk} is equal to h_{jk}^C in the cosine case, and h_{jk}^S in the sine case.

We approximate ρ_j by the geometric mean of ρ_j and ρ_k and hence

$$\begin{aligned}\lambda_j = \rho_j^{-\theta}\gamma_j &= [\rho_j^{-\theta} - (\rho_j\rho_k)^{-\theta/2}]\gamma_j + (\rho_j\rho_k)^{-\theta/2}\gamma_j \\ &= [\rho_j^{-\theta/2} - \rho_k^{-\theta/2}]\rho_j^{-\theta/2}\gamma_j + (\rho_j\rho_k)^{-\theta/2}\gamma_j.\end{aligned}$$

Therefore

$$\begin{aligned}\lambda_j - \lambda_k = {} & [\rho_j^{-\theta/2} - \rho_k^{-\theta/2}]\rho_j^{-\theta/2}\gamma_j + (\rho_j\rho_k)^{-\theta/2}\gamma_j \\ & - [\rho_k^{-\theta/2} - \rho_j^{-\theta/2}]\rho_k^{-\theta/2}\gamma_k - (\rho_j\rho_k)^{-\theta/2}\gamma_k \\ = {} & -[\rho_k^{-\theta/2} - \rho_j^{-\theta/2}][\rho_j^{-\theta/2}\gamma_j + \rho_k^{-\theta/2}\gamma_k] + (\rho_j\rho_k)^{-\theta/2}(\gamma_j - \gamma_k).\end{aligned}$$

We can write

$$\begin{aligned}Q = {} & -\sum_{j\neq k} c_j c_k \frac{\rho_k^{-\theta/2} - \rho_j^{-\theta/2}}{j-k}[\rho_j^{-\theta/2}\gamma_j + \rho_k^{-\theta/2}\gamma_k](\rho_j\rho_k)^{\theta/2} \\ & + \sum_{j\neq k} c_j c_k \frac{\gamma_j - \gamma_k}{j-k} \\ := {} & Q_1 + Q_2. \qquad (3.1)\end{aligned}$$

Now denoting $\gamma_j c_j$ by d_j, we have

$$Q_2 = 2\sum_{j\neq k} c_j c_k \frac{\gamma_j}{j-k} = 2\sum_{j\neq k} d_j c_k \frac{1}{j-k}.$$

Use the Hilbert inequality for $\{(j-k)^{-1}, j\neq k\}$ to get

$$|Q_2| \leqslant 2\pi\left(\sum d_j^2\right)^{1/2}\left(\sum c_j^2\right)^{1/2} \leqslant 2\pi\gamma_* \sum c_j^2\,. \qquad (3.2)$$

Noting that $(\rho_k^{-\theta/2} - \rho_j^{-\theta/2})/(j-k) > 0$ for all $j \neq k$, and denoting $j-1/2$ by j_1, we have

$$\begin{aligned}|Q_1| &\leqslant \sum_{j\neq k} |c_j c_k| \frac{\rho_k^{-\theta/2} - \rho_j^{-\theta/2}}{j-k}\left|\rho_j^{-\theta/2}\gamma_j + \rho_k^{-\theta/2}\gamma_k\right|(\rho_j\rho_k)^{\theta/2} \\ &\leqslant \gamma_* \sum_{j\neq k} |c_j c_k| \frac{\rho_k^{-\theta/2} - \rho_j^{-\theta/2}}{j-k}[\rho_j^{-\theta/2} + \rho_k^{-\theta/2}](\rho_j\rho_k)^{\theta/2} \\ &= \gamma_* \sum_{j\neq k} |c_j c_k| \frac{\rho_k^{-\theta} - \rho_j^{-\theta}}{j-k}(\rho_j\rho_k)^{\theta/2} \\ &= \gamma_* \sum_{j\neq k} |c_j c_k| \frac{k_1^{-\theta} - j_1^{-\theta}}{j-k}(j_1 k_1)^{\theta/2} \\ &= \gamma_* \sum_{j\neq k} |c_j c_k| H(j_1, k_1),\end{aligned}$$

where

$$H(x,y) = \frac{y^{-\theta} - x^{-\theta}}{x-y}(xy)^{\theta/2}$$

Note that H is a nonnegative kernel of order $\beta = -1$ and $H(x,x) = \theta/x$. We will show that H satisfies conditions in part (b) of Theorem 1. In that case we will have

$$|Q_1| \leqslant \gamma_* \int_0^\infty H(1,y) y^{-1/2} dy \sum c_j^2, \tag{3.3}$$

where

$$\begin{aligned}\int_0^\infty H(1,y)y^{-1/2}dy &= \int_0^\infty \frac{y^{-\theta}-1}{1-y} y^{\theta/2-1/2} dy \\ &= \int_0^\infty \frac{y^{-\theta/2-1/2} - y^{\theta/2-1/2}}{1-y} dy \\ &= 2\int_0^1 \frac{y^{-\theta/2-1/2} - y^{\theta/2-1/2}}{1-y} dy := 2\,J.\end{aligned}$$

We now obtain an exact expression for J. Making a variable transformation $y = \exp(-t)$, we have

$$\begin{aligned}J &= \int_0^\infty \frac{\exp((1/2+\theta/2)t) - \exp((1/2-\theta/2)t)}{1-\exp(-t)} \exp(-t)dt \\ &= \int_0^\infty \frac{\exp(-(1/2-\theta/2)t) - \exp(-(1/2+\theta/2)t)}{1-\exp(-t)} dt \\ &= \int_0^\infty \left[\frac{\exp(-t)}{t} - \frac{\exp(-(1/2+\theta/2)t)}{1-\exp(-t)}\right] dt \\ &\quad - \int_0^\infty \left[\frac{\exp(-t)}{t} - \frac{\exp(-(1/2-\theta/2)t)}{1-\exp(-t)} -\right] dt.\end{aligned}$$

Note that each of the two integrals above has Gauss's integral representation of digamma function Ψ. Thus

$$\begin{aligned}J &= \Psi(1/2+\theta/2) - \Psi(1/2-\theta/2) \\ &= \Psi(1-(1/2-\theta/2)) - \Psi(1/2-\theta/2) \\ &= \pi\cot(\pi(1/2-\theta/2)) = \pi\tan(\pi\theta/2),\end{aligned}$$

where the last equality follows because of the reflection principle of digamma function, ie,

$$\Psi(1-x) - \Psi(x) = \pi\cot(\pi x).$$

Therefore

$$\int_0^\infty H(1,y)y^{-1/2}dy = 2\pi\tan(\pi\theta/2). \tag{3.4}$$

Our result follows from (3.2), (3.3) and (3.4).

What remains to show is that $H(1,y)y^{-1/2}$ is decreasing and convex in y. Since $H(x,y) = H(y,x)$, it follows that $H(y,1)y^{-1/2}$ is also decreasing and convex in y.

Note that

$$H(1,y)y^{-1/2} = \frac{y^{-\theta}-1}{1-y}\, y^{\theta/2-1/2} = f(y)g(y), \quad \text{with}$$

$$f(y) = \frac{y^{-\theta}-1}{1-y} \quad \text{and} \quad g(y) = y^{\theta/2-1/2}.$$

Clearly, both f and g are non-negative. If we can show that both f and g are decreasing and convex, then their product fg is also convex by Lemma 1. Clearly, g is decreasing and convex. We now show that f is also decreasing and convex.

The remainder theorem of calculus states that for any differentiable function p with a continuous derivative p',

$$p(x+h) - p(x) = h\int_0^1 p'(t(x+h)+(1-t)x)dt.$$

Let $p(x) = x^{-\theta}$. Taking $x = 1$ and $h = y-1$, we have

$$\begin{aligned} y^{-\theta} - 1 &= (y-1)(-\theta)\int_0^1 [ty+1-t]^{-\theta-1}dt \\ &= (1-y)\theta\int_0^1 [ty+1-t]^{-\theta-1}dt. \end{aligned}$$

Hence

$$f(y) = \frac{y^{-\theta}-1}{1-y} = \theta\int_0^1 [ty+1-t]^{-\theta-1}dt.$$

For each t, $[ty+1-t]^{-\theta-1}$ is decreasing and convex in y, and therefore $\int_0^1 [ty+1-t]^{-\theta-1}dt$ is decreasing and convex in y. This completes the proof of the theorem. □

Proof of Theorem 3. If we follow the same notations in the proof of Theorem 2, and denote $\sum c_j c_k g_{jk}$ by Q and $c_k\gamma_k$ by d_k, then

$$Q = 2\sum c_j d_k \frac{\rho_j^{\theta/2}\rho_k^{-\theta/2}}{j_1+k_1} = 2\sum c_j d_k H(j_1,k_1),$$

and thus

$$|Q| \leqslant 2\gamma_* \sum |c_j||c_k| H(j_1,k_1)$$

where $H(x,y) = x^{\theta/2}y^{-\theta/2}/(x+y)$. It is easy to check that the conditions of part (b) of Theorem 1 hold. The result now follows from Theorem 1b once we use Euler's reflection formula to get

$$\begin{aligned} \int_0^\infty H(1,y)y^{-1/2}dy &= Beta((1-\theta)/2,(1+\theta)/2) = \Gamma((1+\theta)/2)\Gamma((1-\theta)/2) \\ &= \pi/\sin(\pi(1+\theta)/2) = \pi\sec(\pi\theta/2). \quad \square \end{aligned}$$

Proof of Lemma 2. We will give a detailed proof for the cosine case, and indicate how the proof for sines is slightly different. In both cases $\rho_j\lambda_j^C$ and $\rho_j\lambda_j^S$ are split into j integrals. The last integral for the cosine case involves approximation of h by $h(1)$, whereas the first integral in the sine case involves estimating h by $h(0)$. In each case, the remaining $j-1$ integrals are approximated by using Hölder continuity of h.

For the cosine case, note that

$$\rho_j\lambda_j^C = \int_0^{\rho_j} h(x/\rho_j)\cos(x)dx = \sum_{t=1}^{j-1} I_t + I_j, \text{ where}$$
$$I_t = \int_{(t-1)\pi}^{t\pi} h(x/\rho_j)\cos(x)dx, \quad I_j = \int_{(j-1)\pi}^{(j-1/2)\pi} h(x/\rho_j)\cos(x)dx. \tag{3.5}$$

The last integral involves approximating $h(x/\rho_j)$ by $h(1)$.

For the sine case, we split the integral $\rho_j\lambda_j^S$ a bit differently

$$\begin{aligned}\rho_j\lambda_j^S &= \int_0^{\rho_j} h(x/\rho_j)\sin(x)dx\\ &= \int_0^{\rho_1} h(x/\rho_j)\sin(x) + \sum_{t=2}^{j}\int_{\rho_{t-1}}^{\rho_t} h(x/\rho_j)\sin(x)dx.\end{aligned}$$

In the first integral $h(x/\rho_j)$ is approximated by $h(0)$.

We now provide details for the cosine case and write λ_j^C as λ_j for notational simplicity. We prove the case for $j \geqslant 2$ since the case for $j = 1$ is simple.

For any $1 \leqslant t \leqslant j-1$, in (3.5) make a transformation $x \to x - (t-1/2)\pi = x - \rho_t$ to get

$$\begin{aligned}I_t &= \int_{(t-1)\pi}^{t\pi} h(x/\rho_j)\cos(x)dx\\ &= \int_{-\pi/2}^{\pi/2} h(x/\rho_j + \rho_t/\rho_j)\cos(x+\rho_t)dx\\ &= (-1)^t\int_{-\pi/2}^{\pi/2} h(x/\rho_j + \rho_t/\rho_j)\sin(x)dx\\ &= (-1)^t\int_{-\pi/2}^{\pi/2}[h(x/\rho_j + \rho_t/\rho_j) - h(\rho_t/\rho_j)]\sin(x)dx.\end{aligned}$$

Since

$$\left|h(x/\rho_j + \rho_t/\rho_j) - h(\rho_t/\rho_j)\right| \leqslant w(h)|x/\rho_j|^\theta$$

we have

$$|I_t| \leqslant w(h)\rho_j^{-\theta}\int_{-\pi/2}^{\pi/2}|x|^\theta|\sin(x)|dx = 2w(h)\rho_j^{-\theta}\int_0^{\pi/2} x^\theta\sin(x)dx. \tag{3.6}$$

Now consider the last term in (3.5). Making a variable transformation $x \to x - \rho_j$, we get

$$
\begin{aligned}
I_j &= \int_{(j-1)\pi}^{(j-1/2)\pi} h(x/\rho_j)\cos(x)dx \qquad (1)\\
&= \int_{-\pi/2}^{0} h(x/\rho_j+1)\cos(x+\rho_j)dx\\
&= (-1)^j \int_{-\pi/2}^{0} h(x/\rho_j+1)\sin(x)dx\\
&= (-1)^j \int_{-\pi/2}^{0} [h(x/\rho_j+1)-h(1)]\sin(x)dx + (-1)^j h(1)\int_{-\pi/2}^{0}\sin(x)dx\\
&= (-1)^j \int_{-\pi/2}^{0} [h(x/\rho_j+1)-h(1)]\sin(x)dx + (-1)^{j-1}h(1). \qquad (3.7)
\end{aligned}
$$

The integral in the last line of the displayed equation above can be bounded as

$$
\begin{aligned}
&\left|\int_{-\pi/2}^{0} [h(x/\rho_j+1)-h(1)]\sin(x)dx\right|\\
&\leqslant w(h)\rho_j^{-\theta}\int_{-\pi/2}^{0} |x^\theta \sin(x)|dx = w(h)\rho_j^{-\theta}\int_{0}^{\pi/2} x^\theta \sin(x)dx. \qquad (3.8)
\end{aligned}
$$

Thus we have from (3.7) and (3.8)

$$
|I_j| \leqslant w(h)\rho_j^{-\theta}\int_{0}^{\pi/2} x^\theta \sin(x)dx + |h(1)|. \qquad (3.9)
$$

From the upper bounds in (3.6) and (3.9), and the expression in (3.5), and denoting the integral $\int_0^{\pi/2} x^\theta \sin(x)dx$ by S_θ we have

$$
\begin{aligned}
|\rho_j\lambda_j| &\leqslant \sum_{t=1}^{j-1} |I_t| + |I_j|\\
&\leqslant (j-1)2w(h)S_\theta\rho_j^{-\theta} + w(h)S_\theta\rho_j^{-\theta} + |h(1)|\\
&= 2(j-1/2)w(h)S_\theta\rho_j^{-\theta} + |h(1)|\\
&= (2/\pi)w(h)S_\theta\rho_j^{1-\theta} + |h(1)|.
\end{aligned}
$$

Thus

$$
\rho_j^\theta|\lambda_j| \leqslant (2/\pi)w(h)S_\theta + \rho_j^{\theta-1}|h(1)| \leqslant (2/\pi)w(h)S_\theta + (2/\pi)^{1-\theta}|h(1)|. \quad \square
$$

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