Annals of Mathematics **200** (2024), 557–608
https://doi.org/10.4007/annals.2024.200.2.4

Brauer's Height Zero Conjecture

By Gunter Malle, Gabriel Navarro, A. A. Schaeffer Fry, and Pham Huu Tiep

Abstract

We complete the proof of Brauer's Height Zero Conjecture from 1955 by establishing the open implication for all odd primes.

Contents

Keywords: Brauer's Height Zero Conjecture, Brauer blocks, defect groups, finite groups of Lie type

AMS Classification: Primary: 20C20; Secondary: 20C15, 20C33.

The first author gratefully acknowledges financial support by SFB TRR 195 – Project-ID 286237555. The research of the second author is supported by Ministerio de Ciencia e Innovación PID2019-103854GB-I00. The third author is grateful for support from the NSF, under grant DMS-2100912. The fourth author gratefully acknowledges the support of the NSF (grants DMS-1840702 and DMS-2200850), the Simons Foundation, and the Joshua Barlaz Chair in Mathematics. Part of this work was done while the authors participated in the programme "*Groups*, *Representations*, *and Applications*: *New Perspectives*" at the Isaac Newton Institute for Mathematical Sciences in 2022. This work was supported by EPSRC grant number EP/R014604/1.

1. Introduction

Brauer's Height Zero Conjecture (BHZ), formulated in 1955 [7], has been one of the most fundamental and challenging problems in the representation theory of finite groups. Deeply influencing the research in the field, it is also a source of many developments in the theory. If p is a prime and B is a Brauer p-block with defect group D of a finite group G, R. Brauer proved that $|G : D|_p$ is the largest power of p dividing the degrees of all the irreducible complex characters in B. (In this paper, n_p denotes the largest power of p dividing the integer n.) Hence, if $\chi \in \mathrm{Irr}(B)$, the set of irreducible complex characters in B, then $\chi(1)_p = |G : D|_p\, p^{h_\chi}$ for some non-negative integer h_χ called the *height of* χ. The conjecture asserts that $\chi(1)_p = |G : D|_p$ for all $\chi \in \mathrm{Irr}(B)$ if and only if D is abelian. That is, $h_\chi = 0$ for all $\chi \in \mathrm{Irr}(B)$ if and only if D is abelian.

The "if" implication of the Height Zero Conjecture was proven in [26], using the classification of finite simple groups, after decades of contributions by many authors. The "only if" implication was proven for p-solvable groups in [21]; for $p = 2$ and blocks of maximal defect (that is, when D is a Sylow 2-subgroup of G) in [45]; and recently for principal blocks, for every prime, in [36]. Furthermore, building upon work in [46], it was shown in [44] that Brauer's Height Zero Conjecture is implied by the inductive Alperin–McKay condition on simple groups (a strong form of another main conjecture in our field). This has enabled L. Ruhstorfer to recently prove the Height Zero Conjecture for $p = 2$ in [52]. However, the verification of the inductive Alperin–McKay condition on simple groups for odd primes remains an enormous challenge.

In this paper we take a different approach and prove the open direction of Brauer's Height Zero Conjecture in the case that p is odd.

THEOREM A. *Let G be a finite group, let p be an odd prime, and let B be a p-block of G with defect group D. If $\chi(1)_p = |G : D|_p$ for all $\chi \in \mathrm{Irr}(B)$, then D is abelian.*

As discussed above, this implies

COROLLARY. *Brauer's Height Zero Conjecture holds.*

A key novelty of our approach is a combined use of new results on blocks of quasi-simple groups as well as on permutation groups, which allows us to

tightly control the structure of a minimal counterexample to BHZ and overcome certain difficulties in proving extendibility of characters from normal subgroups that were encountered in previous approaches.

There are, at least, two major obstacles for our approach. The first is to prove that irreducible characters in p-blocks of quasi-simple groups lie in sufficiently many distinct orbits under the action by their automorphism groups. We think that the following result has independent interest and that it will be useful in the resolution of other problems.

THEOREM B. *Suppose that p is an odd prime, S is a quasi-simple group, and b is a p-block of S with non-cyclic defect groups. Then at least one of the following statements holds*:

(1) $\mathrm{Irr}(b)$ *contains characters from at least three different* $\mathrm{Aut}(S)$*-orbits*; *or*
(2) *all characters in* $\mathrm{Irr}(b)$ *have the same degree.*

In the presence of blocks with cyclic defect groups, or when all the irreducible characters in b have the same degree (and therefore b is nilpotent in the sense of Broué–Puig), we will instead use deep results by Koshitani–Späth, Broué–Puig and Külshammer–Puig ([30], [9], [32]) in order to prove Theorem A.

The second obstacle to our approach is inherent to Brauer's Height Zero Conjecture and independent of any road that is followed to prove it. Suppose that G is a finite group and σ is an automorphism of order a power of p of G that stabilises a p-block B of G, a defect group D of B, and a p-block b_D of $\mathbf{C}_G(D)$ that induces B. If D is abelian, Brauer's Height Zero conjecture (and the inductive Alperin–McKay condition) implies that all the irreducible characters of B are fixed by σ if and only if σ acts trivially on D. In fact, we will need a more sophisticated version of the following result for quasi-simple groups (see Theorem 4.1).

THEOREM C. *Suppose that p is an odd prime and that S is a quasi-simple group such that $\mathbf{Z}(S)$ is a cyclic p'-group. Let b be a p-block of S with abelian defect group D. Suppose that σ is an automorphism of S of p-power order that fixes all the irreducible characters of b, normalises D, and stabilises a block b_D of $\mathbf{C}_S(D)$ that induces b. Then σ acts trivially on D.*

In Section 2, we prove the needed result on permutation groups and other technical results that we will use in the proof of Theorem A. In Section 3, we prove Theorem B. Section 4 is devoted to the proof of a refined version of Theorem C and its needed generalisations. Finally, in Section 5, Theorem A is proved. As was the case in the proof of the other direction of BHZ, as well as the cases of maximal defect for $p = 2$ and for principal blocks, our proof (including the result on permutations groups that we have mentioned as well as Theorems B and C) relies on the Classification of Finite Simple Groups.

Acknowledgements. We thank Zhicheng Feng, Ruwen Hollenbach, Radha Kessar, Markus Linckelmann, Alexander Moretó, Noelia Rizo, Britta Späth and Lizhong Wang for helpful discussions on aspects of this project. We are grateful to the referee for careful reading and insightful comments that helped improve the paper.

2. **Preliminary results**

In this section, we first prove a consequence of [20] that we will use in the proof of Theorem A.

THEOREM 2.1. *Let $p > 2$ be a prime, $n > 1$, and let $G < \mathrm{Sym}(\Omega) = \mathfrak{S}_n$ be a subgroup such that $G = \mathbf{O}^{p'}(G) \neq 1$. Then there is a partition*

$$\Omega = \Delta_1 \sqcup \Delta_2 \sqcup \Delta_3,$$

with $|\Delta_1|, |\Delta_2| > 0$ and $|\Delta_3| \geq 0$ such that the index of $\cap_{i=1}^3 \mathrm{Stab}_G(\Delta_i)$ in G is divisible by p. Moreover, one can choose this partition to have $\Delta_3 = \varnothing$, unless G has a simple quotient S such that one of the following holds:

(1) *$S = \mathfrak{A}_{ap^s-1}$ with $1 \leq a \leq p-1$, $s \geq 1$ and $(a, s) \neq (1, 1)$; or*
(2) *$p = 3$ and $S = C_3$ or $\mathrm{SL}_3(2)$.*

Proof. (i) First we consider the case that G is primitive on Ω. In this case, applying [20, Th. 2] we obtain a partition $\Omega = \Delta_1 \sqcup \Delta_2$ with $p \mid [G : \cap_{i=1}^2 \mathrm{Stab}_G(\Delta_i)]$, unless G has a simple quotient S and (S, p) are as in (1), in fact with $S = \mathfrak{A}_n$ acting on $n = ap^s - 1$ points, or we are in (2), in fact with $G = \mathrm{ASL}_3(2)$ or $\mathrm{A\Gamma L}_1(8)$, acting on the eight vectors of $\mathbb{F}_2^3 = \langle e_1, e_2, e_3 \rangle_{\mathbb{F}_2}$.

In the former case, choosing

$$\Delta_1 = \{1\}, \quad \Delta_2 = \{2, 3, \ldots, p\}, \quad \Delta_3 = \{p+1, p+2, \ldots, n\},$$

we have

$$[G : \bigcap_{i=1}^{3} \mathrm{Stab}_G(\Delta_i)] = p\binom{n}{p}.$$

In the latter case, choosing

$$\Delta_1 = \{0\}, \quad \Delta_2 = \{e_1, e_2\}, \quad \Delta_3 = \mathbb{F}_2^3 \smallsetminus \{0, e_1, e_2\},$$

we see that $\mathrm{Stab}_G(\Delta_i)$ has order dividing 8 and so its index in G is divisible by $p = 3$.

(ii) Now assume that G is transitive but imprimitive on Ω. Let $\Omega = \cup_{i=1}^m \Omega_i$ be a G-invariant partition of Ω, with $1 \leq |\Omega_i| = n/m < n$, and m chosen to be smallest possible subject to these conditions. Let $B := \cap_{i=1}^m \mathrm{Stab}_G(\Omega_i)$ be the base subgroup. Then G/B permutes the $m > 1$ blocks Ω_i transitively, so $1 \neq G/B$. Since $G = \mathbf{O}^{p'}(G)$, we again have $G/B = \mathbf{O}^{p'}(G/B)$. Now G/B satisfies the assumptions on G, and G/B acts transitively, faithfully, and

primitively (by minimality of m) on $\{\Omega_1, \dots, \Omega_m\}$. A desired partition for G/B on this set gives rise to a desired partition on Ω.

(iii) Finally, we consider the case where G acts intransitively on $\Omega = \{1, \dots, n\}$. Suppose that $\Omega_1, \dots, \Omega_m$ are all G-orbits on Ω, and let K_i denote the kernel of G acting on Ω_i. Since $G \neq 1$, G must act non-trivially on at least one Ω_i. So we may assume that $1 \neq G/K_1$. But $G = \mathbf{O}^{p'}(G)$, so p divides $|G/K_1|$, and in fact $G/K_1 = \mathbf{O}^{p'}(G/K_1)$. Now G/K_1 satisfies the assumptions on G, and G/K_1 acts transitively (and faithfully) on Ω_1. If $\Omega_1 = \Delta_1 \sqcup \Delta_2$ is a desired partition for G/K_1, then

$$\Omega = \Delta_1 \sqcup \big(\Delta_2 \cup (\Omega \smallsetminus \Omega_1)\big)$$

is a desired partition for G. If $\Omega_1 = \Delta_1 \sqcup \Delta_2 \sqcup \Delta_3$ is a desired partition for G/K_1 with $\Delta_3 \neq \varnothing$, then

$$\Omega = \Delta_1 \sqcup \Delta_2 \sqcup \big(\Delta_3 \cup (\Omega \smallsetminus \Omega_1)\big)$$

is a desired partition for G. □

For the first part of Theorem 2.1, see also [16, Lemma 3.2].

Next we study the structure of some almost simple groups. In the case where $\bar{S}$ is a simple group of Lie type, we will use the notation $\mathrm{Inndiag}(\bar{S})$ as described in [22, Th. 2.5.12]; for other simple groups $\bar{S}$, we use the convention that $\mathrm{Inndiag}(\bar{S}) = \bar{S}$.

PROPOSITION 2.2. *Let p be an odd prime, and let S be a quasi-simple group. Let $\bar{S} := S/\mathbf{Z}(S)$, $\bar{S} \leq H \leq \mathrm{Aut}(S)$, and assume that $\mathbf{O}^{p'}(H/\bar{S}) = H/\bar{S}$.*

(a) *Then $H/\bar{S}$ has a normal p-complement.*

(b) *Suppose that $p \geq 5$ and that S is not of Lie type A_n, ${}^2\mathrm{A}_n$. Then $H/\bar{S}$ is a cyclic p-group. The same is true if $p = 3$ but S is not of Lie type A_n, ${}^2\mathrm{A}_n$, D_4 or $\mathrm{E}_6(\epsilon q)$ with $3|(q - \epsilon)$, $\epsilon \in \{\pm 1\}$.*

(c) *In general, if $H \leq \mathrm{Inndiag}(\bar{S})$, then $H/\bar{S}$ is a cyclic p-group.*

Proof. (i) Note that $H/\bar{S}$ is embedded in $\mathrm{Out}(\bar{S})$. If $\bar{S}$ is an alternating or sporadic simple group, then $\mathrm{Out}(\bar{S})$ is a 2-group and hence the statements are obvious. Suppose that $\bar{S}$ is a simple group of Lie type. In this case, the structure of $\mathrm{Out}(\bar{S})$ is described in [22, Th. 2.5.12], and we will now verify (a) and (b).

Assume in addition that S is not of type D_4 when $p = 3$. Then the assumption $\mathbf{O}^{p'}(H/\bar{S}) = H/\bar{S}$ implies that $\bar{H} := H/\bar{S}$ is contained in $O \rtimes A$, where $O = \mathrm{Outdiag}(\bar{S})$ is abelian and A is a cyclic p-group. In particular, (a) holds in this case. Given the assumptions in (b), we have that either O is a cyclic p'-group of order at most 4, or S is of type D_{2m} and O is a Klein 4-group. In the former case, $\mathrm{Aut}(O \cap \bar{H})$ is of order at most 2. Hence $\bar{H}$ centralises $O \cap \bar{H}$, $O \cap \bar{H} \leq \mathbf{Z}(\bar{H})$, and $\bar{H}/(O \cap \bar{H}) \hookrightarrow A$ is cyclic. Thus $\bar{H}$ is abelian, with cyclic Sylow p-subgroup, and the statement follows. In the latter

case, A centralises O (see [22, Th. 2.5.12(h)]), so again $\bar{H}$ is abelian, and we are again done.

Now we complete the proof of (a) in the case where S is of type D_4 over $\mathbb{F}_{r^f}$, where r is any prime, and $p = 3$. In this case, $\mathrm{Out}(\bar{S}) = O \rtimes (C_f \times \mathfrak{S}_3)$, so $\bar{H}$ is contained in $O \rtimes (C_e \times C_3)$, where e is the 3-part of f. Since $O = 1$ or $O = C_2 \times C_2$, the claim follows.

(ii) For (c), just note that $\mathrm{Inndiag}(\bar{S})/\bar{S}$ is either cyclic or of p'-order. □

Next we prove some results on blocks that will be useful later on. Our notation for block theory mostly follows [43].

LEMMA 2.3. *Suppose that $N \trianglelefteq G$ and let b be a G-invariant block of N with defect group D. Let b_D be a block of $D\mathbf{C}_N(D)$ inducing b with defect group D. Let T be the stabiliser of b_D in $\mathbf{N}_G(D)$. If B is a block of G covering b, then there is a defect group D_0 of B such that $D_0 \cap N = D$ and $D_0 \leq T$. Also, $\mathbf{N}_G(D) = \mathbf{N}_N(D)T$. Furthermore, $T = (T \cap N)D_0$ if G/N is a p-group.*

Proof. Since b is G-invariant, we have $G = N\mathbf{N}_G(D)$ by the Frattini argument. Now $b_0 = b_D^{\mathbf{N}_N(D)}$ is the Brauer First Main correspondent of b. Since b is $\mathbf{N}_G(D)$-invariant, it follows that b_0 is $\mathbf{N}_G(D)$-invariant by the uniqueness in the Brauer correspondence. By the Harris–Knörr correspondence [43, Th. 9.28], let B_0 be the unique block of $\mathbf{N}_G(D)$ that induces B and covers b_0. Let D_0 be a defect group of B_0, which by [43, Th. 9.28] is a defect group of B. By Knörr's theorem [43, Th. 9.26], we have that $D_0 \cap \mathbf{N}_N(D)$ is a defect group of b_0, and therefore $D_0 \cap N = D_0 \cap \mathbf{N}_N(D) = D$. Since b_0 is $\mathbf{N}_G(D)$-invariant and b_0 covers an $\mathbf{N}_N(D)$-orbit of blocks of $D\mathbf{C}_N(D)$, we have $\mathbf{N}_G(D) = \mathbf{N}_N(D)T$ by the Frattini argument, where we recall that T is the stabiliser of b_D in $\mathbf{N}_G(D)$. By the Fong–Reynolds Theorem 9.14 of [43], there exists an $\mathbf{N}_G(D)$-conjugate of D_0 contained in T.

Suppose then that $D_0^x \leq T$, where $x \in \mathbf{N}_G(D)$, and D_0^x is a defect group of b_T, the block of T that is the Fong–Reynolds correspondent of B_0 over b_D. Then D_0^x is a defect group of B. Also $D_0^x \cap N = (D_0 \cap N)^x = D^x = D$, and this proves the first part. For the final part, notice that $T/(T \cap N)$ is a p-group. Furthermore, $(b_D)^{T\cap N}$ is the Fong–Reynolds correspondent of b_0 over b_D. By uniqueness, it follows that $(b_D)^{T\cap N}$ is T-invariant (using that b_0 and b_D are T-invariant). Also, notice that $(b_D)^{T\cap N}$ is the only block of $T \cap N$ covering b_D, using [43, Cor. 9.21]. We conclude that b_T covers $(b_D)^{T\cap N}$, a block with defect group D. Since $(b_D)^{T\cap N}$ is T-invariant and b_T is the only block of T covering it ([43, Cor. 9.6]), we have $T = (T \cap N)D_0^x$, by [43, Th. 9.17], for instance. □

PROPOSITION 2.4. *Let $N \trianglelefteq G$, and let B be a p-block of G with abelian defect groups. Suppose that G/N is a p-group and that B covers a G-invariant block b of N with defect group D. Suppose b_D is a block of $\mathbf{C}_N(D)$ with defect*

group D that induces b. If $x \in \mathbf{N}_G(D)$ is a p-element that fixes b_D, then $[x, D] = 1$.

Proof. Since B has abelian defect groups, notice that D is abelian by Knörr's theorem [43, Th. 9.26]. Let T be the stabiliser of b_D in $\mathbf{N}_G(D)$. By Lemma 2.3, there is a defect group D_0 of B such that $D_0 \cap N = D$, $D_0 \leq T$, and $T = (T \cap N)D_0$. Since by hypothesis D_0 is abelian, we have $T = (T \cap N)\mathbf{C}_T(D)$. Then $(T \cap N)/\mathbf{C}_N(D)$ is a p'-group by [43, Th. 9.22]. Therefore $T/\mathbf{C}_T(D)$ is a p'-group, and the result follows. □

Recall that a block B of a finite group G is called *quasi-primitive* if whenever $N \trianglelefteq G$, then B covers a unique block of N.

PROPOSITION 2.5. *Suppose that B is a quasi-primitive p-block of a finite group G such that all the irreducible characters of B have height zero. Let $N \trianglelefteq G$, and let b be a block of N covered by B. If $\theta \in \mathrm{Irr}(b)$, then $|G : G_\theta|$ is not divisible by p.*

Proof. We use Dade's group $K = G[b]$. (See, for instance, [42] for an introduction to this object.) Then all the irreducible characters of b are K-invariant (see, for instance, [29, Lemma 3.2(a)]). In particular, θ is K-invariant. Also, $K \trianglelefteq G_b = G$, where G_b is the stabiliser of b in G. Also if B' is the (unique) block of K covered by B, then $B'^G = B$ and B is the only block of G covering B'. (See [42, Th. 3.5].) Since B covers b, let $\chi \in \mathrm{Irr}(B)$ be over θ. Let $\eta \in \mathrm{Irr}(K)$ be under χ and over θ. Since χ is over η, it follows that the block of η is necessarily B'. By hypothesis, B' is G-invariant. By [43, Cor. 9.18], we have that $|G : G_\eta|$ is prime to p. Since $\eta_N = v\theta$ for some $v \geq 1$, it follows that $G_\eta \leq G_\theta$ and therefore $|G : G_\theta|$ is also not divisible by p. □

In order to prove Theorem A, we will need the block theory *above* a nilpotent block, and *above* a block with cyclic defect group.

THEOREM 2.6. *Let G be a finite group, and let $N \trianglelefteq G$. Let b be a G-invariant nilpotent block of N with defect group Q. Then there exist a p-subgroup P of G such that $Q = P \cap N$, a finite group L with a Sylow p-subgroup P, a central p'-extension L' of L by $Z \leq \mathbf{Z}(L')$ and $\mu \in \mathrm{Irr}(Z)$, such that*

(a) *$PN/N \in \mathrm{Syl}_p(G/N)$.*
(b) *We have $Q \trianglelefteq L$ and $G/N \cong L/Q$. If $|\mathbf{Z}(G)|$ is not divisible by p, $\mathbf{Z}(G) \leq N$ and $Q\mathbf{Z}(G) < N$, then $|L' : \mathbf{Z}(L')| < |G : \mathbf{Z}(G)|$.*
(c) *There is a bijection $B \mapsto B'$ between the blocks of G that cover b and the blocks of L' that cover the block of μ preserving defect groups. Also, there is a height preserving bijection $\mathrm{Irr}(B) \to \mathrm{Irr}(B')$.*

Proof. These are consequences of the theory of blocks above nilpotent blocks developed in [32]. This is also described in Section 8.12 of [33]. (See

also Section 7.2 of [53]). The existence of P, the fact that $P \cap N = Q$ and that PN/N is a Sylow p-subgroup of G/N follow from 8.12.5 and 8.12.6 of [33]. The existence of L and the fact that $P \in \mathrm{Syl}_p(L)$, that $Q \trianglelefteq L$ and that $L/Q \cong G/N$ are part of the statement in 8.12.5. Theorem 8.12.5 also provides a bijection between the blocks of G covering b and the blocks of a certain twisted group algebra $\mathcal{O}_\alpha L$ of L, and corresponding blocks are Morita equivalent (see Remark 8.12.8). In Remark 8.12.8, the existence of L' and the relationship with $\mathcal{O}_\alpha L$ is given.

Corollary 9.2.5 of [33] says that a perfect isometry preserves heights, and any of Corollary 9.3.3 or 9.3.4 imply that a Morita equivalence induces a perfect isometry. Then 9.7.1 of [33] implies that any Morita equivalence given by a bimodule with source of rank prime to p induces isomorphisms between defect groups. This is restated as part of 9.11.2 of [33].

This covers all but the last inequality in (b). So, assume $|\mathbf{Z}(G)|$ is not divisible by p, $\mathbf{Z}(G) \leq N$ and $Q\mathbf{Z}(G) < N$. Then

$$|L' : \mathbf{Z}(L')| \leq |L| = |G/N||Q| = |G/N||Q\mathbf{Z}(G) : \mathbf{Z}(G)| < |G : \mathbf{Z}(G)|. \quad \square$$

THEOREM 2.7. *Suppose $N \trianglelefteq G$ are finite groups and B is a p-block of G with defect group D covering a G-invariant block b_0 of N, where p is odd. Suppose that N is the central product of the G-conjugates of a component S of G. Suppose that the block b_0 covers the block b of S and that $D_0 = D \cap S$ is cyclic and non-central in S. Let $D_1 = D \cap N$, and let b_1 be the block of $\mathbf{N}_N(D_1)$ that induces b_0. Let B_1 be the block of $\mathbf{N}_G(D_1)$ with defect group D that covers b_1 and induces B. If all characters in $\mathrm{Irr}(B)$ have height zero, then all characters in $\mathrm{Irr}(B_1)$ have height zero.*

Proof. By [30, Ths. 1.1 and 7.6], the block b of S satisfies the inductive Alperin–McKay condition. (In fact, the "intermediate subgroup" in [30, Def. 7.2], is the normaliser of the corresponding defect group.) We apply Theorem 6.1 and Proposition 6.2 of [44], noticing that we do not need to assume that the simple group $S/\mathbf{Z}(S)$ satisfies the inductive Alperin–McKay condition, because we know that blocks of S involved in our statement have cyclic defect groups (and therefore satisfy the inductive Alperin–McKay condition). Notice that D_1 is a p-radical subgroup of N since it is the defect group of b_1. If f is any block of N with defect group D_1, using that N is the central product of G-conjugates of S, then f covers a unique block of S with defect group $D_1 \cap S = D_0$, which is cyclic. Hence, by [44, Prop. 6.2], there is an $\mathbf{N}_G(D_1)$-equivariant bijection

$$\Omega : \mathrm{Irr}(N|D_1) \to \mathrm{Irr}(\mathbf{N}_N(D_1)|D_1)$$

such that

$$(G_\theta, N, \theta) \sim_b (\mathbf{N}_G(D_1)_{\theta'}, \mathbf{N}_N(D_1), \theta')$$

for every $\theta \in \mathrm{Irr}(b_0)$, where $\theta' = \Omega(\theta)$. (Here $\mathrm{Irr}(N|D_1)$ is the set of characters of N belonging to blocks with defect group D_1. The definition of block isomorphism of character triples denoted above with $\sim_b$ is given in [44, Def. 3.6].) Now, we apply [44, Cor. 3.10] to construct a height preserving bijection $\mathrm{Irr}(B|\theta) \to \mathrm{Irr}(B_1|\theta')$. Since $\Omega(\mathrm{Irr}(b_0)) = \mathrm{Irr}(b_1)$ by [44, Th. 6.1(b)], we easily conclude that all irreducible characters in B_1 have height zero. □

The following gives a shorter proof of a generalisation of the main result of [31].

THEOREM 2.8. *Suppose that $N, M \trianglelefteq G$ with $G = NM$. Let B be a block of G that covers a unique block $b_{N\cap M}$ of $N \cap M$. Then there is a defect group D of B such that $D = (D \cap N)(D \cap M)$.*

Proof. We argue by induction on $|G : N| + |G : M| + |G|$. We may assume that $N, M < G$. Let b_N be a block of N covered by B and b_M a block of M covered by B, both covering $b_{N\cap M}$. Let $T = G_{b_N}$ be the stabiliser of b_N, and by the Fong–Reynolds theorem, let b_T be the block of T that induces B and covers b_N. If $T < G$, then by induction there is a defect group D of b_T (and therefore of B) such that

$$D = (D \cap N)(D \cap T \cap M) \leq (D \cap N)(D \cap M).$$

We thus may assume that b_N and b_M are G-invariant.

Now, let us fix a defect group D of B. Let $X = M \cap ND$, so that $X/(N \cap M)$ is a p-group. Thus $b_{N\cap M}$ is covered by a unique block b_X of X. Also the unique block b_{ND} that covers b_N has defect group D by Problem 9.4 of [43]. If $ND < G$, by induction there is $n \in ND$ such that

$$D^n = (D^n \cap N)(D^n \cap X) = (D \cap N)^n (D \cap X)^n,$$

using that $X \trianglelefteq ND$. Hence

$$D = (D \cap N)(D \cap X) \leq (D \cap N)(D \cap M).$$

So we may assume $G = ND = MD$; in particular, G/N and G/M are p-groups.

Suppose that $N \leq Y \trianglelefteq G$, where $Y < G$, and let b_Y be the unique block of Y covering b_N. Let $Z = M \cap Y$, and let b_Z be the unique block of Z covering $b_{N\cap M}$. Thus b_Z is G-invariant, and covered by B. By induction, there is a defect group D_1 of B such that $D_1 = (D_1 \cap Y)(D_1 \cap M)$. Now, since b_Y is G-invariant, we have that $D_1 \cap Y$ is a defect group of b_Y. Again by induction, there is $y \in Y$ such that

$$D_1^y \cap Y = (D_1^y \cap N)(D_1^y \cap Z).$$

Hence, $D_1 \cap Y = (D_1 \cap N)(D_1 \cap Z)$. Thus $D_1 = (D_1 \cap N)(D_1 \cap M)$, and we are done.

So we may assume $|G : N| = p = |G : M|$. Also, $G = (N \cap M)D$, using Fong's Theorem 9.17 of [43]. Since $G = (N \cap M)D$, we have that $N = (N \cap M)(N \cap D)$ and $M = (N \cap M)(M \cap D)$. In particular, $N \cap D \neq M \cap D$. Since $D/(D \cap N \cap M)$ is $C_p \times C_p$, we necessarily have $D = (D \cap N)(D \cap M)$. □

COROLLARY 2.9. *Suppose that $G = S_1 * \cdots * S_m$ is a central product of its subgroups S_i, $1 \leq i \leq m$. If B is a block of G with defect group D, then $D = (D \cap S_1) \cdots (D \cap S_m)$.*

Proof. Use induction on m and Theorem 2.8; see also [53, Lemma 7.5]. □

The following elementary result will be used in the final step of our proof of Theorem A.

LEMMA 2.10. *Suppose that $N \trianglelefteq G$ are finite groups with G/N a p-group. Let $Q \trianglelefteq G$ be such that $Q \cap N = 1$. Let b be a G-invariant block of N, and let B be the block of G that covers b. Let $\bar{b}$ be the unique block of NQ/Q that corresponds to b under the natural isomorphism, and let $\bar{B}$ be the unique block of G/Q that covers $\bar{b}$. Then D/Q is a defect group of $\bar{B}$.*

Proof. We know that the block $\bar{B}$ is contained in a unique block $\tilde{B}$ of G. (See the remark before Theorem 7.6 of [43].) Let E/Q be a defect group of $\bar{B}$. Let $\tau \in \mathrm{Irr}(b)$, and consider

$$\gamma := \tau \times 1_Q \in \mathrm{Irr}(N \times Q).$$

The block of $N \times Q$ that contains γ is only covered by B by [43, Cor. 9.6]. Then γ, considered as a character of NQ/Q, lies in $\bar{b}$. Let $\psi \in \mathrm{Irr}(\bar{B})$ over γ. Then ψ, considered as a character of G, lies over τ, and therefore $\psi \in \mathrm{Irr}(B)$. It follows that $\mathrm{Irr}(B) \cap \mathrm{Irr}(\bar{B}) \neq \varnothing$, and hence $\bar{B}$ is contained in B. (See the remark before Theorem 9.9 of [43].) Consequently, $\tilde{B} = B$. By [43, Th. 9.9(a)], we have $E/Q \leq D/Q$. Notice that $\bar{b}$ is G-invariant by uniqueness. Therefore $(E/Q)(NQ/Q) = G/Q$ by [43, Th. 9.17]. By the same reason, $DN = G$. Also, $(E/Q) \cap (NQ/Q)$ is a defect group of $\bar{b}$, and $D \cap N$ is a defect group of b (by [43, Th. 9.26]). In particular, $|E \cap NQ| = |Q||D \cap N|$. Then

$$|E/Q| = |G : NQ||D \cap N| = |D/Q|,$$

and the proof is complete. □

3. **Orbits of characters in a block**

In this section, we prove Theorem B, which we now restate.

THEOREM 3.1. *Suppose that p is an odd prime, S is a quasi-simple group and b is a p-block of S with non-cyclic defect groups. Then at least one of the following statements holds*:

(1) *there exist characters $\alpha, \beta, \gamma \in \mathrm{Irr}(b)$ that belong to three different* $\mathrm{Aut}(S)$*-orbits*; *or*

(2) *all characters in* $\mathrm{Irr}(b)$ *have the same degree.*

We remark that in the proof of Theorem A in Section 5, Theorem 3.1 is only needed for blocks with abelian defect groups. However, the more general statement may be of independent interest, and its proof is not substantially different. We also remark that the assumption of non-cyclic defect groups is needed. For example, for certain values of q with $p \mid\mid (q-1)$, there is a cyclic quasi-isolated block of $\mathrm{SL}_2(q)$ with characters from only two $\mathrm{Aut}(\mathrm{SL}_2(q))$-orbits.

Throughout, for A a group acting on a group G as automorphisms and a block b of G, we write $k_A(b)$ for the number of distinct A-orbits of characters of G whose intersection with $\mathrm{Irr}(b)$ is non-empty. In the situation of Theorem 3.1, our aim will be to show

$$k_{\mathrm{Aut}(S)}(b) \geq 3.$$

We also use $\mathrm{cd}(b)$ to denote the set of distinct character degrees in $\mathrm{Irr}(b)$.

3.1. *Initial considerations.* We begin by considering cases that can be completed more computationally.

PROPOSITION 3.2. *Let $p \geq 3$ be a prime, S a quasi-simple group such that $S/\mathbf{Z}(S)$ is one of the sporadic simple groups, the Tits group ${}^2\mathrm{F}_4(2)'$, $\mathrm{G}_2(2)'$, ${}^2\mathrm{G}_2(3)' = \mathrm{L}_2(8)$, or a simple group of Lie type with exceptional Schur multiplier. Let b be a p-block for S with non-cyclic defect and $|\mathrm{cd}(b)| > 1$. Then $k_{\mathrm{Aut}(S)}(b) \geq 3$. In particular, Theorem 3.1 holds for these groups.*

Proof. This can be seen using the GAP Character see, e.g., Library [57]. We note that the groups with exceptional Schur multipliers are listed in [22, Table 6.1.3]. □

PROPOSITION 3.3. *Let $p \geq 3$ a prime, and let S be quasi-simple such that $S/\mathbf{Z}(S) = \mathfrak{A}_n$, with $5 \leq n \leq 8$. Then $|\mathrm{cd}(b)| \geq 3$ for every non-cyclic p-block b of S. In particular, Theorem 3.1 holds for these groups.*

Proof. This can again be seen using GAP and the GAP character table library. □

PROPOSITION 3.4. *Let $S = \hat{\mathfrak{A}}_n$ be the double cover of the alternating group $\mathfrak{A}_n$, with $n \geq 9$. Suppose that $p \geq 3$ is a prime, and let b be a p-block of S with non-cyclic defect groups. Then $k_{\mathrm{Aut}(S)}(b) \geq 3$. In particular, Theorem 3.1 holds for these groups.*

Proof. Recall that $\mathrm{Aut}(S) = \mathfrak{S}_n$. If b is a p-block of $\mathfrak{A}_n$, then $k_{\mathfrak{S}_n}(b) \geq k(\widetilde{b})/2$, where $\widetilde{b}$ is a p-block of $\mathfrak{S}_n$ above b. By [50, Prop. 11.4], we have $k(\widetilde{b}) = k(p,w)$, where w is the so-called weight of $\widetilde{b}$ and $k(p,w)$ is as in [50, (3.11)]. But note that $k(p,w)/2 > 2$ for $p \geq 3$ and $w \geq 2$. Hence $k_{\mathfrak{S}_n}(b) \geq 3$.

If, instead, b is a block of S lying over the (unique) non-trivial character of $\mathbf{Z}(S)$, then $k_{\hat{\mathfrak{S}}_n}(b) \geq k(\widetilde{b})/2$, where now $\widetilde{b}$ is a so-called spin block of $\hat{\mathfrak{S}}_n$ above

b and $\hat{\mathfrak{S}}_n$ is a double cover of $\mathfrak{S}_n$ inducing all automorphisms of S. In this case, we have an analogous invariant $k(\widetilde{b}) = \hat{k}^{\pm}(\bar{p}, w)$ (see [50, Prop. (13.4)]). Since b is non-cyclic, we have $w > 1$, and using the definitions in [50, §13], we see again that $\hat{k}^{\pm}(\bar{p}, w)/2 > 2$, and we have $k_{\mathrm{Aut}(S)}(b) \geq 3$ as desired. □

We also need to consider the groups of Lie type that arise as the fixed points of a simple, simply connected linear algebraic group under a Steinberg endomorphism but are not quasi-simple. Throughout, we let

$$\mathfrak{E} := \{\mathrm{SL}_2(2), \mathrm{SU}_3(2), \mathrm{Sp}_4(2)\}. \tag{3.4.1}$$

PROPOSITION 3.5. *Let $p \geq 3$, and let B be a p-block of a group $G \in \mathfrak{E}$, with positive defect. Then $|\mathrm{cd}(B)| \geq 2$. If B is non-cyclic, then $|\mathrm{cd}(B)| \geq 3$.*

Proof. This can be seen using the GAP character table library. □

We remark that the groups $\mathrm{SL}_2(3)$, $\mathrm{G}_2(2)$, ${}^2\mathrm{B}_2(2)$, ${}^2\mathrm{G}_2(3)$, and ${}^2\mathrm{F}_4(2)$, which also occur as fixed points of simple, simply connected groups but are not quasi-simple, also satisfy the above statement, with the exception of the cyclic 3-blocks of $\mathrm{SL}_2(3)$. However, we will not need this here.

Next, we deal with the Suzuki, Ree, and triality groups.

PROPOSITION 3.6. *Let S be quasi-simple such that $\bar{S} := S/\mathbf{Z}(S)$ is a Suzuki, Ree, or triality group ${}^2\mathrm{B}_2(q^2)$, ${}^2\mathrm{G}_2(q^2)$, ${}^2\mathrm{F}_4(q^2)$, or ${}^3\mathrm{D}_4(q)$. Let $p \geq 3$ be a prime and let b be a p-block of S with non-cyclic defect groups. Then $k_{\mathrm{Aut}(S)}(b) \geq 3$. In particular, Theorem 3.1 holds for these groups.*

Proof. Note that the Schur multiplier of $\bar{S}$ is trivial or S was considered already in Proposition 3.2, so we assume $\bar{S} = S$. First suppose $p \mid q^2$, so that $S = {}^2\mathrm{G}_2(q^2)$ or ${}^3\mathrm{D}_4(q)$. Then by a theorem of Humphreys [24], S has exactly two blocks, namely the principal block and a block of defect zero containing only the Steinberg character. By observing the list of unipotent character degrees in [14, §13.9], we see that there are at least three distinct character degrees in the principal block.

Now suppose $3 \leq p \nmid q^2$. Then Sylow p-subgroups of ${}^2\mathrm{B}_2(q^2)$ and ${}^2\mathrm{G}_2(q^2)$ are cyclic. Let $S = {}^2\mathrm{F}_4(q^2)$, with $q^2 = 2^{2f+1}$. Here from [34, Bem. 1], if $p \nmid (q^2 - 1)$, then each semisimple p'-element s in the dual group S^* defines a unique block of positive defect containing the Lusztig series $\mathcal{E}(S, s)$. If p divides $(q^2 - 1)$, then such an s defines a unique block of positive defect if s is of class type t_2 or t_3, and three such blocks for $s = 1$ or of type t_1 in the notation of [55]. In the latter two cases, only one of the three blocks has non-cyclic defect. In each case, we can see from the centraliser structures and conjugacy class types in [55], together with the fact that field automorphisms permute Lusztig series via $\mathcal{E}(S, s)^{\alpha} = \mathcal{E}(S, s^{\alpha^*})$ for $\alpha^* \in \mathrm{Aut}(S^*)$ dual to α (see [47, Cor. 2.4]), that there are at least three characters in the relevant blocks

that cannot be conjugate under field automorphisms of S. In the case that $S = {}^3\mathrm{D}_4(q)$, we may argue similarly to above, taking into consideration the results of [15, Lemma 5.9] and the structure of centralisers and tori discussed in loc. cit. □

3.2. *Reducing to quasi-isolated blocks.* Let p be an odd prime, and let $\mathbf{G}$ be a simple algebraic group of simply connected type over $\overline{\mathbb{F}}_r$ for some prime r. Let q be a power of r and $F\colon \mathbf{G}\to\mathbf{G}$ a Frobenius endomorphism with respect to an $\mathbb{F}_q$-structure. Write $G := \mathbf{G}^F$ for the corresponding finite reductive group.

Let $\iota\colon \mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}$ be a regular embedding as in [12, (15.1)] (see also [19, §1.7]), and write $\widetilde{G} := \widetilde{\mathbf{G}}^F$. Let $(\mathbf{G}^*, F)$ be dual to $(\mathbf{G}, F)$, so that $\mathbf{G}^*$ is simple of adjoint type, and let $(\widetilde{\mathbf{G}}^*, F)$ be dual to $(\widetilde{\mathbf{G}}, F)$. We will write $G^* := \mathbf{G}^{*F}$ and $\widetilde{G}^* := \widetilde{\mathbf{G}}^{*F}$. We further have a dual F-equivariant epimorphism $\iota^*\colon \widetilde{\mathbf{G}}^* \twoheadrightarrow \mathbf{G}^*$ induced by ι. Now, with this setting, $\mathrm{Aut}(G)$ is induced by $\widetilde{G} \rtimes \mathcal{D}$, where $\mathcal{D}$ is the group generated by appropriately chosen graph and field automorphisms (see, e.g., [22, Th. 2.5.1]).

The set $\mathrm{Irr}(G)$ is a disjoint union of Lusztig series $\mathcal{E}(G, s)$ (also called rational series), where s runs over G^*-conjugacy class representatives of semisimple elements of G^*. If $\widetilde{s} \in \widetilde{G}^*$ is such that $\iota^*(\widetilde{s}) = s$, then the series $\mathcal{E}(G, s)$ consists of the constituents of the restrictions of the characters in $\mathcal{E}(\widetilde{G}, \widetilde{s})$ to G (see [12, Prop. 15.6]).

We next consider the case of groups of Lie type in defining characteristic, i.e., when $r = p$.

PROPOSITION 3.7. *Assume S is quasi-simple such that $S/\mathbf{Z}(S)$ is a simple group of Lie type defined in characteristic $p \geq 3$. Let b be a p-block of S with non-cyclic defect groups. Then $k_{\mathrm{Aut}(S)}(b) \geq 3$. In particular, Theorem 3.1 holds for these groups.*

Proof. Let $G = \mathbf{G}^F$ be as above such that $\bar{S} := S/\mathbf{Z}(S) = G/\mathbf{Z}(G)$. We may assume G is the full covering group of $\bar{S}$, as the exceptional covers have been discussed in Proposition 3.2, and that $\bar{S}$ is not of Suzuki or Ree type, from Proposition 3.6. By applying [43, Th. 9.9], it suffices to prove the statement for G, since $p \nmid |\mathbf{Z}(G)|$. Now, every p-block of G is either maximal defect or of defect zero [24], and the blocks of maximal defect are in bijection with the characters of $\mathbf{Z}(G)$, via their central character. Let $\theta \in \mathrm{Irr}(\mathbf{Z}(G))$ correspond to b.

Given this, and by inspection of the character tables for $\mathrm{SL}_2(q)$, $\mathrm{SL}_3(\epsilon q)$, and $\mathrm{Sp}_4(q)$ (see [19, see, e.g., 2.6 and references in Table 2.4]), we see there are at least three character degrees for each block of positive defect in these cases, so we further assume that G is not one of these groups. Furthermore, we may assume $\mathbf{Z}(G) \neq 1$ as any non-solvable group has at least four irreducible characters.

Let $T \leq G$ be a maximal torus. We claim that there is $s \in T^*$, a maximal torus of G^* in duality with T, such that the characters in the (rational) Lusztig series $\mathcal{E}(G,s)$ lie above θ. Indeed, as $\mathbf{Z}(G) \leq T$, there is $\tilde{\theta} \in \mathrm{Irr}(T)$ with $\tilde{\theta}|_{\mathbf{Z}(G)} = \theta$, and then by the character formula, also $R_{\mathbf{T}}^{\mathbf{G}}(\tilde{\theta})|_{\mathbf{Z}(G)} = R_{\mathbf{T}}^{\mathbf{G}}(\tilde{\theta})(1)\theta$. So taking (T^*, s) dual to (T, θ) we find s as claimed. Again by the character formula, $R_{\mathbf{T}}^{\mathbf{G}}(\mu\tilde{\theta})|_{\mathbf{Z}(G)} = R_{\mathbf{T}}^{\mathbf{G}}(\mu\tilde{\theta})(1)\theta$ for all $\mu \in \mathrm{Irr}(T)$ with $\mu|_{\mathbf{Z}(G)} = 1$, and so by duality $R_{\mathbf{T}^*}^{\mathbf{G}^*}(st)|_{\mathbf{Z}(G)} = R_{\mathbf{T}^*}^{\mathbf{G}^*}(st)(1)\theta$ for all $t \in T^* \cap [G^*, G^*]$ (see the proof of [46, Lemma 4.4(ii)]). Note that we may choose $s =: s_1$ to have order only divisible by primes dividing $|G^* : [G^*, G^*]| = |\mathbf{Z}(G)|$.

As G is not of types $A_1, A_2, {}^2A_2, B_2$, there are at least two Zsigmondy primes ℓ_2, ℓ_3 dividing $|G^*|$ but not $|\mathbf{Z}(G)|$. Applying the above claim to suitable maximal tori $T_i \leq G$, $i = 2, 3$, we may choose semisimple elements s_i with order divisible by ℓ_i and possibly by some primes dividing $|G^*/[G^*, G^*]|$. This ensures that $|s_i|$, $i = 1, 2, 3$, are pairwise distinct. Hence the disjoint series $\mathcal{E}(G, s_i)$, $i = 1, 2, 3$, contain distinct characters above θ not conjugate under $\mathrm{Aut}(G)$ by [47, Cor. 2.4], completing the proof. □

Given Proposition 3.7, we now assume that $r \neq p$ for the remainder of Section 3.

If $s \in G^*$ is a p'-element, as is customary we write $\mathcal{E}_p(G, s)$ for the union $\bigcup_t \mathcal{E}(G, st)$ where t ranges over the p-elements of G^* commuting with s. Then $\mathcal{E}_p(G, s)$ is a union of p-blocks of G, and for each block B with $\mathrm{Irr}(B) \subseteq \mathcal{E}_p(G, s)$, we have that $\mathrm{Irr}(B) \cap \mathcal{E}(G, s)$ is non-empty (see [12, Th. 9.12]). At times, we will write $\mathcal{E}(G, p')$ to denote the union of the series $\mathcal{E}(G, s)$ for $s \in G^*$ ranging over semisimple p'-elements.

A fundamental result of Bonnafé–Rouquier [6] yields that the blocks in $\mathcal{E}_p(G, s)$ are Morita equivalent to so-called quasi-isolated blocks of suitable Levi subgroups. In the more general setting of a finite reductive group $H := \mathbf{H}^F$, that is, the fixed points of a connected reductive group $\mathbf{H}$ under a Frobenius morphism $F \colon \mathbf{H} \to \mathbf{H}$ endowing $\mathbf{H}$ with an $\mathbb{F}_q$-rational structure, a block of H is called *quasi-isolated* if it lies in $\mathcal{E}_p(H, s)$ for a semisimple p'-element $s \in H^*$ such that $\mathbf{C}_{\mathbf{H}^*}(s)$ is not contained in any proper F-stable Levi subgroup of $\mathbf{H}^*$. (In such a situation, s is also called *quasi-isolated*.)

The following setup will be useful. Let $\mathbf{H}$ be an F-stable Levi subgroup of $\mathbf{G}$ and $H := \mathbf{H}^F$. Let $\mathbf{H}_0 := [\mathbf{H}, \mathbf{H}]$ and $H_0 := \mathbf{H}_0^F$, so that $\mathbf{H}_0$ is semisimple of simply connected type, by [38, Prop. 6.20(c) and 12.14]. Then by [19, Cor. 1.5.16], H_0 is isomorphic to a direct product $\prod_{i=1}^k \mathbf{H}_i^{F_i}$, where each $\mathbf{H}_i$ is simple of simply connected type and F_i is a Frobenius morphism obtained as some power of F. Write $H_i := \mathbf{H}_i^{F_i}$. Let B be a p-block of H, and let B' be a block of H_0 covered by B. Then B' is isomorphic to a tensor product $\otimes_{i=1}^k B_i$, where B_i is a block of H_i for each $1 \leq i \leq k$.

Recall from (3.4.1) that $\mathfrak{E} = \{\mathrm{SL}_2(2), \mathrm{SU}_3(2), \mathrm{Sp}_4(2)\}$. Note that H_i is perfect (and hence quasi-simple) unless $H_i \in \mathfrak{E} \cup \{\mathrm{SL}_2(3)\}$ (see [38, Th. 24.17]). (Indeed, note that the excluded groups $\mathrm{G}_2(2)$, ${}^2\mathrm{B}_2(2)$, ${}^2\mathrm{G}_2(3)$, and ${}^2\mathrm{F}_4(2)$ cannot occur as an H_i since F is a Frobenius endomorphism and G_2 does not occur as a component of a Levi subgroup of the other types.) Since G is defined in characteristic distinct from p and $|\mathrm{SL}_2(3)|$ is only divisible by the primes 2 and 3, we further will not need to consider $\mathrm{SL}_2(3)$ in what follows. In the following, we will write

$$\bar{k}_{\mathrm{Aut}(H_i)}(B_i) := \begin{cases} k_{\mathrm{Aut}(H_i)}(B_i) & \text{if } H_i \notin \mathfrak{E}, \\ |\mathrm{cd}(B_i)| & \text{if } H_i \in \mathfrak{E}. \end{cases}$$

Hypothesis 3.8. Keep the notation and situation of the previous paragraph. For a positive integer c, let B be a p-block of H such that there exists a block B' as above that satisfies at least one of the following three conditions:

(3.8.1) there exists some i with $1 \le i \le k$ such that $\bar{k}_{\mathrm{Aut}(H_i)}(B_i) \ge c$;

(3.8.2) there exist i, j with $1 \le i \ne j \le k$ such that $\bar{k}_{\mathrm{Aut}(H_i)}(B_i) \ge c-1$ and, furthermore, $\bar{k}_{\mathrm{Aut}(H_j)}(B_j) \ge 2$;

(3.8.3) B' is cyclic of positive defect, and there exists some i with $1 \le i \le k$ such that B_i is cyclic and $\bar{k}_{\mathrm{Aut}(H_i)}(B_i) \ge c-1$.

PROPOSITION 3.9. *Let $G := \mathbf{G}^F$ be as above. Let $\mathbf{H}$ be an F-stable, proper Levi subgroup of $\mathbf{G}$ and $H := \mathbf{H}^F$. Let p be an odd prime not dividing q and B be a non-cyclic p-block of H with $|\mathrm{cd}(B)| > 1$. Then*

(a) *in the notation above, any block B' of H_0 covered by B has positive defect;*

(b) *if B satisfies Hypothesis 3.8 for some positive integer c, then $k_{\mathrm{Aut}(H)}(B) \ge c$.*

Proof. Keep the notation above, and let B' be a block of H_0 covered by B. First, assume B' is defect zero. Write $\mathrm{Irr}(B') = \{\theta_1\}$, and let $T := H_{\theta_1}$ be the inertia subgroup for θ_1 in H. Since H/H_0 is abelian and restrictions from H to H_0 are multiplicity-free (a result of Lusztig — see [12, Th. 15.11]), Gallagher's theorem and Clifford correspondence imply that every element of $\mathrm{Irr}(H \mid \theta_1)$ is of the form $(\beta\hat{\theta}_1)^H$, where $\hat{\theta}_1$ is an extension of θ_1 to T and β is a (linear) character of T/H_0. In particular, every member of $\mathrm{Irr}(B)$ is of this form, and hence $|\mathrm{cd}(B)| = 1$, a contradiction. This shows (a).

We now assume that B satisfies Hypothesis 3.8 for c and aim to show

$$k_{\mathrm{Aut}(H)}(B) \ge c.$$

Note that this is trivial when $c = 1$, so we assume throughout that $c \ge 2$. We claim that it suffices to show that

$$k_{\mathrm{Aut}(H_0)}(B') \ge c. \tag{3.9.1}$$

Indeed, if this is the case, write $\theta_1, \dots, \theta_c$ for representatives in $\mathrm{Irr}(B')$ of c distinct orbits. Then by [43, Th. 9.4], there must be at least c characters $\chi_1, \dots, \chi_c$ in B, lying above $\theta_1, \dots, \theta_c$, respectively. Now H_0 is characteristic in

H (indeed, it is generated by all unipotent elements of H by [19, Rem. 1.5.13], as $\mathbf{H}_0$ is simply connected since $\mathbf{H}$ is a Levi subgroup of the simply connected group $\mathbf{G}$), so is stabilised by any automorphism of H. Then, if χ_i and χ_j are $\mathrm{Aut}(H)$-conjugate for $i \neq j$, then θ_i and θ_j are $\mathrm{Aut}(H)$-conjugate and hence $\mathrm{Aut}(H_0)$-conjugate, a contradiction. Thus, we wil prove, at least in the cases (3.8.1) and (3.8.2), that (3.9.1) holds.

Now, in the case of (3.8.1), we may without loss assume $\bar{k}_{\mathrm{Aut}(H_1)}(B_1) \geq c$. Let $\chi_1, \chi_2 \in \mathrm{Irr}(B_1)$ lie in distinct $\mathrm{Aut}(H_1)$-orbits on $\mathrm{Irr}(B_1)$ if $H_i \notin \mathfrak{E}$ or have distinct degrees if $H_i \in \mathfrak{E}$. First assume that H_1 is not isomorphic to any H_i for $2 \leq i \leq k$. Let $X := H_2 \times \cdots \times H_k$, and let $\alpha \in \mathrm{Aut}(H_0)$. By [1], for $(h, 1_X) \in H_1 \times X$ with $h \in H_1$, we have $\alpha(h, 1_X) = (\alpha_1(h), \alpha_1'(h))$, where $\alpha_1 \in \mathrm{Aut}(H_1)$ and $\alpha_1' \in \mathrm{Hom}(H_1, \mathbf{Z}(X))$ are defined by α. Now, assume $(\chi_1 \otimes \varphi)^\alpha = \chi_2 \otimes \varphi$, where $\varphi \in \mathrm{Irr}(B_2 \otimes \cdots \otimes B_k)$. Then considering elements of the form $(h, 1_X)$ with $h \in H_1$, we see that $\chi_1^{\alpha_1}(h)\theta(h) = \chi_2(h)$ for some linear character $\theta \in \mathrm{Irr}(H_1)$. (Namely, θ is the composition of α_1' with the unique irreducible constituent of $\varphi|_{\mathbf{Z}(X)}$.) If $H_1 \notin \mathfrak{E}$, this contradicts that χ_1 and χ_2 are not conjugate under $\mathrm{Aut}(H_1)$, since θ must be trivial. If $H_1 \in \mathfrak{E}$, it contradicts that $\chi_1(1) \neq \chi_2(1)$. Hence in either case $\chi_1 \otimes \varphi$ cannot be $\mathrm{Aut}(H_0)$-conjugate to $\chi_2 \otimes \varphi$. This shows $k_{\mathrm{Aut}(H_0)}(B') \geq c$. In the case that H_0 contains multiple isomorphic copies of H_1, a similar argument holds, taking instead X to be the (possibly trivial) product of those H_i such that $B_i \ncong B_1$ under this isomorphism and $\chi_1 \otimes \cdots \otimes \chi_1$ (one for each copy of B_1) in place of χ_1.

Arguing similarly, in case (3.8.2), we obtain $k_{\mathrm{Aut}(H_0)}(B') \geq c$, and in case (3.8.3) we obtain $k_{\mathrm{Aut}(H_0)}(B') \geq c - 1$.

Now, assume we are in the situation of (3.8.3), so B' is cyclic but B is not, and B' has positive defect. Then p divides $|H/H_0|$, and we let $H_0 \leq H_p \leq H$ be such that $H_p/H_0 \in \mathrm{Syl}_p(H/H_0)$. Let B_p be the (unique) block of H_p above B'. Then we have $c - 1$ non-$\mathrm{Aut}(H)$-conjugate characters in B_p lying above the non-$\mathrm{Aut}(H_0)$-conjugate characters $\theta_1, \ldots, \theta_{c-1}$ of $\mathrm{Irr}(B')$. We claim that there is at least one more character in $\mathrm{Irr}(B_p)$ that is not $\mathrm{Aut}(H)$-conjugate to these.

Let $(\mathbf{H}^*, F)$ be dual to $(\mathbf{H}, F)$, and write $H^* := \mathbf{H}^{*F}$. Let $s \in H^*$ be a semisimple p'-element such that $\mathrm{Irr}(B) \subseteq \mathcal{E}_p(H, s)$. Let $\mathbf{H}_0 \hookrightarrow \mathbf{H}$ be the inclusion map, and let s_0 be the image of s under the induced dual epimorphism $\mathbf{H}^* \twoheadrightarrow \mathbf{H}_0^*$. Then $\mathrm{Irr}(B') \subseteq \mathcal{E}_p(H_0, s_0)$. Then at least one of these $c - 1$ characters, say θ_1, can be assumed to lie in $\mathcal{E}(H_0, s_0)$ using [12, Th. 9.12].

Let $\iota\colon \mathbf{H} \hookrightarrow \widetilde{\mathbf{H}}$ be a regular embedding, as in [12, (15.1)], and write $\widetilde{H} := \widetilde{\mathbf{H}}^F$. Note that $\iota|_{\mathbf{H}_0}$ is also a regular embedding of $\mathbf{H}_0$ into $\widetilde{\mathbf{H}}$ and we have $\mathbf{H}_0 = [\widetilde{\mathbf{H}}, \widetilde{\mathbf{H}}]$ and $H_0 \lhd H \lhd \widetilde{H}$. Let $\widetilde{s} \in \widetilde{H}^*$ be a semisimple p'-element such that $\iota^*(\widetilde{s}) = s$. Then by [12, Prop. 15.6], $\mathcal{E}(H, s)$ is the set of constituents of the restrictions to H of characters in $\mathcal{E}(\widetilde{H}, \widetilde{s})$, and we may further define $\mathcal{E}(H_p, s)$ to be the set of constituents of restrictions of characters from $\mathcal{E}(H, s)$ to H_p.

Similarly, for $\widetilde{x}$ any semisimple element of $\widetilde{H}^*$ and $x = \iota^*(\widetilde{x})$, we define $\mathcal{E}(H_p, x)$ to be the set of constituents of restrictions of characters from $\mathcal{E}(\widetilde{H}, \widetilde{x})$ to H_p.

Then by [11, Prop. 1.3], θ_1 extends to a character χ in $\mathcal{E}(H_p, s)$, which is hence a member of $\mathrm{Irr}(B_p) \cap \mathcal{E}(H_p, s)$. Then $\chi\beta \in \mathrm{Irr}(B_p)$ for every $\beta \in \mathrm{Irr}(H_p/H_0)$. Now, recall that characters of $\widetilde{H}/H_0$ are in bijection with elements of $\mathbf{Z}(\widetilde{H}^*)$ (see [12, (15.2)]). We write $\hat{\widetilde{z}}$ for the character of $\widetilde{H}/H_0$ corresponding to $\widetilde{z} \in \mathbf{Z}(\widetilde{H}^*)$. Choose $\beta := \hat{\widetilde{z}}$ for $1 \neq \widetilde{z} \in \mathbf{Z}(\widetilde{H}^*)$ of p-power order. Then $\chi\beta \in \mathcal{E}(H_p, sz)$, where $z := \iota^*(\widetilde{z})$ and is not $\widetilde{H}$-conjugate to χ by definition of $\mathcal{E}(H_p, sz)$ and $\mathcal{E}(H_p, s)$. Together with the fact that sz cannot be H^*-conjugate to $\varphi^*(s)$ for any automorphism φ^* since s is p' and z is a p-element, this tells us that χ cannot be $\mathrm{Aut}(H)$-conjugate to $\chi\beta$ (see [47, Cor. 2.4]).

Further, since $\chi\beta$ lies above θ_1, it is not $\mathrm{Aut}(H)$-conjugate to a character above θ_i for $i \neq 1$. Then letting $\chi_c \in \mathrm{Irr}(B)$ above $\chi\beta$ and $\chi_1, \ldots, \chi_{c-1} \in \mathrm{Irr}(B)$ above $\chi, \theta_2, \ldots, \theta_{c-1}$, respectively, $\chi_1, \ldots, \chi_c$ are non-$\mathrm{Aut}(H)$-conjugate members of $\mathrm{Irr}(B)$, as desired. □

Remark 3.10. In our application of Proposition 3.9 and Lemma 3.12 below, we really only require $k_{\mathrm{Aut}(G)_H}(B) \geq c$, rather than $k_{\mathrm{Aut}(H)}(B) \geq c$. Then we remark that since the automorphisms of G respect the product structure of H_0, we could replace each $\bar{k}_{\mathrm{Aut}(H_i)}(B_i)$ with $k_{\mathrm{Aut}(H_i)}(B_i)$ in Hypothesis 3.8, and then the statements of Proposition 3.9 and Lemma 3.12 hold with the condition $k_{\mathrm{Aut}(H)}(B) \geq c$ replaced with $k_{\mathrm{Aut}(G)_H}(B) \geq c$.

In the notation above, note that if B is quasi-isolated, then so is each B_i. Indeed, let $(\mathbf{H}^*, F)$ be dual to $(\mathbf{H}, F)$, and write $H^* := \mathbf{H}^{*F}$. Let $\mathrm{Irr}(B) \subseteq \mathcal{E}_p(H, s)$ for a quasi-isolated semisimple p'-element $s \in H^*$. Let $\mathbf{H}_0 = [\mathbf{H}, \mathbf{H}]$, and keep the rest of the notation from the paragraph before Hypothesis 3.8. Note that the inclusion map $\mathbf{H}_0 \hookrightarrow \mathbf{H}$ is a central isotypy in the sense of [3, Def. 2.A] and [19, 1.3.21], and so is the induced dual epimorphism $\mathbf{H}^* \twoheadrightarrow \mathbf{H}_0^*$, by [19, 1.7.11]. Then if $s_0 \in H_0^*$ is the image of s under the latter map, [3, Prop. 2.3] yields that s_0 is also quasi-isolated. In particular, any block B' of H_0 covered by B is quasi-isolated, and hence so are the blocks $B_1, \ldots, B_k$. (Indeed, if s_0 is quasi-isolated, let s_0 correspond to $\prod s_i$ under the isomorphism with $\prod \mathbf{H}_i^{F_i}$. If s_i is not quasi-isolated in $\mathbf{H}_i$, then neither is its preimage in the corresponding F-simple factor of $\mathbf{H}_0$, in the notation of [19, 1.5.14]. But this would contradict that s_0 is quasi-isolated.)

Remark 3.11. With this and Proposition 3.9, note that if each (quasi-isolated) B_i satisfies

$$\bar{k}_{\mathrm{Aut}(H_i)}(B_i) \geq \begin{cases} c & \text{if } B_i \text{ is non-cyclic,} \\ c-1 & \text{if } B_i \text{ is cyclic,} \end{cases} \tag{3.11.1}$$

then our (quasi-isolated) block B will satisfy $k_{\mathrm{Aut}(H)}(B) \geq c$.

Throughout, when $\mathbf{L}$ is an F-stable Levi subgroup of $\mathbf{H}$, we use $R_{\mathbf{L}}^{\mathbf{H}}$ to denote Lusztig's twisted induction with respect to any parabolic subgroup $\mathbf{P}$ of $\mathbf{H}$ containing $\mathbf{L}$ as a Levi complement. In our situation, since we may exclude groups considered in Proposition 3.2 and $\mathrm{E}_7(2)$ and $\mathrm{E}_8(2)$ since they have trivial outer automorphism group, we have that twisted induction is independent of the choice of $\mathbf{P}$ by [19, Th. 3.3.8]. Hence, as is customary, we will suppress the parabolic subgroup from the notation.

LEMMA 3.12. *As above, let $G := \mathbf{G}^F$ and let $p \nmid q$ be an odd prime. Assume that $k_{\mathrm{Aut}(H)}(B) \geq c$ for each F-stable Levi subgroup $\mathbf{H}$ of $\mathbf{G}$ and each non-cyclic quasi-isolated p-block B of $\mathbf{H}^F$ with $|\mathrm{cd}(B)|>1$. Then $k_{\mathrm{Aut}(G)}(b)\geq c$ for each non-cyclic p-block b of G such that $|\mathrm{cd}(b)| > 1$.*

Proof. Let b be a non-cyclic block of G such that $|\mathrm{cd}(b)| > 1$. If b is quasi-isolated, our assumption yields that the statement holds for b, with $H=G$. Otherwise, by [6], b is Morita equivalent to a quasi-isolated block of a proper Levi subgroup $H := \mathbf{H}^F$ of G. In particular, this Bonnafé–Rouquier Morita equivalence is induced by the map $R_{\mathbf{H}}^{\mathbf{G}}$. Let B be the Bonnafé–Rouquier correspondent for b in H. Note that by [26, Th. 1.3], B is non-cyclic if and only if b is. Further, by the character formula [12, Th. 8.16] for $R_{\mathbf{H}}^{\mathbf{G}}$, we have $|\mathrm{cd}(B)| > 1$ if and only if $|\mathrm{cd}(b)| > 1$.

Now, by the proof of [37, Lemma 3.5], we have

$$\mathrm{Aut}(G)_b = \mathrm{Inn}(G)\mathrm{Aut}(G)_{H,B},$$

and $R_{\mathbf{H}}^{\mathbf{G}}$ is $\mathrm{Aut}(G)_{H,B}$-equivariant. By assumption, $k_{\mathrm{Aut}(H)}(B)$, and hence $k_{\mathrm{Aut}(G)_{H,B}}(B)$, is at least c. Then we have $k_{\mathrm{Aut}(G)_{H,B}}(b) \geq c$, proving that also $k_{\mathrm{Aut}(G)}(b) \geq c$. □

3.3. *e-Harish-Chandra theory and blocks.* In this subsection, we allow $\mathbf{G}$ to be any Levi subgroup of a simple algebraic group of simply connected type and $F : \mathbf{G} \to \mathbf{G}$ a Frobenius endomorphism with respect to an $\mathbb{F}_q$-rational structure. Thanks to the work of Broué–Malle–Michel [8], Cabanes–Enguehard [11], and Kessar–Malle [27], we have a parametrisation of blocks of $G := \mathbf{G}^F$ in terms of e-Harish-Chandra theory. (See, e.g., [19, §3.5] for the notions of e-torus, e-split Levi subgroups and e-cuspidal characters of $(\mathbf{G}, F)$.)

Namely, by [27, Th. A], if $p \geq 3$ is a prime not dividing q and e is the order of q modulo p, then there is a bijection from the set of G-conjugacy classes of e-Jordan quasi-central cuspidal pairs $(\mathbf{L}, \lambda)$ of $\mathbf{G}$ with $\lambda \in \mathcal{E}(\mathbf{L}^F, p')$ to the set of p-blocks of G. (See [27, Def. 2.1, 2.12] for the definitions.) We will write $b_G(\mathbf{L}, \lambda)$ for the block corresponding to $(\mathbf{L}, \lambda)$. Then all irreducible constituents of $R_{\mathbf{L}}^{\mathbf{G}}(\lambda)$ lie in $b_G(\mathbf{L}, \lambda)$ by [27, Th. A]. The next lemma allows us to say more about characters lying in $b_G(\mathbf{L}, \lambda)$ and could be useful for other applications. Here we write d^1 for the map on class functions given by composition with the characteristic function on p'-elements of a group.

LEMMA 3.13. *Let $b = b_G(\mathbf{L}, \lambda)$ be a p-block of $G = \mathbf{G}^F$ in $\mathcal{E}_p(G, s)$ for $s \in \mathbf{L}^{*F}$ a semisimple p'-element. Let $t \in \mathbf{Z}(\mathbf{L}^*)^F$ be a p-element. Then* $\mathrm{Irr}(b) \cap \mathcal{E}(G, st) \neq \varnothing$.

Proof. By what we recalled above, all constituents of $R_{\mathbf{L}}^{\mathbf{G}}(\lambda)$ lie in $\mathrm{Irr}(b)$, so $0 \neq d^1(R_{\mathbf{L}}^{\mathbf{G}}(\lambda))$ has some non-zero constituent in b. Let $\hat{t}$ denote the linear character of $\mathbf{L}^F$ corresponding to $t \in \mathbf{Z}(\mathbf{L}^*)^F$ (see [19, Prop. 2.5.20]). Since d^1 commutes with Lusztig induction [19, Prop. 3.3.17] and $|\hat{t}| = |t|$ is a p-power,

$$d^1(R_{\mathbf{L}}^{\mathbf{G}}(\hat{t}\lambda)) = R_{\mathbf{L}}^{\mathbf{G}}(d^1(\hat{t}\lambda)) = R_{\mathbf{L}}^{\mathbf{G}}(d^1(\lambda)) = d^1(R_{\mathbf{L}}^{\mathbf{G}}(\lambda)) \neq 0,$$

so $d^1(R_{\mathbf{L}}^{\mathbf{G}}(\hat{t}\lambda))$ has a component in b as well, which means that $R_{\mathbf{L}}^{\mathbf{G}}(\hat{t}\lambda)$ has a constituent lying in $\mathrm{Irr}(b)$. But all constituents of $R_{\mathbf{L}}^{\mathbf{G}}(\hat{t}\lambda)$ are contained in $\mathcal{E}(\mathbf{G}^F, st)$ (see [19, Prop. 3.3.20]). □

For $\mathbf{G}$ a connected reductive group such that $[\mathbf{G}, \mathbf{G}]$ is simply connected, we say that a block of $G = \mathbf{G}^F$ is of *quasi-central defect* if it covers a block of $[\mathbf{G}, \mathbf{G}]^F$ that has a defect group contained in $\mathbf{Z}([\mathbf{G}, \mathbf{G}]^F)$. (In particular, when $\mathbf{G}$ is simple, being of quasi-central defect is equivalent to having central defect.)

COROLLARY 3.14. *Let $G = \mathbf{G}^F$ be as above, and let b be a block in $\mathcal{E}_p(G, s)$ with non-trivial defect. Then* $\mathrm{Irr}(b)$ *contains at least two characters not conjugate under* $\mathrm{Aut}(G)$, *where one of the characters lies in $\mathcal{E}(G, s)$ and the other one outside of $\mathcal{E}(G, p')$.*

Proof. First, by the result of Hiss [12, Th. 9.12] we have $\mathrm{Irr}(b) \cap \mathcal{E}(G, s) \neq \varnothing$. Now write $b = b_G(\mathbf{L}, \lambda)$ with $\mathbf{L} \leq \mathbf{G}$ an e-split Levi subgroup whose dual $\mathbf{L}^* \leq \mathbf{G}^*$ contains s. If $\mathbf{L}$ is proper in $\mathbf{G}$, it centralises a non-trivial e-torus of $\mathbf{G}$ and thus its dual centralises a non-trivial e-torus of $\mathbf{G}^*$. Thus there is a p-element $1 \neq t \in \mathbf{Z}(\mathbf{L}^*)^F \leq \mathbf{C}_{G^*}(s)$, and $\mathrm{Irr}(b) \cap \mathcal{E}(G, st) \neq \varnothing$ by Lemma 3.13. If $\mathbf{L} = \mathbf{G}$, the quasi-central defect group D of b is normal in G, hence there exist characters in b non-trivial on D (see, e.g., [43, Th. 9.4]) that again cannot lie in $\mathcal{E}(G, p')$ (see, e.g., [11, Prop. 1.2(v)]). Arguing as before, characters in $\mathcal{E}(G, p')$ and $\mathcal{E}(G, st)$ cannot be $\mathrm{Aut}(G)$-conjugate. □

Note that in the context of Theorem 3.1, we are interested in the case $c = 3$ of Hypothesis 3.8, and hence Corollary 3.14 implies that we now only need to deal with non-cyclic blocks.

The next observation deals with unipotent blocks, and may be of interest for other applications. For this statement, we relax the assumption that p is odd, and if $p = 2$, we define e to be the order of q modulo 4. We remark that for $\epsilon \in \{\pm 1\}$, we write $\mathrm{L}_n(\epsilon q)$ for the group $\mathrm{L}_n(q)$ when $\epsilon = +1$ and $\mathrm{U}_n(q)$ when $\epsilon = -1$, with analogous conventions for related groups $\mathrm{SL}_n(\epsilon q)$, $\mathrm{PGL}_n(\epsilon q)$, and $\mathrm{GL}_n(\epsilon q)$.

LEMMA 3.15. *Let* $\mathbf{G}$ *be a connected reductive group in characteristic* r *such that* $[\mathbf{G},\mathbf{G}]$ *is simply connected, and let* $F\colon \mathbf{G}\to\mathbf{G}$ *be a Frobenius endomorphism. Let* p *be a prime, and let* b *be a unipotent* p*-block of* $G := \mathbf{G}^F$ *that is not of quasi-central defect. Then* $\mathrm{Irr}(b)$ *contains two unipotent characters whose degrees have different* r*-parts. In particular, these cannot be* $\mathrm{Aut}(G)$*-conjugate.*

Proof. We first assume that $\mathbf{G} = [\mathbf{G},\mathbf{G}]$ is simple of simply connected type. Suppose G is of exceptional type. Let e be as defined above and let $b = b_G(\mathbf{L},\lambda)$ for some unipotent quasi-central e-cuspidal pair $(\mathbf{L},\lambda)$ (see [27, Th. A]). Note that our assumption b has non-central defect groups and $\mathbf{G}$ is simple means we have $G \neq L := \mathbf{L}^F$. Then [14, §13.9] and [8, Tables 1 and 2] include the decomposition of $R_{\mathbf{L}}^{\mathbf{G}}(\lambda)$ and the relative Weyl group $W_G(\mathbf{L},\lambda)$ for many of the relevant blocks, and [8, Table 3] includes information about remaining cases where $\mathbf{L}$ is a torus and $W_G(\mathbf{L},\lambda)$ is non-cyclic. By studying the characters in these decompositions, whose degrees are available in [14, §13.9], and at times applying Ennola duality (see [8, Th. 3.3]), we see there are at least two characters in $R_{\mathbf{L}}^{\mathbf{G}}(\lambda)$ whose degrees have distinct r-parts, except possibly when $\mathbf{L}$ is a torus. In the latter situation, we have $\mathbf{L}$ is the centraliser of a Sylow e-torus, e is regular for $\mathbf{G}$, and $\lambda = 1$. Then b is the principal block of G and is the unique block containing p'-degree unipotent characters. Hence b contains at least 1_G and St_G, and the claim holds.

Now suppose G is of classical type. Note that by [10, Th. (i)], we may replace G with a group of the same rational type. If $p = 2$, we are done taking 1_G and St_G, since there is a unique unipotent block in this case (see [12, Th. 21.14]). So, we also assume $p \geq 3$.

First, let $G = \mathrm{SL}_n(\epsilon q)$. Let e' be the order of ϵq modulo p. The unipotent blocks of G are parametrised by e'-cores of partitions of n. Assume b is labelled by the e'-core λ, with $n = e'w + |\lambda|$. As b has non-central defect, $w \geq 1$. Let $\mu_1 = (\lambda_1 + e'w, \lambda_2, \ldots, \lambda_k)$ and $\mu_2 = (\lambda_1, \ldots, \lambda_k, 1^{e'w})$, two partitions of n labelling distinct unipotent characters χ_1, χ_2 in b. Then the degree formulas [19, Props. 4.3.1 and 4.3.5] show that $\chi_1(1)$ and $\chi_2(1)$ have distinct r-part.

Now let G be one of $\mathrm{Sp}_{2n}(q)$, $\mathrm{SO}_{2n+1}(q)$, or $\mathrm{SO}^{\epsilon}_{2n}(q)$, and let e' be the order of q^2 modulo p. Now the unipotent blocks of G (or in the case $G = \mathrm{SO}^{\epsilon}_{2n}(q)$, the blocks of $\mathrm{GO}^{\epsilon}_{2n}(q)$ lying above unipotent blocks of G) are parametrised by e'-cores (if $p \mid (q^{e'} - 1)$) or cocores (if $p \mid (q^{e'} + 1)$) of appropriate symbols (see [8, pp. 48–52] and [10, Th.]). Suppose b (or a block above in $\mathrm{GO}^{\epsilon}_{2n}(q)$) is labelled by the e'-core $(X;Y)$, with $n = e'w + \mathrm{rnk}(X;Y)$. (Here $\mathrm{rnk}(X;Y)$ is the rank of the symbol, defined in [19, p. 301].) Let $X = (x_1, \ldots, x_a)$ and $Y = (y_1, \ldots, y_c)$ with $x_i < x_{i+1}$ and $y_i < y_{i+1}$ for each i. Again we have $w \geq 1$,

and we may consider the two unipotent characters in $\mathrm{Irr}(b)$ labelled by

$$(0, 1, 2, \dots, e'w - 1, x_1 + e'w, x_2 + e'w, \dots, x_a + e'w;$$
$$1, 2, \dots, e'w, y_1 + e'w, y_2 + e'w, \dots, y_c + e'w)$$

and

$$(x_1, \dots, x_{a-1}, x_a + e'w; Y).$$

Here the degree formula [19, Prop. 4.4.7] shows that these two characters will have distinct r-part.

If b is instead labelled by an e'-cocore, Olsson's process of e'-twisting of symbols ([49, p. 235]) shows there is a bijection between symbols $(X'; Y')$ with e'-core $(X; Y)$ and symbols $(\widetilde{X}'; \widetilde{Y}')$ with e'-cocore $(\widetilde{X}; \widetilde{Y})$, where $(\widetilde{X}; \widetilde{Y})$ is the e'-twist of $(X; Y)$. Further, e'-twisting does not change the entries of the symbol, i.e., $(X'; Y')$ and its e'-twist $(\widetilde{X}'; \widetilde{Y}')$ satisfy $X' \cup Y' = \widetilde{X}' \cup \widetilde{Y}'$. Hence from the formula [19, Prop. 4.4.7], we see that the r-part of the characters corresponding to the symbols $(X'; Y')$ and $(\widetilde{X}'; \widetilde{Y}')$ are the same, and we are done with the case where $\mathbf{G} = [\mathbf{G}, \mathbf{G}]$ is simple.

In the general case, we have b lies above a block B of $[\mathbf{G}, \mathbf{G}]^F$ whose defect groups are non-central. We may write $[\mathbf{G}, \mathbf{G}]^F$ as a direct product of groups of the form $G_i = \mathbf{G}_i^{F_i}$, where $\mathbf{G}_i$ is simple of simply connected type and F_i is some power of F. Then B is a tensor product of blocks B_i of G_i, at least one of which, say B_j, must have non-central defect groups. Then from above, $\mathrm{Irr}(B_j)$ contains at least two unipotent characters with different r-parts, and therefore the statement also holds for B. Since the unipotent characters in b are extensions of those in B, this completes the proof. □

3.4. *Type* A. We begin by considering the case of finite linear and unitary groups. In this subsection, we fix $\bar{S} = \mathrm{L}_n(\epsilon q)$, $G = \mathrm{SL}_n(\epsilon q)$, $G^* = \mathrm{PGL}_n(\epsilon q)$, and $\widetilde{G} = \mathrm{GL}_n(\epsilon q) \cong \widetilde{G}^*$.

The blocks of $\widetilde{G}$ have been well-studied, with the parametrisation of the blocks of $\widetilde{G}$ given in [18] and a reduction to smaller-rank linear and unitary groups given in [39]. We use these to prove that Theorem 3.1 holds in the case that $\bar{S} = S/\mathbf{Z}(S) = \mathrm{L}_n(\epsilon q)$ and that Hypothesis 3.8 holds for $c = 3$ and blocks of G. Again in this case, we will provide a slightly more general result.

PROPOSITION 3.16. *Let $G = \mathrm{SL}_n(\epsilon q)$ such that $\mathrm{L}_n(\epsilon q)$ is simple, and let p be an odd prime with $p \nmid q$.*

(a) *Let B be a p-block of G with non-cyclic defect. Then $k_{\mathrm{Aut}(G)}(B) \geq 3$.*
(b) *Theorem 3.1 holds when $S/\mathbf{Z}(S) = \mathrm{L}_n(\epsilon q)$.*

Proof. Keep the notation from above. Note that in this case, $\widetilde{G}^* \cong \widetilde{G} = \mathrm{GL}_n(\epsilon q)$ and $G^* = \mathrm{PGL}_n(\epsilon q) \cong \widetilde{G}/\mathbf{Z}(\widetilde{G})$, and recall from the discussion in Section 3.2 that $\widetilde{G} \rtimes \mathcal{D}$ induces all automorphisms on $\mathrm{L}_n(\epsilon q)$. From Section 3.1,

we may assume that G is the (non-exceptional) Schur covering group for $\mathrm{L}_n(\epsilon q)$. Let e' be the order of ϵq modulo p.

Note that in the situation of (b), we have $S \cong G/Z$, where $Z \leq \mathbf{Z}(G)$ is some central subgroup. Further, writing Z_p for the Sylow p-subgroup of Z, we may identify the blocks of G/Z with blocks of G/Z_p, using [43, Th. 9.9]. Hence we may assume $Z = Z_p$ is a p-subgroup in $\mathbf{Z}(G)$. Then it suffices to show (b) in the case that $e' = 1$ with $S = G/Z_p$ a quotient by some non-trivial central p-group and to show (a).

Let B be a non-cyclic block of G, and suppose s is a semisimple p'-element of G^* such that B lies in $\mathcal{E}_p(G, s)$. In the context of (b), we further assume here that $\bar{B}$ is a block of G/Z_p with non-cyclic defect groups dominated by B, so that $\mathrm{Irr}(\bar{B})$ is comprised of those members of $\mathrm{Irr}(B)$ that are trivial on Z_p.

Applying Corollary 3.14, it suffices for (a) to show that there are at least two non-$\mathrm{Aut}(G)$-conjugate members of $\mathrm{Irr}(B) \cap \mathcal{E}(G, s)$ or two non-$\mathrm{Aut}(G)$-conjugate members of $\mathrm{Irr}(B)$ outside of $\mathcal{E}(G, p')$. For (b), note that every member of $\mathcal{E}(G, s)$ lies above the same character ω_s of $\mathbf{Z}(G)$, by [4, 11.1(d)], and that ω_s must be trivial on Z_p (and in fact on $\mathbf{Z}(G)_p \in \mathrm{Syl}_p(\mathbf{Z}(G))$) since s is a p'-element. In particular, every member of $\mathrm{Irr}(B) \cap \mathcal{E}(G, s)$ may be viewed as a character of $\bar{B}$. Hence for (b), it suffices to show that there are at least two non-$\mathrm{Aut}(G)$-conjugate members of $\mathrm{Irr}(B) \cap \mathcal{E}(G, s)$, or that there are at least two non-$\mathrm{Aut}(G)$-conjugate members of $\mathrm{Irr}(B)$ lying outside of $\mathcal{E}(G, p')$ and trivial on $\mathbf{Z}(G)_p$.

Let $\widetilde{s} \in \widetilde{G^*} \cong \widetilde{G}$ be a semisimple p'-element mapping to s under the natural epimorphism $\widetilde{G^*} \twoheadrightarrow G^*$, and let $\widetilde{B}$ be a block of $\widetilde{G}$ in $\mathcal{E}_p(\widetilde{G}, \widetilde{s})$ covering B. (Indeed, such a setup exists using [12, Th. 9.12 and Prop. 15.6].) Let $\widetilde{D}$ and D be defect groups of $\widetilde{B}$ and B, respectively, satisfying $D = \widetilde{D} \cap G$. Now,

$$\mathbf{C}_{\widetilde{G^*}}(\widetilde{s}) \cong \prod_{i=1}^{k} \mathrm{GL}_{m_i}((\epsilon q)^{\delta_i}),$$

where δ_i and m_i are positive integers. (See [18, §1] for details.) Note that $q - \epsilon$ divides $(\epsilon q)^{\delta_i} - 1$ for each i. Write $G_i := \mathrm{GL}_{m_i}((\epsilon q)^{\delta_i})$ for $1 \leq i \leq k$.

According to [18, Th. (7A)], Jordan decomposition maps

$$\mathrm{Irr}(\widetilde{B}) \cap \mathcal{E}(\widetilde{G}, \widetilde{s})$$

to

$$\mathrm{Irr}(B') \cap \mathcal{E}(\mathbf{C}_{\widetilde{G^*}}(\widetilde{s}), 1),$$

where $B' = \prod_{i=1}^{k} B_i$ is some unipotent block of $\mathbf{C}_{\widetilde{G^*}}(\widetilde{s})$ with B_i a unipotent block of G_i. Let e_i be the order of $(\epsilon q)^{\delta_i}$ modulo p. Then $\widetilde{B}$ (and B') is determined by certain e_i-core partitions λ_i. Namely, writing $m_i := e_i w_i + |\lambda_i|$ for each $1 \leq i \leq k$, [39, Th. (1.9)] yields that the number of characters in $\mathrm{Irr}(\widetilde{B})$ and the defect group of $\widetilde{B}$ are the same as for the principal block of $\prod_{i=1}^{k} \mathrm{GL}_{e_i w_i}((\epsilon q)^{\delta_i})$.

In particular, note that if some B_i has non-central defect, then B_i, and hence B', contains at least two unipotent characters of distinct degree, by Lemma 3.15. In this case, $\mathrm{Irr}(\widetilde{B}) \cap \mathcal{E}(\widetilde{G}, \widetilde{s})$ contains at least two characters of distinct degree, which must lie above two characters in $\mathrm{Irr}(B) \cap \mathcal{E}(G, s)$ lying in distinct $\mathrm{Aut}(G)$-orbits.

To complete the proof of (a) and (b) it therefore suffices to assume that each B_i has central defect, which forces $m_i = e_i w_i = 1$ for each $1 \leq i \leq l$, where without loss we assume that B_i has trivial defect groups for $l < i \leq k$. Here note that $G_i \cong C_{(\epsilon q)^{\delta_i}-1}$ is cyclic for $1 \leq i \leq l$ and a defect group for $\widetilde{B}$ is isomorphic to a Sylow p-subgroup of $\prod_{i=1}^{l} C_{(\epsilon q)^{\delta_i}-1}$. Since $\widetilde{B}$ is non-cyclic, note that $l \geq 2$.

Suppose $\chi_i \in \mathrm{Irr}(B)$ lies under members of $\mathcal{E}(\widetilde{G}, \widetilde{s}\widetilde{t}_i)$, $i = 1, 2$, for p-elements $\widetilde{t}_i \in \mathbf{C}_{\widetilde{G}^*}(\widetilde{s})$. Assume χ_1 is $\mathrm{Aut}(G)$-conjugate to χ_2. Then $(\widetilde{s}\widetilde{t}_1)^\alpha$ is $\widetilde{G}^*$-conjugate to $\widetilde{s}\widetilde{t}_2 z$ for some $z \in \mathbf{Z}(\widetilde{G}^*)$ and $\alpha \in \mathcal{D}$. It follows that $\widetilde{t}_1^\alpha$ is conjugate to $\widetilde{t}_2 z_p$, and hence they share the same set of eigenvalues, where z_p is the p-part of z. To complete the proof of (a), we therefore aim to exhibit $\widetilde{t}_1$ and $\widetilde{t}_2$ such that this cannot be the case.

When $l \geq 3$, elements of $\widetilde{G}^*$ corresponding to $(x, 1, \ldots, 1)$ and $(x, y, 1, \ldots, 1)$ for p-elements $1 \neq x \in G_1$, $1 \neq y \in G_2$ have different multiplicities for the eigenvalue 1, and hence we obtain $\widetilde{t}_1, \widetilde{t}_2$ satisfying $\widetilde{t}_1^\alpha$ is not $\widetilde{G}^*$-conjugate to $\widetilde{t}_2 z$ for any $z \in \mathbf{Z}(\widetilde{G}^*)$ and $\alpha \in \mathcal{D}$.

Now suppose $l = 2$. If $p \nmid (q - \epsilon)$, then p-elements corresponding to $(x, 1)$, (x, y) with $1 \neq x \in G_1$, $1 \neq y \in G_2$ again satisfy the claim, since $p \nmid |\mathbf{Z}(\widetilde{G}^*)|$. Hence we assume that $p \mid (q - \epsilon)$. Note that this forces $p \mid ((\epsilon q)^{\delta_i} - 1)$ for each $i = 1, 2$.

If $p^2 \mid ((\epsilon q)^{\delta_1} - 1)$, say, then elements $\widetilde{t}_1, \widetilde{t}_2$ corresponding to $(x, 1)$ and $(x^p, 1)$ with $|x| = p^2$ have the desired property. We therefore can assume that $p \,||\, ((\epsilon q)^{\delta_1} - 1)$, so that also $p \,||\, (q - \epsilon)$, in which case $D = \widetilde{D} \cap G$ is cyclic, and we are done with (a).

To complete the proof of (b), we wish to show that such $\widetilde{t}_1$ and $\widetilde{t}_2$ exist such that $\omega_{\widetilde{t}_i}$ are further trivial on $\mathbf{Z}(\widetilde{G})_p$. For this, it suffices to find $\widetilde{t}_1$ and $\widetilde{t}_2$ lying in $[\widetilde{G}^*, \widetilde{G}^*] \cong G$. Recall that here $p \mid \gcd(n, q - \epsilon)$. In this case, note that $e_i = 1$ for $1 \leq i \leq k$ since $(q - \epsilon) \mid ((\epsilon q)^{\delta_i} - 1)$, so that B_i is the unique block in $\mathcal{E}(G_i, 1)$ for each $1 \leq i \leq k$, and hence $\widetilde{B}$ is the unique block in $\mathcal{E}_p(\widetilde{G}, \widetilde{s})$. This also forces $l = k$, and $\mathcal{E}(\widetilde{G}, \widetilde{s})$ contains only one character.

Recall that $\widetilde{D}$ is isomorphic to a Sylow p-subgroup of the (in this case abelian) group $\mathbf{C}_{\widetilde{G}^*}(\widetilde{s})$ and in fact by [18, Th. (3C)], if we identify $\widetilde{G}$ with $\widetilde{G}^*$, we may take $\widetilde{D}$ to be a Sylow p-subgroup of $\mathbf{C}_{\widetilde{G}^*}(\widetilde{s})$. This way, we also identify G with $[\widetilde{G}^*, \widetilde{G}^*]$ and have that $D = \widetilde{D} \cap G$ is viewed as a subgroup of

$\mathbf{C}_{\widetilde{G}^*}(\widetilde{s}) \cap [\widetilde{G}^*, \widetilde{G}^*]$. Let $\bar{D} = D/Z_p$ be a defect group for $\bar{B}$. It suffices to argue that $D/\mathbf{Z}(G)_p$ contains at least two non-trivial non-$\mathrm{Aut}(G)$-conjugate elements when $|\mathrm{cd}(B)| > 1$. (Indeed, this would yield p-elements of $\mathbf{C}_{\widetilde{G}^*}(\widetilde{s})$ that lie in $G \cong [\widetilde{G}^*, \widetilde{G}^*]$ and are not $\mathrm{Aut}(G)$-conjugate to $\mathbf{Z}(\widetilde{G}^*)_p$-multiples of one another.) Hence we may assume $D/\mathbf{Z}(G)_p$ is elementary abelian, as otherwise two non-identity elements of $D/\mathbf{Z}(G)_p$ with distinct orders satisfy the claim.

If $k \geq 4$, then $D/\mathbf{Z}(G)_p$ is generated by at least two elements, and as before we may find such elements whose eigenvalue structures do not allow them to be $\mathrm{Aut}(G)$-conjugate.

If $k = 2$ and $\delta_1 = \delta_2$, then this forces $p \mid \delta_1$, since $p \mid n = 2^c\delta_1$ for some positive integer c. Further, note that since D is not cyclic, we know $((\epsilon q)^{\delta_1} - 1)_p > (q - \epsilon)_p$. Then choosing the embeddings into $\widetilde{G}$ of elements of the form (y, y^{-1}) with $|y| = ((\epsilon q)^{\delta_1} - 1)_p$ and $(x, 1)$ with $|x| = (q - \epsilon)_p$, we see these elements lie in G (since $(x, 1)$ is embedded into $\widetilde{G}$ with eigenvalues

$$(x, x^{\epsilon q}, \ldots, x^{(\epsilon q)^{\delta_1 - 1}}, 1, \ldots, 1) = (x, \ldots, x, 1, \ldots, 1),$$

and therefore has determinant 1) and cannot be $\mathrm{Aut}(G)$-conjugate to $\mathbf{Z}(G)_p$-multiples of one another. Hence we may assume that $\delta_1 > \delta_2$. Then note that, again studying the embedding of elements $(s_1, s_2) \in C_{(\epsilon q)^{\delta_1} - 1} \times C_{(\epsilon q)^{\delta_2} - 1}$ into $\widetilde{G}^*$, we see that $\widetilde{s}$ cannot be $\widetilde{G}^*$-conjugate to $\widetilde{s}z$ for any $1 \neq z \in \mathbf{Z}(\widetilde{G}^*)$. Hence the unique character of $\mathcal{E}(\widetilde{G}, \widetilde{s})$ restricts irreducibly to G, and similarly every member of $\mathcal{E}(\widetilde{G}, \widetilde{s}\widetilde{t})$ for $\widetilde{t} \in \mathbf{C}_{\widetilde{G}^*}(\widetilde{s})$ a p-element also restricts irreducibly. Since $\mathbf{C}_{\widetilde{G}^*}(\widetilde{s}\widetilde{t}) = \mathbf{C}_{\widetilde{G}^*}(\widetilde{s})$ for each such $\widetilde{t}$, we see every element of $\mathrm{Irr}(\widetilde{B})$, and hence of $\mathrm{Irr}(B)$, will have the same degree, a contradiction.

So, we finally assume $k = 3$. Arguing like above, we may assume $\delta_1 = \delta_2 = \delta_3$, so $\mathbf{C}_{\widetilde{G}^*}(\widetilde{s}) \cong G_1^3$. If $p \mid\mid ((\epsilon q)^{\delta_1} - 1)$, then $p \mid\mid (q - \epsilon)$, so we have that $Z_p = \mathbf{Z}(G)_p$ and $\bar{D}$ is cyclic, a contradiction. So we see $p^2 \mid ((\epsilon q)^{\delta_1} - 1)$. But then we also have $p^2 \mid |\mathbf{Z}(G)_p|$, since otherwise $D/\mathbf{Z}(G)_p$ contains elements of order p^2, contradicting our assumption that $D/\mathbf{Z}(G)_p$ is elementary abelian. In particular, $p^2 \mid n$. Then here since $n = 2^c \cdot 3\delta_1$, we have $p \mid \delta_1$. Hence we are done by considering elements $(y, y^{-1}, 1)$ and $(x, 1, 1)$ with x, y analogous to the case $k = 2$ above. □

3.5. *Other classical groups.* For $(\mathbf{G}, F)$ a connected reductive group and Frobenius morphism F emitting an $\mathbb{F}_q$-rational structure and $s \in \mathbf{G}^*$ semisimple, we write

$$J_s^{\mathbf{G}} \colon \mathbb{Z}\mathcal{E}(\mathbf{G}^F, s) \to \mathbb{Z}\mathcal{E}(\mathbf{C}_{\mathbf{G}^*}(s)^F, 1)$$

for a Jordan decomposition as in [19, Th. 2.6.22]. Recall from [19, Prop. 3.3.20] that for $\mathbf{K} \leq \mathbf{G}$ an F-stable Levi subgroup with dual $\mathbf{K}^* \leq \mathbf{G}^*$ and $s \in \mathbf{K}^*$, Lusztig induction induces a map $R_{\mathbf{K}}^{\mathbf{G}} \colon \mathbb{Z}\mathcal{E}(\mathbf{K}^F, s) \to \mathbb{Z}\mathcal{E}(\mathbf{G}^F, s)$.

We next consider the case of other classical groups, specifically for quasi-isolated blocks.

PROPOSITION 3.17. *Let $\mathbf{G}$ be simple of simply connected type such that $G = \mathbf{G}^F$ is quasi-simple of type $\mathrm{B}_n(q)$ with $n \geq 2$, $\mathrm{C}_n(q)$ with $n \geq 3$, $\mathrm{D}_n(q)$ with $n \geq 4$ or ${}^2\mathrm{D}_n(q)$ with $n \geq 4$. Let p be an odd prime not dividing q. Then if B is a non-cyclic quasi-isolated p-block of G, we have $k_{\mathrm{Aut}(G)}(B) \geq 3$.*

Proof. As before, let $\iota\colon \mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}$ be a regular embedding, even taking the embedding as in [19, Ex. 1.7.4], and write $\widetilde{G} = \widetilde{\mathbf{G}}^F$. Let B be a non-cyclic quasi-isolated block in $\mathcal{E}_p(G, s)$, and let $\widetilde{B} \subseteq \mathcal{E}_p(\widetilde{G}, \widetilde{s})$ be a block of $\widetilde{G}$ lying above B, where $\iota^*(\widetilde{s}) = s$ is a quasi-isolated p'-element of G^*. Let e be the order of q modulo p. By what we recalled from [27], let $(\mathbf{L}, \lambda)$ be the e-Jordan cuspidal pair of $\widetilde{\mathbf{G}}$ such that $\widetilde{B}$ is the unique block of $\widetilde{G}$ containing all constituents of $R_{\mathbf{L}}^{\widetilde{\mathbf{G}}}(\lambda)$.

Note that we may assume $\widetilde{s} \in \mathbf{L}^*$ for a dual Levi subgroup $\mathbf{L}^* \leq \widetilde{\mathbf{G}}^*$ of $\mathbf{L}$. Then since F is a Frobenius endomorphism and $\mathbf{C}_{\widetilde{\mathbf{G}}^*}(\widetilde{s})$ is connected and has only components of classical type, by [19, Th. 3.3.7] the Mackey formula holds, and by [19, Th. 4.7.2],

$$J_{\widetilde{s}}^{\mathbf{K}} \circ R_{\mathbf{K}_1}^{\mathbf{K}} = R_{\mathbf{C}_{\mathbf{K}_1^*}(\widetilde{s})}^{\mathbf{C}_{\mathbf{K}^*}(\widetilde{s})} \circ J_{\widetilde{s}}^{\mathbf{K}_1} \tag{3.17.1}$$

for any F-stable Levi subgroups $\mathbf{K}_1^* \leq \mathbf{K}^* \leq \widetilde{\mathbf{G}}^*$ containing $\widetilde{s}$ so, in particular, for $\mathbf{K}_1^* = \mathbf{L}^*$.

Also note that $\mathbf{C}_{\mathbf{L}^*}(\widetilde{s})$ is an e-split Levi subgroup of $\mathbf{C}_{\widetilde{\mathbf{G}}^*}(\widetilde{s})$ (by [23, Prop. 2.12]) and (by definition of e-Jordan cuspidality), $(\mathbf{C}_{\mathbf{L}^*}(\widetilde{s}), J_{\widetilde{s}}^{\mathbf{L}}(\lambda))$ is an e-cuspidal pair. Write $\mathbf{H}^* := \mathbf{C}_{\widetilde{\mathbf{G}}^*}(\widetilde{s})$ and $\mathbf{M}^* := \mathbf{C}_{\mathbf{L}^*}(\widetilde{s})$, a Levi subgroup of $\mathbf{H}^*$, and let $\mathbf{H}, \mathbf{M} \leq \widetilde{\mathbf{G}}$ be dual to $\mathbf{H}^*$ and $\mathbf{M}^*$, respectively. Then $(\mathbf{M}, \psi)$, with $\psi := J_1^{\mathbf{M}^*} \circ J_{\widetilde{s}}^{\mathbf{L}}(\lambda)$, is an e-Jordan cuspidal pair for $\mathbf{H}$ defining a unipotent block b of $H := \mathbf{H}^F$. Then by [8, Th. 3.11] and [27, Th. A], we have that $\mathrm{Irr}(b) \cap \mathcal{E}(H, 1)$ is the set of constituents of $R_{\mathbf{M}}^{\mathbf{H}}(\psi)$. From this, we see that Jordan decomposition induces an injection

$$\mathrm{Irr}(b) \cap \mathcal{E}(H, 1) \hookrightarrow \mathrm{Irr}(\widetilde{B}) \cap \mathcal{E}(\widetilde{G}, \widetilde{s}).$$

Let $t \in \mathbf{C}_{\widetilde{G}^*}(\widetilde{s}) = (\mathbf{H}^*)^F$ be a p-element, and let $\widetilde{\mathbf{G}}(t) \leq \widetilde{\mathbf{G}}$ and $\mathbf{H}(t) \leq \mathbf{H}$ be F-stable Levi subgroups in duality with $\mathbf{C}_{\widetilde{\mathbf{G}}^*}(t)$ and $\mathbf{C}_{\mathbf{H}^*}(t)$, respectively. Applying [10, Th. (iii)] to both $\widetilde{B}$ and b and applying (3.17.1) to $(\widetilde{\mathbf{G}}, \mathbf{G}(t), \widetilde{s})$ as well as to its analogue for $(\mathbf{H}^*, \mathbf{H}(t)^*, 1)$, we obtain further that $\mathrm{Irr}(\widetilde{B}) \cap \mathcal{E}(\widetilde{G}, \widetilde{s}t)$ is non-empty if and only if $\mathrm{Irr}(b) \cap \mathcal{E}(H, t)$ is non-empty. More specifically, $\chi \in \mathcal{E}(\widetilde{G}, \widetilde{s}t)$ is in $\mathrm{Irr}(\widetilde{B})$ if and only if $\psi \in \mathcal{E}(H, t)$ lies in $\mathrm{Irr}(b)$, where ψ is the character such that $J_{\widetilde{s}t}^{\widetilde{\mathbf{G}}}(\chi) = J_t^{\mathbf{H}}(\psi)$. Further, by [11, Prop. 5.1], b and $\widetilde{B}$

have isomorphic defect groups. Let $D_b \cong D_{\widetilde{B}}$ and D_B be defect groups of $b, \widetilde{B}$ and B, respectively, chosen so that $D_B = D_{\widetilde{B}} \cap G$.

Now, by [3, (2.2)], we have $\iota^*(\mathbf{H}^*) = \mathbf{C}_{\mathbf{G}^*}(s)^\circ$. Since B is quasi-isolated, the latter has only classical components, of the forms listed in [3, Table II]. Hence the components of $\mathbf{H}$ are of the forms dual to these.

By [10, Th. (i)], the unipotent characters in b are independent of isogeny type. Let $H_0 := [\mathbf{H}, \mathbf{H}]^F$, and let b' be the unipotent block of H_0 below b and b'' the corresponding unipotent block of $[\mathbf{H}, \mathbf{H}]^F_{\mathrm{sc}}$, where $[\mathbf{H}, \mathbf{H}]_{\mathrm{sc}}$ denotes the simply connected covering of the semisimple group $[\mathbf{H}, \mathbf{H}]$. Applying Corollary 3.14 and Lemma 3.15, we may assume $\mathbf{Z}([\mathbf{H}, \mathbf{H}]^F_{\mathrm{sc}})_p$ is a defect group of b'', and each of b, b' and b'' contain exactly one unipotent character. Assume $[\mathbf{H}, \mathbf{H}]^F_{\mathrm{sc}}$ has a factor of type $\mathrm{A}_m(\epsilon q^\delta)$ for some $m, \delta \geq 1$ and $\epsilon \in \{\pm 1\}$ with $d_p(\epsilon q^\delta) = 1$, where $d_p(\epsilon q^\delta)$ denotes the order of ϵq^δ modulo p. Then, $\mathrm{A}_m(\epsilon q^\delta)$ has a unique unipotent p-block, which must contain the trivial and Steinberg characters, contradicting that b contains a unique unipotent character. Thus, since all components of $\mathbf{H}$ are classical, $\mathbf{Z}([\mathbf{H}, \mathbf{H}]^F_{\mathrm{sc}})_p$ is trivial, so $p \nmid |\mathbf{Z}(H_0)|$, and we obtain that $D_b = \mathbf{Z}(H)_p$ from [10, Th. (ii)]. Then using [11, Lemma 4.16, Def. 4.3], we see that $D_{\widetilde{B}}$ can be taken to be equal to $D_b = \mathbf{Z}(H)_p$.

Since $\widetilde{G}/G\mathbf{Z}(\widetilde{G})$ is a 2-group, we know $\mathbf{Z}(\widetilde{G})_p \cap G = 1$, so $D_{\widetilde{B}}$ is a direct product of D_B and $\mathbf{Z}(\widetilde{G})_p$, so $D_B \cong D_{\widetilde{B}}/\mathbf{Z}(\widetilde{G})_p$ and $\mathbf{Z}(H)_p/\mathbf{Z}(\widetilde{G})_p$ must be non-cyclic. However, from our list of possible structures in [3, Table II], we see that $\mathbf{Z}(H)_p/\mathbf{Z}(\widetilde{G})_p$ is trivial or cyclic, unless $\mathbf{G}$ is of type D_n and $\mathbf{H}$ is of type A_{n-3}. In this case, although s is quasi-isolated, $\widetilde{s}$ is not, and $\mathbf{H}$ is a Levi subgroup of $\widetilde{\mathbf{G}}$. From our discussions above and using, for example, the descriptions of possible $(\mathbf{C}_{\mathbf{G}^*}(s)^\circ)^F$ in [35, see, e.g., 2], we have $H_0 = \mathrm{A}_{n-3}(\epsilon q)$ with $d_p(\epsilon q) \neq 1$, and $\mathbf{Z}(H)_p/\mathbf{Z}(\widetilde{G})_p \cong C^2_{(q+\epsilon)_p}$ coming from a torus $C^2_{q+\epsilon} \leq H$.

Let θ be the unique unipotent character in b. We have

$$\mathrm{Irr}(b) = \{\theta\hat{t} \mid t \in \mathbf{Z}(\mathbf{H}^*)^F_p\},$$

where $t \mapsto \hat{t}$ is the isomorphism $\mathbf{Z}(\mathbf{H}^*)^F \to \mathrm{Irr}(H/H_0)$ guaranteed by [19, Prop. 2.5.20]. Now, a construction for $\widetilde{\mathbf{G}} \cong \widetilde{\mathbf{G}}^*$ is presented in [19, Ex. 1.7.4] using tori constructed in [19, Ex. 1.5.6]. The element s and its centraliser in G^* are described in terms of root systems in [3]. With this, we can see $\mathbf{Z}(\mathbf{H}^*)$ through this construction, and we calculate that there are elements t_1, t_2 of order p in $\mathbf{Z}(\mathbf{H}^*)^F \setminus \mathbf{Z}(\widetilde{G}^*)$ such that t_1 is not $\widetilde{G}^*\mathcal{D}$-conjugate to $t_2 z$ for any $z \in \mathbf{Z}(\widetilde{G}^*)_p$. Since $\widetilde{G} \rtimes \mathcal{D}$ induces $\mathrm{Aut}(G)$ and [13, Th. 3.1] tells us Jordan decomposition for $\widetilde{G}$ can be chosen to be $\mathcal{D}$-equivariant, then as in the proof of Proposition 3.16 we obtain two non-$\mathrm{Aut}(G)$-conjugate characters in $\widetilde{B} \backslash \mathcal{E}(\widetilde{G}, p')$ lying above non-$\mathrm{Aut}(G)$-conjugate members of $\mathrm{Irr}(B)$. □

3.6. *Remaining quasi-isolated blocks.*

PROPOSITION 3.18. *Let* $\mathbf{G}$ *be such that* $G := \mathbf{G}^F$ *is quasi-simple of type* $\mathrm{E}_n(q)$ *with* $6 \le n \le 8$, ${}^2\mathrm{E}_6(q)$, $\mathrm{F}_4(q)$ *or* $\mathrm{G}_2(q)$. *Let* p *be an odd prime not dividing* q. *Then if* B *is a non-cyclic quasi-isolated* p*-block of* G, *we have* $k_{\mathrm{Aut}(G)}(B) \ge 3$.

Proof. Let B lie in $\mathcal{E}_p(G, s)$, where s is a quasi-isolated semisimple p'-element of G^*. Let e be the order of q modulo p. In our situation, thanks to [27, Rem. 2.2], the notions of e-Jordan cuspidality and e-cuspidality coincide. When p is good for G, the e-Jordan quasi-central cuspidal pairs are the same as the e-Jordan cuspidal pairs by [27, Th. A(d,e)].

Assume $(\mathbf{G}, \chi)$ is quasi-central e-cuspidal. Then since $\mathbf{G} = [\mathbf{G}, \mathbf{G}]$, [26, Prop. 2.5] implies that $b_G(\mathbf{G}, \chi)$ is of central defect. However, then $b_G(\mathbf{G}, \chi)$ is cyclic.

Hence from now on, we may assume $B = b_G(\mathbf{L}, \lambda)$ with $(\mathbf{L}, \lambda)$ an e-Jordan quasi-central cuspidal pair of G with $G \ne \mathbf{L}^F$ and $\lambda \in \mathcal{E}(\mathbf{L}^F, s)$. Further, from Lemma 3.15 and Corollary 3.14, we may assume B is not unipotent. From here, we consider separately the cases that p is bad and that p is good.

First let $p \ge 3$ be bad for $\mathbf{G}$. Here the block distributions for $\mathcal{E}(G, s)$ are given in [26, Tables 2–4, 6–9], or by Ennola duality with those results. (Namely, when $e = 2$, one formally replaces q with $-q$ in the results for $e = 1$ — see [26, p. 16]; also, the results for ${}^2\mathrm{E}_6(q)$ are obtained from those of $\mathrm{E}_6(q)$ by switching the roles of $e = 1$ and $e = 2$ — see [26, pp. 21].) From this and the knowledge of the degrees of characters in $\mathcal{E}(G, s)$ for each $\mathbf{C}_{G^*}(s)$ listed, (obtained by Jordan decomposition from the unipotent character degrees of groups of small rank), we see that $B \cap \mathcal{E}(G, s)$ contains at least two characters that are not $\mathrm{Aut}(G)$-conjugate in the cases with non-cyclic defect. (Here we may use [26, Prop. 2.7] to understand the defect groups.) Hence we are done in this case, by applying Corollary 3.14.

Next assume $p \ge 5$ is a good prime for $\mathbf{G}$. First suppose $e \ge 3$. In this case, the block distributions for $\mathcal{E}(G, s)$ are given in [23, Tables 2, 3, 5, 7, 8], through a description of the decompositions of $R_{\mathbf{L}}^{\mathbf{G}}(\lambda)$. With this information, combined with the knowledge of the character degrees in $\mathbf{C}_{G^*}(s)$ and again using Corollary 3.14, we see there are at least three non-$\mathrm{Aut}(G)$-conjugate characters when the defect group is non-cyclic.

Finally, if $p \ge 5$ is a good prime for G with $e \in \{1, 2\}$, the decompositions of $R_{\mathbf{L}}^{\mathbf{G}}(\lambda)$ and the groups $W_G(\mathbf{L}, \lambda)$ are the same as those given in [26, Tables 2–4, 6–9] for the bad prime case. But, note that now each $(\mathbf{L}, \lambda)$ gives a distinct block by [27, Th. A], and the same considerations as before complete the proof. □

COROLLARY 3.19. *Let* $\mathbf{G}$ *be simple of simply connected type such that* $G := \mathbf{G}^F$ *is quasi-simple. Let* $p \nmid q$ *be an odd prime, and let* B *be a quasi-isolated* p*-block of* G*. Then* $k_{\mathrm{Aut}(G)}(B) \geq 3$ *if* B *is non-cyclic and* $k_{\mathrm{Aut}(G)}(B) \geq 2$ *if* B *is cyclic of positive defect.*

Proof. This now follows from Corollary 3.14 and Propositions 3.6, 3.16, 3.17, and 3.18. □

PROPOSITION 3.20. *Let* $\mathbf{G}$ *be simple of simply connected type such that* $G := \mathbf{G}^F$ *is quasi-simple, and let* $p \nmid q$ *be an odd prime. Let* $\mathbf{H}$ *be an* F*-stable Levi subgroup of* $\mathbf{G}$*, and let* B *be a non-cyclic quasi-isolated* p*-block of* $H := \mathbf{H}^F$ *with* $|\mathrm{cd}(B)| > 1$*. Then* $k_{\mathrm{Aut}(H)}(B) \geq 3$.

Proof. Using Corollary 3.19 and Proposition 3.5, we have that H satisfies Hypothesis 3.8 with respect to $c = 3$ (see also Remark 3.11). Hence by Proposition 3.9, the statement holds. □

3.7. *Proof of Theorem* B. We are now ready to complete the proof of Theorem B. Recall that, thanks to Section 3.1 and Proposition 3.7, we only need to consider groups of Lie type in non-defining characteristic.

THEOREM 3.21. *Theorem* 3.1 *holds when* S *is quasi-simple such that* $S/\mathbf{Z}(S)$ *is a simple group of Lie type defined in characteristic* $r \neq p$.

Proof. By Section 3.1, we may assume that the simple group $\bar{S} := S/\mathbf{Z}(S)$ has a non-exceptional Schur multiplier, and that S is not of Suzuki or Ree type.

Let G be the Schur cover of $\bar{S}$, so that $G = \mathbf{G}^F$ is the group of fixed points of some simple, simply connected algebraic group $\mathbf{G}$ in characteristic r under some Frobenius morphism F. Assume first that $p \nmid |\mathbf{Z}(G)|$. Then it suffices to show the statement for G, since the irreducible characters in a block of S will be the same as those in the block of G dominating it, viewed via inflation. As in the proof of Lemma 3.12, we have that b is Morita equivalent to a quasi-isolated block B of H, where $H = \mathbf{H}^F$ for an F-stable Levi subgroup $\mathbf{H}$ of $\mathbf{G}$. Now, by Proposition 3.20 together with Lemma 3.12, we have $k_{\mathrm{Aut}(G)}(b) \geq 3$.

Now suppose that p divides $|\mathbf{Z}(G)|$. Then since p is odd, we have that G is $\mathrm{SL}_n(\epsilon q)$ or $\mathrm{E}_6(\epsilon q)$ for some $\epsilon \in \{\pm 1\}$ and some power q of r. In the first case, $p \mid \gcd(n, q - \epsilon)$ and Proposition 3.16(b) finishes the proof. In the second case, $p = 3 \mid (q - \epsilon)$.

So, we finally assume that $\bar{S} = \mathrm{E}_6(\epsilon q) = G/\mathbf{Z}(G)$ and $p = 3 \mid (q - \epsilon)$. Let $\bar{b}$ be a 3-block of $\bar{S}$ with non-cyclic defect groups such that $|\mathrm{cd}(\bar{b})| > 1$, and let b be the block of G dominating $\bar{b}$ lying in $\mathcal{E}_3(G, s)$ for some semisimple $3'$-element $s \in G^*$. Let $\mathbf{H}$ be an F-stable Levi subgroup of $\mathbf{G}$ minimal with the property that $\mathbf{C}_{\mathbf{G}^*}(s) \leq \mathbf{H}^*$, so that s is quasi-isolated in $\mathbf{H}^*$. Let B be

the block of $H = \mathbf{H}^F$ in Bonnafé–Rouquier correspondence with b, and let $\bar{B}$ be its image in $H/\mathbf{Z}(G)$. Then $\bar{B}$ is also not cyclic, by [26, Th. 7.16].

Arguing exactly as in the fourth paragraph of [28, p. 13], either $\mathbf{C}_{\mathbf{G}^*}(s) = \mathbf{C}^\circ_{\mathbf{G}^*}(s) = \mathbf{H}^*$ consists only of components of type A, or $\mathbf{H}_0 := [\mathbf{H}, \mathbf{H}]$ is of type D_4 or D_5.

In the first case, B is the tensor product of some unipotent block of H with the linear character $\hat{s}$ corresponding to $s \in \mathbf{Z}(\mathbf{H}^*)^F$. Let B' be the unipotent 3-block of H_0 covered by $B \otimes \hat{s}^{-1}$. Using Lemma 3.15 or in fact checking directly for groups of type A of rank at most 5, we see B' contains at least two unipotent characters of distinct degree unless either $\mathbf{C}_{G^*}(s)$ is abelian or H_0 contains only components of the form $\mathrm{A}_2(-\epsilon q)$. If $\mathbf{C}_{G^*}(s)$ is abelian, then every member of $\mathcal{E}_3(H, s)$, hence $\mathrm{Irr}(B)$, and hence $\mathrm{Irr}(b)$, has the same degree. Now, the unipotent blocks of $\mathrm{A}_2(\epsilon q)$ consist of one defect zero block and the principal block containing the trivial and Steinberg characters. Then if H_0 contains only components of the form $\mathrm{A}_2(-\epsilon q)$ and B' does not contain two characters of distinct degree, then B' has trivial defect groups, which contradicts Proposition 3.9(a) unless again $|\mathrm{cd}(B)| = |\mathrm{cd}(b)| = 1$.

So we may assume B' contains at least two unipotent characters of distinct degree, so $\mathrm{Irr}(B) \cap \mathcal{E}(H, s)$, and hence $\mathrm{Irr}(b) \cap \mathcal{E}(G, s)$, contains at least two characters of distinct degrees. Recall that the members of $\mathrm{Irr}(b) \cap \mathcal{E}(G, s)$ are trivial on $\mathbf{Z}(G)$, since s is $3'$, and can therefore be viewed as characters in $\mathrm{Irr}(\bar{b})$. Further, the images of the characters in $\mathrm{Irr}(B) \cap \mathcal{E}(H, s)$ under d^1 are linearly independent by [11, Th. 1.7]. Hence there must be at least one more member of $\mathrm{Irr}(\bar{B})$, and therefore at least one member of $\mathrm{Irr}(\bar{b})$ lying outside of $\mathcal{E}(G, s)$, which as before is not $\mathrm{Aut}(G)$-conjugate to the members of $\mathcal{E}(G, s)$.

Now consider the case $\mathbf{H}_0 = [\mathbf{H}, \mathbf{H}]$ is of type D_4 or D_5. Note that $H_0 = \mathbf{H}_0^F$ is simply connected of type D_4 or D_5, and hence has centre of 2-power order. Then a (quasi-isolated) block B' of $H_0\mathbf{Z}(G)/\mathbf{Z}(G) \cong H_0$ lying under $\bar{B}$ contains at least three, respectively two, characters in distinct $\mathrm{Aut}(H_0)$-orbits from Proposition 3.17 and Corollary 3.14 if B' is non-cyclic, respectively cyclic. If B' is non-cyclic, then arguing exactly as in the situation of Proposition 3.9(a) and Lemma 3.12, but with $H_0\mathbf{Z}(G)/\mathbf{Z}(G) \lhd H/\mathbf{Z}(G)$ taking the place of $H_0 \lhd H$, completes the proof. So, assume B' is cyclic. Note that $\mathbf{H} = \mathbf{H}_0\mathbf{Z}^\circ(\mathbf{H})$, and $H_0\mathbf{Z}^\circ(\mathbf{H})^F/\mathbf{Z}(G)$ is normal in $H/\mathbf{Z}(G)$ with $3'$-index, so that a block B''' of this group under $\bar{B}$ is non-cyclic. Further, this group can be identified with a $3'$-quotient of $H_0 \times \mathbf{Z}^\circ(\mathbf{H})^F/\mathbf{Z}(G)$, and hence we may identify B''' with a block $B' \otimes B''$ of $H_0 \times \mathbf{Z}^\circ(\mathbf{H})^F/\mathbf{Z}(G)$, where B'' has non-trivial defect groups. Then taking two non-$\mathrm{Aut}(H)$-conjugate members of B'' (these exist since B'' is a tensor product of the unique block of the Sylow 3-subgroup of $\mathbf{Z}^\circ(\mathbf{H})^F/\mathbf{Z}(G)$ with some character of $3'$ order), we obtain $k_{\mathrm{Aut}(H)}(\bar{B}) \geq 3$ and $k_{\mathrm{Aut}(G)}(\bar{b}) \geq 3$, again arguing as in Propositions 3.9 and Lemma 3.12. □

4. Invariant blocks and defect groups

This section is devoted to the proof of the following result on blocks of quasi-simple groups (which will imply Theorem C):

THEOREM 4.1. *Let p be an odd prime and S a quasi-simple group with $Z = \mathbf{Z}(S)$, $\bar{S} := S/Z$. Let b be a p-block of S with an abelian, non-cyclic defect group D, and let b_D be a block of $\mathbf{C}_S(D)$ with defect group D inducing b. For parts* (a) *and* (c), *in the cases $\bar{S} = \mathrm{E}_6(\epsilon q)$ with $p = 3|(q-\epsilon)$ and $\epsilon \in \{\pm 1\}$, assume in addition that BHZ for the prime p holds for all groups of order smaller than $|\bar{S}|$.*

(a) *Suppose that $\bar{S} \le H \le \mathrm{Aut}(S)$, $A := H/\bar{S}$ has a normal p-complement and a cyclic Sylow p-subgroup Q. Assume b is H-invariant, and for every $\chi \in \mathrm{Irr}(b)$, we have that $|A : A_\chi|$ is p'. If $x \in \mathbf{N}_H(D)$ is a p-element that fixes b_D, then $[x, D] = 1$.*

(b) *Suppose that $\bar{S} \le H \le \mathrm{Aut}(S)$ and $\mathbf{O}^{p'}(H/\bar{S}) = H/\bar{S}$. Also assume that every $\chi \in \mathrm{Irr}(b)$ is H-invariant. Set $J := \mathrm{Inndiag}(\bar{S})$ if S is of Lie type, and $J = \bar{S}$ otherwise. Then HJ/J is a p'-group.*

(c) *Suppose $p \nmid |Z|$, and let $S/Z \le K/Z \le \mathrm{Aut}(S)$ with K/S an abelian p-group. If every irreducible character in b extends to K, then the defect groups of the K-block covering b are abelian.*

4.1. *First reductions.* We keep the notation of Theorem 4.1 throughout the section. In particular, p is always an odd prime. For the proof, we discuss the various possibilities for S and p according to the classification of finite simple groups. By assumption, $|D| > 1$ and thus p divides $|S|$.

LEMMA 4.2. *In the situation of Theorem 4.1*(a), *assume Q is normal in A. Then we may assume $A = Q \ne 1$.*

Proof. If Q is normal in A, all orbits of Q on $\mathrm{Irr}(b)$ will have p'-length as well, and all p-elements in A lie in Q. So we are done if we can show the claim when $A = Q$. Furthermore, if $Q = 1$, any p-element $x \in \mathbf{N}_H(D)$ that fixes b_D centralises D by Proposition 2.4. □

LEMMA 4.3. *In the situation of Theorem 4.1*(c) *assume K/S is cyclic. Then the claim in 4.1*(c) *is a consequence of 4.1*(a).

Proof. Since every irreducible character in b extends to K, the block b is K-invariant. Let B be the unique block of K covering b. By Lemma 2.3, let $\hat{D}$ be a defect group of B such that $\hat{D} \cap S = D$ and $\hat{D} \le T$, where T is the stabiliser of b_D in $\mathbf{N}_K(D)$. Let $x \in \hat{D}$. Then $[x, D] = 1$ by (a), and therefore $D \le \mathbf{Z}(\hat{D})$. As $\hat{D}/D$ is cyclic, this shows $\hat{D}$ is abelian. □

PROPOSITION 4.4. *For the proof of Theorem 4.1, we may assume that $S/\mathbf{Z}(S)$ is simple of Lie type in characteristic different from p.*

Proof. If $\bar{S}$ is an alternating group, a sporadic group, or the Tits simple group, then $|\mathrm{Out}(S)|$ is a 2-power. Hence 4.1(a) follows from Proposition 2.4 (applied to $G = N = S$), while 4.1(b) and (c) hold trivially. The same arguments apply whenever $p \nmid |\mathrm{Out}(\bar{S})|$.

Now assume $\bar{S}$ is simple of Lie type in characteristic $p > 2$. If $\bar{S}$ has an exceptional covering group (see [38, Table 24.3]), then $\mathrm{Out}(S)$ is a 2-group, and we can conclude as above. Hence we may assume $|Z|$ is prime to p and p divides $|\mathrm{Out}(S)|$. By [24] any p-block of S has either full defect or defect zero. Thus our assumption on D being abelian forces $\bar{S} = \mathrm{L}_2(q)$ for some $q = p^f$, so we may take $S = \mathrm{SL}_2(q)$. This group has two p-blocks of maximal defect, the principal block $B_0(S)$ and a block B containing all faithful characters (see [24]). Since $|\mathrm{Out}(S)|$ is divisible by p, Q must induce field automorphisms of order p, so $p|f$. By order reasons, the image of Q is central in $\mathrm{Out}(S)$, so by Lemma 4.2 we may assume $A = Q$. Now by inspection of the generic character table given, e.g., in [19, Ex. 2.1.17 and Table 2.6], there exist irreducible characters in $B_0(S)$ as well as in B not stabilised by Q. Hence the theorem holds in this case. □

PROPOSITION 4.5. *Theorem 4.1 holds for S an exceptional covering group of a simple group of Lie type in characteristic different from p.*

Proof. The simple groups with exceptional covering groups are listed in [38, Table 24.3]. Arguing as in the proof of Proposition 4.4, we need only consider the ones with $\mathrm{Out}(S)$ not a p'-group, which are $\mathrm{L}_3(4)$, $\mathrm{U}_6(2)$, $\mathrm{O}_8^+(2)$, ${}^2B_2(8)$ and ${}^2\mathrm{E}_6(2)$, and the only relevant prime is $p = 3$. Assume $\bar{S} = \mathrm{U}_6(2)$ or ${}^2\mathrm{E}_6(2)$ and that 3 divides $|H/\bar{S}|$, respectively $|K/S|$. Since any outer automorphism of order 3 permutes the three involutions in the Schur multiplier of $\bar{S}$ cyclically and the block b is invariant under H, respectively K, the relevant covering groups are $S = \bar{S}$ and $S = 3.\bar{S}$ only, which are not exceptional coverings. In all other cases, using [57], the 3-blocks of an exceptional covering group S turn out to have either cyclic or non-abelian defect, whence the claim follows. □

PROPOSITION 4.6. *Theorem 4.1 holds for S a Suzuki or Ree group.*

Proof. By Proposition 4.4, we may assume p is not the defining characteristic of S and $S \ncong {}^2F_4(2)'$. Now the Sylow p-subgroups of the Suzuki and the small Ree groups are cyclic for any such $p \geq 3$. For $S = {}^2F_4(q^2)$ with $q^2 > 2$, the only p-block with non-cyclic defect groups is the principal block $b = B_0(S)$, and its defect groups are abelian when $p > 3$ (see [34]). Now $\mathrm{Out}(S)$ is cyclic so, in particular, its Sylow p-subgroup is normal; also $J = \bar{S}$. By Lemma 4.2 we may assume $A = Q$, and by Lemma 4.3 it suffices to show 4.1(a) and 4.1(b). But then the only p-block of H above b is the principal block $B = B_0(H)$. By [34, Bem. 3] all characters in b have height 0, and if all orbits of Q on $\mathrm{Irr}(b)$

have p'-size, the same is true for the characters in $\mathrm{Irr}(B)$. But then by the main result of [36], the Sylow p-subgroups of H are abelian. Alternatively, by inspection of the character table [34], the assumption of 4.1(a) is in fact never satisfied for $A \neq 1$; this establishes 4.1(a), as well as 4.1(b). □

4.2. *Some results on p-blocks.* To deal with groups of Lie type for non-defining primes we first observe some general facts on blocks of finite reductive groups that may be of independent interest.

Let $\mathbf{G}$ be a Levi subgroup of a simple linear algebraic group of simply connected type over an algebraically closed field of positive characteristic and $F : \mathbf{G} \to \mathbf{G}$ a Frobenius endomorphism with respect to an $\mathbb{F}_q$-rational structure. Let $\mathbf{G}^*$ be a group in duality with $\mathbf{G}$ with corresponding Frobenius endomorphism again denoted F. We let $p \geq 3$ be a prime not dividing q and denote by e the order of q modulo p.

Recall that any p-block b of $G := \mathbf{G}^F$ has the property that $\mathrm{Irr}(b) \subseteq \mathcal{E}_p(G, s)$ for some semisimple p'-element $s \in G^* := \mathbf{G}^{*F}$ (see, e.g., [12, Th. 9.12]). Furthermore, by [27, Th. A] there is a bijection between p-blocks b of $\mathbf{G}^F$ and $\mathbf{G}^F$-classes of e-Jordan-cuspidal pairs $(\mathbf{L}, \lambda)$ of $\mathbf{G}$ of quasi-central p-defect, with λ lying in a p'-Lusztig series of $\mathrm{Irr}(\mathbf{L}^F)$ such that all constituents of $R_{\mathbf{L}}^{\mathbf{G}}(\lambda)$ are contained in $\mathrm{Irr}(b)$. We write

$$(\mathbf{L}, \lambda) \mapsto b_G(\mathbf{L}, \lambda)$$

for this map.

LEMMA 4.7. *Let $d \geq 1$ and $\mathbf{L}$ be a maximal proper d-split Levi subgroup of $\mathbf{G}$. Then $|\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F/\mathbf{L}^F|$ is not divisible by a prime bigger than* $\max\{2, d\}$.

Proof. The maximal proper d-split Levi subgroups $\mathbf{L}$ of $\mathbf{G}$ above the centraliser of a fixed Sylow d-torus $\mathbf{S}$ of $\mathbf{G}$ are in one-to-one correspondence with the maximal parabolic subgroups of the relative Weyl group

$$W = \mathbf{N}_{\mathbf{G}}(\mathbf{S})^F/\mathbf{C}_{\mathbf{G}}(\mathbf{S})^F$$

of $\mathbf{S}$ [19, Prop. 3.5.12]. Moreover, if $\mathbf{L}$ corresponds to $W_1 \leq W$, then

$$\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F/\mathbf{L}^F \cong \mathbf{N}_W(W_1)/W_1$$

(see [38, Prop. 26.4]). The claim is thus reduced to a question in reflection groups. For these, there is an immediate reduction to the irreducible case. The latter can be checked case by case using the explicit description of the various relative Weyl groups given in [19, 3.5.11–3.5.15]. For example, when $\mathbf{G}$ is of classical type, then $W = G(m, 1, n)$ or $G(m, 2, n)$ for suitable $m \in \{d, 2d\}$ and $n \geq 1$, for which the assertion is easily verified. □

The following somewhat surprising result may be of independent interest. Here, by a *field automorphism of G* we mean any conjugate of an automorphism induced by a Frobenius endomorphism F_0 of $\mathbf{G}$ commuting with F.

PROPOSITION 4.8. *Let σ be a field automorphism of $G = \mathbf{G}^F$ of order p, and let $\gamma = \sigma\tau$, where τ is an inner-diagonal automorphism of G. Then any p-block of G of non-quasi-central defect contains characters in a Lusztig series that is not γ-stable.*

Proof. Let b be a p-block of G of non-quasi-central defect. As above, $b = b_G(\mathbf{L}, \lambda)$ for some e-split Levi subgroup $\mathbf{L}$ of $\mathbf{G}$, proper since b has non-quasi-central defect. By assumption, up to conjugation, σ is induced by a Frobenius endomorphism F_0 of $\mathbf{G}$ commuting with F, with respect to an $\mathbb{F}_{q_0}$-rational structure on $\mathbf{G}$ where $q = q_0^p$. Then q_0 also has order e modulo p. In particular, any e-torus of $(\mathbf{G}, F_0)$ is also an e-torus of $(\mathbf{G}, F)$, and conversely, any F_0-stable e-torus of $(\mathbf{G}, F)$ is also an e-torus of $(\mathbf{G}, F_0)$. The same relations hold for e-split Levi subgroups since these are the centralisers of e-tori.

By [56, 5.6], σ induces a field automorphism σ^* of $G^* = \mathbf{G}^{*F}$ of the same order. Let $\mathbf{L}^* \leq \mathbf{G}^*$ be in duality with $\mathbf{L}$, an e-split Levi subgroup of $(\mathbf{G}^*, F)$ that is F_0-stable, hence e-split in $(\mathbf{G}^*, F_0)$. Let $\mathbf{L}_1^* \geq \mathbf{L}^*$ be a maximal (proper) e-split Levi subgroup of $(\mathbf{G}^*, F_0)$. Thus, $\mathbf{T}_1 := \mathbf{Z}(\mathbf{L}_1^*)_{\Phi_e}$ is an e-torus of $(\mathbf{G}^*, F)$ and $(\mathbf{G}^*, F_0)$ of rank 1, and so $\mathbf{T}_1^F$ and $\mathbf{T}_1^{F_0}$ have cyclic Sylow p-subgroups [38, Prop. 25.7], with $|\mathbf{T}_1^F|_p > |\mathbf{T}_1^{F_0}|_p$ since $|\Phi_e(q)|_p > |\Phi_e(q_0)|_p$. Thus a generator t of $(\mathbf{T}_1^F)_p$ is not fixed by σ^*; that is, $t^{\sigma^*} = t^a \neq t$ for some integer a.

We claim that $\mathbf{C}_{G^*}((\mathbf{T}_1^F)_p) = \mathbf{C}_{G^*}((\mathbf{T}_1)_{\Phi_e})$. Indeed, if p is a good prime for $\mathbf{G}$, then this is [10, Prop. 2.1(ii)+(iii)]. Otherwise, $\mathbf{G}$ has a factor of exceptional type and $p = 3$ (so $e = 1, 2$) or $\mathbf{G} = \mathrm{E}_8$, $p = 5$ and $e = 1, 2, 4$. In the latter cases, the explicit description of maximal e-split Levi subgroups in [19, §3.5] together with the fact [26, Table 1] that there are no quasi-isolated elements of order p^k, $k \geq 2$, in $\mathbf{G}^*$ shows the claim.

Now assume there exists $g \in G^*$ with $t^a = t^g$, so conjugation by g makes an orbit of length p on $\langle t \rangle = (\mathbf{T}_1^F)_p$. Then g normalises $(\mathbf{T}_1^F)_p$ and hence also

$$\mathbf{C}_{\mathbf{G}^*}((\mathbf{T}_1^F)_p)^F = \mathbf{C}_{\mathbf{G}^*}((\mathbf{T}_1)_{\Phi_e})^F = \mathbf{L}_1^{*F}.$$

But by Lemma 4.7, $\mathbf{N}_{\mathbf{G}^*}(\mathbf{L}_1^*)^F/\mathbf{L}_1^{*F}$ has order prime to p (since $p > e$), which contradicts the assumption that g makes an orbit of length p on $\langle t \rangle$. Hence t, t^{σ^*} are not G^*-conjugate.

Now let $s \in G^*$ be a semisimple p'-element such that $\mathrm{Irr}(b) \subseteq \mathcal{E}_p(G, s)$. Since $\mathrm{Irr}(b)$ contains the constituents of $R_{\mathbf{L}}^{\mathbf{G}}(\lambda)$, we may assume $s \in \mathbf{L}^{*F}$. So

$$t \in \mathbf{Z}(\mathbf{L}_1^{*F}) \leq \mathbf{Z}(\mathbf{L}^{*F}) \leq \mathbf{C}_{G^*}(s).$$

Note that st and $(st)^{\sigma^*}$ are not G^*-conjugate, since neither are their p-parts. Thus, by [56, Prop. 7.2] the Lusztig series $\mathcal{E}(G, st) \subset \mathrm{Irr}(G)$ is not σ-invariant. Since $\mathcal{E}(G, st) \cap \mathrm{Irr}(b) \neq \varnothing$ by Lemma 3.13, the statement follows in the case $\gamma = \sigma$.

It remains to consider the case $\gamma = \sigma\tau$ where τ is not an inner automorphism of G. Note that every semisimple conjugacy class of G^* is stable under inner-diagonal automorphisms of G^*. (Indeed, $\mathrm{Inndiag}(G^*)$ is induced by the action of $(\tilde{\mathbf{G}}^*)^F$ on G^* for some regular embedding $\mathbf{G}^* \hookrightarrow \tilde{\mathbf{G}}^* = \mathbf{Z}(\tilde{\mathbf{G}}^*)\mathbf{G}^*$.) It follows that τ fixes the rational series $\mathcal{E}(G, st)$ in the preceding paragraph, and hence γ again does not stabilise $\mathcal{E}(G, st)$, which contains characters from b. □

By Propositions 4.4, 4.5 and 4.6, to complete the proof of Theorem 4.1 we may assume S is a central quotient of a group $G = \mathbf{G}^F$ as above for $\mathbf{G}$ simple. We now distinguish two situations:

(4.8.1) p does not divide $|\mathbf{Z}(G)|$, and $G \ncong \mathrm{D}_4(q)$ with $p = 3$

and

(4.8.2) p divides $|\mathbf{Z}(G)|$, or $G \cong \mathrm{D}_4(q)$ with $p = 3$.

Proposition 4.9. *Theorem 4.1 holds in Case (4.8.1).*

Proof. Let $(\mathbf{G}, F)$ be such that S is a central quotient of $G = \mathbf{G}^F$. Let b be a p-block of S of non-trivial defect. Since in Case (4.8.1) the order of $\mathbf{Z}(G)$ is prime to p, we may consider b as a block of G. For the groups in (4.8.1), the only outer automorphisms of p-power order are field automorphisms modulo $J = \mathrm{Inndiag}(\bar{S})$. So by Lemma 4.3, it suffices to prove parts (a) and (b) of Theorem 4.1, with $Q \neq 1$ in the case of 4.1(a), or p divides $|HJ/J|$ in the case of 4.1(b). Then H contains an element γ as in Proposition 4.8. But then Proposition 4.8 shows that the assumptions of 4.1(a), (b) are not satisfied. □

4.3. *The groups in Case (4.8.2).*

Proposition 4.10. *Theorem 4.1 holds for $\bar{S} = \mathrm{O}_8^+(q)$ and $p = 3$.*

Proof. Let b be a 3-block of S. Since $\mathbf{Z}(\mathbf{G})$ has order prime to 3, we may consider b as a block of G and thus assume $S = G$. To prove any part of Theorem 4.1, it suffices to show that $|H/\bar{S}|$, respectively $|K/S|$, is coprime to 3.

Assume the contrary that H, respectively K, contains an element σ that induces an outer automorphism of order 3 of $\bar{S}$. By Proposition 4.8 we may assume σ induces a triality graph automorphism modulo inner-diagonal and field automorphisms. We have $3|(q^2 - 1)$, so $e \in \{1, 2\}$. Write $b = b_G(\mathbf{L}, \lambda)$. Let $\mathbf{L}^* \leq \mathbf{G}^*$ be dual to $\mathbf{L}$. Let $\mathbf{L}_1^* \geq \mathbf{L}^*$ be a maximal (proper) e-split Levi subgroup of $\mathbf{G}^*$. Then $\mathbf{L}_1^*$ is either of type A_3 or A_1^3. If all maximal e-split Levi subgroups above $\mathbf{L}^*$ are of type A_1^3, then $\mathbf{L}^*$ itself must be of type A_1^3. A computation inside the Weyl group shows that the Levi subgroup of type $W(\mathrm{A}_1^3)$ of $W(\mathrm{D}_4)$ has index 2 in its normaliser. Thus, by [11, Lemma 4.16] a defect group of b is $\mathbf{Z}(\mathbf{L})_3^F$, hence cyclic, which was excluded.

Thus we may choose $\mathbf{L}_1^*$ of type A_3. Let t be a generator of the cyclic group $\mathbf{Z}(\mathbf{L}_1^*)_3^F$. Note that there are three $\mathbf{G}^*$-classes of Levi subgroups of

type A_3 in $\mathbf{G}^*$, each of which is fixed by inner-diagonal and field automorphisms, and permuted transitively by σ^*. It follows that the G^*-class of t is not σ^*-invariant. We can now conclude as in the proof of Proposition 4.8 that b contains characters that are not σ-stable. □

We now show an extension of Proposition 4.8 to groups of type A.

PROPOSITION 4.11. *Let $\mathbf{G} = \mathrm{SL}_n$, and assume that p divides $|\mathbf{Z}(\mathbf{G}^F)|$. Let $b = b_G(\mathbf{L}, \lambda)$ be a p-block of $G = \mathbf{G}^F$ with abelian defect, and assume $\mathbf{Z}^\circ(\mathbf{L})_{\Phi_e}$ has rank at least 2. Suppose $F = F_0^p$ for a Frobenius endomorphism F_0 of $\mathbf{G}$, and let σ be the automorphism of G induced by F_0. Then for any inner-diagonal automorphism τ of G, $\mathrm{Irr}(b)$ contains a character that is not $\sigma\tau$-stable and is trivial on $\mathbf{Z}(G)_p$.*

Proof. We have $G = \mathrm{SL}_n(\epsilon q)$ and $G^* = \mathrm{PGL}_n(\epsilon q)$, with $\epsilon \in \{\pm 1\}$. The assumption on $|\mathbf{Z}(\mathbf{G}^F)|$ forces $p|(q-\epsilon)$. First consider the case $\epsilon = 1$. Then $e = 1$. Let $\mathbf{L}^* \leq \mathbf{G}^*$ be dual to $\mathbf{L}$. Up to replacing $\mathbf{L}^*$ by a $\mathbf{G}^{*F}$-conjugate we may assume that F_0 acts on $\mathbf{Z}(\mathbf{L}^*)_{\Phi_1} = \mathbf{Z}(\mathbf{L}^*)$ by the q_0-power map, where $q = q_0^p$. Now $\mathbf{Z}(\mathbf{L}^*)_{\Phi_1}$ has rank at least two, so there exist at least two maximal proper 1-split Levi subgroups $\mathbf{L}_1^*, \mathbf{L}_2^* \geq \mathbf{L}^*$, with $\mathbf{Z}(\mathbf{L}_i^*)$ a 1-torus of $\mathbf{Z}(\mathbf{L}^*)$ of rank 1. Let t_i be a generator of the cyclic Sylow p-subgroup of $\mathbf{Z}(\mathbf{L}_i^*)^F$. Then preimages $\tilde{t}_i$ of t_i in $\mathrm{GL}_n(q)$ (under the natural quotient map) can be taken as diagonal matrices where $\tilde{t}_i$ has m_i eigenvalues ζ, a primitive kth root of unity with $k = |\mathbf{Z}(\mathbf{L}_i^*)^F|_p = |t_i|$ (a p-power), and all other eigenvalues 1, with $1 \leq m_1 < m_2 < n$ say. We have $t_i^\sigma = F_0(t_i) = t_i^{q_0}$, so t_i is not σ-stable.

Let a_1, a_2 be integers not both divisible by p such that

$$m_1 a_1 + m_2 a_2 \equiv 0 \pmod{k}.$$

Then $\tilde{t} := \tilde{t}_1^{a_1} \tilde{t}_2^{a_2}$ has determinant 1, so lies in $\mathrm{SL}_n(q)$, whence

$$t := t_1^{a_1} t_2^{a_2} \in [G^*, G^*].$$

By [46, Prop. 4.5] this means that any character in $\mathcal{E}(G, st)$, for $s \in L^*$ a semisimple p'-element has $\mathbf{Z}(G)_p$ in its kernel. Furthermore, the class of t is not σ-stable. Now $\tilde{t}$ has eigenvalues $\zeta^{a_1+a_2}, \zeta^{a_2}$ and 1, and at least two of those are distinct by the choice of a_1, a_2. Thus, either

$$\mathbf{C}_{\mathrm{GL}_n(q)}(\tilde{t}) = \mathrm{GL}_{n_1}(q) \times \mathrm{GL}_{n-n_1}(q)$$

for a suitable $1 \leq n_1 < n$, or

$$\mathbf{C}_{\mathrm{GL}_n(q)}(\tilde{t}) = \mathrm{GL}_{m_1}(q) \times \mathrm{GL}_{m_2-m_1}(q) \times \mathrm{GL}_{n-m_2}(q).$$

All of these have automiser of order at most 2 unless $m_1 = m_2 - m_1 = n/3$ when it is $\mathfrak{S}_3$. Assume we are in the latter case and $p = 3$. If $\mathbf{L}^*$ itself is of this form, then the defect groups of b are non-abelian, being a non-trivial extension of $\mathbf{Z}(\mathbf{L}^*)_3^F$ with a Sylow 3-subgroup of the automiser of $\mathbf{L}^F$ [11, Lemma 4.16].

Otherwise, $\mathbf{L}^*$ has strictly smaller rank, and then we may choose the $\mathbf{L}_i^*$ such that the centraliser of $\tilde{t}$ is not of this special form. We may then complete the argument as in the proof of Proposition 4.8.

If $\epsilon = -1$ (so G is unitary), we may argue in an entirely similar fashion. □

The proof actually shows that the assumption on abelian defect groups can be dropped when either $p \geq 5$, or $p = 3$ and in addition $\mathbf{L}$ is not of type $\mathrm{A}^3_{n/3-1}$.

PROPOSITION 4.12. *Theorem 4.1 holds for* $\bar{S} = \mathrm{L}_n(\epsilon q)$, $\epsilon \in \{\pm 1\}$ *and* $p \mid (q - \epsilon)$.

Proof. By our previous reductions and Proposition 4.9 we may assume S is a central quotient of $G = \mathrm{SL}_n(\epsilon q)$. Let b be a p-block of S with abelian non-cyclic defect and $\tilde{b}$ the p-block of G dominating it. Then by the proof of [2, Th. 5], one of the following holds for any block B of $\mathrm{GL}_n(\epsilon q)$ lying above $\tilde{b}$:

(4.12.1) B has abelian defect groups; or

(4.12.2) $p = 3$, $n = 3m$, and $\mathrm{Irr}(B) \subseteq \mathcal{E}_3(\mathrm{GL}_n(\epsilon q), s)$, with $C := \mathbf{C}_{\mathrm{GL}_n(\epsilon q)}(s) \cong \mathrm{GL}_3((\epsilon q)^m)$ for some semisimple $3'$-element $s \in \mathrm{GL}_n(\epsilon q)$, and $((\epsilon q)^m - 1)_3 = 3$.

Now first assume that H or K induce an automorphism $\gamma = \sigma\tau$, where σ is a field automorphism of S of order p, and $\tau \in J = \mathrm{Inndiag}(S)$. That is, σ is induced by a field automorphism F_0 of $\mathbf{G}$, where after conjugation we may assume $F = F_0^p$. As $p|(q - \epsilon)$ and q is a pth power, in fact we have $p^2|(q - \epsilon)$, and so we are in case (4.12.1). Also note that if $\tilde{b} = b_G(\mathbf{L}, \lambda)$, then $\mathbf{Z}^\circ(\mathbf{L})_{\Phi_e}$ has rank at least 2 since $\tilde{b}$ has non-cyclic defect. But then by Proposition 4.11 the block $\tilde{b}$ contains an irreducible character of G not fixed by γ and trivial on $\mathbf{Z}(G)_p$, hence a character of b. Thus the assumptions of any of the parts (a), (b), and (c) in Theorem 4.1 are not satisfied.

So we may now assume that the p-elements in H and K only induce diagonal automorphisms. In particular, this establishes 4.1(b) and shows that K/S is cyclic. Hence by Lemma 4.3 it suffices to prove 4.1(a); moreover, using Lemma 4.2 we may assume $H/\bar{S}$ is a p-group contained in $J/\bar{S}$.

Suppose we are in the situation of (4.12.1). Since $\mathrm{GL}_n(\epsilon q)$ induces all diagonal automorphisms on G and thus on S and since $H/\bar{S}$ is a cyclic p-group, we can find a p-element $g \in \mathrm{GL}_n(\epsilon q)$ such that $G_1 := \langle G, g \rangle$ induces H while acting on S. Furthermore, $G \lhd G_1 \leq \mathrm{GL}_n(\epsilon q)$ and G_1/G is a p-group. Since every character in b is H-invariant, $\tilde{b}$ is G_1-invariant, and since the defect groups of any G_1-block lying above $\tilde{b}$ are abelian, the statement follows from Proposition 2.4.

Finally we consider the situation of (4.12.2). Recall our hypothesis that the defect group D of b is abelian but non-cyclic. As shown in the proof of [2, Th. 5], $|D| \le 3^2$, whence $|D| = 3^2$. On the other hand, B has defect groups of order 3^4, and $(q-\epsilon)_3 = 3$, so the defect groups of $\tilde{b}$ have order (at least) 3^3. It follows that S is a quotient of G by a central subgroup Z of order z divisible by 3. By Lemma 4.2 we may assume $|H/\bar{S}| = (q-\epsilon)_3 = 3$, and that H is induced by the conjugation action of $G_1 := \langle G, g\rangle \le \mathrm{GL}_n(\epsilon q)$ for some 3-element g. As G_1/G is a 3-group, $\tilde{b}$ lies under a unique 3-block B_1 of G_1 and so B_1 lies under B. Since BHZ holds for b by [2], every character $\theta \in \mathrm{Irr}(b)$ has height zero, so the 3-part of its degree is

$$d := |\mathrm{L}_n(\epsilon q)|_3/3^2 = |\mathrm{GL}_n(\epsilon q)|_3/3^4.$$

By assumption, θ is g-invariant, so, viewed as G-character, it extends to a character of G_1, which is still trivial at Z. Thus the degrees of all characters in $\mathrm{Irr}(B_1)$ that are trivial at Z have 3-part d.

By [2, Th. 1], $\mathrm{Irr}(B)$ consists of all characters in $\mathcal{E}(\mathrm{GL}_n(\epsilon q), st)$ for t a 3-element in $C = \mathrm{GL}_3((\epsilon q)^m)$. As θ is trivial at Z and lies under some such character, using [46, Prop. 4.5] we see that the order of $\det(st)$ divides $(q-\epsilon)/z$. But s is $3'$ and t is a 3-element, so the order of $\det(s)$ divides $(q-\epsilon)/z$. Let $\omega \in \mathbb{F}_{q^2}^\times$ be of order 3, and consider

$$t := \mathrm{diag}(1, \omega, \omega^2) \in \mathrm{SL}_3((\epsilon q)^m) < C.$$

Then t centralises s and, as $\det(t) = 1$, $\det(st) = \det(s)$ has order dividing $(q-\epsilon)/z$. Again by [46, Prop. 4.5], the semisimple character $\chi_{st} \in \mathcal{E}(\mathrm{GL}_n(\epsilon q), st)$ is trivial at Z. Any irreducible constituent ψ of the restriction of χ_{st} to G_1 is also trivial at Z. By uniqueness of B_1, we have $\psi \in \mathrm{Irr}(B_1)$. Since G_1 has $3'$-index in $\mathrm{GL}_n(\epsilon q)$,

$$\begin{aligned}\psi(1)_3 &= \chi_{st}(1)_3 = |\mathrm{GL}_n(\epsilon q)|_3/|\mathbf{C}_C(t)|_3\\ &= |\mathrm{GL}_n(\epsilon q)|_3/((\epsilon q)^m - 1)_3^3 = |\mathrm{GL}_n(\epsilon q)|_3/3^3 = 3d,\end{aligned}$$

and this is a contradiction. □

PROPOSITION 4.13. *Let $\bar{S} = \mathrm{E}_6(\epsilon q)$, $\epsilon \in \{\pm 1\}$ with $3|(q-\epsilon)$. Assume that BHZ for $p = 3$ holds for all groups of order smaller than $|\bar{S}|$. Then Theorem 4.1 holds for $\bar{S}$ at $p = 3$.*

Proof. Let b be a 3-block of S with non-cyclic abelian defect groups, and let $\hat{b} = b_G(\mathbf{L}, \lambda)$ be the block of $G = \mathbf{G}^F$ dominating it. So $b = \hat{b}$ if and only if $S = G$. Again we first show that field automorphisms of order 3 modulo $J := \mathrm{Inndiag}(\bar{S})$ do not point-wise fix all irreducible characters in b. First let $\epsilon = 1$, so $G = \mathrm{E}_6(q)$ and $e = 1$. Set $Z := (\mathbf{Z}^\circ(\mathbf{L}^*)^F)_3$, a 3-group of rank at least 2. Assume $F = F_0^3$, and accordingly write $q = q_0^3$. Now F_0 acts by $x \mapsto x^{q_0}$ on Z. Let $\mathbf{L}_i^*$, $i = 1, 2$, be two distinct maximal 1-split Levi subgroups

of $\mathbf{G}^*$ containing the dual Levi $\mathbf{L}^* \leq \mathbf{G}^*$ of $\mathbf{L}$ and $t_i \in \mathbf{Z}(\mathbf{L}_i^*)_3^F \leq Z$ generators of the Sylow 3-subgroups. Let $a_i \in \mathbb{Z}$ not both divisible by 3 be such that $t := t_1^{a_1} t_2^{a_2} \in [G^*, G^*]$. Then we have $F_0(t) = t^{q_0} \neq t$. Note that t has order divisible by 9.

Now $\mathbf{L}' := \mathbf{C}_{\mathbf{G}^*}(t)$ is either a maximal proper 1-split Levi subgroup of $\mathbf{G}^*$ or equal to the intersection $\mathbf{L}_1^* \cap \mathbf{L}_2^*$, a Levi subgroup of semisimple rank 4. Assume for a moment that $\mathbf{L}'$ does not have type D_4. Now $\mathbf{N}_{G^*}(\langle t \rangle)$ normalises $\mathbf{C}_{\mathbf{G}^*}(t) = \mathbf{L}'$. But the automiser of the latter does not contain elements of order 3 by Lemma 4.7, respectively by inspection. Thus, t is not $\mathbf{G}^{*F}$-conjugate to $F_0(t)$, and we may complete the argument as in Proposition 4.11 to see that $\mathrm{Irr}(\hat{b})$ contains characters trivial on $\mathbf{Z}(\mathbf{G})^F$ in Lusztig series not fixed by F_0. If $\mathbf{L}'$ has type D_4, its automiser is the symmetric group of degree 3. By a computation in the reflection representation of the Weyl group using [57], there are generators x, y of the Sylow 3-subgroup of $\mathbf{Z}(\mathbf{L}')^F$ such that an element w of order 3 in the automiser acts by $x \mapsto y \mapsto (xy)^{-1}$. (Note that $\mathbf{Z}(\mathbf{L}')^F$ lies in a maximally split torus on which the Weyl group naturally acts.) Then $t := x^w x^{-1} = x^{-1}y$ is a 3-element in $[G^*, G^*]$. Now w acts by $t^w = x^{-1}y^{-2}$, while F_0 sends every element in $\mathbf{Z}(\mathbf{L}')$ to its qth power. So again, t is not G^*-conjugate to $F_0(t)$, and we conclude as before.

The above result already establishes 4.1(b) and shows that K/S is cyclic in 4.1(c). By Lemma 4.3, it remains to prove 4.1(a), and we may now assume H only induces diagonal automorphisms of S, and in fact $H = G^*$ using Lemma 4.2. First assume $S = G$. Let $\mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}$ be a regular embedding, and let $\tilde{G} := \widetilde{\mathbf{G}}^F$. Then any defect group of b is contained in a maximally split torus of G, and thus defect groups of any block $\tilde{b}$ of $\tilde{G}$ above b are contained in a maximally split torus of $\tilde{G}$ (see [26, Th. 1.2(b)] for quasi-isolated blocks and [11, Lemma 4.16] for the others) and so are abelian. Since $\tilde{G}$ induces all diagonal automorphisms of G, we are done by applying Proposition 2.4 (and arguing as in case (4.12.1) of the proof of Proposition 4.12).

Finally, assume $S = G/\mathbf{Z}(G)$ is the simple group (which is isomorphic to $[G^*, G^*] \leq G^* = H$ as duality keeps the root system of type E_6). Consider the block $\tilde{b}$ of G^* covering b. If $\tilde{b}$ has abelian defect groups, then the statement again follows from Proposition 2.4. Assume $\tilde{b}$ has a non-abelian defect group $\tilde{D}$, and is not quasi-isolated. Then its Morita-equivalent Jordan correspondent block $\tilde{b}_1$, of a group of order less than $|G^*|/3 = |\bar{S}|$, also has non-abelian defect by [5, Th. 1.1]. By assumption, $\tilde{b}_1$ satisfies BHZ, so contains a character of positive height. But then so does $\tilde{b}$ since the Bonnafé–Rouquier Morita equivalence [6] preserves heights. Also by assumption, b has abelian defect, and hence all characters in $\mathrm{Irr}(b)$ possess height zero by the main result of [26]. Now the existence of characters of positive height in $\tilde{b}$ implies that G^* does not fix all characters in $\mathrm{Irr}(b)$. On the other hand, if $\tilde{b}$ is quasi-isolated then

defect groups of b are non-abelian or cyclic by [26, Prop. 4.3 and Th. 1.2(b)] and [17, pp. 353–354].

For the twisted type groups, that is $\epsilon = -1$, entirely similar considerations apply. □

COROLLARY 4.14. *Let p be an odd prime, and let S be a quasi-simple group. Let $\bar{S} := S/\mathbf{Z}(S)$, $\bar{S} \leq H \leq \mathrm{Aut}(S)$, and assume $\mathbf{O}^{p'}(H/\bar{S}) = H/\bar{S}$. If b is a p-block of S with abelian, non-cyclic defect groups such that every character in b is H-invariant, then $H/\bar{S}$ is a cyclic p-group.*

Proof. The statement follows from Proposition 2.2(b) unless $\bar{S}$ is of type A_n, ${}^2\mathrm{A}_n$, or $p = 3$ and $\bar{S}$ is of type D_4 or $\mathrm{E}_6(\epsilon q)$ with $3|(q-\epsilon)$ and $\epsilon \in \{\pm 1\}$. In the latter cases, for $J := \mathrm{Inndiag}(S)$, we have $p \nmid |HJ/J|$ by Propositions 4.9, 4.10, 4.12, and 4.13. By assumption, $H/\bar{S}$, and hence HJ/J, has no non-trivial p'-quotient. Thus $H \leq J$, and the claim follows from Proposition 2.2(c). □

5. **Proofs of Theorems A and C**

5.1. *Proof of Theorem* A. Now we proceed to prove the "only if" implication of Brauer's Height Zero Conjecture for primes $p > 2$. Suppose B is a p-block of G with defect group D, and assume all $\chi \in \mathrm{Irr}(B)$ have height zero. We want to show that D is abelian. We will assume that G is a counterexample to BHZ, first with $|G/\mathbf{Z}(G)|$ smallest possible, and then with $|G|$ smallest possible. By the Gluck–Wolf theorem [21], G is not p-solvable. Recall that if $H \leq G$ and $N \trianglelefteq G$, then $|H : \mathbf{Z}(H)| \leq |G : \mathbf{Z}(G)|$ and $|G/N : \mathbf{Z}(G/N)| \leq |G : \mathbf{Z}(G)|$.

Step 1. B is a quasi-primitive block; that is, if $N \trianglelefteq G$ and e is a block of N covered by B, then e is G-invariant.

This follows by Fong–Reynolds ([43, Th. 9.14]) and induction.

Step 2. If N is a proper normal subgroup of G, then $D \cap N$ is abelian. In particular, $\mathbf{O}^{p'}(G) = G$, D is not contained in any proper normal subgroup of G, and $Q := \mathbf{O}_p(G)$ is abelian.

Suppose N is a proper normal subgroup of G and e is a p-block of N covered by B. By [43, Th. 9.26], we have $D \cap N$ is a defect group of e, using that e is G-invariant (by Step 1). Let $\xi \in \mathrm{Irr}(e)$. By [43, Th. 9.4], there is some $\chi \in \mathrm{Irr}(B)$ (of height zero) lying over ξ. By [41, Lemma 2.2], we have that ξ has height zero. Hence $D \cap N$ is abelian by minimality of G.

Now, if $\mathbf{O}^{p'}(G) < G$, then $D = D \cap \mathbf{O}^{p'}(G)$ is abelian by the previous claim applied to $N = \mathbf{O}^{p'}(G)$, a contradiction. Finally, since $Q = \mathbf{O}_p(G) < G$ and $Q \leq D$, the claim applied to $N = Q$ shows that $Q = Q \cap D$ is abelian.

Now let

$$C := \mathbf{C}_G(Q),$$

so that $Q \leq C$.

Step 3. We have that $\mathbf{Z}(G)$ has p'-order. In particular, we may assume that $C < G$ or that $Q = 1$. Also, if $L \trianglelefteq G$ is a non-trivial p-group, then D/L is abelian.

Suppose that $1 < L$ is a normal p-subgroup of G. Then there is a block $\bar{B}$ of G/L contained in B with defect group D/L by [43, Th. 9.9(b)]. By the definition of heights, all irreducible characters in $\bar{B}$ have height zero. Thus D/L is abelian, by BHZ applied to G/L.

Assume now that $1 < L$ is a central p-subgroup. Let $\delta \in \mathrm{Irr}(D)$, and let $\nu \in \mathrm{Irr}(L)$ be under δ. Let $\chi \in \mathrm{Irr}(B)$ over ν. Since χ has height zero, it follows that ν extends to some linear $\tilde{\nu} \in \mathrm{Irr}(D)$, by [40, Th. 4.4], using that ν is G-invariant. By Gallagher's Corollary 6.17 of [25], we have $\delta = \tilde{\nu}\beta$ for some $\beta \in \mathrm{Irr}(D/L)$. Since D/L is abelian by the previous paragraph, we conclude that $\delta(1) = 1$. Hence D is abelian, contrary to the choice of G. Hence $p \nmid |\mathbf{Z}(G)|$.

Step 4. If $Z := \mathbf{O}_{p'}(G)$, then $Z = \mathbf{Z}(G)$ and Z is cyclic.

Let $\theta \in \mathrm{Irr}(Z)$ such that the p-block $\{\theta\}$ of Z is covered by B. We know that θ is G-invariant. We prove this step using the language of θ-blocks and character triples (see [51].) We have that (G, Z, θ) is a character triple. By Problems 8.12 and 8.13 of [43], there exists an ordinary-modular character triple (G^*, Z^*, θ^*) isomorphic to (G, Z, θ), which we can construct as in [51, Th. 3.4]. Hence Z^* has order not divisible by p and is central in G^*. Since G/Z is isomorphic to G^*/Z^*, we have that $Z^* = \mathbf{O}_{p'}(G^*)$. If

$$\sigma : \mathrm{Irr}(G|\theta) \to \mathrm{Irr}(G^*|\theta^*)$$

denotes the associated standard bijection, we have $\sigma(\mathrm{Irr}(B)) = \mathrm{Irr}(B^*)$ for a unique p-block B^* of G^*. In particular, $\mathrm{Irr}(B) = \mathrm{Irr}(B|\theta)$ is a θ-block. Since $\sigma(\chi)(1) = \chi(1)/\theta(1)$, then $\sigma(\chi)(1)_p = \chi(1)_p$ for $\chi \in \mathrm{Irr}(B)$, and so all characters in $\mathrm{Irr}(B^*)$ have the same height (zero).

Let D^* be a defect group for B^*. We show next that D and D^* are isomorphic, hence if D^* is abelian, then so is D. Since

$$|G : D|_p = \chi(1)_p = \sigma(\chi)(1)_p = |G^* : D^*|_p$$

and $|G|_p = |G^*|_p$, we have that $|D| = |D^*|$. If D_θ/Z is a θ-defect group of B, then, by [51, Def. 4.1], we have $D_\theta/Z \cong D^*Z^*/Z^* \cong D^*$. By [51, Th. 5.1], we have that $D_\theta/Z \leq DZ/Z$, replacing D by a G-conjugate, if necessary. By comparing orders, we have that $D_\theta/Z = DZ/Z$ is isomorphic to D. Therefore D and D^* are isomorphic.

Notice that $|G^* : \mathbf{Z}(G^*)| \leq |G^* : Z^*| = |G : Z| \leq |G : \mathbf{Z}(G)|$, using that $\mathbf{Z}(G) \leq Z$ (by Step 3). Therefore, if $|G^* : \mathbf{Z}(G^*)| < |G : \mathbf{Z}(G)|$, then we are done by applying BHZ to G^*. In the case of equality, we have $Z = \mathbf{Z}(G)$. Finally, we show that Z is cyclic. Let $\{\lambda\}$ be the block of Z covered by B,

where $\lambda \in \mathrm{Irr}(Z)$. Let $K = \ker(\lambda)$. Hence, K is contained in $\ker(\chi)$ for all $\chi \in \mathrm{Irr}(B)$. If $K > 1$, then we apply [43, Th. 9.9(c)] and BHZ to G/K. The choice of G shows that $K = 1$ and therefore, that Z is cyclic.

From now on, let $\mathbf{E}(X)$ denote the layer of a finite group X.

Step 5. We have that C is not p-solvable and $\mathbf{E}(G) \neq 1$.

Assume that C is p-solvable. As $\mathbf{O}_{p'}(C) = Z$ is central, we have $\mathbf{O}_p(C/Z) = QZ/Z$. Since C is p-solvable, we have $\mathbf{C}_{C/Z}(QZ/Z) = QZ/Z$. This implies $C = QZ$. By [45, Lemma 3.4], then $\mathrm{Irr}(B) = \mathrm{Irr}(G|\lambda)$ for some $\lambda \in \mathrm{Irr}(Z)$. Let $P \in \mathrm{Syl}_p(G)$, and let $\hat{\lambda} = 1_P \otimes \lambda \in \mathrm{Irr}(P \times Z)$. Then $\hat{\lambda}^G$ has p'-degree, and therefore it contains some p'-degree irreducible constituent χ. Now, χ lies over λ, and therefore $\chi \in \mathrm{Irr}(B)$. We conclude that P is a defect group of B. By hypothesis, p does not divide $\tau(1)$ for every $\tau \in \mathrm{Irr}(B)$, whence P (and hence D) is abelian by the main result of [46]. Since G is a minimal counterexample, C is not p-solvable.

Next, assume that $\mathbf{E}(G) = 1$. Then $\mathbf{E}(C) = 1$ and

$$\mathbf{F}^*(C) = \mathbf{F}(C) = \mathbf{O}_{p'}(\mathbf{F}(C)) \times \mathbf{O}_p(C).$$

Now, $\mathbf{O}_p(G) = Q \leq \mathbf{O}_p(C) \leq \mathbf{O}_p(G)$, whence $\mathbf{O}_p(C) = Q$ is central in C, and $\mathbf{O}_{p'}(\mathbf{F}(C)) \leq \mathbf{O}_{p'}(G) = Z$ is central in G. It follows that

$$C = \mathbf{C}_C(\mathbf{F}^*(C)) \leq \mathbf{F}^*(C),$$

and so C is p-solvable, a contradiction to the previous conclusion.

Step 6. Let S be a component of G, and let N be the normal subgroup of G generated by the G-conjugates of S. Let e be the block of N covered by B, and let b be the only block of S that is covered by e. Then e is not nilpotent. In particular, there are $\alpha, \beta \in \mathrm{Irr}(b)$ with different degrees.

First notice that b is the only block covered by e, because N is a central product of the different G-conjugates of S. In particular, every irreducible character of S is N-invariant. Suppose that e is nilpotent. Let $N_1 = N\mathbf{Z}(G)$, and let e_1 be the unique block of N_1 covered by B and covering b. It is clear that e_1 is nilpotent, using the definition of nilpotent blocks. (See [53, Lemma 7.5].) We have that $D_1 := D \cap N_1$ is a defect group of e_1 (see [43, Th. 9.26]). By Theorem 2.6, there is a finite group L' with $|L' : \mathbf{Z}(L')| < |G : \mathbf{Z}(G)|$, where L' has a block B' with defect group D and such that all the irreducible characters of B' have height zero. Therefore D is abelian by BHZ applied to L'. Thus e is not nilpotent.

If all the irreducible characters in b have the same degree, then all the irreducible characters in e, a central product of the G-conjugates of b, also have the same degree. Then by [48, Prop. 1 and Th. 3], we have that $D \cap N$ is abelian with inertial index one. By Broué–Puig [9, 1.ex.3], the block e is nilpotent, and we are again done.

Step 7. Let S be any component of G. Suppose B' is the $\mathbf{E}(G)$-block covered by B and b is the S-block covered by B'. Then b has non-cyclic defect groups. In particular, no component of G has cyclic Sylow p-subgroups.

By Step 6, b has non-central defect groups (since any block with central defect groups is nilpotent, by [9, 1.ex.1]). Suppose that b has cyclic defect groups. Consider the central product N of the different G-conjugates of S, so that $N \trianglelefteq G$. Let $D_1 := D \cap N$, so that $G = \mathbf{N}_G(D_1)N$. Let b_0 be the block of N covered by B. Let b_1 be the Brauer first main correspondent of b_0, which is the block of $\mathbf{N}_N(D_1)$ that induces b_0. Let B_1 be the unique block of $\mathbf{N}_G(D_1)$ that covers b_1 and induces B (by the Harris–Knörr Theorem 9.28 of [43]), which is a block with defect group D. By Theorem 2.7, we conclude that all the irreducible characters of B_1 have height zero. If $\mathbf{N}_G(D_1) < G$, by BHZ applied to $\mathbf{N}_G(D_1)$ we will have that D is abelian, a contradiction. Hence $D_1 \trianglelefteq G$. In this case, $D_1 \leq Q$. Since $[Q, N] = 1$, we conclude that $D_1 \leq \mathbf{Z}(N)$, and b_0 is nilpotent, in contradiction with Step 6.

Step 8. All components of G are normal in G.

Suppose that S_1 is a non-normal component, and write the normal subgroup N in Step 6 as $N = S_1 * S_2 * \cdots * S_m$, a central product of $m > 1$ components, where G/N permutes $S_1, \ldots, S_m$ transitively. In particular, $S_1, \ldots, S_m$ are isomorphic to each other, and we fix an isomorphism between S_1 and any S_i.

Again by Step 1, B covers a unique block e of N that is G-invariant, and e is then the central product of blocks b_i of S_i, $1 \leq i \leq m$. By Step 6, each $\mathrm{Irr}(b_i)$ contains characters of different degrees. Furthermore, as before, the block e has defect group $D \cap N$, which is abelian by Step 2. It follows that b_i has abelian defect groups, which are non-cyclic by Step 7. By Theorem 3.1, $\mathrm{Irr}(b_i)$ contains at least three $\mathrm{Aut}(S_i)$-orbits, say of α_i, β_i, and γ_i. For each i and given the fixed isomorphism between S_1 and S_i, we can view α_1, β_1, and γ_1 as S_i-characters. Relabeling α_i, β_i, and γ_i if necessary, we may assume that

$$\begin{aligned} &\alpha_i \text{ is not } \mathrm{Aut}(S_i)\text{-conjugate to } \beta_1 \text{ or } \gamma_1, \\ &\beta_i \text{ is not } \mathrm{Aut}(S_i)\text{-conjugate to } \alpha_1 \text{ or } \gamma_1, \text{ and} \\ &\gamma_i \text{ is not } \mathrm{Aut}(S_i)\text{-conjugate to } \alpha_1 \text{ or } \beta_1. \end{aligned} \tag{5.1.1}$$

Since $m > 1$, the (transitive) permutation action of G on the set $\{S_1, \ldots, S_m\}$ is non-trivial, and so the kernel K of this action is a proper normal subgroup of G containing N, and from Step 2 we have $\mathbf{O}^{p'}(G/K) = G/K$. Now applying Theorem 2.1 to G/K we obtain a partition

$$\{S_1, \ldots, S_m\} = \Delta_1 \sqcup \Delta_2 \sqcup \Delta_3$$

such that $\cap_{i=1}^{3} \mathrm{Stab}_{G/N}(\Delta_i)$ has index divisible by p in G/N. Setting

$$\theta := \theta_1 \otimes \theta_2 \otimes \cdots \otimes \theta_m,$$

where $\theta_i = \alpha_i$ if $S_i \in \Delta_1$, $\theta_i = \beta_i$ if $S_i \in \Delta_2$, and $\theta_i = \gamma_i$ if $S_i \in \Delta_3$, we then see that B covers the N-block of θ.

Consider any $g \in G$ that fixes θ, and suppose that $S_1^g = S_j$. Note that $\theta|_{S_j}$ is a multiple of θ_j, and $\theta^g|_{S_j}$ is a multiple of α_1^g. Again using the fixed isomorphism $S_1 \cong S_j$ and writing every x^g with $x \in S_1$ as x^σ for a suitable $\sigma \in \mathrm{Aut}(S_j)$, we can write α_1^g as α_1^σ, and so $\theta_j = \alpha_1^\sigma$ and thus α_1 and θ_j are $\mathrm{Aut}(S_j)$-conjugate. Since neither β_j nor γ_j are $\mathrm{Aut}(S_j)$-conjugate to α_1 by (5.1.1), θ_j must be α_j, which means $S_j \in \Delta_1$ by the choice of the θ_i's. This argument, applied to any S_i, shows that g stabilises the partition $\Delta_1 \sqcup \Delta_2 \sqcup \Delta_3$. Thus

$$G_\theta/N \leq \bigcap_{i=1}^{3} \mathrm{Stab}_{G/N}(\Delta_i),$$

and so p divides $|G : G_\theta|$. But this contradicts Proposition 2.5.

Step 9. If S is a (normal by Step 8) component of G, then $\mathbf{Z}(S)$ is of p'-order. Also, if D is any defect group of B and $R := D \cap S$, then $G = S\mathbf{C}_G(R)$, and $[D, R] = 1$. Furthermore, every $\alpha \in \mathrm{Irr}(b)$ (where b is the S-block covered by B) is G-invariant, and extends to DS. Moreover, $G/S\mathbf{C}_G(S)$ is a p-group.

First we prove that there is a defect group D of B satisfying that

$$[D, D \cap S] = 1$$

and $G = S\mathbf{C}_G(S \cap D)$. Let b be the block of S covered by B. Notice that $\mathbf{C}_G(S) \leq K := G[b] \trianglelefteq G$, the Dade group, by the definition of $G[b]$. (See [42], for instance.) Also, notice that $R = D \cap S$ is a defect group of b, and that $G = S\mathbf{N}_G(R)$, by the Frattini argument. Let b_R be a block of $\mathbf{C}_S(R)$ with defect group R inducing b, and let $T = \mathbf{N}_G(R)_{b_R}$. By Lemma 2.3, there is a defect group D^* of B such that $D^* \cap S = R$ and $D^* \leq T$. For the sake of notation, we assume that $D^* = D$. Let $H := G/\mathbf{C}_G(S)$ and $\bar{S} = S\mathbf{C}_G(S)/\mathbf{C}_G(S)$. By Proposition 2.2, we have that $G/S\mathbf{C}_G(S)$ has a normal p-complement $U/S\mathbf{C}_G(S)$. Since $|G : U|$ is a power of p, and B covers a unique block of U, we have $G = UD$, by [43, Th. 9.17]. Now, every $\tau \in \mathrm{Irr}(b)$ has the property that $|G : G_\tau|$ is coprime to p, by Proposition 2.5. Let $x \in D$. Let $W = U\langle x \rangle$. Since $|G : G_\tau|$ is p', then $G = UG_\tau$, so $|W : W_\tau|$ is also p' for every $\tau \in \mathrm{Irr}(b)$. So the hypotheses of Theorem 4.1(a) are satisfied with the group $W/\mathbf{C}_G(S)$ and every p-element $x \in D$. Moreover, by the choice of G, BHZ holds for all finite groups X with $|X/\mathbf{Z}(X)| < |G/\mathbf{Z}(G)|$, in particular for all groups of order less than $|S/\mathbf{Z}(S)|$. We conclude by Theorem 4.1(a) that $[x, R] = 1$ for $x \in D$. Hence $D \leq S\mathbf{C}_G(R) \trianglelefteq G$. If $S\mathbf{C}_G(R) < G$, we are done by Step 2.

Now, let $D_1 := D \cap K$. By [42, Th. 3.5(ii)], we have $D_1 = R\mathbf{C}_D(R) = RD$, and we conclude that $D = D_1$. Hence $K = G$ by Step 2. (Recall that K is normal in G.) Therefore every $\alpha \in \mathrm{Irr}(b)$ is G-invariant (by [29, Lemma 3.2(a)], for instance). By Corollary 4.14, we have that $H/\bar{S}$ is a p-group. Thus $G/S\mathbf{C}_G(S)$ is a p-group.

Let $\theta \in \mathrm{Irr}(b)$. Then θ lies under some $\chi \in \mathrm{Irr}(B)$, which by hypothesis has height zero. By [40, Th. 4.4], we have that θ extends to SD^g for some $g \in G$. Therefore $\theta = \theta^{g^{-1}}$ extends to SD, as claimed.

Now, we prove that if $g \in G$, then $[D^g, D^g \cap S] = 1$ and $G = S\mathbf{C}_G(D^g \cap S)$. Indeed, since S is normal in G, we have that $[D^g, D^g \cap S] = [D, D \cap S]^g = 1$. The second part follows because $D^g \cap S$ is a defect group of b, and therefore $D^g \cap S = R^s$ for some $s \in S$. Thus $G = S\mathbf{C}_G(R) = S\mathbf{C}_G(R^s)$.

Suppose $1 < Z_p$ is a Sylow p-subgroup of $\mathbf{Z}(S)$. By Step 3, D/Z_p is abelian. Since $Z_p \leq \mathbf{O}_p(S) \leq D \cap S$, we have that SD/S is abelian. In this case, the block $\tilde{b}$ of SD that covers b has defect group D (by [43, Probl. 9.4], using that b is G-invariant) and, as we saw, all irreducible characters of b extend to SD. Using that and the fact that SD/S is abelian, it follows that all irreducible characters in $\tilde{b}$ restrict irreducibly to S. In this case, we easily check that all the irreducible characters in $\tilde{b}$ have height zero. If $SD < G$, then we are done by minimality of G. So we may assume that $SD = G$. Since $[D, R] = 1$, we have that $Z_p \leq \mathbf{Z}(G)$. By Step 3, this is not possible. We conclude that $Z_p = 1$ and thus $p \nmid |\mathbf{Z}(S)|$.

Final Step. From Step 9, we conclude that

$$G/K \text{ is a } p\text{-group}, \tag{5.1.2}$$

where

$$K := \bigcap_{S \text{ component of } G} S\mathbf{C}_G(S) = \mathbf{E}(G)\mathbf{C}_G(\mathbf{E}(G)) = E\mathbf{C}_G(E),\ E := \mathbf{E}(G)Z.$$

We also have $F = EQ$ for

$$F := \mathbf{F}^*(G) = \mathbf{F}(G)\mathbf{E}(G) = (Q \times Z) * \mathbf{E}(G).$$

If $F = G$, then G is a central product of an abelian group with quasi-simple groups. However, in this case BHZ holds for G by the quasi-simple case [28], using Corollary 2.9. Hence we may assume $F < G$.

Next we show that

$$K/F \text{ is a } p'\text{-group}. \tag{5.1.3}$$

Indeed, let $T := \mathbf{C}_G(E)$, which contains Q. Let c_1 be the block of T covered by B, using Step 1. By minimality of G, c_1 has an abelian defect group D_1. Note that

$$\mathbf{C}_T(Q) = \mathbf{C}_G(F) \leq F \cap T = QZ. \tag{5.1.4}$$

Hence, by [45, Lemma 3.4] we have that $c_1 = \mathrm{Irr}(T|\lambda)$ for the character $\lambda \in \mathrm{Irr}(Z)$ that lies under B. By inducing $1_{P_1} \times \lambda \in \mathrm{Irr}(P_1 Z)$ to T, where $P_1 \in \mathrm{Syl}_p(T)$ contains Q, we see that there is some p'-degree irreducible character ν of T over λ. As ν lies in c_1, c_1 has maximal defect, and hence $D_1 \in \mathrm{Syl}_p(T)$.

Hence the abelian p-group D_1 centralises Q, and so $D_1 \in \mathbf{C}_T(Q) = QZ$, i.e., $D_1 = Q$, proving (5.1.3).

By Step 1, there is a unique block e of E and a unique block f of F covered by B; in particular, e is covered by f. Let D be any defect group of B. Then $D \cap F$ is abelian by Step 2 (since $F < G$). Let

$$R := D \cap E,$$

so that R is a defect group of the E-block e. Write

$$E = Z * S_1 * \cdots * S_n,$$

where S_i, $1 \le i \le n$, are the components of G, which all are normal in G by Step 8. Since $D_i := D \cap S_i$ is a defect group of the unique S_i-block b_i covered by e, by Corollary 2.9 we have $R = D_1 \times \cdots \times D_n$ (being a direct product since $\mathbf{Z}(E) = Z$ is a p'-group). By Step 9, we have that $[D, R] = 1$. Also, since $[Q, E] = 1$ and the $\mathbf{Z}(S_i)$ are p'-groups by Step 9, we have

$$Q \cap E = 1 \text{ and } F = Q \times E. \tag{5.1.5}$$

Let c_2 be the unique block of $C = \mathbf{C}_G(Q)$ covered by B, using Step 1. By [43, Cor. 9.21], we have that B is the unique block of G that covers c_2. By (5.1.4), $C \cap K = \mathbf{C}_K(Q) = F$. Since K/F is a normal Hall p-complement of G/F by (5.1.2) and (5.1.3), we have that C/F is a p-group. Therefore C/E is a p-group by (5.1.5). By [43, Cor. 9.6], c_2 is the only block of C covering e.

Now, fix some $\rho \in \mathrm{Irr}(e)$, and consider any $\chi \in \mathrm{Irr}(G|\rho)$. We claim that χ lies in B. If $\gamma \in \mathrm{Irr}(C)$ lies under χ and over ρ, then we see that γ lies in c_2. In particular, χ lies in a block that covers c_2, and therefore χ lies in B, as claimed. Recall that G/K is a p-group by (5.1.2), and by Step 1, any block c_3 of K covered by B is G-invariant. By [43, Cor. 9.6], B is the unique block of G that covers c_3. By [43, Cor. 9.18], and using the height zero hypothesis, for every $\tau \in \mathrm{Irr}(c_3)$ that lies under χ, we have $p \nmid \chi(1)/\tau(1)$. But K/F is a p'-group by (5.1.3) and $F = E \times Q$ by (5.1.5) with Q abelian. It follows that $p \nmid \chi(1)/\rho(1)$. Hence G/E has abelian Sylow p-subgroups by [46, Th. A], and so

$$D/(D \cap E) \text{ is abelian.} \tag{5.1.6}$$

Now, if $Q > 1$, then D/Q is abelian by Step 3, and since $D/(D \cap E)$ is abelian by (5.1.6), we obtain $[D, D] \le Q \cap E = 1$ using (5.1.5), i.e., D is abelian, and we arrive at a contradiction. Thus we have shown

$$Q = 1. \tag{5.1.7}$$

For each i, recall that b_i is the unique S_i-block covered by e. Then any $\theta_i \in \mathrm{Irr}(b_i)$ is G-invariant and extends to

$$H_i := S_i D$$

by Step 9. We also have that H_i/S_i is a p-group, and $[D,D] \le D \cap E$ by (5.1.6). If $d \in D \cap E$, we can write $d = xs \in E = S_i\mathbf{C}_E(S_i)$ with $s \in S_i$ and $x \in \mathbf{C}_E(S_i)$. Thus $x = ds^{-1} \in H_i$ centralises S_i, i.e., $x \in \mathbf{C}_{H_i}(S_i)$ and $d \in S_i\mathbf{C}_{H_i}(S_i)$. It follows that $[D,D] \le D \cap E \le S_i\mathbf{C}_{H_i}(S_i)$, and so

$$H_i/S_i\mathbf{C}_{H_i}(S_i) \text{ is an abelian } p\text{-group.} \tag{5.1.8}$$

We also note that $\mathbf{C}_{H_i}(S_i)$ has a normal p-complement — indeed,

$$\mathbf{C}_{H_i}(S_i) \cap S_i = \mathbf{Z}(S_i)$$

is a p'-group (by Step 9), and $\mathbf{C}_{H_i}(S_i)/\mathbf{Z}(S_i) \hookrightarrow H_i/S_i$ is a p-group. So $\mathbf{C}_{H_i}(S_i) = \mathbf{Z}(S_i) \rtimes Q_i$ for a Sylow p-subgroup Q_i of $\mathbf{C}_{H_i}(S_i)$. But then $Q_i \le \mathbf{C}_{H_i}(S_i)$ centralises $\mathbf{Z}(S_i)$, so in fact

$$\mathbf{C}_{H_i}(S_i) = \mathbf{Z}(S_i) \times Q_i$$

and $Q_i = \mathbf{O}_p(\mathbf{C}_{H_i}(S_i)) \lhd H_i$. Note that $\mathbf{O}_p(H_i) \cap S_i \le \mathbf{O}_p(S_i) = 1$, implying $\mathbf{O}_p(H_i) = Q_i$. Recall that the unique block $\tilde{b}_i$ of H_i that covers b_i has defect group D, by [40, Lemma 2.2]; in particular, $Q_i \lhd D$.

Now S_i naturally embeds in

$$L_i := H_i/Q_i$$

as a normal subgroup. Our goal now is to apply Theorem 4.1(c) to L_i with respect to the quasi-simple group

$$\bar{S}_i := S_iQ_i/Q_i \cong S_i$$

and the block $\bar{b}_i$ of $\bar{S}_i$ that is naturally isomorphic to b_i. If $yQ_i \in L_i$ centralises S_iQ_i/Q_i (modulo Q_i), then $[y,S_i] \le Q_i$. As $S_i \trianglelefteq H_i$, we must have

$$[y,S_i] \subseteq S_i \cap Q_i = 1,$$

so $y \in \mathbf{C}_{H_i}(S_i) = \mathbf{Z}(S_i) \times Q_i$. Thus $\mathbf{C}_{L_i}(\bar{S}_i) = \mathbf{Z}(\bar{S}_i)$, and

$$L_i/\bar{S}_i \cong H_i/S_iQ_i = H_i/S_i\mathbf{C}_{H_i}(S_i) \text{ is an abelian } p\text{-group} \tag{5.1.9}$$

by (5.1.8).

Next we show that each $\theta_i \in \mathrm{Irr}(\bar{b}_i)$ extends to L_i. By Step 9, θ_i, considered as a character of S_i, has an extension $\hat{\theta}_i$ to H_i. Restricting to $S_iQ_i = S_i \times Q_i$, we have

$$\hat{\theta}_i|_{S_i \times Q_i} = \theta_i \otimes \lambda$$

for a unique linear character $\lambda \in \mathrm{Irr}(Q_i)$. In particular, λ is H_i-invariant. Recall that $D_i = D \cap S_i$, and we have $D_i \cap Q_i \le S_i \cap Q_i = 1$, so we can view λ as a character of D_iQ_i that is trivial at D_i, and D-invariant. Since the characters in $\mathrm{Irr}(b_i)$ are H_i-invariant, they are also H_i-invariant when considered as characters of $\bar{b}_i$. By Corollary 4.14 applied to L_i with respect to the block $\bar{b}_i$, we have that

$$H_i/S_i\mathbf{C}_{H_i}(S_i) = S_iD/(S_i \times Q_i) \cong D/(D_i \times Q_i)$$

is cyclic. Hence λ extends to a (linear) character ν of $D/D_i \cong H_i/S_i$. Now, viewing ν as a linear character of H_i/S_i, we have that $\hat{\theta}_i\bar{\nu}$ restricts to θ_i on S_i and trivial on Q_i, and thus θ_i extends to L_i, as wanted.

Since b_i and $\bar{b_i}$ are isomorphic, in particular, D_iQ_i/Q_i is a defect group of $\bar{b}_i$. Since D is a defect group of $\tilde{b_i}$, by Lemma 2.10, we have that D/Q_i is a defect group of the block $\bar{b}_i$ of $L_i = H_i/Q_i$. As θ_i extends to L_i for every $\theta_i \in \mathrm{Irr}(\bar{b}_i)$, and $L_i/\bar{S}_i$ is an abelian p-group by (5.1.9), we see that every character in $\bar{b}_i$ has height 0. Applying Theorem 4.1(c) to the block $\bar{b}_i$ of $\bar{S}_i$ (which again holds in the case $p = 3$ and S_i is of type E_6 or ${}^2\mathrm{E}_6$, by minimality of G), we see that D/Q_i is abelian, and thus

$$[D, D] \leq Q_i \leq \mathbf{C}_G(S_i).$$

This is true for all components S_i, so $[D, D]$ centralises $E = \mathbf{E}(G)Z$. Since $Q = 1$ by (5.1.7), we have $E = F$. It follows that

$$[D, D] \leq \mathbf{C}_G(E) = \mathbf{C}_G(F) \leq F,$$

and hence $[D, D] \leq \mathbf{Z}(F)$, and the latter group is a p'-group, again because $\mathbf{O}_p(G) = Q = 1$ by (5.1.7). Consequently, $[D, D] = 1$, contrary to the choice of G as a minimal counterexample. □

5.2. *Proof of Theorem* C. Having proved Theorem A, and hence BHZ for the prime p, we now see that the extra assumption for types $\mathrm{E}_6(\epsilon q)$ in Theorem 4.1(a) is always satisfied. Hence Theorem C follows from Theorem 4.1(a) if the defect group D is not cyclic. So suppose now that we have a quasi-simple group S with $\mathbf{Z}(S)$ cyclic and p'-prime, that σ is a p-power order automorphism of S fixing all the irreducible characters of a block b of S with cyclic defect group D, and that σ stabilises a block b_D of $\mathbf{C}_G(D)$ that induces b. We want to show that σ fixes the elements of D. Let $N = \mathbf{N}_G(D)$. Then $b_N = (b_D)^N$ is the Brauer First Main correspondent of b. By [30], we know that b satisfies the Alperin–McKay inductive condition. In particular, there is a bijection $\mathrm{Irr}(b) \to \mathrm{Irr}(b_N)$ that commutes with the action of σ. Hence, all irreducible characters in b_N are σ-invariant too. By [43, Th. 9.12], there is a unique irreducible character $\theta \in \mathrm{Irr}(b_D)$ with $D \subseteq \ker\theta$. In particular, θ is σ-invariant. Also, the stabiliser T of b_D in N is the stabiliser of θ in N. Also $T/\mathbf{C}_G(D)$ is a p'-group by [43, Th. 9.22]. If b_T is the Fong–Reynolds block of T covering b_D corresponding to b_N, then all irreducible characters in b_T are σ-invariant, by the uniqueness in the Fong–Reynolds correspondence. Notice that b_T is the only block of T that covers b_D, by [43, Cor. 9.21]. Finally, let $\lambda \in \mathrm{Irr}(D)$ be with $o(\lambda) = |D|$, and consider the irreducible character $\theta_\lambda \in \mathrm{Irr}(b_D)$, constructed in [43, Th. 9.12]. Let $\eta \in \mathrm{Irr}(T|\theta_\lambda)$, which necessarily belongs to b_T and is therefore σ-invariant. Since $T/\mathbf{C}_G(D)$ is a p'-group and $o(\sigma)$ is a power of p, we have that some T-conjugate of θ_λ is σ-invariant, by a

counting argument. Since θ is T-invariant, then $(\theta_\lambda)^t = \theta_{\lambda^t}$ for $t \in T$. Hence, we deduce that $\mu = \lambda^t$ is σ-invariant for some $t \in T$. Since $o(\mu) = |D|$, we have that μ is faithful. Therefore, $\mu(d^\sigma) = \mu(d)$ for all $d \in D$ implies that $d^\sigma = d$ for all $d \in D$. □

We conclude the paper with two remarks. First, the assumption $p > 2$ is crucial for our approach: the conclusions of Proposition 2.2 and Corollary 4.14, which play a key role at various steps of the proof of Theorem A, do not hold when $p = 2$.

Secondly, we would like to point out that Brauer's Height Zero Conjecture implies its so-called *projective* version, as shown in [54], as well as the version for θ-blocks in [51].

References

[1] J. N. S. Bidwell, M. J. Curran, and D. J. McCaughan, Automorphisms of direct products of finite groups, *Arch. Math. (Basel)* **86** no. 6 (2006), 481–489. MR 2241597. Zbl 1103.20016. https://doi.org/10.1007/s00013-005-1547-z.

[2] H. I. Blau and H. Ellers, Brauer's height zero conjecture for central quotients of special linear and special unitary groups, *J. Algebra* **212** no. 2 (1999), 591–612. MR 1676855. Zbl 0930.20008. https://doi.org/10.1006/jabr.1998.7661.

[3] C. Bonnafé, Quasi-isolated elements in reductive groups, *Comm. Algebra* **33** no. 7 (2005), 2315–2337. MR 2153225. Zbl 1096.20037. https://doi.org/10.1081/AGB-200063602.

[4] C. Bonnafé, Sur les caractères des groupes réductifs finis à centre non connexe: applications aux groupes spéciaux linéaires et unitaires, *Astérisque* no. 306 (2006), vi+165. MR 2274998. Zbl 1157.20022. Available at http://numdam.org/item/AST_2006__306__R1_0/.

[5] C. Bonnafé, J.-F. Dat, and R. Rouquier, Derived categories and Deligne-Lusztig varieties II, *Ann. of Math.* (2) **185** no. 2 (2017), 609–670. MR 3612005. Zbl 1471.20028. https://doi.org/10.4007/annals.2017.185.2.5.

[6] C. Bonnafé and R. Rouquier, Catégories dérivées et variétés de Deligne-Lusztig, *Publ. Math. Inst. Hautes Études Sci.* no. 97 (2003), 1–59. MR 2010739. Zbl 1054.20024. https://doi.org/10.1007/s10240-003-0013-3.

[7] R. Brauer, Number theoretical investigations on groups of finite order, in *Proceedings of the International Symposium on Algebraic Number Theory, Tokyo & Nikko*, 1955, Science Council of Japan, Tokyo, 1956, pp. 55–62. MR 0083490. Zbl 0073.01403.

[8] M. Broué, G. Malle, and J. Michel, Generic blocks of finite reductive groups, in *Représentations Unipotentes Génériques et Blocs des Groupes Réductifs Finis*, *Astérisque* **212**, 1993, pp. 7–92. MR 1235832. Zbl 0843.20012. Available at http://www.numdam.org/item/AST_1993__212__7_0/.

[9] M. Broué and L. Puig, A Frobenius theorem for blocks, *Invent. Math.* **56** no. 2 (1980), 117–128. MR 0558864. Zbl 0425.20008. https://doi.org/10.1007/BF01392547.

[10] M. Cabanes and M. Enguehard, On unipotent blocks and their ordinary characters, *Invent. Math.* **117** no. 1 (1994), 149–164. MR 1269428. Zbl 0817.20046. https://doi.org/10.1007/BF01232237.

[11] M. Cabanes and M. Enguehard, On blocks of finite reductive groups and twisted induction, *Adv. Math.* **145** no. 2 (1999), 189–229. MR 1704575. Zbl 0954.20023. https://doi.org/10.1006/aima.1998.1814.

[12] M. Cabanes and M. Enguehard, *Representation Theory of Finite Reductive Groups*, *New Math. Monogr.* **1**, Cambridge Univ. Press, Cambridge, 2004. MR 2057756. Zbl 1069.20032. https://doi.org/10.1017/CBO9780511542763.

[13] M. Cabanes and B. Späth, Equivariance and extendibility in finite reductive groups with connected center, *Math. Z.* **275** no. 3-4 (2013), 689–713. MR 3127033. Zbl 1328.20019. https://doi.org/10.1007/s00209-013-1156-7.

[14] R. W. Carter, *Finite Groups of Lie Type*, *Wiley Classics Library*, John Wiley & Sons, Ltd., Chichester, 1993, Conjugacy classes and complex characters, Reprint of the 1985 original, A Wiley-Interscience Publication. MR 1266626.

[15] D. I. Deriziotis and G. O. Michler, Character table and blocks of finite simple triality groups ${}^3D_4(q)$, *Trans. Amer. Math. Soc.* **303** no. 1 (1987), 39–70. MR 0896007. Zbl 0628.20014. https://doi.org/10.2307/2000778.

[16] S. Dolfi, A. Moretó, and G. Navarro, The groups with exactly one class of size a multiple of p, *J. Group Theory* **12** no. 2 (2009), 219–234. MR 2502216. Zbl 1166.20018. https://doi.org/10.1515/JGT.2008.074.

[17] M. Enguehard, Sur les l-blocs unipotents des groupes réductifs finis quand l est mauvais, *J. Algebra* **230** no. 2 (2000), 334–377. MR 1775796. Zbl 0964.20020. https://doi.org/10.1006/jabr.2000.8318.

[18] P. Fong and B. Srinivasan, The blocks of finite general linear and unitary groups, *Invent. Math.* **69** no. 1 (1982), 109–153. MR 0671655. Zbl 0507.20007. https://doi.org/10.1007/BF01389188.

[19] M. Geck and G. Malle, *The Character Theory of Finite Groups of Lie Type*, *Cambridge Stud. Adv. Math.* **187**, Cambridge Univ. Press, Cambridge, 2020, A Guided Tour. MR 4211779. Zbl 07162758. https://doi.org/10.1017/9781108779081.

[20] M. Giudici, M. W. Liebeck, C. E. Praeger, J. Saxl, and P. H. Tiep, Arithmetic results on orbits of linear groups, *Trans. Amer. Math. Soc.* **368** no. 4 (2016), 2415–2467. MR 3449244. Zbl 1382.20003. https://doi.org/10.1090/tran/6373.

[21] D. Gluck and T. R. Wolf, Brauer's height conjecture for p-solvable groups, *Trans. Amer. Math. Soc.* **282** no. 1 (1984), 137–152. MR 0728707. Zbl 0543.20007. https://doi.org/10.2307/1999582.

[22] D. Gorenstein, R. Lyons, and R. Solomon, *The Classification of the Finite Simple Groups. Number* 2. *Part I. Chapter G*, *Math. Surveys Monogr.* **40**, Amer. Math. Soc., Providence, RI, 1996, General group theory. MR 1358135. Zbl 0836.20011. https://doi.org/10.1090/surv/040.2.

[23] R. Hollenbach, On e-cuspidal pairs of finite groups of exceptional Lie type, *J. Pure Appl. Algebra* **226** no. 1 (2022), Paper No. 106781, 22. MR 4266316. Zbl 1520.20024. https://doi.org/10.1016/j.jpaa.2021.106781.

[24] J. E. Humphreys, Defect groups for finite groups of Lie type, *Math. Z.* **119** (1971), 149–152. MR 0285623. Zbl 0198.04502. https://doi.org/10.1007/BF01109967.

[25] I. M. Isaacs, *Character Theory of Finite Groups*, AMS Chelsea Publishing, Providence, RI, 2006, corrected reprint of the 1976 original [Academic Press, New York; [MR 0460423]. MR 2270898. Zbl 1119.20005. https://doi.org/10.1090/chel/359.

[26] R. Kessar and G. Malle, Quasi-isolated blocks and Brauer's height zero conjecture, *Ann. of Math.* (2) **178** no. 1 (2013), 321–384. MR 3043583. Zbl 1317.20006. https://doi.org/10.4007/annals.2013.178.1.6.

[27] R. Kessar and G. Malle, Lusztig induction and ℓ-blocks of finite reductive groups, *Pacific J. Math.* **279** no. 1-2 (2015), 269–298. MR 3437779. Zbl 1365.20012. https://doi.org/10.2140/pjm.2015.279.269.

[28] R. Kessar and G. Malle, Brauer's height zero conjecture for quasi-simple groups, *J. Algebra* **475** (2017), 43–60. MR 3612463. Zbl 1437.20006. https://doi.org/10.1016/j.jalgebra.2016.05.010.

[29] S. Koshitani and B. Späth, Clifford theory of characters in induced blocks, *Proc. Amer. Math. Soc.* **143** no. 9 (2015), 3687–3702. MR 3359562. Zbl 1333.20005. https://doi.org/10.1090/proc/12431.

[30] S. Koshitani and B. Späth, The inductive Alperin-McKay and blockwise Alperin weight conditions for blocks with cyclic defect groups and odd primes, *J. Group Theory* **19** no. 5 (2016), 777–813. MR 3545905. Zbl 1370.20009. https://doi.org/10.1515/jgth-2016-0006.

[31] B. Külshammer, An analogue of the p-regular core for p-blocks of finite groups, *Arch. Math.* (*Basel*) **52** no. 1 (1989), 19–21. MR 0980046. Zbl 0632.20006. https://doi.org/10.1007/BF01197965.

[32] B. Külshammer and L. Puig, Extensions of nilpotent blocks, *Invent. Math.* **102** no. 1 (1990), 17–71. MR 1069239. Zbl 0739.20003. https://doi.org/10.1007/BF01233419.

[33] M. Linckelmann, *The Block Theory of Finite Group Algebras. Vol. I*, *London Math. Soc. Student Texts* **91**, Cambridge Univ. Press, Cambridge, 2018. MR 3821516. Zbl 1515.20015. https://doi.org/10.1017/9781108349321.

[34] G. Malle, Die unipotenten Charaktere von ${}^2F_4(q^2)$, *Comm. Algebra* **18** no. 7 (1990), 2361–2381. MR 1063140. Zbl 0721.20008. https://doi.org/10.1080/00927879008824026.

[35] G. Malle, Cuspidal characters and automorphisms, *Adv. Math.* **320** (2017), 887–903. MR 3709124. Zbl 1483.20013. https://doi.org/10.1016/j.aim.2017.09.026.

[36] G. Malle and G. Navarro, Brauer's Height Zero Conjecture for principal blocks, *J. Reine Angew. Math.* **778** (2021), 119–125. MR 4308615. Zbl 07392170. https://doi.org/10.1515/crelle-2021-0024.

[37] G. Malle, G. Navarro, and B. Späth, On blocks with one modular character, *Forum Math.* **30** no. 1 (2018), 57–73. MR 3739327. Zbl 1458.20014. https://doi.org/10.1515/forum-2015-0197.

[38] G. Malle and D. Testerman, *Linear Algebraic Groups and Finite Groups of Lie Type*, *Cambridge Stud. Adv. Math.* **133**, Cambridge Univ. Press, Cambridge, 2011. MR 2850737. Zbl 1256.20045. https://doi.org/10.1017/CBO9780511994777.

[39] G. O. Michler and J. B. Olsson, Character correspondences in finite general linear, unitary and symmetric groups, *Math. Z.* **184** no. 2 (1983), 203–233. MR 0716272. Zbl 0505.20009. https://doi.org/10.1007/BF01252858.

[40] M. Murai, Block induction, normal subgroups and characters of height zero, *Osaka J. Math.* **31** no. 1 (1994), 9–25. MR 1262786. Zbl 0830.20008. https://doi.org/10.18910/5194.

[41] M. Murai, Normal subgroups and heights of characters, *J. Math. Kyoto Univ.* **36** no. 1 (1996), 31–43. MR 1381537. Zbl 0867.20011. https://doi.org/10.1215/kjm/1250518602.

[42] M. Murai, On blocks of normal subgroups of finite groups, *Osaka J. Math.* **50** no. 4 (2013), 1007–1020. MR 3161425. Zbl 1285.20006. https://doi.org/10.18910/26820.

[43] G. Navarro, *Characters and Blocks of Finite Groups*, *London Math. Soc. Lecture Note Ser.* **250**, Cambridge Univ. Press, Cambridge, 1998. MR 1632299. Zbl 0903.20004. https://doi.org/10.1017/CBO9780511526015.

[44] G. Navarro and B. Späth, On Brauer's Height Zero Conjecture, *J. Eur. Math. Soc.* (*JEMS*) **16** no. 4 (2014), 695–747. MR 3191974. Zbl 1353.20006. https://doi.org/10.4171/JEMS/444.

[45] G. Navarro and P. H. Tiep, Brauer's height zero conjecture for the 2-blocks of maximal defect, *J. Reine Angew. Math.* **669** (2012), 225–247. MR 2980457. Zbl 1280.20011. https://doi.org/10.1515/crelle.2011.147.

[46] G. Navarro and P. H. Tiep, Characters of relative p'-degree over normal subgroups, *Ann. of Math.* (2) **178** no. 3 (2013), 1135–1171. MR 3092477. Zbl 1372.20016. https://doi.org/10.4007/annals.2013.178.3.7.

[47] G. Navarro, P. H. Tiep, and A. Turull, Brauer characters with cyclotomic field of values, *J. Pure Appl. Algebra* **212** no. 3 (2008), 628–635. MR 2365337. Zbl 1173.20007. https://doi.org/10.1016/j.jpaa.2007.06.019.

[48] T. Okuyama and Y. Tsushima, Local properties of p-block algebras of finite groups, *Osaka Math. J.* **20** no. 1 (1983), 33–41. MR 0695615. Zbl 0513.20005. https://doi.org/10.18910/8689.

[49] J. B. Olsson, Remarks on symbols, hooks and degrees of unipotent characters, *J. Combin. Theory Ser. A* **42** no. 2 (1986), 223–238. MR 0847553. Zbl 0659.20010. https://doi.org/10.1016/0097-3165(86)90093-2.

[50] J. B. Olsson, *Combinatorics and Representations of Finite Groups*, *Lecture Notes Math. Univ. Essen* **20**, Universität Essen, Fachbereich Mathematik, Essen, 1993, available at author's webpage. MR 1264418. Zbl 0796.05095.

[51] N. Rizo, p-blocks relative to a character of a normal subgroup, *J. Algebra* **514** (2018), 254–272. MR 3853065. Zbl 1446.20019. https://doi.org/10.1016/j.jalgebra.2018.07.035.

[52] L. Ruhstorfer, The Alperin–McKay and Brauer conjecture for the prime 2, 2022, *Ann. of Math* (2) **201**, no. 2 (2025), to appear. arXiv 2204.06373.

[53] B. Sambale, *Blocks of Finite Groups and their Invariants*, *Lecture Notes in Math.* **2127**, Springer, Cham, 2014. MR 3289382. Zbl 1315.20009. https://doi.org/10.1007/978-3-319-12006-5.

[54] B. Sambale, On the projective height zero conjecture, *Proc. Amer. Math. Soc.* **147** no. 2 (2019), 487–488. MR 3894887. Zbl 1458.20010. https://doi.org/10.1090/proc/14134.

[55] K.-i. Shinoda, The conjugacy classes of the finite Ree groups of type (F_4), *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **22** (1975), 1–15. MR 0372064. Zbl 0306.20014.

[56] J. Taylor, Action of automorphisms on irreducible characters of symplectic groups, *J. Algebra* **505** (2018), 211–246. MR 3789911. Zbl 1436.20084. https://doi.org/10.1016/j.jalgebra.2018.03.008.

[57] The GAP group, GAP — Groups, Algorithms, and Programming, 2022, Version 4.12.2. Available at http://www.gap-system.org.

(Received: September 15, 2022)
(Revised: February 21, 2024)

FB Mathematik, TU Kaiserslautern, Kaiserslautern, Germany
E-mail: malle@mathematik.uni-kl.de

Departament de Matemàtiques, Universitat de València, Burjassot, València, Spain
E-mail: gabriel@uv.es

Department of Mathematics and Statistics, MSU Denver, Denver, CO
Current address: Department of Mathematics, University of Denver, Denver, CO, USA
E-mail: mandi.schaefferfry@du.edu

Department of Mathematics, Rutgers University, Piscataway, NJ, USA
E-mail: tiep@math.rutgers.edu