

A FLEXIBLE APPROACH FOR NORMAL APPROXIMATION OF GEOMETRIC AND TOPOLOGICAL STATISTICS

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ABSTRACT. We derive normal approximation results for a class of stabilizing functionals of binomial or Poisson point process, that are not necessarily expressible as sums of certain score functions. Our approach is based on a flexible notion of the add-one cost operator, which helps one to deal with the second-order cost operator via suitably appropriate first-order operators. We combine this flexible notion with the theory of strong stabilization to establish our results. We illustrate the applicability of our results by establishing normal approximation results for certain geometric and topological statistics arising frequently in practice. Several existing results also emerge as special cases of our approach.

1. INTRODUCTION

Let $(\mathbb{X}, \mathcal{F})$ be a metric measure space equipped with a σ -finite measure $\mathbb{Q}$ and a metric $d : \mathbb{X} \times \mathbb{X} \rightarrow [0, \infty)$. For $s \geq 1$, let $\mathcal{P}_s$ denote the canonical Poisson process on $\mathbb{X}$ with intensity measure $\lambda := s\mathbb{Q}$, and for $\mathbb{Q}$ a probability measure, let ξ_n denote the binomial process associated to $\mathbb{Q}$. Let $d_K(Y, Z)$ denote the Kolmogorov distance between two random variables Y, Z , i.e., $d_K(Y, Z) := \sup_{t \in \mathbb{R}} |\mathbb{P}(Y \leq t) - \mathbb{P}(Z \leq t)|$. In this work, we study normal approximation results for real-valued functionals $F_s(\mathcal{P}_s)$ and $F_n(\xi_n)$ respectively of the Poisson and the binomial point processes in the Kolmogorov metric under relatively flexible assumptions on the functionals. In particular, motivated by geometric and topological statistics, we focus on the case when the functionals F_s and F_n are not necessarily expressible as sums of certain score functions, and on obtaining presumably optimal bounds in this case.

Our proof techniques are based on the widely-used concept of *stabilization*. Indeed, since the use of stabilization concepts to establish central limit theorems for Poisson-based minimal spanning tree in [KL96], these concepts have been widely developed as a general tool to establish normal approximation rates for various functionals of Poisson and Binomial point process. We refer the interested reader to [PY01, Pen05, BY05, PY05, Sch10, LPS16, CS17, LRP17] and reference therein for details. In particular, [LPS16] develops normal approximation bounds for a fairly general class of functions of Poisson processes by combining Malliavin-Stein techniques [PSTU10, PR16], second-order Poincaré inequalities [Cha09, NPR09] and stabilization concepts, and by using the iterated add-one cost operator, also called second order cost operator. Considering the case of functionals expressible as a sum of exponentially stabilizing score functions, [LRSY19] establishes user-friendly normal approximation results based on [LPS16]. The work of [LRPY22] introduces bounds for general functionals of Poisson process. Their method does not involve the hard-to-evaluate iterated add-one cost operators but uses the add-one cost operator at two different scales, an approach pioneered by [CS17] for the case of Poisson-based minimal spanning trees. However, their generality

comes at the cost of sub-optimality – in general, the bounds based on [LRPY22] are sub-optimal compared to those of [LPS16]. Furthermore, for the case of functionals that are expressible as a sum of exponentially stabilizing score functions, the bounds of [LRPY22] necessarily lead to sub-optimal rates.

Hence, the following question remains: *Can one obtain presumably optimal bounds for general functionals that automatically result in presumably optimal bounds when specialized to the case of functionals that can be expressed as sums of score functions.* Following [LRSY19], we use the term *presumably optimal* to refer to the case when the order of the normal approximation is the same as that of a sum of i.i.d. random variables. In this work, we answer this question in the affirmative for a class of functionals. Similar to [LRPY22], our approach is based on the idea of using the add-one cost operator at two scales. However, in contrast to their work, we use it to directly simplify the evaluation of the iterated add-one cost operators. When specialized to the case of sums of score functions, such an approach recovers the presumably optimal results of [LRSY19]. To summarize, we make the following contributions:

- In Definition 2.7, we introduce a flexible notion of the add-one cost operator with a general set A_x that allows to (relatively) easily evaluate computations with the iterated add-one cost or second-order difference operators, for general functionals of Poisson and binomial point process that are not necessarily a sum.
- In our main results, Theorems 3.1 and 3.2, we provide normal approximation results for functionals of Poisson and binomial point processes respectively. In particular, the functionals do not necessarily need to be expressible as sums of certain score functions.
- We illustrate the applicability of our approach by deriving normal approximation results for several geometric and topological statistics. Specifically, in Theorem 4.1 and 4.2 we use our approach to derive normal approximation results for the total edge length of k -Nearest Neighbor graph and weighted k -Nearest Neighbor graph based Shannon entropy estimators. In Theorem 4.3, we derive results for Euler Characteristic, which is an elementary statistics widely used in the field of topological data analysis. Finally, we discuss the applicability of our approach for the minimal spanning tree problem in Theorem 4.4, by recovering existing results via our approach.

Organization. The rest of the paper is organized as follows. In Section 2, we introduce the basics of point processes, stabilization concepts and required assumptions. In Section 3, we present our main theorems and discuss relations to existing results. In Section 4, we discuss applications of our results to geometric and topological statistics. The proofs are provided in Section 5 and 6.

2. PRELIMINARIES

2.1. Point Process Basics. Let $(\mathbb{X}, \mathcal{F})$ be a measure space with a σ -finite measure $\mathbb{Q}$ and a metric $d : \mathbb{X} \times \mathbb{X} \rightarrow [0, \infty)$. Let $\mathbf{N}$ be the set of σ -finite counting measures on $\mathbb{X}$, which can be interpreted as point configurations in $\mathbb{X}$. Thus, we treat the elements from $\mathbf{N}$ as sets. The set $\mathbf{N}$ is equipped with the smallest σ -field $\mathcal{N}$ such that the maps $m_A : \mathbf{N} \rightarrow \mathbb{N} \cup \{0, \infty\}$, $\mathcal{M} \mapsto \mathcal{M}(A)$ are measurable for all $A \in \mathcal{F}$; see [KK97] and [LP11]. A point process η is a random element in $\mathbf{N}$. Denote by $\mathbf{F}(\mathbf{N})$ the class of all measurable functions $f : \mathbf{N} \rightarrow \mathbb{R}$, and by $L^0(\mathbb{X}) := L^0(\mathbb{X}, \mathcal{F})$ the class of all real-valued, measurable functions F on $\mathbb{X}$. Note that, as $\mathcal{F}$ is the completion of $\sigma(\eta)$, each $F \in L^0(\mathbb{X})$ can be

written as $F = f(\eta)$ for some measurable function $f \in \mathbf{F}(\mathbb{N})$. Such a mapping f , called a *representative* of F , is $\mathbb{Q} \circ \eta^{-1}$ -a.s. uniquely defined. In order to simplify the discussion, we make this convention: whenever a general function F is introduced, we will select one of its representatives and denote such a representative mapping by the same symbol F . Throughout this paper, we denote by $L^2_\eta(\mathbb{X})$ the space of all square-integrable functions F of a point process η with $\mathbb{E}F^2(\eta) < \infty$. We mainly consider two different classes of point processes: Poisson point process and binomial point process.

Definition 2.1 (Poisson Point Process). *A Poisson point process with intensity measure λ is a point process $\mathcal{P}(\lambda)$ on $\mathbb{X}$ with the following two properties:*

- (1) $\forall B \in \mathcal{F}$, $\mathcal{P}(\lambda)(B)$ is a Poisson random variable with parameter $\lambda(B)$.
- (2) $\forall m \in \mathbb{N}_+$ and for any pairwise disjoint sets $B_1, B_2, \dots, B_m \in \mathcal{F}$, we have that the random variables $\mathcal{P}(\lambda)(B_1), \mathcal{P}(\lambda)(B_2), \dots, \mathcal{P}(\lambda)(B_m)$ are independent.

Definition 2.2 (Binomial Point Process). *Let P be a probability distribution and n be a fixed positive integer. Let $X_1, X_2, \dots, X_n$ be i.i.d. random variables sampled from P . The binomial point process ξ_n based on P and n is defined as $\xi_n := \sum_{i=1}^n \delta_{X_i}$, where δ is the Dirac measure.*

We now describe the setting for developing normal approximations of functionals of Poisson and binomial point processes. For $s \geq 1$, let $\lambda := s\mathbb{Q}$ be the intensity measure of Poisson point process $\mathcal{P}(\lambda) := \mathcal{P}_s$. For the case when $\mathbb{Q}$ is the probability measure, let ξ_n be the binomial point process based on $\mathbb{Q}$ and n . Consider square-integrable functionals of these two point processes, i.e., $F(\mathcal{P}_s) := F_s(\mathcal{P}_s) \in L^2_{\mathcal{P}_s}(\mathbb{X})$ and $F(\xi_n) := F_n(\xi_n) \in L^2_{\xi_n}(\mathbb{X})$. We then seek upper bounds for the following two quantities:

$$d_K \left(\frac{F_s(\mathcal{P}_s) - \mathbb{E}F_s(\mathcal{P}_s)}{\sqrt{\text{Var } F_s(\mathcal{P}_s)}}, N \right), \quad \text{and} \quad d_K \left(\frac{F_n(\xi_n) - \mathbb{E}F_n(\xi_n)}{\sqrt{\text{Var } F_n(\xi_n)}}, N \right),$$

where N is the standard normal random variable.

2.2. Stabilization. The notion of stabilization is widely used in deriving normal approximation rates for functionals of Poisson or binomial point processes [KL96, PY01, Pen05, PY05]. We start with introducing notions of stabilization for functionals which are not necessarily representable as sums of sore functions.

Definition 2.3 (Add-one Cost Operator). *Let F be a measurable functional of a point process η on $(\mathbb{X}, \mathcal{F})$. The family of Add-One Cost Operators, $D = (D_x)_{x \in \mathbb{X}}$, are defined as*

$$D_x F(\eta) := F(\eta \cup \{x\}) - F(\eta).$$

Similarly, we can define a second-order cost operator (also called iterated add-one cost operator): for any $x_1, x_2 \in \mathbb{X}$,

$$D_{x_1, x_2} F(\eta) := F(\eta \cup \{x_1, x_2\}) - F(\eta \cup \{x_1\}) - F(\eta \cup \{x_2\}) + F(\eta).$$

In addition, we define for any $y \in \mathbb{X}$,

$$D_x F^y(\eta) := F(\eta \cup \{y\} \cup \{x\}) - F(\eta \cup \{y\}).$$

Clearly, when $\{y\} = \emptyset$, it degenerates into the add-one cost operator.

Based on the add-one cost operator introduced above, we next introduce weak and strong stabilization in the context of functionals F_s of the Poisson point process $\mathcal{P}_s$. Similar definitions hold automatically for the case of functionals F_n of the binomial Point Process ξ_n .

Definition 2.4 (Weak Stabilization). *The functional F_s is said to be weakly stabilizing at $x \in \mathbb{X}$, if and only if there exists a random variable Δ_x such that for any sequence $(W_m)_{m \geq 1}$ in $\mathcal{F}$ tending to $\mathbb{X}$, as $m \rightarrow \infty$, we have $D_x F_s(W_m) \rightarrow \Delta_x$, almost surely.*

Definition 2.5 (Strong Stabilization). *The functional F_s is said to be strongly stabilizing at $x \in \mathbb{X}$, if and only if there exists an almost surely finite random variable R_x , which is referred to as the radius of stabilization, such that for all finite $\mathbb{A} \subset \mathbb{X} \setminus B_x(R_x)$, with probability 1,*

$$D_x F_s((\mathcal{P}_s \cap B_x(R_x)) \cup \mathbb{A}) = D_x F_s(\mathcal{P}_s \cap B_x(R_x)),$$

where $B_x(R_x) := \{y \in \mathbb{X} : d(x, y) \leq R_x\}$.

Clearly, strong stabilization implies weak stabilization. In some cases, the functionals F_s and F_n can be represented as a sum of the form

$$F_s(\mathcal{P}_s) := \sum_{x \in \mathcal{P}_s} f_s(x, \mathcal{P}_s), \quad \text{and} \quad F_n(\xi_n) := \sum_{x \in \xi_n} f_n(x, \xi_n), \quad (2.1)$$

where f_s, f_n are called *score functions*. In this case, there exists a useful notion of stabilization based on the score functions. For simplicity, we still state the definition for functionals of Poisson point process; the binomial case is defined similarly.

Definition 2.6 (Score-based Stabilization [LRSY19]). *The score function f_s is said to be stabilizing at $x \in \mathbb{X}$, if and only if there exists an almost surely finite random variable R_x (the radius of stabilization) such that for all finite $\mathbb{A} \subset \mathbb{X} \setminus B_x(R_x)$, we have*

$$f_s(x, (\mathcal{P}_s \cap B_x(r_x)) \cup \mathbb{A}) = f_s(x, \mathcal{P}_s \cap B_x(r_x)).$$

Informally speaking, the above definition posits that the value of the score function f_s will not be affected by the points outside the ball centered at x with radius R_x . For discussing the relation between strong stabilization and score-based stabilization, we present the following simple result.

Proposition 2.1. *Given any $F_s = \sum_{y \in \mathcal{P}_s} f_s(y, \mathcal{P}_s)$, we have for all $x \in \mathbb{X}$,*

$$D_x F_s(\mathcal{P}_s) = f_s(x, \mathcal{P}_s \cup \{x\}) + \sum_{y \in \mathcal{P}_s} D_x f_s(y, \mathcal{P}_s). \quad (2.2)$$

Proof of Proposition 2.1. We have

$$\begin{aligned} D_x F_s(\mathcal{P}_s) &= F_s(\mathcal{P}_s \cup \{x\}) - F_s(\mathcal{P}_s) \\ &= \sum_{y \in \mathcal{P}_s \cup \{x\}} f_s(y, \mathcal{P}_s \cup \{x\}) - \sum_{y \in \mathcal{P}_s} f_s(y, \mathcal{P}_s) \\ &= f_s(x, \mathcal{P}_s \cup \{x\}) + \left(\sum_{y \in \mathcal{P}_s} f_s(y, \mathcal{P}_s \cup \{x\}) - \sum_{y \in \mathcal{P}_s} f_s(y, \mathcal{P}_s) \right) \\ &= f_s(x, \mathcal{P}_s \cup \{x\}) + \sum_{y \in \mathcal{P}_s} D_x f_s(y, \mathcal{P}_s), \end{aligned} \quad (2.3)$$

which completes the proof. $\square$

Remark 2.1. *We now make the following remarks.*

- (1) *Strong stabilization focuses on the cost function of the functional F while the score-based stabilization is assumed on the score functions f . Equation (2.3) reveals this relationship. It plays an important role in Section 4.2 and its proof.*
- (2) *Strong stabilization is more general than score-based stabilization in that strong stabilization does not restrict the form of the functional to be expressible as a sum of scores. Furthermore, the same functional might be expressible in multiple ways as sums of scores. Depending on the representation, it might be easier or harder to compute the radius of stabilization and also the moments of the score functions (which also play a crucial role - see below). Strong stabilization, however, provides an approach to directly work with the functional F itself.*

The following example from the literature on Topological Data Analysis (TDA), further illustrates the aforementioned remarks. Readers unfamiliar with the basics of TDA are directed to the elementary definitions provided in Appendix A. We also refer to [EH10, BCY18] for more on the basics of TDA.

Example 2.1 (Euler Characteristic). *Given a simplicial complex K , the Euler characteristic is defined as*

$$\chi(K) := \sum_{k=0}^{\infty} (-1)^k \# \{S_k\},$$

where $\# \{S_k\}$ is the number of simplices of dimension k .

Typically, the simplicial complex K , is taken to be the Vietoris-Rips complex (VR complex) or the Čech complex, constructed over a point cloud sampled from binomial or Poisson point processes ξ_n or $\mathcal{P}_s$, respectively. In this case, we denote the simplicial complex as $K(\xi_n)$ or $K(\mathcal{P}_s)$ to denote the dependency on the underlying point process explicitly. We now discuss the stabilization properties of the above statistic. While it is possible to express the Euler characteristic as a sum of certain score functions, it is not required to do so, as the Euler characteristic is strongly stabilizing with radius of stabilization $R_x = 2r$ for the Čech and the VR-complex; see [KRP21].

The following example, on the total edge length of a k -nearest neighbor graphs, is a canonical example of a geometric statistic that satisfies score-based stabilization and strong stabilization.

Example 2.2 (k -Nearest Neighbor (k -NN) Graphs). *Consider a configuration of a Poisson point process $\mathcal{P}_s$, where here we represent $\mathcal{P}_s$ by a random number of (conditionally) i.i.d. points X_i , i.e. $\mathcal{P}_s = \{X_i\}_{i=1}^{|\mathcal{P}_s|}$. For some $k \in \mathbb{N}_+$, and for every integer $1 \leq j \leq k$, denote by $X_{j,i}$ the j -nearest neighbor of X_i , i.e. $X_{j,i}$ is the j th closest point to X_i . Furthermore, let $\rho_{j,i}$ denote the distance between $X_{j,i}$ and X_i . Then, the (undirected) k -NN graph $NG_k(\mathcal{P}_s)$ is the graph with the vertex set $V := \mathcal{P}_s$ and an edge $x \sim y$ if y is some j -nearest neighbor of x and (or) x is some j -nearest neighbor of y . For $\vartheta > 0$, we define*

$$f_s(x, \mathcal{P}_s) := \begin{cases} \sum_{x \sim y} \frac{1}{2} d(x, y)^\vartheta, & \text{if } x, y \text{ are mutual } k\text{-nearest neighbors,} \\ \sum_{x \sim y} d(x, y)^\vartheta, & \text{if } x, y \text{ are not mutual } k\text{-nearest neighbors.} \end{cases} \quad (2.4)$$

The total edge length is defined as

$$F_s := \sum_{x \in \mathcal{P}_s} f_s(x, \mathcal{P}_s).$$

According to [LRSY19], the total edge length statistic satisfies score-based stabilization. Additionally, by the proof of [PY01, Lemma 6.1], we also have that it satisfies strong stabilization with the radius of stabilization being $R_x = 4R$, where R is defined in the following way: for each $t > 0$, construct six disjoint equilateral triangles $T_j(t)$, $1 \leq j \leq 6$, such that the origin is a vertex of each triangle, such that each triangle has edge length t and such that $T_j(t) \subset T_j(u)$ whenever $t < u$. Then, define R to be the minimum t such that each triangle $T_j(t)$ contains at least $k + 1$ points from $\mathcal{P}_s$.

By definition, strong stabilization only focuses on the first-order add-one cost operator. In order to deal with second-order cost operators, which are also crucial in obtaining our normal approximation results, we introduce the following flexible notion of add-one cost operators.

Definition 2.7 (Flexible Add-One Cost). *For any point process η in $(\mathbb{X}, \mathcal{F})$, any $x \in \mathbb{X}$ and a set $A_x \in \mathcal{F}$ (that may or may not depend on x), the flexible add-one cost operator for the functional F is defined as*

$$D_x F(A_x) := D_x F(A_x)(\eta) := D_x F(\eta|_{A_x}) := F((\eta|_{A_x}) \cup \{x\}) - F(\eta|_{A_x}),$$

where we denote by $\eta|_{A_x}$ the restriction of the point process η to the set A_x (see, for example, [LRPY22]).

Informally speaking, we introduce the flexible add-one cost $D_x F(A_x)$ by only observing the point process in the ‘window’ A_x . Obviously, if one sets $A_x = \mathbb{X}$, the flexible add-one cost function degenerates into the classical add-one cost function in Definition 2.3. The following proposition, whose proof is immediate by simply using the definition of the second-order cost function, provides a way to deal with the second-order cost function by the flexible cost function defined above.

Proposition 2.2. *Under the setting of Definition 2.3 and 2.7, we have*

$$D_{x_1, x_2} F = (D_{x_1} F^{x_2} - D_{x_1} F^{x_2}(A_x)) + (D_{x_1} F^{x_2}(A_x) - D_{x_1} F(A_x)) + (D_{x_1} F(A_x) - D_{x_1} F).$$

Particularly, when $A_x = \mathbb{X}$, we have

$$D_{x_1, x_2} F = D_{x_1} F^{x_2} - D_{x_1} F.$$

2.3. Assumptions. We now discuss the assumptions made in our work to obtain the normal approximation results. On the measure $\mathbb{Q}$, following [Pen07, PY05, Yuk15, LRSY19], we make the following assumption: There exist constants $\kappa > 0, \omega > 1$ such that for $r \geq 0$ and all $x \in \mathbb{X}$,

$$\limsup_{\epsilon \rightarrow 0^+} \frac{\mathbb{Q}(B_x(r + \epsilon)) - \mathbb{Q}(B_x(r))}{\epsilon} \leq \kappa \omega r^{\omega-1}, \quad \mathbb{Q}(\{x\}) = 0. \quad (2.5)$$

For example, one can consider a measure $\mathbb{Q}$ on $\mathbb{X}$, a full dimensional subset of $\mathbb{R}^d$, with a bounded density with respect to the Lebesgue measure, where one can choose $\kappa = \sup f$ and $\omega = d$. We also make the following tail-bound assumption on the radius of strong stabilization.

Assumption 2.1 (Decay of Radius of Stabilization). *Under the setting of strong stabilization and (2.5), we say the radius of stabilization R_x decays exponentially if and only if there exist constants $c_1, c_2, c_3 > 0$ such that for $r \geq 0$,*

$$\mathbb{P}(R_x \geq r) \leq c_1 e^{-c_2 (s^{1/\omega_r})^{c_3}}.$$

If R_x is based on a binomial process ξ_n , then a similar decay holds with s replaced by n .

Yet another reason for why we refer to Definition 2.7 as “flexible” is that even when the tail probability of the radius of stabilization R_x is unknown for a specific functional, it might be possible to pick A_x “strategically” and use our approach to obtain normal approximation bounds. We illustrate this point in Section 4.4 by using our approach to recover existing results on normal approximation for the total edge length of the minimal spanning tree.

We next move on to the assumptions on the (flexible) add-one cost operator. Throughout the paper, we assume that $\mathbb{E} \int (D_x F)^2 \lambda(dx) < \infty$. Furthermore, we assume the following $\mathbb{K}$ -exponential bound assumption.

Assumption 2.2 ($\mathbb{K}$ -exponential bound). *We say the add-one cost function $D_x F$ satisfies a $\mathbb{K}$ -exponential bound, where $\mathbb{K}$ is a measurable subset of $\mathbb{X}$, if and only if for $x, x^* \in \mathbb{X}$, there exist constants $k_1, k_2, k_3 > 0$ such that*

$$\mathbb{P}(D_x F \neq 0) \leq k_1 e^{-k_2 d_s(x, \mathbb{K})^{k_3}}, \quad \text{and} \quad \mathbb{P}(D_x F^{x^*} \neq 0) \leq k_1 e^{-k_2 d_s(x, \mathbb{K})^{k_3}},$$

where $d_s(\cdot, \cdot) := s^{1/\omega} d(\cdot, \cdot)$ and $d(x, \mathbb{K}) := \inf_{y \in \mathbb{K}} d(x, y)$. Similarly, we can assume the above for binomial point processes by changing s as n .

A similar assumption has been made in [LRSY19, Equations (2.8) and (2.9)] on the score functions to capture functionals whose variances exhibit surface area order scaling. Here, we make the assumption directly on the functional F , which captures a more general class of functionals than that considered in [LRSY19].

Assumption 2.3 (Moment Condition). *We say the functional F_s satisfies the moment condition if and only if there exists some $p > 0$ and $H < \infty$ such that*

$$\sup_{s \geq 1} \sup_{x, x^* \in \mathbb{X}} (\mathbb{E} |D_x F_s|^p + \mathbb{E} |D_x F_s^{x^*}|^p) = H. \quad (2.6)$$

If the Poisson process $\mathcal{P}_s$ is replaced by a binomial process ξ_n , then the above suprema are taken over n rather than s as well as the functional F_s is changed to F_n .

Bounded moment conditions are commonly made to derive normal approximation results. For related work in the context of stabilizing functionals of point process, see [LRSY19, Equations (2.6) and (2.7)] and [LRPY22, Equations (1.5) and (1.8)]. While [LRSY19] considers moment conditions on score functions, we directly deal with the functional F so that it fits a more general class.

3. MAIN RESULTS

We now present our two main results on the normal approximation of a certain class of functionals of Poisson and binomial point process, Theorem 3.1 and 3.2 respectively, that are not necessarily expressible as sums of score functions. We discuss several applications in Section 4. Firstly, we introduce a general result for functionals of Poisson point process. We remark that the following theorem does not leverage Assumptions 2.1, 2.2 and 2.3. However, in Corollary 3.1 we present a refined results under the above mentioned set of assumptions.

Theorem 3.1 (Normal Approximation for Functionals of Poisson Point Processes). *Let F be a functional of the Poisson point process $\mathcal{P}(\lambda)$ with $F \in L^2_{\mathcal{P}(\lambda)}$ and $\mathbb{E} \int (D_x F)^2 \lambda(dx) < \infty$. For any $x, x_1, x_2 \in \mathbb{X}$, define*

$$\mathbb{E}|D_x F - D_x F(A_x)|^4 := b_1(x, A_x), \quad \mathbb{E}|D_x F(A_x)|^4 := b_2(x, A_x), \quad (3.1)$$

and

$$\mathbb{E}|D_{x_1} F^{x_2} - D_{x_1} F^{x_2}(A_{x_1})|^4 := b_3(x_1, x_2, A_{x_1}), \quad (3.2)$$

$$\mathbb{E}|D_{x_1} F(A_{x_1}) - D_{x_1} F|^4 := b_4(x_1, x_2, A_{x_1}), \quad (3.3)$$

$$\mathbb{E}|D_{x_1} F^{x_2}(A_{x_1}) - D_{x_1} F(A_{x_1})|^4 := b_5(x_1, x_2, A_{x_1}). \quad (3.4)$$

Then, there is an absolute constant $C^* > 0$ such that

$$d_K\left(\frac{F - \mathbb{E}F}{\sqrt{\text{Var } F}}, N\right) \leq C^* \sum_{i=1}^6 \gamma'_i,$$

where

$$\gamma'_1 := \frac{1}{\text{Var } F} \left(\int \left(\sum_{j=1}^2 b_j(x_1, A_{x_1})^{\frac{1}{4}} \sum_{j=1}^2 b_j(x_2, A_{x_2})^{\frac{1}{4}} \sum_{j=3}^5 b_j(x_3, x_1, A_{x_3})^{\frac{1}{4}} \sum_{j=3}^5 b_j(x_3, x_2, A_{x_3})^{\frac{1}{4}} \right) \lambda^3(d(x_1, x_2, x_3)) \right)^{\frac{1}{2}},$$

$$\gamma'_2 := \frac{1}{\text{Var } F} \left(\int \sum_{j=3}^5 b_j(x_3, x_1, A_{x_3}) \sum_{j=3}^5 b_j(x_3, x_2, A_{x_3}) \lambda^3(d(x_1, x_2, x_3)) \right)^{\frac{1}{2}},$$

$$\gamma'_3 := \frac{1}{(\text{Var } F)^{\frac{3}{2}}} \int \sum_{j=1}^2 b_j(x, A_x)^{\frac{3}{4}} \lambda(dx),$$

$$\gamma'_4 := \frac{\int \sum_{j=1}^2 b_j(x, A_x)^{\frac{3}{4}} \lambda(dx)}{(\text{Var } F)^2} \left(\left(\int \sum_{j=1}^2 b_j(x, A_x)^{\frac{1}{2}} \lambda(dx) \right)^{\frac{1}{2}} + \left(\int \sum_{j=1}^2 b_j(x, A_x) \lambda(dx) \right)^{\frac{1}{4}} + (\text{Var } F)^{\frac{1}{2}} \right),$$

$$\gamma'_5 := \frac{1}{\text{Var } F} \left(\int \sum_{j=1}^2 b_j(x, A_x) \lambda(dx) \right)^{\frac{1}{2}},$$

$$\gamma'_6 := \frac{1}{\text{Var } F} \left(\int \sum_{j=1}^2 b_j(x_1, A_{x_1})^{\frac{1}{2}} \sum_{j=3}^5 b_j(x_1, x_2, A_{x_1})^{\frac{1}{2}} + \sum_{j=3}^5 b_j(x_1, x_2, A_{x_1}) \lambda^2(d(x_1, x_2)) \right)^{\frac{1}{2}}.$$

Remark 3.1. We make some remarks about the general Theorem 3.1 as follows.

- (i) The above Theorem 3.1 generalizes [LPS16, Theorem 1.2] by introducing the flexible cost function $D_x F_s(A_x)$.
- (ii) Theorem 3.1 is valid for deriving normal approximation rates for general functionals or stabilizing functionals not having a known tail probability bound; see Section 4.4. When such tail bounds are known, a more refined result is available (see Corollary 3.1 below).

When we set $A_x = \mathbb{X}$ and $\lambda = s\mathbb{Q}$, Assumptions 2.1, 2.2 and 2.3 could be leveraged to give upper bounds for the following crucial probabilities that appear *implicitly* in the proof of Theorem 3.1:

$$I_s(x) := \mathbb{P}(D_x F_s \neq 0), \quad (3.5)$$

$$J_s(x_1, x_2) := \mathbb{P}(|D_{x_1} F_s - D_{x_2} F_s^{x_2}| \neq 0), \quad (3.6)$$

resulting in the following corollary.

Corollary 3.1. *Suppose $F_s \in L_{\mathcal{P}_s}^2$ and that F_s is strongly stabilizing with the radius of stabilization R_x decaying exponentially (Assumption 2.1). Further suppose its cost function satisfies the $\mathbb{K}$ -exponential bound (Assumption 2.2) and the bounded moment condition for $p > 4$ (Assumption 2.3). Then, there exists a constant $C_0 > 0$ depending only on the constants in (2.5) and (2.6) such that for $s \geq 1$,*

$$d_K \left(\frac{F_s - \mathbb{E}F_s}{\sqrt{\text{Var } F_s}}, N \right) \leq C_0 \left(\frac{\Theta_{\mathbb{K},s}^{\frac{1}{2}}}{\text{Var } F_s} + \frac{\Theta_{\mathbb{K},s}}{(\text{Var } F_s)^{\frac{3}{2}}} + \frac{\Theta_{\mathbb{K},s}^{\frac{5}{4}} + \Theta_{\mathbb{K},s}^{\frac{3}{2}}}{(\text{Var } F_s)^2} \right),$$

where

$$\Theta_{\mathbb{K},s} := s \int_{\mathbb{X}} e^{-C_2 \frac{(p-4)}{4p} \left(\frac{d_s(x, \mathbb{K})}{2} \right)^{C_3}} \mathbb{Q}(dx). \quad (3.7)$$

Corollary 3.2. *Under the conditions of Corollary 3.1, assume there exists a constant $C > 0$ such that*

$$\sup_{s \geq 1} \frac{\Theta_{\mathbb{K},s}}{\text{Var } F_s} \leq C, \quad (3.8)$$

then there exists a constant $C'_0 > 0$ depending on C and (2.5)-(2.6) such that for $s \geq 1$,

$$d_K \left(\frac{F_s - \mathbb{E}F_s}{\sqrt{\text{Var } F_s}}, N \right) \leq C'_0 \frac{1}{\sqrt{\text{Var } F_s}}. \quad (3.9)$$

Next, we introduce the main theorem for binomial point process. The binomial version of Theorem 3.1 is not immediately known. Indeed, for the Poisson case we leverage Theorem 5.1 for our proofs. However, due to the fact that there is no nice counterpart of the second-order Poincaré inequality (see [LPS16]) an analogue of Theorem 5.1 is not known for the binomial case. On the other hand, we point out that it is possible to obtain a similar result based on [LRP17, Theorem 5.1], which serves as a counterpart of Theorem 5.1 for binomial setting. Based on this approach, we now present our result for the Binomial setting.

Theorem 3.2 (Normal Approximation for Functionals of Binomial Point Process). *Suppose $F_n \in L_{\xi_n}^2$ and invoke the binomial version of assumptions in Corollary 3.1. Then, there exists a constant $C_0 > 0$ depending only on the constants in (2.5)-(2.6) such that for $n \geq 2$,*

$$d_K \left(\frac{F_n - \mathbb{E}F_n}{\sqrt{\text{Var } F_n}}, N \right) \leq C_0 \left(\frac{\Theta_{\mathbb{K},n}^{\frac{1}{2}}}{\text{Var } F_n} + \frac{\Theta_{\mathbb{K},n}}{(\text{Var } F_n)^{\frac{3}{2}}} + \frac{\Theta_{\mathbb{K},n} + \Theta_{\mathbb{K},n}^{\frac{3}{2}}}{(\text{Var } F_n)^2} \right), \quad (3.10)$$

where

$$\Theta_{\mathbb{K},n} := n \int_{\mathbb{X}} e^{-C_2 \frac{(p-4)}{4p} \left(\frac{d_n(x, \mathbb{K})}{2} \right)^{C_3}} \mathbb{Q}(dx).$$

Remark 3.2. We now make the following remark on Theorem 3.2. Compared to the Poisson case, the exponent of $\Theta_{\mathbb{K},n}$ in the third component of the sum on the right hand side of (3.10) is different. In essence, this difference can be traced back to a fundamental fact that there is no nice counterpart of the second-order Poincaré inequality (see [LPS16]) for binomial case. Instead, we use the approach taken in [LRSY19, Theorem 4.2] to prove Theorem 3.2.

Corollary 3.3. Under the conditions of Theorem 3.2, assume there exists a constant $C > 0$ such that

$$\sup_{n \geq 1} \frac{\Theta_{\mathbb{K},n}}{\text{Var } F_n} \leq C, \quad (3.11)$$

then there exists a constant $C'_0 > 0$ depending on C and (2.5)-(2.6) such that for $n \geq 2$,

$$d_K \left(\frac{F_n - \mathbb{E}F_n}{\sqrt{\text{Var } F_n}}, N \right) \leq C'_0 \frac{1}{\sqrt{\text{Var } F_n}}. \quad (3.12)$$

Remark 3.3. We make the following remarks about the above results on both Poisson and binomial cases.

- (i) If $\mathbb{K} = \mathbb{X}$ and $\mathbb{Q}(\mathbb{X}) < \infty$ or $\mathbb{Q}$ has a bounded density with respect to the Lebesgue measure on a compact set, the conditions (3.8) and (3.11) can be simplified as

$$\sup_{s \geq 1} \frac{s}{\text{Var } F_s} \leq C, \quad (3.13)$$

$$\sup_{n \geq 1} \frac{n}{\text{Var } F_n} \leq C, \quad (3.14)$$

- (ii) Optimality: Following [LRSY19], we refer to cases where the bounds in (3.9) and (3.12) can be attained, as being presumably optimal. Indeed, Corollary 3.2 and 3.3 show that if the variance of the statistics F_s, F_n are bounded below by $\Theta_{\mathbb{K},s}, \Theta_{\mathbb{K},n}$, respectively, a presumably optimal normal approximation rate is achieved. To give an intuition on why the above situation is referred to as being presumably optimal, note that for the case of sums of i.i.d random variables, non-trivial i.i.d. random variables can be constructed that achieve the upper bounds of the form in (3.9) and (3.12). Formal lower bounds on the optimality are available for the case of integer-valued statistics in [Eng81] and [PRR13]. Furthermore, in a recent work, [SY21] established lower bounds for a large class of statistics.

Comparison to related works. We now provide some comparisons to the related work. Firstly, our proof techniques, similar to [LRSY19], are based on several central ideas proposed in [LPS16]. For the case of functionals that are expressible as sums of score functions, [LRSY19] established presumably optimal bounds under the score-based stabilization assumption, for both the binomial and Poisson cases. While they too use second-order cost operators, our Theorem 3.1 and Theorem 3.2 handle a much larger class of functionals in comparison (not necessarily as sums of scores). The work of [LRPY22] consider general functionals (not necessarily sums) and work under strong stabilization assumption. However, they only consider the Poisson case. To get explicit bounds (e.g., their Corollary 1.5 and Proposition 1.12), they introduce a specific form of A_x in their proofs and their overall approach results in sub-optimal rates in comparison to our results, Corollary 3.1, and to [LRSY19] in the case when the functional is expressible as sums of scores. Our Theorem 3.1 generalizes [LPS16, Theorem 1.2] by introducing the flexible cost function $D_x F_s(A_x)$ for general functionals of Poisson point process. The work of [LRP17] also consider normal

approximations of general functions (not necessarily as sums). However, their approach is only valid for the binomial case. Moreover, a further investigation of their main theorem [LRP17, Theorem 4.2] reveals that instead of introducing stabilization notions, their normal approximation bounds are obtained by computing some quantities (for example, T, T' in [LRP17, Section 4]), which are complicated to deal with for some functionals, e.g., Euler characteristic; see [KRP21, Proof of Theorem 3.2].

Applying our main results. We conclude this section, with the following three-step procedure illustrating how to apply our main theorems, Theorem 3.1 and Theorem 3.2.

- *Step 1:* Check if the functional F is strongly stabilization (i.e., Definition 2.5), if the tail probability of the radius of stabilization (Assumptions 2.1) could be computed, and verify Assumption 2.2 on the cost functions.
 - If the functional is not strongly stabilizing or no upper bound of the radius of stabilization R_x is known, consider the flexible cost functions $D_x F_s(A_x)$ with appropriate choice of A_x and apply Theorem 3.1.
- *Step 2:* Check bounded moment condition, i.e., Assumption 2.3.
- *Step 3:* In order to check for presumable optimality, one can seek to bound the variance, i.e., (3.13) and (3.14).

If the above three steps are satisfied, apply Corollary 3.1 and Theorem 3.2 for the Poisson and binomial settings respectively.

4. APPLICATIONS

In this section, we illustrate the applicability of our bounds in Theorem 3.1, Corollary 3.1 and Theorem 3.2 on several geometric and topological statistics.

4.1. Total Edge Length of k -Nearest Neighbor Graphs. Recall the definition of k -NN Graphs in Example 2.2 and define the total edge length of a k -NN graph as:

$$F_s^{k\text{-NN}}(\mathcal{P}_s) := \sum_{x \in \mathcal{P}_s} f_s(x, \mathcal{P}_s), \quad (4.1)$$

with f_s as defined in (2.4). similarly, we define $F_n^{k\text{-NN}}$ for an underlying binomial point process.

Theorem 4.1. *Assume there exists a constant $c > 0$ such that, for $r \leq \text{diam}(\mathbb{X}) < \infty$,*

$$\inf_{x \in \mathbb{X}} \mathbb{Q}(B_x(r)) \geq cr^\omega. \quad (4.2)$$

If there exists a constant $C > 0$ such that

$$\sup_{s \geq 1} \frac{s}{\text{Var } F_s^{k\text{-NN}}(\mathcal{P}_s)} \leq C, \quad (4.3)$$

then there exists a constant $C_0 > 0$ such that for $s \geq 1$,

$$d_K \left(\frac{F_s^{k\text{-NN}}(\mathcal{P}_s) - \mathbb{E} F_s^{k\text{-NN}}(\mathcal{P}_s)}{\sqrt{\text{Var } F_s^{k\text{-NN}}(\mathcal{P}_s)}}, N \right) \leq C_0 \frac{1}{\sqrt{s}}.$$

And if there exists a constant $C > 0$ such that

$$\sup_{n \geq 1} \frac{n}{\text{Var } F_n^{k\text{-NN}}(\xi_n)} \leq C, \quad (4.4)$$

then for $n \geq 2$,

$$d_K \left(\frac{F_n^{k-NN}(\xi_n) - \mathbb{E}F_n^{k-NN}(\xi_n)}{\sqrt{\text{Var}F_n^{k-NN}(\xi_n)}}, N \right) \leq C_0 \frac{1}{\sqrt{n}}.$$

Remark 4.1. We make the following remarks about the above result.

- (i) Condition (4.2), is required in addition to the (2.5) for the k -NN statistic; see [LRSY19] for details. Note that the total edge length of a k -NN graph (4.1) is expressible as a sum of score functions. Hence, the results in [LRSY19] already provide presumably optimal bounds. Our results above also recover the same bounds.
- (ii) Now we compare our results to [LRPY22] in the Poisson setting. Recall that similar to our work, they considered general functionals (not necessarily expressible as sums of scores). However, their generality comes at the cost of not having presumably optimal bounds in the setting of the total edge length of a k -NN graph. Specifically, [LRPY22, Proposition 1.12], term $\sqrt{b_n/n}$ with $b_n \rightarrow \infty$ implies that it has a slower rate than $1/\sqrt{n}$. This highlights the benefit of our approach: despite its generality, we still obtain presumably optimal bounds for this specific special case.
- (iii) For the binomial setting, [LRP17] obtained rates in the Kolmogorov metric for the same statistic. However, as discussed in [LRSY19, Remark (i) below Theorem 3.1], their results are sub-optimal and involve additional logarithmic factors, that we avoid.
- (iv) When we consider $\mathbb{X}$ as a full-dimensional compact convex subset of $\mathbb{R}^d$, $\omega = d$, as shown in [PY01, Proof of Theorem 6.1]¹, the conditions (4.3) and (4.4) are satisfied.

4.2. Shannon Entropy. Given an i.i.d. sample $X_1, X_2, \dots, X_n$ from a density q on $\mathbb{R}^d$, the differential (Shannon) entropy is defined as $H(q) := -\mathbb{E}_{X \sim q} \log q(X) = -\int_{\mathbb{R}^d} q(x) \log q(x) dx$. The nearest neighbor entropy estimate, also known as the Kozachenko-Leonenko estimator, was first proposed in [KL87] based on the 1-NN density estimator. A generalization of this estimator based on k -NN density estimator is given by

$$\frac{1}{n} \sum_{i=1}^n \log \left(\frac{(n-1)V_d \rho_{k,i}^d}{e^{\Psi(k)}} \right),$$

where $\rho_{k,i}$ is the distance between X_i and its k -nearest neighbor among $X_1, X_2, \dots, X_n$, $V_d := \pi^{\frac{d}{2}}/\Gamma(1 + \frac{d}{2})$ is the volume of a unit d -dimensional Euclidean ball, $\Psi(k) = -\gamma + \sum_{i=1}^{k-1} 1/i$ is the digamma function and γ is the Euler-Mascheroni constant [PY13, BSY19].

The consistency and CLT for the above estimator in a manifold setting were shown in [PY13] by stabilization theory. However, a non-trivial bias term arises for $d \geq 4$, rendering the above estimator asymptotically inefficient (in the sense of [VdV00, page 367]). To have an (asymptotically) unbiased and efficient estimator, the following weighted k -NN estimator was proposed in [BSY19]. Defining ξ_n as the binomial point process associated with $X_1, X_2, \dots, X_n$ the proposed estimator could be viewed as a functional of ξ_n , and is given by

$$F_n^{\text{SE}}(\xi_n) := \sum_{i=1}^n f_n^w(X_i, \xi_n) \quad \text{where} \quad f_n^w(X_i, \xi_n) := \frac{1}{n} \sum_{j=1}^k w_j \log \left(\frac{(n-1)V_d \rho_{j,i}^d}{e^{\Psi(j)}} \right),$$

and w_j are the weights (such that $\sum_{j=1}^n w_j = 1$) that are chosen to cancel the dominant bias term and make F_n^{SE} asymptotically efficient. We now provide our normal approximation

¹[PY01] consider the case of $\vartheta = 1$. However, a closer examination of the proof shows that it can be easily extended for any $\vartheta > 0$.

results for the above estimator, based on a slightly modified set of assumptions considered in [BSY19].

Theorem 4.2. *Consider a density q supported on a compact set $\mathbb{X} \subset \mathbb{R}^d$ with respect to the Lebesgue measure $\mathbb{Q}$ in $\mathbb{R}^d$. Let $\mathcal{A}$ denote the class of all decreasing functions $a : (0, \infty) \rightarrow [1, \infty)$ such that $a(\delta) = o(\delta^{-\epsilon})$ as $\delta \searrow 0$, for every $\epsilon > 0$. For $a \in \mathcal{A}$, let q be $m := \lceil \beta \rceil - 1$ times differentiable (for $\beta > 0$). For $x \in \mathbb{X}$, let $r_a(x) := (8d^{\frac{1}{2}}a(q(x)))^{-\frac{1}{\beta+1}}$ and define:*

$$M_{q,a,\beta}(x) := \max \left\{ \max_{t=1,\dots,m} \frac{\|q^{(t)}(x)\|}{q(x)}, \sup_{y \in B_x^o(r_a(x))} \frac{\|q^{(m)}(y) - q^{(m)}(x)\|}{q(x)\|y - x\|^{\beta-m}} \right\},$$

where $B_x^o(r) := B_x(r) \setminus \{x\}$. Let the density q also satisfy that following conditions:

$$\|q\|_\infty \leq \gamma, \quad \sup_{x:q(x) \geq \delta} M_{q,a,\beta}(x) \leq a(\delta), \quad \forall \delta > 0.$$

Define the class of weights as follows: for $k \in \mathbb{N}$, let

$$\mathcal{W}^k := \left\{ w \in \mathbb{R}^k : \sum_{j=1}^k w_j \frac{\Gamma(j+\frac{2l}{d})}{\Gamma(j)} = 0, \text{ for } l = 1, \dots, \lfloor \frac{d}{4} \rfloor, \sum_{j=1}^k w_j = 1, w_j = 0, \text{ if } j \notin \left\{ \lfloor \frac{k}{d} \rfloor, \lfloor \frac{2k}{d} \rfloor, \dots, k \right\} \right\}. \quad (4.5)$$

Then under the conditions of [BSY19, Theorem 1], that is, for any $\alpha > d$, $\beta > \frac{d}{2}$ and for any two deterministic sequences of positive integers $k_{0,n}^* = k_0^*$, $k_{1,n}^* = k_1^*$ with $k_0^* \leq k_1^*$, $k_0^*/\log^5 n \rightarrow \infty$, $k_1^* = O(n^{\tau_1})$ and $k_1^* = o(n^{\tau_2})$, where, with $\beta^* := \beta \wedge 1$,

$$\tau_1 < \min \left\{ \frac{2\alpha}{5\alpha + 3d}, \frac{\alpha - d}{2\alpha}, \frac{4\beta^*}{4\beta^* + 3d} \right\}, \quad \text{and} \quad \tau_2 := \min \left\{ 1 - \frac{\frac{d}{4}}{1 + \lfloor \frac{d}{4} \rfloor}, 1 - \frac{d}{2\beta} \right\},$$

as well as the assumption (4.2), there exists a constant $C_0 > 0$ (independent of k, n) such that

$$d_K \left(\frac{F_n^{SE}(\xi_n) - H(q)}{\sqrt{\text{Var} F_n^{SE}(\xi_n)}}, N \right) \leq C_0 \sqrt{\frac{k}{n}},$$

for $k_0^* \leq k \leq k_1^*$.

Remark 4.2. We make the following remarks regarding the above result.

- (i) Asymptotic limit theorems for estimators of the Shannon entropy have been obtained, for example, in [PY13], [BSY19]. The result in [PY13] is a non-central limit theorem, as their estimator suffers from bias in higher dimensions. The result in [BSY19], is a central limit theorem, which was established under the case that the density is supported on $\mathbb{R}^d$. However, no normal convergence rate results were provided in the above works. To our best knowledge, the above result, is the first normal convergence rate result with the true center $H(q)$.
- (ii) It is also possible to obtain a similar result using the method in [LRSY19] since the estimator $F_n^{SE}(\xi_n)$ is expressible as a sum of score functions.
- (iii) Furthermore, the result in Theorem 4.2 is provided for the binomial case. The asymptotic unbiasedness and efficiency of the weighted k -NN estimator of the Shannon entropy based on Poisson point process is open, to the best of our knowledge.

4.3. Euler Characteristic. Recall Example 2.1. Following the setting of [KRP21], consider a bounded density q on $[0, 1]^d$. Let ξ_n be a binomial point process associated with n i.i.d. samples according to the density q and let $\mathcal{P}_n$ be a Poisson point process with intensity measure $n\mathbb{Q}$, where $\mathbb{Q}$ has a density q with respect to the Lebesgue measure, i.e., we set $s = n$.

Construct the Čech complex or the Vietoris-Rips complex $K_r(n^{\frac{1}{d}}\mathcal{P}_n), K_r(n^{\frac{1}{d}}\xi_n)$, see Definition A.2 and A.3 based on the Poisson point process $\mathcal{P}_n$ and the binomial point process ξ_n respectively with $r > 0$ as the filtration time. Here, K_r represents both complexes for simplicity. The factor $n^{\frac{1}{d}}$ corresponds to the thermodynamic/critical regime [GTT19, OT20, Tri17] such that this is equivalent to the case $nr_n^d \rightarrow r \in (0, \infty)$ with r_n as the filtration time. With the above construction, the Euler characteristic is given by

$$F_n^{\text{EC}}(\mathcal{P}_n) := \chi(K_r(n^{\frac{1}{d}}\mathcal{P}_n)) \quad \text{and} \quad F_n^{\text{EC}}(\xi_n) := \chi(K_r(n^{\frac{1}{d}}\xi_n))$$

where $\chi(K(\eta))$ for a filtration K constructed from a point cloud sampled from a point process η is defined in Example 2.1.

Theorem 4.3. *Under the above setting, for some $T > 0$ such that $0 < r \leq T$, there exists a constant $C_0 > 0$ such that for $n \geq 1$,*

$$d_K \left(\frac{F_n^{\text{EC}}(\mathcal{P}_n) - \mathbb{E}F_n^{\text{EC}}(\mathcal{P}_n)}{\sqrt{\text{Var}F_n^{\text{EC}}(\mathcal{P}_n)}}, N \right) \leq C_0 \frac{1}{\sqrt{n}}.$$

And for $n \geq 2$,

$$d_K \left(\frac{F_n^{\text{EC}}(\xi_n) - \mathbb{E}F_n^{\text{EC}}(\xi_n)}{\sqrt{\text{Var}F_n^{\text{EC}}(\xi_n)}}, N \right) \leq C_0 \frac{1}{\sqrt{n}}.$$

Remark 4.3. *We make the following remarks about the above result.*

- (1) CLTs and functional limit theorem for Euler characteristic has been studied in [TO21] by viewing Euler characteristic as a process indexed by r . Normal approximation rate of Euler characteristic under binomial and Poisson sampling was obtained in [KRP21], by computing certain geometric quantities appearing in the general result in [LRP17]. Our flexible stabilization method has advantages of avoiding computing several complicated geometric quantities (as done in [KRP21, Proof of Theorem 3.2]).
- (2) For the Poisson case, [LRPY22] require a specific form of A_x , rendering their result sub-optimal, i.e., the term $\sqrt{b_n/n}$ in [LRPY22, Proposition 1.12], leads to a slower rate than $1/\sqrt{n}$ that we obtain above.
- (3) While Euler characteristic could also be expressible as a sum of score functions, one could possibly leverage the results of [LRSY19] to derive normal convergence rate. Our goal in this example is to demonstrate the flexibility of our general result.

All above applications consider stabilizing statistics when there are known tail bounds for the radius of stabilization R_x , i.e., quantities (3.5) to (3.6). Our Theorem 3.1, however, can deal with the case when we do not have immediate bounds for those probabilities based on the flexible cost function $D_x F_s(A_x)$. We illustrate the above mentioned idea by the following application concerning the minimal spanning tree.

4.4. Edge length of the Minimal Spanning Tree. Consider a finite set $V \subset \mathbb{R}^d$ (usually it is embedded in an underlying graph $G := (V, E)$). A minimal spanning tree T of V is a connected graph with the vertex set V . Define

$$M(V) := \min_T \sum_{e \text{ is an edge of } T} |e|,$$

where the minimal is taken over all possible minimal spanning tree T of V . According to [Pen05] and [CS17], the total edge length statistic $M(V)$ does satisfy certain required stabilization properties. However, to the best of our knowledge, there is no result on the rates of stabilization including quantitative bound on the tail probability of the radius of stabilization. This results in a major difficulty of deriving normal approximation rate for $M(V)$ by some classical methods like [LRSY19]. Our flexible stabilization method can yield the following theorem by picking A_x “strategically” to make use of some existing bounds.

Theorem 4.4. *Following the Euclidean setting in [LRPY22], consider B_0 as the unit hypercube in $\mathbb{R}^d$ centered at the origin and let $B_n := nB_0$, $n \in \mathbb{N}_+$. Given a homogeneous Poisson process $\mathcal{P}(\lambda)$ on $\mathbb{R}^d$ with intensity $\lambda > 0$, let*

$$F_{B_n}^{MST}(\mathcal{P}(\lambda)) := M(\mathcal{P}(\lambda)|_{B_n}).$$

Then, there exist constants $C_0 > 0, 1 > D_1 > 0$ and $D_2 > 0$ not depending on n such that

$$d_K \left(\frac{F_{B_n}^{MST}(\mathcal{P}(\lambda)) - \mathbb{E} F_{B_n}^{MST}(\mathcal{P}(\lambda))}{\sqrt{\text{Var } F_{B_n}^{MST}(\mathcal{P}(\lambda))}}, N \right) \leq \begin{cases} C_0 n^{-D_1}, & \text{if } d = 2, \\ C_0 (\log n)^{-D_2}, & \text{if } d \geq 3. \end{cases}$$

Remark 4.4. *The normal approximation rate of the edge length statistic of the minimal spanning tree has been derived previously in [CS17] and in [LRPY22] with similar results as Theorem 4.4. However, [CS17] only focused on the minimal spanning tree therefore it is hard to generalize for other stabilizing functionals. [LRPY22] used the similar idea of introducing the set A_x but their bounds usually give sub-optimal normal convergence rates (for e.g., see Remark 4.1 regarding the total edge length of k -nearest neighbor graphs in Section 4.1) than ours due to the specific form of their set A_x lacking flexibility.*

5. PROOFS FOR SECTION 3

5.1. Proof of Theorem 3.1. We first prove our main theorem for Poisson case, Theorem 3.1, based on the flexible cost function $D_x F(A_x)$. Without loss of generality, all the constant $C > 0$ mentioned in this section refer to universal constants that might take different values in each step. Our key tool for proving Theorem 3.1 is based on [LPS16, Theorem 1.2], which we restate below.

Theorem 5.1 ([LPS16]). *Let F be a measurable functional satisfying the following two conditions:*

$$\mathbb{E} F(\mathcal{P}(\lambda))^2 < \infty \quad \text{and} \quad \mathbb{E} \int (D_x F)^2 \lambda(dx) < \infty.$$

Let N be a standard normal random variable. Then,

$$d_K \left(\frac{F - \mathbb{E} F}{\sqrt{\text{Var } F}}, N \right) \leq \sum_{i=1}^6 \gamma_i,$$

where

$$\begin{aligned}
\gamma_1 &:= \frac{4}{\text{Var } F} \left(\int (\mathbb{E}(D_{x_1} F)^2 (D_{x_2} F)^2)^{\frac{1}{2}} (\mathbb{E}(D_{x_1, x_3} F)^2 (D_{x_2, x_3} F)^2)^{\frac{1}{2}} \lambda^3(d(x_1, x_2, x_3)) \right)^{\frac{1}{2}}, \\
\gamma_2 &:= \frac{1}{\text{Var } F} \left(\int (\mathbb{E}(D_{x_1, x_3} F)^2 (D_{x_2, x_3} F)^2)^2 \lambda^3(d(x_1, x_2, x_3)) \right)^{\frac{1}{2}}, \\
\gamma_3 &:= \frac{1}{(\text{Var } F)^{\frac{3}{2}}} \int \mathbb{E} |D_x F|^3 \lambda(dx), \\
\gamma_4 &:= \frac{1}{2(\text{Var } F)^2} (\mathbb{E}(F - \mathbb{E} F)^4)^{\frac{1}{4}} \int (\mathbb{E}(D_x F)^4)^{\frac{3}{4}} \lambda(dx), \\
\gamma_5 &:= \frac{1}{\text{Var } F} \left(\int \mathbb{E}(D_x F)^4 \lambda(dx) \right)^{\frac{1}{2}}, \\
\gamma_6 &:= \frac{1}{\text{Var } F} \left(\int 6(\mathbb{E}(D_{x_1} F)^4)^{\frac{1}{2}} (\mathbb{E}(D_{x_1, x_2} F)^4)^{\frac{1}{2}} + 3\mathbb{E}(D_{x_1, x_2} F)^4 \lambda^2(d(x_1, x_2)) \right)^{\frac{1}{2}}.
\end{aligned}$$

Proof of Theorem 3.1. The idea is to use the above theorem, and to bound the first and second order cost functions appearing in the quantities $\gamma_i, i = 1, \dots, 6$ by using our flexible approach. To this end, we rewrite the first and second order cost functions as:

$$D_x F_s = (D_x F_s - D_x F_s(A_x)) + D_x F_s(A_x),$$

and

$$D_{x_1, x_3} F_s = (D_{x_3} F_s^{x_1} - D_{x_3} F_s^{x_1}(A_{x_3})) + (D_{x_3} F_s^{x_1}(A_{x_3}) - D_{x_3} F_s(A_{x_3})) + (D_{x_3} F_s(A_{x_3}) - D_{x_3} F_s).$$

We start with γ_3 . By the fact that $(a + b)^3 \leq 4(a^3 + b^3)$ for any $a \geq 0, b \geq 0$, we have

$$\begin{aligned}
(\text{Var } F)^{\frac{3}{2}} \gamma_3 &= \int \mathbb{E} |(D_x F - D_x F(A_x)) + D_x F(A_x)|^3 \lambda(dx) \\
&\leq \int 4\mathbb{E} |D_x F - D_x F(A_x)|^3 \lambda(dx) + \int 4\mathbb{E} |D_x F(A_x)|^3 \lambda(dx).
\end{aligned}$$

By Hölder's inequality and the assumptions in Theorem 3.1, we then have

$$\mathbb{E} |D_x F(A_x)|^3 \leq (\mathbb{E} |D_x F(A_x)|^4)^{\frac{3}{4}} 1^{1-\frac{3}{4}} \leq b_2(x, A_x)^{\frac{3}{4}}. \quad (5.1)$$

Similarly,

$$\mathbb{E} |D_x F - D_x F(A_x)|^3 \leq b_1(x, A_x)^{\frac{3}{4}}. \quad (5.2)$$

Therefore,

$$(\text{Var } F)^{\frac{3}{2}} \gamma_3 \leq C \int \sum_{j=1}^2 b_j(x, A_x)^{\frac{3}{4}} \lambda(dx) := C(\text{Var } F)^{\frac{3}{2}} \gamma'_3.$$

Next, we turn to γ_4 . According to [LPS16, Lemma 4.3], we have

$$\mathbb{E}(F - \mathbb{E} F)^4 \leq \max \left\{ 256 \left(\int (\mathbb{E}(D_x F)^4)^{\frac{1}{2}} \lambda(dx) \right)^2, 4 \int \mathbb{E}(D_x F)^4 \lambda(dx) + 2(\text{Var } F)^2 \right\}.$$

Consequently,

$$\gamma_4 \leq \frac{\int (\mathbb{E}(D_x F)^4)^{\frac{3}{4}} \lambda(dx)}{2(\text{Var } F)^2} \left(4 \left(\int (\mathbb{E}(D_x F)^4)^{\frac{1}{2}} \lambda(dx) \right)^{\frac{1}{2}} + \sqrt{2} \left(\int \mathbb{E}(D_x F)^4 \lambda(dx) \right)^{\frac{1}{4}} + 2^{\frac{1}{4}} (\text{Var } F)^{\frac{1}{2}} \right).$$

By calculations similar to γ_3 , we have

$$\int (\mathbb{E}(D_x F)^4)^{\frac{1}{2}} \lambda(dx) \leq C \int \sum_{j=1}^2 b_j(x, A_x)^{\frac{1}{2}} \lambda(dx).$$

Similarly,

$$\begin{aligned} \int (\mathbb{E}(D_x F)^4)^{\frac{3}{4}} \lambda(dx) &\leq C \int \sum_{j=1}^2 b_j(x, A_x)^{\frac{3}{4}} \lambda(dx), \\ \int \mathbb{E}(D_x F_s)^4 \lambda(dx) &\leq C \int \sum_{j=1}^2 b_j(x, A_x) \lambda(dx). \end{aligned}$$

Combining all above, we have

$$\begin{aligned} \gamma_4 \leq \frac{C \int \sum_{j=1}^2 b_j(x, A_x)^{\frac{3}{4}} \lambda(dx)}{(\text{Var } F)^2} &\left(\left(\int \sum_{j=1}^2 b_j(x, A_x)^{\frac{1}{2}} \lambda(dx) \right)^{\frac{1}{2}} \right. \\ &\left. + \left(\int \sum_{j=1}^2 b_j(x, A_x) \lambda(dx) \right)^{\frac{1}{4}} + (\text{Var } F)^{\frac{1}{2}} \right) := C\gamma'_4. \end{aligned}$$

Furthermore, by similar arguments on bounding γ_3 , it leads to

$$\gamma_5 \leq \frac{C}{\text{Var } F} \left(\int \sum_{j=1}^2 b_j(x, A_x) \lambda(dx) \right)^{\frac{1}{2}} := C\gamma'_5.$$

We next move on to bounding the remaining part: γ_2 , γ_2 and γ_6 , that involve the second order cost functions. Similarly to the first order cost function, we have

$$\mathbb{E}(D_{x_1, x_2} F)^4 \leq C \sum_{j=3}^5 b_j(x_1, x_2, A_{x_1}). \quad (5.3)$$

Using Cauchy-Schwarz inequality, we have

$$(\mathbb{E}(D_{x_1} F)^2 (D_{x_2} F)^2)^{\frac{1}{2}} \leq \mathbb{E}(D_{x_1} F)^4)^{\frac{1}{4}} \mathbb{E}(D_{x_2} F)^4)^{\frac{1}{4}}, \quad (5.4)$$

$$(\mathbb{E}(D_{x_1, x_3} F)^2 (D_{x_2, x_3} F)^2)^{\frac{1}{2}} \leq \mathbb{E}(D_{x_1, x_3} F)^4)^{\frac{1}{4}} \mathbb{E}(D_{x_2, x_3} F)^4)^{\frac{1}{4}}. \quad (5.5)$$

With all results above, according to (5.1) to (5.5), we give an upper bound for γ_1 in a similar way by Hölder's inequality:

$$\begin{aligned} \gamma_1 &\leq \frac{C}{\text{Var } F} \left(\int \left(\sum_{j=1}^2 b_j(x_1, A_{x_1})^{\frac{1}{4}} \sum_{j=1}^2 b_j(x_2, A_{x_2})^{\frac{1}{4}} \right. \right. \\ &\quad \left. \left. \sum_{j=3}^5 b_j(x_3, x_1, A_{x_3})^{\frac{1}{4}} \sum_{j=3}^5 b_j(x_3, x_2, A_{x_3})^{\frac{1}{4}} \right) \lambda^3(d(x_1, x_2, x_3)) \right)^{\frac{1}{2}} \\ &:= C\gamma'_1. \end{aligned}$$

Similarly, we have

$$\begin{aligned}\gamma_2 &\leq \frac{C}{\text{Var } F} \left(\int \sum_{j=3}^5 b_j(x_3, x_1, A_{x_3}) \sum_{j=3}^5 b_j(x_3, x_2, A_{x_3}) \lambda^3(d(x_1, x_2, x_3)) \right)^{\frac{1}{2}} \\ &:= C\gamma'_2, \\ \gamma_6 &\leq \frac{C}{\text{Var } F} \left(\int \sum_{j=1}^2 b_j(x_1, A_{x_1})^{\frac{1}{2}} \sum_{j=3}^5 b_j(x_1, x_2, A_{x_1})^{\frac{1}{2}} + \sum_{j=3}^5 b_j(x_1, x_2, A_{x_1}) \lambda^2(d(x_1, x_2)) \right)^{\frac{1}{2}} \\ &:= C\gamma'_6.\end{aligned}$$

Combining the obtained bounds above for γ_i , $1 \leq i \leq 6$, we complete the proof. $\square$

5.2. Proof of Corollary 3.1. Without loss of generality, we assume $c_1 = k_1 := C_1$, $c_2 = k_2 := C_2$ and $c_3 = k_3 := C_3$. In Theorem 3.1, we set $A_x = \mathbb{X}$ and $\lambda = s\mathbb{Q}$. Then, we can set

$$\begin{aligned}b_1(x, A_x) &= 0, \\ b_3(x_1, x_2, A_{x_1}) &= 0, \\ b_4(x_1, x_{12}, A_{x_1}) &= 0.\end{aligned}$$

According to Assumption 2.3, by Hölder's inequality, we have:

$$b_2(x, \mathbb{X}) := \mathbb{E}|D_x F_s|^4 \leq \mathbb{E}|D_x F_s|^p \mathbb{P}(D_x F_s \neq 0)^{1-\frac{4}{p}} \leq C \mathbb{P}(D_x F_s \neq 0)^{1-\frac{4}{p}},$$

and

$$\begin{aligned}b_5(x_1, x_2, \mathbb{X}) &:= \mathbb{E}|D_{x_1} F_s^{x_2} - D_{x_1} F_s|^4 \leq \mathbb{E}|D_{x_1} F_s^{x_2} - D_{x_1} F_s|^p \mathbb{P}(|D_{x_1} F_s^{x_2} - D_{x_1} F_s| \neq 0)^{1-\frac{4}{p}} \\ &\leq C \mathbb{P}(|D_{x_1} F_s^{x_2} - D_{x_1} F_s| \neq 0)^{1-\frac{4}{p}}.\end{aligned}$$

Define

$$\begin{aligned}\phi_s &:= s \int_{\mathbb{X}} \mathbb{P}(D_x F_s \neq 0)^{\frac{p-4}{2p}} \mathbb{Q}(dx), \\ \psi_s(x_1, x_2) &:= \mathbb{P}(|D_{x_1} F_s^{x_2} - D_{x_1} F_s| \neq 0)^{\frac{p-4}{4p}}.\end{aligned}$$

With all b_i , $1 \leq i \leq 5$, given above, we will bound all γ'_i , $1 \leq i \leq 6$ in Theorem 3.1. We again start with γ'_3 :

$$\begin{aligned}\gamma'_3 &= \frac{1}{(\text{Var } F_s^{\frac{3}{2}})} \int \sum_{j=1}^2 b_j(x, A_x)^{\frac{3}{4}} \lambda(dx) \\ &= C s \int_{\mathbb{X}} \mathbb{P}(D_x F_s \neq 0)^{(1-\frac{4}{p})\frac{3}{4}} \mathbb{Q}(dx) \\ &\leq C \phi_s.\end{aligned}$$

Similarly, as for γ'_3 , it holds that

$$\begin{aligned}\gamma'_5 &= \frac{1}{\text{Var } F_s} \left(\int \sum_{j=1}^2 b_j(x, A_x) \lambda(dx) \right)^{\frac{1}{2}} \\ &= \frac{1}{\text{Var } F_s} \left(s \int_{\mathbb{X}} \mathbb{P}(D_x F_s \neq 0)^{1-\frac{4}{p}} \mathbb{Q}(dx) \right)^{\frac{1}{2}}\end{aligned}$$

$$\leq C\phi_s^{\frac{1}{2}}.$$

For γ'_4 , we have

$$\begin{aligned} \gamma'_4 = \frac{\int \sum_{j=1}^2 b_j(x, A_x)^{\frac{3}{4}} \lambda(dx)}{(\text{Var } F_s)^2} & \left(\left(\int \sum_{j=1}^2 b_j(x, A_x)^{\frac{1}{2}} \lambda(dx) \right)^{\frac{1}{2}} \right. \\ & \left. + \left(\int \sum_{j=1}^2 b_j(x, A_x) \lambda(dx) \right)^{\frac{1}{4}} + (\text{Var } F_s)^{\frac{1}{2}} \right). \end{aligned} \quad (5.6)$$

Respectively,

$$\begin{aligned} \left(\int \sum_{j=1}^2 b_j(x, A_x)^{\frac{1}{2}} \lambda(dx) \right)^{\frac{1}{2}} &= \left(s \int_{\mathbb{X}} \mathbb{P}(D_x F_s \neq 0)^{(1-\frac{4}{p})\frac{1}{2}} \mathbb{Q}(dx) \right)^{\frac{1}{2}} \\ &\leq C\phi_s^{\frac{1}{2}}, \end{aligned}$$

$$\begin{aligned} \left(\int \sum_{j=1}^2 b_j(x, A_x) \lambda(dx) \right)^{\frac{1}{4}} &= \left(s \int_{\mathbb{X}} \mathbb{P}(D_x F_s \neq 0)^{1-\frac{4}{p}} \mathbb{Q}(dx) \right)^{\frac{1}{4}} \\ &\leq C\phi_s^{\frac{1}{4}}, \end{aligned}$$

and

$$\begin{aligned} \int \sum_{j=1}^2 b_j(x, A_x)^{\frac{3}{4}} \lambda(dx) &= s \int_{\mathbb{X}} \mathbb{P}(D_x F_s \neq 0)^{(1-\frac{4}{p})\frac{3}{4}} \mathbb{Q}(dx) \\ &\leq C\phi_s. \end{aligned}$$

From the above calculations, we see that the exponent $(p-4)/2p$ is set so that indeed ϕ_s provides an upper bound for all the terms appearing in the right hand of (5.6). Hence, we have

$$\gamma'_4 \leq C \left(\frac{\phi_s^{\frac{3}{2}} + \phi_s^{\frac{5}{4}}}{(\text{Var } F_s)^2} + \frac{\phi_s}{(\text{Var } F_s)^{\frac{3}{2}}} \right).$$

For terms that include b_j , $3 \leq j \leq 5$, we have

$$\begin{aligned} \gamma'_1 &= \frac{1}{\text{Var } F_s} \left(\int \left(\sum_{j=1}^2 b_j(x_1, A_{x_1})^{\frac{1}{4}} \sum_{j=1}^2 b_j(x_2, A_{x_2})^{\frac{1}{4}} \right. \right. \\ & \quad \left. \left. \sum_{j=3}^5 b_j(x_3, x_1, A_{x_3})^{\frac{1}{4}} \sum_{j=3}^5 b_j(x_3, x_2, A_{x_3})^{\frac{1}{4}} \right) \lambda^3(d(x_1, x_2, x_3)) \right)^{\frac{1}{2}} \\ &\leq C \frac{s^{\frac{3}{2}}}{\text{Var } F_s} \left(\int_{\mathbb{X}^3} \left(\mathbb{P}(|D_{x_1} F_s^{x_2} - D_{x_1} F_s| \neq 0)^{\frac{p-4}{4p}} \right. \right. \\ & \quad \left. \left. \mathbb{P}(|D_{x_2} F_s^{x_3} - D_{x_2} F_s| \neq 0)^{\frac{p-4}{4p}} \right) \mathbb{Q}^3(d(x_1, x_2, x_3)) \right)^{\frac{1}{2}} \end{aligned}$$

$$= C \frac{s^{\frac{3}{2}}}{\text{Var } F_s} \sqrt{\int_{\mathbb{X}} \left(\int_{\mathbb{X}} \psi_s(x_1, x_2) \mathbb{Q}(dx_2) \right)^2 \mathbb{Q}(dx_1)}.$$

Similarly,

$$\begin{aligned} \gamma'_2 &= \frac{C}{\text{Var } F_s} \left(\int \sum_{j=3}^5 b_j(x_3, x_1, A_{x_3}) \sum_{j=3}^5 b_j(x_3, x_2, A_{x_3}) \lambda^3(d(x_1, x_2, x_3)) \right)^{\frac{1}{2}} \\ &\leq C \frac{s^{\frac{3}{2}}}{\text{Var } F_s} \left(\int_{\mathbb{X}^3} \left(\mathbb{P}(|D_{x_1} F_s^{x_2} - D_{x_1} F_s| \neq 0)^{\frac{p-4}{p}} \right. \right. \\ &\quad \left. \left. \mathbb{P}(|D_{x_2} F_s^{x_3} - D_{x_2} F_s| \neq 0)^{\frac{p-4}{p}} \right) \mathbb{Q}^3(d(x_1, x_2, x_3)) \right)^{\frac{1}{2}} \\ &= C \frac{s^{\frac{3}{2}}}{\text{Var } F_s} \sqrt{\int_{\mathbb{X}} \left(\int_{\mathbb{X}} \psi_s(x_1, x_2) \mathbb{Q}(dx_2) \right)^2 \mathbb{Q}(dx_1)}, \end{aligned}$$

and

$$\begin{aligned} \gamma'_6 &= \frac{1}{\text{Var } F_s} \left(\int \sum_{j=1}^2 b_j(x_1, A_{x_1})^{\frac{1}{2}} \sum_{j=3}^5 b_j(x_1, x_2, A_{x_1})^{\frac{1}{2}} + \sum_{j=3}^5 b_j(x_1, x_2, A_{x_1}) \lambda^2(d(x_1, x_2)) \right)^{\frac{1}{2}} \\ &\leq C \frac{s}{\text{Var } F_s} \left(\int_{\mathbb{X}^2} \mathbb{P}(|D_{x_2} F_s^{x_3} - D_{x_2} F_s| \neq 0)^{\frac{p-4}{2p}} \right. \\ &\quad \left. + \mathbb{P}(|D_{x_2} F_s^{x_3} - D_{x_2} F_s| \neq 0)^{\frac{p-4}{p}} \mathbb{Q}^2(d(x_1, x_2)) \right)^{\frac{1}{2}} \\ &\leq C \frac{s}{\text{Var } F_s} \left(\int_{\mathbb{X}^2} \mathbb{P}(|D_{x_2} F_s^{x_3} - D_{x_2} F_s| \neq 0)^{\frac{p-4}{2p}} \mathbb{Q}^2(d(x_1, x_2)) \right)^{\frac{1}{2}} \\ &\leq C \frac{s}{\text{Var } F_s} \sqrt{\int_{\mathbb{X}^2} \psi_s(x_1, x_2)^2 \mathbb{Q}^2(d(x_1, x_2))}. \end{aligned}$$

Therefore, combining all bounds for γ'_i , $1 \leq i \leq 6$, we have by Theorem 3.1,

$$d_K \left(\frac{F_s - \mathbb{E}F_s}{\sqrt{\text{Var } F_s}}, N \right) \leq C(\theta_1 + \theta_2 + \theta_3), \quad (5.7)$$

where

$$\theta_1 := \frac{s}{\text{Var } F_s} \sqrt{\int_{\mathbb{X}^2} \psi_s(x_1, x_2)^2 \mathbb{Q}^2(d(x_1, x_2))}, \quad (5.8)$$

$$\theta_2 := \frac{s^{\frac{3}{2}}}{\text{Var } F_s} \sqrt{\int_{\mathbb{X}} \left(\int_{\mathbb{X}} \psi_s(x_1, x_2) \mathbb{Q}(dx_2) \right)^2 \mathbb{Q}(dx_1)}, \quad (5.9)$$

$$\theta_3 := \frac{(\phi_s)^{\frac{1}{2}}}{\text{Var } F_s} + \frac{\phi_s}{(\text{Var } F_s)^{\frac{3}{2}}} + \frac{\phi_s^{\frac{5}{4}} + \phi_s^{\frac{3}{2}}}{(\text{Var } F_s)^2}. \quad (5.10)$$

We next proceed to obtain refined bounds for ϕ_s and $\psi_s(x_1, x_2)$. Before proceeding, we recall the following definitions from (3.5) to (3.6):

$$I_s(x) := \mathbb{P}(D_x F_s \neq 0),$$

$$J_s(x_1, x_2) := \mathbb{P}(|D_{x_1} F_s - D_{x_1} F_s^{x_2}| \neq 0).$$

Lemma 5.1. *Assume that all conditions in Corollary 3.1 hold, and recall that R_x denotes the radius of stabilization. Then,*

$$\phi_s \leq C s \int_{\mathbb{X}} e^{-C_2 \frac{p-4}{2p} d_s(x, \mathbb{K})^{C_3}} \mathbb{Q}(dx),$$

and

$$\psi_s(x_1, x_2) \leq C e^{-C_2 \frac{p-4}{4p} d_s(x_1, x_2)^{C_3}}.$$

Proof of Lemma 5.1. Based on the Assumptions 2.1 and 2.2, we immediately obtain

$$I_s(x) \leq C_1 e^{-C_2 d_s(x, \mathbb{K})^{C_3}},$$

$$J_s(x_1, x_2) \leq C_1 e^{-C_2 \max\{d_s(x_1, x_2), d_s(x_1, \mathbb{K}), d_s(x_2, \mathbb{K})\}^{C_3}}.$$

Therefore, we have

$$\phi_s \leq C s \int_{\mathbb{X}} e^{-C_2 \frac{p-4}{2p} d_s(x, \mathbb{K})^{C_3}} \mathbb{Q}(dx)$$

Similarly, we obtain the stated upper bound for $\psi_s(x_1, x_2)$. $\square$

Lemma 5.2. *Suppose the condition (2.5) holds. Then, for any $x \in \mathbb{X}$ and $r \geq 0$, we have*

$$\mathbb{Q}(B_x(r)) \leq \kappa r^\omega.$$

Proof of Lemma 5.2. For any $x \in \mathbb{X}$ fixed, consider $Q(r) := \mathbb{Q}(B_x(r))$, $r \geq 0$ as an increasing function of r . According to Lebesgue's theorem for the differentiability of monotone functions, the derivative $Q'(r)$ exists almost everywhere and then with the condition (2.5),

$$\mathbb{Q}(B_x(r)) - \mathbb{Q}(B_x(0)) \leq \int_0^r Q'(u) du \leq \int_0^r \kappa \omega u^{\omega-1} du = \kappa r^\omega.$$

Note that $\mathbb{Q}(B_x(0)) = 0$. Therefore, we obtain the desired result. $\square$

Lemma 5.3. *Suppose the condition (2.5) holds. For any $x \in \mathbb{X}$, $r \geq 0$ and $\alpha > 0$, there exists a constant $C > 0$ such that*

$$\int_{\mathbb{X} \setminus B_x(r)} e^{-\alpha d_s(x, y)^{C_3}} \mathbb{Q}(dy) \leq \frac{C}{s} e^{-\frac{1}{2} \alpha (s^{1/\omega} r)^{C_3}}.$$

Proof of Lemma 5.3. Let $\{r_n\}_{n=1}^\infty$ be an increasing sequence satisfying

$$r_1 := r, \quad \lim_{n \rightarrow \infty} r_n = \infty, \quad \text{and} \quad \lim_{n \rightarrow \infty} \sup_{n \geq 2} |r_n - r_{n-1}| = 0.$$

Then,

$$\int_{\mathbb{X} \setminus B_x(r)} e^{-\alpha d_s(x, y)^{C_3}} \mathbb{Q}(dy) \leq \sum_{n=2}^\infty e^{-\alpha (s^{1/\omega} r_{n-1})^{C_3}} \mathbb{Q}(B_x(r_n) \setminus B_x(r_{n-1}))$$

$$\begin{aligned}
&\leq \sum_{n=2}^{\infty} e^{-\alpha(s^{1/\omega}r_{n-1})^{C_3}} \kappa \omega r_{n-1}^{\omega-1} (r_n - r_{n-1}) \\
&= \int_r^{\infty} e^{-\alpha(s^{1/\omega}u)^{C_3}} \kappa \omega u^{\omega-1} du.
\end{aligned}$$

Therefore, it suffices to show

$$\zeta(s, r) := s e^{\frac{\alpha}{2}(s^{1/\omega}r)^{C_3}} \int_r^{\infty} e^{-\alpha(s^{1/\omega}u)^{C_3}} u^{\omega-1} du$$

is bounded on $\Theta := \{(s, r) : s \geq 1, r \geq 0\}$. Since $\zeta(s, r)$ is a continuous function, then we only need to show

$$\lim_{(s,r) \rightarrow \partial\Theta} \zeta(s, r) < \infty.$$

Note that

$$\zeta(s, r) \leq s e^{(\frac{\alpha}{2} - \frac{2\alpha}{3})(s^{1/\omega}r)^{C_3}} \int_r^{\infty} e^{-\frac{\alpha}{3}(s^{1/\omega}u)^{C_3}} u^{\omega-1} du.$$

Moreover, let

$$\int_r^{\infty} e^{-\frac{\alpha}{3}(s^{1/\omega}u)^{C_3}} u^{\omega-1} du \leq \int_0^{\infty} e^{-\frac{\alpha}{3}(s^{1/\omega}u)^{C_3}} u^{\omega-1} du := \eta(s),$$

with

$$\lim_{s \rightarrow \infty} \eta(s) < \infty.$$

Then,

$$\zeta(s, r) \leq s e^{(\frac{\alpha}{2} - \frac{2\alpha}{3})(s^{1/\omega}r)^{C_3}} \eta(s).$$

Consequently, noting

$$\frac{\alpha}{2} - \frac{2\alpha}{3} = -\frac{1}{6}\alpha < 0,$$

we have

$$\lim_{(s,r) \rightarrow \partial\Theta} \zeta(s, r) < \infty,$$

giving us the desired result. $\square$

Lemma 5.4. *For any $s \geq 1$, $r \geq 0$ and $\beta > 0$, there exists a constant $C > 0$ such that*

$$s r^{\omega} e^{-\beta(s^{1/\omega}r)^{C_3}} \leq C e^{-\frac{1}{2}\beta(s^{1/\omega}r)^{C_3}}.$$

Proof of Lemma 5.4. Let

$$\mu(s, r) := s r^{\omega} e^{-\frac{1}{2}\beta(s^{1/\omega}r)^{C_3}}.$$

It suffices to prove

$$\sup_{s \geq 1, r \geq 0} \mu(s, r) < \infty.$$

Similar to $\zeta(s, r)$ in Lemma 5.3, we only need to show

$$\lim_{(s,r) \rightarrow \partial\Theta} \mu(s, r) < \infty,$$

where $\Theta := \{(s, r) : s \geq 1, r \geq 0\}$. We note that the above claim follows by calculations similar to that in the proof of Lemma 5.3 and the fact that $-\beta/2 < 0$, thus providing the desired result. $\square$

Before proceeding, we recall the definition of $\Theta_{\mathbb{K},s}$ from (3.7) for convenience:

$$\Theta_{\mathbb{K},s} := s \int_{\mathbb{X}} e^{-C_2 \frac{(p-4)}{4p} \left(\frac{d_s(x, \mathbb{K})}{2} \right)^{C_3}} \mathbb{Q}(dx).$$

Note that by Lemma 5.1, we also have that $\phi_s \leq C\Theta_{\mathbb{K},s}$, for a constant $C > 0$.

Lemma 5.5. *Suppose the conditions in Corollary 3.1 hold. Then, there exists a constant $C > 0$ such that*

$$s^2 \int_{\mathbb{X}^2} \psi_s(x_1, x_2)^2 \mathbb{Q}^2(d(x_1, x_2)) \leq C\Theta_{\mathbb{K},s}.$$

Proof of Lemma 5.5. Without loss of generality, we assume $d_s(x_1, \mathbb{K}) \geq d_s(x_2, \mathbb{K})$. Similar reasoning can be used for the other case. According to Lemma 5.1,

$$\psi_s(x_1, x_2) \leq Ce^{-C_2 \frac{p-4}{4p} \max\{d_s(x_1, x_2), d_s(x_1, \mathbb{K})\}^{C_3}}.$$

Let

$$L_{x_2,s} := s \int_{\mathbb{X}} \psi_s(x_1, x_2)^2 \mathbb{Q}(dx_1).$$

It suffices to show there exists a constant $C > 0$ such that

$$L_{x_2,s} \leq Ce^{-C_2 \frac{p-4}{2p} \left(\frac{d_s(x_2, \mathbb{K})}{2} \right)^{C_3}}.$$

Let $r := \frac{1}{2}d(x_2, \mathbb{K})$ and note the fact that $\max\{x, y\} \geq x, \max\{x, y\} \geq y$ for any x, y , then

$$\begin{aligned} L_{x_2,s} &\leq Cs \int_{\mathbb{X}} e^{-C_2 \frac{p-4}{2p} \max\{d_s(x_1, x_2), d_s(x_1, \mathbb{K})\}^{C_3}} \mathbb{Q}(dx_1) \\ &\leq Cs \int_{B_{x_2}(r)} e^{-C_2 \frac{p-4}{2p} d_s(x_1, \mathbb{K})^{C_3}} \mathbb{Q}(dx_1) + Cs \int_{\mathbb{X} \setminus B_{x_2}(r)} e^{-C_2 \frac{p-4}{2p} d_s(x_1, x_2)^{C_3}} \mathbb{Q}(dx_1) \\ &:= CL_1 + CL_2. \end{aligned}$$

By the triangle inequality, $2r \leq d(x_2, \mathbb{K}) \leq d(x_1, x_2) + d(x_1, \mathbb{K})$, then when $d(x_2, x_1) \leq r$, $d(x_1, \mathbb{K}) \geq r$. Therefore, according to Lemma 5.2 and lemma 5.4, there exists a constant $C > 0$ such that

$$L_1 \leq s \int_{B_{x_2}(r)} e^{-C_2 \frac{p-4}{2p} (s^{1/\omega} r)^{C_3}} \mathbb{Q}(dx_1) \leq s\kappa r^\omega e^{-C_2 \frac{p-4}{4p} (s^{1/\omega} r)^{C_3}} \leq Ce^{-C_2 \frac{p-4}{4p} (s^{1/\omega} r)^{C_3}}.$$

According to Lemma 5.3, there exists a constant $C > 0$ such that

$$L_2 \leq s \cdot \frac{C}{s} e^{-C_2 \frac{p-4}{4p} (s^{1/\omega} r)^{C_3}} = Ce^{-C_2 \frac{p-4}{4p} (s^{1/\omega} r)^{C_3}}.$$

Then,

$$L_{x_2,s} \leq Ce^{-C_2 \frac{p-4}{4p} (s^{1/\omega} r)^{C_3}} = Ce^{-C_2 \frac{p-4}{4p} \left(\frac{d_s(x_2, \mathbb{K})}{2} \right)^{C_3}},$$

giving us the desired result. $\square$

Lemma 5.6. *Suppose the conditions in Corollary 3.1 hold. Then, there exists a constant $C > 0$ such that*

$$s^3 \int_{\mathbb{X}} \left(\int_{\mathbb{X}} \psi_s(x_1, x_2) \mathbb{Q}(dx_2) \right)^2 \mathbb{Q}(dx_1) \leq C\Theta_{\mathbb{K},s}.$$

Proof of Lemma 5.6. We can prove this lemma in a similar way as Lemma 5.5. Let

$$L'_{x_1,s} := s \int_{\mathbb{X}} \psi_s(x_1, x_2) \mathbb{Q}(dx_2).$$

Similar to $L_{x_2,s}$, one can show there exists a constant $C > 0$ such that

$$L'_{x_1,s} \leq C e^{-C_2 \frac{p-4}{8p} \left(\frac{d_s(x_1, \mathbb{K})}{2} \right)^{C_3}}.$$

Therefore,

$$\begin{aligned} s^3 \int_{\mathbb{X}} \left(\int_{\mathbb{X}} \psi_s(x_1, x_2) \mathbb{Q}(dx_2) \right)^2 \mathbb{Q}(dx_1) &= s \int_{\mathbb{X}} \left(s \int_{\mathbb{X}} \psi_s(x_1, x_2) \mathbb{Q}(dx_2) \right)^2 \mathbb{Q}(dx_1) \\ &\leq C s \int_{\mathbb{X}} e^{-C_2 \frac{p-4}{4p} \left(\frac{d_s(x_1, \mathbb{K})}{2} \right)^{C_3}} \mathbb{Q}(dx_1) \\ &\leq C \Theta_{\mathbb{K},s}. \end{aligned}$$

□

With the above ingredients in place, we are finally in a position to prove Corollary 3.1.

Proof of Corollary 3.1. Recall the definition of θ_1 , θ_2 and θ_3 respectively in (5.8), (5.9) and (5.10). According to Lemma 5.1, Lemma 5.5 and Lemma 5.6, there exists a constant $C > 0$ such that

$$\theta_1 \leq C \frac{(\Theta_{\mathbb{K},s})^{\frac{1}{2}}}{\text{Var } F_s}, \quad \theta_2 \leq C \frac{(\Theta_{\mathbb{K},s})^{\frac{1}{2}}}{\text{Var } F_s}, \quad \phi_s \leq C \Theta_{\mathbb{K},s}.$$

Therefore, according to (5.7), we complete the proof. □

5.3. Proof of Theorem 3.2. The proof of the Poisson case applies mutatis mutandis to the binomial case. We now highlight the main changes. First, recall that we do not have a similar result like Theorem 3.1 for Poisson case due to the fact that there is no nice counterpart of the second order Poincaré inequality (see [LPS16]). Hence, Theorem 3.2 cannot follow from Theorem 3.1. Instead, we use [LRSY19, Theorem 4.2] that provides an auxiliary result for the binomial case. While [LRSY19] provided a general result for marked binomial point process, we state the following result for the unmarked binomial point process.

Theorem 5.2 ([LRSY19]). *Let $n \geq 3$ and let F be a functional of a binomial point process η_n with $\mathbb{E}F(\xi_n)^2 < \infty$. Assume that there are constants $c, \rho \in (0, \infty)$ such that*

$$\mathbb{E}|D_x F(\xi_{n-1-|\mathbb{A}|} \cup \mathbb{A})|^{4+\rho} \leq c, \quad \mathbb{Q} - a.e., x \in \mathbb{X}, \mathbb{A} \subset \mathbb{X}, |\mathbb{A}| \leq 2.$$

Then, there is a constant $C := C(c, \rho) \in (0, \infty)$ such that

$$d_K \left(\frac{F - \mathbb{E}F}{\sqrt{\text{Var } F}}, N \right) \leq C(S'_1 + S'_2 + S'_3),$$

with

$$\begin{aligned} \Gamma_n' &:= n \int_{\mathbb{X}} \mathbb{P}(D_x F(\xi_{n-1}) \neq 0)^{\frac{\rho}{8+2\rho}} \mathbb{Q}(dx), \\ \psi_n'(x, x') &:= \sup_{\mathbb{A} \subset \mathbb{X}: |\mathbb{A}| \leq 1} \mathbb{P}(D_{x,x'} F(\xi_{n-2-|\mathbb{A}|}) \neq 0)^{\frac{\rho}{8+2\rho}}, \end{aligned}$$

$$\begin{aligned}
 S'_1 &:= \frac{n}{\text{Var } F} \sqrt{\int_{\mathbb{X}^2} \psi'_n(x, x') \mathbb{Q}^2(d(x, x'))}, \\
 S'_2 &:= \frac{n^{\frac{3}{2}}}{\text{Var } F} \sqrt{\int_{\mathbb{X}} \left(\int_{\mathbb{X}} \psi'_n(x, x') \mathbb{Q}(dx') \right)^2 \mathbb{Q}(dx)}, \\
 S'_3 &:= \frac{(\Gamma'_n)^{\frac{1}{2}}}{\text{Var } F} + \frac{\Gamma'_n}{(\text{Var } F)^{\frac{3}{2}}} + \frac{\Gamma'_n + (\Gamma'_n)^{\frac{3}{2}}}{(\text{Var } F)^2}.
 \end{aligned} \tag{5.11}$$

Remark 5.1. The exponent of Γ'_n in the third component of the sum on the right hand side of (5.11) is different than that of Poisson case, due to the fact that it is not derived by the counterpart of the second order Poincaré inequality. In fact, it is obtained by [LRSY19, Theorem 4.3].

Now, based on Theorem 5.2, we start to prove Theorem 3.2.

Proof of Theorem 3.2. The proof of the binomial point process is similar to that of the Poisson case, based on Theorem 5.2 for binomial case. We treat Theorem 5.2 for binomial as a counterpart of (5.7) for Poisson noting $D_{x,x'} F_n = D_x F_n^{x'} - D_x F_n$. Starting with this, one can follow the same procedure to get the required counterparts of Lemma 5.1 to Lemma 5.6 by changing s as n . This provides the desired result. $\square$

6. PROOFS FOR SECTION 4

6.1. Total Edge Length of k -NN.

Proof of Theorem 4.1. We begin with the Poisson point process case and consider the statistic $F_s^{k\text{-NN}}(\mathcal{P}_s) := \sum_{x \in \mathcal{P}_s} f_s(x, \mathcal{P}_s)$, with f_s as defined in (2.4).

Step 1: From Example 2.2, $F_s^{k\text{-NN}}(\mathcal{P}_s)$ is a strongly stabilizing functional with the radius of stabilization $R_x = 4R$, with R being defined as follows: for each $t > 0$, construct six disjoint equilateral triangles $T_j(t)$, $1 \leq j \leq 6$, such that the origin is a vertex of each triangle, such that each triangle has edge length t and such that $T_j(t) \subset T_j(u)$ whenever $t < u$. Then, define R to be the minimum t such that each triangle $T_j(t)$ contains at least $k + 1$ points from $\mathcal{P}_s$. Consequently, we have

$$\mathbb{P}(R > r) \leq \mathbb{P}(\mathcal{P}_s(\cup_{j=1}^6 T_j(r)) \leq 6k) \leq \mathbb{P}(\mathcal{P}_s(B_x(r\sqrt{3}/2)) \leq 6k),$$

where $\mathcal{P}_s(B_x(r\sqrt{3}/2))$ follows Poisson distribution with parameter $s\mathbb{Q}(B_x(r\sqrt{3}/2))$. According to the assumption (4.2) and a Chernoff bound for Poisson tail, [Pen03, Lemma 1.2], we have that there exists a constant $c' > 0$ such that

$$\begin{aligned}
 \mathbb{P}(R > r) &\leq \mathbb{P}\left(\mathcal{P}_s\left(B_x\left(\frac{r\sqrt{3}}{2}\right)\right) \leq 6k\right) \\
 &\leq \mathbb{P}\left(\text{Poi}\left(cs\left(\frac{r\sqrt{3}}{2}\right)^\omega\right) \leq 6k\right) \\
 &\leq 6ke^{-c'sr^\omega}.
 \end{aligned}$$

This implies the radius of stabilization $R_x = 4R$ decays exponentially. Furthermore, we set $\mathbb{K} = \mathbb{X}$ for the $\mathbb{K}$ -exponential bound.

Step 2: As for the bounded moment condition, according to [LRSY19, Lemma 5.5], for some $p > 4$, the bounded moment condition holds.

Step 3: As for the variance condition, by the assumption (4.3), it is satisfied.

Therefore, according to Corollary 3.2, we end the proof for the Poisson case. For the binomial case, it is similar to the Poisson case by considering a Chernoff bound for the binomial distribution [Pen03, Lemma 1.1] and [LRSY19, Lemma 5.6]. $\square$

6.2. Shannon Entropy Estimation.

Proof of Theorem 4.2. We start by replacing the biased center $\mathbb{E}F_n^{\text{SE}}(\xi_n)$ by the true parameter $H(q)$. By triangle inequality, we have that

$$\begin{aligned} d_K \left(\frac{F_n^{\text{SE}}(\xi_n) - H(q)}{\sqrt{\text{Var } F_n^{\text{SE}}(\xi_n)}}, N \right) &\leq \underbrace{d_K \left(\frac{F_n^{\text{SE}}(\xi_n) - \mathbb{E}F_n^{\text{SE}}(\xi_n)}{\sqrt{\text{Var } F_n^{\text{SE}}(\xi_n)}}, N \right)}_{:= d_1} \\ &\quad + \underbrace{d_K \left(\frac{F_n^{\text{SE}}(\xi_n) - H(q)}{\sqrt{\text{Var } F_n^{\text{SE}}(\xi_n)}}, \frac{F_n^{\text{SE}}(\xi_n) - \mathbb{E}F_n^{\text{SE}}(\xi_n)}{\sqrt{\text{Var } F_n^{\text{SE}}(\xi_n)}} \right)}_{:= d_2}. \end{aligned}$$

We first apply Corollary 3.3 to bound d_1 , using the three step approach.

Step 1: Similar to the total edge length of k -NN, following the proof of [PY01, Lemma 6.1], $F_n^{\text{SE}}(\xi_n)$ is strongly stabilizing with the radius of stabilization R_x decaying exponentially and we take $\mathbb{K} = \mathbb{X}$.

Step 2: As for the moment condition, again, by [LRSY19, Lemma 5.6], the bounded moment condition holds for $p > 4$.

Step 3: For the variance, note that for d_1 , we have

$$\frac{F_n^{\text{SE}}(\xi_n) - \mathbb{E}F_n^{\text{SE}}(\xi_n)}{\sqrt{\text{Var } F_n^{\text{SE}}(\xi_n)}} \stackrel{d}{=} \frac{nF_n^{\text{SE}}(\xi_n) - \mathbb{E}(nF_n^{\text{SE}}(\xi_n))}{\sqrt{\text{Var}(nF_n^{\text{SE}}(\xi_n))}},$$

where $\stackrel{d}{=}$ means equal in distribution. Therefore, we can consider the variance condition for the $nF_n^{\text{SE}}(\xi_n)$ instead. According to [BSY19, Lemma 7], there exists a constant $C > 0$ such that for $k_0^* \leq k \leq k_1^*$,

$$\sup_{n>0} \frac{n}{\text{Var}(nF_n^{\text{SE}}(\xi_n))} \leq C. \quad (6.1)$$

By Corollary 3.3, we immediately have that $d_1 \leq C'_0(k, p)/\sqrt{n}$. Since k also diverges as n goes to ∞ , it is necessary to calculate the constant $C'_0(k, p)$ more explicitly. Therefore, we derive the following lemma. Note that the following lemma is a refined version of Theorem 5.2 as it reveals how the constant $C'_0(k, p)$ is related to the constant p and the functional F thus k .

Lemma 6.1. *Assume there are constants $c > 0, p_0 > 0$ such that*

$$\mathbb{E}|D_x F(\xi_{n-1-|\mathbb{A}|} \cup \mathbb{A})|^{4+p_0} \leq c, \quad |\mathbb{A}| \leq 2.$$

Then, there exists some constant C not depending on n nor F such that

$$d_K \left(\frac{F - \mathbb{E}F}{\sqrt{\text{Var } F}}, N \right) \leq C(S_1 + S_2 + S_3 + S_4 + S_5),$$

where

$$\begin{aligned} S_1 &:= c^{\frac{2}{4+p_0}} \frac{n}{\text{Var } F} \sqrt{\int_{\mathbb{X}^2} \psi_n(x, x') \mathbb{Q}^2(d(x, x'))}, \\ S_2 &:= c^{\frac{2}{4+p_0}} \frac{n^{\frac{3}{2}}}{\text{Var } F} \sqrt{\int_{\mathbb{X}} \left(\int_{\mathbb{X}} \psi_n(x, x') \mathbb{Q}(dx') \right)^2 \mathbb{Q}(dx)}, \\ S_3 &:= c^{\frac{2}{4+p_0}} \frac{\sqrt{\Gamma_n}}{\text{Var } F}, \\ S_4 &:= \left(\sqrt{3} \max \left\{ 4\sqrt{2} c^{\frac{1}{4+p_0}} \frac{\Gamma_n^{\frac{1}{2}}}{(\text{Var } F)^{\frac{1}{2}}}, \sqrt{2} c^{\frac{1}{4+p_0}} \frac{\Gamma_n^{\frac{1}{4}}}{(\text{Var } F)^{\frac{1}{2}}} + 1 \right\} \right. \\ &\quad \left. + c^{\frac{1}{4+p_0}} \frac{\Gamma_n^{\frac{1}{4}}}{n^{\frac{1}{4}} (\text{Var } F)^{\frac{1}{2}}} \right) c^{\frac{3}{4+p_0}} \frac{\Gamma_n}{(\text{Var } F)^{\frac{3}{2}}} + c^{\frac{4}{4+p_0}} \frac{\Gamma_n}{(\text{Var } F)^2}, \\ S_5 &:= c^{\frac{3}{4+p_0}} \frac{\Gamma_n}{(\text{Var } F)^{\frac{3}{2}}}. \end{aligned}$$

Here,

$$\begin{aligned} \Gamma_n &:= n \int_{\mathcal{X}} \mathbb{P}(D_x F(\xi_{n-1}) \neq 0)^{\frac{p}{8+2p_0}} \mathbb{Q}(dx), \\ \psi_n(x, x') &:= \sup_{\mathbb{A} \subset \mathbb{X}: |\mathbb{A}| \leq 1} \mathbb{P}(D_{x,x'} F(\xi_{n-2-|\mathbb{A}|} \cup \mathbb{A}) \neq 0)^{\frac{4}{8+2p_0}}. \end{aligned}$$

The proof of Lemma 6.1 follows in a straightforward manner by [LRSY19, Proof of Theorem 4.2]. Now, we proceed to calculate moment bounds to see how k is related to the constant C'_0 . Let

$$\zeta_i := \sum_{j=1}^k w_j \log \left(\frac{(n-1)V_d \rho_{j,i}^d}{e^{\Psi(j)}} \right).$$

Following [LRSY19, Lemma 5.6], by Jensen's inequality, for $p = 4 + p_0$, $p_0 > 0$,

$$\begin{aligned} &\mathbb{E}|D_y F_n^{\text{SE}}(\xi_{n-1-|\mathbb{A}|} \cup \mathbb{A})|^{4+p_0} \\ &= \mathbb{E} \left| \zeta_k(y, \xi_{n-1-|\mathbb{A}|} \cup \{y\} \cup \mathbb{A}) + \sum_{x \in \xi_{n-1-|\mathbb{A}|} \cup \mathbb{A}} D_y \zeta_k(x, \xi_{n-1-|\mathbb{A}|}) \right|^{4+p_0} \\ &\leq 4^{3+p_0} \mathbb{E}|\zeta_k(y, \xi_{n-1-|\mathbb{A}|} \cup \{y\} \cup \mathbb{A})|^{4+p_0} + 4^{3+p_0} \sum_{x \in \mathbb{A}} \mathbb{E}|D_y \zeta_k(x, \xi_{n-1-|\mathbb{A}|})|^{4+p_0} \\ &\quad + 4^{3+p_0} \sum_{x \in \xi_{n-1-|\mathbb{A}|}} \mathbb{E}|D_y \zeta_k(x, \xi_{n-1-|\mathbb{A}|})|^{4+p_0}. \end{aligned}$$

Then, following [LRSY19, Proof of Lemma 5.6], the constant c in Lemma 6.1 satisfies

$$c \lesssim 4^{3+p_0} c_a + 4^{3+p_0} c_b + 4^{5+\frac{3}{2}p_0} c_b^{\frac{4+p_0}{4+2p_0}} k^{\frac{4}{4+2p_0}},$$

where $A \lesssim B$ means there is a constant $C > 0$ such that $A \leq CB$. Note that according to the definition of weights (4.5), there only exist finitely many terms in the sum of ζ_i and according to [SP16, Section 3.2],

$$c_a := \mathbb{E} |\zeta_k(y, \xi_{n-1-|\mathbb{A}|} \cup \{y\} \cup \mathbb{A})|^{4+p_0} \lesssim \|w\|_\infty^{4+p_0} \mathbb{E} \left| \log \left(\frac{(n-1)V_d \rho_{k,i}^d}{e^{\Psi(k)}} \right) \right|^{4+p_0} < \infty,$$

and similarly,

$$c_b := \mathbb{E} |D_y \zeta_k(x, \xi_{n-1-|\mathbb{A}|})|^{4+p_0} < \infty.$$

Then,

$$c \lesssim k^{\frac{p_0}{4+2p_0}},$$

and by [LRSY19, Theorem 4.3] and [LRP17, Theorem 5.1], the constant C in Lemma 6.1 does not depend on either n or k .

Therefore, according to Corollary 3.3 and Lemma 6.1,

$$d_1 \lesssim \frac{1}{\sqrt{n}} \times c^{\frac{4}{4+p_0}} \lesssim \frac{1}{\sqrt{n}} k^{\frac{4p_0}{(4+p_0)(4+2p_0)}} \leq \sqrt{\frac{k}{n}}. \quad (6.2)$$

Next, we focus on d_2 , which is related to the bias, $\mathbb{E} F_n^{\text{SE}}(\xi_n) - H(q)$. Let

$$h^w := \frac{F_n^{\text{SE}}(\xi_n) - \mathbb{E} F_n^{\text{SE}}(\xi_n)}{\sqrt{\text{Var } F_n^{\text{SE}}(\xi_n)}} \quad \text{and} \quad \Delta h := \frac{H(q) - \mathbb{E} F_n^{\text{SE}}(\xi_n)}{\sqrt{\text{Var } F_n^{\text{SE}}(\xi_n)}}.$$

Then we have

$$\begin{aligned} d_2 &:= d_K \left(\frac{F_n^{\text{SE}}(\xi_n) - H(q)}{\sqrt{\text{Var } F_n^{\text{SE}}(\xi_n)}}, \frac{F_n^{\text{SE}}(\xi_n) - \mathbb{E} F_n^{\text{SE}}(\xi_n)}{\sqrt{\text{Var } F_n^{\text{SE}}(\xi_n)}} \right) \\ &= \sup_{t \in \mathbb{R}} \left| \mathbb{P} \left(\frac{F_n^{\text{SE}}(\xi_n) - H(q)}{\sqrt{\text{Var } F_n^{\text{SE}}(\xi_n)}} \leq t \right) - \mathbb{P} \left(\frac{F_n^{\text{SE}}(\xi_n) - \mathbb{E} F_n^{\text{SE}}(\xi_n)}{\sqrt{\text{Var } F_n^{\text{SE}}(\xi_n)}} \leq t \right) \right| \\ &= \sup_{t \in \mathbb{R}} |\mathbb{P}(h^w \leq t + \Delta h) - \mathbb{P}(h^w \leq t)| \\ &\leq 2d_1 + \sup_{t \in \mathbb{R}} |\Phi(t + \Delta h) - \Phi(t)| \\ &= 2d_1 + 2 \left(\Phi \left(\left| \frac{\Delta h}{2} \right| \right) - \frac{1}{2} \right) \\ &\leq 2d_1 + 1 - e^{-\frac{1}{2}(\Delta h)^2 - \sqrt{\frac{2}{\pi}}|\Delta h|} \\ &\lesssim d_1 + |\Delta h|, \end{aligned}$$

where Φ is the c.d.f. of the standard normal distribution and actually, we applied a tight bound for standard normal distribution function according to [MJ13]. Therefore,

$$d_K \left(\frac{F_n^{\text{SE}}(\xi_n) - H(q)}{\sqrt{\text{Var } F_n^{\text{SE}}(\xi_n)}}, N \right) \lesssim \max\{d_1, |\Delta h|\}.$$

The bias of $F_n^{\text{SE}}(\xi_n)$ satisfies the following bound according to [BSY19, Corollary 4]: for every $\epsilon > 0$,

$$\sup_{f \in \mathcal{F}_{d,\theta}} |\mathbb{E}_f \hat{H}_n^w - H(f)| = O \left(\max \left\{ \left(\frac{k}{n} \right)^{\frac{\alpha}{\alpha+d} - \epsilon}, \left(\frac{k}{n} \right)^{\frac{2(\lfloor \frac{d}{4} \rfloor + 1)}{d}}, \left(\frac{k}{n} \right)^{\frac{\beta}{d}} \right\} \right). \quad (6.3)$$

Note that $\alpha > d$ and $\beta > \frac{d}{2}$. With (6.1) and (6.3), by elementary algebraic manipulation we have that

$$|\Delta h| \lesssim n^{-\frac{1}{2}} \times \left(\frac{k}{n} \right)^{\frac{1}{2}} \lesssim \frac{k^{\frac{1}{2}}}{n}. \quad (6.4)$$

Comparing (6.2) and (6.4), we obtain the desired result. $\square$

6.3. Euler Characteristic.

Proof of Theorem 4.3. We follow the 3-step procedure.

Step 1: According to Example 2.1, the Euler characteristic is strongly stabilizing with the radius of stabilization $R_x = 2r$. Clearly, as a constant, R_x can be bounded exponentially as one can choose c_1 large enough and $c_2, c_3 > 0$ such that for $0 \leq t \leq 2r$,

$$1 \leq c_1 e^{-c_2(n^{1/d}t)^{c_3}}. \quad (6.5)$$

The similar argument also holds for Poisson case. Also, we take $\mathbb{K} = [0, 1]^d$.

Step 2: Moreover, for the bounded moment condition, note that for $p > 4$, there exists a constant $C(d, T) > 0$ such that

$$\begin{aligned} \sup_{n>0, x \in [0,1]^d} \mathbb{E} |D_x F_n^{\text{EC}}(\xi_n)|^p &\leq \mathbb{E} \left| \sum_{\ell=0}^n \# \{ \sigma \in K_r \left(n^{\frac{1}{d}}(\xi_n \cup \{x\}) \right) : \right. \\ &\quad \left. \sigma \text{ is an } \ell\text{-simplex intersecting with } \{x\} \} \right|^p \\ &\leq \left| \sum_{\ell=0}^n \sum_{\{j_1, \dots, j_\ell\} \subset \{1, 2, \dots, n\}} \left(\mathbb{P} \left(x_{j_1} \in B_x \left(n^{-\frac{1}{d}} r \right) \right) \right)^\ell \right|^p \\ &\leq \left| \sum_{\ell=0}^n \binom{n}{\ell} \frac{(C(d, T) \|q\|_\infty r^d)^\ell}{n^\ell} \right|^p \\ &\leq \left| \sum_{\ell=0}^n \frac{(C(d, T) \|q\|_\infty r^d)^\ell}{\ell!} \right|^p \\ &\leq (e^{C(d, T) \|q\|_\infty r^d})^p < \infty. \end{aligned} \quad (6.6)$$

Also, by [YSA17, Lemma 4.1] and [KRP21, Proof of Lemma 4.2],

$$\sup_{n>0, x \in [0,1]^d} \mathbb{E} |F_n^{\text{EC}}(\mathcal{P}_n)|^p < \infty.$$

Moreover, consider

$$D_x F_n^{\text{EC}}(\xi_n)^{x^*} = \chi \left(K_r \left(n^{\frac{1}{d}}(\xi_n \cup \{x\} \cup \{x^*\}) \right) \right) - \chi \left(K_r \left(n^{\frac{1}{d}} \xi_n \right) \right)$$

$$+ \chi \left(K_r \left(n^{\frac{1}{d}} \xi_n \right) \right) - \chi \left(K_r \left(n^{\frac{1}{d}} (\xi_n \cup \{x^*\}) \right) \right),$$

and following a similar argument like (6.6), we have

$$\sup_{n>0, x \in [0,1]^d, x^* \in [0,1]^d} \mathbb{E} |D_x F_n^{\text{EC}}(\xi_n)^{x^*}|^p < \infty.$$

The bounded moment condition is satisfied and similar arguments can be utilized to show for Poisson case.

Step 3: According to [PY01, Theorem 2.1] and [KRP21, proposition 4.6], there exists a constant $C > 0$ such that

$$\sup_{n>0} \frac{n}{\text{Var } F_n^{\text{EC}}(\xi_n)} \leq C,$$

and it also holds for Poisson case.

Therefore, according to Corollary 3.2 and Corollary 3.3, we complete the proof. $\square$

6.4. Edge Length Statistic of the Minimum Spanning Tree.

Proof of Theorem 4.4. According to [Pen05, Theorem 3.3], the total edge length $M(V)$ of the minimal spanning tree satisfies the strong stabilization with a radius of stabilization R_x almost surely finite. Without knowing the tail probability of R_x , we need to use Theorem 3.1. Set

$$A_{x_1} = B_n \cap \{x_1 + n^\alpha B_0\},$$

where $0 < \alpha < 1$ and $\{x + A\} := \{x + y : y \in A\}$ for any set A . According to [LRPY22, Proposition 3.7], for any $q_0 > 0$ and $x \in B_n$, there exists a constant $C_{q_0} > 0$ independent of n, x such that uniformly

$$\mathbb{E} |D_x F_{B_n}^{\text{MST}}(\mathcal{P}(\lambda))|^{q_0} \leq C_{q_0}. \quad (6.7)$$

The same bound also holds for $D_x F_{B_n}^{y, \text{MST}}$

As for the second order cost, according to [LRPY22, Proposition 3.11]², for any $q_0 \geq 1$, there exist constants $E_0 > 0, E_1 > 0, E_2 > 0$ such that for any $x_1 \in B_n$ with $d(x_1, \partial B_n) > n^\alpha$,

$$\mathbb{E} |D_{x_1} F_{B_n}^{\text{MST}}(A_{x_1}) - D_{x_1} F_{B_n}^{\text{MST}}(\mathcal{P}(\lambda))|^{q_0} \leq \begin{cases} E_0 n^{-E_1}, & \text{if } d = 2, \\ E_0 (\log n)^{-E_2}, & \text{if } d \geq 3. \end{cases} \quad (6.8)$$

The same bound also holds for $D_{x_1} F_{B_n}^{y, \text{MST}}$. On the other hand, the variance bound is directly from [LRPY22, Propostion 3.9]: there exists a constant $c > 0$,

$$\text{Var } F_{B_n}^{\text{MST}}(\mathcal{P}(\lambda)) \geq c |B_n| \asymp n^d,$$

Then, plugging these two bounds (6.7) and (6.8) in Theorem 3.1, we get that there exist constants $C_{\text{first}} > 0$ and $C_{\text{second}} > 0$ such that,

$$\sum_{j=1}^2 b_j(x_1, A_{x_1})^{\frac{1}{4}} \leq C_{\text{first}},$$

²This proposition is provided for $q_0 = 1$. However, a closer examination of the proof reveals that it can be extended to any $q_0 \geq 1$ in a straightforward manner.

and for $x_3 \in B_n$, $d(x_3, \partial B_n) > n^\alpha$ and $d(x_1, x_3) \geq n^\alpha$,

$$\sum_{j=3}^5 b_j(x_3, x_1, A_{x_3})^{\frac{1}{4}} \leq \begin{cases} 2E_0 n^{-E_1}, & \text{if } d = 2, \\ 2E_0 (\log n)^{-E_2}, & \text{if } d \geq 3, \end{cases}$$

and for other $(x_1, x_3) \in B_n \times B_n$, the points near the boundary of B_n , similar to [LRPY22, proof of Proposition 3.11], we just apply uniform moment bounds for the flexible cost functions and get

$$\sum_{j=3}^5 b_j(x_3, x_1, A_{x_3})^{\frac{1}{4}} \leq C_{\text{second}}.$$

Therefore, by the fact that

$$|\{(x, y) \in B_n^2 : A_x \cap B_n \cap A_y \neq \emptyset\}| \asymp n^d n^{d\alpha},$$

then we have,

$$\gamma'_1 \lesssim \begin{cases} \frac{n^{\frac{d\alpha}{2}}}{n^{\frac{d}{2}}} C_{\text{second}} + 2E_0 n^{-E_1}, & \text{if } d = 2, \\ \frac{n^{\frac{d\alpha}{2}}}{n^{\frac{d}{2}}} C_{\text{second}} + 2E_0 (\log n)^{-E_2}, & \text{if } d \geq 3. \end{cases}$$

Similarly, one can derive

$$\gamma'_2 \lesssim \begin{cases} \frac{n^{\frac{d\alpha}{2}}}{n^{\frac{d}{2}}} C_{\text{second}} + 2E_0 n^{-E_1}, & \text{if } d = 2, \\ \frac{n^{\frac{d\alpha}{2}}}{n^{\frac{d}{2}}} C_{\text{second}} + 2E_0 (\log n)^{-E_2}, & \text{if } d \geq 3, \end{cases}$$

and

$$\gamma'_3, \gamma'_4, \gamma'_5 \lesssim \frac{1}{n^{\frac{d}{2}}},$$

and

$$\gamma'_6 \lesssim \begin{cases} \frac{n^{\frac{d\alpha}{2}}}{n^{\frac{d}{2}}} C_{\text{second}} + 2E_0 n^{-E_1}, & \text{if } d = 2, \\ \frac{n^{\frac{d\alpha}{2}}}{n^{\frac{d}{2}}} C_{\text{second}} + 2E_0 (\log n)^{-E_2}, & \text{if } d \geq 3, \end{cases}$$

Therefore, we complete the proof by invoking Theorem 3.1. $\square$

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REFERENCES

- [BCY18] Jean-Daniel Boissonnat, Frédéric Chazal, and Mariette Yvinec. *Geometric and topological inference*, volume 57. Cambridge University Press, 2018. (Cited on page 5.)
- [BSY19] Thomas Berrett, Richard Samworth, and Ming Yuan. Efficient multivariate entropy estimation via k -nearest neighbour distances. *The Annals of Statistics*, 47(1):288–318, 2019. (Cited on pages 12, 13, 26, and 29.)
- [BY05] Yuliy Baryshnikov and Joesph Yukich. Gaussian limits for random measures in geometric probability. *The Annals of Applied Probability*, 15(1A):213–253, 2005. (Cited on page 1.)
- [Cha09] Sourav Chatterjee. Fluctuations of eigenvalues and second order Poincaré inequalities. *Probability Theory and Related Fields*, 143(1):1–40, 2009. (Cited on page 1.)
- [CS17] Sourav Chatterjee and Sanchayan Sen. Minimal spanning trees and Stein’s method. *The Annals of Applied Probability*, 27(3):1588–1645, 2017. (Cited on pages 1 and 15.)
- [EH10] Herbert Edelsbrunner and John Harer. *Computational topology: An introduction*. American Mathematical Soc., 2010. (Cited on page 5.)
- [Eng81] Gunnar Englund. A remainder term estimate for the normal approximation in classical occupancy. *The Annals of Probability*, pages 684–692, 1981. (Cited on page 10.)
- [GTT19] Akshay Goel, Khanh Duy Trinh, and Kenkichi Tsunoda. Strong law of large numbers for Betti numbers in the thermodynamic regime. *Journal of Statistical Physics*, 174(4):865–892, 2019. (Cited on page 14.)
- [KK97] Olav Kallenberg and Olav Kallenberg. *Foundations of modern probability*, volume 2. Springer, 1997. (Cited on page 2.)
- [KL87] Lyudmyla Kozachenko and Nikolai Leonenko. Sample estimate of the entropy of a random vector. *Problemy Peredachi Informatsii*, 23(2):9–16, 1987. (Cited on page 12.)
- [KL96] Harry Kesten and Sungchul Lee. The central limit theorem for weighted minimal spanning trees on random points. *The Annals of Applied Probability*, 6(2):495–527, 1996. (Cited on pages 1 and 3.)
- [KRP21] Johannes Krebs, Benjamin Roycraft, and Wolfgang Polonik. On approximation theorems for the Euler characteristic with applications to the bootstrap. *Electronic Journal of Statistics*, 15(2):4462–4509, 2021. (Cited on pages 5, 11, 14, 29, and 30.)
- [LP11] Günter Last and Mathew Penrose. Poisson process Fock space representation, chaos expansion and covariance inequalities. *Probability Theory and Related Fields*, 150(3):663–690, 2011. (Cited on page 2.)
- [LPS16] Günter Last, Giovanni Peccati, and Matthias Schulte. Normal approximation on Poisson spaces: Mehler’s formula, second order Poincaré inequalities and stabilization. *Probability Theory and Related Fields*, 165(3):667–723, 2016. (Cited on pages 1, 2, 8, 9, 10, 15, 16, and 24.)
- [LRP17] Raphaël Lachièze-Rey and Giovanni Peccati. New Berry–Esseen bounds for functionals of binomial point processes. *The Annals of Applied Probability*, 27(4):1992–2031, 2017. (Cited on pages 1, 9, 10, 11, 12, 14, and 28.)
- [LRPY22] Raphaël Lachièze-Rey, Giovanni Peccati, and Xiaochuan Yang. Quantitative two-scale stabilization on the Poisson space. *The Annals of Applied Probability*, 32(4):3085–3145, 2022. (Cited on pages 1, 2, 6, 7, 10, 12, 14, 15, 30, and 31.)
- [LRSY19] Raphaël Lachièze-Rey, Matthias Schulte, and Joesph Yukich. Normal approximation for stabilizing functionals. *The Annals of Applied Probability*, 29(2):931–993, 2019. (Cited on pages 1, 2, 4, 6, 7, 10, 12, 13, 14, 15, 24, 25, 26, 27, and 28.)
- [MJ13] Andrew Mastin and Patrick Jaillet. Log-quadratic bounds for the Gaussian Q-function. *arXiv preprint arXiv:1304.2488*, 2013. (Cited on page 28.)
- [NPR09] Ivan Nourdin, Giovanni Peccati, and Gesine Reinert. Second order Poincaré inequalities and CLTs on Wiener space. *Journal of Functional Analysis*, 257(2):593–609, 2009. (Cited on page 1.)
- [OT20] Takashi Owada and Andrew Thomas. Limit theorems for process-level Betti numbers for sparse and critical regimes. *Advances in Applied Probability*, 52(1):1–31, 2020. (Cited on page 14.)
- [Pen03] Mathew Penrose. *Random geometric graphs*, volume 5. OUP Oxford, 2003. (Cited on pages 25 and 26.)

- [Pen05] Mathew Penrose. Multivariate spatial central limit theorems with applications to percolation and spatial graphs. *The Annals of Probability*, 33(5):1945–1991, 2005. (Cited on pages 1, 3, 15, and 30.)
- [Pen07] Mathew Penrose. Gaussian limits for random geometric measures. *Electronic Journal of Probability*, 12:989–1035, 2007. (Cited on page 6.)
- [PR16] Giovanni Peccati and Matthias Reitzner. *Stochastic analysis for Poisson point processes: Malliavin calculus, Wiener-Itô chaos expansions and stochastic geometry*, volume 7. Springer, 2016. (Cited on page 1.)
- [PRR13] Erol A Peköz, Adrian Röllin, and Nathan Ross. Degree asymptotics with rates for preferential attachment random graphs. *The Annals of Applied Probability*, 23(3):1188–1218, 2013. (Cited on page 10.)
- [PSTU10] Giovanni Peccati, Josep Lluís Solé, Murad S Taqqu, and Frederic Utzet. Stein’s method and normal approximation of Poisson functionals. *The Annals of Probability*, 38(2):443–478, 2010. (Cited on page 1.)
- [PY01] Mathew Penrose and Joseph Yukich. Central limit theorems for some graphs in computational geometry. *Annals of Applied probability*, pages 1005–1041, 2001. (Cited on pages 1, 3, 6, 12, 26, and 30.)
- [PY05] Mathew Penrose and Joseph Yukich. Normal approximation in geometric probability. *Stein’s method and applications*, 5:37–58, 2005. (Cited on pages 1, 3, and 6.)
- [PY13] Mathew Penrose and Joseph Yukich. Limit theory for point processes in manifolds. *The Annals of Applied Probability*, 23(6):2161–2211, 2013. (Cited on pages 12 and 13.)
- [Sch10] Tomasz Schreiber. Limit theorems in stochastic geometry. *New perspectives in stochastic geometry*, pages 111–144, 2010. (Cited on page 1.)
- [SP16] Shashank Singh and Barnabás Póczos. Finite-sample analysis of fixed-k nearest neighbor density functional estimators. *Advances in neural information processing systems*, 29, 2016. (Cited on page 28.)
- [SY21] Matthias Schulte and JE Yukich. Rates of multivariate normal approximation for statistics in geometric probability. *arXiv preprint arXiv:2103.00625*, 2021. (Cited on page 10.)
- [TO21] Andrew Thomas and Takashi Owada. Functional limit theorems for the Euler characteristic process in the critical regime. *Advances in Applied Probability*, 53(1):57–80, 2021. (Cited on page 14.)
- [Tri17] Khanh Duy Trinh. A remark on the convergence of Betti numbers in the thermodynamic regime. *Pacific Journal of Mathematics for Industry*, 9(1):1–7, 2017. (Cited on page 14.)
- [VdV00] Aad Van der Vaart. *Asymptotic statistics*, volume 3. Cambridge university press, 2000. (Cited on page 12.)
- [YSA17] D. Yogeshwaran, Eliran Subag, and Robert Adler. Random geometric complexes in the thermodynamic regime. *Probability Theory and Related Fields*, 167(1):107–142, 2017. (Cited on page 29.)
- [Yuk15] Joseph Yukich. Surface order scaling in stochastic geometry. *The Annals of Applied Probability*, 25(1):177–210, 2015. (Cited on page 6.)

APPENDIX A. BACKGROUND ON TDA

Definition A.1 (Simplicial Complexes). *An abstract simplicial complex over a (finite) vertex set $\mathbb{A}_n := \{a_1, \dots, a_n\}$ is a collection S of subsets of $\mathbb{A}_n$ with the properties that*

- (i) $\{a_i\} \in S, i = 1, \dots, n,$
- (ii) $\sigma \in S$ and $\tau \subset \sigma$ implies that $\tau \in S$.

Every $\tau \subset \sigma$ is called a face of σ . Every $\sigma \in S$ with $|\sigma| = \ell + 1, \ell \geq 0$, is called ℓ -simplex.

Note that the vertices do not necessarily have to be elements of a Euclidean space. If they are (affinely independent) elements of $\mathbb{R}^d$, one can think of every simplex of order $\ell \leq d$ as a convex hull of $\ell + 1$ (affinely independent) vertices, so that 0-simplices are points, 1-simplices are lines, 2-simplices are triangles, etc. The following two types of simplicial complexes, the Vietoris-Rips complex (VR complex) and the Čech complex, are widely used in TDA.

Definition A.2 (VR Complex). *Following the definition A.1, let the vertex set $\mathbb{A}_n$ be in a metric space with the metric d . Then, the VR complex $VR_r(\mathbb{A}_n)$ for a given positive real number $r > 0$ is a collection of simplices, where a simplex $\sigma \in VR_r(\mathbb{A}_n)$ if and only if for any pair of vertices $a_i, a_j \in \sigma$, $d(a_i, a_j) < r$.*

Definition A.3 (Čech Complex). *Following the definition A.1, let the vertex set $\mathbb{A}_n$ be in a metric space with the metric d . Then, the Čech complex $C_r(\mathbb{A}_n)$ for a given positive real number $r > 0$ is a collection of simplices, where a simplex $\sigma := \{a_i\}_{i \in I} \in C_r(\mathbb{A}_n)$ for some $I \subset \{1, 2, \dots, n\}$ if and only if for $\bigcap_{i \in I} B_{a_i}(\frac{r}{2}) \neq \emptyset$.*

Definition A.4 (Filtrations). *A filtration $\mathcal{S}$ of a simplicial complex S is a nested sequence of simplicial complexes $\emptyset = S_0 \subset S_1 \subset \dots \subset S_I = S$, where $S_i = S_{i-1} \cup \sigma_i$, $i = 1, \dots, I$ for some $\sigma_i \in S$. A filtration is thus equivalent to an ordering of the simplices in the complex. Usually, a filtration is given in form of a filtration function $\psi : S \rightarrow \mathbb{R}$ that assigns a real value $\psi(\sigma)$ to each simplex $\sigma \in S$. The filtration itself is then defined via $S(r) = \{\sigma \in S : \psi(\sigma) \leq r\}$. Note that while r is a continuous parameter, there are only finitely many values of r at which the complex is changing for a simplex over a finite set of vertices as considered here. Additionally, the parameter r here is called as the filtration parameter or the filtration time.*

Remark A.1. *The real number r in Definition A.2 and A.3 is called the filtration parameter/time for the VR complex and the Čech complex respectively.*