Time-fractional Caputo derivative versus other integrodifferential operators in generalized Fokker-Planck and generalized Langevin equations

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Fractional diffusion and Fokker-Planck equations are widely used tools to describe anomalous diffusion in a large variety of complex systems. The equivalent formulations in terms of Caputo or Riemann-Liouville fractional derivatives can be derived as continuum limits of continuous-time random walks and are associated with the Mittag-Leffler relaxation of Fourier modes, interpolating between a short-time stretched exponential and a long-time inverse power-law scaling. More recently, a number of other integrodifferential operators have been proposed, including the Caputo-Fabrizio and Atangana-Baleanu forms. Moreover, the conformable derivative has been introduced. We study here the dynamics of the associated generalized Fokker-Planck equations from the perspective of the moments, the time-averaged mean-squared displacements, and the autocovariance functions. We also study generalized Langevin equations based on these generalized operators. The differences between the Fokker-Planck and Langevin equations with different integrodifferential operators are discussed and compared with the dynamic behavior of established models of scaled Brownian motion and fractional Brownian motion. We demonstrate that the integrodifferential operators with exponential and Mittag-Leffler kernels are not suitable to be introduced to Fokker-Planck and Langevin equations for the physically relevant diffusion scenarios discussed in our paper. The conformable and Caputo Langevin equations are unveiled to share similar properties with scaled and fractional Brownian motion, respectively.

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I. INTRODUCTION

Power laws and fractional derivatives have a long tradition in the sciences. Thus, Buelffinger modified Hooke's law to a general power exponent in 1729 [1,2]. Weber is credited for having (indirectly) discovered viscoelasticity, following his report of a nonelastic aftereffect in stretched silk threads in 1835 [3], an effect that initiated the development of generalized rheological models [4]. An explicit time dependence deviating from the exponential law was proposed by Kohlrausch in 1847 in the form of a stretched exponential, introducing a fractional power law into an exponential function [3]. Power-law time dependencies in relaxation phenomena were reported by Nutting in 1921 [5]. To grasp such a behavior in a compact dynamic equation, Scott Blair then formulated a relaxation equation with a fractional derivative [6]. Fractional rheological models have since been used widely to describe the viscoelastic or glassy behavior in complex systems [7–11]. Fractional relaxation processes are also relevant in biological contexts [12,13].

Fractional derivatives have also found widespread use in the context of anomalous diffusion, typically defined in terms of the power-law form [14–18]

$$\langle x^2(t)\rangle \simeq K_\alpha t^\alpha \tag{1}$$

of the mean-squared displacement (MSD). Here K_α of physical dimension length^2/time$^\alpha$ is the generalized diffusion coefficient, and the value of the anomalous diffusion exponent α distinguishes subdiffusion ($0 < \alpha < 1$) from superdiffusion ($\alpha > 1$), including the special cases of normal diffusion ($\alpha = 1$) and ballistic, wavelike motion ($\alpha = 2$). A breakthrough in describing anomalous diffusion came with the work of Schneider and Wyss [19], who started from the integral version of the diffusion equation

$$P(x,t) - P_0(x) = K_1 \int_0^t \frac{\partial^2}{\partial x^2} P(x,t')dt' \tag{2}$$

for the probability density function $P(x,t)$ to find the test particle at position x at time t, given the initial condition $P_0(x)$ at $t = 0$. They then replaced the integral by a Riemann-Liouville

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(RL) fractional integral operator defined for a suitable function $f(t)$ as [20,21]

$$ {}_0D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(t')}{(t-t')^{1-\alpha}} dt', \tag{3}$$

which is a direct generalization of the Cauchy multiple integral. After differentiation by time once one obtains the equivalent fractional diffusion equation (FDE) in RL-form [15],

$$ \frac{\partial}{\partial t} P(x,t) = K_\alpha \, {}_0D_t^{1-\alpha} \frac{\partial^2}{\partial x^2} P(x,t), \tag{4}$$

where ${}_0D_t^{1-\alpha} = (d/dt)\, {}_0D_t^{-\alpha}$ is the RL fractional differential operator [20,21]. The solution of the FDE (4) can be obtained in terms of Fox H-functions [15,19,22]. The asymptotic behavior of the PDF $P(x,t)$ encoded in the FDE (4) has a stretched Gaussian shape [15,19,23]. FDEs of the above form can be derived as the continuum limit of continuous-time random walks (CTRWs) with scale-free waiting time densities $\psi(t) \simeq t^{-1-\alpha}$ with $0 < \alpha < 1$ and jump length densities with finite variance [15,23–28]. By a single-particle tracking method, waiting time densities with power-law forms were revealed, i.e., in protein motion in membranes [29], colloidal tracer motion in actin networks [30,31], or for tracers in nonlaminar flows [32]. Power-law waiting time densities were also identified in dynamic maps [33] or simulations of drug molecules diffusing in silica slits [34]. When an external potential influences the particle motion, the FDE (4) can be generalized to the fractional Fokker-Planck equation [15,23,26,27,35–38]. For transport in groundwater and other applications, advection terms as well as mobile-immobile scenarios are considered in generalized versions of FDEs [39–46]. In the context of FDEs and fractional Fokker-Planck equations of RL-type, the relaxation of modes follows the Mittag-Leffler pattern (see below) that interpolates between an initial stretched exponential and a long-time inverse power-law [7,15].

Apart from the mentioned RL fractional derivatives, there exist a wide variety of other types of fractional derivatives [47–50], many of which are being used in engineering and science applications. Possibly the most widely used apart from the RL definition is the Caputo fractional derivative [51,52]. We note that both definitions are in fact equivalent as long as the initial values are properly taken into account. Thus, when we solve the FDE (4) for a specific initial value problem, the Caputo version of the FDE studied below leads to the same result. Caputo fractional derivatives are used to model, inter alia, non-Darcian flow [53], permeability models for rocks [54], contaminant transport [55], or viscoelastic diffusion [56,57]. Apart from the Caputo or RL derivative based on a power-law integral kernel with a (weak) singularity, in the past decade some new nonsingular integrodifferential operators have been proposed. One option is the Caputo-Fabrizio (CF) integrodifferential operator with an exponential kernel [58]. The CF operator was employed in a number of areas, for instance fluid flow [59], virus models [60,61], and a human liver model [62]. Alternatively, the Atangana-Baleanu (AB) integrodifferential operator based on the Mittag-Leffler function for the memory kernel [63] aims to describe the full memory effect in systems since the Mittag-Leffler function combines a stretched exponential shape and a power-law decay at short and long times, respectively. The AB operator is used in FDEs [64], Cauchy and source problems for advection-diffusion [65], optimal control [66], and disease models [67]. Comparisons between the AB and CF integrodifferential operators are investigated in relaxation and diffusion models [68], reaction-diffusion models [69], cancer models [70], heat transfer analysis [71], and for the Casson fluid [72].

Apart from the nonlocal integrodifferential operators mentioned above, a local derivative, the so-called conformable derivative, has been introduced [73–76] and studied from a physical point of view [77]. Applications of the conformable derivative formulation have been discussed for anomalous diffusion [78], advection-diffusion [79,80], non-Darcian flow [81], and other differential equations [82–86]. A similar variant is the Hausdorff derivative proposed earlier by Chen [87]. This derivative is also employed in diffusion scenarios [88–92], anomalous diffusion in magnetic resonance imaging [93], viscoelastic modeling [94], or the Richards' equation [95]. It was shown that the conformable derivative is in fact proportional to the Hausdorff derivative [96,97]. Further discussions of these derivatives can be found in [98–100].

We scrutinize here the integrodifferential operators recently proposed in the framework of FDEs and generalized Fokker-Planck equations as well as their applications in generalizations of the stochastic Langevin equations. Fourier and Laplace transforms are used to obtain analytical solutions for the PDFs and the moments to study the dynamics encoded in these dynamic equations. In particular, we unveil the connections between FDEs and fractional Langevin equations with other well-known stochastic processes, particularly with scaled Brownian motion (SBM, based on a Langevin equation with deterministic power-law time dependence of the diffusion coefficient) and fractional Brownian motion (FBM, based on a Langevin equation driven by zero-mean Gaussian noise with long-range, power-law correlations). Our discussion is based on experimentally measurable quantities. These include the first and second moments, the MSD, and the PDF. Moments can be directly inferred from measured time series as either ensemble or time averages [101]. PDFs can also be reconstructed in many contemporary studies and used, *inter alia*, to check for non-Gaussianity features [102].

The paper is structured as follows. In Sec. II we introduce and briefly discuss different integrodifferential operators and recall the generalized Fokker-Planck and Langevin equations in describing stochastic processes in complex systems. In Sec. III we discuss the results of local and nonsingular integrodifferential operators in Fokker-Planck equations for the force-free case and for a constant drift. Specifically, we discuss the extent to which the CF and AB integrodifferential operators can provide physically meaningful descriptions in the anomalous diffusion context. The results of the conformable and Caputo diffusion equations (with drift) and SBM (with drift) are compared. In Sec. IV we focus on generalized Langevin equations for SBM and FBM, as well as Langevin equations with the four integrodifferential operators introduced in Sec. II. Specifically, we again discuss the physical implications of the CF and AB integrodifferential operators in this context. The related moments, time-averaged MSD (TAMSD), and autocovariance function (ACVF) are considered to assess different Langevin equations. In Sec. V we summarize and discuss our results. We also present two tables with the main results for the generalized Fokker-Planck and Langevin equations discussed in the paper.

II. INTEGRODIFFERENTIAL OPERATORS, GENERALIZED FOKKER-PLANCK AND LANGEVIN EQUATIONS

In this section, we provide a brief introduction to four definitions of integrodifferential operators that are frequently employed in theoretical modeling and engineering, the Caputo derivative (note our remarks to the extent these are equivalent to the RL-derivative) and conformable derivative, as well as the two recently proposed CF and AB integrodifferential operators. We then introduce them to generalized formulations of the Fokker-Planck and Langevin equations.

A. Integrodifferential operators

The Caputo derivative of order $\alpha \in (0, 1]$ for a suitable function $f(t)$ is defined in terms of a power-law kernel [51,52],

$$ {}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(t')}{(t-t')^\alpha} dt', \tag{5} $$

where $f'(t) = df(t)/dt$. The Caputo derivative thus has a (weak) singularity at $t' = t$. The special feature introduced by Caputo in his derivative is the fact that the derivative on the function $f(t)$, $f'(t) = df(t)/dt$, is contained *inside* the integral. This contrasts the definition of the RL fractional derivative, in which the differentiation is taken *after* the fractional integration [15,20]. In the Caputo formulation, e.g., when using Laplace transform methods to derive the solution, the initial conditions thus enter in the traditional way. For $0 < \alpha < 1$, e.g., the initial value of $f(t)$ at $t = 0$ is needed. This contrasts the RL derivative, for which fractional-order initial conditions enter [20]. However, this complication for the RL derivative can be circumvented in the Schneider and Wyss integral formulation presented above [7,13–15,19,26,27,35]. In the solution of time-fractional equations with initial condition given at $t = 0$, the Laplace transform

$$ \mathscr{L}\{f(t)\}(s) \equiv \tilde{f}(s) = \int_0^\infty f(t) \exp(-st) dt \tag{6} $$

is of central importance. For the fractional RL-Integral, the Laplace transform reads $\mathscr{L}\{f(t)\} = s^{-\alpha}\tilde{f}(s)$, and for the Caputo-fractional operator we have

$$ \mathscr{L}\left\{ {}_0^C D_t^\alpha f(t) \right\} = s^\alpha \tilde{f}(s) - s^{\alpha-1} f(0). \tag{7} $$

Recently, two new definitions of integrodifferential operators with a nonsingular kernel were proposed. One is the Caputo-Fabrizio (CF) integrodifferential operator of order $\alpha \in (0, 1]$ defined with an exponential kernel [58],

$$ {}_0^{\rm CF} D_t^\alpha f(t) = \frac{M(\alpha)}{(1-\alpha)\tau^\alpha} \int_0^t f'(t') \exp\left(\frac{-\alpha(t-t')}{(1-\alpha)\tau} \right) dt', \tag{8} $$

where the factor $M(\alpha)$ introduced in [58] is chosen such that $M(0) = M(1) = 1$. Note that we introduced the timescale τ in order to get dimensions correct. The choice τ^α in the factor in front of the exponential allows us to take the limits $\alpha = 0$ [where we get $f(t) - f(0)$] and $\alpha = 1$ (the normal differential) consistently. The Laplace transform of the CF-operator reads

$$ \mathscr{L}\left\{ {}_0^{CF} D_t^\alpha f(t) \right\} = \frac{s\tilde{f}(s) - f(0)}{(1-\alpha)\tau^\alpha s + \alpha\tau^{\alpha-1}}. \tag{9} $$

The other variant for a generalized fractional derivative is given by the Atangana-Baleanu (AB) integrodifferential operator with a Mittag-Leffler kernel [63],

$$ {}_0^{\rm AB} D_t^\alpha f(t) = \frac{B(\alpha)}{(1-\alpha)\tau^\alpha} \int_0^t f'(t') E_\alpha\left(-\alpha \frac{(t-t')^\alpha}{(1-\alpha)\tau^\alpha} \right) dt', \tag{10} $$

where $E_\alpha(-z) = \sum_{k=0}^\infty (-z)^k/\Gamma(1+\alpha k)$ is the one-parameter Mittag-Leffler function with expansion around infinity, $E_\alpha(-z) \sim -\sum_{k=1}^\infty (-z)^{-k}/\Gamma(1-\alpha k)$ [103]. In particular, when $\alpha = 1$, $E_1(z) = e^z$. Note that again we introduced the timescale τ for dimensional consistency. Moreover, we note that here $B(\alpha)$ is a normalization function satisfying $B(0) = B(1) = 1$ and has the same properties as in the CF operator. To simplify our notation in the following, we set $M(\alpha) = B(\alpha) = 1$ [58,63]. The Laplace transform of the AB-operator has the form

$$ \mathscr{L}\left\{ {}_0^{\rm AB} D_t^\alpha f(t) \right\} = \frac{s^\alpha \tilde{f}(s) - s^{\alpha-1} f(0)}{(1-\alpha)\tau^\alpha s^\alpha + \alpha}. \tag{11} $$

All these definitions—Caputo, CF, and AB integrodifferential operators—correspond to convolutions of the derivative $f'(t)$ with different choices for the kernels, i.e., power-law, exponential, and Mittag-Leffler functions, respectively. In contrast, there also exists a local definition of a generalized derivative, namely, the conformable derivative of order $\alpha \in (0, 1]$, defined via [104]

$$ T_\alpha f(t) = \lim_{\bar{\tau} \to 0} \frac{f(t + \bar{\tau}^\alpha t^{1-\alpha}) - f(t)}{\bar{\tau}^\alpha}. \tag{12} $$

Here we used the small variable $\bar{\tau}$ with dimension of time to housekeep physical dimensions. If the conformable derivative of f of order α exists in some interval $(0, a)$, $a > 0$, and $\lim_{t\to 0^+} T_\alpha f(t)$ exists, then we define $T_\alpha f(0) = \lim_{t\to 0^+} T_\alpha f(t)$. The conformable Laplace transform of $f(t)$ is defined by [73]

$$ \mathscr{L}_\alpha\{f(t)\}(s) \equiv \tilde{f}_\alpha(s) = \int_0^\infty f(t) t^{\alpha-1} \exp\left(-s\frac{t^\alpha}{\alpha} \right) dt, \tag{13} $$

generalizing the standard Laplace transform (6). The relationship between the conformable Laplace transform and the ordinary Laplace transform is $\mathscr{L}_\alpha\{f(t)\}(s) = \mathscr{L}\{f([\alpha t]^{1/\alpha})\}(s)$. The conformable Laplace transform of the conformable derivative is

$$ \mathscr{L}_\alpha\{T_\alpha f(t)\}(s) = s\tilde{f}_\alpha(s) - f(0). \tag{14} $$

The Hausdorff derivative (*fractal* derivative) of a suitable function $f(t)$ with respect to t^α [87] is defined as

$$ \frac{df(t)}{dt^\alpha} = \lim_{t'\to t} \frac{f(t) - f(t')}{t^\alpha - (t')^\alpha}. \tag{15} $$

From this definition we see that the Hausdorff derivative is also local in nature. The connection between the Hausdorff derivative and the conformable derivative is given by [96,97]

$$\alpha\frac{df(t)}{dt^{\alpha}}=t^{1-\alpha}\frac{df(t)}{dt}=T_{\alpha}f(t). \tag{16}$$

B. Generalized Fokker-Planck equations

We now use these operators to generalize the Fokker-Planck equation [105],

$$\frac{\partial}{\partial t}P(x,t)=\left(K_1\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial x}\frac{F(x)}{m\eta}\right)P(x,t), \tag{17}$$

in the presence of a general external force field $F(x)$. Here m is the particle mass and η the friction coefficient [105]. For the force, we obtain solutions for vanishing or constant external force F_0. In the latter case, we then use the drift velocity $v=F_0/(m\eta)$. In the generalization based on the RL derivative, the fractional Fokker-Planck equation was analyzed for different linear and nonlinear force fields [14,15,27,35].

We introduce here the four generalized differential operators above and analyze the generalized diffusion equation (GDE) or generalized diffusion equation with drift (drift-GDE),

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}P(x,t)=\left(K_{\alpha}\frac{\partial^2}{\partial x^2}-v_{\alpha}\frac{\partial}{\partial x}\right)P(x,t), \tag{18}$$

with initial condition $P(x,0)=\delta(x)$ for $v_{\alpha}=0$ and $v_{\alpha}\neq 0$ and "natural" boundary conditions $P(|x|\to\infty,t)=0$. These standard initial and boundary conditions will be applied throughout this work. Note that we introduced the α-dependent velocity v_{α} for dimensionality housekeeping purposes. This can be achieved by setting $v_{\alpha}=v\tau^{1-\alpha}$, where τ is a timescale, such that v_{α} has dimension length/time$^{\alpha}$. We seek solutions of Eq. (18) on the infinite line $-\infty<x<\infty$, and the notation $\partial^{\alpha}/\partial t^{\alpha}$ represents our four operators.

For our analysis, we also need to introduce SBM [106,107], whose diffusion coefficient is explicitly time-dependent and evolves as power-law

$$\mathscr{K}_{\alpha}(t)=\alpha K_{\alpha}t^{\alpha-1} \tag{19}$$

with $\alpha>0$. SBM is a Gaussian self-similar Markovian process with independent but nonstationary increments. It finds applications in turbulence [108], stochastic hydrology [109], finance [110], granular gases [111], and magnetic resonance imaging [112], to name a few. The Fokker-Planck equation for SBM in an external force field is given by [107]

$$\frac{\partial}{\partial t}P(x,t)=\left(\mathscr{K}_{\alpha}(t)\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial x}\frac{F(x)}{m\eta}\right)P(x,t). \tag{20}$$

C. Generalized Langevin equations

Fokker-Planck equations are deterministic equations for the PDF $P(x,t)$. A stochastic description of the position of a test particle in the presence of a fluctuating force is the Langevin equation [113], the alternative standard description of diffusive processes [114]. The (overdamped) Langevin equation corresponding to the Fokker-Planck equation (17) with $P_0(x)=\delta(x)$ reads

$$\frac{d}{dt}x(t)=\frac{F(x)}{m\eta}+\sqrt{2K_1}\xi(t), \tag{21}$$

with the initial position $x(0)=0$. Here $\xi(t)$ is zero-mean white Gaussian noise with ACVF $\langle\xi(t)\xi(t')\rangle=\delta(t-t')$.

Then, the *generalized* Langevin equation for the thermalized system in the overdamped approximation reads [115]

$$\int_0^t\gamma(t-t')\frac{dx(t')}{dt'}dt'=\frac{F(x)}{m\eta}+\frac{\zeta(t)}{m\eta}, \tag{22}$$

where the noise autocorrelation function is coupled to the frictional kernel by the Kubo-Zwanzig fluctuation-dissipation theorem (FDT) [116] $\langle\zeta(t_1)\zeta(t_2)\rangle=k_BTm\eta\gamma(t_1-t_2)$, where k_B is the Boltzmann constant and T is the absolute temperature of the environment. The noise $\zeta(t)$ obeying the FDT is called *internal*. Generalized Langevin equations with FDT and different kernels, e.g., of exponential and Mittag-Leffler shapes, have been extensively studied before [117–121].

In what follows, we generalize the Langevin equation (21) via the four operators in Eqs. (5), (8), (10), and (12) in the presence of a constant force, with the unifying notation

$$\frac{d^{\alpha}}{dt^{\alpha}}x(t)=v_{\alpha}+\sqrt{2K_{\alpha}}\xi(t). \tag{23}$$

We note that FBM and the generalized Langevin equations we consider here do not fulfill the generalized fluctuation-dissipation theorem and thus do not describe equilibrium systems. Instead, the noise is considered to be *external* [122]. This is appropriate for active systems, in which energy is dissipated, e.g., living biological cells.

We will compare the resulting dynamics with that encoded by SBM, whose Langevin equation is given by [106,107]

$$\frac{d}{dt}x(t)=\frac{F(x)}{m\eta}+\sqrt{2\mathscr{K}_{\alpha}(t)}\xi(t), \tag{24}$$

which is equivalent to the deterministic equation (20). The noise $\xi(t)$ has the same properties as for the standard Langevin equation (21), i.e., it is zero-mean white Gaussian noise. In the following, we will consider the cases of zero and constant force.

As we will see, it will also be of interest to compare our results to the dynamics of FBM [123], which is stationary in increments and nearly ergodic [124–126]. As a generalization of Brownian motion, FBM is an effective stochastic process to model anomalous diffusion [101]. Its Langevin equation reads [16,123]

$$\frac{d}{dt}x(t)=\frac{F(x)}{m\eta}+\sqrt{2K_{\alpha}}\xi_{\alpha}(t), \tag{25}$$

where $\xi_{\alpha}(t)$ is zero-mean fractional Gaussian noise with the long-range, power-law ACVF $\langle\xi_{\alpha}(t_1)\xi_{\alpha}(t_2)\rangle\sim\frac{1}{2}\alpha(\alpha-1)|t_1-t_2|^{\alpha-2}$ when $|t_1-t_2|\gg 1$. FBM is defined for the range $0<\alpha<2$ of the anomalous diffusion exponent, instead of which the Hurst exponent $H=\alpha/2$ is often used. From the noise ACVF we can see that the noise correlations are positive (persistent) when the motion is superdiffusive, while they are negative (antipersistent) in the subdiffusive case.

We will characterize the dynamics of the processes that we consider the moments, the time-averaged MSD (TAMSD)

[16,101], and the displacement ACVF of the process. The TAMSD is important in the analysis of single-particle trajectories measured in modern tracking experiments; it is defined via

$$\overline{\delta^2(\Delta)} = \frac{1}{T-\Delta}\int_0^{T-\Delta}(x(t+\Delta)-x(t))^2dt, \tag{26}$$

where T is the length of the time series (measurement time) and Δ is called the lag time. The mean TAMSD is obtained from averaging over a number N of individual traces $\overline{\delta_i^2(\Delta)}$,

$$\langle\overline{\delta^2(\Delta)}\rangle = \frac{1}{N}\sum_{i=1}^{N}\overline{\delta_i^2(\Delta)}. \tag{27}$$

In the Birkhoff-Boltzmann sense, a system is considered ergodic when ensemble and time averages are equivalent in the limit of long measurement times. We here consider a stochastic process nonergodic when the ensemble-averaged MSD and the TAMSD are disparate in the limit of long observation times, $\lim_{T\to\infty}\overline{\delta^2(\Delta)} \neq \langle x^2(\Delta)\rangle$. The displacement ACVF is defined as

$$C_\Delta(t) = C_\Delta(t) = \frac{\langle[x(t+\Delta)-x(t)][x(\Delta)-x(0)]\rangle}{\Delta^2}. \tag{28}$$

III. GENERALIZED FOKKER-PLANCK EQUATIONS

A. Generalized Fokker-Planck equations in the force-free case

We now first assess the dynamics encoded in the generalized Fokker-Planck equation (18) in the absence of an external force.

1. Caputo and nonsingular differential operators

When $F(x) = 0$, the generalized Fokker-Planck equation (18) reduces to the GDE

$$\frac{\partial^\alpha}{\partial t^\alpha}P(x,t) = K_\alpha\frac{\partial^2}{\partial x^2}P(x,t), \tag{29}$$

with $0 < \alpha \leqslant 1$, which we will solve on the interval $-\infty < x < \infty$ for the initial condition $P(x,0) = \delta(x)$. Here the operator $\partial^\alpha/\partial t^\alpha$ represents the Caputo, CF, and AB integrodifferential operators, Eqs. (5), (8), and (10).

Applying the Laplace and Fourier transforms to the GDE (29), we find

$$\hat{\tilde{P}}_C(k,s) = \frac{s^{\alpha-1}}{s^\alpha + K_\alpha k^2}, \tag{30}$$

$$\hat{\tilde{P}}_{\rm CF}(k,s) = \frac{1}{s + K_\alpha[\alpha\tau^{\alpha-1} + s(1-\alpha)\tau^\alpha]k^2}, \tag{31}$$

$$\hat{\tilde{P}}_{\rm AB}(k,s) = \frac{s^{\alpha-1}}{s^\alpha + K_\alpha[\alpha + s^\alpha(1-\alpha)\tau^\alpha]k^2}. \tag{32}$$

After inverse Fourier transform, we obtain the PDFs in Laplace space,

$$\tilde{P}(x,s) = \frac{1}{2s}\phi(s)\exp(-\phi(s)|x|), \tag{33}$$

where we define

$$\phi_C(s) = \sqrt{\frac{s^\alpha}{K_\alpha}}, \tag{34}$$

$$\phi_{\rm CF}(s) = \sqrt{\frac{s}{(1-\alpha)\tau^\alpha K_\alpha s + \alpha\tau^{\alpha-1}K_\alpha}}, \tag{35}$$

$$\phi_{\rm AB}(s) = \sqrt{\frac{s^\alpha}{(1-\alpha)\tau^\alpha K_\alpha s^\alpha + \alpha K_\alpha}}, \tag{36}$$

for the three operators, respectively.

We first recall the PDF of the Caputo fractional diffusion equation (FDE) based on the Caputo operator (5) in (x,t)-space,

$$P_C(x,t) = \frac{1}{2\sqrt{K_\alpha t^\alpha}}M_{\alpha/2}\left(\frac{|x|}{\sqrt{K_\alpha t^\alpha}}\right), \tag{37}$$

where M_ν denotes the Mainardi function [127,128], also called the M function (of the Wright type), of order ν. Using the asymptotic representation of $M_{\alpha/2}$, the PDF $P_C(x,t)$ can be shown to have a stretched Gaussian shape [129], corresponding to the above-mentioned results in [15,19,23]. An alternative representation of the PDF is via Fox H-functions [15,19]. The MSD of the Caputo-FDE reads

$$\langle x^2(t)\rangle_C = \frac{2K_\alpha}{\Gamma(\alpha+1)}t^\alpha, \tag{38}$$

and the corresponding kurtosis has the time-independent value

$$\kappa_C = \frac{3\alpha\Gamma(\alpha)^2}{\Gamma(2\alpha)}, \tag{39}$$

demonstrating the non-Gaussian character of $P_C(x,t)$.

To vouchsafe that the solutions of the AB-GDE and the CF-GDE are proper PDFs, they should be completely monotonic in the Laplace domain [130]. That is, we should first check the complete monotonicity of the form (33) together with Eqs. (35) and (36). In Appendix B1, we use the theory of Bernstein functions to demonstrate that the complete monotonicity is indeed ensured.

We now calculate the long-time limit of the PDF for the GDEs of CF- and AB-type from the Laplace representation in Eq. (33) together with Eqs. (35), and (36). In Laplace space, the long-time limit $t\to\infty$ corresponds to $s\to 0$, for which we find the asymptotic behaviors

$$\tilde{P}_{\rm CF}(x,s) \sim \frac{1}{2s}\sqrt{\frac{s}{\alpha\tau^{\alpha-1}K_\alpha}}\exp\left(-\sqrt{\frac{s}{\alpha\tau^{\alpha-1}K_\alpha}}|x|\right), \tag{40}$$

$$\tilde{P}_{\rm AB}(x,s) \sim \frac{1}{2s}\sqrt{\frac{s^\alpha}{\alpha K_\alpha}}\exp\left(-\sqrt{\frac{s^\alpha}{\alpha K_\alpha}}|x|\right). \tag{41}$$

Laplace back-transforming, these forms correspond to the long-time asymptotes

$$P_{\rm CF}(x,t) \sim \frac{1}{\sqrt{4\alpha\tau^{\alpha-1}K_\alpha\pi t}}\exp\left(-\frac{x^2}{4\alpha\tau^{\alpha-1}K_\alpha t}\right), \tag{42}$$

$$P_{\rm AB}(x,t) \sim \frac{1}{\sqrt{4\alpha K_\alpha t^\alpha}}M_{\alpha/2}\left(\frac{|x|}{\sqrt{\alpha K_\alpha t^\alpha}}\right). \tag{43}$$

Equation (42) demonstrates that the CF-GDE describes normal diffusion at long times, with a Gaussian shape and x scaling like $t^{1/2}$. In contrast Eq. (43) for the AB-GDE at long times is similar to the PDF of the Caputo-FDE. Both behaviors are expected from the definitions of the respective integrodifferential operators: at long times, the CF-operator includes an exponential cutoff, whereas the AB-operator has an asymptotic power-law tail equivalent to the kernel in the Caputo operator.

The short-time limit of the PDFs for the CF- and AB-GDE can be similarly calculated from their Laplace representations (33), (35), and (36). For $t \to 0$, corresponding to $s \to \infty$ in the Laplace domain, we have

$$\tilde{P}_{\mathrm{CF}}(x,s), \tilde{P}_{\mathrm{AB}}(x,s) \sim \frac{1}{2s\sqrt{(1-\alpha)\tau^\alpha K_\alpha}} \exp\left(-\frac{|x|}{\sqrt{(1-\alpha)\tau^\alpha K_\alpha}}\right), \tag{44}$$

from which we obtain the asymptotic short-time result in the time domain,

$$P_{\mathrm{CF}}(x,t), P_{\mathrm{AB}}(x,t) \sim \frac{1}{\sqrt{4(1-\alpha)\tau^\alpha K_\alpha}} \exp\left(-\frac{|x|}{\sqrt{(1-\alpha)\tau^\alpha K_\alpha}}\right). \tag{45}$$

This is a remarkable result, showing a finite width of the initial condition, which we will comment on below. The PDFs of the Caputo-, CF-, and AB-GDEs are shown for different times in Figs. 1(a)–1(c). Indeed, for the CF and AB cases the shapes of the limits for $t = 0$ correspond to a Laplace distribution.

We now calculate the MSDs for the CF- and AB-GDE,

$$\langle x^2(t)\rangle_{\mathrm{CF}} = 2\alpha K_\alpha \tau^{\alpha-1} t + 2(1-\alpha)K_\alpha \tau^\alpha \tag{46}$$

and

$$\langle x^2(t)\rangle_{\mathrm{AB}} = \frac{2K_\alpha}{\Gamma(\alpha)} t^\alpha + 2(1-\alpha)K_\alpha \tau^\alpha. \tag{47}$$

While the long-time behaviors $\langle x^2(t)\rangle_{\mathrm{CF}} \simeq t$ and $\langle x^2(t)\rangle_{\mathrm{AB}} \simeq t^\alpha$ produce normal and subdiffusive scaling, the values of the MSDs in the limit $t \to 0$ have finite values, corresponding to the finite width of the limits (45) of the PDFs. We also calculated the kurtosis in Appendix B1, Eqs. (B16) and (B17). At short times, $\kappa_{\mathrm{CF}}, \kappa_{\mathrm{AB}} \sim 6$, this means the CF- and AB-DE describe non-Gaussian process. At long times, $\kappa_{\mathrm{CF}} \sim 3$, i.e., the CF-GDE describes a Gaussian process, while $\kappa_{\mathrm{AB}} \sim 3\alpha\Gamma(\alpha)^2/\Gamma(2\alpha) = \kappa_C$, which shows that the AB-GDE is similar to the PDF of the Caputo-FDE in this long-time limit. The kurtosis of these models is shown in Fig. 2. The MSDs of the GDE with Caputo-, CF-, and AB-operators are shown in Fig. 3.

From this discussion we see that within the framework of the GDE considered here with initial value $P(x,0)$ given at time $t = 0$, the CF- and AB-operators produce inconsistent results in the short-time limit. This observation deserves a separate formal investigation. We note that the same results for the Fourier-Laplace forms Eqs. (31) and (32) can be derived from the corresponding integral formulations (see Appendix A) of these operators.

Next we establish the relation of CF-GDE and AB-GDE in Eq. (29) to the continuous-time random walk.

2. *Relation to the continuous-time random walk*

The generalized diffusion equation arises as a long space-time limit of CTRW, characterized by two PDFs, the distributions of jumps $\lambda(x)$ and waiting times $\psi(t)$. The jump distribution possesses a finite variance, and this property leads to the appearance of the second-order space derivative on the right-hand side of the generalized diffusion equation. The waiting time distributions determines the kernel $\theta(t)$ of the integrodifferential operator on the left-hand side. In Laplace space, this relation obtains a simple form, see, e.g., [131],

$$\tilde{\psi}(s) = \frac{1}{1+\tau^\alpha s\tilde{\theta}(s)}. \tag{48}$$

The waiting time PDF together with a Gaussian jump length PDF with $\hat{\lambda}(k) \sim 1 - \sigma^2 k^2$ yield the Fourier-Laplace form of the Montroll-Weiss relation [132,133], see also [15,134,135]

$$\hat{\tilde{P}}(k,s) = \frac{1-\tilde{\psi}(s)}{s} \frac{1}{1-\tilde{\psi}(s)(1-\sigma^2k^2)} \tag{49}$$

for the PDF. Rewriting Eq. (49) as Eq. (50),

$$\tilde{\theta}(s)[s\hat{\tilde{P}}(k,s) - 1] = -\frac{\sigma^2}{\tau^\alpha} k^2 \hat{\tilde{P}}(k,s), \tag{50}$$

and taking on the inverse Fourier-Laplace transform, we obtain the generalized diffusion equation

$$\int_0^t \theta(t-t')\frac{\partial}{\partial t'}P(x,t')dt' = K_\alpha \frac{\partial^2}{\partial x^2}P(x,t), \tag{51}$$

with the memory kernel $\theta(t)$, and $K_\alpha = \sigma^2/\tau^\alpha$. The initial condition is again of the form $P_0(x) = \delta(x)$, i.e., $\hat{P}_0(k) = 1$.

For the Caputo derivative, CF, and AB operators, see Eqs. (5), (8), and (10), the corresponding kernels in the GDEs have the form

$$\theta_C(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)},$$
$$\theta_{\mathrm{CF}}(t) = \frac{1}{(1-\alpha)\tau^\alpha} \exp\left(-\frac{\alpha t}{(1-\alpha)\tau}\right),$$
$$\theta_{\mathrm{AB}}(t) = \frac{1}{(1-\alpha)\tau^\alpha} E_\alpha\left(-\alpha\frac{t^\alpha}{(1-\alpha)\tau^\alpha}\right). \tag{52}$$

We recall here the waiting time PDF for the Caputo FDE. In Laplace space,

$$\tilde{\psi}_C(s) = \frac{1}{\tau^\alpha s^\alpha + 1}, \tag{53}$$

which is a completely monotonic function [131]. Then the corresponding waiting time PDF is

$$\psi_C(t) = \frac{1}{\tau^\alpha} t^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{t^\alpha}{\tau^\alpha}\right), \tag{54}$$

where $E_{\alpha,\beta}(-z) = \sum_{k=0}^{\infty}(-z)^k/\Gamma(\beta+\alpha k)$ is the two-parameter Mittag-Leffler function with expansion around infinity, $E_{\alpha,\beta}(-z) \sim -\sum_{k=1}^{\infty}(-z)^{-k}/\Gamma(\beta-\alpha k)$ [103]. In particular, when $\beta = 1$, $E_{\alpha,1}(z) = E_\alpha(z)$. We note that $\psi_C(t)$ has a weak singularity at $t = 0$, and in the long-time limit, with $-z\Gamma(-z)\Gamma(z) = \pi\csc(\pi z)$ [149], we have

$$\lim_{t\to+\infty} \psi_C(t) = \lim_{t\to+\infty} \frac{t^{\alpha-1}}{\tau^\alpha} E_{\alpha,\alpha}\left(-\frac{t^\alpha}{\tau^\alpha}\right) \sim \frac{\Gamma(\alpha+1)\sin(\pi\alpha)\tau^\alpha}{\pi} t^{-(1+\alpha)}, \tag{55}$$

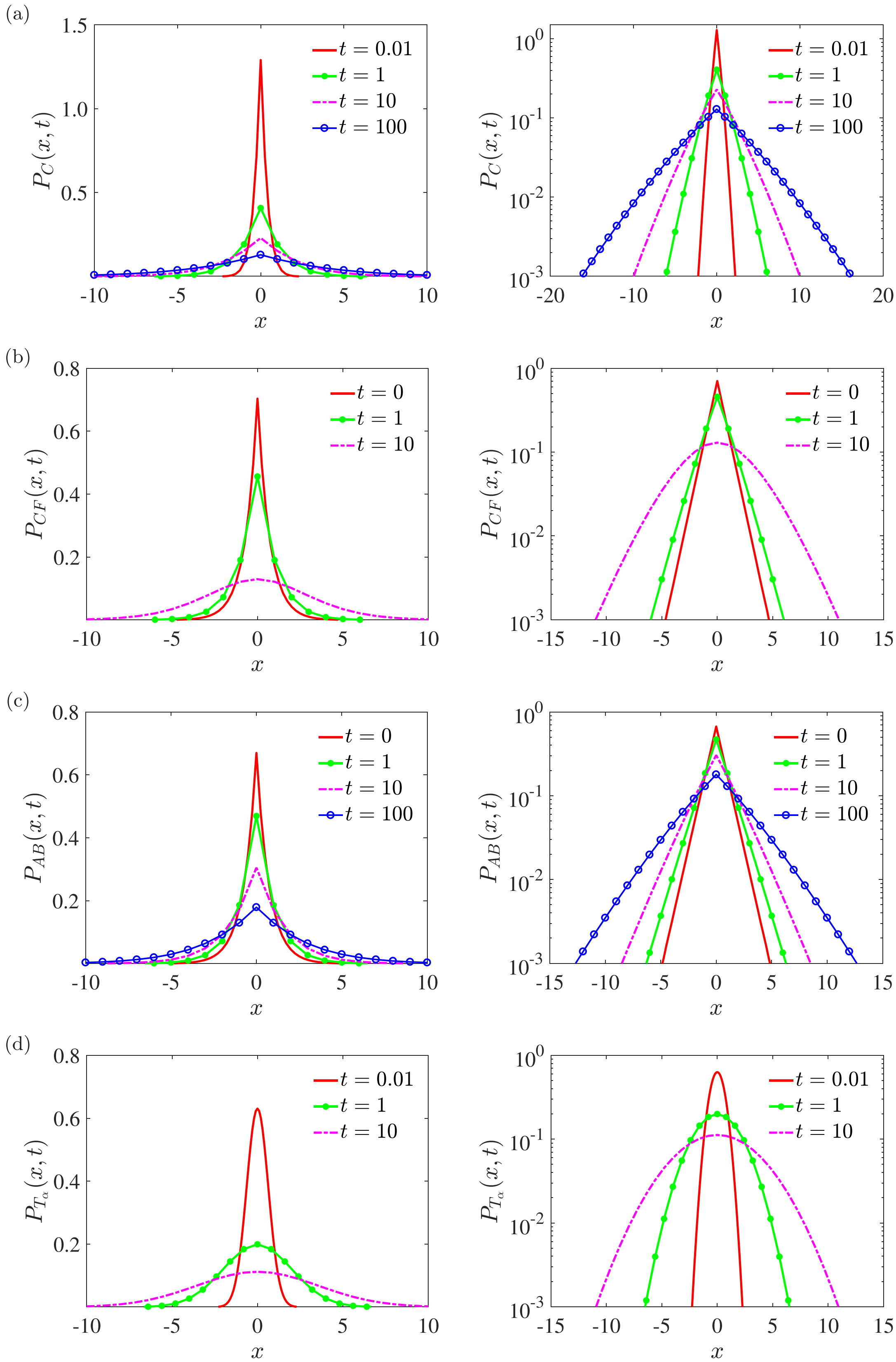

FIG. 1. PDF of diffusion equations with initial condition $P(x, 0) = \delta(x)$ with different integrodifferential operators for $K_\alpha = 1$, $\tau = 1$, and $\alpha = 0.5$: (a) Caputo-fractional, (b) Caputo-Fabrizio, (c) Atangana-Baleanu, and (d) Conformable. The solutions for the Caputo-FDE, CF-GDE, and AB-GDE are obtained by applying an inverse Laplace transform to Eq. (33). Note that both PDFs for the CF- and AB-GDE have a Laplace shape with finite width at $t = 0$. The solution (75) of the conformable diffusion equation is Gaussian at all times. The shapes are produced using MATLAB code.

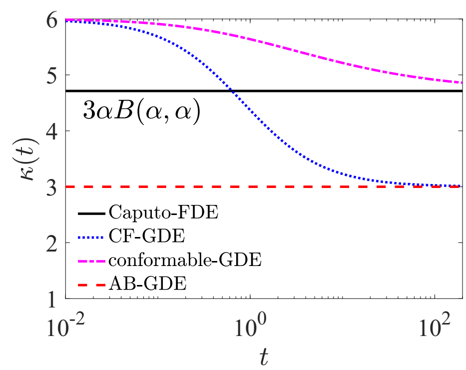

FIG. 2. Kurtosis κ for the GDEs with different integrodifferential operators for $K_\alpha = 1$, $\tau = 1$, and $\alpha = 0.5$.

that is, at long times, $\psi_C(t)$ decays as $t^{-(1+\alpha)}$. We note that $\psi_C(t)$ satisfies normalization, i.e.,

$$\begin{aligned}\int_0^\infty \psi_C(t')dt' &= \frac{1}{\tau^\alpha}\int_0^\infty (t')^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{(t')^\alpha}{\tau^\alpha}\right)dt'\\ &= \lim_{z\to+\infty}\frac{1}{\tau^\alpha}z^\alpha E_{\alpha,\alpha+1}\left(-\frac{z^\alpha}{\tau^\alpha}\right)\\ &= 1. \end{aligned} \tag{56}$$

Now we consider the waiting time PDF for CF and AB cases, first in Laplace space. According to Eqs. (48) and (52), we obtain

$$\tilde{\psi}_{\rm CF}(s) = \frac{1-\alpha}{2-\alpha} + \frac{\alpha}{2-\alpha}\frac{1}{(2-\alpha)\tau s+\alpha}, \tag{57}$$

$$\tilde{\psi}_{\rm AB}(s) = \frac{1-\alpha}{2-\alpha} + \frac{\alpha}{2-\alpha}\frac{1}{(2-\alpha)\tau^\alpha s^\alpha+\alpha}. \tag{58}$$

Notice that $\tilde{\psi}_{\rm CF}(s)$ and $\tilde{\psi}_{\rm AB}(s)$ are completely monotonic functions, therefore by applying an inverse Laplace transform we obtain functions which are proper PDFs in the time domain,

$$\psi_{\rm CF}(t) = \frac{1-\alpha}{2-\alpha}\delta(t) + \frac{\alpha}{\tau(2-\alpha)^2}\exp\left(-\frac{\alpha}{\tau(2-\alpha)}t\right), \tag{59}$$

$$\psi_{\rm AB}(t) = \frac{1-\alpha}{2-\alpha}\delta(t) + \frac{\alpha t^{\alpha-1}}{\tau^\alpha(2-\alpha)^2}E_{\alpha,\alpha}\left(-\frac{\alpha}{\tau^\alpha(2-\alpha)}t^\alpha\right). \tag{60}$$

One can easily check their normalization, $\int_0^\infty \psi_{\rm CF}(t')dt' = 1$, $\int_0^\infty \psi_{\rm AB}(t')dt' = 1$.

From Eqs. (59) and (60), we notice that the waiting time PDFs of both cases contain the term $\delta(t)$, which means that the particle jumps at the initial time $t = 0$, instead of waiting on site. This observation is in line with the property of having a nonzero MSD at $t = 0$; see Eqs. (46) and (47). It may imply that the use of CF and AB operators in anomalous dynamics requires proper initial conditions that are different from those used in standard formulations of the diffusion problem [i.e., $P(x, t = 0) = \delta(x)$] and in standard formulations of the continuous-time random-walk models, in which the particle arrives at a site at $t = 0$, then waits, and then makes a jump. Such a discussion goes beyond the scope of our paper and requires further investigation. Here we only conclude that while CF and AB integrodifferential operators may be useful for the description of other generalized dynamics, we do not consider them for the following discussion of GDEs. Below we show that these two operators also lead to inconsistent formulations of the generalized Langevin equations.

3. Alternative formulation for the CF-GDE

We digress briefly to shed some light on the delicate issue of placing specific forms of exponential tempering or other nonsingular kernels in the integrodifferential operators used above. We consider the case of the CF-operator. Let us use the Schneider-Wyss idea and start with the integral form (2) of the diffusion equation. We naively replace the integral on the right-hand side with the operator (note that an analogous choice was made in Ref. [136])

$${}_0^{CF}D_t^{-\alpha} = \frac{\tau^{\alpha-1}}{1-\alpha}\int_0^t f(t')\exp\left(-\frac{\alpha(t-t')}{(1-\alpha)\tau}\right)dt', \tag{61}$$

which is an integral operator with exponential tempering similar to the formulation of the CF-operator (8). Instead of the solution (31), for the initial condition $P_0(x) = \delta(x)$ we then obtain

$$P(k,s) = \frac{1+\alpha/[(1-\alpha)s\tau]}{\alpha/[(1-\alpha)\tau]+s+K_\alpha\tau^{\alpha-1}k^2}. \tag{62}$$

The normalization is fulfilled, as for $k = 0$ the Laplace transform reads $1/s$. The initial condition can be found from this equation by setting $s \to \infty$, which is $1/s$ to leading order. The inverse Fourier transform indeed reproduces the initial condition $P_0(x) = \delta(x)$.

The second moment is obtained by differentiation twice with respect to k and setting $k = 0$,

$$\langle x^2(t)\rangle = \frac{2K_\alpha\tau^\alpha(1-\alpha)}{\alpha}\left[1-\exp\left(-\frac{\alpha t}{(1-\alpha)\tau}\right)\right]. \tag{63}$$

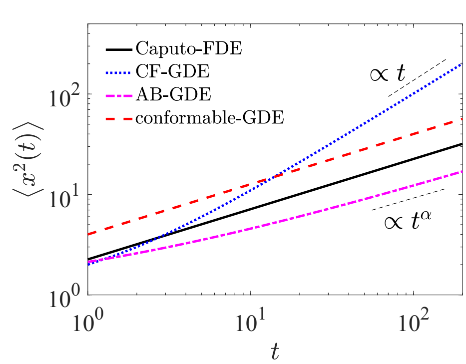

FIG. 3. MSD for the GDEs with different integrodifferential operators for $K_\alpha = 1$, $\tau = 1$, and $\alpha = 0.5$.

At short times we find normal diffusion,

$$\langle x^2(t)\rangle \sim 2K_\alpha\tau^{\alpha-1}t \tag{64}$$

with the effective diffusion coefficient $K_\alpha\tau^{\alpha-1}$, and at long times the saturation value

$$\langle x^2(t)\rangle \sim \frac{2(1-\alpha)K_\alpha\tau^\alpha}{\alpha} \tag{65}$$

is reached. This convergence to a stationary plateau is similar to what was observed previously for the tempering of FBM, which effected a behavior consistent with confinement [137]. The inverse Fourier transform of $P(k,s)$ becomes

$$P(x,s)=\frac{\sqrt{\pi\{\alpha/[(1-\alpha)\tau]+s\}/2}}{s\sqrt{K_\alpha\tau^{\alpha-1}}}\times\exp\left(-\frac{|x|\sqrt{\alpha/[(1-\alpha)\tau]+s}}{\sqrt{K_\alpha\tau^{\alpha-1}}}\right). \tag{66}$$

At $s\to\infty$,

$$P(x,s)\sim\frac{1}{s}\lim_{s\to\infty}\sqrt{\frac{\pi/2}{K_\alpha\tau^{\alpha-1}/s}}\exp\left(-\frac{|x|\sqrt{s}}{\sqrt{K_\alpha\tau^{\alpha-1}}}\right)\sim\frac{1}{s}\delta(x), \tag{67}$$

where we used the limiting form of the δ-function. Thus, indeed, this form leads back to the consistent initial value. In the opposite long-time limit corresponding to $s\to 0$, we have

$$P(x,s)\sim\frac{1}{s}\sqrt{\frac{\pi/2}{K_\alpha\tau^\alpha(1-\alpha)/\alpha}}\exp\left(-\frac{|x|\sqrt{\alpha/(1-\alpha)}}{\sqrt{K_\alpha\tau^\alpha}}\right). \tag{68}$$

We thus have the stationary limit

$$P(x)\sim\sqrt{\frac{\pi/2}{K_\alpha\tau^\alpha(1-\alpha)/\alpha}}\exp\left(-\frac{|x|\sqrt{\alpha/(1-\alpha)}}{\sqrt{K_\alpha\tau^\alpha}}\right). \tag{69}$$

This is a Laplace distribution. We showed here for a simple modification of the integral form of the diffusion equation using a CF-type exponential tempering in the integral how the time evolution of the PDF $P(x,t)$ will reach a nontrivial stationary value. Physically, this can be viewed as a consequence of introducing a finite timescale by the exponential factor in the integral. Beyond this time, the contributions of the integral to the dynamics of $P(x,t)$ become exponentially small.

4. SBM and conformable diffusion equation

SBM in the absence of an external force, $F(x)=0$ in Eq. (20), and for the standard initial condition $P(x,0)=\delta(x)$ has the Gaussian shape [106,107],

$$P(x,t)=\frac{1}{\sqrt{4\pi K_\alpha t^\alpha}}\exp\left(-\frac{x^2}{4K_\alpha t^\alpha}\right), \tag{70}$$

with the associated MSD

$$\langle x^2(t)\rangle = 2K_\alpha t^\alpha. \tag{71}$$

Concurrently, the GDE based on the conformable derivative is

$$T_\alpha P(x,t)=K_\alpha\frac{\partial^2}{\partial x^2}P(x,t). \tag{72}$$

For $P(x,0)=\delta(x)$ and after applying the conformable Laplace transform together with a Fourier transform

$$\mathscr{F}\{g(x)\}\equiv\int_{-\infty}^{\infty}g(x)\exp(ikx)dx, \tag{73}$$

we find

$$\hat{\tilde{P}}_{T_\alpha}(k,s)=\frac{1}{s+K_\alpha k^2}. \tag{74}$$

This directly leads to the PDF in (x,t) space,

$$P_{T_\alpha}(x,t)=\sqrt{\frac{\alpha}{4\pi K_\alpha t^\alpha}}\exp\left(-\frac{\alpha x^2}{4K_\alpha t^\alpha}\right). \tag{75}$$

The Gaussianity of this PDF implies that the kurtosis is $\kappa_{T_\alpha}(t)=3$. The MSD encoded by the PDF (72) reads

$$\langle x^2(t)\rangle_{T_\alpha}=\frac{2K_\alpha}{\alpha}t^\alpha. \tag{76}$$

Thus, the conformable-GDE has the same PDF, MSD, and kurtosis as subdiffusive SBM. The PDF (75) of the conformable-GDE is displayed in Fig. 1(d). The associated MSD and kurtosis are shown in Figs. 2 and 3. We will now show that the addition of a constant force (constant drift) allows one to distinguish between these two models.

B. Generalized Fokker-Planck equation with drift

From Sec. III A 1 we conclude that the CF- and AB-operators in the formulation of the GDE (18) [for $F(x)=0$] do not provide a consistent formulation. We therefore do not consider them further here. We therefore limit our discussion of the anomalous diffusion equation with drift to the Caputo, SBM, and conformable formulations.

1. Caputo diffusion equation with drift

The Caputo Fokker-Planck equation with drift reads (see also Ref. [138])

$${}_0^CD_t^\alpha[P(x,t)]=K_\alpha\frac{\partial^2}{\partial x^2}P(x,t)-v_\alpha\frac{\partial}{\partial x}P(x,t). \tag{77}$$

From the CTRW perspective, Eq. (77) appears as the diffusion (long time and space) limit of a walk with unequal probabilities to jump to the right and to the left [26,139,140]. In Laplace space the solution of Eq. (77) for the initial condition $P_0(x)=\delta(x)$ becomes

$$\tilde{P}_C(x,s)=\frac{s^{\alpha-1}}{\sqrt{v_\alpha^2+4K_\alpha s^\alpha}}\times\exp\left(\frac{v_\alpha x}{2K_\alpha}-|x|\frac{\sqrt{v_\alpha^2+4K_\alpha s^\alpha}}{2K_\alpha}\right). \tag{78}$$

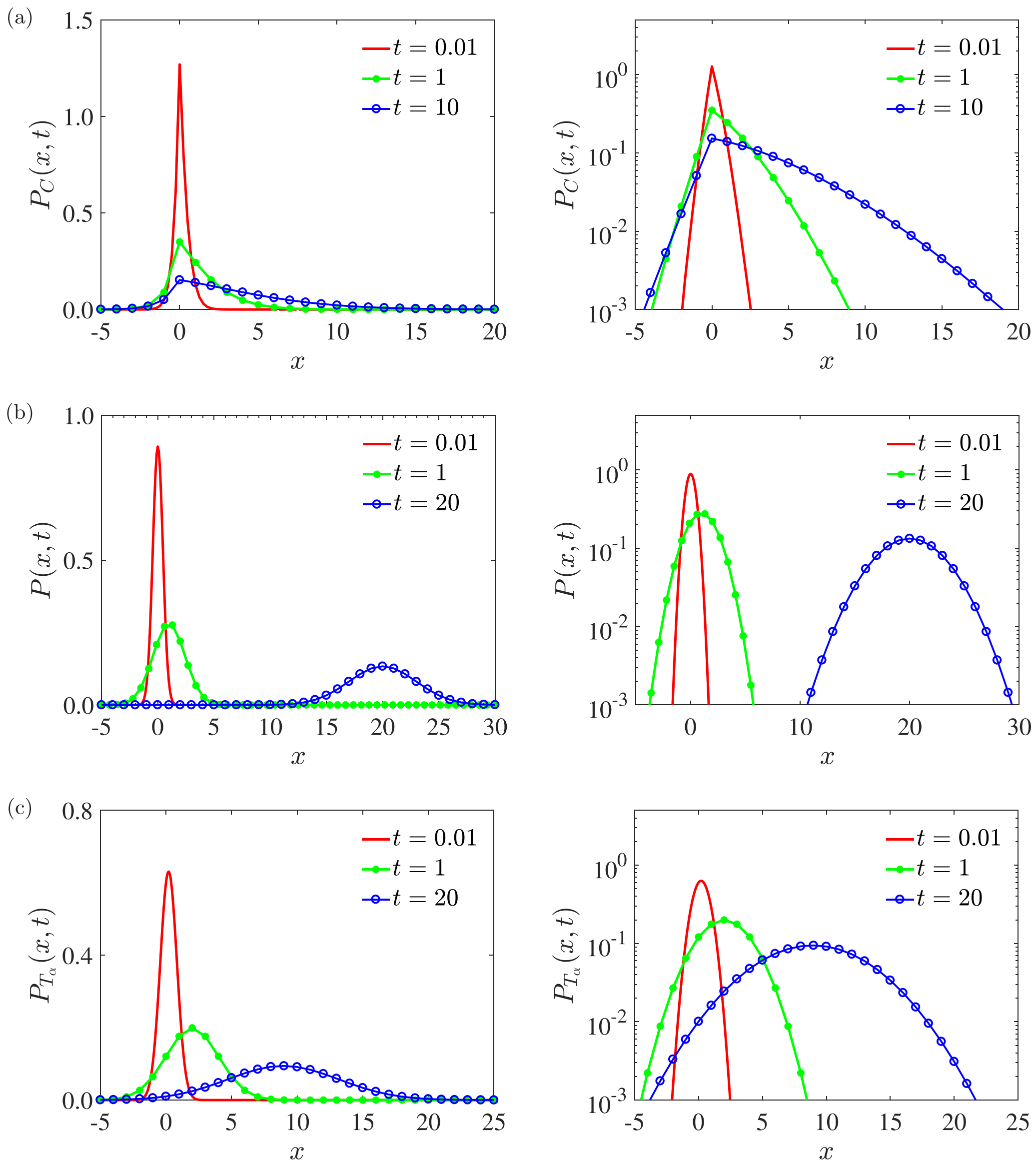

FIG. 4. PDFs for drift-GDEs with initial condition $P(x, 0) = \delta(x)$ of (a) Caputo, (b) SBM, and (c) conformable forms. We use the parameters $\alpha = 0.5$, $v = v_\alpha = 1$, and $K_\alpha = 1$. Note that the position of the maximum of the PDF for the Caputo-FDE is stationary in space, corresponding to particles that have not moved up to time t. The maximum of the PDF for SBM with drift moves faster than that for the conformable-GDE.

Applying a numerical inverse Laplace transformation to Eq. (78), we show this PDF in Fig. 4(a).

The first moment and second moment encoded by the FDE (77) are

$$\langle x(t)\rangle_C = \frac{v_\alpha}{\Gamma(\alpha+1)}t^\alpha \tag{79}$$

and

$$\langle x^2(t)\rangle_C = \frac{2K_\alpha}{\Gamma(\alpha+1)}t^\alpha + \frac{2v_\alpha^2}{\Gamma(2\alpha+1)}t^{2\alpha}. \tag{80}$$

The MSD can then be calculated as

$$\langle(\Delta x)^2\rangle_C = \frac{2K_\alpha}{\Gamma(\alpha+1)}t^\alpha + \left[\frac{2}{\Gamma(2\alpha+1)} - \frac{1}{\Gamma(\alpha+1)^2}\right]v_\alpha^2 t^{2\alpha}. \tag{81}$$

As is well known from CTRW with a drift [15,132,139], the MSD contains a term proportional to $v_\alpha^2 t^{2\alpha}$, i.e., for $1/2 < \alpha < 1$ an effective superdiffusion occurs. This is due to the strong separation of particles stuck at the origin from mobile, advected particles.

2. SBM with drift and conformable diffusion equation with drift

The Fokker-Planck equation for SBM with drift is

$$\frac{\partial}{\partial t}P(x,t) = \mathscr{K}_\alpha(t)\frac{\partial^2}{\partial x^2}P(x,t) - v\frac{\partial}{\partial x}P(x,t), \tag{82}$$

where $\mathscr{K}_\alpha$ is defined in Eq. (19), from which we obtain the PDF

$$P(x,t) = \frac{1}{\sqrt{4\pi K_\alpha t^\alpha}}\exp\left(-\frac{(x-vt)^2}{4K_\alpha t^\alpha}\right), \tag{83}$$

which is shown in Fig. 4(b). The associated moments are

$$\langle x(t)\rangle = vt, \tag{84}$$

$$\langle x^2(t)\rangle = 2K_\alpha t^\alpha + v^2t^2, \tag{85}$$

such that the MSD is

$$\langle(\Delta x)^2\rangle = 2K_\alpha t^\alpha. \tag{86}$$

In contrast to the Caputo-FDE case, for SBM the Galilean invariance is preserved, giving rise to the similarity variable $x - vt$ in the PDF (83).

The GDE with drift based on the conformable derivative has the form

$$T_\alpha P(x,t) = K_\alpha \frac{\partial^2}{\partial x^2}P(x,t) - v_\alpha \frac{\partial}{\partial x}P(x,t). \tag{87}$$

Application of the conformable Laplace transform and Fourier transform yields

$$\hat{\tilde{P}}_{T_\alpha}(k,s) = \frac{1}{s + (K_\alpha k^2 + iv_\alpha k)}. \tag{88}$$

From this form we obtain the PDF in (x,t)-space,

$$P_{T_\alpha}(x,t) = \sqrt{\frac{\alpha}{4\pi K_\alpha t^\alpha}}\exp\left(-\frac{\alpha(x - v_\alpha t^\alpha/\alpha)^2}{4K_\alpha t^\alpha}\right), \tag{89}$$

which is shown in Fig. 4(c).

The associated first moment is

$$\langle x(t)\rangle = \frac{v_\alpha t^\alpha}{\alpha}, \tag{90}$$

and the second moment reads

$$\langle x^2(t)\rangle = \frac{2K_\alpha}{\alpha}t^\alpha + \frac{v_\alpha^2}{\alpha^2}t^{2\alpha}. \tag{91}$$

The MSD is then

$$\langle(\Delta x)^2\rangle = \frac{2K_\alpha}{\alpha}t^\alpha. \tag{92}$$

The behaviors of the moments for the Caputo, SBM, and conformable forms of the generalized motion are shown in Fig. 5, from which one can distinguish SBM with drift from the Caputo and conformable drift-GDEs via the first moment: for SBM the first moment grows linearly in t, while for the other two cases a scaling with t^α occurs. From the MSD one can then distinguish the Caputo and conformable forms, respectively, scaling as $t^{2\alpha}$ and t^α in the long-time limit.

IV. GENERALIZED LANGEVIN EQUATIONS

We now turn to generalizations of the stochastic formulation of diffusive processes based on the Langevin equation; see [119,121,124,137,141] for details.

A. Generalized Langevin equations in the force-free case

In the absence of an external force, $F(x) = 0$, we consider formulations with different Caputo-, CF-, and AB-operators, SBM [107], and FBM [123,142].

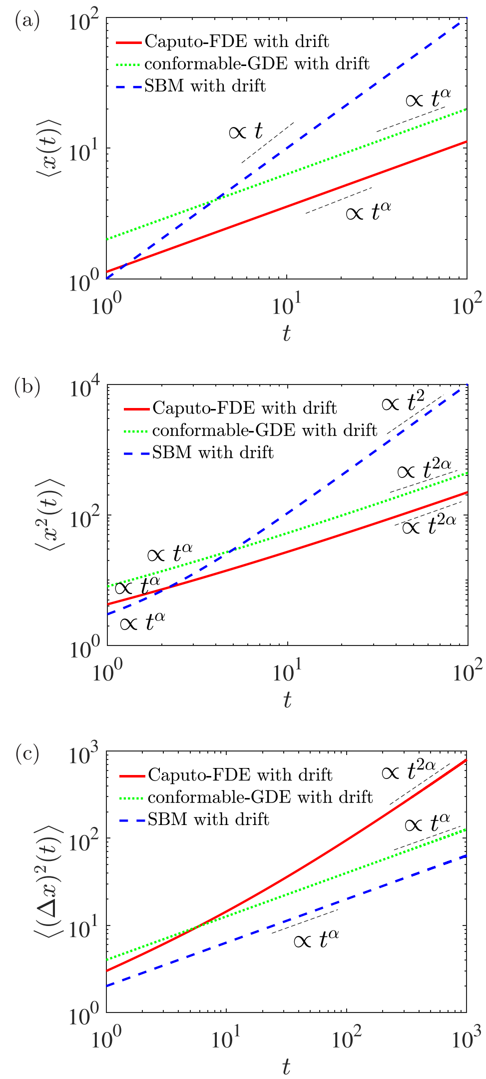

FIG. 5. (a) First and (b) second moments, and (c) MSD for Caputo, conformable, and SBM forms. We chose $\alpha = 0.5$, $v = v_\alpha = 1$, and $K_\alpha = 1$.

1. Caputo and nonsingular integrodifferential operators

Applying the Caputo, CF-, and AB-operators, we have the generalized Langevin equation,

$$\frac{d^\alpha}{dt^\alpha}x(t) = \sqrt{2K_\alpha}\xi(t), \tag{93}$$

for which we impose the initial condition $x(0) = 0$. Here d^α/dt^α represents the Caputo, CF-, and AB-operators. Formal integration produces

$$x(t) = \sqrt{2K_\alpha}\int_0^t H(t - t')\xi(t')dt', \tag{94}$$

where the integral kernel H stands for

$$H_C(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)},$$
$$H_{\rm CF}(t) = (1-\alpha)\tau^\alpha\delta(t) + \alpha\tau^{\alpha-1},$$
$$H_{\rm AB}(t) = (1-\alpha)\tau^\alpha\delta(t) + \alpha\frac{t^{\alpha-1}}{\Gamma(\alpha)} \tag{95}$$

for the Caputo-, CF-, and AB-operators, respectively.

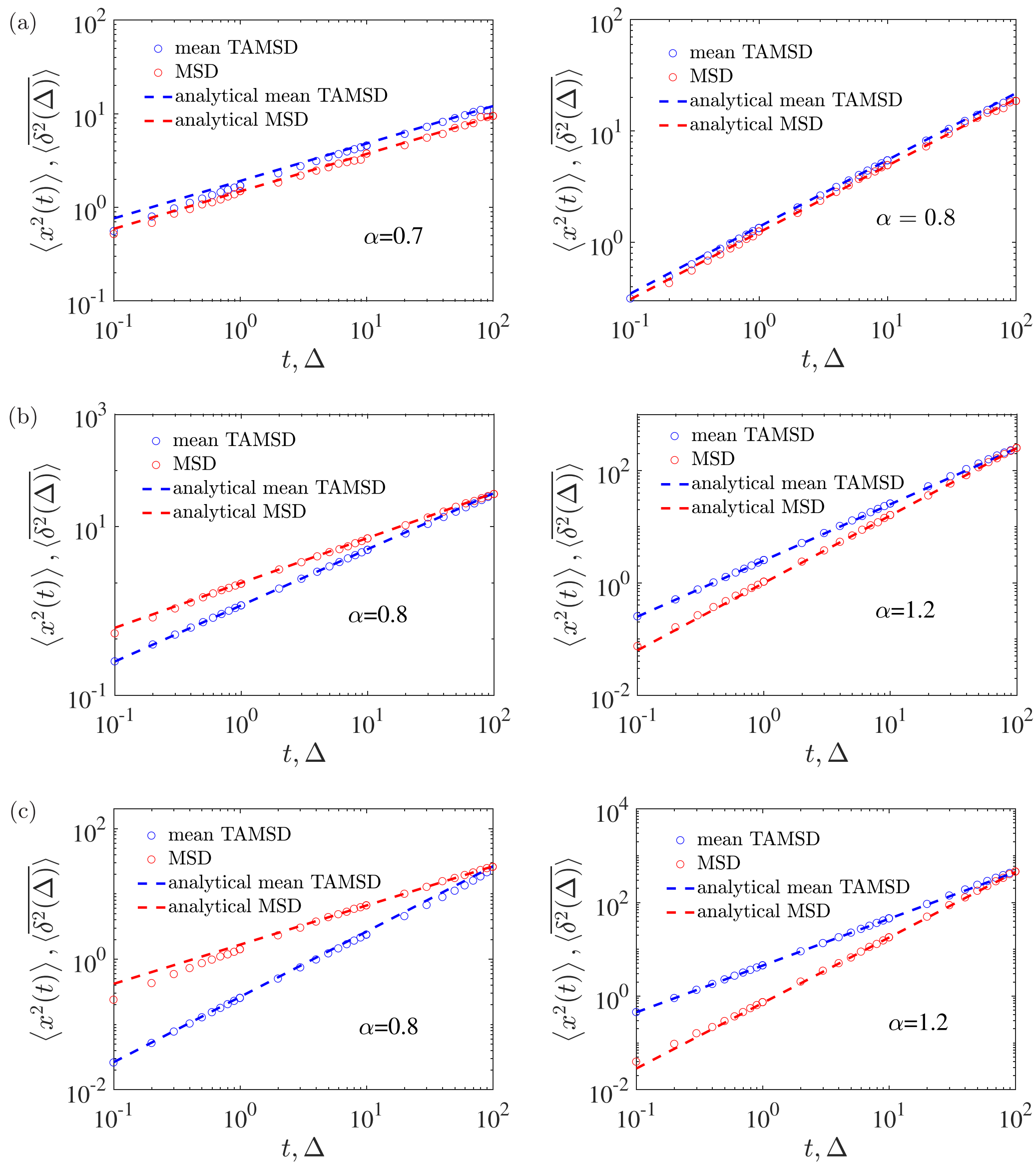

FIG. 6. Simulations and analytical solutions for MSD and mean TAMSD for the (a) Caputo-, (b) SBM-, and (c) conformable-generalized Langevin equations, for $K_\alpha = 0.5$.

For the Caputo-fractional Langevin equation, the two-point correlation function can be obtained in the form

$$\langle x(t_1)x(t_2)\rangle = \frac{2K_\alpha t_2^\alpha t_1^{\alpha-1}}{\alpha\Gamma(\alpha)^2}\,{}_2F_1\left(1-\alpha, 1; \alpha+1; \frac{t_2}{t_1}\right), \quad (96)$$

where we assume (without limitation of generality) that $t_1 \geqslant t_2$, and where

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a}dt \quad (97)$$

is the hypergeometric function [see Eq. (B23) in Appendix B 2 for details of the derivation of Eq. (96)]. From Eq. (96), the MSD follows for $t_1 = t_2 = t$, yielding

$$\langle x^2(t)\rangle = \frac{2K_\alpha}{(2\alpha-1)\Gamma(\alpha)^2}t^{2\alpha-1} \quad (98)$$

for $\alpha > 1/2$ [143]. We also obtain the mean TAMSD (see Appendix C 1 for details) in the limit $\Delta/T \ll 1$,

$$\langle\overline{\delta^2(\Delta)}\rangle \sim \frac{2K_\alpha}{\Gamma(2\alpha)|\cos(\pi\alpha)|}\Delta^{2\alpha-1}. \quad (99)$$

While the MSD and the mean TAMSD for the Caputo-Langevin equation thus have the same scaling exponent $2\alpha - 1$, the two expressions have different prefactors. That implies that the process encoded by the Caputo-Langevin equation is nonergodic in the Birkhoff-Boltzmann sense. In the notation of [144], we call such a case ultraweak ergodicity breaking. Results of stochastic simulations for the MSD and the mean TAMSD for the Caputo-Langevin equation are shown in Fig. 6(a), and we find good agreement with the theoretical results. Details on the discrete simulations scheme for the different operators are provided in Appendix D.

The fact that the Caputo-Langevin equation produces a nonstationary dynamics can be anticipated from the autocovariance function (96); see [145] for further discussion. We finally report the displacement ACVF of this process, which asymptotically reads $C_\Delta(t) \sim [2(\alpha-1)K_\alpha/\{\alpha\Gamma(\alpha)^2\}]\Delta^{\alpha-1}t^{\alpha-2}$; see Appendix C 2 for the derivation. We note that this behavior is different from the Caputo-FDE based on CTRW processes, for which the ACVF is zero beyond $t=\Delta$ [145].

For the case of the CF- and AB-Langevin equations, we focus on their second moment,

$$\langle x^2(t)\rangle = 2K_\alpha\int_0^t H(t')^2dt', \tag{100}$$

where for the kernel H the respective forms from Eq. (95) should be substituted. Noticing that $\int_0^t \delta^2(t')dt' = \infty$, this means that the second moment for both CF- and AB-formulations diverges. Details of the derivations can be found in Appendix B 2. Similar to our observations in the case of the FDE, the formulations in terms of the CF- and AB-operators lead to inconsistent results, and we will not pursue these operators further.

2. SBM- and conformable-generalized Langevin equation

The formal solution of the SBM-Langevin equation (24) when $F(x)=0$ is

$$x(t) = \sqrt{2\alpha K_\alpha}\int_0^t (t')^{\frac{\alpha-1}{2}}\xi(t')dt', \tag{101}$$

and the two-point correlation function reads

$$\langle x(t_1)x(t_2)\rangle = 2K_\alpha[\min\{t_1,t_2\}]^\alpha. \tag{102}$$

The MSD then has the power-law form [107]

$$\langle x^2(t)\rangle = 2K_\alpha t^\alpha. \tag{103}$$

The mean TAMSD grows as [107]

$$\langle\overline{\delta^2(\Delta)}\rangle = \frac{2K_\alpha[T^{\alpha+1}-\Delta^{\alpha+1}-(T-\Delta)^{\alpha+1}]}{(\alpha+1)(T-\Delta)}. \tag{104}$$

In the limit $\Delta/T \ll 1$,

$$\langle\overline{\delta^2(\Delta)}\rangle \sim 2K_\alpha\Delta T^{\alpha-1}, \tag{105}$$

which is linear in the lag time Δ. SBM is thus weakly nonergodic in the above Birkhoff-Boltzmann sense [101]. In contrast to the ultraweak situation above, here the mean TAMSD explicitly depends on the measurement time T. Simulation results for the MSD and the mean TAMSD for SBM are shown in Fig. 6(b).

The conformable Langevin equation

$$T_\alpha x(t) = \sqrt{2K_\alpha}\xi(t) \tag{106}$$

can be rephrased by using the relation $T_\alpha[f(t)] = t^{1-\alpha}df(t)/dt$ between the conformable derivative and the first-order derivative,

$$\frac{d}{dt}x(t) = \sqrt{2K_\alpha}t^{\alpha-1}\xi(t), \tag{107}$$

and thus

$$x(t) = \sqrt{2K_\alpha}\int_0^t (t')^{\alpha-1}\xi(t')dt'. \tag{108}$$

The two-point correlation function is

$$\langle x(t_1)x(t_2)\rangle = \frac{2K_\alpha}{2\alpha-1}[\min\{t_1,t_2\}]^{2\alpha-1} \tag{109}$$

for $\alpha > 1/2$. Finally, the MSD becomes

$$\langle x^2(t)\rangle = \frac{2K_\alpha}{2\alpha-1}t^{2\alpha-1}. \tag{110}$$

For the mean TAMSD we obtain

$$\langle\overline{\delta^2(\Delta)}\rangle = \frac{C_1}{2\alpha}\frac{T^{2\alpha}}{T-\Delta}\left[1-\left(\frac{\Delta}{T}\right)^{2\alpha}-\left(1-\frac{\Delta}{T}\right)^{2\alpha}\right], \tag{111}$$

where $C_1 = 2K_\alpha/[(2\alpha-1)\Gamma(\alpha)^2]$. In the limit $\Delta/T \ll 1$,

$$\langle\overline{\delta^2(\Delta)}\rangle \sim \frac{K_\alpha}{2\alpha-1}\Delta T^{2\alpha-2}. \tag{112}$$

The conformable-Langevin equation encodes a weakly nonergodic and nonstationary dynamic. Simulations results of the MSD and mean TAMSD for the conformable-Langevin equation (106) with different α are shown in Fig. 6(c). From Eqs. (102) and (109) it follows that both processes, SBM and conformable-Langevin equation motion, have independent increments, and thus the ACVFs defined in Eq. (28) vanish for these processes, $C_\Delta(t)=0$. The analysis of the MSDs (103) and (110) for the two processes shows that they have the same time-scaling if we take the exponent α for SBM equal to the exponent $2\alpha-1$ for the conformable-Langevin equation. A similar conclusion follows from Eqs. (105) and (112). Therefore, the information encoded in the MSD and mean TAMSD is insufficient to distinguish between these two processes. Analogously to the situation considered in Sec. III B for the SBM- and conformable-diffusion equations, we will show that adding a constant force (constant drift) allows us to distinguish between these two Langevin models.

3. FBM

The formal solution of FBM for $F(x)=0$ is (25)

$$x(t) = \sqrt{2K_\alpha}\int_0^t \xi_\alpha(t')dt', \tag{113}$$

with the two-point correlation

$$\langle x(t_1)x(t_2)\rangle = K_\alpha\left(t_1^\alpha + t_2^\alpha - |t_1-t_2|^\alpha\right). \tag{114}$$

Thus, the MSD has the power-law form

$$\langle x^2(t)\rangle = 2K_\alpha t^\alpha. \tag{115}$$

The mean TAMSD of FBM is [124]

$$\langle\overline{\delta^2(\Delta)}\rangle = 2K_\alpha\Delta^\alpha. \tag{116}$$

We conclude that this free FBM is ergodic and stationary. Finally, the normalized autocovariance of FBM can be represented as $C_\Delta(t)/C_\Delta(0) = [(t+\Delta)^\alpha - 2t^\alpha + |t-\Delta|^\alpha]/(2\Delta^\alpha)$ for $\Delta \neq 0$ [123]; see also the examples in [146,147] for a comparison with data.

B. Generalized Langevin equations with drift

1. Caputo-fractional Langevin equation with drift

The Caputo-fractional Langevin equation with drift reads

$$ {}_0^C D_t^\alpha x(t) = \sqrt{2K_\alpha}\xi(t) + v_\alpha. \tag{117}$$

After a Laplace transformation,

$$\tilde{x}(s) = \frac{v_\alpha}{s^{1+\alpha}} + \sqrt{2K_\alpha}\frac{\tilde{\xi}(s)}{s^\alpha}. \tag{118}$$

Back-transforming to the time domain,

$$x(t) = \frac{v_\alpha t^\alpha}{\Gamma(1+\alpha)} + \sqrt{2K_\alpha}\int_0^t \frac{(t-t')^{\alpha-1}}{\Gamma(\alpha)}\xi(t')dt'. \tag{119}$$

The first moment is then given by

$$\langle x(t)\rangle = \frac{v_\alpha t^\alpha}{\Gamma(1+\alpha)}, \tag{120}$$

which coincides with the result (79) of the Caputo-FDE. Similar to the derivation of Eq. (96), the two-point correlation function of the Caputo-fractional Langevin equation in the presence of drift is

$$\langle x(t_1)x(t_2)\rangle = \frac{v_\alpha^2}{\Gamma(1+\alpha)^2}(t_1t_2)^\alpha + \frac{2K_\alpha t_2^\alpha t_1^{\alpha-1}}{\alpha\Gamma(\alpha)^2} \times {}_2F_1\left(1-\alpha, 1, \alpha+1, \frac{t_2}{t_1}\right), \tag{121}$$

where we assumed that $t_1 < t_2$. The second moment is

$$\langle x^2(t)\rangle = \frac{v_\alpha^2}{\Gamma(1+\alpha)^2}t^{2\alpha} + \frac{2K_\alpha}{(2\alpha-1)\Gamma(\alpha)^2}t^{2\alpha-1}, \quad \alpha > \frac{1}{2} \tag{122}$$

for $\alpha > 1/2$. We finally obtain the MSD

$$\langle(\Delta x)^2(t)\rangle = \frac{2K_\alpha}{(2\alpha-1)\Gamma(\alpha)^2}t^{2\alpha-1}. \tag{123}$$

The forms of the second moment (122) and the MSD (123) are different from their counterparts (80) and (81) for the Caputo-FDE. In the Caputo-fractional Langevin equation, the drift enters additively and is not affected by the memory in the fractional operator. In contrast, for the Caputo-FDE the particles are immobilized during the waiting times represented by the fractional operator.

2. SBM- and conformable-Langevin equations with drift

The SBM-Langevin equation with drift has the form

$$\frac{d}{dt}x(t) = \sqrt{2\mathscr{K}_\alpha(t)}\xi(t) + v, \tag{124}$$

where $\mathscr{K}_\alpha$ is defined in Eq. (19). The moments are readily calculated, yielding

$$\langle x(t)\rangle = vt \tag{125}$$

and

$$\langle x^2(t)\rangle = 2K_\alpha t^\alpha + v^2t^2. \tag{126}$$

Thus the MSD is

$$\langle(\Delta x)^2\rangle = 2K_\alpha t^\alpha. \tag{127}$$

The conformable-Langevin equation with drift is

$$T_\alpha x(t) = \sqrt{2K_\alpha}\xi(t) + v_\alpha. \tag{128}$$

With the relation $T_\alpha f(t) = t^{1-\alpha}df(t)/dt$, we obtain

$$\frac{d}{dt}x(t) = t^{\alpha-1}(\sqrt{2K_\alpha}\xi(t) + v_\alpha), \tag{129}$$

and thus

$$x(t) = \int_0^t (t')^{\alpha-1}(\sqrt{2K_\alpha}\xi(t') + v_\alpha)dt' = \frac{v_\alpha}{\alpha}t^\alpha + \sqrt{2K_\alpha}\int_0^t (t')^{\alpha-1}\xi(t')dt', \tag{130}$$

and thus the first moment is equivalent to the deterministic form

$$\langle x(t)\rangle = \frac{v_\alpha}{\alpha}t^\alpha. \tag{131}$$

The two-point correlation function becomes

$$\langle x(t_1)x(t_2)\rangle = \frac{v_\alpha^2}{\alpha^2}(t_1t_2)^\alpha + \frac{2K_\alpha}{2\alpha-1}t_2^{2\alpha-1}, \tag{132}$$

valid for $\alpha > 1/2$. The second moment follows as

$$\langle x^2(t)\rangle = \frac{v_\alpha^2}{\alpha^2}t^{2\alpha} + \frac{2K_\alpha}{2\alpha-1}t^{2\alpha-1}. \tag{133}$$

Finally, the MSD has the v_α-independent form

$$\langle(\Delta x)^2(t)\rangle = \frac{2K_\alpha}{2\alpha-1}t^{2\alpha-1} \tag{134}$$

for $\alpha > 1/2$. One can see from Eqs. (125) and (131) that the response to a constant force is different for SBM-Langevin and conformable-Langevin equations, thus allowing us to distinguish between these two anomalous diffusion models.

3. FBM with drift

We finally consider the FBM Langevin equation with drift,

$$x(t) = \sqrt{2K_\alpha}\int_0^t dt'\xi_\alpha(t') + vt, \tag{135}$$

such that

$$\langle x(t)\rangle = vt. \tag{136}$$

The two-point correlation behaves as

$$\langle x(t_1)x(t_2)\rangle = K_\alpha\left(t_1^\alpha + t_2^\alpha - |t_1-t_2|^\alpha\right) + v^2t_1t_2. \tag{137}$$

The second moment encoded in this form is

$$\langle x^2(t)\rangle = 2K_\alpha t^\alpha + v^2t^2. \tag{138}$$

Finally, the MSD reads

$$\langle(\Delta x)^2(t)\rangle = 2K_\alpha t^\alpha. \tag{139}$$

The moments for the different processes are shown in Fig. 7. From the above discussion we see that the first moments of the Caputo-fractional and conformable-Langevin equations with drift have a power-law form, which is distinct from the linear time dependence in the SBM- and FBM-Langevin equations.

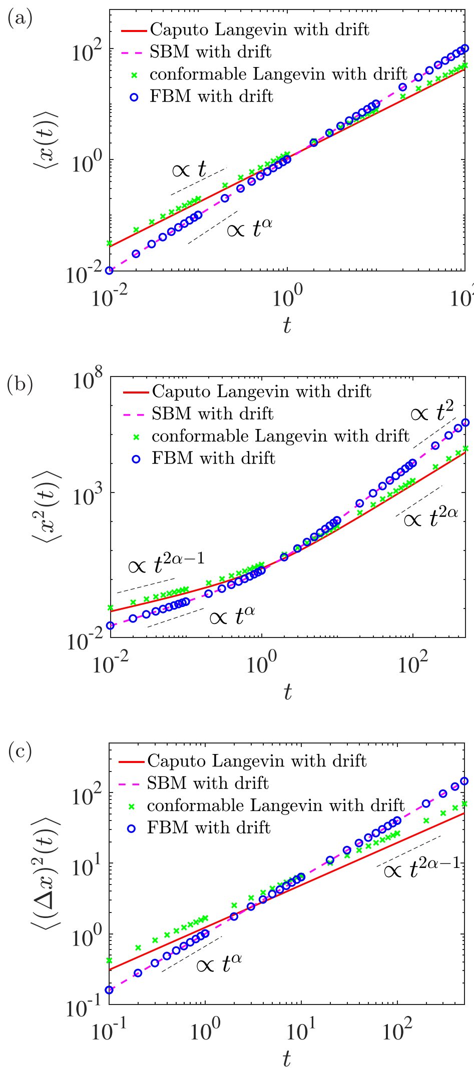

FIG. 7. (a) First and (b) second moments, and (c) MSD for Caputo-fractional, conformable-, SBM-, and FBM-Langevin equations with drift for $v = v_\alpha = 1$, $\alpha = 0.8$, and $K_\alpha = 0.5$. Note that the results of SBM and FBM fully coincide in all three panels.

V. CONCLUSIONS

Fractional dynamic equations of the relaxation and diffusion types have been used in science and engineering for a considerable amount of time. Traditionally, these generalized equations were used in the Riemann-Liouville and Caputo types. These are particularly suitable for the formulation of initial value problems posed at $t = 0$. Other formulations such as the Weyl-Riesz forms have been used as well, e.g., in the context of generalized rheological models for harmonic driving [9,148]. More recently, additional definitions of fractional and conformable operators have been proposed and discussed in the literature. We studied here generalized diffusion and Langevin equations, comparing the classical Caputo-fractional forms with the CF-, AB-, and conformable-generalized differential operators. We also compare these results to two other anomalous diffusion processes, SBM and FBM.

In our analysis, we find that the formulations in terms of the CF- and AB-operators lead to inconsistent results for the PDFs and MSDs in both the GDE and generalized Langevin equation cases. While this point requires further analysis from a more mathematical point of view, here we did not pursue the formulations in terms of these operators further. A possible solution for the incorrect incorporation of the initial values for these two operators in the traditional formulation (note that the integral formulation as outlined in the Introduction for the CF- and AB-cases produces the same results) may be that instead of an initial condition at $t = 0$, the initial condition has to be formulated on an interval. We discussed an alternative formulation similar to the CF-GDE in which the initial condition is consistently incorporated. The latter formulation will deserve further analysis in the future.

Results for the moments, the MSD, and the PDF of the different formulations using Caputo-fractional, conformable-, SBM-, and FBM-dynamic equations are summarized in Tables I and II. Generally we see that the first moments in the presence of drift are identical for both GDE and Langevin formulations for each of the Caputo-, conformable-, and SBM-models, while their higher-order moments are different

TABLE I. Central results for the displacement PDFs and moments of the generalized Fokker-Planck equations with Caputo-fractional and conformable derivatives and SBM for the cases without and with drift.

Fokker-Planck Eq.		Caputo ($0<\alpha\leqslant 1$)	Conformable ($0<\alpha\leqslant 1$)	SBM ($\alpha>0$)
$F(x)=0$	$P(x,t)$	$\frac{1}{2\sqrt{K_\alpha t^\alpha}}M_{\frac{\alpha}{2}}\left(\frac{\lvert x\rvert}{\sqrt{K_\alpha t^\alpha}}\right)$	$\sqrt{\frac{\alpha}{4\pi K_\alpha t^\alpha}}\exp\left(-\frac{\alpha x^2}{4K_\alpha t^\alpha}\right)$	$\frac{1}{\sqrt{4\pi K_\alpha t^\alpha}}\exp\left(-\frac{x^2}{4K_\alpha t^\alpha}\right)$
	$\langle x^2(t)\rangle$	$\frac{2K_\alpha}{\Gamma(\alpha+1)}t^\alpha$	$\frac{2K_\alpha}{\alpha}t^\alpha$	$2K_\alpha t^\alpha$
$F(x)=v$	$P(x,t)$	$\mathscr{L}^{-1}[\tilde{P}_C(x,s)]$[a]	$\sqrt{\frac{\alpha}{4\pi K_\alpha t^\alpha}}\exp\left(-\frac{\alpha(x-\frac{v_\alpha t^\alpha}{\alpha})^2}{4K_\alpha t^\alpha}\right)$	$\frac{1}{\sqrt{4\pi K_\alpha t^\alpha}}\exp\left(-\frac{(x-vt)^2}{4K_\alpha t^\alpha}\right)$
	$\langle x(t)\rangle$	$\frac{v_\alpha}{\Gamma(\alpha+1)}t^\alpha$	$v_\alpha\frac{t^\alpha}{\alpha}$	vt
	$\langle x^2(t)\rangle$	$\frac{2K_\alpha}{\Gamma(\alpha+1)}t^\alpha+\frac{2v_\alpha^2}{\Gamma(2\alpha+1)}t^{2\alpha}$	$\frac{2K_\alpha}{\alpha}t^\alpha+\frac{v_\alpha^2}{\alpha^2}t^{2\alpha}$	$2K_\alpha t^\alpha+v^2t^2$
	$\langle(\Delta x)^2(t)\rangle$	$\frac{2K_\alpha}{\Gamma(\alpha+1)}t^\alpha+at^{2\alpha}$[b]	$\frac{2K_\alpha}{\Gamma(\alpha+1)}t^\alpha$	$2K_\alpha t^\alpha$

[a] $\tilde{P}_C(x,s)=\frac{s^{\alpha-1}}{\sqrt{v_\alpha^2+4K_\alpha s^\alpha}}\exp\left(\frac{v_\alpha x}{2K_\alpha}-\lvert x\rvert\frac{\sqrt{v_\alpha^2+4K_\alpha s^\alpha}}{2K_\alpha}\right)$.
[b] $a=\left(\frac{2}{\Gamma(2\alpha+1)}-\frac{1}{\Gamma(\alpha+1)^2}\right)v_\alpha^2$.

TABLE II. Central results for the moments, mean TAMSD, and autocovariance function of the generalized Langevin equations with Caputo-fractional and conformable derivatives, SBM and FBM, without and with drift.

Langevin Eq.		Caputo ($\frac{1}{2} < \alpha \leqslant 1$)	Conformable ($\frac{1}{2} < \alpha \leqslant 1$)	SBM ($\alpha > 0$)	FBM ($0 < \alpha < 2$)
$F(x) = 0$	$\langle x^2(t)\rangle$	$\frac{2K_\alpha}{(2\alpha-1)\Gamma(\alpha)^2}t^{2\alpha-1}$	$\frac{2K_\alpha}{2\alpha-1}t^{2\alpha-1}$	$2K_\alpha t^\alpha$	$2K_\alpha t^\alpha$
	$\langle\overline{\delta^2(\Delta)}\rangle$	$\sim\frac{2K_\alpha}{\Gamma(2\alpha)\lvert\cos(\pi\alpha)\rvert}\Delta^{2\alpha-1}$	$\sim\frac{K_\alpha}{2\alpha-1}\Delta T^{2\alpha-2}$	$\sim 2K_\alpha\Delta T^{\alpha-1}$	$2K_\alpha\Delta^\alpha$
	$C_\Delta(t)$, $t \gg \Delta$	$\simeq(\alpha-1)K_\alpha\Delta^{\alpha-1}t^{\alpha-2}$	0	0	$\simeq(\alpha-1)\Delta^{\alpha-2}$
$F(x) = v, v_\alpha$	$\langle x(t)\rangle$	$v_\alpha\frac{t^\alpha}{\Gamma(1+\alpha)}$	$v_\alpha\frac{t^\alpha}{\alpha}$	vt	vt
	$\langle x^2(t)\rangle$	$\frac{v_\alpha^2}{\Gamma(1+\alpha)^2}t^{2\alpha}+\frac{2K_\alpha}{(2\alpha-1)\Gamma(\alpha)^2}t^{2\alpha-1}$	$\frac{v_\alpha^2}{\alpha^2}t^{2\alpha}+\frac{2K_\alpha}{2\alpha-1}t^{2\alpha-1}$	$2K_\alpha t^\alpha+v^2t^2$	$2K_\alpha t^\alpha+v^2t^2$
	$\langle(\Delta x)^2(t)\rangle$	$\frac{2K_\alpha}{(2\alpha-1)\Gamma(\alpha)^2}t^{2\alpha-1}$	$\frac{2K_\alpha}{2\alpha-1}t^{2\alpha-1}$	$2K_\alpha t^\alpha$	$2K_\alpha t^\alpha$

between GDE and Langevin descriptions for the Caputo and conformable cases. The Caputo-, conformable-, and SBM-cases exhibit nonstationarity and nonergodic behavior. The PDFs are Gaussian in all cases apart from the Caputo-FDE. We also see that by combining moments in the presence and absence of a constant drift velocity, the three models can be distinguished.

Of particular interest here is the formulation in terms of the conformable derivative. The resulting PDF turns out to be the same as the PDF for SBM in the force-free case, after a renormalization of the generalized diffusion coefficient. In the presence of a drift, both processes differ in the scaling of the first moment and in the form the drift enters the PDF. Despite its "local" definition, the conformable-GDE is weakly nonergodic and shows aging properties. These effects are visible in the comparison of the MSD with the mean TAMSD as well as in the two-point correlation function in the conformable-Langevin equation case.

We also note that we chose to present our analysis in dimensional units. This requires the use of generalized diffusion coefficients and drift velocities. Including dimensionality allows for an explicit extraction of the parameters from measurements. It also demonstrates the different ways (local or with a generalized exponent) the drift enters the different model dynamics.

Our study should help to assess and compare different formulations of GDEs and generalized Langevin equations. A similar analysis should be performed for the case of a harmonic confinement of the test particle. Moreover, different forms of crossovers to normal dynamics should be studied.

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APPENDIX A: INTEGRAL VERSIONS OF CF- AND AB-OPERATORS

To find the inverse operator of Eq. (8), the CF-integral, we take $0 < \alpha \leqslant 1$ and consider the equation

$$ {}^{\rm CF}_{0}D^{\alpha}_{t}f(t) = u(t). \tag{A1}$$

After a Laplace transform, we obtain

$$\mathscr{L}\{f(t)\}(s) = \frac{1}{s}f(0) + \frac{\alpha\tau^{\alpha-1}}{sM(\alpha)}\mathscr{L}\{u(t)\}(s) + \frac{(1-\alpha)\tau^{\alpha}}{M(\alpha)}\mathscr{L}\{u(t)\}(s). \tag{A2}$$

Rearranging and after inverse Laplace transformation, we deduce that

$$f(t) = \frac{(1-\alpha)\tau^{\alpha}}{M(\alpha)}u(t) + \frac{\alpha\tau^{\alpha-1}}{M(\alpha)}\int_0^t u(s)ds + f(0). \tag{A3}$$

Thus, the the CF-integral is defined as

$${}^{\rm CF}_{0}I^{\alpha}_{t}u(t) = \frac{(1-\alpha)\tau^{\alpha}}{M(\alpha)}u(t) + \frac{\alpha\tau^{\alpha-1}}{M(\alpha)}\int_0^t u(s)ds. \tag{A4}$$

Similarly, the modified AB-integral corresponding to the operator (10) is

$${}^{\rm AB}_{0}{\rm I}^{\alpha}_{t}f(t) = \frac{1-\alpha}{B(\alpha)}\tau^{\alpha}f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t f(t')(t-t')^{\alpha-1}dt' \tag{A5}$$

for $0 < \alpha \leqslant 1$.

APPENDIX B: INTEGRODIFFERENTIAL OPERATORS IN DIFFUSION AND LANGEVIN EQUATIONS

1. Integrodifferential operators in diffusion equations

For the Caputo-FDE, the PDF (37) can also be represented in terms of the Fox H-function [22],

$$P_C(x,t) = \frac{1}{\sqrt{4K_\alpha t^\alpha}}H^{1,0}_{1,1}\left[\frac{|x|}{\sqrt{K_\alpha t^\alpha}}\middle|\begin{matrix}(1-\alpha/2,\alpha/2)\\(0,1)\end{matrix}\right]. \tag{B1}$$

For the PDFs of the CF- and AB-GDEs, we check whether their PDFs in Laplace space are completely monotonic [130]. To this end we first check the complete monotonicity of expressions (33), (35), and (36). First, we introduce the completely monotone functions (CMFs) and Bernstein functions (BF), as well as some useful properties of these two types of functions. CMFs can be represented as Laplace transforms of a non-negative function $p(t)$, i.e., $m(x) = \int_0^\infty p(t)\exp(-xt)dt$. They are defined on the non-negative half-axis and have the property that $(-1)^n m^{(n)}(x) \geqslant 0$ for all $n \in \mathbb{N}_0$ and $x \geqslant 0$. The following property holds true for CMFs [130]:

(i) The product $m(x) = m_1(x)m_2(x)$ of two CMFs $m_1(x)$ and $m_2(x)$ is again a CMF.

The Bernstein functions [130] are non-negative functions, whose derivative is completely monotone. They have the property that $(-1)^{(n-1)}b^{(n)}(x) \geqslant 0$ for all $n \in \mathbb{N}$. The Bernstein functions have the following two properties:

(ii) A composition $b_1(b_2(s))$ of Bernstein functions is again a Bernstein function.

(iii) A composition $m(b(x))$ of a CMF $m(x)$ and a Bernstein function $b(x)$ is a CMF.

From these properties it follows that the function $\exp(-ub(x))$ is completely monotone for $u > 0$ if $b(x)$ is a Bernstein function.

Now let $f_1(s) = 1/\sqrt{s(c_1 s + c_2)} = 1/[\sqrt{s}\sqrt{c_1 s + c_2}]$. As $1/\sqrt{s}$ is a CMF, $1/\sqrt{c_1 s + c_2}$ is a CMF, so from property (i) $f_1(s)$ is also a CMF. Let $f_2(s) = \exp(-\sqrt{s/(c_1 s + c_2)}|x|)$. As $\sqrt{s}$ is a BF and $s/(c_1 s + c_2)$ is a BF, then from property (ii), $\sqrt{s/(c_1 s + c_2)}$ is a BF. As $\exp(-s)$ is a CMF, then from property (iii), $f_2(s)$ is a CMF. Consequently, $\tilde{P}_{\rm CF}(x,s) = f_1(s)f_2(s)$ is a CMF. Similarly, $\tilde{P}_{\rm AB}(x,s)$ is a CMF. From the above we can ensure that the PDFs of the CF- and AB-GDEs obtained from Eqs. (33), (35), and (36) represent proper PDFs.

Now we focus on the calculation of the second moment from the GDE (29),

$$\langle x^2(t)\rangle = \int_{-\infty}^{\infty} x^2 P(x,t)dx, \tag{B2}$$

with initial value $\langle x^2(0)\rangle = \int_{-\infty}^{\infty} x^2 P(x,0)dx = 0$ for $P(x,0) = \delta(x)$. It then follows that

$$\frac{d^\alpha}{dt^\alpha}\langle x^2(t)\rangle = K_\alpha \int_{-\infty}^{\infty} x^2\left[\frac{\partial^2}{\partial x^2}P(x,t)\right]dx = 2K_\alpha. \tag{B3}$$

Applying a Laplace transformation to Eq. (B3),

$$\mathscr{L}\left[\frac{d^\alpha}{dt^\alpha}\langle x^2(t)\rangle\right] = 2K_\alpha\frac{1}{s}. \tag{B4}$$

For the Caputo-derivative,

$$s^\alpha\langle \tilde{x}^2(s)\rangle_C = 2K_\alpha\frac{1}{s}, \tag{B5}$$

and the second moment becomes

$$\langle x^2(t)\rangle = \frac{2K_\alpha}{\Gamma(\alpha+1)}t^\alpha, \tag{B6}$$

which is a familiar result [15].

For the CF-GDE,

$$\frac{s}{(1-\alpha)s+\alpha}\langle \tilde{x}^2(s)\rangle_{\rm CF} = 2K_\alpha\frac{1}{s}, \tag{B7}$$

such that

$$\langle \tilde{x}^2(s)\rangle = 2K_\alpha(1-\alpha)\frac{1}{s} + 2K_\alpha\alpha\frac{1}{s^2}, \tag{B8}$$

and then

$$\langle x^2(t)\rangle_{\rm CF} = 2K_\alpha(1-\alpha) + 2K_\alpha\alpha t. \tag{B9}$$

For the AB-GDE,

$$\frac{s^\alpha}{(1-\alpha)s^\alpha+\alpha}\langle \tilde{x}^2(s)\rangle = 2K_\alpha\frac{1}{s}, \tag{B10}$$

such that

$$\langle \tilde{x}^2(s)\rangle = 2K_\alpha(1-\alpha)\frac{1}{s} + 2K_\alpha\alpha\frac{1}{s^{\alpha+1}}, \tag{B11}$$

and then

$$\langle x^2(t)\rangle_{AB} = 2K_\alpha(1-\alpha) + 2K_\alpha\frac{t^\alpha}{\Gamma(\alpha)}. \tag{B12}$$

For the kurtosis, we calculate the third- and fourth-order moments,

$$\langle x^3(s)\rangle = -i\frac{\partial^3}{\partial k^3}\hat{\tilde{P}}(k,s)\bigg|_{k=0}, \tag{B13}$$

$$\langle x^4(s)\rangle = \frac{\partial^4}{\partial k^4}\hat{\tilde{P}}(k,s)\bigg|_{k=0}. \tag{B14}$$

From these we find the kurtosis

$$\kappa(t) = \left\langle\left(\frac{x-\langle x\rangle}{\langle(x-\langle x\rangle)^2\rangle^{1/2}}\right)^4\right\rangle. \tag{B15}$$

The kurtosis for the CF- and AB-GDEs is

$$\kappa_{\rm CF} = 6\left[\frac{(1-\alpha)+\alpha\frac{t}{\sqrt{2}}}{(1-\alpha)+\alpha t}\right]^2 \tag{B16}$$

and

$$\kappa_{\rm AB} = 6\frac{\frac{\alpha t^{2\alpha}}{2\Gamma(2\alpha)} + 2(1-\alpha)\frac{t^\alpha}{\Gamma(\alpha)} + (1-\alpha)^2}{\left[\frac{t^\alpha}{\Gamma(\alpha)} + (1-\alpha)\right]^2}. \tag{B17}$$

2. Integrodifferential operators in the Langevin equation

The generalized Langevin equation with integrodifferential operators is

$$\frac{d^\alpha}{dt^\alpha}x(t) = \sqrt{2K_\alpha}\xi(t), \tag{B18}$$

where $0 < \alpha \leqslant 1$ and d^α/dt^α represents the Caputo-, CF-, and AB-operators. Applying a Laplace transformation,

$$\tilde{x}(s) = \sqrt{2K_\alpha}\frac{1}{s\tilde{\theta}(s)}\tilde{\xi}(s), \tag{B19}$$

where $\theta_C(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$, $\theta_{\rm CF}(t) = \frac{1}{(1-\alpha)\tau^\alpha}\exp(-\frac{\alpha t}{(1-\alpha)\tau})$, $\theta_{\rm AB}(t) = \frac{1}{(1-\alpha)\tau^\alpha}E_\alpha(-\alpha\frac{t^\alpha}{(1-\alpha)\tau^\alpha})$. After inverse Laplace transformation, we obtain

$$x(t) = \sqrt{2K_\alpha}\int_0^t H(t-t')\xi(t')dt'. \tag{B20}$$

The MSD is then

$$\langle x^2(t)\rangle = 2K_\alpha \int_0^t H(t')^2 dt'. \tag{B21}$$

Here, $H_{\rm CF}(t) = (1-\alpha)\tau^\alpha\delta(t) + \alpha\tau^{\alpha-1}$ and $H_{\rm AB}(t) = (1-\alpha)\tau^\alpha\delta(t) + \alpha\frac{t^{\alpha-1}}{\Gamma(\alpha)}$.

For the Caputo derivative,

$$x(t) = \sqrt{2K_\alpha}\int_0^t \frac{(t-t')^{\alpha-1}}{\Gamma(\alpha)}\xi(t')dt'. \tag{B22}$$

The two-point correlation function for the Caputo-fractional Langevin equation is

$$\begin{aligned}\langle x(t_1)x(t_2)\rangle &= 2K_\alpha\int_0^{t_1}\frac{(t_1-t_1')^{\alpha-1}}{\Gamma(\alpha)}dt_1' \\ &\quad\times\int_0^{t_2}\frac{(t_2-t_2')^{\alpha-1}}{\Gamma(\alpha)}dt_2'\langle\xi(t_1')\xi(t_2')\rangle \\ &= \frac{2K_\alpha}{\Gamma(\alpha)^2}\int_0^{t_1}(t_1-t_1')^{\alpha-1}dt_1' \\ &\quad\times\int_0^{t_2}(t_2-t_2')^{\alpha-1}dt_2'\delta(t_1'-t_2') \\ &= \frac{2K_\alpha}{\Gamma(\alpha)^2}\int_0^{t_2}(t_1-t_2')^{\alpha-1}(t_2-t_2')^{\alpha-1}dt_2' \\ &= \frac{2K_\alpha}{\Gamma(\alpha)^2}t_2^\alpha t_1^{\alpha-1}\int_0^1\left(1-\frac{t_2}{t_1}x\right)^{\alpha-1}(1-x)^{\alpha-1}dx \\ &= \frac{2K_\alpha t_2^\alpha t_1^{\alpha-1}}{\alpha\Gamma(\alpha)^2}\,{}_2F_1\left(1-\alpha,1;\alpha+1;\frac{t_2}{t_1}\right).\end{aligned} \tag{B23}$$

Without restricting generality, we assume here that $t_1 \geqslant t_2$, and

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1\frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a}dt \tag{B24}$$

is the hypergeometric function, which for $|z|<1$ is defined by the power series

$${}_2F_1(a,b;c;z) = \sum_{n=0}^\infty\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}. \tag{B25}$$

Here $(q)_n$ is the (rising) Pochhammer symbol

$$(q)_n = \begin{cases}1, & n=0,\\ q(q+1)\cdots(q+n-1), & n>0.\end{cases}$$

Then the MSD is

$$\langle x^2(t)\rangle = C_1 t^{2\alpha-1} \tag{B26}$$

for $\alpha > 1/2$, and where $C_1 = 2K_\alpha/[(2\alpha-1)\Gamma^2(\alpha)]$.

APPENDIX C: TAMSD AND ACVF FOR THE CAPUTO LANGEVIN EQUATION

1. TAMSD

According to the definition (26) of the TAMSD, the mean TAMSD of the Caputo-fractional Langevin equation can be derived in the form

$$\begin{aligned}\langle\overline{\delta^2(\Delta)}\rangle &= \left\langle\frac{1}{T-\Delta}\int_0^{T-\Delta}[x(t+\Delta)-x(t)]^2dt\right\rangle \\ &= \frac{1}{T-\Delta}\int_0^{T-\Delta}[\langle x^2(t+\Delta)\rangle + \langle x^2(t)\rangle - 2I_1]dt \\ &= \frac{C_1}{2\alpha}\frac{1}{T-\Delta}[T^{2\alpha}-\Delta^{2\alpha}+(T-\Delta)^{2\alpha}] - 2I,\end{aligned} \tag{C1}$$

where $C_1 = 2K_\alpha/[(2\alpha-1)\Gamma(\alpha)^2]$ and $I = (T-\Delta)^{-1}\int_0^{T-\Delta}I_1dt$. In this latter expression, we used

$$\begin{aligned}I_1 &= \langle x(t+\Delta)x(t)\rangle \\ &= C_2(t+\Delta)^{\alpha-1}t^\alpha 2F_1\left(1-\alpha,1;1+\alpha;\frac{t}{t+\Delta}\right),\end{aligned} \tag{C2}$$

with $C_2 = 2K_\alpha/[\alpha\Gamma(\alpha)^2]$. We note that in particular when $\Delta=0$, we get $\langle\overline{\delta^2(0)}\rangle = 0$.

Now we focus on the case when $\Delta\neq 0$, and, more specifically, on the limit $\Delta/T \ll 1$. We first calculate I in Eq. (C1), using Eq. (15.3.4) in Ref. [149] for I_1 in Eq. (C2). Then

$$I_1 = C_2\Delta^{\alpha-1}t^\alpha\,{}_2F_1\left(1-\alpha,\alpha;\alpha+1;-\frac{t}{\Delta}\right). \tag{C3}$$

Using the relation between the H-function and the hypergeometric functions, Eq. (1.131) in Ref. [22], we find

$$\begin{aligned}&{}_2F_1\left(1-\alpha,\alpha;\alpha+1;-\frac{t}{\Delta}\right) \\ &\quad= \frac{\alpha}{\Gamma(1-\alpha)}H_{2,2}^{1,2}\left[\frac{t}{\Delta}\middle|\begin{matrix}(\alpha,1),(1-\alpha,1)\\(0,1),(-\alpha,1)\end{matrix}\right].\end{aligned} \tag{C4}$$

Applying relation 1.16.4 in Ref. [150], we obtain

$$\begin{aligned}I &= \frac{C\Delta^{2\alpha}}{(T-\Delta)}\int_0^{T^*}s^\alpha H_{2,2}^{1,2}\left[s\middle|\begin{matrix}(\alpha,1),(1-\alpha,1)\\(0,1),(-\alpha,1)\end{matrix}\right]ds \\ &= \frac{C(T^*)^\alpha}{\Delta^{1-2\alpha}} \\ &\quad\times H_{3,3}^{1,3}\left[T^*\middle|\begin{matrix}(-\alpha,1),(\alpha,1),(1-\alpha,0)\\(0,1),(-\alpha,1),(-\alpha-1,1)\end{matrix}\right],\end{aligned} \tag{C5}$$

where $T^* = (T-\Delta)/\Delta$ and $C = \alpha C_2/\Gamma(1-\alpha)$. Using relation 8.3.2.7 in Ref. [150], we then obtain

$$I = \frac{C(T^*)^\alpha}{\Delta^{1-2\alpha}}H_{3,3}^{3,1}\left[\frac{1}{T^*}\middle|\begin{matrix}(1,1),(1+\alpha,1),(2+\alpha,1)\\(1+\alpha,1),(1-\alpha,1),(\alpha,1)\end{matrix}\right]. \tag{C6}$$

With relation 8.3.2.3 in Ref. [150], in the limit $\Delta/T \ll 1$ we get

$$\begin{aligned}I &\sim \frac{K_\alpha}{\alpha(2\alpha-1)\Gamma(\alpha)^2}(T-\Delta)^{2\alpha-1} \\ &\quad+\frac{K_\alpha}{\Gamma(2\alpha)\cos(\pi\alpha)}\Delta^{2\alpha-1} \\ &\quad+\frac{K_\alpha}{(2\alpha-1)\Gamma(\alpha)^2}\Delta(T-\Delta)^{2\alpha-2}.\end{aligned} \tag{C7}$$

We then obtain the leading behavior of the mean TAMSD of the Caputo-fractional Langevin equation (C1),

$$\langle\overline{\delta^2(\Delta)}\rangle \sim \frac{2K_\alpha}{\Gamma(2\alpha)|\cos(\pi\alpha)|}\Delta^{2\alpha-1}. \tag{C8}$$

2. ACVF

From the two-point correlation function (96) for the Caputo-Langevin equation, here we calculate the ACVF. The general result with $t > 0$ is

$$C_\Delta(t) = \frac{2K_\alpha}{\alpha\Gamma(\alpha)^2\Delta^2} \times \left[\Delta^\alpha(t+\Delta)^{\alpha-1}{}_2F_1\left(1-\alpha, 1, \alpha+1, \frac{\Delta}{t+\Delta}\right) - \Delta^\alpha t^{\alpha-1}{}_2F_1\left(1-\alpha, 1, \alpha+1, \frac{\Delta}{t}\right)\right]. \tag{C9}$$

Using Eq. (15.3.4) in Ref. [149] for the hypergeometric function ${}_2F_1(1-\alpha, 1, \alpha+1, \Delta/[t+\Delta])$, we obtain

$$C_\Delta(t) = \frac{2K_\alpha}{\alpha\Gamma(\alpha)^2\Delta^2}t^{\alpha-1}\Delta^\alpha \times \left[{}_2F_1\left(1-\alpha, \alpha, \alpha+1, -\frac{\Delta}{t}\right) - {}_2F_1\left(1-\alpha, 1, \alpha+1, \frac{\Delta}{t}\right)\right]. \tag{C10}$$

When $t = 0$, $C_\Delta(0) = \langle x^2(\Delta)\rangle/\Delta^2 = \frac{2K_\alpha}{(2\alpha-1)\Gamma(\alpha)^2}\Delta^{2\alpha-3}$. Moreover, when $t \gg \Delta$, using (B25) we have

$${}_2F_1\left(1-\alpha, \alpha, \alpha+1, -\frac{\Delta}{t}\right) \sim 1 - \frac{\alpha(1-\alpha)}{\alpha+1}\frac{\Delta}{t}, \tag{C11}$$

and

$${}_2F_1\left(1-\alpha, 1, \alpha+1, \frac{\Delta}{t}\right) \sim 1 + \frac{1-\alpha}{\alpha+1}\frac{\Delta}{t}, \tag{C12}$$

and then

$$C_\Delta(t) \sim \frac{2(\alpha-1)K_\alpha}{\alpha\Gamma(\alpha)^2}\Delta^{\alpha-1}t^{\alpha-2}. \tag{C13}$$

APPENDIX D: SIMULATIONS

We summarize the discretization scheme for the Langevin equation with Caputo-fractional and conformable derivatives, as well as for SBM.

1. Caputo Langevin equation

We apply the implicit difference method [151]. Let $t = [t_0, t_{n+1}]$ with uniform step size $\Delta t = t_{k+1} - t_k$, $k = 1, 2, \ldots, n$. Then the left side of Eq. (93) with the Caputo derivative is reduced to

$$\frac{d^\alpha x_{n+1}}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t_{n+1}}\frac{dx(\mu)}{d\mu}(t_{n+1}-\mu)^{-\alpha}d\mu \tag{D1}$$

or

$$\frac{d^\alpha x_{n+1}}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)}\sum_{i=0}^{n}\int_{t_i}^{t_{i+1}}\frac{dx(\mu)}{d\mu}(t_{n+1}-\mu)^{-\alpha}d\mu, \tag{D2}$$

where $dx(\mu)/d\mu$ can be approximated by the implicit difference method as

$$\frac{dx(\mu)}{d\mu} = \frac{x_{i+1}-x_i}{\Delta t} + O(\Delta t), \tag{D3}$$

where $\mu \in [t_i, t_{i+1}]$. The remaining integral terms can be solved via

$$\int_{t_i}^{t_{i+1}}(t_{n+1}-\mu)^{-\alpha}d\mu = \frac{(\Delta t)^{1-\alpha}}{1-\alpha}[(n-i+1)^{1-\alpha} - (n-i)^{1-\alpha}]. \tag{D4}$$

Let $a_{n-i} = (n-i+1)^{1-\alpha} - (n-i)^{1-\alpha}$ and $a_0 = 1$. Substituting Eqs. (D3) and (D4) into (D2), one then has

$$\begin{aligned}\frac{d^\alpha x_{n+1}}{dt^\alpha} &\simeq \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}\sum_{i=0}^{n}(x_{i+1}-x_i) \times [(n-i+1)^{1-\alpha} - (n-i)^{1-\alpha}]\\ &= \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}\left[-\sum_{i=1}^{n}x_i(a_{n-i}-a_{n-i-1}) - x_0a_n + a_0x_{n+1}\right].\end{aligned} \tag{D5}$$

The right hand side of Eq. (93) with the Caputo-fractional derivative is $\sqrt{2K_\alpha/\Delta t}\,\eta(i)$, where $\eta(i)$ is a zero-mean Gaussian random variable with unit standard deviation. Then

$$-\frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}\left[\sum_{i=1}^{n}x_i(a_{n-i}-a_{n-i-1}) + x_0a_n + a_0x_{n+1}\right] = \sqrt{\frac{2D_\alpha}{\Delta t}}\eta(i) \tag{D6}$$

and

$$x_{n+1} = \sqrt{2K_\alpha}(\Delta t)^{\alpha-\frac{1}{2}}\Gamma(2-\alpha)\eta(i) + x_0a_n + \sum_{i=1}^{n}x_i(a_{n-i}-a_{n-i-1}). \tag{D7}$$

For x_j, $j = 1, 2, \ldots, n$,

$$x_j = \sqrt{2K_\alpha}(\Delta t)^{\alpha-\frac{1}{2}}\Gamma(2-\alpha)\eta(i) + x_0a_{j-1} + \sum_{i=1}^{j-1}x_i(a_{j-i-1}-a_{j-i-2}). \tag{D8}$$

2. SBM- and conformable-Langevin equation

The Langevin equation of SBM is

$$\frac{dx(t)}{dt} = \sqrt{2\mathscr{K}_\alpha(t)}\xi(t), \tag{D9}$$

where $\mathscr{K}_\alpha$ is defined in Eq. (19). With Eq. (D3) we deduce that

$$\frac{x_{i+1}-x_i}{\Delta t}=\sqrt{\frac{2\alpha K_\alpha i^{\alpha-1}}{\Delta t}}(\Delta t)^{\frac{\alpha-1}{2}}\eta(i), \tag{D10}$$

that is,

$$x_{i+1}=x_i+\sqrt{2\alpha K_\alpha i^{\alpha-1}}(\Delta t)^{\frac{\alpha}{2}}\eta(i). \tag{D11}$$

For the conformable-Langevin equation (106), the finite-difference method produces

$$\alpha\frac{x(t_1)-x(t_2)}{t_1^\alpha-t_2^\alpha}=\sqrt{2K_\alpha}\xi(t_1), \tag{D12}$$

and we finally have

$$x_{i+1}=x_i+\frac{\sqrt{2K_\alpha}}{\alpha}(\Delta t)^{\alpha-\frac{1}{2}}[(i+1)^\alpha-i^\alpha]\eta(i). \tag{D13}$$

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