



An HDG and CG Method for the Indefinite Time-Harmonic Maxwell's Equations Under Minimal Regularity

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Abstract

We propose to use a hybridizable discontinuous Galerkin (HDG) method combined with the continuous Galerkin (CG) method to approximate Maxwell's equations. We make two contributions in this paper. First, even though there are many papers using HDG methods to approximate Maxwell's equations, to our knowledge they all assume that the coefficients are smooth (or constant). Here, we derive optimal convergence estimates for our HDG-CG approximation when the electromagnetic coefficients are *piecewise* $W^{1,\infty}$. This requires new techniques of analysis. Second, we use CG elements to approximate the Lagrange multiplier used to enforce the divergence condition and we obtain a discrete system in which we can decouple the discrete Lagrange multiplier. Because we are using a continuous Lagrange multiplier space, the number of degrees of freedom devoted to this are less than for other HDG methods. We present numerical experiments to confirm our theoretical results.

Keywords Hybridizable discontinuous Galerkin (HDG) method · Maxwell's equations · Piecewise smooth coefficients · Minimal regularity · Error analysis

Mathematics Subject Classification 65N30

1 Introduction

Maxwell's equations govern the propagation of electromagnetic waves and have wide applications in science and technology; such as in the aerospace industry, telecommunications, medicine, and biology. Hence, a large number of computational techniques have been developed for solving Maxwell's equations, including finite difference methods, integral equation

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methods and finite element methods. Amongst these techniques, the finite element method is a popular technique for the solution of time-harmonic electromagnetic problems due to its ability to handle complex geometries and inhomogeneous materials.

The model problem we shall analyze is as follows. Let $\Omega \subset \mathbb{R}^3$ be a simply connected Lipschitz polyhedral domain with connected boundary $\partial\Omega$, we consider the *indefinite* time-harmonic Maxwell equations with a perfectly conducting boundary: find $(\mathbf{u}, p)$ that satisfies

$$\nabla \times (\mu_r^{-1} \nabla \times \mathbf{u}) - \kappa^2 \epsilon_r \mathbf{u} + \overline{\epsilon_r} \nabla p = \mathbf{f} \quad \text{in } \Omega, \quad (1a)$$

$$\nabla \cdot (\epsilon_r \mathbf{u}) = \rho \quad \text{in } \Omega, \quad (1b)$$

$$\mathbf{n} \times \mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega, \quad (1c)$$

$$p = 0 \quad \text{on } \partial\Omega. \quad (1d)$$

Here μ_r and ϵ_r are the relative magnetic permeability and the relative electric permittivity, which may be complex valued (and the overbar denotes complex conjugation). In addition $\mathbf{f} = ik\epsilon_0 \mathbf{j}$, where $\mathbf{j}$ is the given current density and ϵ_0 is the permittivity of vacuum, and $\kappa > 0$ is the wave number. The function ρ denotes the charge density. We note that the topological assumptions can be relaxed [27], but we choose the simplest setting here.

The Lagrange multiplier p is present to stabilize the problem by allowing the explicit imposition of the divergence constraint (1b), and is important in order to stabilize low frequency (small κ) problems [14] including the special case $\kappa = 0$ which corresponds to an electrostatic field. In electromagnetism, the function $\mathbf{f}$ and ρ are not independent, because conservation of charge requires that $\nabla \cdot \mathbf{f} + \kappa^2 \rho = 0$, so that the exact solution $p = 0$. To simplify the presentation, shall assume that $\rho = 0$ in the remainder of the paper.

Among finite element methods, curl-conforming elements (i.e. in $\mathbf{H}(\text{curl}; \Omega)$) have been widely studied, see for example [22, 26–29, 34]. These are often referred to as edge elements, and are known to eliminate the problem of spurious modes which may arise when standard finite elements are used to discretize Maxwell's equations [4]. Low-order edge elements are often used for problems in electromagnetics because they can be easily implemented, however, the use of low-order edge elements often leads to a discrete linear system with a large number of unknowns, especially for electromagnetic problems at high frequencies (large κ). As a result, high-order edge elements have been developed [1, 2, 14] and shown to be more effective than low-order edge elements. However, high-order edge elements introduce extra degrees of freedom in the interior of the elements which increase dramatically with the order of approximation. Typically, these interior degrees of freedom can be eliminated by using a procedure known as static condensation [24]. However, the implementation of high order edge element methods is complicated. Hence, non-conforming methods provide an interesting alternative for this kind of problem and may also be attractive for nonlinear problems.

Interior penalty discontinuous Galerkin (DG) methods have also been used to approximate the solution of the Maxwell's equations for some time. The first DG method for solving Maxwell's equations with high frequency was analyzed in [31] and much improved in [23]. In [20, 21], the local discontinuous Galerkin (LDG) method with high-order nodal elements is used to solve Maxwell's equations. From these studies we see that DG methods have several distinct advantages including their capabilities to handle complex geometries, to provide high-order accurate solutions, to perform *hp* adaptivity, and to retain excellent scalability. However, many existing DG methods are known to be computationally expensive because they have too many degrees of freedom due to nodal duplication on element boundaries.

In part to improve computational efficiency, Hybridizable discontinuous Galerkin (HDG) methods were proposed by Cockburn et al. in [13]. HDG methods are based on a mixed formulation and utilize a numerical flux and a numerical trace to approximate the flux and the trace of the solution. The volume based approximate flux and solution variables can be eliminated by element-by-element condensation. This process leads to a global equation for the approximate boundary traces only. As a result, HDG methods have significantly less globally coupled unknowns, a smaller memory requirement, and lower computational cost compared to other DG methods. HDG methods were first applied to Maxwell's equations in [30] but without an error analysis. Later on, an error analysis was provided in [9, 10, 15] for zero frequency and in [19, 25] for impedance boundary conditions and non-zero wave number. In a very recent paper [8], we used the concept of an M -decomposition, which was proposed by Cockburn et al in [12] for elliptic PDEs to analyze HDG schemes for Maxwell's equations in two dimensions. This analysis provides conditions on the HDG spaces to obtain optimal convergence, and superconvergence of some variables. The extension of this approach to 3D is challenging, and remains to be done. It is worthwhile to mention that all the above works only considered smooth (or even constant) coefficients.

When the material through which the waves propagate is heterogeneous, the functions μ_r and ϵ_r are non constant, and have jumps between materials of different type. In such a situation, the modest regularity pickup of the exact solution in the scale of Sobolev spaces is typically lower than $1/2$ and can be arbitrarily close to 0; see [3, Theorem 5.1]. To the best of our knowledge, all the previously mentioned HDG methods have not been proved to converge for nonsmooth coefficients because the standard analysis uses the solution of a dual problem that must have a sufficiently regular solution to allow for approximation using appropriate interpolation operators. When the coefficients are piecewise smooth, the appropriate regularity estimates are not available. Thus even if the true solution is smooth, the error analysis cannot currently be carried out using the methods considered in the previously mentioned papers. The main novelty of this paper is to prove convergence of our HDG method under realistic assumptions on the coefficients and low regularity for the solution of the adjoint problem. In particular, we follow the main idea in [18] to construct a stable and commuting quasi-interpolation operator, and then adapt the techniques in [5] (developed to analyze standard DG methods) to approximate equation (1) using HDG under weak assumptions on the coefficients, see Assumptions 1.

Furthermore, if we take $q \in H_0^1(\Omega)$ and integrate (1a) multiplied by ∇q we obtain $(\bar{\epsilon}_r \nabla p, \nabla q)_{L^2(\Omega)} = (f, \nabla q)_{L^2(\Omega)}$ giving an independent system for p (under the assumption $p = 0$). Unfortunately, such a decomposition does not hold for the discrete system found in all the above mentioned HDG methods. A second novel contribution of this paper is to mix the use of HDG spaces for $\mathbf{u}$ and a continuous Galerkin (CG) space for p . Our method, which we term HDG-CG, retains the flexibility of HDG for the desired field $\mathbf{u}$, but uses a smaller space for p which is usually easy to approximate (and for which designing hp -spaces is much easier than for edge elements). Then the system of the Lagrange multiplier p is SPD and can be solved very efficiently by MG or AMG. In other words, it is good for designing a block preconditioner and therefore, it is good for solving the whole system.

The outline of the paper is as follows. In Sect. 2, we recall the well-posedness and regularity of solutions of the Maxwell equations (1). In Sect. 3, we set some notation and give the HDG formulation of (1). The error analysis is given in Sect. 4, where we obtain optimal convergence rate for the electric field $\mathbf{u}$ and $\nabla \times \mathbf{u}$. Numerical experiments are provided to illustrate our theoretical results in (5).

2 Well-Posedness and Regularity of Maxwell's Equations

In this section, we first set some notation which will be used through this paper. Second, we discuss the well-posedness and regularity of Maxwell's equations (1). Next, we give the continuous inf-sup condition of the mixed form of (1) since we will use it later in our analysis of the HDG-CG method for approximating (1).

For any bounded domain $\Lambda \subset \mathbb{R}^3$, let $\mathbf{H}^m(\Lambda)$ denote the usual m^{th} -order Sobolev space of vector functions on Λ , and $\|\cdot\|_{m,\Lambda}$, $|\cdot|_{m,\Lambda}$ denote the corresponding norm and semi-norm. We use $(\cdot, \cdot)_{\Lambda}$ to denote the complex inner product on $L^2(\Lambda)$. Similarly, for the boundary $\partial\Lambda$ of Λ , we use $\langle \cdot, \cdot \rangle_{\partial\Lambda}$ to denote the L^2 inner product on $\partial\Lambda$. We define

$$\begin{aligned}\mathbf{H}(\text{curl}; \Lambda) &:= \{\mathbf{u} \in L^2(\Lambda) : \nabla \times \mathbf{u} \in L^2(\Lambda)\}, \\ \mathbf{H}_0(\text{curl}; \Lambda) &:= \{\mathbf{u} \in \mathbf{H}(\text{curl}; \Lambda) : \mathbf{n} \times \mathbf{u} = \mathbf{0} \text{ on } \partial\Lambda\}, \\ \mathbf{H}(\text{div}; \Lambda) &:= \{\mathbf{u} \in L^2(\Lambda) : \nabla \cdot \mathbf{u} \in L^2(\Lambda)\}, \\ \mathbf{H}(\text{div}_{\epsilon_r}^0; \Lambda) &:= \{\mathbf{u} \in L^2(\Lambda) : \nabla \cdot (\epsilon_r \mathbf{u}) = 0\},\end{aligned}$$

where $\mathbf{n}$ is the unit outward normal vector on $\partial\Lambda$.

In this paper, we allow the coefficients μ_r and ϵ_r of the Maxwell's equations (1) to be non-smooth. More precisely, we assume the following: The domain Ω can be decomposed into N subdomains denoted Ω_j , $j = 1, 2, \dots, N$ such that $\bar{\Omega} = \bigcup_{j=1}^N \bar{\Omega}_j$, $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$, and each subdomain Ω_j , $j = 1, 2, \dots, N$, is connected and has a Lipschitz boundary. Moreover,

Assumption 1 (A) $\mu_r, \epsilon_r \in W_{\Sigma}^{1,\infty}(\Omega) := \{v \in L^{\infty}(\Omega); \nabla(v|_{\Omega_i}) \in L^{\infty}(\Omega_i), i = 1, 2, \dots, N\}$.
(B) There exist constants $\bar{\mu}_r, \bar{\epsilon}_r > 0$ such that $\text{Re}(\mu_r) > \bar{\mu}_r$ and $\text{Re}(\epsilon_r) > \bar{\epsilon}_r$ a.e. in Ω .

We don't require any positivity on the imaginary part of the coefficients in the Assumption (B) although generally $\Im(\epsilon_r) \geq 0$.

2.1 Regularity Results for Maxwell's Equations

To investigate the regularity of the Maxwell's equations (1), one needs to characterize the space $\mathbf{H}_0(\text{curl}; \Omega)$, $\mathbf{H}(\text{div}_{\epsilon_r}^0; \Omega)$ and $L^2(\Omega)$. The next three lemmas give several properties of these spaces, and the proofs of these lemmas can be found in [22].

Lemma 2.1 ([22, Lemma 4.2]) *There exists $s_0 > 0$ such $\mathbf{H}_0(\text{curl}; \Omega) \cap \mathbf{H}(\text{div}_{\epsilon_r}^0; \Omega)$ is continuously embedded in $\mathbf{H}^s(\Omega)$ for all $s < s_0$, i.e., the following estimate holds*

$$\|\mathbf{u}\|_{\mathbf{H}^s(\Omega)} \leq C \left(\|\mathbf{u}\|_{L^2(\Omega)} + \|\nabla \times \mathbf{u}\|_{L^2(\Omega)} \right).$$

Lemma 2.2 ([22, Corollary 4.3]) *The embedding $\mathbf{H}_0(\text{curl}; \Omega) \cap \mathbf{H}(\text{div}_{\epsilon_r}^0; \Omega) \hookrightarrow L^2(\Omega)$ is compact.*

Lemma 2.3 ([22, Poincaré-Friedrichs-type inequality, Corollary 4.4]) *There is a constant $C > 0$ depending on Ω only, such that for any $\mathbf{u} \in \mathbf{H}_0(\text{curl}; \Omega) \cap \mathbf{H}(\text{div}_{\epsilon_r}^0; \Omega)$, we have*

$$\|\mathbf{u}\|_{L^2(\Omega)} \leq C \|\nabla \times \mathbf{u}\|_{L^2(\Omega)}.$$

The rigorous proof of the next theorem giving the regularity of solutions of (1) is based on [3, Theorem 5.1] and the details can be found in the arXiv preprint of this paper [7].

Theorem 2.4 Suppose that μ_r and ϵ_r satisfy Assumptions 1 and that $\kappa^2 > 0$ is not an eigenvalue of the problem of finding $\mathbf{w} \in \mathbf{H}_0(\text{curl}; \Omega)$, $\mathbf{w} \neq 0$, such that

$$\nabla \times (\mu_r^{-1} \nabla \times \mathbf{w}) = \kappa^2 \epsilon_r \mathbf{w}. \quad (2)$$

Then (1) has a unique solution $\mathbf{u} \in \mathbf{H}_0(\text{curl}; \Omega)$ and there exists $s > 0$ such that

$$\|\mathbf{u}\|_{\mathbf{H}^s(\Omega)} + \|\nabla \times \mathbf{u}\|_{\mathbf{H}^s(\Omega)} \leq C \|\mathbf{f}\|_{L^2(\Omega)},$$

where the constant C depends on s , Ω , μ_r , ϵ_r and κ .

We test (1a) with ∇q ($q \in H_0^1(\Omega)$) and use the usual regularity result for a second order elliptic problem to get the following result (recall we assume $p = 0$).

Corollary 2.5 If μ_r and ϵ_r satisfy the Assumption 1, $\mathbf{f} \in \mathbf{H}(\text{div}; \Omega)$. Then there is $s_0 > 0$ such that for $0 < s < s_0$, (1) has a unique solution $p \in H_0^1(\Omega)$. Moreover, the following regularity result holds,

$$\|p\|_{H^{1+s}(\Omega)} \leq C \|\nabla \cdot \mathbf{f}\|_{L^2(\Omega)}, \quad (3)$$

where the constant C depends on s , Ω and ϵ_r .

2.2 Mixed Formulation of Maxwell's Equations

To give our HDG formulation for Maxwell's equations, we need to rewrite them into a mixed form. This is rather standard, and we introduce a variable $\mathbf{q}$ which is just the scaled magnetic field in electromagnetism. Let $\mathbf{q} = \mu_r^{-1} \nabla \times \mathbf{u}$ in (1) to get the following mixed form

$$\mu_r \mathbf{q} - \nabla \times \mathbf{u} = \mathbf{0} \quad \text{in } \Omega, \quad (4a)$$

$$\nabla \times \mathbf{q} - \kappa^2 \epsilon_r \mathbf{u} + \bar{\epsilon}_r \nabla p = \mathbf{f} \quad \text{in } \Omega, \quad (4b)$$

$$\nabla \cdot (\epsilon_r \mathbf{u}) = 0 \quad \text{in } \Omega, \quad (4c)$$

$$\mathbf{n} \times \mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega, \quad (4d)$$

$$p = 0 \quad \text{on } \partial\Omega. \quad (4e)$$

Then the mixed weak form of (4) is given as follows: find $(\mathbf{q}, \mathbf{u}, p) \in L^2(\Omega) \times \mathbf{H}_0(\text{curl}; \Omega) \times H_0^1(\Omega)$ such that

$$(\mu_r \mathbf{q}, \mathbf{r})_{L^2(\Omega)} - (\nabla \times \mathbf{u}, \mathbf{r})_{L^2(\Omega)} = 0, \quad (5a)$$

$$(\mathbf{q}, \nabla \times \mathbf{v})_{L^2(\Omega)} - (\kappa^2 \epsilon_r \mathbf{u}, \mathbf{v})_{L^2(\Omega)} + (\nabla p, \epsilon_r \mathbf{v})_{L^2(\Omega)} = (\mathbf{f}, \mathbf{v})_{L^2(\Omega)}, \quad (5b)$$

$$(\epsilon_r \mathbf{u}, \nabla \chi)_{L^2(\Omega)} = 0 \quad (5c)$$

for all $(\mathbf{r}, \mathbf{v}, \chi) \in L^2(\Omega) \times \mathbf{H}_0(\text{curl}; \Omega) \times H_0^1(\Omega)$.

To shorten lengthy equations, for all $(\mathbf{q}, \mathbf{u}, p), (\mathbf{r}, \mathbf{v}, \chi) \in L^2(\Omega) \times \mathbf{H}_0(\text{curl}; \Omega) \times H_0^1(\Omega)$, we define the sesquilinear form $\mathcal{B}^-$ by

$$\begin{aligned} \mathcal{B}^-(\mathbf{q}, \mathbf{u}, p; \mathbf{r}, \mathbf{v}, \chi) &= (\mu_r \mathbf{q}, \mathbf{r})_{L^2(\Omega)} - (\nabla \times \mathbf{u}, \mathbf{r})_{L^2(\Omega)} + (\mathbf{q}, \nabla \times \mathbf{v})_{L^2(\Omega)} \\ &\quad + (\bar{\epsilon}_r \nabla p, \mathbf{v})_{L^2(\Omega)} - (\epsilon_r \mathbf{u}, \nabla \chi)_{L^2(\Omega)} - (\kappa^2 \epsilon_r \mathbf{u}, \mathbf{v})_{L^2(\Omega)}. \end{aligned} \quad (6)$$

By the definition of $\mathcal{B}^-$ in (6), we can write the mixed weak form of (5) as follows: find $(\mathbf{q}, \mathbf{u}, p) \in \mathbf{L}^2(\Omega) \times \mathbf{H}_0(\text{curl}; \Omega) \times H_0^1(\Omega)$ such that

$$\mathcal{B}^-(\mathbf{q}, \mathbf{u}, p; \mathbf{r}, \mathbf{v}, \chi) = (\mathbf{f}, \mathbf{v})_{\mathbf{L}^2(\Omega)} \quad (7)$$

for all $(\mathbf{r}, \mathbf{v}, \chi) \in \mathbf{L}^2(\Omega) \times \mathbf{H}_0(\text{curl}; \Omega) \times H_0^1(\Omega)$.

Next, for all $(\mathbf{q}, \mathbf{u}, p) \in \mathbf{H}(\text{curl}; \Omega) \times \mathbf{H}_0(\text{curl}; \Omega) \times H_0^1(\Omega)$, we define the semi-norm $\|\cdot\|$ by

$$\begin{aligned} \|(\mathbf{q}, \mathbf{u}, p)\|^2 &= \|\sqrt{\text{Re}(\mu_r)} \mathbf{q}\|_{\mathbf{L}^2(\Omega)}^2 + \|\sqrt{\text{Re}(\epsilon_r)} \mathbf{u}\|_{\mathbf{L}^2(\Omega)}^2 \\ &\quad + \|\nabla \times \mathbf{u}\|_{\mathbf{L}^2(\Omega)}^2 + \|\sqrt{\text{Re}(\epsilon_r)} \nabla p\|_{\mathbf{L}^2(\Omega)}^2, \end{aligned} \quad (8)$$

where μ_r , and ϵ_r satisfy Assumption 1 and $\kappa > 0$. It is easy to prove that the semi-norm $\|\cdot\|$ is actually a norm.

Lemma 2.6 $\|\cdot\|$ defines a norm on the space $\mathbf{L}^2(\Omega) \times \mathbf{H}_0(\text{curl}; \Omega) \times H_0^1(\Omega)$.

For all $(\mathbf{q}, \mathbf{u}, p), (\mathbf{r}, \mathbf{v}, \chi) \in \mathbf{L}^2(\Omega) \times \mathbf{H}_0(\text{curl}; \Omega) \times H_0^1(\Omega)$, we define the sesquilinear form $\mathcal{B}^+$ by

$$\mathcal{B}^+(\mathbf{q}, \mathbf{u}, p; \mathbf{r}, \mathbf{v}, \chi) = \mathcal{B}^-(\mathbf{q}, \mathbf{u}, p; \mathbf{r}, \mathbf{v}, \chi) + (\kappa^2 + 1)(\epsilon_r \mathbf{u}, \mathbf{v})_{\mathbf{L}^2(\Omega)}. \quad (9)$$

The proof of the following continuous inf-sup condition for the coercive form $\mathcal{B}^+$ can be founded in the arXiv preprint of this paper [7].

Lemma 2.7 (Continuous inf-sup condition) *Let $\sigma = (\mathbf{q}, \mathbf{u}, p), \tau = (\mathbf{r}, \mathbf{v}, \chi) \in \mathbf{L}^2(\Omega) \times \mathbf{H}_0(\text{curl}; \Omega) \times H_0^1(\Omega)$, we have the following inf-sup condition*

$$\sup_{\sigma \neq \tau} \frac{\text{Re}[\mathcal{B}^+(\sigma; \tau)]}{\|\tau\|} \geq C \|\sigma\|. \quad (10)$$

3 The HDG Method

To describe the HDG method, we first define some notation. Let $\mathcal{T}_h := \{K\}$ denote a conforming and regular mesh of Ω , where each element K is a tetrahedron. For each $K \in \mathcal{T}_h$, we let h_K be the infimum of the diameters of balls containing K and denote the mesh size $h := \max_{K \in \mathcal{T}_h} h_K$. Let $\partial\mathcal{T}_h$ denote the set of faces $F \subset \partial K$ of the elements $K \in \mathcal{T}_h$ (i.e. faces of distinct elements are counted separately) and let $\mathcal{F}_h$ denote the set of faces in the mesh $\mathcal{T}_h$. We denote by h_F the diameter of the face F . We abuse notation by using $\nabla \times$, $\nabla \cdot$ and ∇ for broken curl, div and gradient operators with respect to the mesh partition $\mathcal{T}_h$, respectively. To simplify the notation, we also define a function $\mathbf{h}$ on $\mathcal{T}_h$, $\partial\mathcal{T}_h$ and $\mathcal{F}_h$ which depending on circumstances is defined by:

$$\mathbf{h}|_K = h_K, \quad \forall K \in \mathcal{T}_h, \quad \mathbf{h}|_{\partial K} = h_K, \quad \forall K \in \mathcal{T}_h, \quad \mathbf{h}|_F = h_F, \quad \forall F \in \mathcal{F}_h.$$

Next, we list some formulas which will be frequently used in this paper.

- (1) Let $F \in \mathcal{F}_h$, and let $\nabla_F \cdot$ denote the surface divergence on F where the definition of $\nabla_F \cdot$ can be found in [27, Section 3.4 (page 48)] then the following identity holds for all sufficiently smooth vector functions $\mathbf{v}$ defined in a neighborhood of F :

$$\nabla_F \cdot (\mathbf{n} \times \mathbf{v}) = -\mathbf{n} \cdot (\nabla \times \mathbf{v})|_F. \quad (11)$$

Table 1 Comparison of different HDG methods

Type	$\mathbf{q}_h$	$\mathbf{u}_h$	$\widehat{\mathbf{u}}_h$	p_h	$\widehat{p}_h$	Stabilization for p_h
I [10]	$k - 1$	k	k^-	$k - 1$	k	$\mathbf{h}(p_h - \widehat{p}_h)$
II [10]	$k - 1$	k	k^-	$k - 1$	k	None
III [9]	k	k	k	$k + 1$	$k + 1$	$\mathbf{h}^{-1}(p_h - \widehat{p}_h)$
IV [6]	$k - 1$ or k	k	k	k	k	$\mathbf{h}^{\pm 1}(p_h - \widehat{p}_h)$
V [25]	$k - 1$ or k	k	k	k	k	$\mathbf{h}(p_h - \widehat{p}_h)$
VI [15]	$k - 1$	k	k	$k - 1$	k	$\mathbf{h}(p_h - \widehat{p}_h)$
VII [15]	$k - 1$	k	k^-	k	k	$\mathbf{h}(p_h - \widehat{p}_h)$

(2) Let K be an element in the mesh $\mathcal{T}_h$, $\mathbf{u}, \mathbf{v} \in \mathbf{H}(\text{curl}; K)$, $\mathbf{w} \in \mathbf{H}(\text{div}; F)$, $p \in H^1(\partial F)$ and $\langle \cdot, \cdot \rangle_{\partial F}$ be the standard conjugate-linear $H^{1/2} - H^{-1/2}$ duality pairing on ∂F . In addition let ∇_F denote the surface gradient on F . Then we have

$$(\nabla \times \mathbf{u}, \mathbf{v})_K = \langle \mathbf{n} \times \mathbf{u}, \mathbf{v} \rangle_{\partial K} + (\mathbf{u}, \nabla \times \mathbf{v})_K, \quad (12a)$$

$$-\langle \nabla_F \cdot \mathbf{w}, p \rangle_F = \langle \mathbf{w}, \nabla_F p \rangle_F - \langle \mathbf{n}_E \cdot \mathbf{w}, p \rangle_{\partial F}, \quad (12b)$$

where $\mathbf{n}$ and $\mathbf{n}_E$ are the unit normal to each face of ∂K and each edge of ∂F .

Next, to give the HDG fomulation of (1). First we define the following finite element spaces. Let $k \geq 1$, $m = k - 1$ or $m = k$,

$$\mathcal{Q}_h := \{\mathbf{q}_h \in \mathbf{L}^2(\Omega) : \mathbf{q}_h|_K \in [\mathcal{P}_m(K)]^3, \forall K \in \mathcal{T}_h\},$$

$$\mathcal{U}_h := \{\mathbf{u}_h \in \mathbf{L}^2(\Omega) : \mathbf{u}_h|_K \in [\mathcal{P}_k(K)]^3, \forall K \in \mathcal{T}_h\},$$

$$\widehat{\mathcal{U}}_h := \{\widehat{\mathbf{u}}_h \in \mathbf{L}^2(\mathcal{F}_h) : \widehat{\mathbf{u}}_h|_F \in [\mathcal{P}_k(F)]^3, \widehat{\mathbf{u}}_h \cdot \mathbf{n}|_F = 0, \forall F \in \mathcal{F}_h, \mathbf{n} \times \widehat{\mathbf{u}}_h|_{\partial\Omega} = \mathbf{0}\},$$

$$\mathcal{P}_h := \{p_h \in H_0^1(\Omega) : p_h|_K \in \mathcal{P}_{k+1}(K), \forall K \in \mathcal{T}_h\}.$$

We can now derive the HDG method for (4) by multiplying each equation by the appropriate discrete test function, integrating element by element and using integration by parts element by element in the usual way (c.f. [13]). Summing the results over all elements, the HDG methods seeks $(\mathbf{q}_h, \mathbf{u}_h, \widehat{\mathbf{u}}_h, p_h) \in \mathcal{Q}_h \times \mathcal{U}_h \times \widehat{\mathcal{U}}_h \times \mathcal{P}_h$, such that

$$(\mu_r \mathbf{q}_h, \mathbf{r}_h)_{\mathcal{T}_h} - (\mathbf{u}_h, \nabla \times \mathbf{r}_h)_{\mathcal{T}_h} - \langle \mathbf{n} \times \widehat{\mathbf{u}}_h, \mathbf{r}_h \rangle_{\partial\mathcal{T}_h} = 0, \quad (13a)$$

$$(\mathbf{q}_h, \nabla \times \mathbf{v}_h)_{\mathcal{T}_h} + \langle \mathbf{n} \times \widehat{\mathbf{q}}_h, \mathbf{v}_h \rangle_{\partial\mathcal{T}_h} - (\kappa^2 \epsilon_r \mathbf{u}_h, \mathbf{v}_h)_{\mathcal{T}_h} + (\epsilon_r \nabla p_h, \mathbf{v}_h)_{\mathcal{T}_h} = (\mathbf{f}, \mathbf{v}_h)_{\mathcal{T}_h}, \quad (13b)$$

$$(\epsilon_r \mathbf{u}_h, \nabla \chi_h)_{\mathcal{T}_h} = 0, \quad (13c)$$

$$\langle \mathbf{n} \times \widehat{\mathbf{q}}_h, \widehat{\mathbf{v}}_h \rangle_{\mathcal{F}_h/\partial\Omega} = 0 \quad (13d)$$

for all $(\mathbf{r}_h, \mathbf{v}_h, \widehat{\mathbf{v}}_h, \chi_h) \in \mathcal{Q}_h \times \mathcal{U}_h \times \widehat{\mathcal{U}}_h \times \mathcal{P}_h$, and the choice of $\mathbf{n} \times \widehat{\mathbf{q}}_h$ follows the usual HDG pattern,

$$\mathbf{n} \times \widehat{\mathbf{q}}_h = \mathbf{n} \times \mathbf{q}_h + \mathbf{h}^{-1} \mathbf{n} \times (\mathbf{u}_h - \widehat{\mathbf{u}}_h) \times \mathbf{n}. \quad (13e)$$

It is obvious to see that we can decouple the pressure p_h from the system (13) if we take $\mathbf{v}_h = \nabla \chi_h$. It is worth mentioning that such a decomposition does not hold for the discrete system of other HDG methods in the literature. We list most of them in Table 1, where, in the table, k and k^- is used as a compact way to denote the spaces $\mathcal{P}_k(K)$ and

$\mathcal{P}_{k-1}(F) \oplus \nabla \tilde{\mathcal{P}}_{k+1}(F)$, respectively. Here $\tilde{\mathcal{P}}_k(F)$ is the spaces of homogeneous polynomials of degree k on F .

Following the definition of $\mathcal{B}^\pm$, we define the discrete sesquilinear form $\mathcal{B}_h^\pm$ on the space $\mathcal{Q}_h \times \mathbf{U}_h \times \widehat{\mathbf{U}}_h \times P_h$ by

$$\begin{aligned} \mathcal{B}_h^-(\mathbf{q}_h, \mathbf{u}_h, \widehat{\mathbf{u}}_h, p_h; \mathbf{r}_h, \mathbf{v}_h, \widehat{\mathbf{v}}_h, \chi_h) &= (\mu_r \mathbf{q}_h, \mathbf{r}_h)_{\mathcal{T}_h} - (\mathbf{u}_h, \nabla \times \mathbf{r}_h)_{\mathcal{T}_h} - \langle \mathbf{n} \times \widehat{\mathbf{u}}_h, \mathbf{r}_h \rangle_{\partial \mathcal{T}_h} \\ &\quad + (\nabla \times \mathbf{q}_h, \mathbf{v}_h)_{\mathcal{T}_h} + \langle \mathbf{q}_h, \mathbf{n} \times \widehat{\mathbf{v}}_h \rangle_{\partial \mathcal{T}_h} + (\bar{\epsilon}_r \nabla p_h, \mathbf{v}_h)_{\mathcal{T}_h} \\ &\quad + \langle \mathbf{h}^{-1} \mathbf{n} \times (\mathbf{u}_h - \widehat{\mathbf{u}}_h), \mathbf{n} \times (\mathbf{v}_h - \widehat{\mathbf{v}}_h) \rangle_{\partial \mathcal{T}_h} - (\epsilon_r \mathbf{u}_h, \nabla \chi_h)_{\mathcal{T}_h} - (\kappa^2 \epsilon_r \mathbf{u}_h, \mathbf{v}_h)_{\mathcal{T}_h}, \\ \mathcal{B}_h^+(\mathbf{q}_h, \mathbf{u}_h, p_h; \mathbf{r}_h, \mathbf{v}_h, \chi_h) &= \mathcal{B}_h^-(\mathbf{q}_h, \mathbf{u}_h, p_h; \mathbf{r}_h, \mathbf{v}_h, \chi_h) + (\kappa^2 + 1)(\epsilon_r \mathbf{u}_h, \mathbf{v}_h)_{\mathcal{T}_h}. \end{aligned} \quad (14)$$

Then, we can rewrite the HDG formulation (13) in a compact form: find $(\mathbf{q}_h, \mathbf{u}_h, \widehat{\mathbf{u}}_h, p_h) \in \mathcal{Q}_h \times \mathbf{U}_h \times \widehat{\mathbf{U}}_h \times P_h$ such that

$$\mathcal{B}_h^-(\mathbf{q}_h, \mathbf{u}_h, \widehat{\mathbf{u}}_h, p_h; \mathbf{r}_h, \mathbf{v}_h, \widehat{\mathbf{v}}_h, \chi_h) = (\mathbf{f}, \mathbf{v}_h)_{\mathcal{T}_h} \quad (15)$$

for all $(\mathbf{r}_h, \mathbf{v}_h, \widehat{\mathbf{v}}_h, \chi_h) \in \mathcal{Q}_h \times \mathbf{U}_h \times \widehat{\mathbf{U}}_h \times P_h$.

Although $\mathcal{B}_h^\pm$ is defined on the space $\mathcal{Q}_h \times \mathbf{U}_h \times \widehat{\mathbf{U}}_h \times P_h$, the notation $\mathcal{B}_h^\pm(\mathbf{r}, \mathbf{v}, \mathbf{v}, \chi; \mathbf{r}_h, \mathbf{v}_h, \widehat{\mathbf{v}}_h, \chi_h)$ is meaningful if $(\mathbf{r}, \mathbf{v}, \chi) \in \mathbf{H}(\text{curl}; \Omega) \times \mathbf{H}_0(\text{curl}; \Omega) \times H_0^1(\Omega)$. Moreover, if $(\mathbf{q}, \mathbf{u}, p) \in \mathbf{H}(\text{curl}; \Omega) \times \mathbf{H}_0(\text{curl}; \Omega) \times H_0^1(\Omega)$ is the solution of (4), then

$$\mathcal{B}_h^-(\mathbf{q}, \mathbf{u}, \mathbf{u}, p; \mathbf{r}_h, \mathbf{v}_h, \widehat{\mathbf{v}}_h, \chi_h) = (\mathbf{f}, \mathbf{v}_h)_{\mathcal{T}_h}. \quad (16)$$

The proof of the following lemma is very simple and we omit it here.

Lemma 3.1 For any $(\mathbf{q}_h, \mathbf{u}_h, \widehat{\mathbf{u}}_h, p_h; \mathbf{r}_h, \mathbf{v}_h, \widehat{\mathbf{v}}_h, \chi_h) \in [\mathcal{Q}_h \times \mathbf{U}_h \times \widehat{\mathbf{U}}_h \times P_h]^2$, we have

$$\mathcal{B}_h^-(\mathbf{q}_h, \mathbf{u}_h, \widehat{\mathbf{u}}_h, p_h; -\mathbf{r}_h, \mathbf{v}_h, \widehat{\mathbf{v}}_h, -\chi_h) = \overline{\mathcal{B}_h^-(\mathbf{r}_h, \mathbf{v}_h, \widehat{\mathbf{v}}_h, \chi_h; -\mathbf{q}_h, \mathbf{u}_h, \widehat{\mathbf{u}}_h, -p_h)}. \quad (17)$$

3.1 Preliminary Material

The approximation of Maxwell's equations by DG methods is studied in [9, 10, 12, 15, 19, 25, 31] where the coefficients are assumed smooth so that the solution is regular enough for a duality based error analysis, i.e., $\mathbf{u} \in H^s(\Omega)$ with $s > 1/2$. However, by Theorem 2.4, we only have in general that $\mathbf{u} \in H^s(\Omega)$ with $1/2 > s > 0$ if the coefficients satisfy the Assumption 1 and the traditional approach will fail. Recently, Ern et al. [17] derived an error estimate for conforming methods under the Assumption 1 by constructing stable, commuting quasi-interpolation projectors. The idea is to compose the canonical finite element interpolation operators with a mollification technique; also see Schöberl [32] and Christiansen [11] for more details.

In this section, we follow [27, Chapter 5] to construct standard interpolation operators $\mathcal{I}_h^{\text{div}}$ and $\mathcal{I}_h^{\text{curl}}$, which are only defined on $H^s(\Omega)$ with $s > 1/2$; then by a modification operator $\mathcal{K}_h$ which was constructed in [16, 17], we can smooth functions in the space $H^s(\Omega)$ with $s > 0$ into $H^{s'}(\Omega)$ with $s' > 1/2$; finally define the quasi-interpolations by

$$\mathcal{J}_h^{\text{div}} = ((\mathcal{I}_h^{\text{div}} \mathcal{K}_h|_{V_h^{\text{div}}})^{-1} \mathcal{I}_h^{\text{div}} \mathcal{K}_h, \quad \mathcal{J}_h^{\text{curl}} = (\mathcal{I}_h^{\text{curl}} \mathcal{K}_h|_{V_h^{\text{curl}}})^{-1} \mathcal{I}_h^{\text{curl}} \mathcal{K}_h.$$

First, we define some spaces which will be useful in our analysis. For the convenience, we define

$$\begin{aligned} \mathbf{V}_h^{\text{curl}} &:= \mathbf{U}_h \cap \mathbf{H}(\text{curl}; \Omega), \quad \mathbf{V}_h^{\text{div}} := \mathcal{D}_k(\mathcal{T}_h) \cap \mathbf{H}(\text{div}; \Omega), \\ \mathcal{D}_k(K) &:= [\mathcal{P}_{k-1}(K)]^3 + \tilde{\mathcal{P}}_{k-1}(K)\mathbf{x}, \quad \mathcal{D}_k(F) := [\mathcal{P}_{k-1}(F)]^3 + \tilde{\mathcal{P}}_{k-1}(F)\mathbf{x}. \end{aligned}$$

where $\tilde{\mathcal{P}}_k(K)$ and $\tilde{\mathcal{P}}_k(F)$ are the spaces of polynomials homogeneous of degree k on K and F , respectively. It's worth noting that $\mathcal{D}_k(K)$ and $\mathcal{D}_k(F)$ correspond to the RT space in three dimensions on K and in two dimensions on F , respectively.

Next, we define $\mathcal{I}_h^{\text{div}}$ to be the divergence conforming interpolation of the first family from $\mathbf{H}^s(\Omega) \rightarrow \mathbf{V}_h^{\text{div}}$ by [28, Page 328, Definition 5] and [27, Page 119, Definition 5.14] which is defined element by element via

$$(\mathcal{I}_h^{\text{div}} \mathbf{v}, \mathbf{q}_{k-2})_K = (\mathbf{v}, \mathbf{q}_{k-2})_K \quad \forall \mathbf{q}_{k-2} \in \mathcal{P}_{k-2}(K), \quad (18a)$$

$$\langle \mathbf{n} \cdot \mathcal{I}_h^{\text{div}} \mathbf{v}, q_{k-1} \rangle_F = \langle \mathbf{n} \cdot \mathbf{v}, q_{k-1} \rangle_F \quad \forall q_{k-1} \in \mathcal{P}_{k-1}(F) \quad (18b)$$

for all faces $F \subset \partial K$, where $s > 1/2$ and $\mathbf{v} \in \mathbf{H}^s(\text{div}; \Omega)$. Moreover, we define $\mathcal{I}_h^{\text{curl}}$ be the curl conforming interpolation of the second family from $\mathbf{H}^s(\text{curl}; \Omega) \rightarrow \mathbf{V}_h^{\text{curl}}$ element by element by

$$(\mathcal{I}_h^{\text{curl}} \mathbf{v}, \mathbf{q}_{k-2})_K = (\mathbf{v}, \mathbf{q}_{k-2})_K \quad \forall \mathbf{q}_{k-2} \in \mathcal{D}_{k-2}(K), \quad (19a)$$

$$\langle \mathbf{n} \times \mathcal{I}_h^{\text{curl}} \mathbf{v} \times \mathbf{n}, \mathbf{q}_{k-1} \rangle_F = \langle \mathbf{n} \times \mathbf{v} \times \mathbf{n}, \mathbf{q}_{k-1} \rangle_F \quad \forall \mathbf{q}_{k-1} \in \mathcal{D}_{k-1}(F), \quad (19b)$$

$$\langle \langle \mathcal{I}_h^{\text{curl}} \mathbf{v} \cdot \boldsymbol{\tau}, q_k \rangle \rangle_E = \langle \langle \mathbf{v} \cdot \boldsymbol{\tau}, q_k \rangle \rangle_E \quad \forall q_k \in \mathcal{P}_k(E) \quad (19c)$$

for all faces $F \subset \partial K$ and all edges $E \subset \partial F$, and $\mathbf{v} \in \mathbf{H}^s(\text{curl}; \Omega)$ with $s > 1/2$.

The following lemma shows that the usual commutativity properties hold for the combined first and second kind interpolants used here.

Lemma 3.2 [18, (2.17)] *For $s > 1/2$ and $\mathbf{v} \in \mathbf{H}^s(\text{curl}; K)$, let $\mathcal{I}_h^{\text{div}}$ and $\mathcal{I}_h^{\text{curl}}$ define in (18) and (19), respectively. Then we have the following commutativity property*

$$\nabla \times (\mathcal{I}_h^{\text{curl}} \mathbf{v}) = \mathcal{I}_h^{\text{div}} (\nabla \times \mathbf{v}).$$

By Lemma 3.2 and [16, Theorem 6.5], [17, Corollary 5.4] we now get the following lemma.

Lemma 3.3 ([Stable, commuting projection]) *There exist quasi-interpolation operator $\mathcal{J}_h^{\text{curl}} : \mathbf{L}^1(\Omega) \rightarrow \mathbf{V}_h^{\text{curl}}$ and $\mathcal{J}_h^{\text{div}} : \mathbf{L}^1(\Omega) \rightarrow \mathbf{V}_h^{\text{div}}$ such that*

- (1) $\mathbf{U}_h \cap \mathbf{H}_0(\text{curl}; \Omega)$ is pointwise invariant under $\mathcal{J}_h^{\text{curl}}$.
- (2) $\mathcal{J}_h^{\text{curl}} \mathbf{v} \in \mathbf{H}_0(\text{curl}; \Omega)$ if $\mathbf{v} \in \mathbf{H}_0(\text{curl}; \Omega)$.
- (3) For $p \in [1, \infty]$, there holds the stability

$$\|\mathcal{J}_h^{\text{curl}}\|_{\mathcal{L}(\mathbf{L}^p(\Omega); \mathbf{L}^p(\Omega))} \leq C, \quad \|\mathcal{J}_h^{\text{div}}\|_{\mathcal{L}(\mathbf{L}^p(\Omega); \mathbf{L}^p(\Omega))} \leq C.$$

- (4) For $p \in [1, \infty]$ and $s \in [0, k+1]$, we have the following estimates

$$\|\mathcal{J}_h^{\text{div}} \mathbf{v} - \mathbf{v}\|_{\mathbf{L}^p(\Omega)} \leq C \inf_{\mathbf{v}_h \in \mathbf{V}_h^{\text{div}}} \|\mathbf{v}_h - \mathbf{v}\|_{\mathbf{L}^p(\Omega)} \leq Ch^s |\mathbf{v}|_{s,p}, \quad (20a)$$

$$\|\mathcal{J}_h^{\text{curl}} \mathbf{v} - \mathbf{v}\|_{\mathbf{L}^p(\Omega)} \leq C \inf_{\mathbf{v}_h \in \mathbf{V}_h^{\text{curl}}} \|\mathbf{v}_h - \mathbf{v}\|_{\mathbf{L}^p(\Omega)} \leq Ch^s |\mathbf{v}|_{s,p}. \quad (20b)$$

- (5) For $\mathbf{v} \in \mathbf{H}(\text{curl}; \Omega)$, the following commute property holds

$$\nabla \times (\mathcal{J}_h^{\text{curl}} \mathbf{v}) = \mathcal{J}_h^{\text{div}} (\nabla \times \mathbf{v}). \quad (21)$$

Lemma 3.4 (Discrete Helmholtz decomposition) *For all $\mathbf{v}_h \in \mathbf{U}_h \cap \mathbf{H}_0(\text{curl}; \Omega)$, there exist unique functions $\mathbf{z}_h \in \mathbf{U}_h \cap \mathbf{H}_0(\text{curl}; \Omega)$ and $\xi_h \in P_h$ such that for all $\chi_h \in P_h$ we have*

$$\mathbf{v}_h = \mathbf{z}_h + \nabla \xi_h, \quad (\epsilon_r \mathbf{z}_h, \nabla \chi_h)_{\mathcal{T}_h} = 0. \quad (22)$$

Moreover, the following stability results hold

$$\|\nabla \xi_h\|_{\mathcal{T}_h} \leq C \|\mathbf{v}_h\|_{\mathcal{T}_h}, \quad \|\mathbf{z}_h\|_{\mathcal{T}_h} \leq C \|\mathbf{v}_h\|_{\mathcal{T}_h}. \quad (23)$$

Proof For all $\chi_h \in P_h$, let $\xi_h \in P_h$ be the solution of

$$(\epsilon_r \nabla \xi_h, \nabla \chi_h)_{\mathcal{T}_h} = (\epsilon_r \mathbf{v}_h, \nabla \chi_h)_{\mathcal{T}_h}. \quad (24)$$

The system (24) is well-defined for any $\mathbf{v}_h \in \mathbf{U}_h$, hence we have

$$\|\nabla \xi_h\|_{\mathcal{T}_h} \leq C \|\mathbf{v}_h\|_{\mathcal{T}_h}. \quad (25)$$

We take $\mathbf{z}_h = \mathbf{v}_h - \nabla \xi_h$, then (22) and $\|\mathbf{z}_h\|_{\mathcal{T}_h} \leq C \|\mathbf{v}_h\|_{\mathcal{T}_h}$ hold. Finally, $\mathbf{z}_h$ is unique since ξ_h is the unique solution of (24). $\square$

The proof of the following lemma with smooth coefficients μ_r and ϵ_r is given in [22, Lemma 4.5] and [23, Corollary 4.4]. We extend the result to allow piecewise smooth coefficients. Its proof can be found in the arXiv preprint of this paper [7].

Lemma 3.5 *Assume that $\mathbf{v}_h \in \mathbf{U}_h \cap \mathbf{H}_0(\text{curl}; \Omega)$ satisfies $(\epsilon_r \mathbf{v}_h, \nabla \chi_h)_{L^2(\Omega)} = 0$ for all $\chi_h \in P_h$. Let $\boldsymbol{\Theta} \in \mathbf{H}_0(\text{curl}; \Omega) \cap \mathbf{X}_0$ be the solution of $\nabla \times \boldsymbol{\Theta} = \nabla \times \mathbf{v}_h$, then, we have the following stability estimate and the approximation property for some $s > 0$:*

$$\|\boldsymbol{\Theta}\|_{\mathbf{H}^s(\Omega)} \leq C \|\nabla \times \mathbf{v}_h\|_{L^2(\Omega)}, \quad (26a)$$

$$\|\mathbf{v}_h - \boldsymbol{\Theta}\|_{L^2(\Omega)} \leq Ch^s \|\nabla \times \mathbf{v}_h\|_{L^2(\Omega)}. \quad (26b)$$

The final result of this section is ubiquitous in the analysis of DG methods for Maxwell's equations:

Lemma 3.6 (c.f [23, Proposition 4.5]) *For all $\mathbf{u}_h \in \mathbf{U}_h$, there exists a $\mathbf{u}_h^c \in \mathbf{U}_h \cap \mathbf{H}_0(\text{curl}; \Omega)$ such that*

$$\|\mathbf{u}_h - \mathbf{u}_h^c\|_{\mathcal{T}_h} + \|\mathbf{h} \nabla \times (\mathbf{u}_h - \mathbf{u}_h^c)\|_{\mathcal{T}_h} \leq C \|\mathbf{h}^{\frac{1}{2}} \mathbf{n} \times \llbracket \mathbf{u}_h \rrbracket\|_{\mathcal{F}_h}. \quad (27)$$

3.2 Stability of the Coercive Discrete Problem

Next, for all $(\mathbf{q}_h, \mathbf{u}_h, \widehat{\mathbf{u}}_h, p_h) \in \mathcal{Q}_h \times \mathbf{U}_h \times \widehat{\mathbf{U}}_h \times P_h$, we define

$$\begin{aligned} \|(\mathbf{q}_h, \mathbf{u}_h, \widehat{\mathbf{u}}_h, p_h)\|_h^2 &= \|\sqrt{\text{Re}(\mu_r)} \mathbf{q}_h\|_{\mathcal{T}_h}^2 + \|\sqrt{\text{Re}(\epsilon_r)} \mathbf{u}_h\|_{\mathcal{T}_h}^2 \\ &\quad + \|\nabla \times \mathbf{u}_h\|_{\mathcal{T}_h}^2 + \|\mathbf{h}^{-\frac{1}{2}} \mathbf{n} \times (\mathbf{u}_h - \widehat{\mathbf{u}}_h)\|_{\partial \mathcal{T}_h}^2 \\ &\quad + \|\sqrt{\text{Re}(\epsilon_r)} \nabla p_h\|_{\mathcal{T}_h}^2, \end{aligned} \quad (28)$$

where μ_r, κ and ϵ_r were defined in the Assumption 1. The proof of the following lemma is trivial, hence we omit it here.

Lemma 3.7 $\|\cdot\|_h$ defines a norm on the space $\mathcal{Q}_h \times \mathbf{U}_h \times \widehat{\mathbf{U}}_h \times P_h$.

By the definition of $\|\cdot\|_h$ in (28), for all $(\mathbf{r}, \mathbf{v}, \chi) \in \mathbf{L}^2(\Omega) \times \mathbf{H}_0(\text{curl}; \Omega) \times H_0^1(\Omega)$, it is easy to see that $\|(\mathbf{r}, \mathbf{v}, \mathbf{n} \times \mathbf{v} \times \mathbf{n}, p)\|_h$ is well defined since $\mathbf{n} \times (\mathbf{v} - \mathbf{n} \times \mathbf{v} \times \mathbf{n}) = \mathbf{0}$ on $\partial\mathcal{T}_h$. This will be used frequently in the error analysis.

Next, by the Cauchy-Schwarz inequality and the triangle inequality we have the following result showing the boundedness of $\mathcal{B}_h^\pm$.

Lemma 3.8 (Boundedness of $\mathcal{B}_h^\pm$) *Let $\sigma_h = (\mathbf{q}_h, \mathbf{u}_h, \widehat{\mathbf{u}}_h, p_h)$, $\tau_h = (\mathbf{r}_h, \mathbf{v}_h, \widehat{\mathbf{v}}_h, \chi_h) \in \mathcal{Q}_h \times \mathbf{U}_h \times \widehat{\mathbf{U}}_h \times P_h$, then we have*

$$|\mathcal{B}_h^\pm(\sigma_h; \tau_h)| \leq C \|\sigma_h\|_h \|\tau_h\|_h. \quad (29)$$

It is worth mentioning that the boundedness of $\mathcal{B}_h^\pm$ also holds if σ_h is replaced by $\sigma = (\mathbf{q}, \mathbf{u}, \mathbf{n} \times \mathbf{u} \times \mathbf{n}, p)$ where $(\mathbf{q}, \mathbf{u}, p) \in \mathbf{H}(\text{curl}; \Omega) \times \mathbf{H}_0(\text{curl}; \Omega) \times H_0^1(\Omega)$, because $\mathbf{n} \times (\mathbf{u} - \mathbf{n} \times \mathbf{u} \times \mathbf{n}) = \mathbf{0}$ on each face $F \in \mathcal{F}_h$. Then we have the following inequality

$$|\mathcal{B}_h^\pm(\sigma; \tau_h)| \leq C (\|\mathbf{q}\|_{\mathcal{T}_h} + \|\mathbf{u}\|_{\mathcal{T}_h} + \|\nabla \times \mathbf{u}\|_{\mathcal{T}_h} + \|\nabla p\|_{\mathcal{T}_h}) \|\tau_h\|_h. \quad (30)$$

By the inequalities (29) and (30), provided $\mathbf{u}_h \in \mathbf{H}_0(\text{curl}; \Omega) \cap \mathbf{U}_h$, we have

$$\begin{aligned} |\mathcal{B}_h^\pm(\sigma - \sigma_h; \tau_h)| &\leq C (\|\mathbf{q} - \mathbf{q}_h\|_{\mathcal{T}_h} + \|\mathbf{u} - \mathbf{u}_h\|_{\mathcal{T}_h} \\ &\quad + \|\nabla \times (\mathbf{u} - \mathbf{u}_h)\|_{\mathcal{T}_h} + \|\nabla(p - p_h)\|_{\mathcal{T}_h}) \|\tau_h\|_h. \end{aligned} \quad (31)$$

In preparation for the proof of the following lemma, for any $\mathbf{u}_h \in \mathbf{U}_h$ let $\mathbf{u}_h^c \in \mathbf{U}_h \cap \mathbf{H}_0(\text{curl}; \Omega)$ by the conforming function in Lemma 3.6, then by Lemma 3.4, there exist $\mathbf{y}_h \in \mathbf{U}_h \cap \mathbf{H}_0(\text{curl}; \Omega)$ and $\eta_h \in P_h$, such that for all $\chi_h \in P_h$ we have

$$\mathbf{u}_h^c = \mathbf{y}_h + \nabla \eta_h, \quad (\epsilon_r \mathbf{y}_h, \nabla \chi_h)_{\mathcal{T}_h} = 0. \quad (32)$$

We can now prove the discrete analogue of Lemma 2.7.

Lemma 3.9 (Discrete inf-sup condition) *For all $\sigma_h = (\mathbf{q}_h, \mathbf{u}_h, \widehat{\mathbf{u}}_h, p_h) \in \Sigma_h = \mathcal{Q}_h \times \mathbf{U}_h \times \widehat{\mathbf{U}}_h \times P_h$, we have the following inf-sup condition*

$$\sup_{\mathbf{0} \neq \tau_h \in \Sigma_h} \frac{\text{Re} [\mathcal{B}_h^+(\sigma_h; \tau_h)]}{\|\tau_h\|_h} \geq C \|\sigma_h\|_h. \quad (33)$$

Proof First, we take $\tau_1 = (\mathbf{q}_h, \mathbf{u}_h, \widehat{\mathbf{u}}_h, -p_h)$, then by the definition of $\mathcal{B}_h^+$ in (6), integration by parts and the definition of $\|\cdot\|_h$ in (28) we get

$$\begin{aligned} \text{Re} [\mathcal{B}_h^+(\sigma_h; \tau_1)] &= \|\sqrt{\text{Re}(\mu_r)} \mathbf{q}_h\|_{\mathcal{T}_h}^2 + \|\mathbf{h}^{-\frac{1}{2}} \mathbf{n} \times (\mathbf{u}_h - \widehat{\mathbf{u}}_h)\|_{\partial\mathcal{T}_h}^2 \\ &\quad + \|\kappa \sqrt{\text{Re}(\epsilon_r)} \mathbf{u}_h\|_{\mathcal{T}_h}^2, \end{aligned} \quad (34)$$

$$\|\tau_1\|_h = \|\sigma_h\|_h.$$

Second, we take $\tau_2 = (-\nabla \times \mathbf{u}_h, \mathbf{0}, \mathbf{0}, 0)$ and use Young's inequality and trace inequality to get

$$\begin{aligned} \text{Re} [\mathcal{B}_h^+(\sigma_h; \tau_2)] &= \text{Re} [-(\mu_r \mathbf{q}_h, \nabla \times \mathbf{u}_h)_{\mathcal{T}_h} + \|\nabla \times \mathbf{u}_h\|_{\mathcal{T}_h}^2 - \langle \mathbf{n} \times (\mathbf{u}_h - \widehat{\mathbf{u}}_h), \nabla \times \mathbf{u}_h \rangle_{\partial\mathcal{T}_h}] \\ &\geq -|(\mu_r \mathbf{q}_h, \nabla \times \mathbf{u}_h)_{\mathcal{T}_h}| + \|\nabla \times \mathbf{u}_h\|_{\mathcal{T}_h}^2 - |\langle \mathbf{n} \times (\mathbf{u}_h - \widehat{\mathbf{u}}_h), \nabla \times \mathbf{u}_h \rangle_{\partial\mathcal{T}_h}| \\ &\geq \frac{1}{2} \|\nabla \times \mathbf{u}_h\|_{\mathcal{T}_h}^2 - C_1 \left(\|\sqrt{\text{Re}(\mu_r)} \mathbf{q}_h\|_{\mathcal{T}_h}^2 + \|\mathbf{h}^{-\frac{1}{2}} \mathbf{n} \times (\mathbf{u}_h - \widehat{\mathbf{u}}_h)\|_{\partial\mathcal{T}_h}^2 \right), \end{aligned} \quad (35)$$

$$\|\tau_2\|_h \leq C_2 \|\sigma_h\|_h.$$

Third, we take $\tau_3 = (\mathbf{0}, \nabla p_h, \mathbf{n} \times \nabla p_h \times \mathbf{n}, 0)$ and use Young's inequality to get

$$\begin{aligned} \operatorname{Re} [\mathcal{B}_h^+(\sigma_h; \tau_3)] &= \|\sqrt{\operatorname{Re}(\epsilon_r)} \nabla p_h\|_{\mathcal{T}_h}^2 + \operatorname{Re}[(\kappa^2 \epsilon_r \mathbf{u}_h, \nabla p_h)_{\mathcal{T}_h}] \\ &\geq \frac{1}{2} \|\sqrt{\operatorname{Re}(\epsilon_r)} \nabla p_h\|_{\mathcal{T}_h}^2 - C_3 \|\kappa \sqrt{\operatorname{Re}(\epsilon_r)} \mathbf{u}_h\|_{\mathcal{T}_h}^2, \\ \|\tau_3\|_h &\leq C_4 \|\sigma_h\|_h. \end{aligned} \quad (36)$$

Next, we take $\tau_4 = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \eta_h)$ to get

$$\mathcal{B}_h^+(\sigma_h; \tau_4) = (\epsilon_r \mathbf{u}_h, \nabla \eta_h)_{\mathcal{T}_h}.$$

By (32), Young's inequality and the Lemma 3.6, we have

$$\begin{aligned} \operatorname{Re} [\mathcal{B}_h^+(\sigma_h; \tau_4)] &= \operatorname{Re}(\epsilon_r \mathbf{u}_h^c, \nabla \eta_h)_{\mathcal{T}_h} + \operatorname{Re}(\epsilon_r (\mathbf{u}_h - \mathbf{u}_h^c), \nabla \eta_h)_{\mathcal{T}_h} \\ &= \operatorname{Re}[(\epsilon_r (\mathbf{y}_h + \nabla \eta_h), \nabla \eta_h)_{\mathcal{T}_h}] + \operatorname{Re}[(\epsilon_r (\mathbf{u}_h - \mathbf{u}_h^c), \nabla \eta_h)_{\mathcal{T}_h}] \\ &= \operatorname{Re}[(\epsilon_r \nabla \eta_h, \nabla \eta_h)_{\mathcal{T}_h}] + \operatorname{Re}[(\epsilon_r (\mathbf{u}_h - \mathbf{u}_h^c), \nabla \eta_h)_{\mathcal{T}_h}] \\ &\geq \frac{1}{2} \|\sqrt{\operatorname{Re}(\epsilon_r)} \nabla \eta_h\|_{\mathcal{T}_h}^2 - C_5 \|\mathbf{h}^{\frac{1}{2}} \mathbf{n} \times \llbracket \mathbf{u}_h \rrbracket\|_{\mathcal{F}_h}^2 \\ &\geq \frac{1}{2} \|\sqrt{\operatorname{Re}(\epsilon_r)} \nabla \eta_h\|_{\mathcal{T}_h}^2 - C_5 \|\mathbf{h}^{\frac{1}{2}} \mathbf{n} \times (\mathbf{u}_h - \widehat{\mathbf{u}}_h)\|_{\partial \mathcal{T}_h}^2, \\ \|\tau_4\|_h &\leq C_6 \|\sigma_h\|_h. \end{aligned} \quad (37)$$

Finally, we take $\tau_h = (1 + C_1 + C_3 + C_5)\tau_1 + \sum_{i=2}^4 \tau_i$. By (34)-(37) we have

$$\begin{aligned} \operatorname{Re} [\mathcal{B}_h^+(\sigma_h; \tau_h)] &\geq C_7 \|\sigma_h\|_h^2, \\ \|\tau_h\|_h &\leq C_8 \|\sigma_h\|_h. \end{aligned} \quad (38)$$

The above two inequalities give the desired result (33). $\square$

4 Error Analysis

In this section, we give an error analysis of the HDG approximation to Maxwell's equations given by (13). First, we state main results, i.e., Theorem 4.1, Corollary 4.2. Second, we define a continuous operator $\mathcal{A}$ and discrete operator $\mathcal{A}_h$ as in [5], which is a crucial step to get the error estimate without a duality argument. In the end, we provide the proof of Theorem 4.1.

4.1 Main Result

Before stating our main result, we introduce the standard L^2 -orthogonal projection operator $\Pi_m^o : L^2(K) \rightarrow \mathcal{P}_m(K)$, which satisfies

$$(\Pi_m^o \mathbf{q}, \mathbf{r})_K = (\mathbf{q}, \mathbf{r})_K, \quad \forall \mathbf{r} \in \mathcal{P}_m(K). \quad (39)$$

In the error analysis, we shall use the following well-known bound:

$$\|\mathbf{q} - \Pi_m^o \mathbf{q}\|_{\mathcal{T}_h} \leq Ch^s \|\mathbf{q}\|_{H^s(\Omega)}, \quad (40)$$

where $\mathbf{q} \in H^s(\Omega)$ and $s > 0$ is the index guaranteed by Theorem 2.4.

Furthermore, $p \in H^{1+s}(\Omega)$, which means p may not be continuous on Ω . Therefore, the standard Lagrange interpolation operator is not applicable; hence we utilize the Scott-Zhang

interpolation operator [33] $\mathcal{I}_h : H^1(\Omega) \rightarrow P_h$. For the Scott-Zhang interpolant we have the following bound bound:

$$\|\nabla(p - \mathcal{I}_h p)\|_{\mathcal{T}_h} \leq Ch^s \|p\|_{H^{1+s}(\Omega)}. \quad (41)$$

Now we can state the main result:

Theorem 4.1 Suppose that μ_r and ϵ_r satisfy Assumption 1 and κ^2 is not an eigenvalue of (2). Then for h small enough,

- (A) The HDG formulation has a unique solution $(\mathbf{q}_h, \mathbf{u}_h, \widehat{\mathbf{u}}_h, p_h) \in \mathbf{Q}_h \times \mathbf{U}_h \times \widehat{\mathbf{U}}_h \times P_h$;
 (B) Let $(\mathbf{q}, \mathbf{u}, p)$ be the solution of (7), then we have

$$\begin{aligned} & \|\mathbf{q} - \mathbf{q}_h\|_{\mathcal{T}_h} + \|\mathbf{u} - \mathbf{u}_h\|_{\mathcal{T}_h} + \|\nabla(p - p_h)\|_{\mathcal{T}_h} \\ & \leq C \left(\|\Pi_m^o \mathbf{q} - \mathbf{q}\|_{\mathcal{T}_h} + \|\mathcal{J}_h^{\text{curl}} \mathbf{u} - \mathbf{u}\|_{\mathcal{T}_h} + \|\mathcal{J}_h^{\text{div}} (\nabla \times \mathbf{u}) - \nabla \times \mathbf{u}\|_{\mathcal{T}_h} \right. \\ & \quad \left. + \|\nabla(\mathcal{I}_h p - p)\|_{\mathcal{T}_h} \right), \end{aligned}$$

where C depends on κ , Ω , ϵ_r and μ_r . Here Π_m^o denotes L^2 projection (see (39)).

By the approximation properties of $\mathcal{J}_h^{\text{div}}$, $\mathcal{J}_h^{\text{curl}}$, Π_m^o and $\mathcal{I}_h$ in (20a), (20b), (40) and (41) and using the regularity result in Theorem 2.4, we have:

Corollary 4.2 For general coefficients satisfying Assumption 1, and assuming κ^2 is not a Maxwell eigenvalue, then

$$\|\mathbf{q} - \mathbf{q}_h\|_{\mathcal{T}_h} + \|\mathbf{u} - \mathbf{u}_h\|_{\mathcal{T}_h} + \|\nabla(p - p_h)\|_{\mathcal{T}_h} \leq Ch^s (\|\mathbf{f}\|_{L^2(\Omega)} + \|\nabla \cdot \mathbf{f}\|_{L^2(\Omega)}),$$

where $s \in (0, 1/2)$ and C depends on s , κ , Ω , ϵ_r and μ_r .

4.2 Preliminary Estimates

Definition 4.3 Let $\mathbf{g} \in L^2(\Omega)$, we define the operator $\mathcal{A} := (\mathcal{A}^q, \mathcal{A}^u, \mathcal{A}^p) : L^2(\Omega) \rightarrow \mathbf{H}(\text{curl}; \Omega) \times \mathbf{H}_0(\text{curl}; \Omega) \times H_0^1(\Omega)$ such that for all $\boldsymbol{\tau} = (\mathbf{r}, \mathbf{v}, \chi) \in \mathbf{H}(\text{curl}; \Omega) \times \mathbf{H}_0(\text{curl}; \Omega) \times H_0^1(\Omega)$ we have

$$\mathcal{B}^+(\mathcal{A}\mathbf{g}; \boldsymbol{\tau}) = (\epsilon_r \mathbf{g}, \mathbf{v})_{L^2(\Omega)}. \quad (42)$$

By Definition 4.3 and Lemma 2.6 in [7] we get

$$\|\mathcal{A}^q \mathbf{g}\|_{\mathbf{H}^s(\Omega)} + \|\mathcal{A}^u \mathbf{g}\|_{\mathbf{H}^s(\Omega)} \leq C \|\epsilon_r \mathbf{g}\|_{L^2(\Omega)}. \quad (43)$$

Moreover, if $\mathbf{g} \in \mathbf{H}(\text{div}_{\epsilon_r}, \Omega)$, then we have

$$\|\mathcal{A}^p \mathbf{g}\|_{H^{1+s}(\Omega)} \leq C \|\nabla \cdot (\epsilon_r \mathbf{g})\|_{L^2(\Omega)}. \quad (44)$$

Definition 4.4 Let $\mathbf{g} \in L^2(\Omega)$, we define the operator $\mathcal{A}_h := (\mathcal{A}_h^q, \mathcal{A}_h^u, \mathcal{A}_h^{\widehat{u}}, \mathcal{A}_h^p) : L^2(\Omega) \rightarrow \mathbf{Q}_h \times \mathbf{U}_h \times \widehat{\mathbf{U}}_h \times P_h$ such that

$$\mathcal{B}_h^+(\mathcal{A}_h \mathbf{g}; \boldsymbol{\tau}_h) = (\epsilon_r \mathbf{g}, \mathbf{v}_h)_{\mathcal{T}_h} \quad (45)$$

for all $\boldsymbol{\tau}_h = (\mathbf{r}_h, \mathbf{v}_h, \widehat{\mathbf{v}}_h, \chi_h)$. Moreover, we define $\mathbf{g}_h \in \mathbf{U}_h$ to be the unique solution of

$$(\epsilon_r \mathbf{g}_h, \mathbf{v}_h)_{\mathcal{T}_h} = (\mathbf{g}, \mathbf{v}_h)_{\mathcal{T}_h}.$$

Lemma 4.5 (Stability of $\mathcal{A}$ and $\mathcal{A}_h$) *Let $\mathbf{g} \in \mathbf{L}^2(\Omega)$, then we have*

$$\|\mathcal{A}\mathbf{g}\| \leq C \|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}, \quad (46a)$$

$$\|\mathcal{A}_h\mathbf{g}\|_h \leq C \|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}. \quad (46b)$$

where $\|\cdot\|$ and $\|\cdot\|_h$ were defined in (8) and (28), respectively.

Proof Let $\boldsymbol{\tau} = (\mathbf{r}, \mathbf{v}, \chi) \in \mathbf{L}^2(\Omega) \times \mathbf{H}_0(\text{curl}; \Omega) \times H_0^1(\Omega)$, by Lemma 2.7 and Definition 4.3 we get

$$\|\mathcal{A}\mathbf{g}\| \leq C \sup_{\boldsymbol{\tau} \neq \mathbf{0}} \frac{\text{Re} [\mathcal{B}^+(\mathcal{A}\mathbf{g}; \boldsymbol{\tau})]}{\|\boldsymbol{\tau}\|} = C \sup_{\boldsymbol{\tau} \neq \mathbf{0}} \frac{\text{Re} [(\epsilon_r \mathbf{g}, \mathbf{v})_{\mathbf{L}^2(\Omega)}]}{\|\boldsymbol{\tau}\|} \leq C \|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}.$$

Next, let $\boldsymbol{\tau}_h = (\mathbf{r}_h, \mathbf{v}_h, \widehat{\mathbf{v}}_h, \chi_h) \in \mathbf{Q}_h \times \mathbf{U}_h \times \widehat{\mathbf{U}}_h \times P_h$. We use Lemma 3.9 and Definition 4.3 to get

$$\|\mathcal{A}_h\mathbf{g}\|_h \leq C \sup_{\boldsymbol{\tau}_h \neq \mathbf{0}} \frac{\text{Re} [\mathcal{B}_h^+(\mathcal{A}_h\mathbf{g}; \boldsymbol{\tau}_h)]}{\|\boldsymbol{\tau}_h\|_h} = C \sup_{\boldsymbol{\tau}_h \neq \mathbf{0}} \frac{\text{Re} [(\epsilon_r \mathbf{g}; \mathbf{v}_h)_{\mathcal{I}_h}]}{\|\boldsymbol{\tau}_h\|_h} \leq C \|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}.$$

□

Next, we define

$$\widetilde{\mathcal{A}}\mathbf{g} = (\mathcal{A}^q \mathbf{g}, \mathcal{A}^u \mathbf{g}, \mathcal{A}^p \mathbf{g}), \quad (47)$$

where $\mathcal{A}\mathbf{g}$ was defined in (42). Then for all $\boldsymbol{\tau}_h = (\mathbf{r}_h, \mathbf{v}_h, \widehat{\mathbf{v}}_h, \chi_h) \in \mathbf{Q}_h \times \mathbf{U}_h \times \widehat{\mathbf{U}}_h \times P_h$, by the definition of $\mathcal{B}_h^+$ in (14), (42) and the definition of $\mathcal{B}^+$ in (9) we have

$$\mathcal{B}_h^+(\widetilde{\mathcal{A}}\mathbf{g}; \boldsymbol{\tau}_h) = (\epsilon_r \mathbf{g}, \mathbf{v}_h)_{\mathcal{I}_h} = \mathcal{B}_h^+(\mathcal{A}_h\mathbf{g}; \boldsymbol{\tau}_h). \quad (48)$$

Lemma 4.6 *For any $\mathbf{g} \in \mathbf{H}(\text{div}_{\epsilon_r}; \Omega)$, we have*

$$\|\widetilde{\mathcal{A}}\mathbf{g} - \mathcal{A}_h\mathbf{g}\|_h \leq Ch^s (\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)} + \|\nabla \cdot (\epsilon_r \mathbf{g})\|_{\mathbf{L}^2(\Omega)}). \quad (49)$$

Proof First, let $\mathcal{J}_h(\widetilde{\mathcal{A}}\mathbf{g}) = (\Pi_m^o(\mathcal{A}^q \mathbf{g}), \mathcal{J}_h^{\text{curl}}(\mathcal{A}^u \mathbf{g}), \mathbf{n} \times (\mathcal{J}_h^{\text{curl}}(\mathcal{A}^u \mathbf{g})) \times \mathbf{n},$

$\mathcal{I}_h(\mathcal{A}^p \mathbf{g})) \in \mathbf{Q}_h \times \mathbf{U}_h \times \widehat{\mathbf{U}}_h \times P_h$, where Π_m^o was defined in (39) and $\mathcal{I}_h$ is the Scott-Zhang interpolant. Then

$$\begin{aligned} & \|\mathcal{J}_h(\widetilde{\mathcal{A}}\mathbf{g}) - \mathcal{A}_h\mathbf{g}\|_h \\ & \leq C \sup_{\boldsymbol{\tau}_h \neq \mathbf{0}} \frac{\text{Re} [\mathcal{B}_h^+(\mathcal{J}_h(\widetilde{\mathcal{A}}\mathbf{g}) - \mathcal{A}_h\mathbf{g}; \boldsymbol{\tau}_h)]}{\|\boldsymbol{\tau}_h\|_h} && \text{by (33)} \\ & \leq C \sup_{\boldsymbol{\tau}_h \neq \mathbf{0}} \frac{\text{Re} [\mathcal{B}_h^+(\mathcal{J}_h(\widetilde{\mathcal{A}}\mathbf{g}) - \widetilde{\mathcal{A}}\mathbf{g}; \boldsymbol{\tau}_h)]}{\|\boldsymbol{\tau}_h\|_h} && \text{by (48)} \\ & \leq C \|\mathcal{J}_h(\widetilde{\mathcal{A}}\mathbf{g}) - \widetilde{\mathcal{A}}\mathbf{g}\|_h && \text{by (31).} \end{aligned} \quad (50)$$

Next by the triangle inequality we have

$$\begin{aligned} & \|\widetilde{\mathcal{A}}\mathbf{g} - \mathcal{A}_h\mathbf{g}\|_h \\ & \leq \|\widetilde{\mathcal{A}}\mathbf{g} - \mathcal{J}_h(\widetilde{\mathcal{A}}\mathbf{g})\|_h + \|\mathcal{J}_h(\widetilde{\mathcal{A}}\mathbf{g}) - \mathcal{A}_h\mathbf{g}\|_h \\ & \leq C \|\widetilde{\mathcal{A}}\mathbf{g} - \mathcal{J}_h(\widetilde{\mathcal{A}}\mathbf{g})\|_h && \text{by (50)} \\ & \leq C (\|\Pi_m^o(\mathcal{A}^q \mathbf{g}) - \mathcal{A}^q \mathbf{g}\|_{\mathcal{I}_h} + \|\mathcal{J}_h^{\text{curl}}(\mathcal{A}^u \mathbf{g}) - \mathcal{A}^u \mathbf{g}\|_{\mathcal{I}_h} \end{aligned}$$

$$\begin{aligned}
& + \|\nabla \times (\mathcal{J}_h^{\text{curl}}(\mathcal{A}^u \mathbf{g}) - \mathcal{A}^u \mathbf{g})\|_{\mathcal{T}_h} + \|\nabla(\mathcal{I}_h(\mathcal{A}^p \mathbf{g}) - \mathcal{A}^p \mathbf{g})\|_{\mathcal{T}_h} \quad \text{by (31)} \\
& = C \left(\|\Pi_m^o(\mathcal{A}^q \mathbf{g}) - \mathcal{A}^q \mathbf{g}\|_{\mathcal{T}_h} + \|\mathcal{J}_h^{\text{curl}}(\mathcal{A}^u \mathbf{g}) - \mathcal{A}^u \mathbf{g}\|_{\mathcal{T}_h} \right. \\
& \quad \left. + \|\mathcal{J}_h^{\text{div}}(\nabla \times (\mathcal{A}^u \mathbf{g})) - \nabla \times (\mathcal{A}^u \mathbf{g})\|_{\mathcal{T}_h} + \|\nabla(\mathcal{I}_h(\mathcal{A}^p \mathbf{g}) - \mathcal{A}^p \mathbf{g})\|_{\mathcal{T}_h} \right) \quad \text{by (21)}.
\end{aligned}$$

By the approximation properties of Π_m^o , $\mathcal{I}_h$ and $\mathcal{J}_h^{\text{div}}$ in (40), (41) and (20a) we get

$$\begin{aligned}
& \|\tilde{\mathcal{A}}\mathbf{g} - \mathcal{A}_h \mathbf{g}\|_h \\
& \leq Ch^s (\|\mathcal{A}^q \mathbf{g}\|_{H^s(\Omega)} + \|\mathcal{A}^u \mathbf{g}\|_{H^s(\Omega)} + \|\nabla \times (\mathcal{A}^u \mathbf{g})\|_{H^s(\Omega)} + \|\mathcal{A}^p \mathbf{g}\|_{H^{1+s}(\Omega)}) \\
& \leq Ch^s (\|\mathbf{g}\|_{L^2(\Omega)} + \|\nabla \cdot (\epsilon_r \mathbf{g})\|_{L^2(\Omega)}),
\end{aligned}$$

where we used the regularity results (43) and (44). $\square$

Next, we define the following norm on the space $\mathbf{W} = \mathbf{H}_0(\text{curl}; \Omega) + \mathbf{U}_h$ by

$$\|\mathbf{w}\|_{\mathbf{W}}^2 := \|\sqrt{\text{Re}(\epsilon_r)} \mathbf{w}\|_{\mathcal{T}_h}^2 + \|\nabla \times \mathbf{w}\|_{\mathcal{T}_h}^2 + \|\mathbf{h}^{-\frac{1}{2}} \mathbf{n} \times [\![\mathbf{w}]\!]\|_{\mathcal{F}_h}^2. \quad (51)$$

Lemma 4.7 For any $\mathbf{u}_h \in \mathbf{U}_h$, we have

$$\|\mathcal{A}^u \mathbf{u}_h - \mathcal{A}_h^u \mathbf{u}_h\|_{\mathbf{W}} \leq Ch^s \|\mathbf{u}_h\|_{\mathbf{W}}.$$

Proof Let $\mathbf{u}_h^c$ be defined as in Lemma 3.6, then by Lemma 3.4 we have

$$\mathbf{u}_h^c = \mathbf{z}_h + \nabla \xi_h, \quad (\epsilon_r \mathbf{z}_h, \nabla \eta_h)_{\mathcal{T}_h} = 0, \quad (52)$$

where $\eta_h \in P_h$ and $\|\nabla \xi_h\|_{\mathcal{T}_h} \leq C \|\mathbf{u}_h^c\|_{\mathcal{T}_h}$, $\|\mathbf{z}_h\|_{\mathcal{T}_h} \leq C \|\mathbf{u}_h^c\|_{\mathcal{T}_h}$. Let $\Theta \in \mathbf{H}_0(\text{curl}; \Omega) \cap \mathbf{H}(\text{div}_{\epsilon_r}^0; \Omega)$ be the solution of

$$\nabla \times \Theta = \nabla \times \mathbf{z}_h.$$

Then by (26a) in Lemma 3.5 we have

$$\|\Theta - \mathbf{z}_h\|_{\mathcal{T}_h} \leq Ch^s \|\nabla \times \mathbf{z}_h\|_{\mathcal{T}_h}. \quad (53)$$

Moreover, for all $\xi_h \in P_h$ we have $(\epsilon_r \mathbf{u}_h, \nabla \xi_h)_{\mathcal{T}_h} = 0$ and $(\epsilon_r \Theta, \nabla \xi_h)_{\mathcal{T}_h} = 0$, then

$$\begin{aligned}
\|\Theta - \mathbf{u}_h\|_{\mathcal{T}_h}^2 & \leq C \text{Re} [(\epsilon_r (\Theta - \mathbf{u}_h), \Theta - \mathbf{u}_h)_{\mathcal{T}_h}] \\
& \leq C \text{Re} [(\epsilon_r (\Theta - \mathbf{u}_h), \Theta - \mathbf{z}_h - \nabla \xi_h + \mathbf{u}_h^c - \mathbf{u}_h)_{\mathcal{T}_h}] \\
& \leq C \text{Re} [(\epsilon_r (\Theta - \mathbf{u}_h), \Theta - \mathbf{z}_h + \mathbf{u}_h^c - \mathbf{u}_h)_{\mathcal{T}_h}].
\end{aligned}$$

This gives

$$\|\Theta - \mathbf{u}_h\|_{\mathcal{T}_h} \leq C (\|\Theta - \mathbf{z}_h\|_{\mathcal{T}_h} + \|\mathbf{u}_h^c - \mathbf{u}_h\|_{\mathcal{T}_h}). \quad (54)$$

By the definitions of $\|\cdot\|_{\mathbf{W}}$ and $\|\cdot\|_h$ in (51) and (28), we have, for h small enough and $s \leq 1/2$,

$$\begin{aligned}
& \|(\mathcal{A}^u - \mathcal{A}_h^u) \mathbf{u}_h\|_{\mathbf{W}} \\
& \leq \|(\tilde{\mathcal{A}} - \mathcal{A}_h) \mathbf{u}_h\|_h \quad \text{by (47)} \\
& \leq \|(\tilde{\mathcal{A}} - \mathcal{A}_h)(\mathbf{u}_h - \Theta)\|_h + \|(\tilde{\mathcal{A}} - \mathcal{A}_h)\Theta\|_h \\
& \leq C \left(\|\mathbf{u}_h - \Theta\|_{L^2(\Omega)} + h^s \|\Theta\|_{L^2(\Omega)} \right) \quad \text{by (46a) and (49)} \\
& \leq C \left(\|\mathbf{u}_h - \Theta\|_{L^2(\Omega)} + h^s (\|\mathbf{u}_h - \mathbf{u}_h^c\|_{L^2(\Omega)} + \|\mathbf{u}_h^c\|_{L^2(\Omega)}) \right)
\end{aligned}$$

$$\begin{aligned}
&\leq C \left(\|\Theta - \mathbf{z}_h\|_{\mathcal{T}_h} + \|\mathbf{u}_h^c - \mathbf{u}_h\|_{\mathcal{T}_h} + h^s \|\mathbf{u}_h^c\|_{L^2(\Omega)} \right) && \text{by (54)} \\
&\leq Ch^s \left(\|\nabla \times \mathbf{z}_h\|_{L^2(\Omega)} + \|\mathbf{h}^{-\frac{1}{2}} \mathbf{n} \times \llbracket \mathbf{u}_h \rrbracket \|_{\mathcal{F}_h} + \|\mathbf{u}_h^c\|_{L^2(\Omega)} \right) && \text{by (53) and (27)} \\
&\leq Ch^s \left(\|\mathbf{h}^{-\frac{1}{2}} \mathbf{n} \times \llbracket \mathbf{u}_h \rrbracket \|_{\mathcal{F}_h} + \|\nabla \times \mathbf{u}_h^c\|_{\mathcal{F}_h} + \|\mathbf{u}_h^c\|_{L^2(\Omega)} \right) && \text{by (52)} \\
&\leq Ch^s \|\mathbf{u}_h\|_W && \text{by (27).}
\end{aligned}$$

□

Lemma 4.8 *If κ^2 is not an eigenvalue of the problem (2), then for all $\mathbf{w} \in \mathbf{W} = \mathbf{U}_h + \mathbf{H}_0(\text{curl}; \Omega)$, there exists a positive constant C only depending on Ω and κ such that,*

$$\left\| \frac{\mathbf{w}}{1 + \kappa^2} - \mathcal{A}^u \mathbf{w} \right\|_W \geq C \|\mathbf{w}\|_W.$$

Proof Let $z = 1/(1 + \kappa^2)$ and $\mathbf{g} = (z - \mathcal{A}^u)\mathbf{w}$, then $\mathbf{g} \in \mathbf{H}_0(\text{curl}; \Omega) + \mathbf{U}_h$. This implies $z\mathbf{w} - \mathbf{g} = \mathcal{A}^u \mathbf{w} \in \mathbf{H}_0(\text{curl}; \Omega)$. By the definition of $\mathcal{A}^u$ (see (4.3)), we know $\mathcal{A}^u \mathbf{w}$ satisfies the following equation:

$$\nabla \times (\mu_r^{-1} \nabla \times (\mathcal{A}^u \mathbf{w})) + \epsilon_r (\mathcal{A}^u \mathbf{w}) + \bar{\epsilon}_r \mathcal{A}^p \mathbf{w} = \epsilon_r \mathbf{w}.$$

Setting $\mathbf{w} = (1/z)(\mathcal{A}^u \mathbf{w} + \mathbf{g})$ on the right hand side gives

$$\nabla \times (\mu_r^{-1} \nabla \times (\mathcal{A}^u \mathbf{w})) - \kappa^2 \epsilon_r (\mathcal{A}^u \mathbf{w}) + \bar{\epsilon}_r \mathcal{A}^p \mathbf{w} = \frac{1}{z} \epsilon_r \mathbf{g},$$

and so $z\mathbf{w} - \mathbf{g} \in \mathbf{H}_0(\text{curl}; \Omega)$ satisfies the following equation:

$$\nabla \times (\mu_r^{-1} \nabla \times ((z\mathbf{w} - \mathbf{g}))) - \kappa^2 \epsilon_r (z\mathbf{w} - \mathbf{g}) + \bar{\epsilon}_r \mathcal{A}^p \mathbf{w} = \frac{1}{z} \epsilon_r \mathbf{g}. \quad (55)$$

Since κ^2 is not an eigenvalue of problem (2), then by the Theorem 2.4 we have

$$\|z\mathbf{w} - \mathbf{g}\|_{\mathbf{H}(\text{curl}; \Omega)} \leq \frac{C}{|z|} \|\epsilon_r \mathbf{g}\|_{L^2(\Omega)} \leq \frac{C}{|z|} \|\mathbf{g}\|_W. \quad (56)$$

Since $\|z\mathbf{w} - \mathbf{g}\|_{\mathbf{H}(\text{curl}; \Omega)} = \|z\mathbf{w} - \mathbf{g}\|_W$, then by (56) we have

$$\|\mathbf{w}\|_W \leq \frac{1}{|z|} (\|z\mathbf{w} - \mathbf{g}\|_W + \|\mathbf{g}\|_W) \leq C \|\mathbf{g}\|_W = C \|(z - \mathcal{A}^u)\mathbf{w}\|_W.$$

□

Lemma 4.9 *If κ^2 is not an eigenvalue of problem (2) and h is small enough, then for all $\mathbf{w} \in \mathbf{W} = \mathbf{U}_h + \mathbf{H}_0(\text{curl}; \Omega)$, there exists a positive constant C only depending on Ω and κ such that,*

$$\left\| \frac{\mathbf{w}}{1 + \kappa^2} - \mathcal{A}_h^u \mathbf{w} \right\|_W \geq C \|\mathbf{w}\|_W.$$

Proof Let $z = 1/(1 + \kappa^2)$, by Lemma 4.7 and 4.8 and the triangle inequality we have

$$\|(z - \mathcal{A}_h^u)\mathbf{w}\|_W \geq \|(z - \mathcal{A}^u)\mathbf{w}\|_W - \|(\mathcal{A}^u - \mathcal{A}_h^u)\mathbf{w}\|_W \geq (C - h^s) \|\mathbf{w}\|_W.$$

The desired result holds if h small enough. □

The next result follows from the coercivity proved in the previous lemma.

Corollary 4.10 *If κ^2 is not an eigenvalue of problem (2) and h is small enough, then for all $\mathbf{w}_h \in \mathbf{U}_h$, there exists a positive constant C only depending on Ω and κ such that,*

$$\left\| \left(\frac{1}{1+\kappa^2} - \mathcal{A}_h^u \right)^{-1} \mathbf{w}_h \right\|_{\mathbf{W}} \leq C \|\mathbf{w}_h\|_{\mathbf{W}}. \quad (57)$$

4.3 Proof of (A) in Theorem 4.1

Lemma 4.11 *If κ^2 is not an eigenvalue of problem (2) and h is small enough, then the HDG scheme (7) has a unique solution $\boldsymbol{\sigma}_h = (\mathbf{q}_h, \mathbf{u}_h, \widehat{\mathbf{u}}_h, p_h) \in \mathcal{Q}_h \times \mathbf{U}_h \times \widehat{\mathbf{U}}_h \times P_h$. Moreover, we have*

$$\|\boldsymbol{\sigma}_h\|_h \leq C \|\mathbf{f}\|_{L^2(\Omega)}. \quad (58)$$

Proof For any $\boldsymbol{\tau}_h = (\mathbf{r}_h, \mathbf{v}_h, \widehat{\mathbf{v}}_h, \chi_h) \in \mathcal{Q}_h \times \mathbf{U}_h \times \widehat{\mathbf{U}}_h \times P_h$, we have

$$\mathcal{B}_h^-(\boldsymbol{\sigma}_h; \boldsymbol{\tau}_h) = \mathcal{B}_h^+(\boldsymbol{\sigma}_h; \boldsymbol{\tau}_h) - (1+\kappa^2)(\epsilon_r \mathbf{u}_h, \mathbf{v}_h)_{\mathcal{T}_h}. \quad (59)$$

Then the HDG scheme (7) is equivalent to finding $\boldsymbol{\sigma}_h = (\mathbf{q}_h, \mathbf{u}_h, \widehat{\mathbf{u}}_h, p_h) \in \mathcal{Q}_h \times \mathbf{U}_h \times \widehat{\mathbf{U}}_h \times P_h$ such that

$$\frac{1}{1+\kappa^2} \mathcal{B}_h^+(\boldsymbol{\sigma}_h; \boldsymbol{\tau}_h) - (\epsilon_r \mathbf{u}_h, \mathbf{v}_h)_{\mathcal{T}_h} = \frac{1}{1+\kappa^2} (\mathbf{f}, \mathbf{v}_h)_{\mathcal{T}_h}. \quad (60)$$

By Definition 4.4, we have the following equations:

$$\mathcal{B}_h^+(\mathcal{A}_h \mathbf{u}_h; \boldsymbol{\tau}_h) = (\epsilon_r \mathbf{u}_h, \mathbf{v}_h)_{\mathcal{T}_h}, \quad (61a)$$

$$\frac{1}{1+\kappa^2} (\mathbf{f}, \mathbf{v}_h)_{\mathcal{T}_h} = \frac{1}{1+\kappa^2} (\epsilon_r \mathbf{f}_h, \mathbf{v}_h)_{\mathcal{T}_h} = \frac{1}{1+\kappa^2} \mathcal{B}_h^+(\mathcal{A}_h \mathbf{f}_h; \boldsymbol{\tau}_h). \quad (61b)$$

By (61), we can rewrite (60) as follows:

$$\mathcal{B}_h^+ \left(\frac{1}{1+\kappa^2} \boldsymbol{\sigma}_h - \mathcal{A}_h \mathbf{u}_h - \frac{1}{1+\kappa^2} \mathcal{A}_h \mathbf{f}_h; \boldsymbol{\tau}_h \right) = 0.$$

Due to the coercivity of $\mathcal{B}_h^+$ in Lemma 3.9, we have:

$$\frac{1}{1+\kappa^2} \boldsymbol{\sigma}_h - \mathcal{A}_h \mathbf{u}_h - \frac{1}{1+\kappa^2} \mathcal{A}_h \mathbf{f}_h = \mathbf{0},$$

i.e.,

$$\left(\frac{1}{1+\kappa^2} - \mathcal{A}_h^u \right) \mathbf{u}_h = \frac{1}{1+\kappa^2} \mathcal{A}_h^u \mathbf{f}_h, \quad (62)$$

which is uniquely solvable by Lemma 4.9. Moreover, by (57), we have:

$$\begin{aligned} \|\mathbf{u}_h\|_{\mathcal{T}_h} &\leq C \|\mathbf{u}_h\|_{\mathbf{W}} && \text{by (51)} \\ &\leq C \|\mathcal{A}_h^u \mathbf{f}_h\|_{\mathbf{W}} && \text{by (57) and (62)} \\ &\leq C \|\mathcal{A}_h \mathbf{f}_h\|_h && \text{by (28)} \\ &\leq C \|\mathbf{f}_h\|_{L^2(\Omega)} && \text{by (46b)} \\ &\leq C \|\mathbf{f}\|_{L^2(\Omega)} && \text{by (61b)}. \end{aligned} \quad (63)$$

Next, by (59) we get

$$\mathcal{B}_h^+(\sigma_h; \tau_h) = (1 + \kappa^2)(\epsilon_r \mathbf{u}_h, \mathbf{v}_h)_{\mathcal{T}_h} + (\mathbf{f}, \mathbf{v}_h)_{\mathcal{T}_h} \quad (64)$$

for all $\tau_h = (\mathbf{r}_h, \mathbf{v}_h, \widehat{\mathbf{v}}_h, \chi_h) \in \mathcal{Q}_h \times \mathbf{U}_h \times \widehat{\mathbf{U}}_h \times P_h$. We can now prove the uniqueness of any solution to the discrete HDG problem. Suppose $\mathbf{f} = \mathbf{0}$, then $\mathbf{u}_h = \mathbf{0}$ by (63), and so $\sigma_h = \mathbf{0}$ by (64) and Lemma 3.9. Since the linear system corresponding to the discrete HDG problem is square, uniqueness implies existence. Therefore, the HDG scheme (7) has a unique solution. Furthermore,

$$\begin{aligned} \|\sigma_h\|_h &\leq C \sup_{\tau_h \neq \mathbf{0}} \frac{\operatorname{Re} [\mathcal{B}_h^+(\sigma_h; \tau_h)]}{\|\tau_h\|_h} && \text{by (33)} \\ &= C \sup_{\tau_h \neq \mathbf{0}} \frac{\operatorname{Re} [(1 + \kappa^2)(\epsilon_r \mathbf{u}_h, \mathbf{v}_h)_{\mathcal{T}_h} + (\mathbf{f}, \mathbf{v}_h)_{\mathcal{T}_h}]}{\|\tau_h\|_h} && \text{by (64)} \\ &\leq C(\|\mathbf{u}_h\|_{L^2(\Omega)} + \|\mathbf{f}\|_{L^2(\Omega)}) \\ &\leq C\|\mathbf{f}\|_{L^2(\Omega)} && \text{by (63)}. \end{aligned}$$

□

4.4 Proof of (B) in Theorem 4.1

We first prove that a discrete inf-sup condition holds.

Lemma 4.12 (Discrete inf-sup conditions on $\mathcal{B}_h^-$) *Let $\sigma_h = (\mathbf{q}_h, \mathbf{u}_h, \widehat{\mathbf{u}}_h, p_h)$, $\tau_h = (\mathbf{r}_h, \mathbf{v}_h, \widehat{\mathbf{v}}_h, \chi_h) \in \Sigma_h = \mathcal{Q}_h \times \mathbf{U}_h \times \widehat{\mathbf{U}}_h \times P_h$, and suppose κ^2 is not a Maxwell eigenvalue and h small enough. Then we have following inf-sup condition*

$$\sup_{\mathbf{0} \neq \sigma_h \in \Sigma_h} \frac{\operatorname{Re} [\mathcal{B}_h^-(\sigma_h; \tau_h)]}{\|\tau_h\|_h} \geq C \|\sigma_h\|_h. \quad (65)$$

Proof By (38) in the proof of Lemma 3.9, there exists $\tau_1 = (\mathbf{r}_1, \mathbf{v}_1, \widehat{\mathbf{v}}_1, \chi_1) \in \Sigma_h$ such that

$$\begin{aligned} \operatorname{Re} [\mathcal{B}_h^-(\sigma_h; \tau_1)] &\geq C_1 \|\sigma_h\|_h^2, \\ \|\tau_1\|_h &\leq C_2 \|\sigma_h\|_h. \end{aligned}$$

This is equivalent to

$$\operatorname{Re} [\mathcal{B}_h^-(\sigma_h; \tau_1) + (1 + \kappa^2)(\epsilon_r \mathbf{u}_h, \mathbf{v}_1)_{\mathcal{T}_h}] \geq C_1 \|\sigma_h\|_h^2.$$

By the Cauchy-Schwarz inequality we have

$$\operatorname{Re} [\mathcal{B}_h^-(\sigma_h; \tau_1)] \geq C_3 \|\sigma_h\|_h^2 - C_4(1 + \kappa^2) \|\epsilon_r\|_{L^\infty(\Omega)} \|\mathbf{u}_h\|_{\mathcal{T}_h}^2. \quad (66)$$

For $\mathbf{u}_h \in \mathbf{U}_h$, we may choose $\mathbf{u}_h^c \in \mathbf{U}_h \cap \mathbf{H}_0(\operatorname{curl}; \Omega)$ so that the estimate in Lemma 3.6, is satisfied. Then by Lemma 3.4, there exist $\mathbf{z}_h \in \mathbf{U}_h \cap \mathbf{H}_0(\operatorname{curl}; \Omega)$ and $\xi_h \in P_h$, such that for all $\chi_h \in P_h$ we have

$$\mathbf{u}_h^c = \mathbf{z}_h + \nabla \xi_h, \quad (\epsilon_r \mathbf{z}_h, \nabla \chi_h)_{\mathcal{T}_h} = 0. \quad (67)$$

Let $\Theta \in \mathbf{H}_0(\operatorname{curl}; \Omega) \cap \mathbf{H}(\operatorname{div}_{\epsilon_r}^0; \Omega)$ be the solution of

$$\nabla \times \Theta = \nabla \times \mathbf{z}_h.$$

Then by (26a) we have

$$\begin{aligned}
 \|\Theta - \mathbf{z}_h\|_{\mathcal{T}_h} &\leq Ch^s \|\nabla \times \mathbf{z}_h\|_{\mathcal{T}_h} \\
 &= Ch^s \|\nabla \times \mathbf{u}_h^c\|_{\mathcal{T}_h} && \text{by (67)} \\
 &\leq Ch^s (\|\nabla \times (\mathbf{u}_h^c - \mathbf{u}_h)\|_{\mathcal{T}_h} + \|\nabla \times \mathbf{u}_h\|_{\mathcal{T}_h}) \\
 &\leq Ch^s (\|\mathbf{h}^{-\frac{1}{2}} \mathbf{n} \times \llbracket \mathbf{u}_h \rrbracket\|_{\mathcal{F}_h} + \|\nabla \times \mathbf{u}_h\|_{\mathcal{T}_h}) && \text{by Lemma 3.6.}
 \end{aligned} \tag{68}$$

Let $\tau_2 = (\mathbf{r}_2, \mathbf{v}_2, \widehat{\mathbf{v}}_2, \chi_2) \in \Sigma_h$ be the solution of

$$\mathcal{B}_h^-(\tau_2; \tau_h) = (\Theta, \epsilon_r \mathbf{v}_h)_{\mathcal{T}_h}$$

guaranteed by Lemma 4.11. This implies

$$\mathcal{B}_h^-(\tau_2; \sigma_h) = (\Theta, \epsilon_r \mathbf{u}_h)_{\mathcal{T}_h}. \tag{69}$$

Furthermore, by (58) we have

$$\|\tau_2\|_h \leq C \|\Theta\|_{\mathcal{T}_h}. \tag{70}$$

Next, we take $\tau_2^* = (-\mathbf{r}_2, \mathbf{v}_2, \widehat{\mathbf{v}}_2, -\chi_2)$ and $\tau_3 = (-\mathbf{q}_h, \mathbf{u}_h, \widehat{\mathbf{u}}_h, -p_h)$ to get

$$\begin{aligned}
 \mathcal{B}_h^-(\sigma_h, \tau_2^*) &= \overline{\mathcal{B}_h^-(\tau_2; \tau_3)} && \text{by (17)} \\
 &= \overline{(\Theta, \epsilon_r \mathbf{u}_h)_{\mathcal{T}_h}} && \text{by (69)} \\
 &= (\epsilon_r \mathbf{u}_h, \Theta)_{\mathcal{T}_h} \\
 &= (\epsilon_r \mathbf{u}_h, (\Theta - \mathbf{z}_h - \nabla \xi_h))_{\mathcal{T}_h} + (\epsilon_r \mathbf{u}_h, \mathbf{u}_h^c - \mathbf{u}_h)_{\mathcal{T}_h} + (\epsilon_r \mathbf{u}_h, \mathbf{u}_h)_{\mathcal{T}_h} && \text{by (67)} \\
 &= (\epsilon_r \mathbf{u}_h, \Theta - \mathbf{z}_h)_{\mathcal{T}_h} + (\epsilon_r \mathbf{u}_h, (\mathbf{u}_h^c - \mathbf{u}_h))_{\mathcal{T}_h} + (\epsilon_r \mathbf{u}_h, \mathbf{u}_h)_{\mathcal{T}_h} && \text{by (13c).}
 \end{aligned}$$

Then by the Lemma 3.6, (68), the Cauchy-Schwarz inequality and Young's inequality, we have

$$\begin{aligned}
 \text{Re} [\mathcal{B}_h^-(\sigma_h; \tau_2^*)] &\geq \frac{1}{2} \|\sqrt{\text{Re}(\epsilon_r)} \mathbf{u}_h\|_{\mathcal{T}_h}^2 - Ch^{2s} \left(\|\mathbf{h}^{-\frac{1}{2}} \mathbf{n} \times \llbracket \mathbf{u}_h \rrbracket\|_{\mathcal{F}_h}^2 + \|\nabla \times \mathbf{u}_h\|_{\mathcal{T}_h}^2 \right) \\
 &= \frac{1}{2} \|\sqrt{\text{Re}(\epsilon_r)} \mathbf{u}_h\|_{\mathcal{T}_h}^2 - Ch^{2s} \left(\|\mathbf{h}^{-\frac{1}{2}} \mathbf{n} \times \llbracket \mathbf{u}_h - \widehat{\mathbf{u}}_h \rrbracket\|_{\partial \mathcal{T}_h}^2 + \|\nabla \times \mathbf{u}_h\|_{\mathcal{T}_h}^2 \right) \\
 &\geq \frac{1}{2} \|\sqrt{\text{Re}(\epsilon_r)} \mathbf{u}_h\|_{\mathcal{T}_h}^2 - Ch^{2s} \|\sigma_h\|_h^2,
 \end{aligned} \tag{71}$$

where we used the definition of $\|\cdot\|_h$ in (28).

Finally, we take $\tau_h = \tau_1 + \frac{2C_4(1+\kappa^2)\|\epsilon_r\|_{L^\infty(\Omega)}}{\bar{\epsilon}_r} \tau_2^*$. Then by (66), (71) and letting h be small enough we get the desired result. $\square$

Our final lemma gives the desired error estimate:

Lemma 4.13 *Let μ_r and ϵ_r satisfy Assumption 1. Suppose κ^2 is not an eigenvalue of problem (2). Let $(\mathbf{q}, \mathbf{u}, p) \in \mathbf{H}(\text{curl}; \Omega) \times \mathbf{H}_0(\text{curl}; \Omega) \times H_0^1(\Omega)$ and $\sigma_h = (\mathbf{q}_h, \mathbf{u}_h, \widehat{\mathbf{u}}_h, p_h) \in \Sigma_h = \mathcal{Q}_h \times \mathbf{U}_h \times \widehat{\mathbf{U}}_h \times P_h$ be the solution of (7) and (15), respectively. Then for h small enough, we have*

$$\begin{aligned}
 &\|\mathbf{q} - \mathbf{q}_h\|_{\mathcal{T}_h} + \|\mathbf{u} - \mathbf{u}_h\|_{\mathcal{T}_h} + \|\nabla(p - p_h)\|_{\mathcal{T}_h} \\
 &\leq C \left(\|\Pi_m^o \mathbf{q} - \mathbf{q}\|_{\mathcal{T}_h} + \|\mathcal{J}_h^{\text{curl}} \mathbf{u} - \mathbf{u}\|_{\mathcal{T}_h} + \|\mathcal{J}_h^{\text{div}} (\nabla \times \mathbf{u}) - \nabla \times \mathbf{u}\|_{\mathcal{T}_h} \right. \\
 &\quad \left. + \|\nabla(\mathcal{I}_h p - p)\|_{\mathcal{T}_h} \right),
 \end{aligned}$$

where C depends on κ , Ω , ϵ_r and μ_r .

Proof First, let $\sigma = (\mathbf{q}, \mathbf{u}, p)$ and $\mathcal{J}_h \sigma = (\Pi_m^o \mathbf{q}, \mathcal{J}_h^{\text{curl}} \mathbf{u}, \mathbf{n} \times \mathcal{J}_h^{\text{curl}} \mathbf{u} \times \mathbf{n}, \mathcal{I}_h p)$, where Π_m^o and $\mathcal{I}_h$ denote the standard L^2 projection and the Scott-Zhang interpolation, respectively. Then

$$\|\mathcal{J}_h \sigma - \sigma_h\|_h \leq C \sup_{\mathbf{0} \neq \boldsymbol{\tau}_h \in \Sigma_h} \frac{\text{Re} [\mathcal{B}_h^-(\mathcal{J}_h \sigma - \sigma_h; \boldsymbol{\tau}_h)]}{\|\boldsymbol{\tau}_h\|_h} \quad \text{by (65)}$$

$$= C \sup_{\mathbf{0} \neq \boldsymbol{\tau}_h \in \Sigma_h} \frac{\text{Re} [\mathcal{B}_h^-(\mathcal{J}_h \sigma - \sigma; \boldsymbol{\tau}_h)]}{\|\boldsymbol{\tau}_h\|_h} \quad \text{by (16)}$$

$$\leq C (\|\Pi_m^o \mathbf{q} - \mathbf{q}\|_{\mathcal{T}_h} + \|\mathcal{J}_h^{\text{curl}} \mathbf{u} - \mathbf{u}\|_{\mathcal{T}_h} + \|\nabla \times (\mathcal{J}_h^{\text{curl}} \mathbf{u} - \mathbf{u})\|_{\mathcal{T}_h} + \|\nabla(\mathcal{I}_h p - p)\|_{\mathcal{T}_h}) \quad \text{by (31)}$$

$$= C (\|\Pi_m^o \mathbf{q} - \mathbf{q}\|_{\mathcal{T}_h} + \|\mathcal{J}_h^{\text{curl}} \mathbf{u} - \mathbf{u}\|_{\mathcal{T}_h} + \|\mathcal{J}_h^{\text{div}}(\nabla \times \mathbf{u}) - \nabla \times \mathbf{u}\|_{\mathcal{T}_h} + \|\nabla(\mathcal{I}_h p - p)\|_{\mathcal{T}_h}) \quad \text{by (21)}.$$

By the definition of $\|\cdot\|_h$ in (28) we have

$$\begin{aligned} & \|\Pi_m^o \mathbf{q} - \mathbf{q}_h\|_{\mathcal{T}_h} + \|\mathcal{J}_h^{\text{curl}} \mathbf{u} - \mathbf{u}_h\|_{\mathcal{T}_h} + \|\nabla(\mathcal{I}_h p - p)\|_{\mathcal{T}_h} \\ & \leq C (\|\Pi_m^o \mathbf{q} - \mathbf{q}\|_{\mathcal{T}_h} + \|\mathcal{J}_h^{\text{curl}} \mathbf{u} - \mathbf{u}\|_{\mathcal{T}_h} + \|\mathcal{J}_h^{\text{div}}(\nabla \times \mathbf{u}) - \nabla \times \mathbf{u}\|_{\mathcal{T}_h} \\ & \quad + \|\nabla(\mathcal{I}_h p - p)\|_{\mathcal{T}_h}). \end{aligned}$$

Combined with the triangle inequality we get the desired result. $\square$

5 Numerical Experiments

In this section, we present two numerical tests of the HDG-CG method for Maxwell's equations. The domain of the following two examples is the unit cube $\Omega = (0, 1) \times (0, 1) \times (0, 1)$.

Example 1 We first test the convergence rate of the method for Maxwell's equations with wave number $\kappa = 1$ when the coefficients are piecewise smooth but the solution is smooth. More specifically, the data is chosen as

$$\mu_r = \begin{cases} 0.2 - 0.4i, & x < 0.5, \\ 0.25 - 0.25i, & x \geq 0.5, \end{cases} \quad \epsilon_r = \begin{cases} 1 + 2i, & x < 0.5, \\ 2 + 2i, & x \geq 0.5, \end{cases}$$

$$\mathbf{u} = [u_1, u_2, u_3]^T, \quad u_1 = 1, \quad u_2 = (x - 0.5)^2 z, \quad u_3 = (x - 0.5)^2 y, \quad p = 0.$$

The source term is chosen to match the exact solution of Eq. 1 and the approximation errors are listed in Table 2. In this case the duality approach would need to handle discontinuous coefficients which limit the regularity of the dual solution to H^s with $s < 1/2$. Our analysis covers this case. Since the chosen true solution is smooth, we expect optimal order convergence as is seen in Table 2.

Example 2 Next, we test the convergence rate of the method for the Maxwell's equations with wave number $\kappa = 1$, when both the coefficients and the exact solution are piecewise smooth. More specifically, the data is chosen as

Table 2 History of convergence for Example 1

k	$\frac{h}{\sqrt{3}}$	$\ \mathbf{q} - \mathbf{q}_h\ _{L^2(\Omega)} / \ \mathbf{q}\ _{L^2(\Omega)}$		$\ \mathbf{u} - \mathbf{u}_h\ _{L^2(\Omega)} / \ \mathbf{u}\ _{L^2(\Omega)}$	
		Error	Rate	Error	Rate
1	1/2	5.10E-01		3.05E-01	
	1/4	2.75E-01	0.89	8.74E-02	1.80
	1/8	1.42E-01	0.95	2.32E-02	1.91
	1/16	7.22E-02	0.98	6.02E-03	1.95
	1/20	5.79E-02	0.99	3.88E-03	1.96
2	1/2	8.05E-02		2.45E-02	
	1/4	2.12E-02	1.93	3.05E-03	3.00
	1/8	5.42E-03	1.96	3.84E-04	2.99
	1/12	2.43E-03	1.98	1.14E-04	2.99

Table 3 History of convergence for Example 2

k	$\frac{h}{\sqrt{3}}$	$\ \mathbf{q} - \mathbf{q}_h\ _{L^2(\Omega)} / \ \mathbf{q}\ _{L^2(\Omega)}$		$\ \mathbf{u} - \mathbf{u}_h\ _{L^2(\Omega)} / \ \mathbf{u}\ _{L^2(\Omega)}$	
		Error	Rate	Error	Rate
1	2	5.10E-01		1.53E-01	
	4	2.75E-01	0.89	4.38E-02	1.80
	8	1.42E-01	0.95	1.16E-02	1.91
	16	7.22E-02	0.98	3.02E-03	1.95
	20	5.79E-02	0.99	1.95E-03	1.96
2	2	8.05E-02		1.23E-02	
	4	2.12E-02	1.93	1.53E-03	3.00
	8	5.42E-03	1.96	1.93E-04	2.99
	12	2.43E-03	1.98	5.73E-05	2.99

$$\mu_r = \begin{cases} 0.2 - 0.4i, & x < 0.5, \\ 0.25 - 0.25i, & x \geq 0.5, \end{cases} \quad \epsilon_r = \begin{cases} 1 + 2i, & x < 0.5, \\ 2 + 2i, & x \geq 0.5, \end{cases}$$

$$\mathbf{u} = [u_1, u_2, u_3]^T, \quad u_1 = \begin{cases} 2 & x < 0.5, \\ 1, & x \geq 0.5, \end{cases}, \quad u_2 = (x - 0.5)^2 z, \\ u_3 = (x - 0.5)^2 y, \quad p = 0.$$

and the source term is chosen to match the exact solution of Eq. 1 and the approximation errors are listed in Table 3. In this case the solution is piecewise analytic and the mesh is chosen so that the surface of discontinuity $x = 0.5$ is a union of faces in the mesh. Thus standard error estimates for polynomial interpolation applied tetrahedron by tetrahedron give an optimal error estimate. This is confirmed in Table 3.

6 Conclusion

We have proved that the HDG-CG method for the time harmonic Maxwell system converges even in the presence of general piecewise smooth coefficients, as are usually encountered in practical applications. Our numerical results (see [7]) suggest that the method is stable even when $\kappa = 0$ so that the use of expanded HDG spaces for the Lagrange multiplier p is not needed, and a CG space is sufficient.

We expect that the method of proof given in our paper will be useful for other HDG methods which are intended for use on heterogeneous media. The dependence of the coefficients in the estimates on the wave number κ was not traced, and this should be done in the future. However the simple model problem used here would need to be revised to have Robin type boundary conditions (or other boundary conditions) in which the dependency of the solution of continuous problem on κ is known.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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