SOME SMOOTH CIRCLE AND CYCLIC GROUP ACTIONS ON EXOTIC SPHERES

J.D. QUIGLEY

ABSTRACT. Classical work of Lee, Schultz, and Stolz relates the smooth transformation groups of exotic spheres to the stable homotopy groups of spheres. In this note, we apply recent progress on the latter to deduce the existence of smooth circle and cyclic group actions on certain exotic spheres.

CONTENTS

1. INTRODUCTION

One appealing feature of spheres is their high degree of symmetry. For example, for all $n \geq 1$, the n-sphere S^n equipped with its standard smooth structure has smooth rotational symmetry, i.e., it supports a nontrivial smooth action of the circle group $\mathbb{T}$. More generally, by regarding S^n as the unit sphere in $\mathbb{R}^{n+1}$, we see that S^n admits a smooth $SO(n+1)$-action. In a precise sense, cf. [Str94, Sec. 0], this implies that spheres are the "most symmetric" of all simply connected smooth manifolds.

However, if we consider *exotic* n-spheres, i.e., spheres which are homeomorphic but not diffeomorphic to S^n, then this high degree of symmetry is usually not present, cf. [Hsi67, HH67, HH69, LY74, Str94]. In 1985, Schultz [Sch85] highlighted the following questions concerning the rotational symmetry of exotic spheres:

Question 1.1 ([Sch85])**.** Let Σ^n be an exotic n-sphere, $n \geq 5$. Does Σ^n support a nontrivial smooth $\mathbb{T}$-action? Does Σ^n support a nontrivial smooth $\mathbb{Z}/p$-action for every prime p?

Bredon [Bre67], Schultz [Sch75], and Joseph [Jos81] have produced examples of nontrivial smooth $\mathbb{T}$-actions on certain $(8k+1)$- and $(8k+2)$-dimensional exotic spheres. Schultz has also shown that every 8- and 10-dimensional exotic sphere admits a smooth semi-free $\mathbb{T}$-action with fixed point set S^4 [Sch72] and that for each odd prime p, a certain $(2p^2-2p-2)$-dimensional exotic sphere admits a nontrivial smooth $\mathbb{T}$-action [Sch73]; related examples are discussed in [Sch85], cf. Corollary 3.3. Nonexistence results, as well as the existence of some nontrivial smooth $\mathbb{Z}/p$-actions, are discussed in [Sch78].

The purpose of this note is to provide some positive answers to Question 1.1. Our approach is as follows. Classical results of Lee and Schultz (Theorem 1.2) and Schultz and Stolz (Theorem 1.5) link Question 1.1 to the stable homotopy groups of spheres, the Kervaire–Milnor sequences [KM63], and the Mahowald invariant [MR93]. Using recent progress on the stable homotopy groups of spheres and Mahowald invariants, we are able to prove the existence of nontrivial smooth $\mathbb{T}$- and $\mathbb{Z}/p$-actions on many exotic spheres up to dimension 100. We also describe arithmetic conditions on the pair (n,p) under which every exotic n-sphere admits a smooth free $\mathbb{Z}/p$-action.

The first result we will use was proven by C.N. Lee in [Lee68] and refined by Schultz in [Sch85]:

Theorem 1.2 ([Lee68], [Sch85, Thm. 1.9])**.** *Let Σ^n be an exotic sphere. For all but finitely many primes p, Σ^n supports a nontrivial smooth $\mathbb{Z}/p$-action. More precisely, Σ^n admits a free smooth $\mathbb{Z}/p$-action whenever p is prime to the order of Σ^n in Θ_n.*

Here, Θ_n is the group of h-cobordism classes of homotopy n-spheres introduced by Kervaire and Milnor in [KM63]. The group Θ_n sits in exact sequences with Θ_n^{bp}, the subgroup of h-cobordism classes of homotopy n-spheres which bound stably parallelizable manifolds, and $\operatorname{coker}(J_n)$, the cokernel of the J-homomorphism.

In Section 2, we apply results on the order of Θ_n^{bp} from [KM63, Lev85] and $\operatorname{coker} J_n$ from [BHHM20, BMQ22, IWX20, Rav86] to study the order of Θ_n. In Proposition 2.6 and Corollary 2.9, we provide arithmetic conditions on the dimension n and odd prime p under which every exotic n-sphere admits a smooth free $\mathbb{Z}/p$-action. In Appendix A, we compute the prime factors of Θ_n for all $n \leq 100$; combined with Theorem 1.2, this allows us to deduce the existence of nontrivial smooth $\mathbb{Z}/p$-actions on many exotic spheres:

Theorem 1.3. *The prime factors of Θ_n^{bp} and $\operatorname{coker} J_n$ are listed for $n \leq 100$ in Appendix A. Every exotic n-sphere admits a smooth free $\mathbb{Z}/p$-action for each prime p which does not appear in row n.*

Example 1.4. The prime factors of $\operatorname{coker}(J_{23})$ are 2 and 3, and the prime factors of Θ_{23}^{bp} are 2, 23, 89, and 691. Therefore each exotic 23-sphere admits a smooth free $\mathbb{Z}/p$-action for all primes $p \notin \{2, 3, 23, 89, 691\}$.

Theorem 1.2 leaves open the question of nontrivial smooth $\mathbb{Z}/p$-actions on exotic n-spheres whose order in Θ_n is divisible by p. Using beautiful results of Schultz and Stolz, it is sometimes possible to prove that such exotic spheres admit nontrivial smooth $\mathbb{T}$- or $\mathbb{Z}/p$-actions. To state their results, let $\mathcal{P}(-)$ denote the Pontryagin–Thom construction identifying framed bordism classes of stably framed smooth n-manifolds with classes in the n-th stable homotopy group of spheres π_n^s (cf. [Kos93, Sec. IX.5]), and let $M(-)$ be the Mahowald invariant [MR93] which associates a nontrivial coset in the stable homotopy groups of spheres to any nontrivial element in the stable homotopy groups of spheres.

Theorem 1.5 ([Sch85, Thm. 3.7] for p odd, [Sto88, Thm. D] for $p = 2$)**.**

(1) *Let p be an odd prime. Let $\alpha \neq 0 \in (\pi_n^s)_{(p)}$ and $\beta \in M(\alpha) \in (\operatorname{coker} J_m)_{(p)}$. Suppose Σ_0 is a framed sphere with $\mathcal{P}(\Sigma_0) = \alpha$.*
   (a) *If $m - n$ is even, then there is an exotic sphere Σ_1 such that $\mathcal{P}(\Sigma_1) = \beta$ and Σ_1 admits a smooth $\mathbb{Z}/p$-action with fixed point set Σ_0.*
   (b) *If $m - n$ is odd, then there is an exotic sphere Σ_1 such that $\mathcal{P}(\Sigma_1) = \beta$ and Σ_1 admits a smooth $\mathbb{T}$-action with an $(n-1)$-dimensional fixed point set.*

(2) *Let* $\alpha \neq 0 \in (\pi_n^s)_{(2)}$ *and* $\beta \in M(\alpha) \in (\operatorname{coker} J_m)_{(2)}$. *Suppose* Σ_0 *is a framed sphere with* $\mathcal{P}(\Sigma_0) = \alpha$ *and let* Σ_1 *be an exotic sphere with* $\mathcal{P}(\Sigma_1) = \beta$. *If* $m \geq 2n+1$ *and either* m *and* $m-n$ *are both odd or* $m-n$ *is even and* $m \equiv 1 \mod 4$, *then there exists a smooth* $\mathbb{Z}/2$*-action on the connected sum* $\Sigma_1 \# \Sigma'$, $\Sigma' \in \Theta_{m+1}^{bp}$, *with fixed point set* Σ_0.

Remark 1.6. We believe there is a slight ambiguity in [Sch85, Thm. 3.7]. On page 248 of *loc. cit.*, Schultz takes the Pontryagin–Thom construction to be a map from Θ_n/Θ_n^{bp} to $\operatorname{coker} J_n$ and defines the Mahowald invariant (page 259, *loc. cit*) only on elements in the quotient group $\operatorname{coker} J$. However, the Pontryagin–Thom isomorphism and Mahowald invariant apply to elements in the stable homotopy groups of spheres before quotienting.

We believe a more applicable version of [Sch85, Thm. 3.7] appears as [Sto88, Thm. C]. There, one begins with a pair of framed spheres Σ_0 and Σ_1, identifies them with elements in the stable homotopy groups of spheres via the Pontryagin–Thom construction, and assumes that $\Sigma_1 \in M(\Sigma_0)$. In particular, Schultz's principal example [Sch85, Ex. 3.8] works under these weaker hypotheses.

Remark 1.7. We will not use it here, but it is worth mentioning that a partial converse to Theorem 1.5 also appears in the work of Stolz [Sto88, Thm. D]. If Σ^m is a homotopy sphere with a smooth $\mathbb{Z}/2$-action with fixed point set some homotopy sphere Σ^n, then there exist framings a and b of Σ^m and Σ^n, respectively, such that $\mathcal{P}([\Sigma^m, a]) \in M(\mathcal{P}([\Sigma^n, b]))$ or $[\Sigma^m, a] = 0$.

In Section 3, we recall many Mahowald invariant computations which were made after the original appearance of Theorem 1.5 and apply them to prove the existence of nontrivial smooth $\mathbb{T}$- and $\mathbb{Z}/p$-actions on certain exotic spheres whose order in Θ_n is divisible by p.

Theorem 1.8. *Some combinations of dimensions* n *and primes* p *for which Theorem 1.5 implies the existence of a nontrivial smooth* $\mathbb{T}$*- or* $\mathbb{Z}/p$*-action on an exotic* n*-sphere whose order in* Θ_n *is divisible by* p *are marked with an asterisk* $*$ *in Appendix A.*

Example 1.9. In Example 1.4, we mentioned that Theorem 1.2 cannot be used to produce a nontrivial smooth $\mathbb{Z}/3$-action on certain exotic 23-spheres since 3 divides the order of Θ_{23}. However, Behrens [Beh06] showed that

$$M(\alpha_2) \doteq \beta_1^2 \alpha_1,$$

where $\alpha_2 \in (\pi_7^s)_{(3)}$ and $\beta_1^2\alpha_1 \in (\pi_{23}^s)_{(3)}$. Applying the odd-primary part of Theorem 1.5, we find that the exotic 23-sphere corresponding to $\beta_1^2\alpha_1$, whose order in Θ_{23} is divisible by 3, supports a nontrivial smooth $\mathbb{Z}/3$-action with fixed points the standard 7-sphere.

Remark 1.10. Even in low dimensions, Theorem 1.2 and Theorem 1.5 cannot be used to deduce the existence of smooth nontrivial $\mathbb{Z}/p$-actions on every exotic sphere. For example, we cannot deduce the existence of a smooth nontrivial $\mathbb{Z}/3$-action on certain exotic 10-spheres using these results. Interestingly, Schultz suggests in [Sch78, Sec. 4] that *every* exotic sphere admits a smooth nontrivial $\mathbb{Z}/p$-action for every prime p.

Some suggestions for future work and further remarks on the limitations of these techniques appear in Section 3.3.

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2. Smooth free $\mathbb{Z}/p$-actions via the Kervaire–Milnor sequences

In order to apply Theorem 1.2, we need to analyze the orders of the groups Θ_n of homotopy n-spheres. Kervaire and Milnor showed in [KM63] that there is an isomorphism

$$\Theta_{4k} \cong \operatorname{coker} J_{4k}$$

and exact sequences

$$0 \to \Theta^{bp}_{2k+1} \to \Theta_{2k+1} \to \operatorname{coker} J_{2k+1} \to 0,$$

$$0 \to \Theta_{4k+2} \to \operatorname{coker} J_{4k+2} \xrightarrow{\Phi} \mathbb{Z}/2 \to \Theta^{bp}_{4k+1} \to 0,$$

where Φ is the Kervaire invariant [Ker60]. In this section, we will apply information about Θ^{bp}_n and coker J to analyze the order of Θ_n. Our main result (Proposition 2.6) provides arithmetic conditions on the natural number n and odd prime p under which every exotic n-sphere admits a smooth free $\mathbb{Z}/p$-action.

Our first lemma allows us to ignore contributions from coker J, provided that n is sufficiently small.

Lemma 2.1. *Let p be an odd prime. Then*

$$(\operatorname{coker} J_n)_{(p)} = 0$$

for $n < 2p^2 - 2p - 2$.

Proof. By [Rav86, Sec. 5.3], the first nontrivial element in $(\operatorname{coker} J_*)_{(p)}$ is the Greek letter element β_1, which appears in stem $2p^2 - 2p - 2$. □

We now turn out attention to Θ^{bp}_n. These groups were first studied by Kervaire–Milnor in [KM63], but we refer the reader to Levine [Lev85] for results closer to ours which incorporate the solution of the Adams Conjecture. The restrictions in the case $n = 4k + 1$ follow from the solution of the Kervaire invariant one problem [HHR16].

Theorem 2.2. *The subgroup $\Theta^{bp}_n \subseteq \Theta_n$ of h-cobordism classes of homotopy n-spheres which bound a stably parallelizable manifold is given by*

$$\Theta^{bp}_n \cong \begin{cases} 0 & \text{if } n \equiv 0 \mod 2, \\ 0 & \text{if } n = 4k+1 \in \{1,5,13,29,61, \text{ and possibly } 125\}, \\ \mathbb{Z}/2 & \text{if } n = 4k+1 \notin \{1,5,13,29,61, \text{ and possibly } 125\}, \\ \mathbb{Z}/t_k & \text{if } n = 4k-1, \end{cases}$$

where

$$t_k := \frac{3-(-1)^k}{2} 2^{2k-2}(2^{2k-1}-1) \cdot \operatorname{num}\left(\frac{B_{2k}}{4k}\right)$$

with B_j the j-th Bernoulli number.

Corollary 2.3. *Let p be an odd prime. For $n < 2(p^2-1) - 2(p-1) - 2$,*

$$(\Theta_n)_{(p)} \cong \begin{cases} (\mathbb{Z}/t_k)_{(p)} & \text{if } n = 4k-1,\ k \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.4. *Let p be an odd prime. Then $(\mathbb{Z}/t_k)_{(p)} \neq 0$ if and only if $p | 2^{2k-1} - 1$ or $p | \operatorname{num}\left(\frac{B_{2k}}{2k}\right)$.*

Proof. We have $(\mathbb{Z}/t_k)_{(p)} \neq 0$ if and only if $p|t_k$, and since p is odd,

$$\begin{aligned} p \mid t_k &\iff p \mid \frac{3-(-1)^k}{2} 2^{2k-2}(2^{2k-1}-1) \operatorname{num}\left(\frac{B_{2k}}{4k}\right) \\ &\iff p | (2^{2k-1}-1) \operatorname{num}\left(\frac{B_{2k}}{2k}\right) \\ &\iff p | 2^{2k-1} - 1 \text{ or } p | \operatorname{num}\left(\frac{B_{2k}}{2k}\right). \end{aligned}$$

□

Example 2.5. Since $3 \nmid 2^{2k-1} - 1$ and $3 \nmid \operatorname{num}(\frac{B_{2k}}{2k})$, we have $(\mathbb{Z}/t_k)_{(3)} = 0$ for all k. The first claim follows from elementary modular arithmetic and the second claim follows from the fact that 3 is a regular prime (cf. Remark 2.8).

Putting these observations together, we have shown:

Proposition 2.6. *Let p be an odd prime and let $n < 2p^2 - 2p - 2$. If $n \equiv 0, 1, 2 \mod 4$, or if $n = 4k - 1$ with $p \nmid 2^{2k-1} - 1$ and $p \nmid \operatorname{num}(\frac{B_{2k}}{2k})$, then every exotic n-sphere admits a smooth free $\mathbb{Z}/p$-action.*

In the following two remarks, we give some examples of when the conditions in Proposition 2.6 are satisfied.

Remark 2.7. We first consider the condition $p \nmid 2^{2k-1} - 1$. If $2k - 1$ is prime, then quadratic reciprocity implies that the prime factors of $2^{2k-1} - 1$ must be congruent to ± 1 modulo 8. Thus if $2k - 1$ is prime and $p \not\equiv \pm 1 \mod 8$, the condition $p \nmid 2^{2k-1} - 1$ is satisfied.

Remark 2.8. We can also say something about the condition $p \nmid \operatorname{num}(\frac{B_{2k}}{2k})$. As explained in the proof of [Tha12, Thm. 1], the prime factors of $\operatorname{num}(\frac{B_{2k}}{2k})$ must be irregular.[1] Thus if p is a regular prime, the condition $p \nmid \operatorname{num}(\frac{B_{2k}}{2k})$ is satisfied.

Corollary 2.9. *Let $p \not\equiv \pm 1 \mod 8$ be an odd regular prime and let $n = 4k - 1 < 2p^2 - 2p - 2$ with $2k - 1$ prime. Then every exotic n-sphere admits a smooth free $\mathbb{Z}/p$-action.*

3. Nontrivial smooth $\mathbb{T}$- and $\mathbb{Z}/p$-actions via the Mahowald invariant

As discussed in the introduction, we can find nontrivial smooth $\mathbb{T}$- and $\mathbb{Z}/p$-actions on exotic spheres whose order in Θ_n is divisible by p using the *Mahowald invariant*. We refer the reader to [MR93] for a definition, since for our purposes, it suffices to know that for each prime p, the Mahowald invariant is a construction which assigns a nontrivial coset in the p-local stable homotopy groups of spheres to each nontrivial element in the p-local stable homotopy groups of spheres. We

[1] Recall that a prime q is irregular if $q|B_j$ for some even $j \leq p - 3$.

will freely use the names of elements in the stable homotopy groups of spheres from [Rav86] (e.g., for elements like α_i, β_j) and [IWX20] (e.g., for elements like $\eta\eta_4$, κ^2).

Our recollection of Mahowald invariant computations is divided into two sections. In Section 3.1, computations of infinite families of Mahowald invariants are discussed, and in Section 3.2, additional low-dimensional computations at small primes are recalled. After each result is stated, its consequences for transformation groups of exotic spheres via Theorem 1.5 are given. Directions for future work and a discussion of the limitations of these techniques appear in Section 3.3.

3.1. **Mahowald invariants of infinite families.** Computing the Mahowald invariants of infinite families of elements in the stable homotopy groups of spheres is a difficult problem in stable homotopy theory. This section recalls almost all of the existing computations in this direction.

Theorem 3.1 ([MR93, Thm. 3.5], [Sad92, Cor. 1.4])**.** *For all $i > 0$ and primes $p \geq 5$,*
$$\beta_i \in M(\alpha_i).$$

Theorem 3.2 ([Beh06, Thm. 15.7])**.** *For all $i > 0$ with $i \equiv 0, 1, 5 \mod 9$ and $p = 3$,*
$$(-1)^{i+1}\beta_i \in M(\alpha_i).$$

We note that the p-primary Mahowald invariant is natural in the action of $\mathbb{Z}_p^\times$ on the stable stems, so if $\beta \in M(\alpha)$, then $\ell\beta \in M(\ell\alpha)$ for $\ell \in \mathbb{Z}_p^\times$. In particular, the signs above don't pose an issue for the actions on exotic spheres deduced below.

Corollary 3.3 (Compare with [Sch85, Ex. 3.8] for $p \geq 5$)**.** *For all $i \geq 1$ and primes $p \geq 5$, and for all $i \geq 1$ with $i \equiv 0, 1, 5 \mod 9$ and $p = 3$, every exotic sphere $\Sigma^{2(p^2-1)i-2(p-1)-2}$ corresponding to $\beta_i \neq 0 \in \operatorname{coker} J_{2(p^2-1)i-2(p-1)-2}$ supports a smooth $\mathbb{T}$-action with a $(2(p-1)i-2)$-dimensional fixed point set.*

Remark 3.4. The case $i = 1$ was proven by more geometric methods in [Sch73].

Theorem 3.5 ([Sad92, Sec. 6])**.** *For all primes $p \geq 5$,*
$$\beta_{p/2} \in M(\alpha_{p/2}).$$

Corollary 3.6. *For all primes $p \geq 5$, every exotic sphere $\Sigma^{2(p^2-1)p-4(p-1)-2}$ corresponding to $\beta_{p/2} \neq 0 \in \operatorname{coker} J$ supports a smooth $\mathbb{T}$-action with a $(2(p-1)p-2)$-dimensional fixed point set.*

3.2. **Low-dimensional Mahowald invariants at small primes.** The difficulty of computing p-primary Mahowald invariants increases when p is a small prime. In [Beh06, Beh07], Behrens introduced new techniques which allowed for the computation of new 3- and 2-primary Mahowald invariants in low dimensions.

Proposition 3.7 ([Beh06, Prop. 12.1])**.** *The following Mahowald invariants hold at $p = 3$:*

$$M(\alpha_2) \doteq \beta_1^2\alpha_1; \qquad M(\alpha_{3/2}) = -\beta_{3/2}; \qquad M(\alpha_3) = \beta_3;$$
$$M(\alpha_4) \doteq \beta_1^5; \qquad M(\alpha_{6/2}) = \beta_{6/2}; \qquad M(\alpha_6) = -\beta_6.$$

Here, we write '$\doteq$' for equations which hold up to multiplication by a unit.

Corollary 3.8. *Let $p = 3$.*

(1) *Every exotic sphere Σ^{23} corresponding to $\beta_1^2\alpha_1 \neq 0 \in \operatorname{coker} J_{23}$ supports a smooth $\mathbb{Z}/3$-action wtih fixed points S^7.*

(2) *Every exotic sphere* Σ^{38}, Σ^{42}, Σ^{50}, Σ^{86}, *and* Σ^{90} *corresponding to the nontrivial elements* $\beta_{3/2}$, β_3, β_1^5, $\beta_{6/2}$, *and* β_6 *in* coker J, *respectively, supports a smooth* $\mathbb{T}$*-actions with* 10-, 10-, 14-, 22-, *and* 22*-dimensional fixed point sets, respectively.*

Theorem 3.9 (Part of [Beh07, Thm. 11.1]). *The following Mahowald invariants hold at* $p = 2$*:*

$$M(\eta^2) = \nu^2; \quad M(\eta^3) = \nu^3; \quad M(2\nu) = \sigma\eta; \quad M(\sigma) = \sigma^2;$$
$$M(2\sigma) = \eta_4; \quad M(4\sigma) = \eta\eta_4; \quad M(8\sigma) = \eta^2\eta_4; \quad M(\eta\sigma) = \nu^*;$$
$$M(\eta^2\sigma) = \nu\nu^*; \quad M(v_1^4\eta) = \nu\bar{\kappa}; \quad M(v_1^4\eta^2) = \kappa^2; \quad M(v_1^4\eta^3) = \eta q;$$
$$M(v_1^4\nu) = \nu^2\bar{\kappa}; \quad M(v_1^4 2\nu) = q.$$

Corollary 3.10. *Let* $p = 2$*. Then, potentially after taking the connected sum with elements in* Θ_*^{bp}*, every exotic sphere* Σ^9, Σ^{17}, Σ^{21}, *and* Σ^{33} *corresponding to the nontrivial elements* ν^3, $\eta\eta_4$, $\nu\nu^*$, *and* ηq *in coker* J*, respectively, supports a smooth involution with fixed points* S^3, M^7, S^9, *and* M^{11}*, where* M^n *is any homotopy* n*-sphere which maps to zero in* coker J*.*

3.3. **Further remarks.** We close by mentioning some directions for follow-up work and limitations of these techniques.

Remark 3.11. Schultz mentions [Sch85, Pg. 260] the possibility of applying Theorem 1.5 to Mahowald invariants at higher chromatic heights. Mahowald and Ravenel state [MR93] that $M(\beta_1) = \beta_1^p$, and outline an approach to showing $M(\beta_i) = \gamma_i$ for $i \geq 2$ and $p \geq 7$.[2] Assuming this is true, one obtains nontrivial smooth S^1-actions on some additional exotic spheres.

Remark 3.12. Many divided Greek letter elements $\beta_{kp/i}$ in coker J are represented by exotic spheres (see, for instance, [Beh07, Sec. 3]). For example, the elements $\beta_{6/3} \in \text{coker}(J_{82})_{(3)}$ and $\beta_{6/2} \in \text{coker}(J_{86})_{(3)}$ are represented by exotic 82- and 86-spheres, respectively, for which a nontrivial $\mathbb{Z}/3$-action is not guaranteed by Theorem 1.2. As mentioned in Theorem 3.5, Sadofsky showed in [Sad92] that $M(\alpha_{p/2}) = \beta_{p/2}$ for all primes $p \geq 5$. It seems plausible that $M(\alpha_{kp/i}) = \beta_{kp/i}$ for larger k and i; if this were true, one could deduce the existence of additional infinite families of nontrivial $\mathbb{Z}/p$-actions on exotic spheres whose order in Θ_n is divisible by p.

Remark 3.13. Belmont and Isaksen [BI22] have recently introduced some promising techniques for computing 2-primary Mahowald invariants. It would be interesting to see how far these ideas can be pushed and their consequences for nontrivial smooth involutions on exotic spheres.

Remark 3.14 (Limitations). Fix a prime p. Let $G_k := (\pi_k^s)_{(p)}$, and let $R_k \subseteq G_k$ denote the subgroup generated by classes which are Mahowald invariants. In [MR93, Conj. 1.13], Mahowald and Ravenel conjecture that

$$\lim_{k\to\infty} \frac{\log_p |R_k|}{\log_p |G_k|} = \frac{1}{p^2},$$

assuming that $\log_p |G_k|$ grows linearly in k. Burklund [BHS22, Bur22] has recently shown that $\log_p |G_k|$ grows subexponentially, so one might expect that the limit

[2] Here, the restriction $p \geq 7$ ensures that γ_i is defined via the work of Miller–Ravenel–Wilson [MRW77].

above approaches $1/p$ instead of $1/p^2$. If this is true, then Theorem 1.5 can be applied to roughly 1 out of every p exotic spheres from coker J if p is odd (and some smaller proportion if $p = 2$).

Appendix A. Tables of nontrivial $\mathbb{T}$- and $\mathbb{Z}/p$-actions in low dimensions

In this appendix, we compute the prime factors of $\text{coker}(J_n)$ and Θ_n^{bp} in all dimensions $n \leq 100$ where exotic spheres are known to exist (see [BHHM20] for a list up to dimension 140).

The prime factors of $\text{coker}(J_n)$ follow directly from inspection of the p-local stable homotopy groups of spheres. Note that $\text{coker}(J_n)_{(p)} = 0$ for $n \leq 100$ if $p \geq 11$, so we only need to examine $p \in \{2, 3, 5, 7\}$:

- For $p = 2$, this follows from recent work of Isaksen–Wang–Xu [IWX20], $n \leq 95$, and from [BHHM20, BMQ22] for $96 \leq n \leq 100$.
- For $p \in \{3, 5\}$, we use Ravenel's extensive Adams–Novikov spectral sequence computations [Rav86, Thms. 7.5.3, 7.6.5].
- For $p = 7$, the only element in $\text{coker}(J)_n$ with $n \leq 100$ is $\beta_1 \in \text{coker}(J_{82})$.

The prime factors of Θ_n^{bp} follow from Theorem 2.2. Since $\Theta_n^{bp} = 0$ if n is even, prime factors only appear when n is odd. We can further reduce to the study of odd prime factors, since 2 divides $|\Theta_n^{bp}|$ for all odd $n \geq 7$, except in the exceptional cases $n \in \{1, 5, 13, 29, 61\}$. The odd prime factors of t_k were determined using Mathematica.

We add an asterisk whenever some exotic sphere whose order is divisible by p admits a $\mathbb{T}$- or $\mathbb{Z}/p$-action via Theorem 1.5. For example, for $n = 9$, the prime 2 divides the order of $\text{coker}(J_9)$, but the element $\nu^3 \in \text{coker}(J_9)$ is a Mahowald invariant (Theorem 3.9) for which Theorem 1.5 applies. Thus we have '2*' in the row $n = 9$ and second column, instead of just '2'.

Finally, we added two asterisks to the '2' in the $n = 30$ row, since the only nontrivial element in $\text{coker}(J_{30})_{(2)}$ is θ_4, which has Kervaire invariant one and thus does not detect an exotic sphere.

n	prime factors of coker(J_n)	prime factors of Θ_n^{bp}
7	2	2
8	2	
9	2*	2
10	2, 3*	
11	2	2, 31
13	3	
14	2	
15	2	2, 127
16	2	
17	2*	2
18	2	
19	2	2, 7, 73
20	2, 3	
21	2*	2
22	2	
23	2, 3*	2, 23, 89, 691
24	2	
25	2	2
26	2, 3	
27	2	2, 8191
28	2	
29	3	
30	2**, 3	
31	2	2, 7, 31, 151, 3617
32	2	
33	2*	2
34	2	
35	2	2, 43867
36	2, 3	
37	2, 3	2
38	2, 3*, 5*	
39	2, 3	2, 283, 617
40	2, 3	
41	2	2
42	2, 3*	
43	2	2, 7, 127, 131, 337, 593
44	2	
45	2, 3, 5	2
46	2, 3	
47	2, 3	2, 47, 103, 178481, 2294797
48	2	
49	2, 3	2
50	2, 3*	

FIGURE 1. The prime factors of Θ_n^{bp}, $7 \leq n \leq 50$. The trivial group Θ_{12} is omitted.

n	prime factors of coker(J_n)	prime factors of Θ_n^{bp}
51	2	2, 657931
52	2, 3	
53	2	2
54	2	
55	2, 3	2, 7, 73, 9349, 262657, 362903
57	2	2
58	2	
59	2	2, 233, 1103, 1721, 2089, 1001259881
60	2	
62	2, 3	
63	2	2, 37, 683, 305065927, 2147493647
64	2	
65	2, 3	2
66	2	
67	2	2, 7, 23, 89, 5999479, num($B_{34}/34$)
68	2, 3	
69	2	2
70	2	
71	2	2, 31, 71, 127, 122921, num($B_{36}/36$)
72	2, 3	
73	2	2
74	2, 3*	
75	2, 3	2, 223, 616318177, num($B_{38}/38$)
76	2, 5	
77	2	2
78	2, 3	
79	2	2, 7, 79, 8191, 121369, 137616929, 1897170067619
80	2	
81	2, 3	2
82	2, 3, 7*	
83	2, 5	2, 13367, 164511353, num($B_{42}/42$)
84	2, 3	
85	2, 3	2
86	2, 3*, 5*	
87	2	2, 59, 431, 8089, 9719, 2099863, 2947939, 1798482437
88	2	
89	2	2
90	2, 3*	
91	2, 3	2, 7, 31, 73, 151, 631, 23311 383799511, 67568238839737
92	2, 3	
93	2, 3, 5	2
94	2, 3	

FIGURE 2. The prime factors of Θ_n^{bp}, $51 \leq n \leq 94$. The trivial groups Θ_{56} and Θ_{61} are omitted.

n	prime factors of coker(J_n)	prime factors of Θ_n^{bp}
95	2, 3	2, 653, 2351, 4513, 56039, 10610063, 13264529, 31184907679, 59862819377, 140737488355327, 153298748932447906241
96	2	
97	2	2
98	2	
99	2, 3	2, 127, 417202699, 4432676798593, 562949953421311, 47464429777438199
100	2, 3	

FIGURE 3. The prime factors of Θ_n^{bp}, $90 \leq n \leq 100$.

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Max Planck Institute for Mathematics
Current address: University of Virginia
Email address: mbp6pj@virginia.edu