WEIGHTED COMPOSITION OPERATORS ON THE MITTAG-LEFFLER SPACES OF ENTIRE FUNCTIONS

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Abstract. The subject of this manuscript is the investigation of (possibly unbounded) weighted composition operators arising from the formal expression $E(\psi,\varphi)f = \psi \cdot f \circ \varphi$ over Mittag-Leffler spaces of entire functions. In this context, the functions ψ and φ are entire functions, and this manuscript presents basic operator theoretic properties such as closability, invertibility, cyclicity, complex symmetry, boundedness, compactness, and the essential norm.

Significantly, a characterization of ψ and φ are obtained for bounded weighted composition operators over the Mittag-Leffler space of entire functions for parameter $0 < \alpha < 2$, and many extant results in the Fock space are reproduced in the more general context of Mittag-Leffler spaces.

1. Introduction

Let $\mathbb{X}$ be a Banach space of holomorphic functions on some domain G. Consider the formal expression of the following form

$$E(\psi,\varphi)f = \psi \cdot f \circ \varphi,$$

where ψ, φ are holomorphic functions on G and $\varphi : G \to G$. The *maximal weighted composition operator* corresponding to ψ and φ over $\mathbb{X}$ is defined as follows

$$\mathrm{dom}(W_{\psi,\varphi,\max}) = \{f \in \mathbb{X} : E(\psi,\varphi)f \in \mathbb{X}\}, \text{and}$$
$$W_{\psi,\varphi,\max} f = E(\psi,\varphi)f, \quad \forall f \in \mathrm{dom}(W_{\psi,\varphi,\max}).$$

The domain, $\mathrm{dom}(W_{\psi,\varphi,\max})$, is called *maximal* and the operator $W_{\psi,\varphi,\max}$ is "maximal" for the reason that it cannot be extended as an operator in $\mathbb{X}$ generated by the expression $E(\psi,\varphi)$. The domain of an unbounded operator is integral to the definition of the operator, where an expression studied on different domains may generate operators with different properties [19]. This dependence on the domain motivates the consideration of the formal expression $E(\psi,\varphi)$ on subspaces of the maximal domain $\mathrm{dom}(W_{\psi,\varphi,\max})$. The operator $W_{\psi,\varphi}$ is called an *unbounded weighted composition operator* if $W_{\psi,\varphi} \preceq W_{\psi,\varphi,\max}$. In other words, the domain $\mathrm{dom}(W_{\psi,\varphi})$ is a subspace of the maximal domain $\mathrm{dom}(W_{\psi,\varphi,\max})$, and the operator $W_{\psi,\varphi}$ is the restriction of the maximal operator $W_{\psi,\varphi,\max}$ on $\mathrm{dom}(W_{\psi,\varphi})$. A weighted composition operator can be regarded as a generalization of the *multiplication operator* $M_\psi f = \psi \cdot f$ (cf. [28, 22, 21]) and the *composition operator* $C_\varphi f = f \circ \varphi$ (cf. [3, 14, 6]).

The principle avenue of the investigation of weighted composition operators is in the interaction of operator-theoretic properties, such as boundedness or invertibility, and function-theoretic properties of the symbols, ψ and φ, such as growth rates and zeros. Much of the literature investigates weighted composition operators on Banach spaces of holomorphic functions in the unit disc or the unit ball (cf. [13]). More recent studies have extended the research to include such operators over Fock spaces of entire functions, which bears relevance to the current study. Many operator-theoretic properties of $W_{\psi,\varphi}$ have been characterized

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completely, such as: boundedness, isometry, compactness, selfadjointness, normality, cohyponormality, complex symmetry, etc (cf. [12, 11, 16]). The investigation of weighted composition operators over the Mittag-Leffler spaces of entire functions has not been considered and is the subject of the current manuscript.

The *Mittag-Leffler space* of entire functions of order $\alpha > 0$, denoted as $ML^2(\mathbb{C}, \alpha)$ (cf. [24]), is the reproducing kernel Hilbert space (RKHS) associated with the kernel functions given as

$$K_{\alpha,w}(z) = \sum_{j\geq 0} \frac{z^j \bar{w}^j}{\Gamma(\alpha j + 1)}, \quad w \in \mathbb{C} \tag{1.1}$$

where Γ is the Gamma function, $\Gamma(\alpha) = \int_0^\infty \tau^{\alpha-1} e^{-\tau}\, d\tau$. The Aronszajn theorem (cf. [1]) affords the following representation of the Mittag-Leffler space of entire functions of order $\alpha > 0$,

$$ML^2(\mathbb{C}, \alpha) := \left\{ f(z) = \sum_{j\geq 0} f_j z^j \in \mathcal{E}(\mathbb{C}) : \sum_{j\geq 0} |f_j|^2 \Gamma(\alpha j + 1) < \infty \right\},$$

where $\mathcal{E}(\mathbb{C})$ is the collection of holomorphic functions over $\mathbb{C}$. The space $ML^2(\mathbb{C}, \alpha)$ inherits its name from Gösta Magnus Mittag-Leffler who introduced his eponymous function,

$$E_\alpha(u) = \sum_{j\geq 0} \frac{u^j}{\Gamma(\alpha j + 1)}, \quad u \in \mathbb{C}. \tag{1.2}$$

Basic properties of the space $ML^2(\mathbb{C}, \alpha)$ are investigated in a series of manuscripts [23, 24]. Consequently, the kernel function (1.1) may be rewritten as $K_{\alpha,w}(z) = E_\alpha(z\overline{w})$. The Mittag-Leffler function generalizes the exponential function through the replacement of the factorial in the power series expansion of the exponential function by the Gamma function whereby $E_1(u) = e^u$.

The Mittag-Leffler function has been of growing interest for several decades due to its direct involvement in the fractional calculus as well as problems of physics, biology, chemistry, engineering and other applied sciences (cf. [5, 17, 10]). However, the study of Mittag-Leffler RKHSs were initiated in [23] to facilitate numerical methods for the Caputo fractional derivative by leveraging the eigenfunction relationship $D_*^\alpha E_\alpha(\lambda t^\alpha) = \lambda E_\alpha(\lambda t^\alpha)$, where D_*^α is the Caputo fractional derivative (cf. [5]). In [23], it was shown that for RKHSs of continuously differentiable functions that norm convergence implies convergence of Caputo fractional derivatives, and a kernelized predictor-corrector method was introduced for initial value problems of Caputo type.

It was proved in [24], that the norm for the space $ML^2(\mathbb{C}, \alpha)$ can be expressed as

$$\|f\| := \left(\frac{1}{\alpha\pi} \int_{\mathbb{C}} |f(z)|^2 |z|^{\frac{2}{\alpha}-2} e^{-|z|^{\frac{2}{\alpha}}}\, dz \right)^{1/2}.$$

Mittag-Leffler spaces give a one parameter generalization of the Fock space [24]. More precisely, Fock space can be obtained from Mittag-Leffler space by setting $\alpha = 1$. Moreover, $ML^2(\mathbb{C}, 1) \subseteq ML^2(\mathbb{C}, \alpha), \quad \forall \alpha \in (0, 1]$, and the containment is proper for $\alpha \neq 1$ [24].

In this paper, we study the basic operator properties of weighted composition operators on $ML^2(\mathbb{C}, \alpha)$. This investigation includes the closability, invertibility, cyclicity, complex symmetry, boundedness, compactness, and the essential norm. In Proposition 4.1, we show that every maximal weighted composition operator is closed on the Mittag-Leffler spaces. We characterize those weighted composition operators on $ML^2(\mathbb{C}, \alpha)$ that are invertible in the sense of unbounded operators (Theorems 5.1-5.2), bounded (Theorem 7.1), and compact (Corollary 8.5 and Theorem 8.6). In addition to these studies, we characterize all functions which are non-vanishing in $ML^2(\mathbb{C},\alpha)$ (Proposition 5.3). In Section 6, we establish some conditions for a complex symmetric operator to be cyclic on Mittag-Leffler spaces.

Notations

Throughout the paper, we denote by $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ by the sets of integers, real numbers, complex numbers, respectively. For a set $A \subseteq \mathbb{R}$, the symbol $A_{\geq\delta}$ stands for the set $\{x \in A : x \geq \delta\}$. We denote by $\mathbb{C}[z]$ the polynomial ring. For a set G, $\mathbf{Span}(G)$ denotes the linear span of all elements in G and $\mathbf{Cl}(G)$ denotes the closure of G. For two unbounded operators X, Y, the notation $X \preceq Y$ means that X is the *restriction* of Y on the domain $\mathrm{dom}(X)$; namely

$$\mathrm{dom}(X) \subseteq \mathrm{dom}(Y), \quad Xz = Yz, \forall z \in \mathrm{dom}(X).$$

The product XY is defined as

$$\mathrm{dom}(XY) = \{z \in \mathrm{dom}(Y) : Yz \in \mathrm{dom}(X)\}, \quad XYz = X(Yz), \forall z \in \mathrm{dom}(XY).$$

For an unbounded operator T, $\mathrm{dom}(T)$ denotes the domain of T,

$$\mathrm{dom}(T^\infty) = \bigcap_{n\geq 0} \mathrm{dom}(T^n)$$

and $\mathcal{O}(T, x) = \{T^n x : n \in \mathbb{Z}_{\geq 0}\}$, where $x \in \mathrm{dom}(T^\infty)$.

2. Preliminaries

Definition 2.1. A RKHS, H, over a set G is a Hilbert space of functions from G to $\mathbb{C}$ such that for each $x \in G$, the evaluation functional $f \mapsto f(x)$ is bounded.

By the Riesz representation theorem, for every $x \in G$ there is a corresponding function $k_x \in H$ such that $\langle f, k_x\rangle = f(x)$ for all $f \in H$. The kernel function corresponding to H over the set G is given by $K(x, y) = \langle k_y, k_x\rangle$.

This section presents some results established in [10] for Mittag-Leffler functions and [24] for Mittag-Leffler spaces of entire functions, which are RKHSs over $\mathbb{C}$. Lemma 2.2 and Remark 2.3 are fundamental inequalities that characterize the growth of Mittag-Leffler functions, and will be later utilized in characterizing bounded weighted composition operators over Mittag-Leffler spaces.

Lemma 2.2 ([10, Corollary 3.8])**.** *Let $0 < \alpha < 2$. Then the following limit holds:*

$$\lim_{t\to\infty, t\in\mathbb{R}} e^{-t^{1/\alpha}} |E_\alpha(t)| = \frac{1}{\alpha}.$$

Remark 2.3. By Lemma 2.2, there exist positive constants C_1, C_2 such that

$$C_1 e^{t^{1/\alpha}} \leq |E_\alpha(t)| \leq C_2 e^{t^{1/\alpha}}, \quad \forall t \geq 0.$$

For $\alpha > 0$ and $z \in \mathbb{C}$, we define the normalized kernel function as

$$k_{\alpha,z} = \frac{K_{\alpha,z}}{\|K_{\alpha,z}\|}, \tag{2.1}$$

where

$$\|K_{\alpha,z}\|^2 = \langle K_{\alpha,z}, K_{\alpha,z}\rangle = K_{\alpha,z}(z) = E_\alpha(|z|^2) = \sum_{j\geq 0} \frac{|z|^{2j}}{\Gamma(\alpha j + 1)}.$$

As the norm of the Mittag-Leffler spaces of entire functions can be expressed as an integral over $\mathbb{C}$ with a radially symmetric weight, the monomials can be demonstrated to be mutually orthogonal with respect to the Mittag-Leffler space's inner product. Therefore the following proposition holds.

Proposition 2.4 ([24])**.** *Let $\alpha > 0$. The following properties hold.*

(1) *The set* $\{\mathbf{e}_j\}_{j\geq 0}$, *where*

$$\mathbf{e}_j(z) = \frac{z^j}{\sqrt{\Gamma(\alpha j+1)}},$$

is an orthonormal basis for $ML^2(\mathbb{C},\alpha)$.

(2) $|f(z)|^2 \leq E_\alpha(|z|^2)\|f\|^2$ *for every* $f \in ML^2(\mathbb{C},\alpha)$.

Finally, the follow proposition specializes [4, Proposition 2.1] to the Mittag-Leffler space of entire functions.

Proposition 2.5 ([4, Proposition 2.1])**.** *Let* $\alpha > 0$. *For each* $f \in ML^2(\mathbb{C},\alpha)$, *we have*

$$\lim_{|z|\to\infty} \langle f, k_{\alpha,z}\rangle = 0.$$

3. Some initial properties

This section demonstrates several technical details concerning weighted composition operators over Mittag-Leffler spaces of entire functions.

3.1. **Kernel** $\ker(W_{\psi,\varphi})$**.** We are first interested in the kernel and image of an unbounded weighted composition operator acting on Mittag-Leffler spaces. Proposition 3.1 applies to RKHS of entire functions that have strictly positive definite kernels [20], of which the Mittag-Leffler space is a special case.

Proposition 3.1. *Let* H *be a RKHS of entire functions with a strictly positive definite kernel function* $K(z,w)$, *and let* $W_{\psi,\varphi}$ *be a densely defined weighted composition operator over* H *induced by two entire functions* ψ *and* φ. *If* $\psi \not\equiv \mathbf{0}$, *then either* φ *is nonconstant and*

$$\ker(W_{\psi,\varphi}) = \{0\} \text{ and } \mathbf{Cl}[\operatorname{Im}(W^*_{\psi,\varphi})] = H,$$

or $\varphi \equiv \phi_0 \in \mathbb{C}$ *and*

$$\ker(W_{\psi,\varphi}) = \{K(\cdot,\phi_0)\}^\perp \text{ and } \mathbf{Cl}[\operatorname{Im}(W^*_{\psi,\varphi})] = K(\cdot,\phi_0)\mathbb{C}.$$

Proof. Suppose that $f \in \ker(W_{\psi,\varphi})$, then for all $z \in \mathbb{C}$, $\psi(z)f(\varphi(z)) = 0$. Let $\omega \in \mathbb{C}$ be a nonvanishing point for ψ and let $\{\omega_m\} \in \mathbb{C}$ be a sequence of points converging to ω such that $\psi(\omega_m) \neq 0$ for all m. Hence, $f(\varphi(\omega_m)) = 0$ for each m. Since ω_m has a cluster point, $f \circ \varphi \equiv 0$.

Either, φ is nonconstant and $f \equiv 0$ or $\varphi \equiv \phi_0 \in \mathbb{C}$. In the first case, this means that $\ker(W_{\psi,\varphi}) = \{0\}$. Hence,

$$H = \mathbf{Cl}[\operatorname{Im}(W^*_{\psi,\varphi})] \oplus \ker(W_{\psi,\varphi}) = \mathbf{Cl}[\operatorname{Im}(W^*_{\psi,\varphi})].$$

In the second case $\psi \in H$, since if $g \in \operatorname{dom}(W_{\psi,\varphi})$ is such that $g(\varphi(z)) \equiv g(\phi_0) \in \mathbb{C}\setminus\{0\}$ (the density of the domain guarantees such a g), then $W_{\psi,\varphi}g = g(\phi_0)\cdot\psi \in H$. Therefore, $W_{\psi,\varphi}$ maps to the one dimensional subspace spanned by ψ. The kernel is given as

$$\ker(W_{\psi,\varphi}) = \{g \in H : g(\phi_0) = 0\} = \{K(\cdot,\phi_0)\}^\perp.$$

□

3.2. **Reproducing kernel algebra.** The lemma below shows kernel functions always belong to the domain $\operatorname{dom}[(W^*_{\psi,\varphi})^n]$. Moreover, the action $(W^*_{\psi,\varphi})^n K_{\alpha,z}$ can be written explicitly.

Lemma 3.2. *For every* $\alpha > 0$, $n \in \mathbb{Z}_{\geq 1}$ *and* $z \in \mathbb{C}$, $K_{\alpha,z} \in dom[(W^*_{\psi,\varphi})^n]$, *and*

$$(W^*_{\psi,\varphi})^n K_{\alpha,z} = \overline{\psi(z)\psi(\varphi(z))\cdots\psi(\varphi_{n-1}(z))}K_{\alpha,\varphi_n(z)},$$

where $\varphi_\ell = \varphi\circ\cdots\circ\varphi$ *denotes the* ℓ*-th iteration of the function* φ.

Proof. Let $z \in \mathbb{C}$ and $\alpha > 0$. For every $f \in \mathrm{dom}(W_{\psi,\varphi})$, we have

$$\langle W_{\psi,\varphi} f, K_{\alpha,z}\rangle = W_{\psi,\varphi} f(z) = \psi(z)\langle f, K_{\alpha,\varphi(z)}\rangle = \langle f, \overline{\psi(z)} K_{\alpha,\varphi(z)}\rangle,$$

which provides the conclusion for $n = 1$. Now suppose that the conclusion holds for $n = \kappa$. By induction, we have $K_{\alpha,z} \in \mathrm{dom}[(W^*_{\psi,\varphi})^\kappa]$ and

$$(W^*_{\psi,\varphi})^\kappa K_{\alpha,z} = \overline{\psi(z)\psi(\varphi(z))\cdots\psi(\varphi_{\kappa-1}(z))} K_{\alpha,\varphi_\kappa(z)}.$$

It follows from $K_{\alpha,\varphi_\kappa(z)} \in \mathrm{dom}(W^*_{\psi,\varphi})$, that

$$\begin{aligned}(W^*_{\psi,\varphi})^{\kappa+1} K_{\alpha,z} &= W^*_{\psi,\varphi}\left(\overline{\psi(z)\psi(\varphi(z))\cdots\psi(\varphi_{\kappa-1}(z))} K_{\alpha,\varphi_\kappa(z)}\right)\\ &= \overline{\psi(z)\psi(\varphi(z))\cdots\psi(\varphi_{\kappa-1}(z))} W^*_{\psi,\varphi}\left(K_{\alpha,\varphi_\kappa(z)}\right)\\ &= \overline{\psi(z)\psi(\varphi(z))\cdots\psi(\varphi_{\kappa}(z))} K_{\alpha,\varphi_{\kappa+1}(z)}.\end{aligned}$$

□

The following quantities play an important role in the present manuscript:

$$M_{\alpha,z}(\psi,\varphi) = |\psi(z)|^2 e^{|\varphi(z)|^{2/\alpha} - |z|^{2/\alpha}}, \quad M_\alpha(\psi,\varphi) = \sup_{z\in\mathbb{C}} M_{\alpha,z}(\psi,\varphi).$$

The following lemma will be used in characterizing the boundedness of a weighted composition operator acting on Mittag-Leffler spaces in Theorem 7.1.

Lemma 3.3. *Let $2 > \alpha > 0$. If $k_{\alpha,\varphi(z)} \in dom(W_{\psi,\varphi})$ for some $z \in \mathbb{C}$, then*

$$\left\|W_{\psi,\varphi} k_{\alpha,\varphi(z)}\right\| \ge C_1^{1/2} C_2^{-1/2} M_{\alpha,z}(\psi,\varphi)^{1/2},$$

where C_1, C_2 are the positive constants mentioned in Remark 2.3.

Proof. By Proposition 2.4(2),

$$\begin{aligned}\left\|W_{\psi,\varphi} k_{\alpha,\varphi(z)}\right\|^2 &\ge \left|W_{\psi,\varphi} k_{\alpha,\varphi(z)}(z)\right|^2 E_\alpha(|z|^2)^{-1}\\ &= |\psi(z)|^2 E_\alpha(|\varphi(z)|^2) E_\alpha(|z|^2)^{-1}\\ &\ge C_1 C_2^{-1} M_{\alpha,z}(\psi,\varphi).\end{aligned}$$

□

The following result is similar to the idea of [16, Proposition 2.1], but for a completeness of exposition we give a proof.

Proposition 3.4. *Let $0 < \alpha < 2$. Let ψ and φ be entire functions on $\mathbb{C}$ such that $\psi \not\equiv \mathbf{0}$. If there exists a positive constant C such that*

$$M_{\alpha,z}(\psi,\varphi) \le C, \quad \forall z \in \mathbb{C}, \tag{3.1}$$

then the function φ takes the form $\varphi(z) = Az + B$, where A and B are complex constants, with $|A| \le 1$.

Proof. Since $\psi \not\equiv \mathbf{0}$, there is an integer $\kappa \in \mathbb{Z}_{\ge 0}$ and an entire function ζ with $\zeta(0) \ne 0$ such that $\psi(z) = z^\kappa \zeta(z)$. Taking logarithms on both sides of (3.1), we obtain

$$|\varphi(z)|^{2/\alpha} - |z|^{2/\alpha} + 2\kappa\log|z| + 2\log|\zeta(z)| \le \log C, \quad \forall z \in \mathbb{C},$$

the convention that $\log 0 = -\infty$ is employed. Through the change of variables $z = re^{i\theta}$ and then integrating with respect to θ on $[-\pi,\pi]$ the following is obtained

$$\begin{aligned}\log C &\ge \int_{-\pi}^{\pi} |\varphi(re^{i\theta})|^{2/\alpha} \frac{d\theta}{2\pi} - r^{2/\alpha} + 2\kappa\log r + 2\int_{-\pi}^{\pi} \log\left|\zeta(re^{i\theta})\right| \frac{d\theta}{2\pi}\\ &\ge \int_{-\pi}^{\pi} |\varphi(re^{i\theta})|^{2/\alpha} \frac{d\theta}{2\pi} - r^{2/\alpha} + 2\kappa\log r + 2\log|\zeta(0)|, \end{aligned} \tag{3.2}$$

where the last inequality uses Jensen's inequality for harmonic functions.

If $0 < \alpha \le 1$, then by Jensen's inequality (see [18, Theorem 1.7.3]), we can further estimate

$$\log C \ge \left(\int_{-\pi}^{\pi} |\varphi(re^{i\theta})|^2 \frac{d\theta}{2\pi} \right)^{1/\alpha} - r^{2/\alpha} + 2\kappa \log r + 2\log|\zeta(0)|. \tag{3.3}$$

Now consider the power expansion

$$\varphi(z) = \sum_{j\ge 0} \varphi_j z^j, \quad z \in \mathbb{C},$$

where $\varphi_0, \varphi_1, \cdots$ are complex constants. Then

$$\int_{-\pi}^{\pi} |\varphi(re^{i\theta})|^2 \frac{d\theta}{2\pi} = \sum_{j\ge 0} |\varphi_j|^2 r^{2j},$$

and hence the inequality (3.3) is rewritten as

$$\log C \ge \left(\sum_{j\ge 0} |\varphi_j|^2 r^{2j} \right)^{1/\alpha} - r^{2/\alpha} + 2\kappa \log r + 2\log|\zeta(0)|.$$

Since this inequality holds for all $r > 0$, we must have $|\varphi_1| \le 1$, and $\varphi_j = 0$ for all $j \ge 2$. This implies that $\varphi(z) = Az + B$, where $|A| \le 1$.

If $1 < \alpha < 2$, then $1 < \frac{2}{\alpha} < 2$, and a lower bound on (3.2) can be realized via the Hausdorff-Young inequality (cf. [15, Theorem IV.2.1]) as

$$\log(C) \ge \left(\sum_{j\ge 0} |\varphi_j|^q r^{qj} \right)^{\frac{1}{q}\cdot\frac{2}{\alpha}} - r^{2/\alpha} + 2\kappa \log r + 2\log|\zeta(0)|,$$

where $q = \frac{2}{2-\alpha}$. Since this inequality holds for all $r > 0$, the conclusion follows as with the case of $0 < \alpha \le 1$. □

3.3. **The operator Ω_m.** For $f(z) = \sum_{k\ge 0} f_k z^k$, we define $\Omega_m f(z) = \sum_{k\ge m} f_k z^k$ acting on $ML^2(\mathbb{C}, \alpha)$. This operator enjoys the following properties.

Proposition 3.5. *Let $\alpha > 0$, then $\|\Omega_m\| = 1$ for all $m \in \mathbb{Z}_{\ge 0}$, and*

$$|\Omega_m f(z)| \le \|f\| \sqrt{\sum_{n=m}^{\infty} \frac{|z|^2}{\Gamma(\alpha n + 1)}}$$

for all $m \in \mathbb{Z}_{\ge 0}$, $z \in \mathbb{C}$, and $f \in ML^2(\mathbb{C}, \alpha)$.

Proof. Since $\mathbf{e}_j(z) = \frac{z^j}{\sqrt{\Gamma(\alpha j+1)}}$ is an orthonormal basis for $ML^2(\mathbb{C}, \alpha)$, it follows immediately that Ω_m is the projection onto $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{m-1}\}^{\perp}$. Thus, $\|\Omega_m\| = 1$, and Ω_m is selfadjoint and idempotent. Moreover, let $f \in ML^2(\mathbb{C}, \alpha)$, then

$$\begin{aligned}
|\Omega_m f(z)| &= |\langle \Omega_m f, K_{\alpha,z} \rangle| \\
&= |\langle f, \Omega_m K_{\alpha,z} \rangle| \\
&\leq \|f\| \|\Omega_m K_{\alpha,z}\| \\
&= \|f\| \sqrt{\langle \Omega_m K_{\alpha,z}, \Omega_m K_{\alpha,z} \rangle} \\
&= \|f\| \sqrt{\langle \Omega_m K_{\alpha,z}, K_{\alpha,z} \rangle} \\
&= \|f\| \sqrt{\Omega_m K_{\alpha,z}(z)} \\
&= \|f\| \sqrt{\sum_{n=m}^{\infty} \frac{|z|^{2n}}{\Gamma(\alpha n + 1)}}.
\end{aligned}$$

The last equality follows as

$$\Omega_m K_{\alpha,w}(z) = \Omega_m \sum_{n=0}^{\infty} \frac{\bar{w}^n z^n}{\Gamma(\alpha n + 1)} = \sum_{n=m}^{\infty} \frac{\bar{w}^n z^n}{\Gamma(\alpha n + 1)}$$

for $w \in \mathbb{C}$. □

4. Dense Domain & Closed Graph

The following result may be well-known, but a proof is included here for the completeness of our exposition.

Proposition 4.1. *Let $\psi : G \to \mathbb{C}$ and $\varphi : G \to G$, and suppose that H is a RKHS. Given the domain*

$$dom(W_{\psi,\varphi,max}) := \{f \in H : \psi(\cdot) \cdot f(\varphi(\cdot)) \in H\}, \tag{4.1}$$

the maximal weighted composition operator, $W_{\psi,\varphi,max} : dom(W_{\psi,\varphi,max}) \to H$, given as $W_{\psi,\varphi,max} f := \psi(\cdot) \cdot f(\varphi(\cdot))$ is closed.

Proof. Suppose that $\{f_n\}_{n=1}^{\infty} \in \text{dom}(W_{\psi,\varphi,max})$ such that $f_n \to f \in H$ and $W_{\psi,\varphi} f_n \to g \in H$. The goal of this proof is to establish that $f \in \text{dom}(W_{\psi,\varphi,max})$ and $W_{\psi,\varphi,max} f = g$. Since, norm convergence implies pointwise convergence in a RKHS, it can be seen that $\psi(x) f_n(\varphi(x)) = W_{\psi,\varphi,max} f_n(x) \to g(x)$ for all $x \in G$, but also for each $x \in G$, $\psi(x) f_n(\varphi(x)) \to \psi(x) f(\varphi(x))$. Hence, $g(x) = \psi(x) f(\varphi(x))$ for all x. Since $g \in H$, it follows that $f \in \text{dom}(W_{\psi,\varphi,max})$ by the definition given in (4.1). □

We say that a linear operator T is *bounded* on a Banach space W if the domain $\text{dom}(T) = W$ and there exists a constant C for which $\|Tx\| \leq C\|x\|$ for all $x \in W$. Proposition 4.1 contains the following corollary.

Corollary 4.2. *A maximal weighted composition operator is bounded on $ML^2(\mathbb{C}, \alpha)$ if and only if its domain is the whole space.*

The result below offers an alternate description of the maximal weighted composition operators.

Proposition 4.3. *Let L be the linear operator given by*

$$dom(L) = \textbf{Span}(\{K_{\alpha,z} : z \in \mathbb{C}\}), \quad LK_{\alpha,z} = \overline{\psi(z)} K_{\alpha,\varphi(z)}.$$

Then $W_{\psi,\varphi,\max} = L^*$. Moreover, the operator $W_{\psi,\varphi,\max}$ is densely defined if and only if the operator L is closable.

Proof. Let $f = \sum_{j=1}^n \lambda_j K_{\alpha,z_j} \in \mathrm{dom}(L)$. For every $g \in ML^2(\mathbb{C},\alpha)$, we have

$$\langle Lf, g\rangle = \sum_{j=1}^n \lambda_j \overline{\psi(z_j) g(\varphi(z_j))} = \sum_{j=1}^n \lambda_j \overline{E(\psi,\varphi) g(z_j)}.$$

By the Riesz lemma, $g \in \mathrm{dom}(L^*)$ if and only if there exists $\Delta > 0$ such that

$$|\langle Lf, g\rangle| \le \Delta \|f\|, \quad \forall f \in \mathrm{dom}(L),$$

or equivalently,

$$|\sum_{j=1}^n E(\psi,\varphi) g(z_j)\overline{\lambda_j}|^2 \le \Delta^2 \sum_{j,\ell=1}^n \lambda_j \overline{\lambda_\ell} K_{\alpha,z_j}(z_\ell).$$

In view of [27], the latter is equivalent to the fact that $E(\psi,\varphi)g \in ML^2(\mathbb{C},\alpha)$. This gives $\mathrm{dom}(L^*) = \mathrm{dom}(W_{\psi,\varphi,\max})$. Moreover,

$$\langle Lf, g\rangle = \langle f, E(\psi,\varphi)g\rangle = \langle f, W_{\psi,\varphi,\max} g\rangle, \quad \forall f \in \mathrm{dom}(L), \forall g \in \mathrm{dom}(W_{\psi,\varphi,\max}),$$

which implies $W_{\psi,\varphi,\max} = L^*$. The conclusion follows from [25, Proposition 1.8(i)]. □

4.1. **Product of two operators.** The following proposition is a necessary condition for an unbounded weighted composition operator to be invertible.

Proposition 4.4. *Let $\alpha > 0$, and let $W_{\psi,\varphi}$ be a densely defined, unbounded weighted composition operator, induced by the symbols ψ and φ. If there exists a bounded, linear operator S on $ML^2(\mathbb{C},\alpha)$ such that*

$$W_{\psi,\varphi} S = I, \tag{4.2}$$

then the following holds:

1. *The function ψ is non-vanishing.*
2. *The function φ takes the form $\varphi(z) = Az + B$ with $A \neq 0$.*
3. *The operator $W_{\zeta,\phi,\max}$ is bounded on $ML^2(\mathbb{C},\alpha)$, where*

$$\zeta(z) = \frac{1}{\psi((z-B)A^{-1})}, \quad \phi(z) = (z-B)A^{-1}. \tag{4.3}$$

4. *The identity $S = W_{\zeta,\phi,\max}$ holds.*

Proof. (1) By (4.2) and [25, Propositions 1.6(iv)-1.7], we have

$$S^* W_{\psi,\varphi}^* \preceq (W_{\psi,\varphi} S)^* = I,$$

and by Lemma 3.2, it follows that

$$\overline{\psi(z)} S^* K_{\alpha,\varphi(z)} = S^* W_{\psi,\varphi}^* K_{\alpha,z} = K_{\alpha,z}, \forall z \in \mathbb{C}.$$

Hence, conclusion (1) is established. Moreover,

$$S^* K_{\alpha,\varphi(z)} = (\overline{\psi(z)})^{-1} K_{\alpha,z}, \forall z \in \mathbb{C}. \tag{4.4}$$

(2) By [26, Exercise 14, Chapter 3], it is enough to show that the function φ is injective. Indeed, if there exist z_1, z_2 such that $\varphi(z_1) = \varphi(z_2)$, then $S^* K_{\alpha,\varphi(z_1)} = S^* K_{\alpha,\varphi(z_2)}$, and so, by (4.4),

$$(\overline{\psi(z_1)})^{-1} K_{\alpha,z_1}(u) = (\overline{\psi(z_2)})^{-1} K_{\alpha,z_2}(u), \forall u \in \mathbb{C}.$$

In particular, with $u = 0$, we get $\psi(z_1) = \psi(z_2) \neq 0$. Substitute back into the above identity to get $K_{\alpha,z_1} = K_{\alpha,z_2}$ or equivalently

$$\sum_{j\ge 0} \frac{u^j \overline{z_1}^j}{\Gamma(\alpha j + 1)} = \sum_{j\ge 0} \frac{u^j \overline{z_2}^j}{\Gamma(\alpha j + 1)}, \quad \forall u \in \mathbb{C}.$$

Differentiating the above equality with respect to the variable u and evaluating it at the point $u = 0$, we obtain $z_1 = z_2$ as desired.

(3) The identity in (4.4) may be expressed as

$$S^*K_{\alpha,u} = \overline{\zeta(u)}K_{\alpha,\phi(u)},\ \forall u \in \mathbb{C}.$$

By Proposition 4.3, we have $W_{\zeta,\phi,\max} = S^{**} = S$, where the last equality uses the fact that the operator S is bounded. □

The following is an observation related to the product of two unbounded weighted composition operators and its proof is left to the reader.

Lemma 4.5. *For entire functions $\psi_1, \psi_2, \varphi_1, \varphi_2$, the product $W_{\psi_1,\varphi_1}W_{\psi_2,\varphi_2}$ is the weighted composition operator given by*

$$W_{\psi_1,\varphi_1}W_{\psi_2,\varphi_2}f = E(\xi,\eta)f, \quad \forall f \in dom(W_{\psi_1,\varphi_1}W_{\psi_2,\varphi_2}),$$

where $\xi = \psi_1 \cdot (\psi_2 \circ \varphi_1)$, $\eta = \varphi_2 \circ \varphi_1$, and the domain is given as

$$dom(W_{\psi_1,\varphi_1}W_{\psi_2,\varphi_2}) = \{f \in dom(W_{\psi_2,\varphi_2}) : W_{\psi_2,\varphi_2}f \in dom(W_{\psi_1,\varphi_1})\}.$$

5. Invertibility

Recall that an unbounded linear operator T is called *invertible* if there exists a bounded linear operator S such that $TS = I$ and $ST \preceq I$. In this section, we characterize *unbounded* weighted composition operators which are invertible. It turns out that the hypothesis of Proposition 4.4 provides a sufficient condition for a maximal weighted composition operator to be invertible.

The first result is devoted to characterizing maximal weighted composition operators which are invertible on $ML^2(\mathbb{C}, \alpha)$.

Theorem 5.1. *Let $\alpha > 0$, and let $W_{\psi,\varphi,\max}$ be a densely defined maximal weighted composition operator induced by the symbols ψ and φ with $\psi \not\equiv \mathbf{0}$. The following assertions are equivalent.*

1. *The operator $W_{\psi,\varphi,\max}$ is invertible.*
2. *There exists a bounded, linear operator S such that identity (4.2) holds, that is $W_{\psi,\varphi,\max}S = I$.*
3. *The symbols satisfy the following conditions:*
   - (a) *The function ψ is non-vanishing.*
   - (b) *The function φ takes the form $\varphi(z) = Az + B$ with $A \neq 0$.*
   - (c) *The operator $W_{\zeta,\phi,\max}$ is bounded on $ML^2(\mathbb{C}, \alpha)$, where the symbols ζ and ϕ are expressed as in (4.3).*

Furthermore, in this case, $W^{-1}_{\psi,\varphi,\max} = W_{\zeta,\phi,\max}$.

Proof. It is clear that (1)⟹(2), while the implication (2)⟹(3) holds by Proposition 4.4.

Hence, the implication (3)⟹(1) remains to be verified. Since the operator $W_{\zeta,\phi,\max}$ is bounded, by Lemma 4.5, it can be seen that

$$W_{\psi,\varphi,\max}W_{\zeta,\phi,\max} = I, \quad W_{\zeta,\phi,\max}W_{\psi,\varphi,\max} \preceq I,$$

which yields (1). □

The next result shows that an invertible weighted composition operator must be maximal.

Theorem 5.2. *Let $\alpha > 0$, and let $W_{\psi,\varphi}$ be a densely defined unbounded weighted composition operator induced by the symbols ψ and φ with $\psi \not\equiv \mathbf{0}$. The following assertions are equivalent.*

1. *The operator $W_{\psi,\varphi}$ is invertible.*
2. *The identity $W_{\psi,\varphi,\max} = W_{\psi,\varphi}$ holds, and*
   - (a) *The function ψ is non-vanishing.*
   - (b) *The function φ takes the form $\varphi(z) = Az + B$ with $A \neq 0$.*
   - (c) *The operator $W_{\zeta,\phi,\max}$ is bounded on $ML^2(\mathbb{C}, \alpha)$, where the symbols ζ, ϕ are of forms (4.3).*

Proof. It is clear that (2)$\Longrightarrow$(1). We prove (1)$\Longrightarrow$(2) as follows. Suppose that the operator $W_{\psi,\varphi}$ is invertible. Hence by Proposition 4.4, the conclusions 2(a)-2(c) hold, and

$$W_{\psi,\varphi}W_{\zeta,\phi,\max} = I, \quad W_{\zeta,\phi,\max}W_{\psi,\varphi} \preceq I.$$

By Theorem 5.1, it follows that

$$W_{\psi,\varphi,\max}W_{\zeta,\phi,\max} = I, \quad W_{\zeta,\phi,\max}W_{\psi,\varphi,\max} \preceq I.$$

Thus,

$$W_{\psi,\varphi,\max} = W_{\psi,\varphi}W_{\zeta,\phi,\max}W_{\psi,\varphi,\max} \preceq W_{\psi,\varphi},$$

which gives $W_{\psi,\varphi,\max} = W_{\psi,\varphi}$. □

An invertible weighted composition operator $W_{\psi,\varphi}$ gives rise to a symbol ψ which is non-vanishing over $\mathbb{C}$ in $ML^2(\mathbb{C},\alpha)$. Thus, it is necessary to discover the structure of ψ in our framework.

Proposition 5.3. *Let $0<\alpha<2$ and let $h \in \mathcal{E}(\mathbb{C})$. The following assertions are equivalent.*

(1) $e^h \in ML^2(\mathbb{C},\alpha)$.
(2) *The function h must be a polynomial with degree at most $[2/\alpha]$, and the leading coefficient of h has modulus less than $1/2$ if $2/\alpha \in \mathbb{Z}$.*

In this case, the inclusion $e^h\mathbb{C}[z] \subset ML^2(\mathbb{C},\alpha)$ holds.

Proof. The implication (2)$\Longrightarrow$(1) was proved in [24, Proposition 4.4]. The rest task is to show that (1)$\Longrightarrow$(2). Recall that C_1 and C_2 denote positive constants mentioned in Remark 2.3. By Proposition 2.5, it follows that

$$0 = \lim_{|z|\to\infty} \left\langle e^h, k_{\alpha,z} \right\rangle = \lim_{|z|\to\infty} \frac{e^{h(z)}}{\|K_{\alpha,z}\|} = \lim_{|z|\to\infty} \frac{e^{h(z)}}{E_\alpha(|z|^2)^{1/2}}.$$

By the definition of a limit, for z with sufficiently large modulus we have

$$e^{\operatorname{Re} h(z)} \le C_2^{-1/2} E_\alpha(|z|^2)^{1/2} \le e^{\frac{1}{2}|z|^{2/\alpha}}.$$

Then the order ρ of e^h is

$$\begin{aligned} \rho &= \limsup_{\delta\to\infty} \frac{\log[\max_{|z|=\delta} \operatorname{Re} h(z)]}{\log \delta} \\ &\le \limsup_{\delta\to\infty} \frac{\log[2^{-1}\delta^{2/\alpha}]}{\log\delta} \\ &\le \frac{2}{\alpha}. \end{aligned}$$

By [2, Corollary 4.5.11], the function h must be a polynomial with degree at most $[2/\alpha]$.

Consider the case when $2/\alpha \in \mathbb{Z}$. In this case, we will demonstrate, via a proof by contradiction, that the leading coefficient of h has modulus less than $1/2$. Indeed, assume that

$$h(z) = \sum_{j=0}^{\kappa} h_j z^j, \quad z \in \mathbb{C},$$

where $\kappa = 2/\alpha$ and $h_0, \cdots, h_\kappa$ are complex constants with $|h_\kappa| \ge 1/2$. Since $h_\kappa = |h_\kappa| e^{i\arg(h_\kappa)}$, without loss of generality, we may assume that $h_\kappa \in \mathbb{R}$ with $h_\kappa \ge 1/2$. For $t \in \mathbb{R}$, we have

$$\left\langle e^h, k_{\alpha,t} \right\rangle = \|K_{\alpha,t}\|^{-1} \left\langle e^h, K_{\alpha,t} \right\rangle = \|K_{\alpha,t}\|^{-1} e^{h(t)} = E_\alpha(t^2)^{-1/2} e^{\sum_{j=0}^{\kappa} h_j t^j}.$$

Hence,

$$\left| \left\langle e^h, k_{\alpha,t} \right\rangle \right| \ \ge\ C_2^{-1/2} e^{\sum_{j=0}^{\kappa-1} t^j \operatorname{Re} h_j + (h_\kappa - \frac{1}{2}) t^\kappa}.$$

Since $h_\kappa \geq 1/2$, the right-hand side cannot tend to 0 as $t \to \pm\infty$, but this is impossible by Proposition 2.5.

Here the inclusion $e^h\mathbb{C}[z] \subset ML^2(\mathbb{C},\alpha)$ will be demonstrated for the case when $2/\alpha \in \mathbb{Z}$, and the remaining case is left to the reader. As proved above, the function h takes the following form $h(z) = \sum\limits_{j=0}^{\kappa} h_j z^j$, where $\kappa = 2/\alpha$, and $|h_\kappa| < 1/2$. Let $g \in \mathbb{C}[z]$ and ε be a positive number such that

$$0 < \varepsilon < \frac{1}{2}\left(\frac{1}{2} - |h_\kappa|\right).$$

There exists $R > 0$ such that for every $|z| \geq R$ we have

$$|g(z)| \leq e^{\varepsilon|z|^\kappa}, \quad \left|\sum_{j=0}^{\kappa-1} h_j z^j\right| \leq \varepsilon |z|^\kappa,$$

and then

$$\left|g(z)e^{h(z)}\right| = |g(z)|\, e^{\operatorname{Re} h(z)} \leq |g(z)|\, e^{|h(z)|} \leq e^{(|h_\kappa|+2\varepsilon)|z|^\kappa}, \quad \forall |z| \geq R.$$

Thus,

$$\begin{aligned}
&\int\limits_{\mathbb{C}} \left|g(z)e^{h(z)}\right|^2 |z|^{\kappa-2} e^{-|z|^\kappa}\, dz \\
&\leq \left(\int\limits_{|z|\leq R} + \int\limits_{|z|>R}\right) \left|g(z)e^{h(z)}\right|^2 |z|^{\kappa-2} e^{-|z|^\kappa}\, dz \\
&\leq \int\limits_{|z|\leq R} \left|g(z)e^{h(z)}\right|^2 |z|^{\kappa-2} e^{-|z|^\kappa}\, dz + \int\limits_{|z|>R} |z|^{\kappa-2} e^{-[1-2(|h_\kappa|+2\varepsilon)]\cdot|z|^\kappa}\, dz \\
&< \infty.
\end{aligned}$$

□

6. Cyclicity & Complex Symmetry

A closed densely defined linear operator T is called (i) *cyclic* if there exists an element $x \in \mathrm{dom}(T^\infty)$ such that $\mathbf{Span}\,\mathcal{O}(T,x)$ is dense; (ii) $\mathcal{C}$*-self adjoint* (or simply: *complex symmetric*) if there exists a conjugation $\mathcal{C}$ (i.e. an anti-linear isometric involution) such that $T = \mathcal{C}T^*\mathcal{C}$. The theory of complex symmetric operators has developed over the last decade since being initiated by Garcia and Putinar in [8, 9]. Due to potential applications, complex symmetric operators are of great importance in quantum mechanics (cf. [7]). In this section, we establish some conditions for a complex symmetric operator to be cyclic on Mittag-Leffler spaces.

Theorem 6.1. *Let $\alpha > 0$, and let $W_{\psi,\varphi}$ be a densely defined unbounded weighted composition operator induced by two entire functions $\psi \not\equiv \mathbf{0}$ and $\varphi \not\equiv \mathbf{const}$. Suppose that there exists an involutive mapping $S : ML^2(\mathbb{C},\alpha) \to ML^2(\mathbb{C},\alpha)$, such that*

$$dom(W^*_{\psi,\varphi}) \subseteq dom(W_{\psi,\varphi}S) \tag{6.1}$$

and

$$\|W_{\psi,\varphi}Sf\| \leq \|W^*_{\psi,\varphi}f\|, \ \forall f \in dom(W^*_{\psi,\varphi}). \tag{6.2}$$

The following conclusions hold.

1. *The function ψ is non-vanishing.*
2. *The function φ takes the form $\varphi(z) = Az+B$, where A and B are complex constants, with $A \neq 0$.*

(3) *If* $|A|<1$, *then* $W^*_{\psi,\varphi}$ *is cyclic.*

Proof. (1) Assume to the contrary that $\psi(z_0)=0$ for some $z_0\in\mathbb{C}$. Then there is a neighbourhood V of z_0 such that $\psi(z)\neq 0$ for every $z\in V\setminus\{z_0\}$. Lemma 3.2 shows that $K_{\alpha,z_0}\in \mathrm{dom}(W^*_{\psi,\varphi})$ and $W^*_{\psi,\varphi}K_{\alpha,z_0}=\overline{\psi(z_0)}K_{\alpha,\varphi(z_0)}=0$.

By assumptions (6.1)-(6.2), we have $SK_{\alpha,z_0}\in\mathrm{dom}(W_{\psi,\varphi})$ and $W_{\psi,\varphi}SK_{\alpha,z_0}=0$. Consequently, taking into account the structure of the operator $W_{\psi,\varphi}$, it follows that

$$\psi(z)SK_{\alpha,z_0}(\varphi(z))=W_{\psi,\varphi}SK_{\alpha,z_0}(z)=0,\quad \forall z\in\mathbb{C},$$

which implies that $SK_{\alpha,z_0}\circ\varphi\equiv 0$ on $V\backslash\{z_0\}$. Since φ is a non-constant function, $SK_{\alpha,z_0}\equiv 0$, $K_{\alpha,z_0}\equiv 0$ (because S is involutive). This is a contradiction.

(2) By [26, Exercise 14, Chapter 3], it is enough to show that the function φ is injective.

Suppose that $\varphi(z_1)=\varphi(z_2)$, for some $z_1,z_2\in\mathbb{C}$. Since K_{α,z_1} and K_{α,z_2} both belong to the domain $\mathrm{dom}(W^*_{\psi,\varphi})$, so do their linear combinations. Lemma 3.2(1) gives

$$W^*_{\psi,\varphi}(\overline{\psi(z_2)}K_{\alpha,z_1}-\overline{\psi(z_1)}K_{\alpha,z_2})=\overline{\psi(z_1)\psi(z_2)}K_{\alpha,\varphi(z_1)}-\overline{\psi(z_1)\psi(z_2)}K_{\alpha,\varphi(z_2)}=\mathbf{0}.$$

This implies, again by assumption (6.2), that $W_{\psi,\varphi}S(\overline{\psi(z_2)}K_{\alpha,z_1}-\overline{\psi(z_1)}K_{\alpha,z_2})=\mathbf{0}$. Which means that $S(\overline{\psi(z_2)}K_{\alpha,z_1}-\overline{\psi(z_1)}K_{\alpha,z_2})\in\ker(W_{\psi,\varphi})$. Hence, by Proposition 3.1, it must be a zero function. Since the operator S is involutive, we get

$$(\overline{\psi(z_2)}K_{\alpha,z_1}-\overline{\psi(z_1)}K_{\alpha,z_2})(u)=0,\ \forall u\in\mathbb{C},$$

which gives $z_1=z_2$.

(3) Let $d\neq B(1-A)^{-1}$ and

$$S:=\mathbf{Cl}\left[\mathbf{Span}\{(W^*_{\psi,\varphi})^nK_{\alpha,d}:\quad n\in\mathbb{Z}_{\geq 0}\}\right].$$

Assume to the contrary that there exists an element $g\perp S$. Hence, Lemma 3.2 gives

$$\begin{aligned}0 &= \left\langle g,(W^*_{\psi,\varphi})^nK_{\alpha,d}\right\rangle=\psi(d)\psi(\varphi(d))\cdots\psi(\varphi_{n-1}(d))\left\langle g,K_{\alpha,\varphi_n(d)}\right\rangle\\ &= \psi(d)\psi(\varphi(d))\cdots\psi(\varphi_{n-1}(d))g(\varphi_n(d)),\end{aligned}$$

which implies, by the first part, that $g(\varphi_n(d))=0$ for all $n\in\mathbb{Z}_{\geq 0}$. Inductive arguments show that $\varphi_n(d)=A^nd+(1-A^n)B(1-A)^{-1}$, which implies, as $|A|<1$, that $\lim_{n\to\infty}g(\varphi_n(d))=B(1-A)^{-1}$. Thus, $g\equiv\mathbf{0}$ by the identity theorem, and so we have $S=ML^2(\mathbb{C},\alpha)$. ☐

Remark 6.2. Note that the class of operators satisfying the conditions (6.1)-(6.2) is quite diverse. In particular, it contains many well-known operators, such as selfadjoint operators, normal operators, $\mathcal{C}$-self adjoint operators, and cohyponormal operators, etc.

Corollary 6.3. *Let* $\alpha>0$, *and let* $W_{\psi,\varphi}$ *be a cohyponormal weighted composition operator induced by two entire functions* $\psi\not\equiv\mathbf{0}$ *and* $\varphi\not\equiv\mathbf{const}$. *If* $|\varphi'(0)|<1$, *then* $W_{\psi,\varphi}$, $W^*_{\psi,\varphi}$ *are cyclic operators on* $ML^2(\mathbb{C},\alpha)$.

Corollary 6.4. *Let* $\alpha>0$, *and let* $W_{\psi,\varphi}$ *be a* $\mathcal{C}$*-self adjoint weighted composition operator, induced by two entire functions* $\psi\not\equiv\mathbf{0}$ *and* $\varphi\not\equiv\mathbf{const}$. *Then the following conclusions hold:*

(1) *The function* ψ *is non-vanishing.*
(2) *The function* φ *takes the form* $\varphi(z)=Az+B$, *where* A,B *are complex constants, with* $A\neq 0$.
(3) *If* $|A|<1$, *then* $W_{\psi,\varphi}$, $W^*_{\psi,\varphi}$ *are cyclic operators on* $ML^2(\mathbb{C},\alpha)$.

Proof. By the definition of a complex symmetric operator, there exists a conjugation $\mathcal{C}$ such that

$$\mathrm{dom}(W^*_{\psi,\varphi})=\mathrm{dom}(W_{\psi,\varphi}\mathcal{C}),\quad W_{\psi,\varphi}\mathcal{C}f=\mathcal{C}W^*_{\psi,\varphi}f,\ \forall f\in\mathrm{dom}(W^*_{\psi,\varphi}),$$

which implies, as $\mathcal{C}$ is isometric, that

$$\mathrm{dom}(W^*_{\psi,\varphi})=\mathrm{dom}(W_{\psi,\varphi}\mathcal{C}),\quad \|W_{\psi,\varphi}\mathcal{C}f\|=\left\|W^*_{\psi,\varphi}f\right\|,\ \forall f\in\mathrm{dom}(W^*_{\psi,\varphi}).$$

Hence, we can use Theorem 6.1 to demonstrate that the operator $W_{\psi,\varphi}^*$ is cyclic, where $\varphi(z) = Az + B$ with $|A| < 1$, and the linear span of the following elements

$$(W_{\psi,\varphi}^*)^n K_{\alpha,d}, \quad \forall n \in \mathbb{Z}_{\geq 0}$$

is dense in $ML^2(\mathbb{C}, \alpha)$. Here, $d \neq B(1-A)^{-1}$. We prove by induction on $n \in \mathbb{Z}_{\geq 0}$, that $\mathcal{C}K_{\alpha,d} \in \text{dom}[(W_{\psi,\varphi})^n]$ and

$$(W_{\psi,\varphi})^n \mathcal{C}K_{\alpha,d} = \mathcal{C}(W_{\psi,\varphi}^*)^n K_{\alpha,d}.$$

The proofs for $n = 0$ and $n = 1$ are left to the reader. Now we suppose that the conclusion holds for $n = \kappa$, and we will demonstrate it for $n = \kappa + 1$. By Lemma 3.2, we have $K_{\alpha,d} \in \text{dom}[(W_{\psi,\varphi}^*)^\kappa]$,

$$(W_{\psi,\varphi}^*)^\kappa K_{\alpha,d} = \overline{\psi(d)\psi(\varphi(d))\cdots\psi(\varphi_{\kappa-1}(d))} K_{\alpha,\varphi_\kappa(d)},$$

and hence

$$\mathcal{C}(W_{\psi,\varphi}^*)^\kappa K_{\alpha,d} = \psi(d)\psi(\varphi(d))\cdots\psi(\varphi_{\kappa-1}(d))\mathcal{C}K_{\alpha,\varphi_\kappa(d)}.$$

By the inductive assumption with $n = \kappa$, we get $\mathcal{C}K_{\alpha,d} \in \text{dom}[(W_{\psi,\varphi})^\kappa]$ and

$$(W_{\psi,\varphi})^\kappa \mathcal{C}K_{\alpha,d} = \mathcal{C}(W_{\psi,\varphi}^*)^\kappa K_{\alpha,d} = \psi(d)\psi(\varphi(d))\cdots\psi(\varphi_{\kappa-1}(d))\mathcal{C}K_{\alpha,\varphi_\kappa(d)}.$$

Again by Lemma 3.2, $K_{\alpha,\varphi_\kappa(d)} \in \text{dom}(W_{\psi,\varphi}^*)$, which implies, due to the complex symmetry of $W_{\psi,\varphi}$, that $\mathcal{C}K_{\alpha,\varphi_\kappa(d)} \in \text{dom}(W_{\psi,\varphi})$ and

$$W_{\psi,\varphi}\mathcal{C}K_{\alpha,\varphi_\kappa(d)} = \mathcal{C}W_{\psi,\varphi}^* K_{\alpha,\varphi_\kappa(d)}.$$

Thus, we conclude that $\mathcal{C}K_{\alpha,d} \in \text{dom}[(W_{\psi,\varphi})^{\kappa+1}]$ and

$$\begin{aligned}
(W_{\psi,\varphi})^{\kappa+1}\mathcal{C}K_{\alpha,d} &= W_{\psi,\varphi}\mathcal{C}(W_{\psi,\varphi}^*)^\kappa K_{\alpha,d} \\
&= \psi(d)\psi(\varphi(d))\cdots\psi(\varphi_{\kappa-1}(d))W_{\psi,\varphi}\mathcal{C}K_{\alpha,\varphi_\kappa(d)} \\
&= \psi(d)\psi(\varphi(d))\cdots\psi(\varphi_{\kappa-1}(d))\mathcal{C}W_{\psi,\varphi}^* K_{\alpha,\varphi_\kappa(d)} \\
&= \psi(d)\psi(\varphi(d))\cdots\psi(\varphi_{\kappa-1}(d))\psi(\varphi_\kappa(d))\mathcal{C}K_{\alpha,\varphi_{\kappa+1}(d)} \\
&= \mathcal{C}(W_{\psi,\varphi}^*)^{\kappa+1}K_{\alpha,d},
\end{aligned}$$

where the fourth and fifth equalities use Lemma 3.2. These show that the set

$$\mathbf{Cl}\,[\mathbf{Span}\{(W_{\psi,\varphi})^n\mathcal{C}K_{\alpha,d} : \quad \forall n \in \mathbb{Z}_{\geq 0}\}]$$

is dense in $ML^2(\mathbb{C}, \alpha)$. In other words, the operator $W_{\psi,\varphi}$ is cyclic. □

7. Boundedness

In this section, we give characterizations for the boundedness of $W_{\psi,\varphi} : ML^2(\mathbb{C}, \alpha) \to ML^2(\mathbb{C}, \alpha)$, where $0 < \alpha < 2$. Our results involve the estimate for the operator norm $\|W_{\psi,\varphi}\|$.

Theorem 7.1. *Let $0 < \alpha < 2$, and let $\varphi : \mathbb{C} \to \mathbb{C}$ and $\psi : \mathbb{C} \to \mathbb{C}$ be two functions such that φ is nonconstant and ψ is not identically zero. Then the following assertions are equivalent.*

(1) *The maximal weighted composition operator $W_{\psi,\varphi,\max}$ is bounded on $ML^2(\mathbb{C}, \alpha)$.*
(2) *The domain dom$(W_{\psi,\varphi,\max}) = ML^2(\mathbb{C}, \alpha)$.*
(3) *The symbols are entire and satisfy $M_\alpha(\psi, \varphi) < \infty$.*

In this case, the function φ takes the form $\varphi(z) = Az + B$, where A, B are complex constants with $0 < |A| \leq 1$, and the operator norm satisfies

$$\|W_{\psi,\varphi}f\| \leq M_\alpha(\psi, \varphi)^{1/2}|A|^{-1/\alpha}2^{1/\alpha-1}(C_2|B|^{2/\alpha}\alpha^{-1} + 1)^{1/2} \cdot \|f\|, \quad \forall f \in ML^2(\mathbb{C}, \alpha),$$

when $0 < \alpha \leq 1$, and

$$\|W_{\psi,\varphi}f\| \leq M_\alpha(\psi, \varphi)^{\frac{1}{2}}|A|^{-1/\alpha}\left(C_2\pi\alpha(3|B|)^{\frac{2}{\alpha}} + \frac{\alpha}{2^{\frac{2}{\alpha}-2}}\right)^{\frac{1}{2}} \cdot \|f\|,$$

when $1 < \alpha < 2$.

Proof. The equivalence (1)$\Longleftrightarrow$(2) was discussed in Corollary 4.2. Recall that C_1 and C_2 denote positive constants mentioned in Remark 2.3.

To prove that (1)$\Longrightarrow$(3), suppose that the weighted composition operator $W_{\psi,\varphi,\max} : ML^2(\mathbb{C},\alpha) \to ML^2(\mathbb{C},\alpha)$ is bounded.

Polynomials are contained in $ML^2(\mathbb{C};\alpha)$, and since e^z has order 1, it is contained in $ML^2(\mathbb{C};\alpha)$ for $0 < \alpha < 2$. It can then be seen that $W_{\psi,\varphi}1 = \psi(z)$, and $\psi(z) \in ML^2(\mathbb{C};\alpha)$. Also, $W_{\psi,\varphi}z = \psi(z)\varphi(z) \in ML^2(\mathbb{C};\alpha)$, so φ is analytic everywhere except possibly where ψ is zero. Moreover, $\varphi(z)$ has at worst poles of finite multiplicity. However, $e^{\varphi(z)}$ has essential singularities everywhere that φ has a pole. Since $W_{\psi,\varphi}e^z = \psi(z)e^{\varphi(z)} \in ML^2(\mathbb{C};\alpha)$ and ψ has zeros of finite multiplicity wherever $e^{\varphi(z)}$ has an essential singularity, the function $z \mapsto \psi(z)e^{\varphi(z)}$ would not be entire if φ had a singularity. Thus, φ is analytic throughout $\mathbb{C}$ and is entire.

Let $\Delta := \|W_{\psi,\varphi,\max}\|$. By Lemma 3.3, we have

$$M_{\alpha,z}(\psi,\varphi) \le \Delta^2 C_2 C_1^{-1}, \quad \forall z \in \mathbb{C}.$$

Taking the supremum with respect to $z \in \mathbb{C}$ yields $M_\alpha(\psi,\varphi) < \infty$. Hence, by Proposition 3.4 the function φ takes the form $\varphi(z) = Az + B$, where A and B are complex constants, with $|A| \le 1$. Thus, we have (1)$\Longrightarrow$(3).

Conversely, to prove (3)$\Longrightarrow$(2), suppose that $M_\alpha(\psi,\varphi) < \infty$. Let $f \in ML^2(\mathbb{C},\alpha)$, then by Proposition 2.4 and Remark 2.3

$$|f(z)|^2 \le E_\alpha(|z|^2)\|f\|^2 \le C_2 e^{|z|^{2/\alpha}}\|f\|^2, \quad \forall z \in \mathbb{C}. \tag{7.1}$$

We have

$$\begin{aligned}
&\int_{\mathbb{C}} |\psi(z)f(\varphi(z))|^2 |z|^{\frac{2}{\alpha}-2} e^{-|z|^{\frac{2}{\alpha}}} \frac{dz}{\alpha\pi} \\
&\le M_\alpha(\psi,\varphi) \int_{\mathbb{C}} |f(\varphi(z))|^2 |z|^{\frac{2}{\alpha}-2} e^{-|\varphi(z)|^{\frac{2}{\alpha}}} \frac{dz}{\alpha\pi} \\
&= |A|^{-2/\alpha} M_\alpha(\psi,\varphi) \int_{\mathbb{C}} |f(u)|^2 |u-B|^{\frac{2}{\alpha}-2} e^{-|u|^{\frac{2}{\alpha}}} \frac{du}{\alpha\pi}, \qquad (7.2)
\end{aligned}$$

where in the last equality was obtained via a change of variables $u = \varphi(z) = Az + B$.

For $0 < \alpha \le 1$, the integral on the right-hand side is written as

$$\begin{aligned}
&\left(\int_{|u|\le|B|} + \int_{|u|\ge|B|} \right) |f(u)|^2 |u-B|^{\frac{2}{\alpha}-2} e^{-|u|^{\frac{2}{\alpha}}}\, du \\
&\le (2|B|)^{\frac{2}{\alpha}-2} \int_{|u|\le|B|} |f(u)|^2 e^{-|u|^{\frac{2}{\alpha}}}\, du + \int_{|u|\ge|B|} |f(u)|^2 (2|u|)^{\frac{2}{\alpha}-2} e^{-|u|^{\frac{2}{\alpha}}}\, du \\
&\le \pi C_2 2^{\frac{2}{\alpha}-2} |B|^{2/\alpha}\|f\|^2 + \int_{|u|\ge|B|} |f(u)|^2 (2|u|)^{\frac{2}{\alpha}-2} e^{-|u|^{\frac{2}{\alpha}}}\, du \\
&\le \pi 2^{\frac{2}{\alpha}-2}(C_2|B|^{2/\alpha} + \alpha) \cdot \|f\|^2,
\end{aligned}$$

where the second inequality uses (7.1). We conclude that

$$\begin{aligned}
&\int_{\mathbb{C}} |\psi(z)f(\varphi(z))|^2 |z|^{\frac{2}{\alpha}-2} e^{-|z|^{\frac{2}{\alpha}}} \frac{dz}{\alpha\pi} \\
&\le M_\alpha(\psi,\varphi)|A|^{-2/\alpha} 2^{2/\alpha-2} \alpha^{-1}(C_2|B|^{2/\alpha} + \alpha) \cdot \|f\|^2,
\end{aligned}$$

which means that $\mathrm{dom}(W_{\psi,\varphi,\max}) = ML^2(\mathbb{C},\alpha)$.

If $1<\alpha<2$, note that $\frac{|u|}{2} < |u-B| < 2|u|$ for $|u|>2|B|$. Hence, for $1<\alpha<2$, $|u-B|^{\frac{2}{\alpha}-2} < \frac{1}{2^{\frac{2}{\alpha}-2}}|u|^{\frac{2}{\alpha}-2}$. Thus, the integral in (7.2) may be bounded as

$$\begin{aligned}\int_{\mathbb{C}} |f(u)|^2|u-B|^{\frac{2}{\alpha}-2}e^{-|u|^{\frac{2}{\alpha}}}du &= \left(\int_{|u|\le 2B} + \int_{|u|>2B}\right)|f(u)|^2|u-B|^{\frac{2}{\alpha}-2}e^{-|u|^{\frac{2}{\alpha}}}du\\ &\le \sup_{|u|\le 2B}\left(|f(u)|e^{-|u|^{\frac{2}{\alpha}}}\right)\int_{|u|\le 2B}|u-B|^{\frac{2}{\alpha}-2}du + \frac{1}{2^{\frac{2}{\alpha}-2}}\int_{|u|>2B}|f(u)|^2|u|^{\frac{2}{\alpha}-2}e^{-|u|^{\frac{2}{\alpha}}}du\\ &\le \|f\|^2_{ML^2(\mathbb{C},\alpha)}\left(C_2\int_{|u|\le 2B}|u-B|^{\frac{2}{\alpha}-2}du + \frac{\alpha}{2^{\frac{2}{\alpha}-2}}\right)\\ &\le \|f\|^2_{ML^2(\mathbb{C},\alpha)}\left(C_2\pi\alpha(3|B|)^{\frac{2}{\alpha}} + \frac{\alpha}{2^{\frac{2}{\alpha}-2}}\right),\end{aligned}$$

where C_2 is given in Remark 2.3. Therefore,

$$\begin{aligned}&\int_{\mathbb{C}} |\psi(z)f(\varphi(z))|^2|z|^{\frac{2}{\alpha}-2}e^{-|z|^{\frac{2}{\alpha}}}\frac{dz}{\alpha\pi}\\ &\le M_\alpha(\psi,\varphi)|A|^{-2/\alpha}\left(C_2\pi\alpha(3|B|)^{\frac{2}{\alpha}} + \frac{\alpha}{2^{\frac{2}{\alpha}-2}}\right)\cdot\|f\|^2\end{aligned}$$

for the case $1<\alpha<2$.

Thus, we have (3)$\Longrightarrow$(2). □

Theorem 7.2. *Let $0<\alpha<2$, and let $\varphi \equiv B$ be a constant function, and let $\psi \not\equiv \mathbf{0}$ be an entire function. Then the following assertions are equivalent:*

1. *The maximal weighted composition operator $W_{\psi,\varphi,\max}$ is bounded on $ML^2(\mathbb{C},\alpha)$.*
2. *The domain $\mathrm{dom}(W_{\psi,\varphi,\max}) = ML^2(\mathbb{C},\alpha)$.*
3. *The function ψ satisfies $\psi \in ML^2(\mathbb{C},\alpha)$.*

In this case, the operator norm satisfies

$$\|W_{\psi,\varphi}f\| \le \|\psi\|E_\alpha(|B|^2)^{1/2}\|f\|, \quad \forall f \in ML^2(\mathbb{C},\alpha).$$

Proof. The equivalence (1)$\Longleftrightarrow$(2) was discussed in Corollary 4.2. Let $\varphi \equiv B$, where B is a complex constant.

For (1)$\Longleftrightarrow$(3), suppose that $W_{\psi,\varphi}$ is bounded. Then $\psi = W_{\psi,\varphi}(\mathbf{1}) \in ML^2(\mathbb{C},\alpha)$.

Conversely, for (3)$\Longleftrightarrow$(1), suppose $\psi \in ML^2(\mathbb{C},\alpha)$. For every $f \in ML^2(\mathbb{C},\alpha)$, we have $E(\psi,\varphi)f = \psi\cdot f(B)$, and hence

$$\|E(\psi,\varphi)f\| = |f(B)|\cdot\|\psi\| \le \|\psi\|E_\alpha(|B|^2)^{1/2}\|f\| < \infty.$$

The proof is complete. □

8. Compactness & Essential norm

Upon the resolution of the characterization of bounded weighted composition operators over the Mittag-Leffler space, a remaining challenge is the study of the compactness and essential norm of weighted composition operators.

In a general setting, for two Banach spaces $\mathbb{X}$ and $\mathbb{V}$ we denote by $\mathcal{C}(\mathbb{X},\mathbb{V})$ the set of all compact operators from $\mathbb{X}$ into $\mathbb{V}$. The essential norm of a bounded, linear operator $A:\mathbb{X}\to\mathbb{V}$, denoted as $\|A\|_e$, is defined as

$$\|A\|_e := \inf\{\|A-T\| : T \in \mathcal{C}(\mathbb{X},\mathbb{V})\}.$$

It is easy to check that A is compact if and only if $\|A\|_e = 0$.

As introduced in Preliminaries, $ML^2(\mathbb{C},\alpha)$ is a Hilbert space. So, we can make use of the following criterion for compactness (cf. [19]).

Proposition 8.1. *Let $0<\alpha<2$. A bounded, linear operator V is compact on $ML^2(\mathbb{C},\alpha)$ if and only if $\lim\limits_{m\to\infty}\|Vg_m - Vg\| \to 0$ whenever $g_m \to g$ weakly.*

The following result is a criterion for the weak convergence on the Mittag-Leffler space, $ML^2(\mathbb{C},\alpha)$, with $\alpha>0$.

Proposition 8.2. *Let $0<\alpha<2$. A sequence (g_m) converges weakly to 0 in $ML^2(\mathbb{C},\alpha)$ if and only if it is*

1. *bounded in the norm of $ML^2(\mathbb{C},\alpha)$;*
2. *uniformly convergent to 0 on compact subsets of $\mathbb{C}$.*

An immediate consequence of Proposition 8.2 is the following.

Corollary 8.3. *Let $0<\alpha<2$, $A\in\mathbb{C}\setminus\{0\}$, $B\in\mathbb{C}$ and (w_m) be a sequence in $\mathbb{C}$ with $\lim\limits_{m\to\infty}|w_m| = \infty$. Then the sequence (g_m), where $g_m = k_{\alpha,Aw_m+B}, \forall m\geq 1$, converges weakly in $ML^2(\mathbb{C},\alpha)$ to 0.*

We are now in a position to estimate $\|W_{\psi,\varphi}\|_e$.

Theorem 8.4. *Let $0<\alpha<2$, and $W_{\psi,\varphi}$ be a bounded weighted composition operator on $ML^2(\mathbb{C},\alpha)$ induced by two entire functions $\varphi\not\equiv$ **const** and $\psi\not\equiv$ **0** (that is $\varphi(z)=Az+B$ with $0<|A|\leq 1$). Then the essential norm satisfies the following estimate*

$$C_1^{1/2}C_2^{-1/2}\limsup_{|z|\to\infty} M_{\alpha,z}(\psi,\varphi)^{1/2} \leq \|W_{\psi,\varphi}\|_e \leq |A|^{-1/\alpha}2^{1/\alpha-1}\limsup_{|z|\to\infty} M_{\alpha,z}(\psi,\varphi)^{1/2},$$

where C_1 and C_2 are the positive constants mentioned in Remark 2.3.

Proof. First, we establish an estimate on the lower bound. Let (z_n) be a sequence such that $|z_n|\to\infty$, and F be a compact operator on $ML^2(\mathbb{C},\alpha)$. By Corollary 8.3, the sequence $(k_{\alpha,\varphi(z_n)})$ converges weakly to 0 as $n\to\infty$ and hence, $\|Fk_{\alpha,\varphi(z_n)}\|\to 0$ as $n\to\infty$. Therefore,

$$\begin{aligned}
\|W_{\psi,\varphi}-F\| &\geq \limsup_{n\to\infty}\|(W_{\psi,\varphi}-F)k_{\alpha,\varphi(z_n)}\| \\
&\geq \limsup_{n\to\infty}(\|W_{\psi,\varphi}k_{\alpha,\varphi(z_n)}\| - \|Fk_{\alpha,\varphi(z_n)}\|) \\
&\geq \limsup_{n\to\infty}\|W_{\psi,\varphi}k_{\alpha,\varphi(z_n)}\| \\
&\geq C_1^{1/2}C_2^{-1/2}\limsup_{n\to\infty} M_{\alpha,z_n}(\psi,\varphi)^{1/2},
\end{aligned}$$

where the last inequality uses Lemma 3.3.

In order to estimate the upper bound, we need the following estimate

$$\|W_{\psi,\varphi}\|_e \leq \liminf_{m\to\infty}\|W_{\psi,\varphi}\Omega_m\|, \tag{8.1}$$

where Ω_m is the operator defined in Section 3.3. Note that since $I-\Omega_m$ is compact on $ML^2(\mathbb{C},\alpha)$, so is $W_{\psi,\varphi}(I-\Omega_m)$, and $\|W_{\psi,\varphi}(I-\Omega_m)\|_e = 0$. For any compact operator A on $ML^2(\mathbb{C},\alpha)$, we have

$$\|W_{\psi,\varphi}-A\| \leq \|W_{\psi,\varphi}\Omega_m\| + \|W_{\psi,\varphi}(I-\Omega_m)-A\|.$$

Taking the infimum over compact operators A and letting $m\to\infty$ in the above inequality, we obtain (8.1).

Setting

$$F(u) := M_{\alpha,(u-B)A^{-1}}(\psi,\varphi),\ u\in\mathbb{C}.$$

Then $F(\varphi(z)) = M_{\alpha,z}(\psi,\varphi)$, and $F(u) \le M_\alpha(\psi,\varphi)$. Fix $R > 0$ and take $f \in ML^2(\mathbb{C},\alpha)$. By setting $u = Az + B$,

$$\begin{aligned}\|W_{\psi,\varphi}\Omega_m f\|^2 &= \frac{|A|^{-2/\alpha}}{\alpha\pi}\int\limits_{\mathbb{C}} F(u)|\Omega_m f(u)|^2|u-B|^{2/\alpha-2}e^{-|u|^{2/\alpha}}\,du\\ &= \frac{|A|^{-2/\alpha}}{\alpha\pi}\left(\int\limits_{|u|\le R} + \int\limits_{|u|>R}\right) F(u)|\Omega_m f(u)|^2|u-B|^{2/\alpha-2}e^{-|u|^{2/\alpha}}\,du\\ &= I_1 + I_2,\end{aligned}$$

where $R > |B|$. Note that by Proposition 3.5,

$$|\Omega_m f(z)| \le S(z) := \|f\|\sum_{k\ge m}|z|^k\Gamma(\alpha k+1)^{-1/2}.$$

Hence,

$$I_1 \le \frac{1}{\alpha\pi}|A|^{-2/\alpha}M_\alpha(\psi,\varphi)S(R)^2\int\limits_{|u|\le R}|u-B|^{2/\alpha-2}e^{-|u|^{2/\alpha}}\,du,$$

which implies that $\lim\limits_{m\to\infty} I_1 = 0$. Meanwhile,

$$I_2 \le \frac{1}{\alpha\pi}|A|^{-2/\alpha}\sup_{|u|>R}|F(u)|\int\limits_{|u|>R}|\Omega_m f(u)|^2|u-B|^{2/\alpha-2}e^{-|u|^{2/\alpha}}\,du.$$

Since $|u|/2 \le |u-B| \le 2|u|$ for R sufficiently large, we have

$$|u-B|^{2/\alpha-2} \le \begin{cases} 2^{2/\alpha-2}|u|^{2/\alpha-2} & \text{if } 0<\alpha\le 1,\\ 2^{-2/\alpha+2}|u|^{2/\alpha-2} & \text{if } 0<\alpha\le 1,\end{cases}$$

and so there is a constant C with the property that

$$\begin{aligned} I_2 &\le C\sup_{|u|>R}|F(u)|\int\limits_{|u|>R}|\Omega_m f(u)|^2|u|^{2/\alpha-2}e^{-|u|^{2/\alpha}}\,du\\ &\le C\sup_{|u|>R}|F(u)|\cdot\|\Omega_m f\|^2\\ &\le C\|f\|^2\sup_{|u|>R}|F(u)|.\end{aligned}$$

By (8.1), we have, for every $R > 0$,

$$\begin{aligned}\|W_{\psi,\varphi}\|_e &\le \liminf_{m\to\infty}\|W_{\psi,\varphi}\Omega_m f\|\\ &\le C^{1/2}\|f\|\cdot\left[\sup_{|u|>R}|F(u)|\right]^{1/2}\\ &= C^{1/2}\|f\|\sup_{|u|>R}|F(u)|^{1/2}.\end{aligned}$$

Letting $R\to\infty$ in the above inequality, we get the desired conclusion. □

Corollary 8.5. *Let $0<\alpha<2$, and $W_{\psi,\varphi}$ be a bounded weighted composition operator on $ML^2(\mathbb{C},\alpha)$ induced by two entire functions $\varphi \not\equiv$* **const** *and $\psi \not\equiv$* **0***, where $\varphi(z) = Az + B$ with $0 < |A| \le 1$. Then the weighted composition operator is compact if and only if $\lim\limits_{|z|\to\infty} M_{\alpha,z}(\psi,\varphi) = 0$.*

When φ is a constant function, a bounded weighted composition operator $W_{\psi,\varphi}$ must necessarily be compact.

Theorem 8.6. *Let $0 < \alpha < 2$, and $W_{\psi,\varphi}$ a bounded weighted composition induced by an entire function $\psi \not\equiv \mathbf{0}$ and a constant function φ.*

(1) *The operator $W_{\psi,\varphi} : ML^2(\mathbb{C}, \alpha) \to ML^2(\mathbb{C}, \alpha)$ is compact,*
(2) *the operator $W_{\psi,\varphi} : ML^2(\mathbb{C}, \alpha) \to ML^2(\mathbb{C}, \alpha)$ is bounded, and*
(3) *the function ψ satisfies $\psi \in ML^2(\mathbb{C}, \alpha)$.*

In this case, $\|W_{\psi,\varphi}\|_e = 0$.

Proof. Let $\varphi \equiv B$, where B is a complex constant. Clearly, (1) $\Longrightarrow$ (2) by definition, while (2) $\Longleftrightarrow$ (3) by Theorem 7.2. To prove that (3) $\Longrightarrow$ (1), we suppose that (3) holds. Let $(g_m) \subset ML^2(\mathbb{C}, \alpha)$ be a sequence converging weakly to 0 in $ML^2(\mathbb{C}, \alpha)$. By Proposition 8.2, it converges to 0 pointwise. Since $\psi \in ML^2(\mathbb{C}, \alpha)$, we have

$$\|W_{\psi,\varphi} g_m\| = |g_m(B)| \cdot \|\psi\|,$$

which tends to 0 as $m \to \infty$. The proof is complete. □

9. Data Availability

Complex Analysis and Operator Theory requires a data availability statement in their manuscripts. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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