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Spatial movement with temporally distributed memory and Dirichlet boundary condition ☆

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Abstract

In this paper, a reaction-diffusion population model with Dirichlet boundary condition and a directed movement oriented by a temporally distributed delay is proposed to describe the lasting memory of animals moving over space. The temporal kernel of the memory is taken as Gamma distribution function, among which there are two biologically meaningful cases: one is the weak kernel which implies that animals can immediately acquire knowledge and memory decays over time, the other is the strong kernel by which we assume that animals' memory undergoes learning and memory decay stages. It is shown that the population stabilizes to a positive steady state and aggregates in the interior of the territory when the delay kernel is the weak type. In the strong kernel case, oscillatory patterns can first arise via a Hopf bifurcation with a small memory delay and then vanish when the system undergoes a Hopf bifurcation with a large memory delay, which implies that stability switch occurs and spatial-temporal patterns emerge for intermediate value of delays.

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1. Introduction

The effect of animals' memory and cognition on their diffusive spatial movement recently has received much attention in characterizing complex animal movement patterns [8]. In [23], a diffusive animal movement model with explicit spatial memory is proposed by assuming that animals have information gained via their long-distance sight or social communications with their conspecifics [17]:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = d_1 \Delta u(x,t) + d_2 div(u(x,t)\nabla u(x,t-\tau)) + g(u(x,t)), & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial n}(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,t) = \eta(x,t), & x \in \Omega,\ t \in (-\tau, 0], \end{cases} \tag{1.1}$$

where the memory-based diffusion is related to the memory of a particular moment in the past, which induces a discrete delay. However, a spatially and/or temporally distributed delay for memory variation is more realistic as highly developed animals can remember the historic distribution or clusters of the species in space. Such delays may include decreases in intensity and spatial precision [8,23]. Based on this work, we formulate a scalar reaction-diffusion equation with a spatiotemporally distributed memory-based diffusion term in [25] to model the diffusive movement of animals who can memorize past information as well as the information from their surroundings. This new distributed memory-based model provides a more realistic quantitative framework for characterizing complicated memory waning and gaining processes in a relatively simple self-contained way.

In the natural world, there are also some solitary animals who spend a majority of their lives without others of their species, with possible exceptions for mating and raising their young. In particular, most carnivores are accounted as solitary and asocial because the costs of intraspecific competition outweigh the benefits accrued with group living [7]. There are a lot of examples for solitary animals, such as tigers (Panthera tigris), pumas (Puma con-color), and jaguars (Panthera onca) [7,16,18], and so on. In such situation, the influence of the spatial variation from other individuals of the species is negligible, thus we may reasonably assume that animals' memory mainly makes a temporal contribution to the spatial movement.

In this paper, we formulate a partial differential equation model for a single species animal movement with the explicit incorporation of spatial memory via a temporally distributed delayed diffusion. Throughout the paper, we use $u(x,t)$ to denote the population density of a biological species in spatial location x at time t. It is also assumed that the population is in a spatial habitat Ω, an open, bounded, and connected subset of $\mathbb{R}^m$ with $m \geq 1$. The boundary $\partial\Omega$ is assumed to be smooth. The population density function $u(x,t)$ satisfies

$$\begin{cases} u_t(x,t) = d_1 \Delta u(x,t) + d_2 \nabla \cdot (u(x,t)\nabla v(x,t)) + \lambda u(x,t)(1-u(x,t)), & x \in \Omega,\ t > 0, \\ v(x,t) = g * u(x,t) = \int_{-\infty}^{t} g(\tau, t-s) u(x,s) ds, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,t) = \eta(x,t), & x \in \Omega,\ t \in (-\infty, 0]. \end{cases} \tag{1.2}$$

Here the parameters $d_1 > 0$ and $d_2 \in \mathbb{R}$ are the random diffusion coefficient and the memory-based diffusion coefficient, respectively; the population satisfies a logistic growth law with a growth parameter $\lambda > 0$; the function $u(x,t)$ satisfies a Dirichlet boundary condition $u(x,t) = 0$, which means the environment of the boundary is hostile; and $\eta(x,t) \geq 0$ is the initial condition.

The function $v(x,t)$ is the weighted temporal average of the population density before time t and at location x, which reflects the animal's memory and knowledge of their previous spatial distributions. The memory can cause the animals to possess a repulsive movement to past history when $d_2 > 0$ and a attractive movement to their past when $d_2 < 0$. The temporal kernel function $g(\tau,t)$ reflects the dependence on the distribution of memory in the past time. From the mathematical perspective, $g : [0,\infty) \to \mathbb{R}^+$ is a probability distribution function satisfying

$$\int_0^\infty g(\tau,t)dt = 1. \tag{1.3}$$

In our model, the temporal kernel is taken as Gamma distribution function of order n (with $n \in \mathbb{N} \cup \{0\}$):

$$g(\tau,t) = g_n(\tau,t) = \frac{t^n e^{-t/\tau}}{\tau^{n+1}\Gamma(n+1)}. \tag{1.4}$$

In particular, we mainly consider the following two specific cases which are commonly employed in the biological modeling [5,9,13,24]:

$$g_0(\tau,t) = \frac{1}{\tau}e^{-\frac{t}{\tau}}, \quad g_1(\tau,t) = \frac{t}{\tau^2}e^{-\frac{t}{\tau}}, \tag{1.5}$$

which are referred to as the weak kernel and the strong kernel, respectively. The weak kernel function $g_0(\tau,t)$ is strictly decreasing in t, which biologically reflects one of the common ways of memory decay: the longer time goes by, the dimmer memories become. While the strong kernel $g_1(\tau,t)$ is increasing first and then decreasing, which corresponds to the knowledge acquisition phase and the knowledge decay phase, respectively. The mean and variance of $g_n(\tau,\cdot)$ are given by $\mathbb{E}(g_n(\tau,\cdot)) = (n+1)\tau$ and $\mathbb{V}ar(g_n(\tau,\cdot)) = (n+1)\tau^2$, from which we see that τ is related to the average of delay kernels. In this sense, we take τ as the bifurcation parameter to measure the influence of spatial memory on the dynamics.

The movement of population in (1.2) can be derived from mass conservation law and a modified Fick's law following [23]:

$$\mathbf{J}(x,t) = -d_1\nabla_x u(x,t) - d_2\mathbf{w}(x,t) \cdot u(x,t),$$

where $\mathbf{w}(x,t)$ is a vector field indicating animal movement direction and strength, and we assume that

$$\mathbf{w}(x,t) = \nabla_x \left(\int_{-\infty}^t g_n(\tau,t-s)u(x,s)ds \right), \tag{1.6}$$

where, in addition to the random diffusion with the coefficient d_1, the flux is proportional to the negative gradient of a weighted average historic density distribution. In [9,40], some scalar diffusive equations with distributed delay in the reaction term under Neumann boundary condition were investigated. The instability and Hopf bifurcation in a scalar reaction-diffusion model with Dirichlet boundary problem and distributed delay included in growth function are studied in [24]. A two-species diffusive population model with Dirichlet boundary problem and distributed delay is considered in [10]. In all these existing works, the distributed delay is used to describe the growth of the population, while in our model (1.2), the distributed delay describes the directed diffusion of the population as a result of spatial memory. About the study of the effect of the memory-base diffusion, analysis of a reaction-diffusion model with discrete delay memory-based movement was conducted in [21] for the Neumann boundary condition case (see also [34] for a model with spatial heterogeneity). In [23], a model with both discrete delay memory-based movement and maturation delay for Neumann boundary case was considered, and a more general model for Dirichlet boundary condition was studied in [1]. A model with additional nonlocal effect was also studied in [29]. Memory-based movement with spatial-temporal distributed delays was considered in [25,30]. Effect of memory-based cross-diffusion on systems of reaction-diffusion equations was investigated in [22,28].

Our results show that Eq. (1.2) has a positive steady state solution under certain conditions on d_1, d_2 and λ. In particular, the positive steady state exists and is unique when $d_1, d_2 > 0$ and $\lambda > \lambda_* = d_1\lambda_1$ where λ_1 is the principal eigenvalue of $-\Delta$. We then consider the stability of the unique positive steady state u_λ when λ is slightly larger than λ_* with the increase of the average time delay τ. It is shown that the non-homogeneous steady state remains locally asymptotically stable in the case of weak kernel. For the strong kernel case, u_λ loses its stability via a Hopf bifurcation so that a spatially non-homogeneous time-periodic pattern emerges. If we continue to increase the delay value, the non-constant steady state can gain its stability again, so a stability switch occurs in this system. Note that this phenomenon is different from the reaction-diffusion systems with time delay incorporated in the reaction terms ([3,24,31]).

This paper is organized as follows. In Section 2, we show the existence and uniqueness of the non-homogeneous steady state of Eq. (1.2) via a bifurcation approach. The Hopf bifurcations near the non-homogeneous steady state are investigated for a general delay kernel parameter n in the system in Section 3. Particularly, for the weak and strong kernel cases which have biological meanings, we carry out a detailed bifurcation analysis. Some numerical simulations are shown in Section 4, some further remarks and comments about our work and possible future works in Section 5. In the paper, the space of measurable functions for which the p-th power of the absolute value is Lebesgue integrable defined on a bounded and smooth domain $\Omega \subseteq \mathbb{R}^m$ is denoted by $L^p(\Omega)$ and we use $W^{k,p}(\Omega)$ to denote the real-valued Sobolev space based on $L^p(\Omega)$ space. Denote $X = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and $Y = L^p(\Omega)$, where $p > m$. For a linear vector space Z, we define its complexification to be $Z_{\mathcal{C}} = \{x_1 + ix_2 : x_1, x_2 \in Z\}$. Also, we denote by $\mathbb{N}$ the set of all the positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2. Spatially non-homogeneous steady states

The steady state solutions of Eq. (1.2) satisfy

$$\begin{cases} d_1\Delta u(x) + d_2\nabla \cdot (u(x)\nabla u(x)) + \lambda u(x)(1 - u(x)) = 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2.1}$$

where $d_1 > 0,\ d_2 \in \mathbb{R}$ and $\lambda > 0$.

First we have the following *a priori* estimates for the solutions of Eq. (2.1).

Lemma 2.1. *Suppose that $d_1 > 0$ and $d_2 \in \mathbb{R}$. If $u(x)$ is a nonnegative solution of Eq. (2.1) satisfying $d_1 + d_2 u(x) > 0$ for any $x \in \Omega$, then either $u \equiv 0$ or $u > 0$ in Ω. In the latter case, $0 < u(x) < 1,\ x \in \Omega$.*

Proof. Suppose that $u(x)$ is a nonnegative solution of Eq. (2.1) and $u(x) \not\equiv 0$. From (2.1), we know that $u(x)$ satisfies

$$(d_1 + d_2 u(x))\Delta u(x) + d_2|\nabla u(x)|^2 + \lambda u(x)(1 - u(x)) = 0. \tag{2.2}$$

Let $u(x_0) = \max_{x\in\bar{\Omega}} u(x)$, from the maximum principle, we have $u(x_0) \le 1$. Then, we apply the strong maximum principle and obtain that $u(x_0) < 1$. Thus, it is true that $u(x) < 1$ for all $x \in \Omega$. Similarly, $u(x) > 0$ holds for $x \in \Omega$ by the strong maximum principle. □

We have the following results regrading the existence, uniqueness and bifurcation of positive solutions of Eq. (2.1).

Theorem 2.2. *Suppose that $d_1 > 0$ and $d_2 \in \mathbb{R}$.*

(i) Let λ_1 be the principal eigenvalue of $-\Delta$ and let ϕ be the corresponding positive eigenfunction, then $\lambda = \lambda_* \triangleq d_1\lambda_1$ is a bifurcation point for Eq. (2.1). More precisely, near $(\lambda_*, 0)$, there is a smooth curve Γ_1 of positive solutions of Eq. (2.1) bifurcating from the line of constant solutions $\Gamma_0 = \{(\lambda, 0) : \lambda > 0\}$ with the following form:

$$\Gamma_1 = \{(\lambda(s), u(s)) = (\lambda_* + \lambda'(0)s + o(s), s\phi + o(s)) : 0 < s < \delta\}, \tag{2.3}$$

where

$$\lambda'(0) = \frac{(2d_1 + d_2)\lambda_* \int_\Omega \phi^3 dx}{2d_1 \int_\Omega \phi^2 dx}; \tag{2.4}$$

(ii) if $d_2 > -2d_1$, the bifurcation at $\lambda = \lambda_*$ is forward, and there exists a positive solution u_λ for $\lambda \in (\lambda_*, \lambda^*)$, where λ^* is a threshold value such that $d_1 + d_2 u_\lambda > 0$ holds for $\lambda \in (\lambda_*, \lambda^*)$; if $d_2 < -2d_1$, the bifurcation at $\lambda = \lambda_*$ is backward, and there exists a positive solution u_λ for $\lambda \in (\lambda^{**}, \lambda_*)$, where λ^{**} is a threshold value such that $d_1 + d_2 u_\lambda > 0$ holds for $\lambda \in (\lambda^{**}, \lambda_*)$;

(iii) if $d_2 > -d_1$, Eq. (2.1) has a positive solution u_λ for all $\lambda > \lambda_*$;

(iv) if $d_2 > 0$, the positive solution u_λ of Eq. (2.1) for $\lambda > \lambda_*$ is unique.

Proof. To prove (i), for fixed $d_1 > 0$ and $d_2 \in \mathbb{R}$, we define a nonlinear mapping $F : \mathbb{R}^+ \times X \to Y$:

$$F(\lambda, u) = d_1\Delta u + d_2\nabla\cdot(u\nabla u) + \lambda u(1 - u), \tag{2.5}$$

where $u \in X$, and it is obvious that $u = 0$ is a trivial solution of $F(\lambda, u) = 0$ for all $\lambda > 0$. Then, we take the Fréchet derivative of F with respect to u and obtain

$$F_u(\lambda, u)[w] = d_1\Delta w + d_2 u\Delta w + d_2 w\Delta u + 2d_2\nabla u \cdot \nabla w + \lambda(1 - 2u)w.$$

At $u = 0$, we have $F_u(\lambda, 0)[w] = d_1\Delta w + \lambda w$.

We determine the null space and range space of the operator $F_u(\lambda_*, 0)$. We know that the operator $-d_1\Delta$ has a principle eigenvalue $\lambda = \lambda_*$ corresponding to a positive eigenfunction $\phi > 0$. Hence the null space and the range space of $F_u(\lambda_*, 0)$ are $\mathcal{N}(F_u(\lambda_*, 0)) = Span\{\phi\}$ and $\mathcal{R}(F_u(\lambda_*, 0)) = \{h \in L^p(\Omega) : \int_\Omega h\phi dx = 0\}$ (we denote it as Y_1), respectively. Thus $dim\,(\mathcal{N}(F_u(\lambda_*, 0))) = 1$ and $codim\,(\mathcal{R}(F_u(\lambda_*, 0))) = 1$. Next we show that $F_{\lambda u}(\lambda_*, 0)[\phi] \notin \mathcal{R}(F_u(\lambda_*, 0))$. From (2.5), we have $F_{\lambda u}(\lambda, u)[w] = w(1 - 2u)$, thus $F_{\lambda u}(\lambda_*, 0)[\phi] = \phi$. It is clear that $F_{\lambda u}(\lambda_*, 0)[\phi] \notin \mathcal{R}(F_u(\lambda_*, 0))$ as $\int_\Omega \phi^2 dx > 0$. Therefore, the bifurcation from simple eigenvalue theorem [6] can be applied at $(\lambda, u) = (\lambda_*, 0)$, thus there is a smooth curve Γ_1 of positive solutions of Eq. (2.1) bifurcating from the line of constant solutions Γ_0, and Γ_1 is in form of (2.3).

For the bifurcation direction of Γ_1, from [20], we know that the bifurcation direction can be determined by

$$\lambda'(0) = -\frac{\langle l, F_{uu}(\lambda_*, 0)[\phi, \phi]\rangle}{2\langle l, F_{\lambda u}(\lambda_*, 0)[\phi]\rangle}, \tag{2.6}$$

where $l \in Y^*$ (the dual space of Y) and $\langle l, f\rangle = \int_\Omega f\phi dx$. From

$$F_{uu}(\lambda, u)[w_1, w_2] = d_2 w_1\Delta w_2 + d_2 w_2\Delta w_1 + 2d_2\nabla w_1\nabla w_2 - 2\lambda w_1 w_2,$$

we have

$$\begin{aligned} F_{uu}(\lambda_*, 0)[\phi, \phi] &= 2d_2\phi\Delta\phi + 2d_2|\nabla\phi|^2 - 2\lambda_*\phi^2 \\ &= -2d_2\lambda_1\phi^2 + 2d_2|\nabla\phi|^2 - 2d_1\lambda_1\phi^2 = -2\lambda_1(d_1 + d_2)\phi^2 + 2d_2|\nabla\phi|^2, \end{aligned} \tag{2.7}$$

where $\Delta\phi = -\lambda_1\phi$ is applied. Substituting (2.7) into (2.6), we obtain

$$\begin{aligned} \lambda'(0) &= -\frac{-2\lambda_*(d_1 + d_2)\int_\Omega \phi^3 dx + 2d_2\lambda_1\int_\Omega |\nabla\phi|^2\phi dx}{2d_1\int_\Omega \phi^2 dx} \\ &= -\frac{-2\lambda_*(d_1 + d_2)\int_\Omega \phi^3 dx + d_2\lambda_*\int_\Omega \phi^3 dx}{2d_1\int_\Omega \phi^2 dx} \\ &= \frac{(2d_1 + d_2)\lambda_*\int_\Omega \phi^3 dx}{2d_1\int_\Omega \phi^2 dx}. \end{aligned} \tag{2.8}$$

Note that

$$d_1\int_\Omega \phi^2\Delta\phi dx = -2d_1\int_\Omega \phi|\nabla\phi|^2 dx = -\lambda_*\int_\Omega \phi^3 dx \tag{2.9}$$

holds because of the fact that $d_1\phi^2\Delta\phi = -\lambda_*\phi^3$ according to $d_1\Delta\phi = -\lambda_*\phi$. Also we can conclude that the bifurcation occurs at $\lambda = \lambda_*$ is forward when $d_2 > -2d_1$ ($\lambda'(0) > 0$) and is backward when $d_2 < -2d_1$ ($\lambda'(0) < 0$). When $\lambda'(0) > 0$, there exists a threshold value $\lambda^* > \lambda_*$ such that the solution $u_\lambda = u(s)$ satisfies $d_1 + d_2u_\lambda > 0$ when $s = \lambda - \lambda_*$ is small enough. A similar threshold value $\lambda^{**} < \lambda_*$ exists when $\lambda'(0) < 0$. This completes the proof for (i) and (ii).

For the proof of (iii), we use the method of upper-lower solution to prove the existence of a positive solution of Eq. (2.1) for all $\lambda > \lambda_*$. Let $\underline{u} = \epsilon\phi$ with $\epsilon > 0$, then

$$\begin{aligned}
&d_1\Delta(\epsilon\phi) + d_2\nabla\cdot(\epsilon\phi\nabla\epsilon\phi) + \lambda\epsilon\phi(1-\epsilon\phi)\\
=&\epsilon d_1\Delta\phi + d_2\epsilon^2\phi\Delta\phi + d_2\epsilon^2|\nabla\phi|^2 + \lambda\epsilon\phi - \lambda\epsilon^2\phi^2\\
=&\epsilon(\lambda-\lambda_*)\phi + \epsilon^2(d_2\phi\Delta\phi + d_2|\nabla\phi|^2 - \lambda\phi^2) \ge 0
\end{aligned}$$

when $\lambda > \lambda_*$ for sufficiently small $\epsilon > 0$. It is known that $\underline{u} = 0$ for $x \in \partial\Omega$. This means that $\underline{u}$ is a lower solution for Eq. (2.1). Also, one can easily verify that $\bar{u} = 1$ is an upper solution for Eq. (2.1). By choosing $\epsilon > 0$ small enough, we have $\underline{u} < \bar{u}$. Under the condition $d_2 > -d_1$, we claim that $d_1 + d_2u(x) > 0$ holds. We explain it in two cases: (1) if $-d_1 < d_2 < 0$, then $d_1 + d_2u(x) > d_1 + d_2 > 0$ as $0 < u(x) < 1$; (2) if $d_2 \ge 0$, then $d_1 + d_2u(x) > 0$ holds for sure. Then, we know that Eq. (2.1) is elliptic type for any of its positive solution. From [15, Theorem 3.1], there exists a minimal solution u_λ^m and a maximal solution u_λ^M of Eq. (2.1) satisfying $\underline{u} < u_\lambda^m \le u_\lambda^M < \bar{u} = 1$, which ensures the existence of steady state for $\lambda > \lambda_*$.

At the end, we prove the uniqueness of u_λ when $d_2 > 0$. Let u_λ^M be the maximal solution obtained above. Suppose that there exists another positive solution of Eq. (2.1) $\tilde{u}_\lambda \not\equiv u_\lambda^M$. Then from Lemma 2.1, we must have $\tilde{u}_\lambda(x) \le u_\lambda^M(x)$ for $x \in \Omega$. Then, u_λ^M is a positive solution for the following equation:

$$\nabla\cdot\Big((d_1 + d_2u_\lambda^M)\nabla\varphi\Big) + \lambda(1-u_\lambda^M)\varphi = \sigma\varphi,\ x\in\Omega,\ \ \varphi = 0,\ x\in\partial\Omega, \tag{2.10}$$

for $\sigma = 0$, thus $\sigma = 0$ is the principal eigenvalue for Eq. (2.10). Similarly, we know that $\tilde{\sigma} = 0$ is the principal eigenvalue for the following equation:

$$\nabla\cdot((d_1 + d_2\tilde{u}_\lambda)\nabla\varphi) + \lambda(1-\tilde{u}_\lambda)\varphi = \tilde{\sigma}\varphi,\ x\in\Omega,\ \ \varphi = 0,\ x\in\partial\Omega. \tag{2.11}$$

From the variational characterization of σ and $\tilde{\sigma}$, we have

$$\begin{aligned}
0 = \sigma =& -\inf_{\varphi\in X}\frac{\int_\Omega(d_1 + d_2u_\lambda^M)|\nabla\varphi|^2 - \lambda\int_\Omega(1-u_\lambda^M)\varphi^2}{\int_\Omega\varphi^2}\\
\le& -\inf_{\varphi\in X}\frac{\int_\Omega(d_1 + d_2\tilde{u}_\lambda)|\nabla\varphi|^2 - \lambda\int_\Omega(1-\tilde{u}_\lambda)\varphi^2}{\int_\Omega\varphi^2} = \tilde{\sigma} = 0,
\end{aligned} \tag{2.12}$$

as we have $d_1 + d_2u_\lambda^M \ge d_1 + d_2\tilde{u}_\lambda$ and $1 - u_\lambda^M \le 1 - \tilde{u}_\lambda$ when $d_2 > 0$. Since the equality in (2.12) only holds when $\tilde{u}_\lambda \equiv u_\lambda^M$, we must have $\tilde{u}_\lambda \equiv u_\lambda^M$. This completes the proof of (iv). □

Remark 2.3.

1. When $d_2 > -2d_1$, the positive solution u_λ of Eq. (2.1) for $\lambda \in [\lambda_*, \lambda^*]$ can also be expressed in the following form from (2.3) and (2.4):

$$u_\lambda = (\lambda - \lambda_*)\alpha_\lambda[\phi + (\lambda - \lambda_*)\xi_\lambda],\ \alpha_\lambda = \frac{2d_1 \int_\Omega \phi^2 dx}{(2d_1 + d_2)\lambda \int_\Omega \phi^3 dx}, \tag{2.13}$$

where $\xi_{\lambda_*} \in X_1$ with $X_1 = \left\{h \in X : \int_\Omega h\phi dx = 0\right\}$ is the unique solution of the following equation:

$$(d_1\Delta + \lambda_*)\xi_{\lambda_*} + \phi + \alpha_{\lambda_*}[d_2\nabla \cdot (\phi\nabla\phi) - \lambda_*\phi^2] = 0. \tag{2.14}$$

Note that the form (2.13) will be used to analyze the stability of u_λ in the following section.
2. The condition $d_1 + d_2u(x) > 0$ is required to guarantee the ellipticity of Eq. (2.1), and this condition holds automatically when $d_2 \geq 0$. When $d_2 < 0$, there may exist solutions of Eq. (2.1) not satisfying $d_1 + d_2u(x) > 0$.

3. Stability and Hopf bifurcations

In this section we always assume that the shape parameter $n \in \mathbb{N}^+ \cup \{0\}$ is fixed and $d_2 > 0$ so that Eq. (1.2) has a unique positive steady state u_λ according to Theorem 2.2 (iv). Define

$$\hat{\lambda} = \min\left\{\lambda^*, \lambda_* + \frac{d_1}{d_2\alpha_{\lambda_*}M}\right\}, \tag{3.1}$$

where $\lambda_*,\ \lambda^*$ is defined in Theorem 2.2, α_{λ_*} is defined as in (2.13) when $\lambda = \lambda_*$ and $M = \max\limits_{\lambda\in(\lambda_*,\lambda^*]} \|\phi + (\lambda - \lambda_*)\xi_\lambda\|_\infty$. Now we consider the stability and associated bifurcations at u_λ when $\lambda \in (\lambda_*, \hat{\lambda}]$.

By letting $u(x,t) = u_\lambda(x) + \tilde{u}(x,t)$, $v(x,t) = u_\lambda(x) + \tilde{v}(x,t)$, we obtain the linearization of Eq. (1.2) at u_λ:

$$\begin{cases} \tilde{u}_t = d_1\Delta\tilde{u} + d_2\nabla \cdot (u_\lambda\nabla\tilde{v}) + d_2\nabla \cdot (\tilde{u}\nabla u_\lambda) + \lambda(1 - 2u_\lambda)\tilde{u}, & x \in \Omega,\ t > 0, \\ \tilde{v}(x,t) = \displaystyle\int_{-\infty}^{t} g_n(\tau, t-s)\tilde{u}(x,s)ds = \int_{-\infty}^{0} g_n(\tau, -s)\tilde{u}(x,t+s)ds, & x \in \Omega,\ t > 0, \\ \tilde{u}(x,t) = \tilde{v}(x,t) = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{3.2}$$

From [36] Chapter 3, the semigroup induced by the solutions of Eq. (3.2) has an infinitesimal generator $A_{n\tau}(\lambda)$ which is given by

$$A_{n\tau}(\lambda)\varphi_n = \dot{\varphi}_n, \tag{3.3}$$

and the domain of $A_{n\tau}(\lambda)$ is

$$\mathfrak{D}(A_{n\tau}(\lambda)) = \left\{\varphi_n \in \mathcal{C}_{\mathbb{C}} \cap \mathcal{C}^1_{\mathbb{C}} : \dot{\varphi}_n(0) = A(\lambda)\varphi_n + d_2\nabla \cdot \left(u_\lambda \int_{-\infty}^{0} g_n(\tau, -s)\nabla\varphi_n(s)ds\right) - \lambda u_\lambda\varphi_n\right\},$$

where

$$
\begin{gathered}
A(\lambda)\varphi_n = d_1\Delta\varphi_n + d_2\nabla\cdot(\varphi_n\nabla u_\lambda) + \lambda(1-u_\lambda)\varphi_n,\\
\mathcal{C}_{\mathbb{C}} = C((-\infty,0],Y_{\mathbb{C}}),\ \mathcal{C}^1_{\mathbb{C}} = C^1((-\infty,0],Y_{\mathbb{C}}),\ \varphi_n\in X_{\mathbb{C}}.
\end{gathered}
$$

The spectral set of $A_{n\tau}(\lambda)$ is

$$
\sigma(A_{n\tau}(\lambda)) = \{\mu\in\mathbb{C} : \Lambda(\lambda,\mu,\tau)\psi_n = 0,\ \text{for some } \psi_n\in X_{\mathbb{C}}\backslash\{0\}\}, \tag{3.4}
$$

where

$$
\begin{aligned}
\Lambda(\lambda,\mu,\tau) &= A(\lambda) + d_2\nabla\cdot(u_\lambda\nabla)\int_{-\infty}^{0} g_n(\tau,-s)e^{\mu s}ds - \lambda u_\lambda - \mu\\
&= A(\lambda) + \frac{d_2}{(1+\mu\tau)^{n+1}}\nabla\cdot(u_\lambda\nabla) - \lambda u_\lambda - \mu.
\end{aligned} \tag{3.5}
$$

Note that (3.5) holds from the integral

$$
\int_{-\infty}^{0} g_n(\tau,-s)e^{\mu s}ds = \frac{1}{\tau^{n+1}\Gamma(n+1)}\int_{-\infty}^{0} s^n e^{s/\tau}e^{\mu s}ds = \frac{1}{(1+\mu\tau)^{n+1}}. \tag{3.6}
$$

When $\tau\to 0$, the stability of steady state of Eq. (1.2) is determined by the limiting operator

$$
A_0(\lambda) = A(\lambda) + d_2\nabla\cdot(u_\lambda\nabla) - \lambda u_\lambda, \tag{3.7}
$$

and we have the following conclusion.

Proposition 3.1. *When $d_2>0$ and $\lambda\in(\lambda_*,\hat{\lambda})$, for sufficiently small $\tau\geq 0$, the positive steady state u_λ is locally asymptotically stable with respect to Eq. (1.2). Moreover, for any $\tau>0$, $0\notin\sigma(A_{n\tau}(\lambda))$.*

Proof. We first prove $\sigma(A_0(\lambda))\subseteq\{\mu\in\mathbb{C} : \mathcal{R}e(\mu)<0\}$, from which the stability of u_λ when $\tau=0$ can be achieved. We write $A_0(\lambda)\psi=\mu\psi$ as follows

$$
d_1\Delta\psi + d_2\nabla\cdot(\psi\nabla u_\lambda) + d_2\nabla\cdot(u_\lambda\nabla\psi) + \lambda(1-2u_\lambda)\psi = \mu\psi, \tag{3.8}
$$

which can be rewritten as

$$
(d_1+d_2u_\lambda)\Delta\psi + 2d_2\nabla u_\lambda\nabla\psi + d_2\Delta u_\lambda\psi + \lambda(1-2u_\lambda)\psi = \mu\psi.
$$

Because $d_2>0$, so there must be a constant η such that $d_1+d_2u_\lambda\geq\eta>0$, which makes (3.8) a strongly elliptic equation. From [2], we know that $A_0(\lambda)$ has a principal eigenvalue $\mu_1\in\mathbb{R}$ with its corresponding real eigenfunction $\psi_1>0$. Also we know that u_λ satisfies

$$d_1 \Delta u_\lambda + d_2 \nabla \cdot (u_\lambda \nabla u_\lambda) + \lambda(1-u_\lambda)u_\lambda = 0. \tag{3.9}$$

Multiplying (3.8) by u_λ, multiplying (3.9) by ψ_1, integrating them over Ω and subtracting each other, we obtain

$$\begin{aligned}
\mu_1 \int_\Omega \psi_1 u_\lambda dx =& d_1 \int_\Omega (\Delta\psi_1 u_\lambda - \Delta u_\lambda \psi_1) dx - \lambda \int_\Omega u_\lambda^2 \psi_1 dx + d_2 \int_\Omega \nabla \cdot (\psi_1 \nabla u_\lambda) u_\lambda dx \\
& + d_2 \int_\Omega \nabla \cdot (u_\lambda \nabla \psi_1) u_\lambda dx - d_2 \int_\Omega \nabla \cdot (u_\lambda \nabla u_\lambda) \psi_1 dx \\
=& -\lambda \int_\Omega u_\lambda^2 \psi_1 dx - d_2 \int_\Omega \psi_1 |\nabla u_\lambda|^2 dx \\
& - d_2 \int_\Omega u_\lambda \nabla \psi_1 \nabla u_\lambda dx + d_2 \int_\Omega u_\lambda \nabla u_\lambda \nabla \psi_1 dx \\
=& -\lambda \int_\Omega u_\lambda^2 \psi_1 dx - d_2 \int_\Omega \psi_1 |\nabla u_\lambda|^2 dx,
\end{aligned}$$

which implies that $\mu_1 < 0$ as $u_\lambda > 0$, $\psi_1 > 0$ and $d_2 > 0$. Therefore, all the eigenvalues of $A_0(\lambda)$ have negative real parts, which implies that the steady state of Eq. (1.2) is locally asymptotically stable when $\tau = 0$.

Next we claim that $\sup\limits_{\mu \in \sigma(A_{n\tau}(\lambda))} \mathcal{R}e(\mu) < 0$ holds for $\tau > 0$ sufficiently small. Similar to Lemma 2.2 (ii) in [24], we know that $\lim\limits_{\tau \to 0} \sigma_b(A_{n\tau}(\lambda)) = \sigma_b(A_0(\lambda))$ holds, where $\sigma_b(A_{n\tau}(\lambda)) := \sigma(A_{n\tau}(\lambda)) \cap \{\mu \in \mathbb{C} : \mathcal{R}e(\mu) > b\}$. Together with $\sigma(A_0(\lambda)) \subseteq \{\mu \in \mathbb{C} | \mathcal{R}e(\mu) < 0\}$, the claim is proved. Hence we know that all the eigenvalues of $A_{n\tau}(\lambda)$ have negative real part for sufficient small $\tau > 0$, which implies that u_λ is locally asymptotically stable with respect to Eq. (1.2) for sufficiently small $\tau \geq 0$.

Finally if $0 \in \sigma(A_{n\tau}(\lambda))$ for any $\tau > 0$, then 0 is an eigenvalue of $A_0(\lambda)$, but we have shown that all the eigenvalues of $A_0(\lambda)$ have negative real parts. That is a contradiction. Hence for any $\tau > 0$, $0 \notin \sigma(A_{n\tau}(\lambda))$. □

From Proposition 3.1, we see that u_λ is locally asymptotically stable for sufficiently small $\tau \geq 0$. Next we show that u_λ loses its stability when τ increases and Hopf bifurcations occur for the system (1.2). First the following boundedness will be needed later.

Lemma 3.2. *For any $(\lambda, \mu, \tau, \psi_n) \in (\lambda_*, \hat{\lambda}] \times \mathbb{C} \times \mathbb{R}^+ \times Y_{\mathbb{C}}$ satisfying $A_{n\tau}(\lambda)\psi_n = \mu \psi_n$ which is defined as in (3.3), there exists a constant $M_1 > 0$ depending on d_1, d_2 such that $\|\nabla \psi_n\|_{Y_{\mathbb{C}}} \leq M_1 \|\psi_n\|_{Y_{\mathbb{C}}}$ when $\mathcal{R}e(\mu) \geq 0$.*

Proof. Since $u_\lambda \in X$ is the unique solution of (2.1) and $0 < u_\lambda < 1$ from Lemma 2.1, we have $|u_\lambda|_{1+\gamma} \leq M_2$ from the Sobolev embedding theorem, where $\gamma \in (0, 1/2)$, M_2 is a constant depending on γ, λ^*, Ω and $|\cdot|_{1+\gamma}$ is the norm in $C^{1+\gamma}(\bar{\Omega})$. Moreover from the regularity theory for elliptic equations, we can obtain $u_\lambda \in C^{2+\beta}(\bar{\Omega})$ with $0 < \beta < \gamma$ and $|u_\lambda|_{2+\beta} \leq M_3$ with M_3 depending on β, d_1, d_2, M_2 and Ω. By the definition of $A_{n\tau}(\lambda)$ in (3.3), we have

$$\left\langle \left[A(\lambda) + \frac{d_2}{(1+\mu\tau)^{n+1}} \nabla \cdot (u_\lambda \nabla) - \lambda u_\lambda - \mu \right] \psi_n, \psi_n \right\rangle = 0. \tag{3.10}$$

Taking the real part of (3.10), we obtain

$$\begin{aligned} d_1 \|\nabla \psi_n\|^2_{Y_{\mathbb{C}}} =& d_2 \mathcal{R}e \left\{ \int_\Omega \nabla \cdot (\psi_n \nabla u_\lambda) \bar{\psi}_n dx \right\} + \lambda \int_\Omega (1 - 2u_\lambda) |\psi_n|^2 dx \\ &+ \mathcal{R}e \left\{ \frac{d_2}{(1+\mu\tau)^{n+1}} \int_\Omega \nabla \cdot (u_\lambda \nabla \psi_n) \bar{\psi}_n dx \right\} - \mathcal{R}e(\mu) \|\psi_n\|^2_{Y_{\mathbb{C}}}. \end{aligned} \tag{3.11}$$

Since

$$\begin{aligned} &\int_\Omega \nabla \cdot (\psi_n \nabla u_\lambda) \bar{\psi}_n dx = \int_\Omega (\nabla \psi_n \nabla u_\lambda + \psi_n \Delta u_\lambda) \bar{\psi}_n dx \\ =& - \int_\Omega \psi_n \nabla \cdot (\bar{\psi}_n \nabla u_\lambda) dx + \int_\Omega |\psi_n|^2 \Delta u_\lambda dx, \end{aligned}$$

so we have

$$\mathcal{R}e \left\{ \int_\Omega \nabla \cdot (\psi_n \nabla u_\lambda) \bar{\psi}_n dx \right\} = \frac{1}{2} \int_\Omega \Delta u_\lambda |\psi_n|^2 dx. \tag{3.12}$$

Also one can verity that

$$\mathcal{R}e \left\{ \int_\Omega \nabla \cdot (u_\lambda \nabla \psi_n) \bar{\psi}_n dx \right\} = -\mathcal{R}e \left\{ \int_\Omega u_\lambda |\nabla \psi_n|^2 dx \right\} = - \int_\Omega u_\lambda |\nabla \psi_n|^2 dx. \tag{3.13}$$

Combining Eqs. (3.11), (3.12), (3.13) and the fact that $\mathcal{R}e(\mu) \geq 0$, $\|u_\lambda\|_\infty \geq 0$, we obtain

$$\begin{aligned} d_1 \|\nabla \psi_n\|^2_{Y_{\mathbb{C}}} =& \frac{d_2}{2} \int_\Omega \Delta u_\lambda |\psi_n|^2 dx + \lambda \int_\Omega (1 - 2u_\lambda) |\psi_n|^2 dx \\ &- \mathcal{R}e \left(\frac{d_2}{(1+\mu\tau)^{n+1}} \right) \int_\Omega u_\lambda |\nabla \psi_n|^2 dx - \mathcal{R}e(\mu) \|\psi_n\|^2_{Y_{\mathbb{C}}}, \\ \leq& \frac{|d_2|}{2} \|\Delta u_\lambda\|_\infty \|\psi_n\|^2_{Y_{\mathbb{C}}} + \lambda \|\psi_n\|^2_{Y_{\mathbb{C}}} + \left| \mathcal{R}e \left(\frac{d_2}{(1+\mu\tau)^{n+1}} \right) \right| \|u_\lambda\|_\infty \|\nabla \psi_n\|^2_{Y_{\mathbb{C}}}. \end{aligned}$$

By the fact that $\mathcal{R}e(\mu) \geq 0$ and $\tau > 0$, it can be verified that

$$\left| \mathcal{R}e \left(\frac{1}{(1+\mu\tau)^{n+1}} \right) \right| = \left| |1+\mu\tau|^{-(n+1)} \cos(\arg((1+\mu\tau)^{n+1})) \right| \leq |1+\mu\tau|^{-(n+1)} \leq 1,$$

where arg(·) stands for the argument of a complex number. Together with the boundedness of ξ_λ, u_λ and ψ_n for $\lambda \in (\lambda_*, \hat{\lambda}]$ and $|u_\lambda|_{2+\beta} \leq M_3$, we know that there exists a constant $M_4 > 0$ such that

$$\|\nabla\psi_n\|^2_{Y_\mathbb{C}} \leq \frac{\left(\frac{d_2}{2}\|\Delta u_\lambda\|_\infty + \lambda\right)\|\psi_n\|^2_{Y_\mathbb{C}}}{d_1 - d_2\|u_\lambda\|_\infty} \leq M_4\|\psi_n\|^2_{Y_\mathbb{C}},$$

where

$$d_1 - d_2\|u_\lambda\|_\infty = d_1 - d_2(\lambda-\lambda_*)\alpha_\lambda\|\phi + (\lambda-\lambda_*)\xi_\lambda\|_\infty > d_1 - d_2(\lambda-\lambda_*)\alpha_{\lambda_*}M > 0$$

holds from the definition of $\hat{\lambda}$ in (3.1). Thus, $\|\nabla\psi_n\|_{Y_\mathbb{C}} \leq M_1\|\psi_n\|_{Y_\mathbb{C}}$ holds with $M_1 = \sqrt{M_4}$. □

From Proposition 3.1, u_λ loses its stability and a Hopf bifurcation occurs when $A_{n\tau}(\lambda)$ has a pair of purely imaginary eigenvalues $\mu = \pm i\omega_n$ ($\omega_n > 0$). From (3.5), we know that the operator $A_{n\tau}(\lambda)$ has an eigenvalue $i\omega_n$ is equivalent to

$$\left[A(\lambda) + d_2\nabla\cdot(u_\lambda\nabla)\frac{1}{(1+i\theta_n)^{n+1}} - \lambda u_\lambda - i\omega_n\right]\psi_n = 0, \quad \psi_n \in X_\mathbb{C}\setminus\{0\}, \tag{3.14}$$

where $\theta_n := \omega_n\tau$. Next we will show that there exist some triples $(\omega_n, \theta_n, \psi_n)$ which solve Eq. (3.14) for $n \geq 0$. For further discussion, we need the following lemma.

Lemma 3.3. *Recall that λ_1 is the principal eigenvalue of $-\Delta$, we have*

(i) if $z \in X_\mathbb{C}$ and $\langle\phi, z\rangle = 0$, then $|\langle(d_1\Delta+\lambda_*)z, z\rangle| \geq d_1(\lambda_2-\lambda_1)\|z\|^2_{Y_\mathbb{C}}$ with $\lambda_* = d_1\lambda_1$, where λ_2 is the second eigenvalue of $-\Delta$ on $H^1_0(\Omega)$;
(ii) for each $n \geq 0$, if there exist some $(\omega_n, \theta_n, \psi_n)$ satisfying Eq. (3.14) with $\psi_n \in X_\mathbb{C}$, then $\omega_n/(\lambda-\lambda_*)$ is uniformly bounded for $\lambda \in (\lambda_*, \hat{\lambda}]$.

Proof. The proof of part (*i*) is similar to that of Lemma 3.2 in [3], thus we omit it here and mainly prove part (ii). By Eq. (3.14), we have

$$\left\langle\left[A(\lambda) + d_2\nabla\cdot(u_\lambda\nabla)\frac{1}{(1+i\theta_n)^{n+1}} - \lambda u_\lambda - i\omega_n\right]\psi_n, \psi_n\right\rangle = 0. \tag{3.15}$$

From $1 + i\theta_n = \sqrt{1+\theta_n^2}\, e^{i\eta_n}$ with $\tan\eta_n = \theta_n$, Eq. (3.15) can be rewritten as

$$\left\langle\left[A(\lambda) + d_2\nabla\cdot(u_\lambda\nabla)(1+\theta_n^2)^{-(n+1)/2}e^{-i(n+1)\eta_n} - \lambda u_\lambda - i\omega_n\right]\psi_n, \psi_n\right\rangle = 0. \tag{3.16}$$

From the imaginary part of Eq. (3.16), we have

$$\begin{aligned}\omega_n\langle\psi_n,\psi_n\rangle &= d_2(1+\theta_n^2)^{-(n+1)/2}\sin((n+1)\eta_n)\,\langle\nabla\cdot(u_\lambda\nabla\psi_n),\psi_n\rangle\\ &= -d_2(1+\theta_n^2)^{-(n+1)/2}\sin((n+1)\eta_n)\,\langle u_\lambda\nabla\psi_n,\nabla\psi_n\rangle.\end{aligned}$$

Let $m(\lambda, \xi_\lambda) = \phi + (\lambda - \lambda_*)\xi_\lambda$, we obtain

$$\begin{aligned}\frac{|\omega_n|}{\lambda - \lambda_*} =& \frac{d_2\alpha_\lambda \left|(1+\theta_n^2)^{-(n+1)/2}\sin((n+1)\eta_n)\langle m(\lambda,\xi_\lambda)\nabla\psi_n, \nabla\psi_n\rangle\right|}{\|\psi_n\|^2_{Y_{\mathbb{C}}}} \\ <& \frac{d_2\alpha_\lambda \|m(\lambda,\xi_\lambda)\|_\infty \|\nabla\psi_n\|^2_{Y_{\mathbb{C}}}}{\|\psi_n\|^2_{Y_{\mathbb{C}}}},\end{aligned}$$

as $\left|((1+\theta_n^2)^{-(n+1)/2})\right| < 1$. From the boundedness of $u_\lambda = (\lambda - \lambda_*)\alpha_\lambda m(\lambda, \xi_\lambda)$ proved in Lemma 2.1 $m(\lambda, \xi_\lambda)$ is bounded in X. Together with the boundedness of $\|\nabla\psi_n\|^2_{Y_{\mathbb{C}}}$ obtained in Lemma 3.2, we can obtain the boundedness of $\omega_n/(\lambda - \lambda_*)$ by the continuity of $\lambda \mapsto (\alpha_\lambda, \xi_\lambda)$. $\square$

We know that $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$ can be decomposed as

$$X_{\mathbb{C}} = Span\{\phi\} \oplus X_{1\mathbb{C}}, Y_{\mathbb{C}} = Span\{\phi\} \oplus Y_{1\mathbb{C}}, \tag{3.17}$$

where

$$X_{1\mathbb{C}} = \left\{h \in X_{\mathbb{C}} : \int_\Omega h\phi dx = 0\right\}, \; Y_{1\mathbb{C}} = \left\{h \in Y_{\mathbb{C}} : \int_\Omega h\phi dx = 0\right\}.$$

Suppose that $(\omega_n, \theta_n, \psi_n)$ is a solution of Eq. (3.14) with $\psi_n \in X_{\mathbb{C}}$, then ψ_n can be decomposed and normalized as

$$\begin{aligned}&\psi_n = \beta_n\phi + (\lambda - \lambda_*)z_n, \quad \langle\phi, z_n\rangle = 0, \\ &\|\psi_n\|^2_{Y_{\mathbb{C}}} = \beta_n^2\|\phi\|^2_{Y_{\mathbb{C}}} + (\lambda-\lambda_*)^2\|z_n\|^2_{Y_{\mathbb{C}}} = \|\phi\|^2_{Y_{\mathbb{C}}}.\end{aligned} \tag{3.18}$$

Substituting Eqs. (2.13), (3.18) and $\omega_n = (\lambda - \lambda_*)h_n$ into Eq. (3.14), we get the following equivalent system:

$$\begin{aligned}g_1(z_n, \beta_n, h_n, \theta_n, \lambda) :=& (d_1\Delta + \lambda_*)z_n + (\beta_n\phi + (\lambda-\lambda_*)z_n)\{1 - ih_n - 2\lambda\alpha_\lambda[\phi + (\lambda-\lambda_*)\xi_\lambda]\} \\ &+ d_2\alpha_\lambda\nabla\cdot\{[\beta_n\phi + (\lambda-\lambda_*)z_n]\nabla[\phi + (\lambda-\lambda_*)\xi_\lambda]\} \\ &+ \frac{d_2\alpha_\lambda}{(1+i\theta_n)^{n+1}}\nabla\cdot\{[\phi + (\lambda-\lambda_*)\xi_\lambda]\nabla[\beta_n\phi + (\lambda-\lambda_*)z_n]\} = 0, \\ g_2(z_n, \beta_n, h_n, \theta_n, \lambda) :=& \left(\beta_n^2 - 1\right)\|\phi\|^2_{Y_{\mathbb{C}}} + (\lambda-\lambda_*)^2\|z_n\|^2_{Y_{\mathbb{C}}} = 0.\end{aligned} \tag{3.19}$$

We define $G : X_{1\mathbb{C}} \times \mathbb{R}^3 \times \mathbb{R} \to Y_{\mathbb{C}} \times \mathbb{R}$ as

$$G(z_n, \beta_n, h_n, \theta_n, \lambda) := (g_1, g_2).$$

We will show that $G = 0$ can be solved for $\lambda \to \lambda_*$, and we first solve the limiting equation when $\lambda = \lambda_*$ in the following lemma.

Lemma 3.4. *When $\lambda = \lambda_*$, define*

$$m_n = \cos^{n+2}\left(\frac{\pi}{n+2}\right), \quad n \in \mathbb{N}. \tag{3.20}$$

Then the following statements are true:

(i) if $n = 0$, there is no solution for $G(z_n, \beta_n, h_n, \theta_n, \lambda_*) = 0$;
(ii) if $n \geq 1$, m_n is positive and increasing in n. When $d_2 \geq 2d_1/m_n$, $G(z_n, \beta_n, h_n, \theta_n, \lambda_*) = 0$ has at least one solution which can be expressed as

$$(z_n, \beta_n, h_n, \theta_n) = W_{n\lambda_*} \triangleq (z_{n\lambda_*}, \beta_{n\lambda_*}, h_{n\lambda_*}, \theta_{n\lambda_*}) \tag{3.21}$$

with

$$\beta_{n\lambda_*} = 1, \; \theta_{n\lambda_*} = \tan(\eta_{n\lambda_*}), \; h_{n\lambda_*} = \frac{d_2}{2d_1 + d_2} \sin((n+1)\eta_{n\lambda_*}) \cos^{n+1} \eta_{n\lambda_*},$$

where $\eta_{n\lambda_*}$ satisfies

$$\cos((n+1)\eta_{n\lambda_*}) \cos^{n+1} \eta_{n\lambda_*} = -\frac{2d_1}{d_2}, \tag{3.22}$$

and $z_{n\lambda_*}$ is the unique solution of the following equation

$$(d_1\Delta + \lambda_*)z_{n\lambda_*} + (1 - ih_{n\lambda_*})\phi + d_2\alpha_{\lambda_*}\left(1 + \frac{1}{(1 + i\theta_{n\lambda_*})^{n+1}}\right)\nabla\cdot(\phi\nabla\phi) - 2\lambda_*\alpha_{\lambda_*}\phi^2 = 0. \tag{3.23}$$

Proof. We solve $G(z_n, \beta_n, h_n, \theta_n, \lambda) = (g_1, g_2) = 0$ when $\lambda = \lambda_*$. Firstly, we have $\beta_n = \beta_{n\lambda_*} = 1$ through solving $g_2|_{\lambda=\lambda_*} = 0$. When $\lambda = \lambda_*$, $g_1 = 0$ is equivalent to

$$(d\Delta + \lambda_*)z_n + (1 - ih_n)\phi + d_2\alpha_{\lambda_*}\left(1 + \frac{1}{(1 + i\theta_n)^{n+1}}\right)\nabla\cdot(\phi\nabla\phi) - 2\lambda_*\alpha_{\lambda_*}\phi^2 = 0. \tag{3.24}$$

Multiplying (3.24) by ϕ and integrating over Ω, we have

$$\begin{aligned}
&(1 - ih_n)\int_\Omega \phi^2 dx + d_2\alpha_{\lambda_*}\left(1 + \frac{1}{(1 + i\theta_n)^{n+1}}\right)\int_\Omega \nabla\cdot(\phi\nabla\phi)\phi dx - 2\lambda_*\alpha_{\lambda_*}\int_\Omega \phi^3 dx \\
=&(1 - ih_n)\int_\Omega \phi^2 dx - \frac{d_2\lambda_*\alpha_{\lambda_*}}{2d_1}\left(1 + \frac{1}{(1 + i\theta_n)^{n+1}}\right)\int_\Omega \phi^3 dx - 2\lambda_*\alpha_{\lambda_*}\int_\Omega \phi^3 dx \\
=&\left[1 - ih_n - \frac{d_2}{2d_1 + d_2}\left(1 + \frac{1}{(1 + i\theta_n)^{n+1}}\right) - \frac{4d_1}{2d_1 + d_2}\right]\int_\Omega \phi^2 dx = 0,
\end{aligned}$$

which can be inferred from (2.9) and the definition of α_{λ_*} in (2.13). Then, we have

$$ih_n = -\frac{d_2}{2d_1+d_2}\frac{1}{(1+i\theta_n)^{n+1}} - \frac{2d_1}{2d_1+d_2}. \tag{3.25}$$

Separating the real and imaginary parts of Eq. (3.25), we have

$$\begin{cases} \sin((n+1)\eta_n) = \dfrac{(2d_1+d_2)h_n}{d_2}(1+\theta_n^2)^{(n+1)/2}, \\ \cos((n+1)\eta_n) = -\dfrac{2d_1}{d_2}(1+\theta_n^2)^{(n+1)/2}, \end{cases}$$

where $\tan\eta_n = \theta_n$ and $\eta_n \in (0, \pi/2]$ as $\theta_n > 0$. Since $\tan^2\eta_n + 1 = \sec^2\eta_n$, then we have $(1+\theta_n^2)^{(n+1)/2} = \sec^{n+1}\eta_n$, then the above equations are equivalent to

$$\begin{cases} \sin((n+1)\eta_n) = \dfrac{(2d_1+d_2)h_n}{d_2}\dfrac{1}{\cos^{n+1}\eta_n}, \\ \cos((n+1)\eta_n) = -\dfrac{2d_1}{d_2}\dfrac{1}{\cos^{n+1}\eta_n}. \end{cases} \tag{3.26}$$

By the second equation of (3.26), we obtain Eq. (3.22) from which η_n can be solved. Once η_n is solved, then by the first equation of (3.26), h_n can be solved as in Eq. (3.21). Finally, from (3.22) and (3.25), we obtain that $z_{n\lambda_*}$ satisfies Eq. (3.23).

Now we consider the solvability of Eq. (3.22). When $n = 0$, Eq. (3.22) becomes $\cos^2\eta_n = -\dfrac{d_2}{d_1}$ which is not solvable as $d_1, d_2 > 0$, thus the conclusion in (i) is proved. For a general $n \geq 1$, by the boundedness of cosine function, we know that Eq. (3.22) is not solvable if $d_2 < 2d_1$. When $d_2 \geq 2d_1$, let

$$f_n(\eta_n) = \cos((n+1)\eta_n)\cos^{n+1}\eta_n, \ \ \eta_n \in \left(0, \frac{\pi}{2}\right], \tag{3.27}$$

then we have the following properties of the function $f_n(\eta_n)$:

(a) $f_n(0) = 1, \ f_n\left(\dfrac{\pi}{2}\right) = 0$;

(b) the zeros of f_n are $\eta_{nk} = \dfrac{(2k-1)\pi}{2(n+1)}, \ k = 1, 2, \cdots, \left[\dfrac{n+1}{2}\right]$, where $[\cdot]$ denotes the integer part of a real number;

(c) treating η_n as a continuous variable, we have $f_n'(\eta_n) = -(n+1)\cos^n\eta_n \sin((n+2)\eta_n)$, thus the critical points of $f_n(\eta_n)$ are $\tilde{\eta}_{nk} = \dfrac{k\pi}{n+2}, \ k = 1, 2, \cdots, \left[\dfrac{n+1}{2}\right]$;

(d) $f_n(\eta_n)$ reaches its global minimum at $\eta_n = \tilde{\eta}_{n1} = \dfrac{\pi}{n+2}$, and $\min\limits_{\eta_n \in [0,\pi/2]} f_n(\eta_n) = -m_n$ with m_n defined by (3.20) which is increasing in n.

The results in (a), (b), (c) can be obtained by direct calculations. In the following, we prove (d) in two steps.

Firstly, we will show that the global minimum of $f_n(\eta_n), \ \eta_n \in (0, \pi/2]$ exists. From (c), we know that $f_n(\tilde{\eta}_{nk})$ are extreme values of $f_n(\eta_n)$, and we claim that $|f_n(\tilde{\eta}_{nk})|$ is decreasing in k. By the definition of f_n, we have

$$|f_n(\tilde{\eta}_{nk})| = \left|\cos\left(\frac{(n+1)k\pi}{n+2}\right)\cos^{n+1}\left(\frac{k\pi}{n+2}\right)\right| = \left|\cos\left(\frac{k\pi}{n+2}\right)\right|^{n+2},$$

which is decreasing in k when $k \le [(n+1)/2]$. Also, it is not difficult to verify that $f_n'(\eta_n) < 0$ for $0 < \eta_n < \eta_{n1}$ and $f_n'(\eta_n) > 0$ for $\eta_{n1} < \eta_n < \eta_{n2}$, which implies that η_{n1} is a local minimum value of $f_n(\eta_n)$. Together with the fact that $|f_n(\tilde{\eta}_{n1})|$ is the largest one in all of the extreme values of $f_n(\eta_n)$, we can draw the conclusion that $f_n(\eta_n)$ reaches its global minimum at $\eta_n = \tilde{\eta}_{n1} = \dfrac{\pi}{n+2}$.

The second step, we prove that $m_n = -\min\limits_{\eta_n \in [0,\pi/2]} f_n$ is increasing in n. By a direct calculation, we know that

$$m_n = -f_n\left(\frac{\pi}{n+2}\right) = -\cos\frac{(n+1)\pi}{n+2}\cos^{n+1}\left(\frac{\pi}{n+2}\right) = \cos^{n+2}\left(\frac{\pi}{n+2}\right).$$

By letting $x = n+2 > 2$, we have $m_n(x) = \left(\cos\dfrac{\pi}{x}\right)^x$, where x is a continuous variable, thus $m_n(x) = e^{x\ln(\cos(\pi/x))}$. Then, we take the derivative of $m_n(x)$ with respect to x and obtain

$$m_n'(x) = \left(\cos\frac{\pi}{x}\right)^x\left[\ln\left(\cos\frac{\pi}{x}\right) + \frac{\pi}{x}\tan\frac{\pi}{x}\right].$$

Define $q(x) = \ln\left(\cos\dfrac{\pi}{x}\right) + \dfrac{\pi}{x}\tan\dfrac{\pi}{x}$, then we have

$$q'(x) = -\frac{\pi^2}{x^3}\sec^2\frac{\pi}{x} < 0, \ \ x > 2.$$

Therefore, $q(x) > q_{min} = \lim\limits_{x\to+\infty} q(x) = 0$. Because $\cos\dfrac{\pi}{x} > 0$ for $x > 2$, so we obtain $m_n'(x) > 0$, which implies that $m_n(x)$ is increasing with respect to x for $x > 2$. According to the relationship that $x = n+2$, we know that m_n is also increasing with respect to n.

At the end, we prove that $G(z_n, \beta_n, h_n, \theta_n, \lambda_*) = 0$ has at least one valid solution. From the above discussion, we know that $f_n(\eta_n) \ge -m_n$, thus Eq. (3.22) can be solved when $-2d_1/d_2 \ge -m_n$ which is equivalent to $d_2 \ge 2d_1/m_n$. By the properties of the function $f_n(\eta_n)$, it can be inferred that Eq. (3.22) has at least one solution $\eta_n = \eta_{n\lambda_*}$ satisfying $\dfrac{\pi}{2(n+1)} < \eta_{n\lambda_*} \le \dfrac{\pi}{n+2}$ as $\eta_n = \dfrac{\pi}{2(n+1)}$ is the first zero of $f_n(\eta_n)$ and $f_n(\eta_n)$ reaches its minimum $-m_n$ at $\eta_n = \dfrac{\pi}{n+2}$. From (3.26), we see that $\sin((n+1)\eta_n) > 0, \ \cos((n+1)\eta_n) < 0$, which implies that $\dfrac{\pi}{2(n+1)} < \eta_n < \dfrac{\pi}{n+1}$ and $\eta_{n\lambda_*}$ is a valid solution. Therefore, it can be inferred that $G(z_n, \beta_n, h_n, \theta_n, \lambda_*) = 0$ has at least one solution. This completes the proof. □

Remark 3.5. Lemma 3.4 gives the necessary conditions for Hopf bifurcations to occur in system (1.2):

(i) For $n = 0$, $G(z_n, \beta_n, h_n, \theta_n, \lambda_*) = 0$ has no solution, which implies that it is impossible to have Hopf bifurcations for the weak kernel case.

(ii) For $n \geq 1$, $G(z_n, \beta_n, h_n, \theta_n, \lambda_*) = 0$ has at least one solution when $d_2 \geq 2d_1/m_n$ where m_n is defined as (3.20), thus it is possible for Hopf bifurcations to occur when $d_2 \geq 2d_1/m_n$. In what follows, we give two examples: there are two Hopf bifurcation values for strong kernel ($n = 1$) case in Proposition 3.7 and there is a unique Hopf bifurcation critical value for $n = 4$ case in Example 4.2. Moreover, by the monotonicity of m_n with respect to n, we know that there is a larger range of d_2 (for a fixed $d_1 > 0$) for Hopf bifurcations to occur when n is larger. Also when d_2 is larger, the number of solutions of (3.22) could be more than two for larger n.

Now by applying the implicit function theorem, we obtain the following result regarding the eigenvalue problem (3.14) for $\lambda \in [\lambda_*, \hat{\lambda}]$.

Theorem 3.6. *For each $n \in \mathbb{N}$ and $W_{n\lambda_*}$ defined in Eq. (3.21), we have the following results:*

(i) there is a unique continuously differentiable map $W_n(\lambda) : [\lambda_*, \hat{\lambda}] \to X_{1\mathbb{C}} \times \mathbb{R}^3$ defined by $W_n(\lambda) := (z_{n\lambda}, \beta_{n\lambda}, h_{n\lambda}, \theta_{n\lambda})$ such that $G(W_n(\lambda), \lambda) = 0$ and $W_n(\lambda_*) = W_{n\lambda_*}$;
(ii) for $\lambda \in (\lambda_*, \hat{\lambda}]$, the eigenvalue problem

$$\Lambda(\lambda, i\omega_n, \tau_n)\psi_n = 0,\ \ \tau_n > 0,\ \ \psi_n \in X_{\mathbb{C}} \setminus \{0\}$$

is solvable with Λ defined in (3.5), that is, $i\omega_n \in \sigma(A_{n\tau}(\lambda))$ if and only if

$$\begin{aligned} &\omega_n = \omega_{n\lambda} := (\lambda - \lambda_*)h_{n\lambda},\ \ \tau_n = \tau_{n\lambda} := \theta_{n\lambda}/\omega_{n\lambda}, \\ &\psi_n = r_n\psi_{n\lambda} \text{ with } \psi_{n\lambda} := \beta_{n\lambda}\phi + (\lambda - \lambda_*)z_{n\lambda}, \end{aligned} \tag{3.28}$$

where r_n is a nonzero constant.

Proof. We define $T_n = (T_{n1}, T_{n2}) : X_{1\mathbb{C}} \times \mathbb{R}^3 \mapsto Y_{\mathbb{C}} \times \mathbb{R}$ by $T_n := D_{(z_n,\beta_n,h_n,\theta_n)}G\left(W_{n\lambda_*}, \lambda_*\right)$, which is the Fréchet derivative of G with respect to $(z_n, \beta_n, h_n, \theta_n)$ at $(z_{n\lambda_*}, \beta_{n\lambda_*}, h_{n\lambda_*}, \theta_{n\lambda_*})$. Then we have

$$\begin{aligned} T_{n1}(\chi, \kappa, \epsilon, \vartheta) =& (d\Delta + \lambda_*)\chi + \kappa\Bigg[(1 - ih_{n\lambda_*} - 2\lambda_*\phi)\phi \\ &+ d_2\alpha_{\lambda_*}\left(1 + \frac{1}{(1 + i\theta_{n\lambda_*})^{n+1}}\right)\nabla\cdot(\phi\nabla\phi)\Bigg] \\ &- i\epsilon\phi - \vartheta\frac{i(n+1)\lambda_*\alpha_{\lambda_*}\nabla\cdot(\phi\nabla\phi)}{\left(1 + i\theta_{n\lambda_*}\right)^{n+2}}, \\ T_{n2}(\chi, \kappa, \epsilon, \vartheta) =& 2\kappa\|\phi\|^2_{Y_{\mathbb{C}}}, \end{aligned}$$

where $\alpha_{\lambda*}$ is defined in (2.13). We can verify that T_n is bijective from $X_{1\mathbb{C}} \times \mathbb{R}^3$ to $Y_{\mathbb{C}} \times \mathbb{R}$ in the following two steps.
Step 1. We prove that T_n is injective. Suppose $T_n(\chi, \kappa, \epsilon, \vartheta) = (T_{n1}, T_{n2}) = (0, 0)$, we immediately obtain $\kappa = 0$ by solving $T_{n2} = 2\kappa\|\phi\|^2_{Y_{\mathbb{C}}} = 0$ as $\|\phi\|^2_{Y_{\mathbb{C}}} > 0$. Substituting $\kappa = 0$ into $T_{n1} = 0$, we have

$$(d\Delta+\lambda_*)\chi - i\epsilon\phi - \vartheta \frac{i(n+1)\lambda_*\alpha_{\lambda_*}\nabla\cdot(\phi\nabla\phi)}{\left(1+i\theta_{n\lambda_*}\right)^{n+2}} = 0. \tag{3.29}$$

Letting $\chi = \chi_r + i\chi_i$, we obtain the following equation by separating the real part of (3.29):

$$(d\Delta+\lambda_*)\chi_r - \vartheta \frac{(n+1)\lambda_*\alpha_{\lambda_*}\nabla\cdot(\phi\nabla\phi)\sin\left((n+2)\eta_{n\lambda_*}\right)}{\left(1+\theta_{n\lambda_*}^2\right)^{\frac{n+2}{2}}} = 0, \tag{3.30}$$

where $1+i\theta_{n\lambda_*} = \sqrt{1+\theta_{n\lambda_*}^2}e^{i\eta_{n\lambda_*}}$ is applied. Multiplying both sides of Eq. (3.30) by ϕ and integrating over Ω, we obtain

$$-\vartheta \frac{(n+1)\lambda_*\alpha_{\lambda_*}\sin\left((n+2)\eta_{n\lambda_*}\right)}{\left(1+\theta_{n\lambda_*}^2\right)^{\frac{n+2}{2}}}\int_\Omega \nabla\cdot(\phi\nabla\phi)\phi dx = 0,$$

as $\int_\Omega (d\Delta+\lambda_*)\chi_r\phi dx = \int_\Omega (d\Delta+\lambda_*)\phi\chi_r dx = 0$. By the fact that $\int_\Omega \nabla\cdot(\phi\nabla\phi)\phi dx = -\int_\Omega \phi|\nabla\phi|^2 dx \neq 0$, we know that $\vartheta = 0$ as $\int_\Omega (d\Delta+\lambda_*)\chi_i\phi dx = \int_\Omega (d\Delta+\lambda_*)\phi\chi_i dx = 0$. From (3.29) and $\vartheta = 0$, we have $(d\Delta+\lambda_*)\chi_i - \epsilon\phi = 0$, which implies that $\epsilon = 0$. Since $\epsilon = 0$, $\vartheta = 0$ and $\kappa = 0$, it can be inferred that $\chi = 0$, thus T_n is injective.

Step 2. Here we show that T_n is surjective. Suppose $(\xi, a) \in Y_{\mathbb{C}} \times \mathbb{R}$ and $T_n(\chi, \kappa, \epsilon, \vartheta) = (\xi, a)$, then we have $\kappa = a/\|\phi\|_{Y_{\mathbb{C}}}^2$ by solving $T_{n2} = a$. As $\chi \in X_{1\mathbb{C}}$, then we know that $\int_\Omega \chi\phi dx = 0$. Multiplying both sides of $T_{n1} = 0$ by ϕ and integrating it over Ω, it can be obtained that

$$\begin{aligned}&\frac{a}{\|\phi\|_{Y_{\mathbb{C}}}^2}\left[(1-ih_{n\lambda_*})\int_\Omega \phi^2 dx - 2\lambda_*\int_\Omega \phi^3 dx + d_2\alpha_{\lambda_*}\left(1+\frac{1}{(1+i\theta_{n\lambda_*})^{n+1}}\right)\int_\Omega \nabla\cdot(\phi\nabla\phi)\phi dx\right]\\&-i\epsilon\int_\Omega \phi^2 dx - \vartheta\frac{i(n+1)\lambda_*\alpha_{\lambda_*}}{\left(1+i\theta_{n\lambda_*}\right)^{n+2}}\int_\Omega \nabla\cdot(\phi\nabla\phi)\phi dx = 0.\end{aligned}$$

From the above equation, we obtain the following two equations by separating the real and imaginary parts:

$$\begin{aligned}&\frac{a}{\|\phi\|_{Y_{\mathbb{C}}}^2}\left[\int_\Omega \phi^2 dx - 2\lambda_*\int_\Omega \phi^3 dx + d_2\alpha_{\lambda_*}\left(1+\frac{\cos((n+1)\eta_{n\lambda_*})}{(1+\theta_{n\lambda_*}^2)^{\frac{n+1}{2}}}\right)\int_\Omega \nabla\cdot(\phi\nabla\phi)\phi dx\right]\\&+\vartheta\frac{(n+1)\lambda_*\alpha_{\lambda_*}\sin((n+2)\eta_{n\lambda_*})}{\left(1+\theta_{n\lambda_*}^2\right)^{\frac{n+2}{2}}}\int_\Omega \nabla\cdot(\phi\nabla\phi)\phi dx = 0,\end{aligned} \tag{3.31}$$

and

$$\frac{a}{\|\phi\|_{Y_{\mathbb{C}}}^2}\left[h_{n\lambda_*})\int_\Omega \phi^2 dx + d_2\alpha_{\lambda_*}\left(1+\frac{\sin((n+1)\eta_{n\lambda_*})}{(1+\theta_{n\lambda_*}^2)^{\frac{n+1}{2}}}\right)\int_\Omega \nabla\cdot(\phi\nabla\phi)\phi dx\right] - \epsilon\int_\Omega \phi^2 dx$$
$$+\vartheta\frac{(n+1)\lambda_*\alpha_{\lambda_*}\cos((n+2)\eta_{n\lambda_*})}{\left(1+\theta_{n\lambda_*}^2\right)^{\frac{n+2}{2}}}\int_\Omega \nabla\cdot(\phi\nabla\phi)\phi dx = 0. \tag{3.32}$$

Then, ϑ can be solved from Eq. (3.31). Substituting the obtained value of ϑ into (3.32), we can get the value for ϵ. Finally, we put all the values of $\kappa,\ \vartheta$ and ϵ into $T_{n1}=0$ and we can uniquely solve χ. Thus, T_n is surjective.

By the implicit function theorem, there exists a continuously differentiable mapping $W_n(\lambda):[\lambda_*,\hat{\lambda}]\to X_{1\mathbb{C}}\times\mathbb{R}^3$ such that $G(W_n(\lambda),\lambda)=0$ with $W_n(\lambda_*)=W_{n\lambda_*}$. This completes the proof of existence.

And we need also to prove the uniqueness of the solution, that is, we need to verify that if there exists another mapping $\tilde{W}_n(\lambda)$ such that $G\left(\tilde{W}_n(\lambda),\lambda\right)=0$, then $\tilde{W}_n(\lambda)\to W_{n\lambda_*}$ as $\lambda\to\lambda_*$ in the norm of $X_{1\mathbb{C}}\times\mathbb{R}^3$. From Lemma 3.3, we see that $\{\tilde{h}_{n\lambda}\}$ is bounded as $\tilde{h}_{n\lambda}=\tilde{\omega}_{n\lambda}/(\lambda-\lambda_*)$ for each n. And $\{\tilde{\beta}_{n\lambda}\}$ is also bounded according to the second equation of Eq. (3.19). For the boundedness of $\{\tilde{z}_{n\lambda}\}$, we first prove that $\nabla\tilde{z}_{n\lambda}$ is bounded. Multiplying the first equation of Eq. (3.19) by $\{\tilde{z}_{n\lambda}\}$ and integrating over Ω, we obtain

$$d_1\|\nabla\tilde{z}_{n\lambda}\|_{Y_{\mathbb{C}}}^2 \le \lambda_*\|\tilde{z}_{n\lambda}\|_{Y_{\mathbb{C}}}^2 + M_5\|\tilde{z}_{n\lambda}\|_{Y_{\mathbb{C}}} + (\lambda-\lambda_*)M_6\|\nabla\tilde{z}_{n\lambda}\|_{Y_{\mathbb{C}}}^2 + (\lambda-\lambda_*)M_7\|\tilde{z}_{n\lambda}\|_{Y_{\mathbb{C}}}^2,$$

where

$$\begin{aligned}M_5 =& d_2\alpha_\lambda\left(\|\nabla\phi\|_{Y_{\mathbb{C}}}\|\nabla m(\lambda,\xi_\lambda)\|_\infty + \|\phi\|_{Y_{\mathbb{C}}}\|\Delta m(\lambda,\xi_\lambda)\|_\infty\right)\\ &+\frac{d_2\alpha_\lambda}{|(1+i\theta_{n\lambda})^{n+1}|}\left(\|\nabla\phi\|_{Y_{\mathbb{C}}}\|\nabla m(\lambda,\xi_\lambda)\|_\infty + \|\Delta\phi\|_{Y_{\mathbb{C}}}\|m(\lambda,\xi_\lambda)\|_\infty\right)\\ M_6 =& \frac{d_2\alpha_\lambda\|m(\lambda,\xi_\lambda)\|_\infty}{|(1+i\theta_{n\lambda})^{n+1}|},\ \ M_7=\frac{d_2\alpha_\lambda\|\Delta m(\lambda,\xi_\lambda)\|_\infty}{2},\end{aligned}$$

with $m(\lambda,\xi_\lambda)=\phi+(\lambda-\lambda_*)\xi_\lambda$ and Hölder inequality being applied. Also, according to the boundedness of $\alpha_\lambda,\ m(\lambda,\xi_\lambda),\ \frac{1}{|(1+i\theta_{n\lambda})^{n+1}|}$ for $\lambda\in(\lambda_*,\hat{\lambda}]$, we can obtain the boundedness of $M_5,\ M_6$ and M_7. As $\lambda\in(\lambda_*,\hat{\lambda}]$, we know that $d_1-(\lambda-\lambda_*)M_6>0$ by the definition of $\hat{\lambda}$ defined as (3.1), thus we have

$$\|\nabla\tilde{z}_{n\lambda}\|_{Y_{\mathbb{C}}}^2 \le \frac{M_5}{d_1-(\lambda-\lambda_*)M_6}\|\tilde{z}_{n\lambda}\|_{Y_{\mathbb{C}}} + \frac{\lambda_*+(\lambda-\lambda_*)M_7}{d_1-(\lambda-\lambda_*)M_6}\|\tilde{z}_{n\lambda}\|_{Y_{\mathbb{C}}}^2 \tag{3.33}$$

On the other hand, from Lemma 3.3, we know that

$$|\langle(d_1\Delta+\lambda_*)\tilde{z}_{n\lambda},\tilde{z}_{n\lambda}\rangle| \ge d_1(\lambda_2-\lambda_1)\|\tilde{z}_{n\lambda}\|_{Y_{\mathbb{C}}}^2,$$

where λ_2 is the second eigenvalue of operator $-\Delta$. Together with the first equation of Eq. (3.19) and (3.33), we obtain

$$(d_1\lambda_2-\lambda_*)\|\tilde z_{n\lambda}\|^2_{Y_{\mathbb{C}}} \le M_5\|\tilde z_{n\lambda}\|_{Y_{\mathbb{C}}}+(\lambda-\lambda_*)M_6\|\nabla\tilde z_{n\lambda}\|^2_{Y_{\mathbb{C}}}+(\lambda-\lambda_*)M_7\|\tilde z_{n\lambda}\|^2_{Y_{\mathbb{C}}}$$
$$\le \frac{d_1M_5}{d_1-(\lambda-\lambda_*)M_6}\|\tilde z_{n\lambda}\|_{Y_{\mathbb{C}}}+\frac{(\lambda-\lambda_*)(\lambda_*M_6+d_1M_7)}{d_1-(\lambda-\lambda_*)M_6}\|\tilde z_{n\lambda}\|^2_{Y_{\mathbb{C}}},$$

which implies that

$$\|\tilde z_{n\lambda}\|_{Y_{\mathbb{C}}} \le \frac{d_1M_5}{(d_1\lambda_2-\lambda_*)(d_1-(\lambda-\lambda_*)M_6)}+\frac{(\lambda-\lambda_*)(\lambda_*M_6+d_1M_7)}{(d_1\lambda_2-\lambda_*)(d_1-(\lambda-\lambda_*)M_6)}\|\tilde z_{n\lambda}\|_{Y_{\mathbb{C}}}.$$

Hence, $\{\tilde z_{n\lambda}\}$ is bounded in $Y_{\mathbb{C}}$ when $\lambda\in[\lambda_*,\hat\lambda]$. Since the operator $d_1\Delta+\lambda_*:(X_1)_{\mathbb{C}}\mapsto(Y_1)_{\mathbb{C}}$ has a bounded inverse, by applying $(d_1\Delta+\lambda_*)^{-1}$ on $g_1\left(\tilde z_{n\lambda},\tilde\beta_{n\lambda},\tilde h_{n\lambda},\tilde\theta_{n\lambda},\lambda\right)=0$, we find that $\{\tilde z_{n\lambda}\}$ is also bounded in $X_{\mathbb{C}}$, and hence $\left\{\tilde W_n(\lambda):\lambda\in(\lambda_*,\lambda^*]\right\}$ is precompact in $X_{1\mathbb{C}}\times\mathbb{R}^3$. Therefore, there is a subsequence $\left\{\tilde W_n(\lambda^j):=\left(\tilde z_{n\lambda^j},\tilde\beta_{n\lambda^j},\tilde h_{n\lambda^j},\tilde\theta_{n\lambda^j}\right)\right\}$ such that

$$\tilde W_n(\lambda^j)\to\tilde W_n(\lambda_*),\quad \lambda^j\to\lambda_* \text{ as } j\to\infty.$$

By taking the limit of the equation $(d_1\Delta+\lambda_*)^{-1}G\left(\tilde W_n(\lambda^j),\lambda^j\right)=0$ as $j\to\infty$, we have that $G\left(\tilde W_n(\lambda_*),\lambda_*\right)=0$. Also, by Lemma 3.4, we know that $G(z_n,\beta_n,h_n,\theta_n,\lambda_*)=0$ has a unique solution given by $(z_n,\beta_n,h_n,\theta_n)=W_{n\lambda_*}$, thus $\tilde W_n(\lambda_*)=W_{n\lambda_*}$. Because $\tilde W_n(\lambda)\to W_{n\lambda_*}$ as $\lambda\to\lambda_*$ in the norm of $X_{1\mathbb{C}}\times\mathbb{R}^3$, so $\tilde W_n(\lambda)=W_n(\lambda)$ by the continuity of $W_n(\lambda)$ in λ. This proves part (i), and part (ii) is immediately observed from part (i). □

Although Eq. (3.22) is solvable for a general $n\ge1$, it is still difficult to give an explicit solution. For the strong kernel case ($n=1$), we can explicitly solve Eq. (3.22), and the results are stated as follows.

Proposition 3.7. *When $n=1$ and $d_2>16d_1$, Eq. (3.22) has exactly two solutions:*

$$\eta^{(1)}_{1\lambda_*}=\arccos\frac12\sqrt{1+\sqrt{\frac{d_2-16d_1}{d_2}}},\ \ \eta^{(2)}_{1\lambda_*}=\arccos\frac12\sqrt{1-\sqrt{\frac{d_2-16d_1}{d_2}}},\tag{3.34}$$

and

$$h^{(1)}_{1\lambda_*}=\frac{2d_2}{2d_1+d_2}\sin\eta^{(1)}_{1\lambda_*}\cos^3\eta^{(1)}_{1\lambda_*},\ \ h^{(2)}_{1\lambda_*}=\frac{2d_2}{2d_1+d_2}\sin\eta^{(2)}_{1\lambda_*}\cos^3\eta^{(2)}_{1\lambda_*}.\tag{3.35}$$

Thus there are exactly two possible critical delay values $\tau^{(i)}_{1\lambda}$ $(i=1,2)$ for Hopf bifurcation satisfying

$$\lim_{\lambda\to\lambda_*}(\lambda-\lambda_*)\tau^{(1)}_{1\lambda}=\frac{\theta^{(1)}_{1\lambda_*}}{h^{(1)}_{1\lambda_*}},\ \ \lim_{\lambda\to\lambda_*}(\lambda-\lambda_*)\tau^{(2)}_{1\lambda}=\frac{\theta^{(2)}_{1\lambda_*}}{h^{(2)}_{1\lambda_*}}\tag{3.36}$$

where $\theta^{(i)}_{1\lambda_*}=\tan\eta^{(i)}_{1\lambda_*}$, $i=1,2$ and $0<\tau^{(1)}_{1\lambda}<\tau^{(2)}_{1\lambda}$.

Proof. We put $n = 1$ into Eq. (3.22) and obtain

$$\cos(2\eta_1)\cos^2\eta_1 = (2\cos^2\eta_1 - 1)\cos^2\eta_1 = \frac{-2d_1}{d_2}. \tag{3.37}$$

Let $\rho = \cos^2\eta_1 \in (0, 1]$, Eq. (3.37) can be rewritten as

$$2\rho^2 - \rho + \frac{2d_1}{d_2} = 0,$$

which has two roots

$$\rho_1 = \frac{1}{4} - \sqrt{\frac{1}{16} - \frac{d_1}{d_2}},\ \rho_2 = \frac{1}{4} + \sqrt{\frac{1}{16} - \frac{d_1}{d_2}}$$

satisfying $0 < \rho_1 < \frac{1}{4} < \rho_2 < \frac{1}{2}$. Thus, η_1 can be solved as

$$\eta_{1\lambda_*}^{(1)} = \arccos\sqrt{\rho_2},\ \eta_{1\lambda_*}^{(2)} = \arccos\sqrt{\rho_1},$$

satisfying $\frac{\pi}{4} < \eta_{1\lambda_*}^{(1)} < \frac{\pi}{3} < \eta_{1\lambda_*}^{(2)} < \frac{\pi}{2}$, from which Eq. (3.34) can be derived. Substituting (3.34) into (3.21), we obtain the values of h_1 as in (3.35). At the end, by the fact that $\theta_n = \omega_n\tau_n = (\lambda - \lambda_*)h_n\tau_n$, we know that

$$\tau_n = \tau_{n\lambda} = \frac{\theta_n}{(\lambda - \lambda_*)h_n} = \frac{\tan\eta_n}{(\lambda - \lambda_*)h_n},$$

thus we can calculate the possible critical values for $n = 1$ case as in Eq. (3.36). By the definition in (3.34), we see that $\eta_{1\lambda_*}^{(1)} < \pi/3$ and $\eta_{1\lambda_*}^{(2)} > \pi/3$ as $d_2 - 16d_1 > 0$, thus we immediately obtain that $h_{1\lambda_*}^{(1)} > h_{1\lambda_*}^{(2)}$ and $\theta_{1\lambda_*}^{(1)} < \theta_{1\lambda_*}^{(2)}$. By (3.36), we reach our conclusion that $0 < \tau_{1\lambda}^{(1)} < \tau_{1\lambda}^{(2)}$ when λ is close to λ_*. □

Next we verify the transversality conditions for Hopf bifurcation when $n = 1$.

Lemma 3.8. *When $n = 1$ and $d_2 > 16d_1$, for each $\lambda \in (\lambda_*, \hat{\lambda}]$, let $\tau_{1\lambda}^{(i)}$, $i = 1, 2$ be defined as (3.36), we have*

(i) $\mu = \mu\left(\tau_{1\lambda}^{(i)}\right) := i\omega_{1\lambda}^{(i)}$ is a simple eigenvalue of $A_{1\tau}(\lambda)$ when $\tau = \tau_{1\lambda}^{(i)}$;
(ii) $\mathcal{R}e\left(\frac{d\mu}{d\tau}\left(\tau_{1\lambda}^{(1)}\right)\right) > 0$ and $\mathcal{R}e\left(\frac{d\mu}{d\tau}\left(\tau_{1\lambda}^{(2)}\right)\right) < 0$.

Proof. The proof of part (i) is similar to that of Theorem 3.5 in [31], so we omit it here. For part (ii), by applying the implicit function theorem, we know that there exists a neighborhood $O \times D \times H \subset \mathbb{R} \times \mathbb{C} \times X_{\mathbb{C}}$ of $\left(\tau_{1\lambda}^{(i)},\ i\omega_{1\lambda}^{(i)},\ \psi_{1\lambda}^{(i)}\right)$ and a continuous differential function (μ, ψ) :

$O \to D \times H$ such that, for each $\tau \in O$, $\mu(\tau)$ is the only eigenvalue of $A_{1\tau}(\lambda)$ with its associated eigenfunction $\psi(\tau)$ and the following equalities hold:

$$\begin{aligned} &\mu\left(\tau_{1\lambda}^{(i)}\right) = i\omega_{1\lambda}^{(i)}, \quad \psi\left(\tau_{1\lambda}^{(i)}\right) = \psi_{1\lambda}^{(i)}, \\ &\Lambda(\lambda, \mu(\tau), \tau) = \left[A(\lambda) + \frac{d_2}{(1+\mu(\tau)\tau)^2}\nabla\cdot(u_\lambda\nabla) - \lambda u_\lambda - \mu(\tau)\right]\psi(\tau) = 0, \quad \tau \in O. \end{aligned} \tag{3.38}$$

Differentiating Eq. (3.38) with respect to τ at $\tau = \tau_{1\lambda}^{(i)}$, we get

$$\begin{aligned} &-\frac{d\mu\left(\tau_{1\lambda}^{(i)}\right)}{d\tau}\left[\frac{2d_2\tau_{1\lambda}^{(i)}}{\left(1+\mu\left(\tau_{1\lambda}^{(i)}\right)\tau_{1\lambda}^{(i)}\right)^3}\nabla\cdot\left(u_\lambda\nabla\psi_{1\lambda}^{(i)}\right) + \psi_{1\lambda}^{(i)}\right] \\ &-\frac{2d_2\mu\left(\tau_{1\lambda}^{(i)}\right)}{\left(1+\mu\left(\tau_{1\lambda}^{(i)}\right)\tau_{1\lambda}^{(i)}\right)^3}\nabla\cdot\left(u_\lambda\nabla\psi_{1\lambda}^{(i)}\right) + \Lambda\left(\lambda, i\omega_{1\lambda}^{(i)}, \tau_{1\lambda}^{(i)}\right)\frac{d\psi}{d\tau}\left(\tau_{1\lambda}^{(i)}\right) = 0. \end{aligned} \tag{3.39}$$

Multiplying Eq. (3.39) by $\overline{\psi_{1\lambda}^{(i)}}$ and integrating over Ω, we obtain

$$\begin{aligned} \frac{d\mu}{d\tau}\left(\tau_{1\lambda}^{(i)}\right) &= \frac{-\dfrac{2d_2\mu\left(\tau_{1\lambda}^{(i)}\right)}{\left(1+\mu\left(\tau_{1\lambda}^{(i)}\right)\tau_{1\lambda}^{(i)}\right)^3}\displaystyle\int_\Omega \nabla\cdot\left(u_\lambda\nabla\psi_{1\lambda}^{(i)}\right)\overline{\psi_{1\lambda}^{(i)}}dx}{\dfrac{2d_2\tau_{1\lambda}^{(i)}}{\left(1+\mu\left(\tau_{1\lambda}^{(i)}\right)\tau_{1\lambda}^{(i)}\right)^3}\displaystyle\int_\Omega \nabla\cdot\left(u_\lambda\nabla\psi_{1\lambda}^{(i)}\right)\overline{\psi_{1\lambda}^{(i)}}dx + \int_\Omega\left|\psi_{1\lambda}^{(i)}\right|^2dx} \\ &= \frac{\dfrac{2id_2\omega_{1\lambda}^{(i)}}{\left(1+i\theta_{1\lambda}^{(i)}\right)^3}\displaystyle\int_\Omega u_\lambda\left|\nabla\psi_{1\lambda}^{(i)}\right|^2dx}{\displaystyle\int_\Omega\left(\psi_{1\lambda}^{(i)}\right)^2dx - \dfrac{2d_2\tau_{1\lambda}^{(i)}}{\left(1+i\theta_{1\lambda}^{(i)}\right)^3}\int_\Omega u_\lambda\left|\nabla\psi_{1\lambda}^{(i)}\right|^2dx}. \end{aligned} \tag{3.40}$$

When $\lambda \to \lambda_*$, we can obtain the following results

$$\lim_{\lambda\to\lambda_*}\mathcal{R}e\left(\frac{1}{(\lambda-\lambda_*)^2}\frac{d\mu}{d\tau}\left(\tau_{1\lambda}^{(i)}\right)\right) = \mathcal{R}e\left(\frac{\dfrac{2id_2h_{1\lambda_*}^{(i)}\alpha_{\lambda_*}}{\left(1+i\theta_{1\lambda_*}^{(i)}\right)^3}\displaystyle\int_\Omega\phi\left|\nabla\phi\right|^2dx}{\displaystyle\int_\Omega\phi^2dx - \dfrac{2d_2\theta_{1\lambda_*}^{(i)}\alpha_{\lambda_*}}{h_{1\lambda_*}\left(1+i\theta_{1\lambda_*}^{(i)}\right)^3}\int_\Omega\phi\left|\nabla\phi\right|^2dx}\right)$$

$$
\begin{aligned}
&= \mathcal{R}e\left(\frac{2id_2\left(h^{(i)}_{1\lambda_*}\right)^2}{(2d_1+d_2)h^{(i)}_{1\lambda_*}\left(1+i\theta^{(i)}_{1\lambda_*}\right)^3-2d_2\theta^{(i)}_{1\lambda_*}}\right)\\
&= \frac{2d_2(2d_1+d_2)\left(h^{(i)}_{1\lambda_*}\right)^3\left(1+\left(\theta^{(i)}_{1\lambda_*}\right)^2\right)^{3/2}\sin\left(3\eta^{(i)}_{1\lambda_*}\right)}{R_n^2+I_n^2},
\end{aligned}
$$

where

$$
R_n = (2d_1+d_2))h^{(i)}_{1\lambda_*}\left(1+\left(\theta^{(i)}_{1\lambda_*}\right)^2\right)^{3/2}\cos\left(3\eta^{(i)}_{1\lambda_*}\right)-2d_2\theta^{(i)}_{1\lambda_*},
$$

$$
I_n = (2d_1+d_2))h^{(i)}_{1\lambda_*}\left(1+\left(\theta^{(i)}_{1\lambda_*}\right)^2\right)^{3/2}\sin\left(3\eta^{(i)}_{1\lambda_*}\right),
$$

and the fact that $\int_\Omega \phi\,|\nabla\phi|^2\,dx = \frac{\lambda_*}{2d_1}\int_\Omega \phi^3 dx = \frac{1}{\alpha_{\lambda_*}(2d_1+d_2)}\int_\Omega \phi^2 dx$ from (2.9) and (2.13) is applied. Therefore, we see that the sign of $\mathcal{R}e\left(\frac{d\mu}{d\tau}\left(\tau^{(i)}_{1\lambda_*}\right)\right)$ depends on the sign of $\sin\left(3\eta^{(i)}_{1\lambda_*}\right)$. From (3.34), it is known that $\frac{\pi}{4} < \eta^{(1)}_{1\lambda_*} < \frac{\pi}{3}$ and $\frac{\pi}{3} < \eta^{(2)}_{1\lambda_*} < \frac{\pi}{2}$, thus we have

$$
\frac{3\pi}{4} < 3\eta^{(1)}_{1\lambda_*} < \pi < 3\eta^{(2)}_{1\lambda_*} < \frac{3\pi}{2},
$$

which implies that $\sin\left(3\eta^{(1)}_{1\lambda_*}\right) > 0$ and $\sin\left(3\eta^{(2)}_{1\lambda_*}\right) < 0$. Hence the results in part (ii) follow from the continuous differentiability of $\mu(\tau)$ with respect to λ. □

Based on the discussion above, we give the results about the stability of steady state of Eq. (1.2) in the following two theorems for weak kernel and strong kernel cases, respectively.

Theorem 3.9. *When $n = 0$ (weak kernel), all the eigenvalues of $A_{0\tau}(\lambda)$ have negative real parts for all $\tau > 0$, and the positive steady state u_λ of (1.2) is locally asymptotically stable for all $\tau > 0$.*

Proof. To prove our conclusion, we use a similar method as Proposition 2.9 in [4]. Assume that $A_{0\tau}(\lambda)$ has eigenvalues with positive real parts, then there exists a sequence $\{\lambda^j\}_{j=1}^{\infty}$, satisfying $\lambda^j > \lambda_*$ for $j \geq 1$, $\lim_{j\to\infty}\lambda^j = \lambda_*$, and for each j, the eigenvalue problem

$$
\begin{cases}
A(\lambda^j)\psi_{\lambda^j} + \dfrac{d_2}{1+\mu_{\lambda^j}\tau}\nabla\cdot(u_{\lambda^j}\nabla\psi_{\lambda^j}) - \lambda^j u_{\lambda^j}\psi_{\lambda^j} = \mu\psi_{\lambda^j}, & x\in\Omega,\\
\psi_{\lambda^j} = 0, & x\in\partial\Omega,
\end{cases}
\tag{3.41}
$$

has an eigenvalue μ_{λ^j} with $\mathcal{R}e(\mu_{\lambda^j}) \geq 0$ and the corresponding eigenfunction ψ_{λ^j} satisfying $\|\psi_{\lambda^j}\|_{Y_{\mathbb{C}}} = 1$. Define $\tilde{A}(\lambda) \triangleq d_1\Delta + d_2\nabla \cdot (u_\lambda \nabla) + \lambda(1-u_\lambda)$, then one can verify that $A(\lambda^j)u_{\lambda^j} = \tilde{A}(\lambda^j)u_{\lambda^j} = 0$. Thus, Eq. (3.41) can be rewritten as

$$\begin{cases} \tilde{A}(\lambda^j)\psi_{\lambda^j} + d_2\nabla \cdot (\psi_{\lambda^j}\nabla u_{\lambda^j}) - \dfrac{d_2\mu_{\lambda^j}\tau}{1+\mu_{\lambda^j}\tau}\nabla \cdot (u_{\lambda^j}\nabla\psi_{\lambda^j}) - \lambda^j u_{\lambda^j}\psi_{\lambda^j} = \mu_{\lambda^j}\psi_{\lambda^j}, & x \in \Omega, \\ \psi_{\lambda^j} = 0, & x \in \partial\Omega. \end{cases} \tag{3.42}$$

Then, we write ψ_{λ^j} as $\psi_{\lambda^j} = c_{\lambda^j}u_{\lambda^j} + \phi_{\lambda^j}$, where $c_{\lambda^j} \in \mathbb{C}$ and $c_{\lambda^j} = \langle u_{\lambda^j}, \psi_{\lambda^j}\rangle / \langle u_{\lambda^j}, u_{\lambda^j}\rangle$. Here u_{λ^j} is the positive steady state of Eq. (1.2) for $\lambda = \lambda^j$, and $\phi_{\lambda^j} \in X_{\mathbb{C}}$ satisfies $\langle \phi_{\lambda^j}, u_{\lambda^j}\rangle = 0$.

From the fact that $\tilde{A}(\lambda^j)u_{\lambda^j} = A(\lambda^j)u_{\lambda^j} = 0$ and $u_{\lambda^j} > 0$, we know that 0 is the principal eigenvalue of $\tilde{A}(\lambda^j)$ and u_{λ^j} is the corresponding eigenfunction. Also, it is not difficult to verify that $\tilde{A}(\lambda^j)$ is a self-adjoint operator, thus

$$\langle \tilde{A}(\lambda^j)\phi_{\lambda^j}, u_{\lambda^j}\rangle = \langle \tilde{A}(\lambda^j)u_{\lambda^j}, \phi_{\lambda^j}\rangle = 0.$$

Substituting $\psi_{\lambda^j} = c_{\lambda^j}u_{\lambda^j} + \phi_{\lambda^j}$ and $\mu = \mu_{\lambda^j}$ into Eq. (3.42), innerproducting with ψ_{λ^j}, we have

$$\begin{aligned} &\langle \tilde{A}(\lambda^j)\phi_{\lambda^j}, \phi_{\lambda^j}\rangle \\ =&\mu_{\lambda^j} + \left\langle \lambda^j u_{\lambda^j}\psi_{\lambda^j} - d_2\nabla \cdot (\psi_{\lambda^j}\nabla u_{\lambda^j}) + \frac{d_2\mu_{\lambda^j}\tau}{1+\mu_{\lambda^j}\tau}\nabla \cdot (u_{\lambda^j}\nabla\psi_{\lambda^j}), \psi_{\lambda^j}\right\rangle. \end{aligned} \tag{3.43}$$

As $\tilde{A}(\lambda^j)$ is a self-adjoint operator and 0 is its principal eigenvalue, thus $\langle \tilde{A}(\lambda^j)\phi_{\lambda^j}, \phi_{\lambda^j}\rangle \leq 0$ holds for any $\phi_{\lambda^j} \in X_{\mathbb{C}}$. Define

$$D_j = \left\langle \lambda^j u_{\lambda^j}\psi_{\lambda^j} - d_2\nabla \cdot (\psi_{\lambda^j}\nabla u_{\lambda^j}) + \frac{d_2\mu_{\lambda^j}\tau}{1+\mu_{\lambda^j}\tau}\nabla \cdot (u_{\lambda^j}\nabla\psi_{\lambda^j}), \psi_{\lambda^j}\right\rangle,$$

then we can obtain that

$$\begin{aligned} |D_j| &\leq \sigma\|u_{\lambda^j}\|_\infty + d_2\|\nabla u_{\lambda^j}\|_\infty\|\psi_{\lambda^j}\|_{Y_{\mathbb{C}}}\|\nabla\psi_{\lambda^j}\|_{Y_{\mathbb{C}}} + d_2\|u_{\lambda^j}\|_\infty\|\nabla\psi_{\lambda^j}\|^2_{Y_{\mathbb{C}}} \\ &\leq \sigma\|u_{\lambda^j}\|_\infty + d_2M_1\|\nabla u_{\lambda^j}\|_\infty\|\psi_{\lambda^j}\|^2_{Y_{\mathbb{C}}} + d_2M_1^2\|u_{\lambda^j}\|_\infty\|\psi_{\lambda^j}\|^2_{Y_{\mathbb{C}}}, \end{aligned} \tag{3.44}$$

where $\|\nabla\psi_{\lambda^j}\|_{Y_{\mathbb{C}}} \leq M_1\|\psi_{\lambda^j}\|_{Y_{\mathbb{C}}}$ from Lemma 3.2 is applied, $\sigma = \max\limits_{j\geq 1}\lambda^j$, and

$$\left|\frac{\mu_{\lambda^j}\tau}{1+\mu_{\lambda^j}\tau}\right|^2 = \left|\frac{\mathcal{R}e^2(\mu_{\lambda^j}\tau) + \mathcal{R}e(\mu_{\lambda^j}) + \mathcal{I}m^2(\mu_{\lambda^j}) + i\mathcal{I}m(\mu_{\lambda^j})}{\mathcal{R}e^2(\mu_{\lambda^j}\tau) + 2\mathcal{R}e(\mu_{\lambda^j}) + 1 + \mathcal{I}m^2(\mu_{\lambda^j})}\right|^2 < 1,$$

which can be verified through elementary calculation. According to the assumption that $\mathcal{R}e(\mu_{\lambda^j}) \geq 0$, Eq. (3.43) and the fact that $\langle \tilde{A}(\lambda^j)\psi_{\lambda^j}, \psi_{\lambda^j}\rangle \leq 0$, it can be inferred that

$$0 \leq \mathcal{R}e(\mu_{\lambda^j}) \leq |D_j|, \quad 0 \leq \left|\mathcal{I}m(\mu_{\lambda^j})\right| \leq |D_j|,$$

together with (3.44), $\lim_{j\to\infty}\|u_{\lambda^j}\|_\infty = 0$, and $\lim_{j\to\infty}\|\nabla u_{\lambda^j}\|_\infty = 0$ which can be inferred from (2.13), we have

$$\lim_{j\to\infty}\mathcal{R}e(\mu_{\lambda^j}) = \lim_{j\to\infty}\left|\mathcal{I}m(\mu_{\lambda^j})\right| = 0.$$

From (3.43) and using similar argument as in the proof of Lemma 2.3 in [3], we have

$$\left|D_j\right| + |\mu_{\lambda^j}| \geq |\langle \tilde{A}(\lambda^j)\phi_{\lambda^j}, \phi_{\lambda^j}\rangle| \geq |\lambda_2(\lambda^j)| \cdot \|\phi_{\lambda^j}\|^2_{Y_{\mathbb{C}}}, \tag{3.45}$$

where $\lambda_2(\lambda^j)$ is the second eigenvalue of $\tilde{A}(\lambda^j)$. When $j \to \infty$, both $|D_j|$ and $|\mu_{\lambda^j}|$ go to zero as $\lim_{j\to\infty}\|u_{\lambda^j}\|_\infty = 0$ and $\lim_{j\to\infty}\|\nabla u_{\lambda^j}\|_\infty = 0$ hold, so the inequality (3.45) implies that $\lim_{j\to\infty}\|\phi_{\lambda^j}\|_{Y_{\mathbb{C}}} = 0$.

Since $\psi_{\lambda^j} = c_{\lambda^j}u_{\lambda^j} + \phi_{\lambda^j}$ and $\|\psi_{\lambda^j}\|_{Y_{\mathbb{C}}} = 1$, then we obtain

$$\lim_{n\to\infty}|c_{\lambda^j}|(\lambda^j - \lambda_*)\lim_{j\to\infty}\left\|\frac{u_{\lambda^j}}{\lambda^j - \lambda_*}\right\|_{Y_{\mathbb{C}}} = \alpha_{\lambda^j}\lim_{n\to\infty}|c_{\lambda^j}|(\lambda^j - \lambda_*) = 1,$$

and hence $\lim_{j\to\infty}|c_{\lambda^j}|(\lambda^j - \lambda_*) = \dfrac{1}{\alpha_{\lambda^j}} > 0$. Now we calculate that

$$\begin{aligned}\frac{D_j}{\lambda^j - \lambda_*} =& \frac{1}{\lambda^j - \lambda_*}\Big\langle \lambda^j u_{\lambda^j}(c_{\lambda^j}u_{\lambda^j} + \phi_{\lambda^j}) - d_2\nabla\cdot((c_{\lambda^j}u_{\lambda^j} + \phi_{\lambda^j})\nabla u_{\lambda^j}) \\ &+ \frac{d_2\mu_{\lambda^j}\tau}{1 + \mu_{\lambda^j}\tau}\nabla\cdot(u_{\lambda^j}\nabla(c_{\lambda^j}u_{\lambda^j} + \phi_{\lambda^j})), (c_{\lambda^j}u_{\lambda^j} + \phi_{\lambda^j})\Big\rangle \\ =& J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8 - J_9 - J_{10} - J_{11} - J_{12},\end{aligned} \tag{3.46}$$

where

$$J_1 = \lambda^j|c_{\lambda^j}|^2(\lambda^j - \lambda_*)^2\int_\Omega \frac{u^3_{\lambda^j}}{(\lambda^j - \lambda_*)^3}dx,\ J_2 = \lambda^j c_{\lambda^j}(\lambda^j - \lambda_*)\int_\Omega \frac{u^2_{\lambda^j}\overline{\phi_{\lambda^j}}}{(\lambda^j - \lambda_*)^2}dx,$$

$$J_3 = \lambda^j\overline{c_{\lambda^j}}(\lambda^j - \lambda_*)\int_\Omega \frac{u^2_{\lambda^j}\phi_{\lambda^j}}{(\lambda^j - \lambda_*)^2}dx,\ J_4 = \lambda^j\int_\Omega \frac{|\phi_{\lambda^j}|^2 u_{\lambda^j}}{\lambda^j - \lambda_*}dx,$$

$$J_5 = d_2|c_{\lambda^j}|^2(\lambda^j - \lambda_*)^2\int_\Omega \frac{u_{\lambda^j}|\nabla u_{\lambda^j}|^2}{(\lambda^j - \lambda_*)^3}dx,\ J_6 = d_2c_{\lambda^j}(\lambda^j - \lambda_*)\int_\Omega \frac{u_{\lambda^j}\nabla u_{\lambda^j}\overline{\nabla\phi_{\lambda^j}}}{(\lambda^j - \lambda_*)^2}dx,$$

$$J_7 = d_2\overline{c_{\lambda^j}}(\lambda^j - \lambda_*)\int_\Omega \frac{|\nabla u_{\lambda^j}|^2\phi_{\lambda^j}}{(\lambda^j - \lambda_*)^2}dx,\ J_8 = d_2\int_\Omega \frac{\nabla u_{\lambda^j}\phi_{\lambda^j}\overline{\nabla\phi_{\lambda^j}}}{\lambda^j - \lambda_*}dx,$$

$$J_9 = \frac{d_2\mu_{\lambda^j}\tau|c_{\lambda^j}|^2(\lambda^j-\lambda_*)^2}{1+\mu_{\lambda^j}\tau}\int_\Omega \frac{u_{\lambda^j}|\nabla u_{\lambda^j}|^2}{(\lambda^j-\lambda_*)^3}dx,$$

$$J_{10} = \frac{d_2\mu_{\lambda^j}\tau c_{\lambda^j}(\lambda^j-\lambda_*)}{1+\mu_{\lambda^j}\tau}\int_\Omega \frac{u_{\lambda^j}\nabla u_{\lambda^j}\overline{\nabla\phi_{\lambda^j}}}{(\lambda^j-\lambda_*)^2}dx,$$

$$J_{11} = \frac{d_2\mu_{\lambda^j}\tau\overline{c_{\lambda^j}}(\lambda^j-\lambda_*)}{1+\mu_{\lambda^j}\tau}\int_\Omega \frac{u_{\lambda^j}\nabla\phi_{\lambda^j}\nabla u_{\lambda^j}}{(\lambda^j-\lambda_*)^2}dx,\ \ J_{12} = \frac{d_2\mu_{\lambda^j}\tau}{1+\mu_{\lambda^j}\tau}\int_\Omega \frac{u_{\lambda^j}\phi_{\lambda^j}\overline{\nabla\phi_{\lambda^j}}}{\lambda^j-\lambda_*}dx.$$

Since $\lim_{j\to\infty}\|\phi_{\lambda^j}\|_{Y_{\mathbb{C}}}=0$, then $\lim_{j\to\infty}\|\phi_{\lambda^j}\|_{L^1}=0$. Also we know that $\|\nabla\psi_{\lambda^j}\|_{Y_{\mathbb{C}}}$ is bounded, so $\|\nabla\phi_{\lambda^j}\|_{Y_{\mathbb{C}}}$ is also bounded. Thus, together with $\lim_{j\to\infty}|c_{\lambda^j}|(\lambda^j-\lambda_*)=\dfrac{1}{\alpha_{\lambda^j}}>0$, we have

$$\lim_{j\to\infty} J_1 = \lambda_*\alpha_{\lambda_*}\int_\Omega \phi^3 dx, \quad \lim_{j\to\infty} J_5 = d_2\alpha_{\lambda_*}\int_\Omega \phi|\nabla\phi|^2dx,$$

$$\lim_{j\to\infty} J_9 = \frac{d_2\mu_{\lambda^j}\tau}{1+\mu_{\lambda^j}\tau}\alpha_{\lambda_*}\int_\Omega \phi|\nabla\phi|^2dx, \quad \lim_{j\to\infty} J_i = 0,\ \ i=2,3,4,6,7,8,10,11,12.$$

Therefore, we have

$$\begin{aligned}\lim_{j\to\infty}\frac{D_j}{\lambda^j-\lambda_*} &= \alpha_{\lambda_*}\left(\lambda_*\int_\Omega\phi^3dx + d_2\int_\Omega\phi|\nabla\phi|^2dx - \frac{d_2\mu_{\lambda^j}\tau}{1+\mu_{\lambda^j}\tau}\int_\Omega\phi|\nabla\phi|^2dx\right)\\ &= \alpha_{\lambda_*}(2d_1+d_2)\int_\Omega\phi|\nabla\phi|^2dx - \frac{\alpha_{\lambda_*}d_2\mu_{\lambda^j}\tau}{1+\mu_{\lambda^j}\tau}\int_\Omega\phi|\nabla\phi|^2dx\end{aligned}$$

where $2d_1\int_\Omega\phi|\nabla\phi|^2dx = \lambda_*\int_\Omega\phi^3dx$ is applied. Thus,

$$\begin{aligned}\lim_{j\to\infty}\frac{\mathcal{R}e(D_j)}{\lambda^j-\lambda_*} &= \alpha_{\lambda_*}\int_\Omega\phi|\nabla\phi|^2dx\left(2d_1+d_2-d_2\mathcal{R}e\left(\frac{\mu_{\lambda^j}\tau}{1+\mu_{\lambda^j}\tau}\right)\right)\\ &> 2d_1\alpha_{\lambda_*}\int_\Omega\phi|\nabla\phi|^2dx > 0.\end{aligned}$$

Again by (3.43), it can be obtained that

$$\mathcal{R}e(\mu_{\lambda^j}) = \mathcal{R}e\left(\langle\tilde{A}(\lambda^j)\psi_{\lambda^j},\psi_{\lambda^j}\rangle\right) - \mathcal{R}e(D_j) < 0. \tag{3.47}$$

That is a contradiction with $\mathcal{R}e(\mu_{\lambda^j})\geq 0$ for $j\geq 1$. Then we know that all the eigenvalues of $A_{0\tau}(\lambda)$ have negative real parts and thus u_λ is locally asymptotically stable for Eq. (1.2) when $n=0$. □

For the strong kernel case $n = 1$, from Theorem 3.6, Proposition 3.7 and Lemma 3.8, we have the following results about the Hopf bifurcations near the positive steady state u_λ of (1.2) and the stability of u_λ.

Theorem 3.10. *Suppose that $d_1 > 0$, $d_2 > 16d_1$ and $\lambda \in (\lambda_*, \hat{\lambda}]$, let $\tau_{1\lambda}^{(i)}$ be defined by (3.36) with $i = 1, 2$ and $A_{n\lambda}(\tau)$ in (3.3) with $n = 1$, then we have the following results:*

(i) there exists exactly two critical points $\tau_{1\lambda}^{(1)}$ and $\tau_{1\lambda}^{(2)}$ such that all the eigenvalues of $A_{1\tau}(\lambda)$ have negative real parts when $\tau \in \left(0, \tau_{1\lambda}^{(1)}\right)$. $A_{1\tau}(\lambda)$ has a pair of purely imaginary eigenvalues $\pm i\omega_{1\lambda}^{(i)}$ $\left(\omega_{1\lambda}^{(i)} > 0\right)$ when $\tau = \tau_{1\lambda}^{(i)}$ for $i = 1, 2$, $A_{1\tau}(\lambda)$ has two eigenvalues with positive real parts when $\tau \in \left(\tau_{1\lambda}^{(1)}, \tau_{1\lambda}^{(2)}\right)$, and when $\tau \in \left(\tau_{1\lambda}^{(2)}, +\infty\right)$, all the eigenvalues of $A_{1\tau}(\lambda)$ have negative real parts;

(ii) Hopf bifurcations occur at $\tau = \tau_{1\lambda}^{(1)}$ and $\tau = \tau_{1\lambda}^{(2)}$ for Eq. (1.2) so that there is a continuous family of periodic solutions when τ is in a neighborhood of $\tau_{1\lambda}^{(1)}$ and $\tau_{1\lambda}^{(2)}$ in the form of

$$\{(\tau_1(s), u_1(x, t, s), T_1(s)) : s \in (0, \delta_1)\}$$

where $u_1(x, t, s)$ is a $T_1(s)$-periodic solution of (1.2) with $\tau = \tau_1(s)$, and $\tau_1(0) = \tau_{1\lambda}^{(1)}$ or $\tau_1(0) = \tau_{1\lambda}^{(2)}$, $\lim_{s \to 0^+} u_1(x, t, s) = u_\lambda(x)$ and $\lim_{s \to 0^+} T_1(s) = 2\pi/\omega_{1\lambda}$;

(iii) the positive steady state u_λ of Eq. (1.2) is locally asymptotically stable for $\tau \in \left(0, \tau_{1\lambda}^{(1)}\right) \cup \left(\tau_{1\lambda}^{(2)}, +\infty\right)$, and it is unstable for $\tau \in \left(\tau_{1\lambda}^{(1)}, \tau_{1\lambda}^{(2)}\right)$.

In Theorems 3.9 and 3.10, the stability/instability of positive steady state u_λ of Eq. (1.2) is given for the weak kernel ($n = 0$) and strong kernel ($n = 1$) cases, respectively. In particular, it is shown that a delay-induced instability occurs for the positive steady state in a window of delay values $\tau \in \left(\tau_{1\lambda}^{(1)}, \tau_{1\lambda}^{(2)}\right)$, and the positive steady state regains the stability for $\tau > \tau_{1\lambda}^{(2)}$. This is an example of stability switch.

For a general $n \geq 2$, we can also analyze the occurrence of Hopf bifurcation and the stability of the positive steady state in a similar way. For large n and large d_2, from Lemma 3.4, we may have more than two bifurcation points and multiple stability switches, as long as the bifurcation value $\tau_{n\lambda}$ satisfies

$$\lim_{\lambda \to \lambda_*} (\lambda - \lambda_*)\tau_{n\lambda} = \frac{\theta_{n\lambda_*}}{h_{n\lambda_*}} = \frac{\sin \eta_{n\lambda_*}}{\sin(n+1)\eta_{n\lambda_*} \cos^{n+1} \eta_{n\lambda_*}} > 0. \tag{3.48}$$

4. Numerical simulations

Here we show some numerical simulations of Eq. (1.2) to verify our theoretical results in Section 3. Numerical simulations of Eq. (1.2) is challenging as the distributed delay is an integral over an infinite interval. By a similar method as in [39] and [25], Eq. (1.2) with $n = 0$ (weak kernel) can be converted into an equivalent new system without time delays:

$$\begin{cases} u_t(x,t) = d_1 \Delta u(x,t) + d_2 div(u(x,t)\nabla v(x,t)) + \lambda u(x,t)(1-u(x,t)), & x \in \Omega,\ t > 0, \\ v_t(x,t) = \tau^{-1}(u(x,t) - v(x,t)), & x \in \Omega,\ t > 0, \\ u(x,t) = v(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x,0),\ \ v(x,0) = \tau^{-1} \int\limits_{-\infty}^{0} e^{\frac{s}{\tau}} u_0(x,s) ds, & x \in \Omega. \end{cases} \tag{4.1}$$

And when $n = 1$ (strong kernel), Eq. (1.2) is equivalent to

$$\begin{cases} u_t(x,t) = d_1 \Delta u(x,t) + d_2 div(u(x,t)\nabla v(x,t)) + \lambda u(x,t)(1-u(x,t)), & x \in \Omega,\ t > 0, \\ v_t(x,t) = \tau^{-1} w(x,t) - v(x,t)), & x \in \Omega,\ t > 0, \\ w_t(x,t) = \tau^{-1}(u(x,t) - w(x,t)), & x \in \Omega,\ t > 0, \\ u(x,t) = v(x,t) = w(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x,0),\ \ v(x,0) = -\tau^{-2} \int\limits_{-\infty}^{0} s e^{\frac{s}{\tau}} u_0(x,s) ds, & x \in \Omega, \\ w(x,0) = \tau^{-1} \int\limits_{-\infty}^{0} e^{\frac{s}{\tau}} u_0(x,s) ds, & x \in \Omega. \end{cases} \tag{4.2}$$

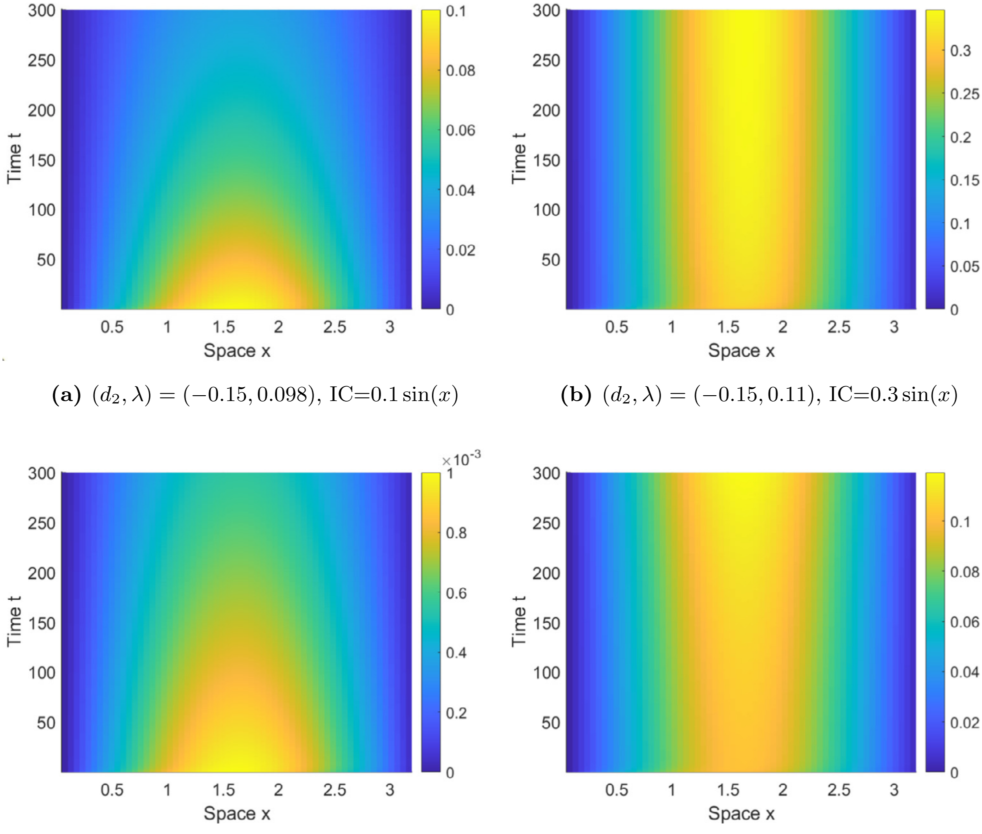
We use the equivalent systems (4.1) and (4.2) for the numerical simulations of (1.2), and a similar system with $n + 2$ equations can be used for the simulations when $n \geq 2$. In all simulations, we take $d_1 = 0.1$ and $\Omega = (0, \pi)$, then we have $\lambda_* = 0.1$.

In Fig. 1, we show the convergence of solutions of Eq. (4.1) for different d_2 and λ values to verify Theorem 2.2. When $d_2 > -2d_1 = -0.2$, a forward bifurcation of steady state occurs and a positive steady state exists for $\lambda > \lambda_*$: the solution of Eq. (4.1) converges to zero when $\lambda = 0.098 < \lambda_*$ (see (a)) and to a spatially non-homogeneous steady state when $\lambda = 0.11 > \lambda_*$ (see (b)). When $d_2 < -2d_1 = -0.2$, a backward bifurcation occurs: when $\lambda = 0.098 < \lambda_*$, the solution of system (4.1) converges to a spatially non-homogeneous steady state for a large initial value (see (d)), while it comes to zero for a small initial value, which shows a bistable dynamics.

Example 4.1. When $n = 1$ (strong kernel case), from Proposition 3.7, we know that the condition for Hopf bifurcation to occur is $d_2 > 16d_1$. Taking $d_1 = 0.1$ and the spatial domain as $\Omega = (0, \pi)$, we obtain $\lambda_* = d_1\lambda_1 = 0.1$ and $d_2^* = 16d_1 = 1.6$. If $d_2 = 1.4 < 1.6$, Hopf bifurcation will not occur; if $d_2 = 2 > 1.6$, according to Proposition 3.7, the Hopf bifurcation values are

$$\tau_{1\lambda}^{(1)} \approx 140,\ \ \tau_{1\lambda}^{(2)} \approx 960.$$

In Fig. 2, we use system (4.2) to simulate the dynamical behavior of Eq. (1.2) for the strong kernel case. When we set $\lambda = 0.13 > \lambda_* = 0.1$, we take $d_2 = 1.4 < 16d_1$ and $d_2 = 2 > 16d_1$ for the numerical simulations, respectively. When $d_2 = 1.4$, we see that the solution of Eq. (4.2) converges to the positive steady state for any $\tau > 0$. For $d_2 = 2$ case, a stability switch phenomenon occurs by taking $\tau = 100$, $\tau = 400$, and $\tau = 1500$: the solution converges to a stable steady state when $\tau = 100 < \tau_{1\lambda}^{(1)} = 140$ (see Fig. 2 (d)), then the steady state loses its stability and a spatially non-homogeneous oscillatory pattern arises when we increase τ value such that



(a) $(d_2, \lambda) = (-0.15, 0.098)$, IC=$0.1\sin(x)$ **(b)** $(d_2, \lambda) = (-0.15, 0.11)$, IC=$0.3\sin(x)$

(c) $(d_2, \lambda) = (-0.25, 0.098)$, IC=$0.001\sin(x)$ **(d)** $(d_2, \lambda) = (-0.25, 0.098)$, IC=$0.1\sin(x)$

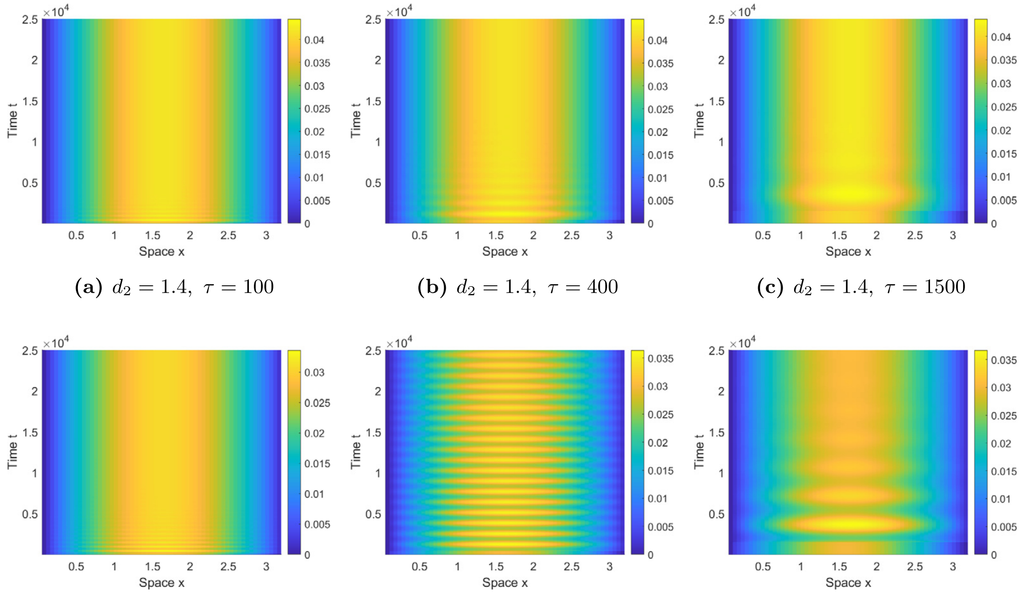
Fig. 1. Numerical simulations of system (1.2) for the weak kernel case when the parameters are $d_1 = 0.1$, $\tau = 10$ and $\Omega = (0, \pi)$, where "IC" stands for "Initial condition". In each figure, the color indicates the value of $u(x,t)$ according to the colorbar. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

$\tau_{1\lambda}^{(1)} < \tau = 400 < \tau_{1\lambda}^{(2)}$ (see Fig. 2 (e)). However, if we continue to increase τ to $\tau = 1500 > \tau_{1\lambda}^{(2)}$, we see from Fig. 2 (f) that the solution of Eq. (4.2) converges again to the positive steady state.

When we increase n, Hopf bifurcation can still occur, however the dynamics of the system (1.2) may differ from the strong kernel case, for example, there is only one Hopf bifurcation value when $n = 4$.

Example 4.2. When $n = 4$, we have $m_4 = \cos^6\left(\frac{\pi}{6}\right) = 0.4219$ according to Lemma 3.4, and Hopf bifurcation can occur in system (1.2) when $d_2 \geq 2d_1/m_4 = 4.741d_1$. Taking $d_1 = 0.1$ and the spatial domain as $\Omega = (0, \pi)$, we obtain $\lambda_* = d_1\lambda_1 = 0.1$ and $d_2^* = 2d_1/m_4 = 0.4741$, Eq. (3.22) becomes $\cos(5\eta_{5\lambda})\cos^5\eta_{5\lambda} = -0.2$ and has two solutions $\eta_{4\lambda}^{(1)} = 0.3720$, $\eta_{4\lambda}^{(2)} = 0.7305$. By solving (3.35), we obtain $h_{4\lambda}^{(1)} = 0.5605$ and $h_{4\lambda}^{(2)} = -0.0934 < 0$ which is not valid. Finally, the unique Hopf bifurcation values can be computed as

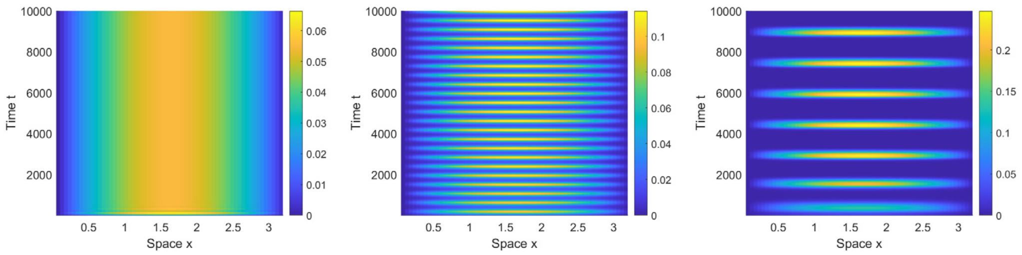
$$\tau_{4\lambda}^{(1)} = \frac{\tan\left(\eta_{4\lambda}^{(1)}\right)}{(\lambda - \lambda_*)h_{4\lambda}^{(1)}} \approx 23.$$



(a) $d_2 = 1.4,\ \tau = 100$ **(b)** $d_2 = 1.4,\ \tau = 400$ **(c)** $d_2 = 1.4,\ \tau = 1500$

(d) $d_2 = 2,\ \tau = 100$ **(e)** $d_2 = 2,\ \tau = 400$ **(f)** $d_2 = 2,\ \tau = 1500$

Fig. 2. Numerical simulations of system (1.2) for the strong kernel case when the parameters are $d_1 = 0.1,\ \lambda = 0.13$ and $\Omega = (0, \pi)$. (Upper row): $d_2 = 1.4 < 16d_1$, Hopf bifurcation will not occur and the steady state remains stable for any $\tau > 0$; (Lower row): $d_2 = 2 > 16d_1$, the steady state is stable for small and large τ value, and it is unstable for intermediate τ values (it converges to a spatially non-homogeneous time-periodic solution). In each figure, the color indicates the value of $u(x,t)$ according to colorbar, and the initial value is taken as $u_0(x) = 0.04\sin(x)$ for (a)-(c) and $u_0(x) = 0.03\sin(x)$ for (d)-(e).



(a) $\tau = 10$ **(b)** $\tau = 30$ **(c)** $\tau = 100$

Fig. 3. Numerical simulations of system (1.2) for $n = 4$ case when the parameters are $d_1 = 0.1,\ d_2 = 1,\ \lambda = 0.13$ and $\Omega = (0, \pi)$. In each figure, the color indicates the value of $u(x,t)$ according to colorbar, and the initial value is taken as $u_0(x) = 0.03\sin(x)$.

Then, we take different τ values to perform simulations for $n = 4$ case of system (1.2): $\tau = 10$, $\tau = 30$, and $\tau = 100$ to perform the simulations: the solutions converge to a stable positive steady state when $\tau = 10 < \tau_{4\lambda}$ (see Fig. 3 (a)), then the steady state loses its stability and a spatially non-homogeneous oscillatory pattern arises when we increase τ value such that $\tau = 30 > \tau_{4\lambda}$ (see Fig. 3 (b)). If we continue to increase τ to $\tau = 100$, we see from Fig. 3 (c) that the time-periodic pattern still exists but with a larger period.

5. Discussion

In the past decade, spatial memory and cognition drew much attention in the mechanistic modeling of animal movements [8,14]. In the present work, we formulate a reaction-diffusion equation with a diffusive temporally distributed delay term. This work is inspired by the recent work [25] where a spatiotemporal distributed delay is considered to model the effect of social animals' memory, and the memory of animals is assumed to rely on the whole historic information as well as the spatial territory conditions. While, it is also known that many animals, especially most carnivores, are solitary and asocial because the costs of intraspecific competition outweigh the benefits accrued with group living [7]. Therefore it is of interest to study the role of spatial memory of solitary type animals in their spatial movement. As such animals have very weak connections with others of their species, the influence of the spatial variation from other individuals is negligible. The diffusive delay term caused by animals' memory can be modeled by a weighted integral of the population density over all the past time, which is named as temporal distributed delay according to the previous study [13]. It is also known that many solitary animals, such as tiger, are usually very territorial, which means that they will not allow others to step in their territory and they will be driven away when they invade others' land as well. In this situation, we may reasonably assume that the boundary environment of the animals' territory is hostile.

For the existence of spatially non-homogeneous steady state in the temporally distributed memory model (1.2), it is shown that the spatially non-homogeneous steady state is generated via a forward steady-state bifurcation when $d_2 > -2d_1$ and a backward steady-state bifurcation when $d_2 < -2d_1$. Also, the spatially non-homogeneous steady state is unique when $d_2 > 0$. Under the condition that $d_2 > 0$, we investigate the stability of the unique spatially non-homogeneous steady state. When the temporal kernel is take as weak kernel, the spatially non-homogeneous steady state remains stable for any τ which is the average delay of the temporal time delay in model (1.2). The result is similar to the weak kernel case of a spatiotemporal delay model [25]. In the strong kernel case, we show that when the strength of the memory-based movement is strong enough (d_2 large), the positive non-homogeneous steady state solution could lose its stability if the average delay τ is in an intermediate range, and two Hopf bifurcations occur when τ increases: the first one to destabilize the steady state, and the second one to regain the stability. This shows that a temporally distributed memory-based delay with a large average delay value is a stabilizing force, which is different from the cases of discrete delay [21] or spatiotemporal delay [25]. From the biological perspective, the oscillatory patterns generated via Hopf bifurcations reflect the periodically temporal distribution of the species, which is reasonable as the animals will periodically move in their habitat to gain better resources according to their memory and experience. Our study shows that this situation usually happens when the average memory is not too large or small.

In [33], Wang and Salmaniw summarize the study about the spatial memory models and leave some open problems. To be more concrete, we refer the readers to the works in [1,11,21–23,26–29,32,34,35,37] for the discrete spatial memory and [12,19,25,30,38] for distributed spatial memory (with Neumann boundary condition). Our work in the present paper contributes to the study of the effects of memory on animals' spatial movements by proposing a temporally distributed delay term to model the diffusive memory under a hostile boundary condition. In the future, it is natural to extend the modeling idea to interacting species in an ecosystem, for instance, spatial memory of resource distribution by consumers and spatial memory of predator distribution by prey. The aggregated research efforts in this direction will contribute to the ecological theory of cognitive animal movements.

Data availability

No data was used for the research described in the article.

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