A LOCALIZATION-DELOCALIZATION TRANSITION FOR NONHOMOGENEOUS RANDOM MATRICES

LAURA SHOU AND RAMON VAN HANDEL

Abstract. We consider $N \times N$ self-adjoint Gaussian random matrices defined by an arbitrary deterministic sparsity pattern with d nonzero entries per row. We show that such random matrices exhibit a canonical localization-delocalization transition near the edge of the spectrum: when $d \gg \log N$ the random matrix possesses a delocalized *approximate* top eigenvector, while when $d \ll \log N$ any approximate top eigenvector is localized. The key feature of this phenomenon is that it is universal with respect to the sparsity pattern, in contrast to the delocalization properties of *exact* eigenvectors which are sensitive to the specific sparsity pattern of the random matrix.

1. Introduction

Understanding the eigenvectors of large random matrices, particularly whether they are delocalized or localized, is of interest in many areas including mathematical physics, computer science, and combinatorics. A delocalized vector is one with roughly equal mass spread throughout its coordinates, while a localized vector has much of its mass concentrated on relatively few coordinates. The guiding example of delocalization arises in rotationally invariant ensembles such as the classical Gaussian orthogonal ensemble (GOE): their eigenvectors are uniformly distributed on the unit sphere, and are therefore always delocalized. Properties of uniform random vectors on the sphere can therefore be used as a benchmark for measuring delocalization. Much work in this direction has been done for general *Wigner-type matrices* [10, 13], for a variety of indicators of delocalization.

On the other hand, the most localized vectors are simply the coordinate directions, with all mass concentrated on a single coordinate. These arise as eigenvectors of diagonal matrices, such as a diagonal matrix with i.i.d. diagonal Gaussian entries. To interpolate between the two extremes of a diagonal matrix and a Gaussian Wigner matrix, one can consider models of varying degrees of sparseness. One such model of interest in mathematical physics is random band matrices, which are zero outside of a band around the diagonal, and whose eigenvectors are conjectured to undergo a phase transition from localization to delocalization depending on the band width. See, e.g., [6] for a survey of this topic.

In this paper we will consider Gaussian random matrices with an *arbitrary* sparsity pattern, which is only assumed to be d-regular in the sense that there are d nonzero entries in each row and column. This model includes the above mentioned Gaussian models (GOE, diagonal, and band matrices), as well as many other matrices that may be highly nonhomogeneous. As will be illustrated below using simple examples, the delocalization of eigenvectors is sensitive to the choice of sparsity

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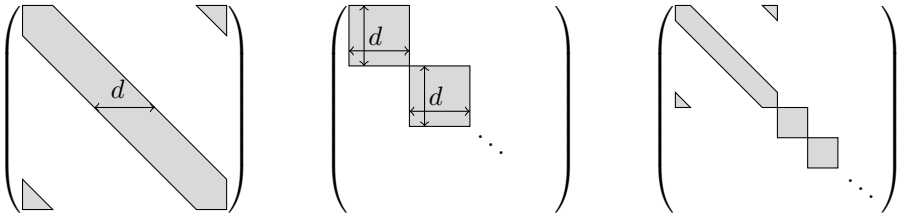

FIGURE 1. Band matrix, block matrix, and their direct sum

pattern; an understanding of such questions for arbitrary sparsity patterns is far beyond current technology. In contrast, we will show that a much simpler phenomenon arises in this general setting near the edge of the spectrum: the *approximate* top eigenvectors (by which we will mean unit vectors v with $\|Xv\|_2$ close to $\|X\|$) exhibit a canonical localization-delocalization transition at $d \sim \log N$. This shows in particular that while the behavior of the *exact* eigenvectors may be sensitive to the structure of the model, a weaker notion of (de)localization can nonetheless arise universally with respect to the sparsity pattern.

1.1. **Sparse Gaussian matrices.** The following general model will be considered throughout this paper. Fix an arbitrary d-regular undirected graph $G = ([N], E)$ with N vertices, which may contain self-loops but no multiple edges between the same pair of vertices. We now define an $N \times N$ self-adjoint random matrix X_N by setting $(X_N)_{xy} = 1_{x \sim y} g_{xy}$, where g_{xy} are independent standard (real) Gaussian variables modulo symmetry $g_{yx} = g_{xy}$, and $x \sim y$ denotes that x and y are connected by an edge in G. From now on, we fix a sequence of such matrices X_N indexed by N; it is implicit in the notation that d and G depend on N.

The graph G is used here merely as a convenient way to encode an arbitrary sparsity pattern of the entries of X_N. For example, the complete graph with N vertices yields a Wigner matrix with i.i.d. standard Gaussian entries (modulo symmetry), while the graph consisting of N isolated points with self-loops corresponds to the diagonal matrix with i.i.d. standard Gaussians on the diagonal. Two intermediate examples are periodic band matrices with band width d, which are generated by the graph on $\mathbb{Z}/N\mathbb{Z}$ with edges between nodes within distance $\frac{d-1}{2}$ of each other, and block Wigner matrices with block size d, which are generated by the disjoint union of $\frac{N}{d}$ complete graphs with d vertices (cf. Figure 1). Let us emphasize that while these simple examples possess many special symmetries that could potentially facilitate their analysis, a general d-regular graph can be highly nonhomogeneous and need not possess any tractable structure.

Before we turn to the delocalization phenomenon that will be studied in this paper, let us emphasize that delocalization of eigenvectors in the the classical sense cannot be universal with respect to the sparsity pattern.

Example 1.1. Consider the three examples illustrated in Figure 1: an $N \times N$ band matrix or block matrix with d nonzero entries per row, or their direct sum.

The situation for band matrices is subject to deep conjectures arising from mathematical physics. In particular, it is believed (cf. [15, 6] and the references therein) that this model should exhibit a localization-delocalization phase transition at $d \sim N^{5/6}$ for eigenvectors near the edge of the spectrum: these eigenvectors are

expected to be delocalized when $d \gg N^{5/6}$, and to be localized when $d \ll N^{5/6}$.[1] These conjectures remain largely open to date.

On the other hand, it is clear that *every* eigenvector of a random block matrix must be supported in one of the blocks. Thus the block matrix model has localized eigenvectors whenever $d = o(N)$. In fact, [3, Theorem 2.9] shows that the eigenvectors in this model are nearly maximally localized (the latter result states that for any model of the kind considered in this paper, the mass of eigenvectors near the edge must be spread over at least $\sim \frac{d}{\log N}$ coordinates).

Finally, the set eigenvectors of the direct sum of a block matrix and a band matrix of width d is the union of the eigenvectors of a block and a band matrix. In this example, it may be the case that half the eigenvectors are localized and half are delocalized. While contrived, such very simple examples already show that when we admit *arbitrary* sparsity patterns, it is not even clear whether classical eigenvector delocalization questions may be meaningfully formulated.

As general d-regular graphs can exhibit arbitrarily complicated nonhomogeneities, the above examples suggest that the study of classical delocalization questions at this level of generlity is likely to be a hopeless task. Instead, the aim of this paper is to exhibit a new type of (de)localization phenomenon that is universal with respect to sparsity pattern. This phenomenon is necessary of a fundamentally different nature than classical eigenvector delocalization questions. In order to explain the nature of our main results, let us first recall that the macroscopic behavior of the *eigenvalues* is in fact universal with respect to sparsity pattern.

Theorem 1.2 (Eigenvalues)**.** *The following hold whenever $d, N \to \infty$.*

a. The empirical eigenvalue distribution μ_N of $\frac{X_N}{\sqrt{d}}$ converges weakly in probability to the standard semicircle distribution:

$$\mu_N \Longrightarrow \rho_{\mathrm{sc}}(x)\,dx := \frac{1}{2\pi}\sqrt{4-x^2}\,1_{|x|\le 2}\,dx \quad \textit{in probability.}$$

b. For every $\varepsilon > 0$, we have

$$\max\Big\{(2-o(1))\sqrt{d}, C\sqrt{\log N}\Big\} \le \mathbf{E}\|X_N\| \le (2+\varepsilon)\sqrt{d} + K_\varepsilon\sqrt{\log N},$$

where C is a universal constant and K_ε depends on ε only.

Proof. Part a. and the upper bound of part b. are given in [3, Theorem 2.3] and [2, Theorem 1.1], respectively. The $(2-o(1))\sqrt{d}$ lower bound of part b. follows directly from part a., while the $C\sqrt{\log d}$ lower bound is given in [2, Corollary 3.15]. □

Theorem 1.2 shows that the extreme eigenvalues of X_N exhibit the following phase transition. When $d \gg \log N$, we have

$$\mathbf{E}\|X_N\| = (2+o(1))\sqrt{d},$$

that is, the extreme eigenvalues of X_N converge to the edge of the bulk (semicircle) eigenvalue distribution. This behavior is analogous to that of Wigner matrices. In contrast, when $d \ll \log N$, we have

$$c\sqrt{\log N} \le \mathbf{E}\|X_N\| \le C\sqrt{\log N},$$

[1] We use the notation $A \ll B$ to mean $A/B \to 0$, and $A \gg B$ to mean $A/B \to \infty$, as $N \to \infty$.

that is, the extreme eigenvalues are of the same order as the size of the largest entry of X_N (as the maximum of n independent standard Gaussian variables is of order $\sqrt{\log n}$) and are separated from the bulk.

The above interpretation of the eigenvalue phase transition hints at an associated (de)localization phenomenon. When $d \gg \log N$, the behavior of the extreme eigenvalues is "Wigner-like", and one may expect to inherit some delocalization properties of Wigner matrices. In contrast, when $d \ll \log N$ the magnitude of the extreme eigenvalues is explained by the presence of exceptionally large matrix entries, which suggests that the extreme eigenvectors should be localized on the coordinates associated to these entries. Our main result will show that this phenomenon is captured not by (de)localization of the *exact* top eigenvectors, which is ruled out by Example 1.1, but by that of *approximate* top eigenvectors.

Remark 1.3. Physics heuristics suggest that the transition between localization and delocalization of the exact eigenvectors coincides with a transition between Poissonian and random matrix statistics of the associated eigenvalues [15, 6]. In contrast, the localization-delocalization transition of this paper coincides with the transition where outlier eigenvalues detach from the bulk of the spectrum. As these outliers appear at the macroscopic scale, this phenomenon is much simpler and more robust than the fluctuations of the eigenvalues at the local scale.

1.2. **Main result.** Localization and delocalization of vectors can be described by various non-equivalent notions, such as the ℓ^∞ norm or other ℓ^p norms [9, 8], joint distribution of coordinates [16], and no-gaps delocalization [14]; the survey [13] includes results on several different notions of delocalization. Here we will use the notion of (L,κ)-delocalization used in [9, §7]. A vector delocalized in this sense is one that has no "peaks" of mass $> \kappa^2$ in any subset of $\le L$ coordinates.

Definition 1.4 (Delocalization)**.** A real vector $v \in \mathbb{S}^{N-1}$ is (L,κ)-*delocalized* if for every set $A \subseteq [N]$ of size $|A| \le L$, we have $\sum_{j\in A} v_j^2 \le \kappa^2$. The set of (L,κ)-delocalized vectors will be denoted by

$$D_{L,\kappa} := \left\{ v \in \mathbb{S}^{N-1} : \sum_{j\in A} |v_j|^2 \le \kappa^2 \text{ for all } A \subseteq [N], |A| \le L \right\}.$$

The (L,κ)-delocalization condition becomes stricter for smaller κ and larger L. We are primarily interested in the situation where L is proportional to N. We will colloquially refer to a vector, or more precisely a sequence of vectors of increasing dimension $N \to \infty$, as *delocalized* if it is $(\nu N,\kappa)$-delocalized for some $0 < \nu, \kappa < 1$ independent of N, and otherwise we will refer to it as *localized*. In other words, a delocalized vector is one that is not concentrated in a vanishing fraction of the coordinates, while a localized vector is one that has a constant fraction of the mass concentrated in just a vanishing fraction of the coordinates.

Next, recall that any *exact* top eigenvector v (i.e., an eigenvector whose eigenvalue has the largest magnitude) of a self-adjoint matrix X satisfies

$$\|Xv\|_2 = \sup_{w\in\mathbb{S}^{N-1}} \|Xw\|_2 = \|X\|.$$

Thus we define an *approximate* top eigenvector as follows.

Definition 1.5 (Approximate top eigenvector)**.** $v \in \mathbb{S}^{N-1}$ is an $(1-\varepsilon)$-*approximate top eigenvector* of a self-adjoint matrix X if $\|Xv\|_2 \ge (1-\varepsilon)\|X\|$.

We can now formulate our main result.

Theorem 1.6 (Localization-delocalization transition)**.** *Let $N \to \infty$, and fix parameters ε, κ, ν that are independent of N. Then the following hold:*

(i) Localized regime $d \ll \log N$: *for any $0 < \varepsilon < 1$ and $0 < \kappa < 1 - \varepsilon$*

$$\mathbf{P}[\textit{every } (1-\varepsilon)\textit{-approximate top eigenvector of } X_N \textit{ is in } \mathbb{S}^{N-1} \backslash D_{o(1)N,\kappa}] = 1 - o(1).$$

(ii) Delocalized regime $d \gg \log N$: *for any $0 < \kappa < 1$ and $0 < \nu < \frac{c\kappa^2}{\log \frac{e}{\kappa}}$*

$$\mathbf{P}[\textit{there exists a } (1-o(1))\textit{-approximate top eigenvector of } X_N \textit{ in } D_{\nu N,\kappa}] = 1 - o(1).$$

Qualitatively, Theorem 1.6 yields a strong dichotomy between the localized and delocalized regimes. In the delocalized regime, there exists a $(1-o(1))$-approximate, that is, a "nearly exact," top eigenvector that is delocalized. On the other hand, in the localized regime, every approximate top eigenvector that is even within a constant fraction of the edge of the spectrum must be localized.

A more careful interpretation of the quantitative aspect of Theorem 1.6 further strenghtens this dichotomy. Recall that the benchmark example of a delocalized vector is a uniform random vector in $\mathbb{S}^{N-1}$; such random vectors arise as eigenvectors of GOE matrices, and therefore possess the strongest form of delocalization one could hope for. In Appendix A, we will show that a uniform random vector in $\mathbb{S}^{N-1}$ is $(\nu N, \kappa)$-delocalized with high probability if and only if $\nu \lesssim \frac{\kappa^2}{\log \frac{e}{\kappa}}$. Thus the delocalized regime of Theorem 1.6 yields an $(1 - o(1))$-approximate top eigenvector that exhibits nearly the same degree of delocalization as a uniform random vector in the sphere, up to the value of the universal constant c.

It should be emphasized that Theorem 1.6 sheds little light on classical questions in random matrix theory surrounding *exact* eigenvectors: as is illustrated by Example 1.1, such questions are not even meaningful at the level of generality of this paper. While part (i) of Theorem 1.6 yields localization of exact as well as approximate eigenvectors, it captures only a small subset of the regime in which some classical random matrix models (such as band matrices) exhibit eigenvector localization. On the other hand, part (ii) does not provide any information at all on individual eigenvectors, while Example 1.1 illustrates that delocalization of approximate eigenvectors can arise even when all the exact eigenvectors are localized. The aim of Theorem 1.6 is not to address such classical random matrix questions, but rather to exhibit a fundamentally different delocalization phenomenon that captures a nontrivial feature of a much larger class of random matrix models.

1.3. **Outline.** This paper is organized as follows. In section 2, we prove the localized regime of Theorem 1.6. The basis of the proof is to show, using subgaussian estimates, that the existence of a delocalized approximate top eigenvector implies $\|X_N\| = O(\sqrt{d})$; consequently, no delocalized approximate top eigenvector can exist in the regime where $\|X_N\| \sim \sqrt{\log N} \gg \sqrt{d}$.

The remainder of the paper is devoted to the delocalized regime. The basic idea of the proof is to construct an approximate top eigenvector by taking a random superposition of many exact eigenvectors near the edge of the spectrum. In order to show such a vector is delocalized, we must establish that even though each exact eigenvector may be localized, the subspace spanned by sufficiently many of these eigenvectors is delocalized in an appropriate sense (in particular, this rules out that the exact eigenvectors are all simultaneously localized in a small subset of the

coordinates). In section 3, we first explain what property of a linear subspace is needed to ensure that it contains delocalized vectors. We then show in section 4 that this property is indeed satisfied for an appropriate eigenspace of X_N; this is accomplished by approximating the projection matrix of the eigenspace in terms of the resolvent, whose behavior is goverened by a mesoscopic form of the semicircle law. Combining all the above ingredients concludes the proof of Theorem 1.6.

Finally, Appendix A shows that the delocalization provided by Theorem 1.6(ii) agrees quantitatively with that of a uniform random vector in $\mathbb{S}^{N-1}$. This result is included to clarify the meaning of our main result, and is not used elsewhere.

2. Proof of localization for $d \ll \log N$

The proof of localization Theorem 1.6(i) is based on Gaussian concentration. Let us recall the general principle for future reference, cf. [4, §5.4].

Theorem 2.1 (Gaussian concentration)**.** *Let Z be a standard Gaussian vector in $\mathbb{R}^n$, and let $f : \mathbb{R}^n \to \mathbb{R}$ be L-Lipschitz with respect to the Euclidean norm. Then*

$$\mathbf{P}\left[|f(Z) - \mathbb{E}[f(Z)]| \geq t\right] \leq 2e^{-t^2/2L^2} \quad \text{for all } t \geq 0.$$

We need the following corollary.

Corollary 2.2. *For every $w \in \mathbb{R}^N$ such that $\|w\|_\infty \leq 1$, we have*

$$\mathbf{P}[\|X_N w\|_2 \geq \sqrt{dN} + t] \leq 2e^{-t^2/4d}.$$

Proof. By Cauchy-Schwarz, $|\langle z, w\rangle| \leq \sqrt{d}\|z\|_2$ for any z with d nonzero entries. Thus any self-adjoint matrices X, Y with d nonzero entries in each row satisfy

$$|\|Xw\|_2 - \|Yw\|_2| \leq \|(X-Y)w\|_2 \leq \sqrt{2d}\left(\sum_{i\geq j}(X_{ij} - Y_{ij})^2\right)^{1/2}.$$

We can therefore view $\|X_N w\|_2$ as a $\sqrt{2d}$-Lipschitz function of the i.i.d. standard Gaussian variables $(g_{xy})_{x\geq y, x\sim y}$ that appear in its definition. The conclusion follows from Theorem 2.1 and $\mathbf{E}\|X_N w\|_2 \leq (\mathbf{E}\|X_N w\|_2^2)^{1/2} = \sqrt{d}\|w\|_2 \leq \sqrt{dN}$. □

The key idea behind the localized regime is the following.

Lemma 2.3. *Let $0 < \varepsilon < 1$, $0 < \kappa < 1 - \varepsilon$, $\frac{1}{N} \leq \nu < 1$. Define the event*

$$\Omega_{\nu,\kappa,\varepsilon,N} := \{\textit{there exists a } (1-\varepsilon)\textit{-approximate top eigenvector of } X_N \textit{ in } D_{\nu N,\kappa}\}.$$

Then

$$\mathbf{P}[\Omega_{\nu,\kappa,\varepsilon,N} \cap \{\|X_N\| > \tfrac{8}{1-\varepsilon-\kappa}(\tfrac{d}{\nu})^{1/2}\}] \leq 2e^{-3N}.$$

Proof. Let $L = \lfloor \nu N \rfloor$, and define for any $v \in \mathbb{S}^{N-1}$ the vectors $v^+, v^- \in \mathbb{R}^N$ as

$$v_i^+ := v_i \mathbb{1}_{|v_i| > L^{-1/2}}, \qquad v_i^- := v_i \mathbb{1}_{|v_i| \leq L^{-1/2}}.$$

As $|\{i : |v_i| > L^{-1/2}\}| \leq L$ for every $v \in \mathbb{S}^{N-1}$, we can estimate

$$\|v^+\|_2^2 = \sum_{i:|v_i|>L^{-1/2}} |v_i|^2 \leq \kappa^2, \qquad \|v^-\|_\infty \leq L^{-1/2}$$

for all $v \in D_{L,\kappa} = D_{\nu N,\kappa}$. We readily obtain

$$\begin{aligned}\sup_{v\in D_{L,\kappa}} \|X_N v\|_2 &\le \sup_{v\in D_{L,\kappa}} \|X_N v^-\|_2 + \sup_{v\in D_{L,\kappa}} \|X_N v^+\|_2 \\ &\le L^{-1/2} \sup_{\|w\|_\infty\le 1} \|X_N w\|_2 + \kappa\|X_N\|.\end{aligned}$$

In particular, as $\sup_{v\in D_{L,\kappa}} \|X_N v\|_2 \ge (1-\varepsilon)\|X_N\|$ on $\Omega_{\nu,\kappa,\varepsilon,N}$, we can estimate

$$\mathbf{P}[\Omega_{\nu,\kappa,\varepsilon,N} \cap \{\|X_N\| > t\}] \le \mathbf{P}\left[\frac{L^{-1/2}}{1-\varepsilon-\kappa} \sup_{\|w\|_\infty\le 1} \|X_N w\|_2 > t\right].$$

Now note that as $w \mapsto \|X_N w\|_2$ is convex, the supremum over w is attained at one of the extreme points $\{-1,+1\}^N$ of the unit cube. Thus a union bound yields

$$\mathbf{P}[\Omega_{\nu,\kappa,\varepsilon,N} \cap \{\|X_N\| > t\}] \le \sum_{w\in\{-1,+1\}^N} \mathbf{P}[\|X_N w\|_2 > (1-\varepsilon-\kappa)L^{1/2}t],$$

and the conclusion follows from Corollary 2.2 and $\lfloor \nu N\rfloor \ge \frac{\nu N}{2}$ for $\nu \ge \frac{1}{N}$. □

We can now complete the proof of the localized regime.

Proof of Theorem 1.6(i). Assume that $d \ll \log N$, and fix $\beta > 0$ independent of N. Let $\nu = \nu(N) := \frac{d}{\beta^2 \log N} = o(1)$, and note that $\frac{1}{N} \le \nu < 1$ for N sufficiently large as $d \ge 1$. We can therefore apply Lemma 2.3 to estimate

$$\begin{aligned}&\mathbf{P}[\text{there exists a } (1-\varepsilon)\text{-approximate top eigenvector of } X_N \text{ in } D_{\nu(N)N,\kappa}] \\ &\qquad\le 2e^{-3N} + \mathbf{P}\big[\|X_N\| \le \tfrac{8}{1-\varepsilon-\kappa}\beta\sqrt{\log N}\big].\end{aligned}$$

The proof is therefore complete once we show that the probability on the right-hand side is $o(1)$ for a sufficiently small choice of β. To this end, note that as $\|X-Y\| \le \sqrt{2}\,(\sum_{i\ge j}(X_{ij}-Y_{ij})^2)^{1/2}$ for any self-adjoint matrices X, Y, the random variable $\|X_N\|$ may be viewed as a $\sqrt{2}$-Lipschitz function of the underlying i.i.d. standard Gaussian variables $(g_{xy})_{x\ge y}$. Thus Theorem 2.1 yields

$$\mathbf{P}[|\|X_N\| - \mathbf{E}\|X_N\|| \ge t] \le 2e^{-t^2/4} \quad \text{for all } t > 0. \tag{2.1}$$

As $\mathbf{E}\|X_N\| \ge C\sqrt{\log N}$ by Theorem 1.2, it follows that

$$\mathbf{P}\big[\|X_N\| \le \tfrac{1}{2}C\sqrt{\log N}\big] \le 2N^{-C^2/16} = o(1),$$

concluding the proof. □

3. Delocalization of uniform random vectors in a subspace

We now turn to the delocalized regime of Theorem 1.6(ii), whose proof will occupy the remainder of this paper. In the regime $d \gg \log N$, Theorem 1.2 ensures that the largest eigenvalue of X_N sticks to the bulk of the spectrum. Consequently, any superposition of the top $o(N)$ exact eigenvectors of X_N will yield a $(1-o(1))$-approximate top eigenvector. The basic idea behind the proof is to show that even though each exact eigenvector may itself be localized, we can always find a superposition of $o(N)$ exact eigenvectors that is delocalized.

In order for such a strategy to succeed, the exact eigenvectors must exhibit at least the following two qualitative features:

1. Each exact eigenvector must be at least somewhat delocalized: if each exact eigenvector were concentrated on $O(1)$ coordinates, then any superposition of $o(N)$ such eigenvectors would always be localized.
2. While individual exact eigenvectors may be localized, the top $o(N)$ exact eigenvectors cannot be simultaneously localized in the same subset of coordinates. In other words, the locations where different exact eigenvectors are localized must be spread out across all the coordinates.

Property 1. was previously established in [3, Theorem 2.9], which states that the exact eigenvectors near the edge of the spectrum must be spread over at least $\sim \frac{d}{\log N}$ coordinates (this result will not be used in our proofs). However, this does not suffice to ensure Property 2., which requires us to understand delocalization of the entire space spanned by the top $o(N)$ eigenvectors.

In this section, we begin our analysis by explaining what property of a linear subspace $E \subseteq \mathbb{R}^N$ is needed to ensure that E contains a delocalized unit vector. The main result of this section is the following.

Proposition 3.1 (Delocalized subspace)**.** *Let $E \subseteq \mathbb{R}^N$ be a linear subspace of dimension m, and denote by P_E the orthogonal projection onto E. Suppose that*

$$\max_{x\in[N]} (P_E)_{xx} \le \frac{Cm}{N}$$

for some constant $C > 0$. Then there exists a constant $c > 0$ that depends only on C such that $E \cap D_{\nu N,\kappa} \neq \varnothing$ for every $0 < \kappa < 1$ and $0 < \nu < \frac{c\kappa^2}{\log \frac{e}{\kappa}}$.

Proposition 3.1 shows that the relevant delocalization property of a linear subspace is control of the diagonal entries of its projection matrix. The remainder of the proof of Theorem 1.6 will then aim to show that this property holds when E is taken to be the linear span of the top $o(N)$ exact eigenvectors of X_N.

Let us first turn to the proof of Proposition 3.1. Rather than establish delocalization in the sense of Definition 1.4 directly, it will be more convenient to establish ℓ^q-bounds. A simple lemma shows that the former is implied by the latter.

Lemma 3.2. *For all $v \in \mathbb{S}^{N-1}$, $L \le N$, and $q \ge 2$, we have $v \in D_{L,L^{1/2-1/q}\|v\|_q}$.*

Proof. It suffices to note that for any $A \subseteq [N]$ with $|A| \le L$, we have

$$\sum_{j\in A} |v_j|^2 \le L^{1-2/q} \left(\sum_{j\in A} |v_j|^q \right)^{2/q} \le L^{1-2/q} \|v\|_q^2$$

by Hölder's inequality. □

To prove Proposition 3.1, we will bound the ℓ^q-norm of a uniformly chosen random vector in $E \cap \mathbb{S}^{N-1}$; the conclusion then follows from Lemma 3.2 by optimizing over q. Proposition 3.1 may therefore be viewed as a complement to the delocalization established in Appendix A for a uniformly chosen random vector in the entire sphere $\mathbb{S}^{N-1}$: here we show that a uniformly chosen unit vector in a subspace E is still delocalized when E satisfies the requisite assumption. (We do not develop high probablity results as in Appendix A, as these are not needed in the sequel.)

Proof of Proposition 3.1. To estimate the ℓ^q-norm of a uniformly chosen random vector in $E \cap \mathbb{S}^{N-1}$, let $Z \sim N(0, P_E)$ be a Gaussian random vector in $\mathbb{R}^N$ with zero mean and covariance matrix P_E, i.e., a standard Gaussian vector in E. Then

$V := \frac{Z}{\|Z\|_2}$ is uniformly distributed on $E \cap \mathbb{S}^{N-1}$ and $\|Z\|_2$ and V are independent (as the law of Z is rotationally invariant in E). Therefore

$$\mathbf{E}\|Z\|_2^q \cdot \mathbf{E}\|V\|_q^q = \mathbf{E}\|Z\|_q^q = \sum_{x=1}^N \mathbf{E}|Z_x|^q.$$

Now note that $Z_x \sim N(0, (P_E)_{xx})$ for every x, so that $\mathbf{E}|Z_x|^q \le 2q^{q/2}(P_E)_{xx}^{q/2}$ for all $q \ge 2$ [4, Theorem 2.1]. On the other hand, we have by Jensen's inequality

$$(\mathbf{E}\|Z\|_2^q)^{2/q} \ge \mathbf{E}\|Z\|_2^2 = \mathrm{Tr}(P_E) = m.$$

Combining the above estimates with the assumption on P_E yields

$$\mathbf{E}\|V\|_q^q \le 2(Cq)^{q/2}N^{1-q/2} \quad \text{for all } q \ge 2.$$

In particular, there exists $v \in E \cap \mathbb{S}^{N-1}$ so that $\|v\|_q^q \le 2(Cq)^{q/2}N^{1-q/2}$.

Now fix $0 < \nu < 1$ and let $q = 2\log\frac{e}{\nu}$. Then there exists $v \in E \cap \mathbb{S}^{N-1}$ so that

$$(\nu N)^{1/2-1/q}\|v\|_q \le 2^{1/q}(Cq)^{1/2}\nu^{1/2-1/q} \le (4eC)^{1/2}\sqrt{\nu\log\frac{e}{\nu}}.$$

Applying Lemma 3.2 shows that whenever $0 < \nu < 1$ satisfies $4eC\nu\log\frac{e}{\nu} \le \kappa^2$, there exists a vector $v \in E \cap D_{\nu N,\kappa}$. The conclusion follows readily. □

4. Proof of delocalization for $d \gg \log N$

To complete the proof of Theorem 1.6(ii), it remains to show that the condition of Proposition 3.1 holds for the space E spanned by the top $o(N)$ eigenvectors of X_N. To this end, we first approximate the projection matrix P_E in terms of the resolvent of X_N. We can then apply resolvent estimates for nonhomogeneous random matrices to deduce the requisite delocalization property.

Let us begin by formalizing the projection matrix approximation.

Lemma 4.1 (Projection matrix approximation)**.** *Let X be a self-adjoint matrix, and let $E_{[a,b]}$ be the space spanned by the eigenvectors of X with eigenvalues in $[a,b]$. Then for any $a < b$ and $\gamma \ge \delta > 0$, we can estimate*

$$P_{E_{[a,b]}} \le \left(1 + \frac{2\delta}{\gamma}\right)\frac{1}{\pi}\,\mathrm{Im}\int_{a-\gamma}^{b+\gamma}(X - \lambda - i\delta)^{-1}\,d\lambda$$

and

$$\frac{1}{\pi}\,\mathrm{Im}\int_{a+\gamma}^{b-\gamma}(X - \lambda - i\delta)^{-1}\,d\lambda \le P_{E_{[a,b]}} + \frac{\delta}{\pi\gamma}\mathbb{1}$$

in the positive semidefinite order (here $\mathrm{Im}\,X := \frac{X-X^*}{2i}$).

Proof. We begin by estimating

$$\begin{aligned}\int_{a-\gamma}^{b+\gamma}\mathrm{Im}\,\frac{1}{x-\lambda-i\delta}\,d\lambda &= \int_{a-\gamma}^{b+\gamma}\frac{\delta}{(x-\lambda)^2+\delta^2}\,d\lambda \\ &= \tan^{-1}\left(\frac{b+\gamma-x}{\delta}\right) - \tan^{-1}\left(\frac{a-\gamma-x}{\delta}\right) \ge 2\tan^{-1}\left(\frac{\gamma}{\delta}\right)1_{[a,b]}(x)\end{aligned}$$

for all $x \in \mathbb{R}$, where we used that $b+\gamma-x \geq \gamma$ and $x+\gamma-a \geq \gamma$ for $a \leq x \leq b$. Now note that as $g(s) := \tan^{-1}(\frac{1}{s})$ is convex for $s \geq 0$ and satisfies $g(0^+) = \frac{\pi}{2}$, $g'(0^+) = -1$, we can estimate $\tan^{-1}(t) \geq \frac{\pi}{2} - \frac{1}{t}$ for all $t \geq 0$. Thus

$$1_{[a,b]}(x) \leq \left(1 + \frac{2\delta}{\gamma}\right) \frac{1}{\pi} \operatorname{Im} \int_{a-\gamma}^{b+\gamma} \frac{1}{x - \lambda - i\delta}\, d\lambda$$

for all $x \in \mathbb{R}$, where we used $1 - \frac{2}{\pi}u \geq \frac{1}{1+2u}$ for $|u| \leq 1$.

In the opposite direction, an analogous computation yields

$$\begin{aligned}\int_{a+\gamma}^{b-\gamma} \operatorname{Im} \frac{1}{x - \lambda - i\delta}\, d\lambda &= \tan^{-1}\left(\frac{b-\gamma-x}{\delta}\right) - \tan^{-1}\left(\frac{a+\gamma-x}{\delta}\right) \\ &\leq \pi 1_{[a,b]}(x) + \left(\frac{\pi}{2} - \tan^{-1}\left(\frac{\gamma}{\delta}\right)\right) 1_{[a,b]^c}(x) \leq \pi 1_{[a,b]}(x) + \frac{\delta}{\gamma}\end{aligned}$$

for all $x \in \mathbb{R}$, where we used $\tan^{-1}(t) \leq \frac{\pi}{2}$ and $\tan^{-1}(t) \geq \frac{\pi}{2} - \frac{1}{t}$, respectively.

Applying these inequalities to X yields the conclusion by functional calculus. □

The reason Lemma 4.1 is useful for our purposes is that we can compute the resolvent of X_N by means of a mesoscopic semicircle law. The proof of the following result is an elementary application of the "intrinsic freeness" theory of [1]. (In the special case of interest here such a bound could alternatively be obtained, albeit with considerably more effort, by adapting the methods of [7, §3].)

Lemma 4.2 (Mesoscopic semicircle law)**.** *Denote by*

$$m_{\mathrm{sc}}(z) := -\frac{z}{2} + \frac{\sqrt{z^2-4}}{2}$$

the Stieltjes transform of the standard semicircle law. Then

$$\|\mathbf{E}[(d^{-1/2}X_N - z)^{-1}] - m_{\mathrm{sc}}(z)\mathbb{1}_N\| \leq \frac{2}{d\,(\operatorname{Im} z)^5}$$

for every $z \in \mathbb{C}$ with $\operatorname{Im} z > 0$.

Proof. The theory of [1] enables us to compare the spectral statistics of a random matrix X_N with those of a certain deterministic operator $X_{N,\mathrm{free}}$ that arises from free probability theory.[2] In particular, [1, Theorem 2.8 and Lemma 3.1] yield

$$\|\mathbf{E}[(d^{-1/2}X_N - z)^{-1}] - (\mathrm{id} \otimes \tau)[(d^{-1/2}X_{N,\mathrm{free}} - z)^{-1}]\| \leq \frac{2}{d} \frac{1}{(\operatorname{Im} z)^5}.$$

It remains to compute $G(z) := (\mathrm{id} \otimes \tau)[(d^{-1/2}X_{N,\mathrm{free}} - z)^{-1}]$.

To this end, note that by [11, eq. (1.5)], $G(z)$ satisfies the matrix Dyson equation

$$\frac{1}{d} \sum_{\{i,j\}:i\sim j} E_{ij} G(z) E_{ij} + G(z)^{-1} + z\mathbb{1}_N = 0,$$

where $E_{ij} := e_i e_j^* + 1_{j\neq i} e_j e_i^*$. Moreover, [12, Theorem 2.1] states that for any $z \in \mathbb{C}$ with $\operatorname{Im} z > 0$, the matrix Dyson equation has a unique solution with positive imaginary part. As $m_{\mathrm{sc}}(z)$ satisfies the equation $m_{\mathrm{sc}}(z) + m_{\mathrm{sc}}(z)^{-1} + z = 0$ and $\operatorname{Im} m_{\mathrm{sc}}(z) > 0$ whenever $\operatorname{Im} z > 0$, it is readily verified that $G(z) = m_{\mathrm{sc}}(z)\mathbb{1}_N$ is the unique solution of the equation in the present setting. □

[2] More precisely, $X_{N,\mathrm{free}}$ may be defined as a matrix whose entries are $(X_{N,\mathrm{free}})_{xy} = 1_{x\sim y} s_{xy}$, where s_{xy} are freely independent semicircular variables modulo symmetry $s_{yx} = s_{xy}$.

At this point we can clearly see the origin of the delocalization phenomenon: Lemma 4.2 shows that the resolvent behaves to leading order as a multiple of the identity matrix; by Lemma 4.1, this will ensure that all diagonal entries of the projection matrix of a suitable eigenspace of X_N are roughly of the same order, which is precisely what is needed to apply Proposition 3.1.

Before we proceed to implementing this program, we must establish concentration of the diagonal entries of the resolvent. This will be needed in order to upgrade the expected resolvent bound of Lemma 4.2 to a high probability bound.

Lemma 4.3 (Resolvent concentration)**.** *Define* $G_N(z) := (d^{-1/2}X_N - z)^{-1}$*. Then for any* $a < b$ *and* $t, \delta > 0$*, we have*

$$\mathbf{P}\left[\left| \operatorname{Im} \int_a^b G_N(\lambda + i\delta)_{xx} d\lambda - \operatorname{Im} \int_a^b \mathbf{E} G_N(\lambda + i\delta)_{xx} d\lambda \right| \ge \frac{t}{\delta^2} \frac{b-a}{\sqrt{d}}\right] \le 2e^{-t^2/4}.$$

Proof. Using the resolvent identity, we have

$$\|(X-z)^{-1} - (Y-z)^{-1}\| = \|(X-z)^{-1}(Y-X)(Y-z)^{-1}\| \le \frac{\|Y-X\|}{(\operatorname{Im} z)^2}.$$

Thus

$$f(X) := \operatorname{Im} \int_a^b [(d^{-1/2}X - \lambda - i\delta)^{-1}]_{xx}\, d\lambda$$

is $(b-a)d^{-1/2}\delta^{-2}$-Lipschitz with respect to the operator norm. In particular, as $\|Z\| \le (\sum_{ij} |Z_{ij}|^2)^{1/2} \le \sqrt{2}\,(\sum_{i \ge j} |Z_{ij}|^2)^{1/2}$ for any self-adjoint matrix Z, we may view $f(X_N)$ as a $(b-a)\sqrt{2}\,d^{-1/2}\delta^{-2}$-Lipschitz function of the i.i.d. standard Gausian variables $(g_{xy})_{x \ge y}$ that define X_N. The conclusion follows by Theorem 2.1. □

We can now combine the above results.

Corollary 4.4 (Projection matrix estimate)**.** *Let* $0 \le a < b \le 3$ *and* $0 < \delta \le \gamma \le 1$*. Denote by* P *the projection matrix of the space spanned by the eigenvectors of* $d^{-1/2}X_N$ *with eigenvalues in* $[a, b]$*. Then we have*

$$\mathbf{P}\left[\max_{x \in [N]} P_{xx} \ge \left(1 + \frac{2\delta}{\gamma}\right) \frac{1}{\pi} \left\{ \operatorname{Im} \int_{a-\gamma}^{b+\gamma} m_{\mathrm{sc}}(\lambda + i\delta) d\lambda + \frac{10}{\delta^5 d} + \frac{20\sqrt{\log N}}{\delta^2 \sqrt{d}} \right\}\right] \le \frac{2}{N^3}$$

and

$$\mathbf{P}\left[\min_{x \in [N]} P_{xx} \le \frac{1}{\pi} \left\{ \operatorname{Im} \int_{a+\gamma}^{b-\gamma} m_{\mathrm{sc}}(\lambda + i\delta) d\lambda - \frac{\delta}{\gamma} - \frac{6}{\delta^5 d} - \frac{12\sqrt{\log N}}{\delta^2 \sqrt{d}} \right\}\right] \le \frac{2}{N^3}.$$

Proof. Note first that Lemmas 4.1 and 4.2 yield

$$P_{xx} \le \left(1 + \frac{2\delta}{\gamma}\right) \frac{1}{\pi} \operatorname{Im} \int_{a-\gamma}^{b+\gamma} G_N(\lambda + i\delta)_{xx} d\lambda,$$

$$\operatorname{Im} \int_{a-\gamma}^{b+\gamma} \mathbf{E} G_N(\lambda + i\delta)_{xx} d\lambda \le \operatorname{Im} \int_{a-\gamma}^{b+\gamma} m_{\mathrm{sc}}(\lambda + i\delta) d\lambda + \frac{10}{\delta^5 d},$$

where we used that $b - a + 2\gamma \le 5$. Thus Lemma 4.3 yields

$$\mathbf{P}\left[P_{xx} \ge \left(1 + \frac{2\delta}{\gamma}\right) \frac{1}{\pi} \left\{ \operatorname{Im} \int_{a-\gamma}^{b+\gamma} m_{\mathrm{sc}}(\lambda + i\delta) d\lambda + \frac{10}{\delta^5 d} + \frac{5t}{\delta^2 \sqrt{d}} \right\}\right] \le 2e^{-t^2/4}$$

for all $x \in [N]$ and $t \ge 0$. The first inequality now follows by taking a union bound and choosing $t = 4\sqrt{\log N}$. The second inequality is derived analogously. □

The final ingredient that will be needed in the proof of Theorem 1.6(ii) is an estimate on the integrals that appear in Corollary 4.4.

Lemma 4.5 (Semicircle law estimate)**.** *For any $0 < \delta < \varepsilon < 1$ and $c \ge 2 + \varepsilon$*

$$\left(1 - \frac{2\delta}{\varepsilon}\right)\frac{\sqrt{3}}{3\pi}\varepsilon^{3/2} \le \frac{1}{\pi}\operatorname{Im}\int_{2-2\varepsilon}^{c} m_{\rm sc}(\lambda + i\delta)d\lambda \le 2\varepsilon^{3/2} + \frac{\delta}{\varepsilon}.$$

Proof. As $m_{\rm sc}(z) = \int (x-z)^{-1}\rho_{\rm sc}(x)\,dx$ is the Stieltjes transform of the standard semicircle law (as defined in Theorem 1.2), we can apply precisely the same estimates as in Lemma 4.1 to estimate for any $a < b$

$$\frac{1}{\pi}\operatorname{Im}\int_{a+\varepsilon}^{b-\varepsilon} m_{\rm sc}(\lambda+i\delta)d\lambda - \frac{\delta}{\varepsilon} \le \int_a^b \rho_{\rm sc}(x)dx \le \left(1+\frac{2\delta}{\varepsilon}\right)\frac{1}{\pi}\operatorname{Im}\int_{a-\varepsilon}^{b+\varepsilon} m_{\rm sc}(\lambda+i\delta)d\lambda.$$

Choosing $a = 2-3\varepsilon$, $b = c+\varepsilon$ and $a = 2-\varepsilon$, $b = c-\varepsilon$, respectively, yields

$$\left(1 - \frac{2\delta}{\varepsilon}\right)\int_{2-\varepsilon}^{2} \rho_{\rm sc}(x)dx \le \frac{1}{\pi}\operatorname{Im}\int_{2-2\varepsilon}^{c} m_{\rm sc}(\lambda+i\delta)d\lambda \le \int_{2-3\varepsilon}^{2}\rho_{\rm sc}(x)dx + \frac{\delta}{\varepsilon},$$

where we used that $\rho_{\rm sc}$ is supported on $[-2,2]$ and that $1-u \le \frac{1}{1+u}$ for $u \ge 0$. It remains to note that we can estimate

$$\frac{2\sqrt{1-u/4}}{3\pi}u^{3/2} \le \int_{2-u}^{2}\rho_{\rm sc}(x)dx = \frac{1}{2\pi}\int_{2-u}^{2}\sqrt{(2-x)(2+x)}dx \le \frac{2}{3\pi}u^{3/2}$$

for $u \le 4$, and the proof is readily completed. □

We are now ready to complete the proof of Theorem 1.6(ii).

Proof of Theorem 1.6(ii). We will assume throughout the proof that $d \gg \log N$. Let E_N be the space spanned by the eigenvectors of $d^{-1/2}X_N$ with eigenvalues in the interval $[2-\varepsilon, 3]$, and denote by P_N its projection matrix. Here $\varepsilon = \varepsilon_N = o(1)$ will depend on N in a manner that will be chosen below. As

$$\mathbf{P}\left[d^{-1/2}\|X_N\| \ge (2+s) + K_s\sqrt{\frac{\log N}{d}} + t\right] \le 2e^{-dt^2/4}$$

for any $s, t > 0$ by (2.1) and Theorem 1.2, we have

$$\mathbf{P}[d^{-1/2}\|X_N\| \le 2 + o(1)] = 1 - o(1).$$

Thus every unit vector in E_N is a $(1-o(1))$-approximate top eigenvector of X_N with probability $1-o(1)$. It remains to show that E_N contains delocalized vectors.

To this end, we begin by applying Corollary 4.4 and Lemma 4.5 with $a = 2-\varepsilon$, $b = 3$, $\gamma = \frac{\varepsilon}{2}$, and $\delta = \varepsilon^3$. This yields

$$\mathbf{P}\left[\max_{x\in[N]}(P_N)_{xx} \ge C\varepsilon^{3/2} + \frac{C}{\varepsilon^{15}d} + \frac{C}{\varepsilon^6}\sqrt{\frac{\log N}{d}}\right] \le \frac{2}{N^3}$$

and

$$\mathbf{P}\left[\min_{x\in[N]}(P_N)_{xx} \le c\varepsilon^{3/2} - \frac{C}{\varepsilon^{15}d} - \frac{C}{\varepsilon^6}\sqrt{\frac{\log N}{d}}\right] \le \frac{2}{N^3}$$

for sufficiently large N, where $c, C > 0$ are universal constants. In particular, if we choose $\varepsilon = (\frac{\log N}{d})^{1/17}$, the last two terms inside each of the above probabilities become negligible, and we can conclude that

$$\mathbf{P}\left[\max_{x\in[N]}(P_N)_{xx} \le C' \min_{x\in[N]}(P_N)_{xx}\right] = 1 - o(1)$$

for a universal constant C'.

Now let $m_N = \dim E_N$ (note that this is a random variable). Then

$$\min_{x\in[N]} (P_N)_{xx} \le \frac{1}{N}\sum_{x\in[N]} (P_N)_{xx} = \frac{\mathrm{Tr}(P_N)}{N} = \frac{m_N}{N}.$$

Thus we have shown that the assumption of Proposition 3.1 holds with probability $1 - o(1)$, completing the proof of the theorem. □

Appendix A. Delocalization of uniform random vectors in $\mathbb{S}^{N-1}$

The aim of this section is to elucidate the delocalization properties of uniform random vectors in $\mathbb{S}^{N-1}$. The main result of this section is the following.

Proposition A.1 (Unit sphere delocalization)**.** *Let V_N be a random vector that is uniformly distributed on $\mathbb{S}^{N-1}$, and fix $0 < \kappa < 1$ that is independent of N. Then there exist universal constants $c_1, c_2 > 0$ such that the following hold as $N \to \infty$.*

a. If $\nu \le \frac{c_1\kappa^2}{\log\frac{e}{\kappa}}$, then $\mathbf{P}[V_N \in D_{\nu N,\kappa}] = 1 - o(1)$.

b. If $\nu \ge \frac{c_2\kappa^2}{\log\frac{e}{\kappa}}$, then $\mathbf{P}[V_N \in D_{\nu N,\kappa}] = o(1)$.

In particular, this result shows that the approximate top eigenvector provided by Theorem 1.6 in the delocalized regime is essentially as delocalized as a uniform random vector, up to to the value of the universal constant.

We begin by introducing some notation. For any vector $z \in \mathbb{R}^N$, we denote by $z_{(1)} \ge z_{(2)} \ge \cdots \ge z_{(N)} \ge 0$ the decreasing rearrangement of z, that is, the absolute values of the entries of z sorted in decreasing order. We define the norm

$$\|z\|_{(L)} := \sup_{A\subseteq[N]:|A|\le L} \left(\sum_{j\in A} |z_j|^2\right)^{1/2} = \left(\sum_{k=1}^{\lfloor L\rfloor} z_{(k)}^2\right)^{1/2}.$$

Thus $z \in D_{L,\kappa}$ if and only if $\|z\|_{(L)} \le \kappa$. The proof of Proposition A.1 requires us to estimate $\|V_N\|_{(\nu N)}$. Let us first investigate its expectation.

Lemma A.2. *There exist universal constants $C_1, C_2 > 0$ so that*

$$C_1\nu\log\frac{e}{\nu} \le \mathbf{E}\|V_N\|_{(\nu N)}^2 \le C_2\nu\log\frac{e}{\nu}.$$

Proof. As $\frac{1}{3} \le \|z\|_{(N/2)}^2 \le \|z\|_{(\nu N)}^2 \le 1$ for all $\nu \ge \frac{1}{2}$ and $z \in \mathbb{S}^{N-1}$, it suffices to consider $\nu < \frac{1}{2}$. Let Z be a standard Gaussian vector in $\mathbb{R}^N$. Then well-known estimates on order statistics (see, e.g., [5, Theorem 2.5]) yield

$$c\log\frac{eN}{k} \le \mathbf{E}Z_{(k)}^2 \le C\log\frac{eN}{k}$$

for all $1 \le k \le \frac{N}{2}$, where $c, C > 0$ are universal constants. Stirling's formula yields

$$c'\nu N\log\frac{e}{\nu} \le \mathbf{E}\|Z\|_{(\nu N)}^2 \le C'\nu N\log\frac{e}{\nu}$$

for universal constants $c', C' > 0$. It remains to note that $Z \overset{\mathrm{d}}{=} \|Z\|_2 V_N$ where V_N is independent of Z, so that $\mathbf{E}\|Z\|_{(\nu N)}^2 = N\,\mathbf{E}\|V_N\|_{(\nu N)}^2$. □

We can now complete the proof of Proposition A.1.

Proof of Proposition A.1. It is shown in [13, Theorem 2.10] that

$$\mathbf{P}\big[\big|\|V_N\|_{(\nu N)} - \mathbf{E}\|V_N\|_{(\nu N)}\big| > t\big] \le Ce^{-cNt^2},$$

where $c, C > 0$ are universal constants. This implies that $\mathrm{Var}(\|V_N\|_{(\nu N)}) = O(\frac{1}{N})$, so that $\mathbf{E}\|V_N\|_{(\nu N)} = (\mathbf{E}\|V_N\|^2_{(\nu N)})^{1/2} + o(1)$ as $N \to \infty$. Using Lemma A.2 and applying the above tail estimate again yields

$$\begin{aligned}\mathbf{P}[V_N \in D_{\nu N,\kappa}] &= 1 - o(1) \quad \text{when } C'\nu \log \tfrac{e}{\nu} \le \kappa^2,\\ \mathbf{P}[V_N \in D_{\nu N,\kappa}] &= o(1) \qquad \text{when } c'\nu \log \tfrac{e}{\nu} \ge \kappa^2\end{aligned}$$

for some universal constants $c', C' > 0$. The conclusion follows readily. □

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School of Mathematics, University of Minnesota, 206 Church St SE, Minneapolis, MN 55455, USA
Email address: shou0039@umn.edu

Fine Hall 207, Princeton University, Princeton, NJ 08544, USA
Email address: rvan@math.princeton.edu