A VOLUME = MULTIPLICITY FORMULA FOR p-FAMILIES OF IDEALS

SUDIPTA DAS

Abstract. In this paper, we work with certain families of ideals called p-families in rings of prime characteristic. This family of ideals is present in the theories of tight closure, Hilbert-Kunz multiplicity, and F-signature. For each p-family of ideals, we attach a Euclidean object called p-body, which is analogous to the Newton Okounkov body associated with a graded family of ideals. Using the combinatorial properties of p-bodies and algebraic properties of the Hilbert-Kunz multiplicity, we establish in this paper a Volume = Multiplicity formula for p-families of $\mathfrak{m}_R$-primary ideals in a Noetherian local ring R.

1. Introduction

Let $(R, \mathfrak{m}, \mathbb{K})$ be a Noetherian local ring of dimension d, with prime characteristic $p > 0$, I be an $\mathfrak{m}$-primary ideal and $q = p^e$ for some $e \in \mathbb{N}$, and $I^{[q]} = (x^q \mid x \in I)$, the q-th Frobenius power of I. We denote the $\mathfrak{m}$-adic completion of R by $\hat{R}$ and $\ell_R(-)$ denotes the length as an R-module.

The Hilbert-Kunz theory deals with the question of how $\ell_R\left(R/I^{[q]}\right)$ behaves as a function on q and how understanding this behavior leads us to have a better understanding of the singularities of the ring. In his work, Kunz [9] introduced this study to measure how close the ring R is to be regular. Later P. Monsky [11] showed that

$$e_{HK}(I, R) = \lim_{e \to \infty} \frac{\ell_R(R/I^{[q]})}{q^d}$$

exists for any $\mathfrak{m}$-primary ideal I, this positive real number is called the *Hilbert-Kunz Multiplicity* of I. Contrary to the Hilbert-Samuel multiplicity of an $\mathfrak{m}$-primary ideal I, which is given by $e(I, R) = \lim_{k \to \infty} \dfrac{\ell_R(R/I^k)}{k^d/d!}$, the Hilbert-Kunz Multiplicity demonstrates much more complicated behavior [6]

In this paper, we work with certain families of ideals called p-families. A p-family of ideals is a sequence of ideals $I_\bullet = \{I_q\}_{q=1}^{\infty}$ with $I_q^{[p]} \subseteq I_{pq}$ for all q. Given a p-family of $\mathfrak{m}$-primary ideals, we define the following limit as the *volume* of the p-family.

$$\mathrm{Vol}_R(I_\bullet) = \lim_{q \to \infty} \frac{\ell_R(R/I_q)}{q^d}$$

In [7], Jack Jeffries and Daniel J Hernández showed that $\mathrm{Vol}_R(I_\bullet)$ exists as a limit if and only if $\dim N(\hat{R}) < d$, where $N(\hat{R})$ denotes the nilradical of $\hat{R}$. This provides an alternating proof of the existence of Hilbert Kunz multiplicity and F-signature for a complete local domain ([7], Corollary 6.1,6.8).

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This article aims to prove the following general *Volume = Multiplicity* formula for p-families of $\mathfrak{m}$-primary ideals.

Theorem 5.3. Let $(R, \mathfrak{m}, \mathbb{K})$ be a d-dimensional local ring of characteristic $p > 0$. If $\dim N(\hat{R}) < d$, then for any p-family of $\mathfrak{m}$-primary ideals $I_\bullet = \{I_q\}_{q=1}^{\infty}$

$$\mathrm{Vol}_R(I_\bullet) = \lim_{q\to\infty} \frac{\ell_R(R/I_q)}{q^d} = \lim_{q\to\infty} \frac{e_{HK}(I_q, R)}{(q)^d}$$

To prove our main result, we have followed the approach of Cutkosky's existence theorem ([2] [3]), which can be outlined as follows:

(1) Reduce to the case of a complete local domain.

(2) Using a suitable valuation on this complete local domain R, we attach to every p-family of ideals a combinatorial structure, called p-system of ideals (see Definition 3.11) and associate an Euclidean object called p-body (see Definition 3.12).

(3) To compute the relevant R-module lengths, we prove an approximation Theorem for p-systems of ideals. Since Lazarsfeld and Mustaţă in their paper [10, Proposition 3.1] have proved similar approximation theorem and using it recovered the Fujita Approximation Theorem [10, Theorem 3.3], we call it *Fujita Type Approximation Theorem for p-systems of ideals* (see Theorem 3.18).

Although the Hilbert-Kunz multiplicity behaves much differently compared to the Hilbert-Samuel multiplicity, a parallel analog of Theorem 5.3 for every graded family $J_\bullet = \{J_n\}_{n\in\mathbb{N}}$ (i.e., a sequence of ideals $J_\bullet$ such that $J_m.J_n \subseteq J_{m+n}$) of $\mathfrak{m}$-primary ideals and Hilbert-Samuel multiplicities has been proven for valuation ideals associated to an Abhyankar valuation in a regular local ring which is essentially of finite type over a field in [5] when R is a local domain which is essentially of finite type over an algebraically closed field $\mathbb{K}$ with $R/\mathfrak{m} = \mathbb{K}$ in [10], it is proven when R is analytically unramified with perfect residue field in [3]. Finally in [4] Cutkosky settled this for any d-dimensional Noetherian local ring R with $\dim N(\hat{R}) < d$.

2. Preliminaries

This section reviews a few basic notions from semigroup theory, convex geometry, and valuation theory.

Let U and V be arbitrary subsets of $\mathbb{R}^d$. If U is Lebesgue measurable, its measure $\mathrm{Vol}_{\mathbb{R}^d}(U)$ is called the *volume* of U. We define the Minkowski sum of two sets as $U + V := \{\ \boldsymbol{u} + \boldsymbol{v} \mid \boldsymbol{u} \in U, \boldsymbol{v} \in V \}$.

A *convex cone* is any subset of $\mathbb{R}^d$ that is closed under taking an $\mathbb{R}$-linear combination of points with non-negative coefficients. Let $Cone(U) \subseteq \mathbb{R}^d$ be the convex cone which is the closure of the set of all linear combinations $\sum_i \lambda_i u_i$, with $u_i \in U$ and $\lambda_i \in \mathbb{R}_{\geqslant 0}$. Note that a cone in $\mathbb{R}^d$ has a non-empty interior if and only if the real vector space it generates has dimension d. We call such a cone *full-dimensional*.

A cone C is *pointed* if it is closed and if there exists a vector $\boldsymbol{a} \in \mathbb{R}^d$ such that $(\boldsymbol{u}, \boldsymbol{a}) > 0$, for all $\boldsymbol{u} \in C \setminus \{\boldsymbol{0}\}$, where $(-,-)$ is the inner product in $\mathbb{R}^d$.

If C is a pointed cone and α is a non-negative real number, we define:

$$H = H_\alpha := \{\ \boldsymbol{u} \in \mathbb{R}^d \mid (\boldsymbol{u}, \boldsymbol{a}) < \alpha \ \}$$

a *truncating half-space* for C.

Remark 2.1. Let C be a pointed cone. If $\alpha > 0$, then a *truncation* of C is defined as $C \cap H_\alpha$. It is a non-empty, bounded subset of C.

Remark 2.2. If H is any truncating halfspace of a pointed cone C then so is qH, for all $q > 0$.

We define a *semigroup* to be any subset of $\mathbb{Z}^d$ that contains $\mathbf{0}$ and it is closed under addition; e.g. $\mathbb{N}$ is the semigroup of non-negative integers. A semigroup S is *finitely generated* if there exist a finite subset $S_0 \subseteq S$ such that every element of S can be written as an $\mathbb{N}$-linear combination of elements of S_0. A subset $T \subseteq S$ of a semigroup S, is an *ideal* of S whenever $S + T$ is contained in T. A semigroup S is called *pointed* if $\boldsymbol{a} \in S$ and $-\boldsymbol{a} \in S$ then $\boldsymbol{a}$ is $\mathbf{0}$.

Remark 2.3. If $\operatorname{Cone}(S)$ is pointed, then so is S, but the converse is generally false. However, it does hold if S is assumed to be finitely generated.

Remark 2.4. For any $\mathbb{Z}$-linear embedding $i : \mathbb{Z}^d \hookrightarrow \mathbb{R}$, there exists some unique $\boldsymbol{a} \in \mathbb{R}^d$, whose coordinates are linearly independent over $\mathbb{Q}$, such that $i(\boldsymbol{u}) = (\boldsymbol{a}, \boldsymbol{u})$ for all $u \in \mathbb{Z}^d$. We define $\boldsymbol{u} \leqslant_{\boldsymbol{a}} \boldsymbol{v}$ whenever $(\boldsymbol{a}, \boldsymbol{u}) \leqslant (\boldsymbol{a}, \boldsymbol{v})$.

Let $\mathbb{F}^\times = \mathbb{F} \setminus \{0\}$. Fix, $\boldsymbol{a} \in \mathbb{R}^d$, and fix the embedding $\mathbb{Z}^d \hookrightarrow \mathbb{R}$ given by $\boldsymbol{v} \to (\boldsymbol{a}, \boldsymbol{v})$, defined by $\boldsymbol{a}$. An *$\boldsymbol{a}$-valuation* on a field $\mathbb{F}$ with value group $\mathbb{Z}^d$ is a surjective group homomorphism: $\vartheta : \mathbb{F}^\times \to \mathbb{Z}^d$ with the property that

$$\vartheta(xy) = \vartheta(x) + \vartheta(y),$$
$$\vartheta(x + y) \geqslant_{\boldsymbol{a}} \min\{\vartheta(x), \vartheta(y)\}.$$

For any subset $N \subseteq \mathbb{F}$, with $N^\times = N \setminus \{0\}$ we define $\vartheta(N) := \vartheta(N^\times)$, and this is called the *image* of N under ϑ.
Given a point $\boldsymbol{u} \in \mathbb{Z}^d$, we define:

$$\mathbb{F}_{\geqslant \boldsymbol{u}} := \{\ x \in \mathbb{F} \mid \vartheta(x) \geqslant_{\boldsymbol{a}} \boldsymbol{u}\ \} \cup \{0\} \quad \text{and} \quad \mathbb{F}_{> \boldsymbol{u}} := \{\ x \in \mathbb{F} \mid \vartheta(x) >_{\boldsymbol{a}} \boldsymbol{u}\ \} \cup \{0\}$$

The *local ring* of ϑ is the subring $(V_\vartheta, m_\vartheta, k_\vartheta)$ of $\mathbb{F}$, such that $V_\vartheta = \mathbb{F}_{\geqslant 0}$, and $m_\vartheta = \mathbb{F}_{>\mathbf{0}}$. A local domain (D, m, k) is *dominated by* $(V_\vartheta, m_\vartheta, k_\vartheta)$, if (D, m, k) is a local subring of $(V_\vartheta, m_\vartheta, k_\vartheta)$, i.e., $\mathfrak{m} \subseteq \mathfrak{m}_\vartheta$.

Remark 2.5. If $\boldsymbol{u} \in \mathbb{Z}^d$, then both $\mathbb{F}_{\geqslant \boldsymbol{u}}$ and $\mathbb{F}_{> \boldsymbol{u}}$ are modules over V_ϑ (as $V_\vartheta = \mathbb{F}_{\geqslant 0}$ and $\vartheta(xy) = \vartheta(x) + \vartheta(y)$) and $\mathbb{F}_{\geqslant \boldsymbol{u}}/\mathbb{F}_{> \boldsymbol{u}}$ is a vector space over k_ϑ. Note that, if $\boldsymbol{w} = \vartheta(f)$, for some non-zero $f \in \mathbb{F}$, then:

$$\dim_{k_\vartheta} (\mathbb{F}_{\geq \boldsymbol{w}}/\mathbb{F}_{> \boldsymbol{w}}) = 1.$$

Indeed if f is as above then for any g such that $\vartheta(g) = \boldsymbol{w}$, we have $g = \left(\dfrac{g}{f}\right) \cdot f$, where $\dfrac{g}{f} \in k_\vartheta$ because $\vartheta\left(\dfrac{g}{f}\right) = 0$.

Remark 2.6. Let (D, m, k) be a local domain of dimension d with fractional field $\mathbb{F}$, and let $\vartheta : \mathbb{F}^\times \twoheadrightarrow \mathbb{Z}^d$ be an *$\boldsymbol{a}$-valuation* with value group $\mathbb{Z}^d$. Note that $S = \vartheta(D)$ is a subsemigroup of $\mathbb{Z}^d$, and the image of any ideal I in D is a semigroup ideal of S. Moreover as ϑ is surjective, the Minkowski sum $S + (-S)$ is equal to $\mathbb{Z}^d$, as ϑ is surjective. Therefore, the real vector space generated by $\operatorname{Cone}(S)$ is $\mathbb{R}^d$, i.e., $\operatorname{Cone}(S)$ is full dimensional.

Observe that D is dominated by V_ϑ if and only if for all $\boldsymbol{x} \in \vartheta(m)$, we have $\boldsymbol{x} >_{\boldsymbol{a}} \mathbf{0}$ and $S = \vartheta(m) \cup \{\mathbf{0}\}$, i.e., any non-zero $\boldsymbol{u} \in S$, satisfies $(\boldsymbol{a}, \boldsymbol{u}) > 0$.

Definition 2.7. Following Remark 2.6, if D is dominated by V_ϑ and $(\boldsymbol{a}, \boldsymbol{u}) > 0$, for every $\boldsymbol{u} \in \mathrm{Cone}(S) \setminus \{\mathbf{0}\}$, then we say D is *strongly dominated* by ϑ.

3. OK-valuation, OK-domain, p-systems and p-bodies

In this section, we review the constructions of OK-valuation, OK-domain, p-systems, and p-bodies following the work of Cutkosky ([2],[3]) and Jeffries and Hernández [7]. The main result of this section is Theorem 3.18.

Definition 3.1. Let (D, m, k) be a d-dimensional local domain with fractional field $\mathbb{F}$. Fix the embedding $\mathbb{Z}^d \hookrightarrow \mathbb{R}$ defined by $\boldsymbol{a} \in \mathbb{R}^d$.
If an $\boldsymbol{a}$-valuation

$$\vartheta : \mathbb{F}^\times \to \mathbb{Z}^d$$

with value group $\mathbb{Z}^d$, and local ring $(V_\vartheta, m_\vartheta, k_\vartheta)$ satisfies the following conditions:
i. D is strongly dominated by V_ϑ, (see Definition 2.7)
ii. the resulting extension of residue fields $k \hookrightarrow k_\vartheta$ is finite, and
iii. there exists a point $\boldsymbol{v} \in \mathbb{Z}^d$ such that:

$$D \cap \mathbb{F}_{\geqslant n\boldsymbol{v}} \subseteq m^n, \ \forall n \in \mathbb{N}$$

then we say that $(V_\vartheta, m_\vartheta, k_\vartheta)$ is *OK-relative* to D.

Definition 3.2. Let D be a d-dimensional local domain with fraction field $\mathbb{F}$. If there exists a valuation ϑ on $\mathbb{F}$ with value group $\mathbb{Z}^d$, and valuation ring $(V_\vartheta, m_\vartheta, k_\vartheta)$ that is *OK-relative* to D, we say that D is an *OK-domain*.

Example 3.3. Let $R = \mathbb{K}[\![x_1, ..., x_d]\!]$ is a power series ring of dimension d, containing a field $\mathbb{K}$, $\mathfrak{m} = (x_1, ..., x_d)$ is the maximal ideal and $\mathbb{F}$ is the quotient field of R. Take $a_1, ..., a_d$ be rationally independent real numbers and without loss of generality assume $a_i > 0$ for all $1 \leqslant i \leqslant d$. Define an ordering on $\mathbb{Z}^d$ by $\boldsymbol{a} = (a_1, ..., a_d)$. Let $f \in R$; then we can write,

$$f = \sum_{s_{i_1,...,i_d} \in \mathbb{K}, (i_1,...,i_d) \in \mathbb{N}^d} s_{i_1,...,i_d} x_1^{i_1} ... x_d^{i_d}.$$

Since $\boldsymbol{a}$ has positive entries, this will imply that every subset of $\mathbb{N}^d$ has a least element, i.e., there exist a well defined least exponent vector $(i_1^{'}, ..., i_d^{'})$ among its terms and we define, $\vartheta(f) = (i_1^{'}, ..., i_d^{'})$.
We claim that the induced valuation $\vartheta : \mathbb{F}^\times \to \mathbb{Z}^d$ is OK relative to R.
Let $S = \vartheta(R)$, therefore from the above definition it is clear that $C = \mathrm{Cone}(S)$ is contained in the non-negative octant, then $(\boldsymbol{u}, \boldsymbol{a}) > 0$ for every nonzero $\boldsymbol{u}$ in C. This shows that R is strongly dominated by ϑ, satisfying condition *i.* Let $(V, \mathfrak{m}_\vartheta, \mathbb{K}_\vartheta)$ is the valuation ring $\mathbb{F}_{\geqslant \mathbf{0}}$ with it's maximal ideal $\mathbb{F}_{>\mathbf{0}}$, then clearly $\mathbb{K}_\vartheta = V/\mathfrak{m}_\vartheta \cong R/\mathfrak{m} \cong \mathbb{K}$, which satisfies condition *ii.* Moreover, let $a_j = \max\{a_1, ..., a_d\}$, then we claim that $\vartheta(x_j) = (0, .., 1, 0, .., 0) = \boldsymbol{v}$ is the vector for condition *iii.* Indeed let,

$$f \in R \cap \mathbb{F}_{\geqslant a\boldsymbol{v}} \implies \vartheta(f) \geqslant a\boldsymbol{v} \implies (i_1, ..., i_d) \geqslant (0, .., a, 0, .., 0)$$

$$\implies a_1 i_1 + ... + a_d i_d \geqslant a a_j \implies a a_j \leqslant a_j i_1 + ... + a_j i_d (\because a_j = \max\{a_1, ..., a_d\})$$

$$\implies a \leqslant i_1 + ... + i_d \implies R \cap \mathbb{F}_{\geqslant a\boldsymbol{v}} \subseteq \mathfrak{m}^a$$

This finishes the proof that every power series ring containing a field is an OK-domain.

Example 3.4. [7, Example 3.5]) This immediately implies a regular local ring R containing a field is an OK-domain.

Example 3.5. [7, Corollary 3.7] An interesting class of OK-domain examples is excellent local domains containing a field. In particular, a complete local domain containing a field is an OK-domain.

For the rest of this section, we fix a d-dimensional local domain D of characteristic $p > 0$ with residue field k and fraction field $\mathbb{F}$, a $\mathbb{Z}$-linear embedding of $\mathbb{Z}^d$ into $\mathbb{R}$ induced by a vector $\boldsymbol{a}$ in $\mathbb{R}^d$, and a valuation $\vartheta : \mathbb{F}^\times \twoheadrightarrow \mathbb{Z}^d$ that is OK relative to D. We use S to denote the semigroup $\vartheta(D)$ in $\mathbb{Z}^d$, and C to denote the closed cone in $\mathbb{R}^d$ generated by S. We follow the notation established in Definition 3.1.

Remark 3.6. If M is a D-submodule of $\mathbb{F}$, then the D-module structure on M induces a k-vector space structure on the quotient $\dfrac{M \cap \mathbb{F}_{\geqslant \boldsymbol{u}}}{M \cap \mathbb{F}_{>\boldsymbol{u}}}$ and by definition this space is non-zero if and only if there exists an element $x \in M$ with $\vartheta(x) = \boldsymbol{u}$.

Definition 3.7. For a D-submodule M of $\mathbb{F}$, we define:

$$\vartheta^{(h)}(M) = \left\{ \boldsymbol{u} \in \mathbb{Z}^d \mid \dim_k \left(\frac{M \cap \mathbb{F}_{\geqslant \boldsymbol{u}}}{M \cap \mathbb{F}_{>\boldsymbol{u}}} \right) \geqslant h \right\},$$

where $1 \leqslant h \leqslant [k_\vartheta : k]$.

Remark 3.8. Let $g \in D^\times$, with $\boldsymbol{v} = \vartheta(g)$, and let $\boldsymbol{u} \in \mathbb{Z}^d$. Let $I_\bullet = \{I_q\}_{q=1}^\infty$ be a sequence of ideals indexed by $q = p^e$, then the map:

$$\frac{I_q \cap \mathbb{F}_{\geqslant \boldsymbol{u}}}{I_q \cap \mathbb{F}_{>\boldsymbol{u}}} \to \frac{I_q \cap \mathbb{F}_{\geqslant \boldsymbol{u}+\boldsymbol{v}}}{I_q \cap \mathbb{F}_{>\boldsymbol{u}+\boldsymbol{v}}} \ : \ [x] \mapsto [gx]$$

is a k-linear injection for all $\boldsymbol{u} \in \mathbb{Z}^d$ and for all I_q in that sequence. Therefore using Definition 3.7, we have $\vartheta^{(h)}(I_q) + S \subseteq \vartheta^{(h)}(I_q)$,i.e., $\vartheta^{(h)}(I_q)$ is an ideal of $S = \vartheta(D)$ for all $1 \leqslant h \leqslant [k_\vartheta : k]$.

Lemma 3.9. [7, Lemma 3.11] *For a D-submodule M of $\mathbb{F}$ and $\boldsymbol{v} \in \mathbb{Z}^d$, we have:*

$$\ell_D \left(M/(M \cap \mathbb{F}_{\geqslant \boldsymbol{v}}) \right) = \sum_{h=1}^{[k_\vartheta : k]} \#(\vartheta^{(h)}(M) \cap H)$$

where $H = \left\{ \boldsymbol{u} \in \mathbb{R}^d \mid (\boldsymbol{a}, \boldsymbol{u}) < (\boldsymbol{a}, \boldsymbol{v}) \right\}$.

We now discuss the notions of *p-system* and *p-bodies*.

Definition 3.10. A semigroup S is called *standard* if $S - S = \mathbb{Z}^d$, and the full dimensional cone generated by S in $\mathbb{R}^d$ is pointed. Following the discussion in Remark 2.6 and from Definition 3.1 the main example of a standard semigroup in $\mathbb{Z}^d$ is $\vartheta(D)$ associated to an $\boldsymbol{a}$-valuation ϑ that is *OK* relative to some d-dimensional local domain D.

Definition 3.11. A collection of subsets $T_\bullet = \{T_q\}_{q=1}^\infty$ of a semigroup S indexed by $q = p^e$ satisfying
i. T_q is an ideal of S. $(T_q + S \subseteq S)$ and *ii.* $pT_q \subseteq T_{pq}$ is called a *p-system.*

Definition 3.12. The p-body associated to a given p-system of ideals $T_\bullet$ of a semigroup in $\mathbb{Z}^d$ is defined by
$\Delta(S, T_\bullet) = \bigcup_{q=1}^{\infty} \Delta_q(S, T_\bullet)$ where $\Delta_q(S, T_\bullet) = \frac{1}{q}T_q + \mathrm{Cone}(S)$

Remark 3.13. Although we have defined the notion of p-body for a p-system of ideals in any semigroup, in our situation, we are mostly concerned with p-system of ideals in the standard semigroup. Moreover, let $q_1 = p^{e_1}$ and $q_2 = p^{e_2}$ and $e_1 \leqslant e_2$, then $p^{e_2 - e_1} T_{p^{e_1}} \subseteq T_{p^{e_2}}$ (using Definition 3.12) this implies $\frac{T_{p^{e_1}}}{p^{e_1}} \subseteq \frac{T_{p^{e_2}}}{p^{e_2}}$. Therefore the p-body associated to a given p-system of ideals is an ascending union of sets.

Example 3.14. [7, Example 4.5] Let T be any subset of S, define $T_q = T + S$, for all $q = p^e$, then:
i. Clearly $T_q + S \subseteq T_q$
ii. $pT_q = (pT + pS) \subseteq (pT + S) \subseteq (T + (p-1)T + S) \subseteq T + S$ which is equal to T_{pq}.
Therefore T_q is a p-system of ideals for all $q = p^e$. Moreover, we have $\Delta_q(S, T_\bullet) = \left(\frac{1}{q}T + \mathrm{Cone}(S)\right)$ and the closure of $\Delta(S, T_\bullet)$ equals $\mathrm{Cone}(S)$.

Example 3.15. Let D be an OK domain for an OK valuation ϑ and assume $S = \vartheta(D)$ is the corresponding semigroup. Take an ideal I in D and define $T_q = q\vartheta(I) + S$. Following the same argument as in Example 3.14, one can show that $\{T_q\}_{q=1}^{\infty}$ is a p-system of ideals. Moreover $\Delta(S, T_\bullet) = \Delta_q(S, T_\bullet) = \vartheta(I) + \mathrm{Cone}(S)$.

In the previous example, one can replace $\vartheta(I)$ with any arbitrary subset T of S, and the corresponding p-body would be $T + \mathrm{Cone}(S)$. This shows that p bodies need not be convex, and in practical applications, mostly, they are not.

Remark 3.16. Following Definition 3.12 one can see that every p-body Δ is the union of countably many translates of $C = \mathrm{Cone}(S)$ therefore, it is Lebesgue measurable. If H is a truncating halfspace of C, then the volume of $(\Delta \cap H)$ is a well-defined real number.

The following Theorem describes $\mathrm{Vol}_{\mathbb{R}^d}(\Delta \cap H)$ for any truncating halfspace H of $\mathrm{Cone}(S)$. This Theorem is one of the key ingredients in the proof of our main Theorem.

Theorem 3.17. [7, Theorem 4.10] *For a standard semigroup S in $\mathbb{Z}^d$, a p-system $T_\bullet$ in S, and a truncating halfspace H for* $\mathrm{Cone}(S)$, *we have:*

$$\lim_{q \to \infty} \frac{\#(T_q \cap qH)}{q^d} = \mathrm{Vol}_{\mathbb{R}^d}(\Delta(S, T_\bullet) \cap H)$$

Lazarsfeld and Mustaţă in their paper [10, Proposition 3.1] have proved an approximation theorem similar to the following Theorem and using this in a global setup with D a big divisor on an irreducible projective variety; they have recovered the Fujita Approximation Theorem [10, Theorem 3.3]. Since our approximation is similar to a different set of properties on the semigroup, we call it *Fujita Type Approximation Theorem for p system of ideals.*

Theorem 3.18. *Let S be a standard semigroup in $\mathbb{Z}^d$. If $T_\bullet$ is a p-system of ideals in S, and let H is any truncating halfspace for* $\mathrm{Cone}(S)$, *then for any given $\epsilon > 0$, there exists q_0 such that if $q \geqslant q_0$ the following inequality hold.*

$$\lim_{e \to \infty} \frac{\#((p^e T_q + S) \cap p^e qH)}{p^{ed} q^d} \geqslant \mathrm{Vol}_{\mathbb{R}^d}(\Delta(S, T_\bullet) \cap H) - \epsilon$$

Proof. By Example 3.15 we notice that $T_\bullet^{'} = \{p^eT_q + S\}_{e=1}^{\infty}$ is a p-system of ideals. Therefore using Theorem 3.17 and Remark 2.2 we have,

$$\lim_{e\to\infty} \frac{\#((p^eT_q + S)\cap p^eqH)}{p^{ed}} = \mathrm{Vol}_{\mathbb{R}^d}((T_q + \mathrm{Cone}(S))\cap qH) \tag{3.1}$$

Now note that $\frac{\mathrm{Vol}_{\mathbb{R}^d}((T_q+\mathrm{Cone}(S)\cap qH)}{q^d} = \mathrm{Vol}_{\mathbb{R}^d}((\frac{T_q}{q}+\mathrm{Cone}(S))\cap H)$ and from Remark 3.13 and Remark 3.16, we know that these are an ascending union of measurable sets. Therefore,

$$\lim_{q\to\infty} \frac{\mathrm{Vol}_{\mathbb{R}^d}((T_q + \mathrm{Cone}(S))\cap qH)}{q^d} = \mathrm{Vol}_{\mathbb{R}^d}(\Delta(S, T_\bullet)\cap H) \tag{3.2}$$

We get Equation 3.2 by using the sub-additivity and continuity from below properties of the Lebesgue measure. Now combining Equation 3.1, Equation 3.2 for a chosen $\epsilon > 0$, we can find q_0 such that for all $q \geqslant q_0$
$\lim_{e\to\infty} \frac{\#((p^eT_q+S)\cap p^eqH)}{p^{ed}q^d} \geqslant \mathrm{Vol}_{\mathbb{R}^d}(\Delta(S, T_\bullet)\cap H) - \epsilon$ □

4. p-families and a brief introduction to Hilbert Kunz Multiplicity

In this section, all rings will be commutative, Noetherian, and of prime characteristic, $p > 0$.

Let I be an ideal of a ring R, generated by $\{x_1, ..., x_r\}$ then p^eth Frobenius power of I denoted by $I^{[p^e]}$ is generated by $\{x_1^{p^e}, ..., x_r^{p^e}\}$.

Definition 4.1. A sequence of ideals $I_\bullet = \{I_{p^e}\}_{e=0}^{\infty}$ is called a p-family whenever $I_q^{[p]} \subseteq I_{pq}$, where $q = p^e$ for some $e \geqslant 0$.

Remark 4.2. We note that if $(R, \mathfrak{m})$ is local, then every term in this family is $\mathfrak{m}$-primary if and only if I_1 is $\mathfrak{m}$ primary.

Example 4.3. Let $I_\bullet = \{J^{[p^e]}\}_{e=0}^{\infty}$ for some fixed ideal J, then $I_{p^e}^{[p]} = J^{[p^{e+1}]} = I_{p^{e+1}}$, therefore $I_\bullet$ is a p-family.

Lemma 4.4. *Let $(R, \mathfrak{m})$ be a local ring and let $I_\bullet = \{I_q\}_{q=1}^{\infty}$ be a p-family of $\mathfrak{m}$-primary ideals. Then there exist a $c > 0$ such that $\mathfrak{m}^{cq} \subseteq I_q$ and $\mathfrak{m}^{cp^eq} \subseteq I_q^{[p^e]}$ for all q a power of p and for all $e \geqslant 0$.*

Proof. We know from Remark 4.2 that there exist c_1 such that $(\mathfrak{m}^{c_1})^{[q]} \subseteq I_q$ for all q, a power of p. Let $\mathfrak{m}$ be generated by b elements then using pigeon-hole principle, $\mathfrak{m}^{bc_1p^eq} \subseteq (\mathfrak{m}^{c_1})^{[p^eq]} \subseteq ((\mathfrak{m}^{c_1})^{[q]})^{[p^e]} \subseteq I_q^{[p^e]}$, for all q a power of p and for all $e \geqslant 0$. So, letting $c = bc_1$ we get our desired conclusion. □

We define the Frobenius map $F : R \to R$ by $F(r) = r^p$. This map turns R into an R-module with a nonstandard action $r.x = r^px$, and we call this module F_*R. Inductively one can define F_*^eR. We define an R-linear map $\phi_e : F_*^eR \to R$ which is a set map from R to R such that ϕ_e is additive and $\phi_e(x^{p^e}y) = x\phi(y)$.
The following is an interesting example of p-families.

Example 4.5. Assume R is reduced and I is an R-ideal. Define for all $q = p^e$, $J_q = \{x \in R \mid \phi(x) \in I, \forall\phi \in Hom_R(F_*^e(R), R)\}$. Notice that J_q is an ideal for each $q = p^e$. Since R is reduced we can identify $F_*^e(R)$ as $R^{\frac{1}{p^e}}$, the ring of p^e-th roots of the elements in R. We claim that this is a p-family of ideals.

Indeed, let $\phi \in Hom_R(F_*^{e+1}R, R)$ and choose $r \in J_q$ then $\psi(r^{\frac{1}{p^e}}) \in I$ for all $\psi \in Hom_R(F_*^e R, R)$. Now $\phi\left((r^p)^{\frac{1}{p^{e+1}}}\right) = \phi_e\left(r^{\frac{1}{p^e}}\right) \in I$, as $r \in J_q$. This implies $J_q^{[p]} \subseteq J_{pq}$.

Remark 4.6. If in the previous example we choose $I = \mathfrak{m}$, then we get a p-family of m-primary ideals because $\mathfrak{m}^{[q]} \subseteq J_q$. Use of this sequence of ideals was present in the work of Y. Yao [13] as well as F. Enescu and I. Aberbach [1]. Later this sequence of ideals was used in the work of K. Tucker [12] for proving that F-signature exists.

Let (D, m, k) be a d-dimensional local OK-domain with OK-valuation ϑ. Choose any p family $I_\bullet$ of ideals, then
i. $\vartheta(I_q)$ is an ideal of the semigroup $\vartheta(D) = S$, and,
ii. $p\vartheta(I_q) \subseteq \vartheta(I_q^{[p]}) \subseteq \vartheta(I_{pq})$. Therefore, $\{\vartheta(I_q)\}_{q=1}^{\infty}$ is a p-system of ideals in S.

Remark 4.7. Following Definition 3.7 and Remark 3.8, one can observe that although $\vartheta^{(h)}(I_q)$ is an ideal but we need the extra condition that $\{m_1, ..., m_h\}$ are k-linear independent in $\vartheta^{(h)}(I_q)$ implies $\{m_1^p, ..., m_h^p\}$ are k-linear independent in $\vartheta^{(h)}(I_{pq})$ but it doesn't always true unless we assume that k is perfect. In ([7, Lemma 3.12]) they get around it by uniformly approximating $\vartheta^h(I_q)$ with $\vartheta(I_q)$ and proving the following result:

Proposition 4.8. [7, Corollary 5.10] Let D be a d-dimensional local OK-domain with OK-valuation ϑ. For a p-family of ideals $I_\bullet$ in D we have:

$$\lim_{q\to\infty} \frac{(\vartheta^{(h)}(I_q) \cap qH)}{q^d} = \mathrm{Vol}_{\mathbb{R}^d}(\Delta(S, \vartheta(I_\bullet)) \cap H)$$

where $S = \vartheta(D)$, $C = \mathrm{Cone}(S)$ and H is any truncating halfspace of C.

We will now discuss Hilbert-Kunz Multiplicity and some important properties.

For the rest of this section let $(R, \mathfrak{m}, \mathbb{K})$ be a d-dimensional local ring and $q = p^e$, for some $e \in \mathbb{N}$.

Definition 4.9. Let I be an $\mathfrak{m}$-primary ideal of R. We define Hilbert-Kunz Multiplicity of I by $e_{HK}(I, R) = \lim_{q\to\infty} \dfrac{\ell\left(R/I^{[q]}\right)}{q^d}$

Remark 4.10. Let J be an $\mathfrak{m}$-primary ideal of R, then $\ell(R/J)$ is unaffected by completion. Therefore using the above definition, we have

$$e_{HK}(J, R) = e_{HK}(J\hat{R}, \hat{R})$$

Lemma 4.11. *Let I be an $\mathfrak{m}$-primary ideal of R and M be a finitely generated R-module. Then there exists a constant $\alpha > 0$ such that*

$$\ell_R\left(\frac{M}{I^{[p^e]}M}\right) \leqslant \alpha p^{e.\dim M}$$

Proof. Since I is $\mathfrak{m}$-primary there exist $c > 0$ such that $\mathfrak{m}^c \subseteq I$. Let b is the minimal number of generators of I, then $\mathfrak{m}^{p^e bc} \subseteq I^{p^e b} \subseteq I^{[p^e]}$ (using pigeon-hole principle). Therefore $\ell_R\left(\frac{M}{I^{[p^e]}}\right) \leqslant \ell_R\left(\frac{M}{\mathfrak{m}^{p^e bc}}\right)$ and the later agrees with a polynomial in $p^e bc$ of degree $\dim M$. If the leading coefficient of this polynomial is a then choose $a_0 >> a$ implies $\ell_R(\frac{M}{\mathfrak{m}^{p^e bc}})$ is bounded above by , $a_0(b^{\dim M} c^{\dim M} p^{e\dim M})$ taking $\alpha = a_0 b^{\dim M} c^{\dim M}$ will finish the proof. ☐

Corollary 4.12. Let N be an ideal of R and $A = R/N$, then for any $\mathfrak{m}$-primary ideal I of R there exist $\beta > 0$ such that

$$0 \leqslant \ell_R\left(R/I^{[p^e]}\right) - \ell_A\left(A/I^{[p^e]}\right) \leqslant \beta . p^{e.\dim N}$$

Proof. Take the following exact sequence

$$0 \to N/(N \cap I^{[p^e]}) \to R/I^{[p^e]} \to A/I^{[p^e]}A \to 0$$

Using the length formula, we have

$$\ell_R\left(R/I^{[p^e]}\right) - \ell_A\left(A/I^{[p^e]}\right) = \ell_R\left(N/(N \cap I^{[p^e]})\right) \tag{4.1}$$

Note that $\ell_R\left(A/I^{[p^e]}\right) = \ell_A\left(A/I^{[p^e]}\right)$ and $I^{[p^e]}N \subseteq I^{[p^e]} \cap N$. Now using Lemma 4.11, we have from Equation 4.1, $\ell_R\left(R/I^{[p^e]}\right) - \ell_A\left(A/I^{[p^e]}\right) \leqslant \ell_R\left(N/I^{[p^e]}N\right) \leqslant \beta p^{e.\dim N}$ □

Definition 4.13. Let $f, g : \mathbb{N} \to \mathbb{R}$ be real valued functions from the set of non-negative integers. We say $f(n) = O(g(n))$ if there exists a positive constant C such that $|f(n)| \leqslant Cg(n)$ for all $n \gg 0$ and we say $f(n) = o(g(n))$ if $\lim_{n\to\infty} \dfrac{f(n)}{g(n)} = 0$

Now we will mention some results on Hilbert Kunz Multiplicity without proof; for a detailed verification, we refer readers to [8].

Proposition 4.14. Let

$$0 \to N \to M \to K \to 0$$

be a short exact sequence of finitely generated R-modules. Then

$$\ell_R\left(M/I^{[q]}M\right) = \ell_R\left(N/I^{[q]}N\right) + \ell_R\left(K/I^{[q]}K\right) + O(q^{d-1}) \tag{4.2}$$

Now dividing Equation 4.2 by q^d and taking $q \to \infty$ we have

$$e_{HK}(I, M) = e_{HK}(I, K) + e_{HK}(I, N)$$

Theorem 4.15. *Let I be an $\mathfrak{m}$-primary ideal of R and M is a finitely generated R-module. Let Γ be the set of minimal prime ideals P of R such that* $\dim(R/P) = \dim(R)$. *Then*

$$e_{HK}(I, M) = \sum_{p\in\Gamma} e_{HK}(I, R/P)\,\ell(M_P)$$

The main idea to prove this result is to take a prime filtration of M, use the fact that $e_{HK}(I, R/Q) = 0$ if $\dim R/Q < \dim R$ and then use Proposition 4.14.

Remark 4.16. In the Theorem 4.15, if we choose $M = R$ and further assume R is reduced. Then

$$e_{HK}(I, R) = \sum_{p\in\Gamma} e_{HK}(I, R/P)$$

because a reduced local ring with Krull dimension 0 is a field, therefore $\ell(R_P) = 1$

5. Volume=Multiplicity Formula

In this section we prove our main result, a general *Volume=Multiplicity* formula for p-families of $\mathfrak{m}$-primary ideals. We begin with the following important lemma.

Lemma 5.1. [7, Lemma 5.21] *Let $(R, \mathfrak{m}, \mathbb{K})$ be a d-dimensional reduced local ring of positive characteristic, and $I_\bullet$ be a sequence of ideals of R indexed by the powers of p such that $\mathfrak{m}^{cq} \subseteq I_q$ for some positive integer c and for all $q = p^e, e \in \mathbb{N}$. Let $P_1, ..., P_n$ be the minimal primes of R, then there exists $\delta > 0$ such that for all q,*

$$\left| \sum_{i=1}^{n} \ell_{R_i}\left(R_i/I_q R_i\right) - \ell_R\left(R/I_q\right) \right| \leqslant \delta . q^{d-1}$$

where $R_i = R/P_i$, for all $i \in \{1, ..., n\}$

Remark 5.2. Suppose $(R, \mathfrak{m})$ be a d-dimensional reduced local ring of chracteristic $p > 0$ and I is an $\mathfrak{m}$-primary ideal. Since R is reduced and $I_\bullet = \{I^{[p^e]}\}_{e=0}^{\infty}$ is a p-family of $\mathfrak{m}$ primary ideals, using Lemma 5.1, Lemma 4.4 one can give an alternative proof of Remark 4.16.

The following is the Main Theorem of this paper.

Theorem 5.3. *Let $(R, \mathfrak{m}, \mathbb{K})$ be a d-dimensional local ring of characteristic $p > 0$. If the R-module dimension of the nilradical of $\hat{R}$ is less than d, then for any p-family of $\mathfrak{m}$-primary ideals $I_\bullet = \{I_q\}_{q=1}^{\infty}$, we have*

$$\lim_{q\to\infty} \frac{\ell_R(R/I_q)}{q^d} = \lim_{q\to\infty} \frac{e_{HK}(I_q, R)}{(q)^d}$$

Proof. **Step 1: The case of OK-Domain:**

Assume R is an OK-domain and let $\vartheta : \mathbb{F}^\times \to \mathbb{Z}^d$ be the valuation that is OK-relative to R with respect to a $\mathbb{Z}$-linear embedding of $\mathbb{Z}^d$ into $\mathbb{R}$ induced by a vector $\boldsymbol{a}$. Let $(V, \mathfrak{m}_\vartheta, \mathbb{K}_\vartheta)$ be the associated valuation ring, $S = \vartheta(R)$ be the corresponding semigroup in $\mathbb{Z}^d$ and $C = \mathrm{Cone}(S)$.
Let $\boldsymbol{u} \in \mathbb{Z}^d$; then we have the following two exact sequences:

$$0 \to \frac{I_q}{I_q \cap \mathbb{F}_{\geq \boldsymbol{u}}} \to \frac{R}{I_q \cap \mathbb{F}_{\geq \boldsymbol{u}}} \to \frac{R}{I_q} \to 0 \tag{5.1}$$

$$0 \to \frac{R \cap \mathbb{F}_{\geqslant \boldsymbol{u}}}{I_q \cap \mathbb{F}_{\geqslant \boldsymbol{u}}} \to \frac{R}{I_q \cap \mathbb{F}_{\geqslant \boldsymbol{u}}} \to \frac{R}{R \cap \mathbb{F}_{\geqslant \boldsymbol{u}}} \to 0 \tag{5.2}$$

From Equation 5.1, 5.2, we have

$$\ell_R\left(R/I_q\right) = \ell_R\left(\frac{R \cap \mathbb{F}_{\geqslant \boldsymbol{u}}}{I_q \cap \mathbb{F}_{\geqslant \boldsymbol{u}}}\right) + \ell_R\left(\frac{R}{R \cap \mathbb{F}_{\geqslant \boldsymbol{u}}}\right) - \ell_R\left(\frac{I_q}{I_q \cap \mathbb{F}_{\geqslant \boldsymbol{u}}}\right) \tag{5.3}$$

Let $\boldsymbol{v} \in S$ satisfy the last condition in Definition 3.1. Choose the number $c > 0$ from Lemma 4.4, define $\boldsymbol{w}$=$c\boldsymbol{v}$, then $R \cap \mathbb{F}_{\geqslant q\boldsymbol{w}} \subseteq \mathfrak{m}^{cq} \subseteq I_q$. Therefore

$$I_q \cap \mathbb{F}_{\geqslant q\boldsymbol{w}} = R \cap \mathbb{F}_{\geqslant q\boldsymbol{w}} \tag{5.4}$$

Let H be the halfspace $\{\boldsymbol{u}' \mid (\boldsymbol{a}, \boldsymbol{u}') < (\boldsymbol{a}, \boldsymbol{w})\}$. Then using Equation 5.3, 5.4 and Lemma 3.9 we have,

$$\ell_R\left(R/I_q\right) = \sum_{h=1}^{[\mathbb{K}_\vartheta:\mathbb{K}]} \#(\vartheta^h(R) \cap qH) - \sum_{h=1}^{[\mathbb{K}_\vartheta:\mathbb{K}]} \#(\vartheta^h(I_q) \cap qH) \tag{5.5}$$

Fix $q^{'}$ a power of p. I do the same calculation with respect to the p-system of ideals $J_\bullet = \{I_{q^{'}}^{[p^e]}\}_{e=0}^{\infty}$ then using Lemma 4.4 and Lemma 3.9, we have

$$\ell_R\left(R/I_{q^{'}}^{[p^e]}\right) = \sum_{h=1}^{[\mathbb{K}_\vartheta:\mathbb{K}]} \#(\vartheta^h(R)\cap p^e q^{'} H) - \sum_{h=1}^{[\mathbb{K}_\vartheta:\mathbb{K}]} \#(\vartheta^h(I_{q^{'}}^{[p^e]})\cap p^e q^{'} H) \tag{5.6}$$

Note that we can consider $J_q^{'} = R$ for all $q = p^e$. It is clearly a p-family of ideals. Therefore using Proposition 4.8 and Equation 5.5, we have

$$\lim_{q\to\infty} \frac{\ell_R(R/I_q)}{q^d} = [\mathbb{K}_\vartheta:\mathbb{K}].(\mathrm{Vol}_{\mathbb{R}^d}(\Delta(S,\vartheta(J_\bullet^{'})\cap H) - \mathrm{Vol}_{\mathbb{R}^d}(\Delta(S,\vartheta(I_\bullet)\cap H)) \tag{5.7}$$

Similarly applying Proposition 4.8 and Equation 5.6 for the p-system of ideals $J_\bullet = \{I_{q^{'}}^{[p^e]}\}_{e=0}^{\infty}$, we have

(5.8)

$$\lim_{e\to\infty} \frac{\ell_R\left(R/I_{q^{'}}^{[p^e]}\right)}{p^{ed}(q^{'})^d} = [\mathbb{K}_\vartheta:\mathbb{K}].(\mathrm{Vol}_{\mathbb{R}^d}(\Delta(S,\vartheta(J_\bullet^{'})\cap H) - \mathrm{Vol}_{\mathbb{R}^d}(\Delta(S,\vartheta(J_\bullet)\cap H))$$

Using Definition 3.12, we have

$$\Delta(S,\vartheta(J_\bullet)) = \bigcup_{e=0}^{\infty} \frac{1}{p^e}\vartheta\left(I_{q^{'}}^{[p^e]}\right) + \mathrm{Cone}(S). \tag{5.9}$$

Now $\vartheta\left(I_{q^{'}}^{[p^e]}\right) \subseteq \vartheta(I_{q^{'}p^e})$, as $\{I_q\}$ is a p-system of ideals and $\frac{1}{p^e}\vartheta\left(I_{q^{'}}^{[p^e]}\right) \subseteq \frac{1}{p^e}\vartheta(I_{q^{'}p^e}) \subseteq \frac{1}{p^e q^{'}}\vartheta(I_{q^{'}p^e})$. Therefore $\bigcup_{e=0}^{\infty}\frac{1}{p^e}\vartheta\left(I_{q^{'}}^{[p^e]}\right) + \mathrm{Cone}(S) \subseteq \bigcup_{e=0}^{\infty}\frac{1}{p^e q^{'}}\vartheta(I_{q^{'}p^e}) + \mathrm{Cone}(S) \subseteq \bigcup_{q=1}^{\infty}\frac{1}{q}\vartheta(I_q) + \mathrm{Cone}(S) = \Delta(S,\vartheta(I_\bullet)\cap\mathbb{H})$. From this, we conclude

$$\mathrm{Vol}_{\mathbb{R}^d}(\Delta(S,\vartheta(J_\bullet)\cap H)) \leqslant \mathrm{Vol}_{\mathbb{R}^d}(\Delta(S,\vartheta(I_\bullet)\cap H)) \tag{5.10}$$

Moreover $p^e\vartheta(I_{q^{'}}) + S \subseteq \vartheta(I_{q^{'}}^{[p^e]})$. Indeed for $\boldsymbol{s}_1 \in \vartheta(I_{q^{'}})$ and $\boldsymbol{s}_2 \in S$, we have $\vartheta(x) = \boldsymbol{s}_1$ for some $x \in I_{q^{'}}$ and $\vartheta(y) = \boldsymbol{s}_2$ for some $y \in R$. Therefore $p^e\boldsymbol{s}_1 + \boldsymbol{s}_2 = \vartheta(x^{[p^e]}\times y) =$ and $x^{[p^e]}\times y \in I_{q^{'}}^{[p^e]}$.

Using the observation above and Theorem 3.18, we have for any given $\epsilon > 0$, there exist q_0 such that for all $q^{'} \geqslant q_0$,

(5.11)

$$\mathrm{Vol}_{\mathbb{R}^d}(\Delta(S,\vartheta(I_\bullet)\cap H) - \epsilon \leqslant \lim_{e\to\infty}\frac{\#(p^e\vartheta(I_{q^{'}}) + S)\cap p^e q^{'} H)}{p^{ed}(q^{'})^d} \leqslant \lim_{e\to\infty}\frac{\#(\vartheta(I_{q^{'}}^{[p^e]}\cap p^e q^{'} H)}{p^{ed}(q^{'})^d}$$

Now using Equation 5.10 and 5.11 in Equation 5.7 and 5.8 we have for all $q^{'} \geqslant q_0$,

$$\lim_{q\to\infty}\frac{\ell_R(R/I_q)}{q^d} \leqslant \lim_{e\to\infty}\frac{\ell_R(R/I_{q^{'}}^{[p^e]})}{p^{ed}(q^{'})^d} = \frac{e_{HK}(I_{q^{'}},R)}{(q^{'})^d} \leqslant \lim_{q\to\infty}\frac{\ell_R(R/I_q)}{q^d} + \epsilon$$

Therefore letting $q^{'} \to \infty$, we obtain the result in OK-domain case.

Step 2: Reduction to the OK-domain case:

Since $\ell(R/J)$ is unaffected by completion for any $\mathfrak{m}$-primary ideal J, we may assume that R is complete.

Using [7, Corollary 5.18] and Corollary 4.12 by choosing $R = \hat{R}$ and $N = N(\hat{R})$

with the condition that $\dim N < d$ we can therefore assume that R is complete and reduced.
Now suppose that the minimal primes of the complete reduced ring R are $\{P_1, ..., P_n\}$. Let $R_i = R/P_i$ be complete local domain for all $1 \leqslant i \leqslant n$. By Remark 4.16, we have

$$\frac{e_{HK}(I_q, R)}{(q)^d} = \sum_{i=1}^{n} \frac{e_{HK}(I_q R_i, R_i)}{(q)^d} \tag{5.12}$$

We also have from Lemma 5.1

$$\lim_{q\to\infty} \frac{\ell_R(R/I_q)}{q^d} = \sum_{i=1}^{n} \lim_{q\to\infty} \frac{\ell_{R_i}(R_i/I_q R_i)}{q^d} \tag{5.13}$$

Since each R_i is a complete local domain and therefore an OK-domain by Example 3.5, using Equation 5.12 and 5.13 we have

$$\begin{aligned}
\lim_{q\to\infty} \frac{\ell_R(R/I_q)}{q^d} &= \sum_{i=1}^{n} \lim_{q\to\infty} \frac{\ell_{R_i}(R_i/I_q R_i)}{q^d} \\
&= \sum_{i=1}^{n} \lim_{q\to\infty} \frac{e_{HK}(I_q R_i, R_i)}{(q)^d} \\
&= \lim_{q\to\infty} \frac{e_{HK}(I_q, R)}{(q)^d}
\end{aligned}$$

□

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1290 Frenger Mall, Las Cruces, New Mexico 88003-8001
Email address: sudiptad@nmsu.edu