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Blowup algebras of determinantal ideals in prime characteristic

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In memory of Professor Wolmer Vasconcelos

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Abstract

We study when blowup algebras are F-split or strongly F-regular. Our main focus is on algebras given by symbolic and ordinary powers of ideals of minors of a generic matrix, a symmetric matrix, and a Hankel matrix. We also study ideals of Pfaffians of a skew-symmetric matrix. We use these results to obtain bounds on the degrees of the defining equations for these algebras. We also prove that the limit of the normalized regularity of the symbolic powers of these ideals exists and that their depth stabilizes. Finally, we show that, for determinantal ideals, there exists a monomial order for which taking initial ideals commutes with taking symbolic powers. To obtain these results, we develop the notion of F-split filtrations and symbolic F-split ideals.

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Contents

1 | INTRODUCTION

Let R be a Noetherian ring. A *filtration* $\mathbb{I} = \{I_n\}_{n\in\mathbb{Z}_{\geqslant 0}}$ is a sequence of ideals such that $I_0 = R$, $I_{n+1} \subseteq I_n$ for every $n \in \mathbb{Z}_{\geqslant 0}$, and $I_m I_n \subseteq I_{m+n}$ for every $n, m \in \mathbb{Z}_{\geqslant 0}$. Classical examples of filtrations include ordinary and symbolic powers. By taking the initial ideals of a filtration under a monomial order, one obtains a filtration of monomial ideals. Given a filtration, one can construct its Rees algebra $\mathscr{R}(\mathbb{I})$ and associated algebra $\operatorname{gr}(\mathbb{I})$. Notably, the Rees algebra of the ordinary powers of an ideal I gives the coordinate ring of the blowup of $\operatorname{Spec}(R)$ along the variety defined by I.

In this paper, we provide several results regarding ordinary and symbolic powers of determinantal ideals, and their Rees and associated graded algebras. Specifically, we study ideals of minors of generic, symmetric, and Hankel matrices of variables. We also study ideals of Pfaffians of a skew-symmetric matrix of variables. These objects have been intensively studied together with the varieties that they define, and they have connections with other areas of mathematics. For more information on this topic, we refer the interested reader to Bruns and Vetter's book [14], and to the more recent book of Bruns, Conca, Raicu, and Varbaro [11].

In what follows, $I_t(-)$ denotes the ideal generated by t-minors, and $P_{2t}(-)$ the ideal generated by $2t$-Pfaffians. In our first set of results, we show that the Rees and associated graded algebras of determinantal ideals have mild singularities from the perspective of Frobenius [48–51]. In particular, we show that several of them are strongly F-regular, or at least F-split. These singularities are regarded as the characteristic p analog of log-terminal and log-canonical singularities [39, 40, 68, 88–90] (see also [80]). We recall that strongly F-regular rings are Cohen–Macaulay and normal [48]. They are also simple as modules over their ring of differential operators [87]. We point out that the local cohomology modules of F-split rings satisfy desirable vanishing theorems [28, 54] and their defining ideals satisfy Harbourne's conjecture on symbolic powers [3, 36, 41]. In the following result, we denote by $\mathscr{R}(I)$ the Rees algebra corresponding to the ordinary powers of the ideal I, and by $\mathscr{R}^s(I)$ and $\operatorname{gr}^s(I)$ the Rees and associated graded algebras corresponding to the symbolic powers of the ideal I.

Theorem A. *Let K be an F-finite field of prime characteristic $p > 0$. Let X be a generic matrix, Y be a generic symmetric matrix, Z be a generic skew-symmetric matrix, and W be a generic Hankel matrix. For an integer $t > 0$, we have the following:*

(1) *$\mathscr{R}^s(I_t(X))$ and $\operatorname{gr}^s(I_t(X))$ are strongly F-regular (Theorem 6.7).*
(2) *If $p \gg 0$, then $\mathscr{R}(I_t(X))$ is F-split (Theorem 6.8).*
(3) *$\mathscr{R}^s(I_t(Y))$ and $\operatorname{gr}^s(I_t(Y))$ are F-split (Theorem 6.13).*
(4) *If $p \gg 0$, then $\mathscr{R}(I_t(Y))$ is F-split (Theorem 6.17).*
(5) *$\mathscr{R}^s(P_{2t}(Z))$ and $\operatorname{gr}^s(P_{2t}(Z))$ are strongly F-regular (Theorem 6.25).*
(6) *If $p \gg 0$, then $\mathscr{R}(P_t(Z))$ is F-split (Theorem 6.26).*
(7) *$\mathscr{R}^s(I_t(W))$ and $\operatorname{gr}^s(I_t(W))$ are F-split (Theorem 6.33).*
(8) *$\mathscr{R}(I_t(W))$ is F-split (Theorem 6.35).*

The wide variety of determinantal objects that we are able to cover in Theorem A highlights the fact that our techniques have a broad range of applications.

Note that, since $K[X]/I_t(X)$ and $K[Z]/P_t(Z)$ are direct summands of $\mathrm{gr}^s(I_t(X))$ and $\mathrm{gr}^s(P_{2t}(Z))$, respectively, Theorem A(1) and (5) imply the known results that $K[X]/I_t(X)$ and $K[Z]/P_t(Z)$ are strongly F-regular. In fact, the proofs of Theorem A(1) and (5) can be specialized to give alternative proofs for the strong F-regularity of $K[X]/I_t(X)$ and $K[Z]/P_t(Z)$.

Although it was already known that $\mathscr{R}^s(I_t(X))$ is F-rational [8], hence Cohen–Macaulay and normal, F-rationality does not imply that the ring is F-split. Therefore, Theorem A(1) improves this result, as strong F-regularity implies that the ring is both F-rational and F-split. Cohen–Macaulayness was also known for symbolic Rees algebras of ideals of Pfaffians of a generic skew-symmetric matrix [2], and this is now also a consequence of Theorem A(5). We point out that the new techniques we use to study F-singularities of blowup algebras are neither based on the theory of Sagbi bases [16, 63, 78] nor on that of straightening laws [8, 29]. We only invoke known results that use Sagbi bases in order to have that some blowup algebras we consider are Noetherian. Our strategy uses the new notion of F-split filtrations (Definition 4.2), classical methods in tight closure theory [48], and the choice of certain polynomials inspired by Seccia's work on Knutson ideals [81, 82].

Since all the Rees and associated graded algebras in Theorem A are F-split, their a-invariants are not positive [54]. As a consequence, we obtain bounds for the Castelnuovo–Mumford regularity and the degrees of the defining equations of such algebras; see Theorems 6.9, 6.10, 6.18, 6.19, 6.27, 6.28, 6.36, and 6.37. We point out that, even for monomial ideals, it was generally not known how to bound the degrees of the defining equations of these Rees algebras in terms of the generators of the ideal. Significant work has been done over the years in order to find such equations via different methods [34, 38, 55, 56, 64, 67, 70–72, 83, 84, 93, 96].

A related question is whether the limit of normalized Castelnuovo–Mumford regularities, $\lim_{n\to\infty} \frac{\mathrm{reg}(R/I^{(n)})}{n}$, always exists [45]. See also the work of Cutkosky on the subject [20]. Several authors have approached this question in a variety of cases; however, it remains widely open in general. Some classes of ideals for which this limit is known to exist are square-free monomial ideals [46] and ideals of small dimension [45]. We obtain this property for determinantal ideals in prime characteristic.

Theorem B. *Let K be an F-finite field of prime characteristic. Let X be a generic matrix, Y be a generic symmetric matrix, Z be a generic skew-symmetric matrix, and W be a generic Hankel matrix. For an integer $t > 0$, we have the following:*

(1) $\lim_{n\to\infty} \frac{\mathrm{reg}(K[X]/I_t(X)^{(n)})}{n}$ *exists (Theorem 6.6).*
(2) $\lim_{n\to\infty} \frac{\mathrm{reg}(K[Y]/I_t(Y)^{(n)})}{n}$ *exists (Theorem 6.15).*
(3) $\lim_{n\to\infty} \frac{\mathrm{reg}(K[Z]/P_{2t}(Z)^{(n)})}{n}$ *exists (Theorem 6.24).*
(4) $\lim_{n\to\infty} \frac{\mathrm{reg}(K[W]/I_t(W)^{(n)})}{n}$ *exists (Theorem 6.34).*

If the ground field is the field of complex numbers, there are linear formulas for $\mathrm{reg}(R/I^{(n)})$ when I is the ideal of t-minors of a generic matrix [76] or $2t$-Pfaffians of a generic skew-symmetric matrix [75]. These results were obtained using representation theory in characteristic zero. The case of ideals of t-minors of a generic matrix was recently further extended to fields of any characteristic [11]. It is worth mentioning that, in general, the function $\operatorname{reg}(R/I^{(n)})$ is not eventually linear, not even for square-free monomial ideals [30]. In particular, this linearity may fail even if the symbolic Rees algebra, $\mathscr{R}^s(I)$, is Noetherian.

We also obtain that the depth of symbolic powers of determinantal ideals stabilizes, and in some cases, we obtain the stable value. Our approach shows that the stable value equals the minimum of the depths among all the symbolic powers. This minimum value was already computed [14]; however, to the best of our knowledge, it was not shown that the stable and minimum values coincide.

Theorem C. *Let K be an F-finite field of prime characteristic. Let X be a generic matrix, Y be a generic symmetric matrix, Z be a generic skew-symmetric matrix, and W be a generic Hankel matrix. For an integer $t > 0$, we have the following:*

(1) $\lim_{n\to\infty} \operatorname{depth}(K[X]/I_t(X)^{(n)}) = t^2 - 1$ *(Theorem 6.6).*
(2) $\operatorname{depth}(K[Y]/I_t(Y)^{(n)})$ *stabilizes for $n \gg 0$ (Theorem 6.15).*
(3) $\lim_{n\to\infty} \operatorname{depth}(K[Z]/P_{2t}(Z)^{(n)}) = t(2t-1) - 1$ *(Theorem 6.24).*
(4) $\operatorname{depth}(K[W]/I_t(W)^{(n)})$ *stabilizes for $n \gg 0$ (Theorem 6.34).*

It is known that the initial ideals of the determinantal ideals treated in this work are radical with respect certain monomial orders (see Section 6.1). Then, it is natural to compare the initial ideal of their symbolic powers and the symbolic powers of their initial ideals. Sullivant showed that $\operatorname{in}_<(I^{(n)}) \subseteq \operatorname{in}_<(I)^{(n)}$ if K is algebraically closed and $\operatorname{in}_<(I)$ is radical [92]. In the case of ideals of minors of generic matrices [10], and of Hankel matrices of variables [15], not only the containment, but in fact equality is known to hold. As a consequence of the techniques introduced in this paper, we recover these results, and we also obtain equality in the case of Pfaffians.

Theorem D. *Let K be a perfect field of prime characteristic. Let X be a generic matrix, Z be a generic skew-symmetric matrix, and W be a generic Hankel matrix. In each case, let $<$ be the monomial order introduced in Section 6.1. For an integer $t > 0$, we have that*

(1) $\operatorname{gr}\left(\{\operatorname{in}_<\left(I_t(X)^{(n)}\right)\}_{n\in\mathbb{Z}_{\geqslant 0}}\right)$ *is F-split, therefore*

$$\operatorname{in}_<\left(I_t(X)^{(n)}\right) = \operatorname{in}_<(I_t(X))^{(n)}$$

for every $n \in \mathbb{Z}_{\geqslant 0}$ (Theorem 7.16).

(2) $\operatorname{gr}\left(\{\operatorname{in}_<\left(P_{2t}(Z)^{(n)}\right)\}_{n\in\mathbb{Z}_{\geqslant 0}}\right)$ *is F-split, therefore*

$$\operatorname{in}_<\left(P_{2t}(Z)^{(n)}\right) = \operatorname{in}_<(P_{2t}(Z))^{(n)}$$

for every $n \in \mathbb{Z}_{\geqslant 0}$ (Theorem 7.18).

(3) $\operatorname{gr}\left(\{\operatorname{in}_<\left(I_t(W)^{(n)}\right)\}_{n\in\mathbb{Z}_{\geqslant 0}}\right)$ *is F-split, therefore*

$$\operatorname{in}_<\left(I_t(W)^{(n)}\right) = \operatorname{in}_<(I_t(W))^{(n)}$$

for every $n \in \mathbb{Z}_{\geqslant 0}$ (Theorem 7.20).

If K is a field of characteristic zero, then we obtain the equalities $\mathrm{in}_<(I_t(X)^{(n)}) = \mathrm{in}_<(I_t(X))^{(n)}$, $\mathrm{in}_<(P_{2t}(Z)^{(n)}) = \mathrm{in}_<(P_{2t}(Z))^{(n)}$, and $\mathrm{in}_<(I_t(W)^{(n)}) = \mathrm{in}_<(I_t(W))^{(n)}$ for every $n \in \mathbb{Z}_{\geqslant 0}$ (see Corollaries 7.17, 7.19, and 7.21), via reduction to prime characteristic. We point out that we do not obtain analogous results for generic symmetric matrices Y because we do not know whether the Rees algebra associated to the filtration $\{\mathrm{in}_<(I_t(Y)^{(n)})\}_{n\in\mathbb{Z}_{\geqslant 0}}$ is Noetherian.

Theorem D allows us to find bounds for numerical invariants of determinantal ideals in terms of their initial ideals (see Remarks 7.23 and 7.24). We also obtain that the regularity and depth of the initial ideals of such determinantal ideals satisfy desirable properties (see Corollary 7.25). In this context, we provide new examples of existence of limits of normalized Castelnuovo–Mumford regularities for filtrations given by initial ideals. This is closely related to questions previously asked by Herzog, Hoa, and Trung [45].

We stress that our strategy to show Theorem D makes no use of the standard techniques employed before to obtain results about initial ideals of determinantal rings and their ordinary and symbolic powers [2, 10, 17, 24, 25, 91]. In particular, we use neither the straightening laws [29] nor the Knuth–Robinson–Schensted correspondence. Indeed, our techniques to prove Theorem D rely on methods in prime characteristic, and a test for the equality $\mathrm{in}_<(I^{(n)}) = \mathrm{in}_<(I)^{(n)}$ (see Theorem 7.9 and Corollary 7.10) inspired by the work of Huneke, Simis, and Vasconcelos [57].

Our main tool in this paper is our new notion of F-split filtration (Definition 4.2). If the F-split filtration is given by symbolic powers of an ideal, we say that the ideal is symbolic F-split (Definition 5.2). Ideals that are symbolic F-split produce symbolic Rees algebras and symbolic associated graded algebras that are F-split (see Theorem 4.7). As for the classical notion of F-purity, there exists a criterion that allows us to test when an ideal is symbolic F-split (see Theorem 5.8), which resembles the one given by Fedder [33]. We note that if an ideal is symbolic F-split, then its quotient ring is F-split. However, the converse is not true: In Example 5.13, we show that even strong F-regularity does not imply that the ideal is symbolic F-split. Examples of symbolic F-split ideals include square-free monomial ideals (see Example 5.11) and determinantal ideals (see Theorems 6.5, 6.13, 6.23, and 6.33). We refer to Corollary 5.10 and Example 5.16 for additional examples. Using these ideas, we are able to answer a question raised by Huneke[†] regarding F-König ideals (see Example 5.18), which arose in connection to the Conforti–Cornuéjols conjecture [18]. We also show that a-invariants and depths of symbolic F-split ideals have good behavior (see Proposition 4.9). In addition, there is a finite test to verify that their symbolic and ordinary powers coincide (see Theorem 5.7).

2 | NOTATIONS AND PRELIMINARIES

Throughout this paper, all rings are commutative with identity. We begin this section by recalling some notation and preliminary results that we use in the paper.

2.1 | Graded algebras

A $\mathbb{Z}_{\geqslant 0}$-graded ring is a ring A, which admits a direct sum decomposition $A = \bigoplus_{n\geqslant 0} A_n$ of Abelian groups, with $A_i \cdot A_j \subseteq A_{i+j}$ for all i and j.

[†] BIRS-CMO workshop on *Ordinary and Symbolic Powers of Ideals* Summer of 2017, Casa Matemática Oaxaca, Mexico.

Assume that A is a $\mathbb{Z}_{\geqslant 0}$-graded algebra over a Noetherian ring A_0 and let $M = \bigoplus_{n\in\mathbb{Z}} M_n$ and $N = \bigoplus_{n\in\mathbb{Z}} N_n$ be graded A-modules. An A-homomorphism $\varphi : M \to N$ is called *homogeneous of degree c* if $\varphi(M_n) \subseteq N_{n+c}$ for all $n \in \mathbb{Z}$. The set of all graded homomorphisms $M \to N$ of all degrees form a graded submodule of $\mathrm{Hom}_R(M,N)$. In general, these two modules are not the same, but they coincide when M is finitely generated [12].

Given a Noetherian $\mathbb{Z}_{\geqslant 0}$-graded algebra A, there exist $f_1, \dots, f_r \in A$ homogeneous elements such that $A = A_0[f_1, \dots, f_r]$, which is equivalent to $\oplus_{n>0} A_n = (f_1, \dots, f_r)$ [12, Proposition 1.5.4]. Therefore, if A_0 is local, or $\mathbb{Z}_{\geqslant 0}$-graded over a field, there is a minimal set of integers $d_1, \dots, d_r$ such that there exist such $f_1, \dots, f_r$ of degree $d_1, \dots, d_r$, respectively. We call these numbers the *generating degrees of A as A_0-algebra*.

Let $S = A_0[y_1, \dots, y_r]$ be a polynomial ring over A_0 with $\deg(y_i) = d_i$ for $1 \leqslant i \leqslant r$, and let $\phi : S \to A$ be an A_0-algebra homomorphism defined by $\phi(y_i) = f_i$ for $1 \leqslant i \leqslant r$. Consider the ideal $\mathcal{J} = \mathrm{Ker}(\phi)$. We call any minimal set of homogeneous generators of $\mathcal{J}$ the *defining equations of A over A_0*.

2.2 | Methods in prime characteristic

In this subsection, we assume that A is reduced and that it has prime characteristic $p > 0$. For $e \in \mathbb{Z}_{\geqslant 0}$, let $F^e : A \to A$ denote the e-th iteration of the Frobenius endomorphism on A. If A^{1/p^e} denotes the ring of p^e-th roots of A taken in the total field of fractions of A, we can identify F^e with the natural inclusion $\iota : A \hookrightarrow A^{1/p^e}$. Throughout this paper, any A-linear map $\phi : A^{1/p^e} \to A$ such that $\phi \circ \iota = \mathrm{id}_A$ is called a splitting of Frobenius, or just a splitting.

Given an A-module M, we let M^{1/p^e} denote the A-module, which has the same additive structure as M and scalar multiplication defined by $a \cdot m^{1/p^e} := (a^{p^e} m)^{1/p^e}$, for all $a \in A$ and $m^{1/p^e} \in M^{1/p^e}$.

For an ideal I generated by $\{f_1, \dots, f_u\}$, we denote by $I^{[p^e]}$ the ideal generated by $\{f_1^{p^e}, \dots, f_u^{p^e}\}$. We note that $IA^{1/p^e} = (I^{[p^e]})^{1/p^e}$.

In the case in which $A = \oplus_{n\geqslant 0} A_n$ is $\mathbb{Z}_{\geqslant 0}$-graded, we can view A^{1/p^e} as a $\frac{1}{p^e}\mathbb{Z}_{\geqslant 0}$-graded module in the following way: We write $f \in A$ as $f = f_{d_1} + \dots + f_{d_n}$, with $f_{d_j} \in A_{d_j}$. Then, $f^{1/p^e} = f_{d_1}^{1/p^e} + \dots + f_{d_n}^{1/p^e}$, where each $f_{d_j}^{1/p^e}$ has degree d_j/p^e. Similarly, if M is a $\mathbb{Z}$-graded A-module, we have that M^{1/p^e} is a $\frac{1}{p^e}\mathbb{Z}$-graded A-module. As a submodule of A^{1/p^e}, A inherits a natural $\frac{1}{p^e}\mathbb{Z}_{\geqslant 0}$ grading, which is compatible with its original grading. In other words, if $f \in A$ is homogeneous of degree d with respect to its original grading, then it has degree $d = dp^e/p^e$ with respect to the inherited $\frac{1}{p^e}\mathbb{Z}_{\geqslant 0}$ grading.

Definition 2.1. Let A be a Noetherian ring of positive characteristic p. We say that A is *F-finite* if it is a finitely generated A-module via the action induced by the Frobenius endomorphism $F : A \to A$ or, equivalently, if $A^{1/p}$ is a finitely generated A-module. If $(A, \mathfrak{m}, K)$ is a $\mathbb{Z}_{\geqslant 0}$-graded K-algebra, then A is F-finite if and only if K is F-finite, that is, if and only if $[K : K^p] < \infty$. A ring A is called *F-pure* if F is a pure homomorphism, that is, if and only if the map $A \otimes_A M \to A^{1/p} \otimes_A M$ induced by the inclusion ι is injective for all A-modules M. A ring A is called *F-split* if F is a split monomorphism. Finally, an F-finite ring A is called *strongly F-regular* if for every $c \in A$ not in any minimal prime, the map $A \to A^{1/p^e}$ sending $1 \mapsto c^{1/p^e}$ splits for some (equivalently, all) $e \gg 0$.

Remark 2.2. We have that A is F-split if and only if A is a direct summand of $A^{1/p}$. If A is an F-finite ring, then A is F-pure if and only if A is F-split [54, Corollary 5.3].

Remark 2.3. Assume A is an F-finite regular local ring, or a polynomial ring over an F-finite field, then $\operatorname{Hom}_A(A^{1/p^e}, A)$ is a free A^{1/p^e}-module [33, Lemma 1.6]. If Φ is a generator (homogeneous in the graded case) of this module as an A^{1/p^e}-module, then for ideals $I, J \subset A$ we have that the map $\phi := f^{1/p^e} \cdot \Phi = \Phi(f^{1/p^e} -)$ satisfies $\phi\left(J^{1/p^e}\right) \subseteq I$ if and only if $f^{1/p^e} \in \left(IA^{1/p^e} :_{A^{1/p^e}} J^{1/p^e}\right)$ or, equivalently, $f \in \left(I^{[p^e]} :_A J\right)$ [33, Proposition 1.6]. In particular, ϕ is surjective if and only if $f^{1/p^e} \notin \mathfrak{m}A^{1/p^e}$, that is, $f \notin \mathfrak{m}^{[p^e]}$.

Now, assume $A = K[x_1, \dots, x_d]$ is a polynomial ring and $\gamma : K^{1/p^e} \to K$ is a splitting. Let $\Phi : A^{1/p^e} \to A$ be the A-linear map defined by

$$\Phi\left(c^{1/p^e} x_1^{\alpha_1/p^e} \cdots x_d^{\alpha_d/p^e}\right) = \begin{cases} \gamma\left(c^{1/p^e}\right) x_1^{(\alpha_1 - p^e + 1)/p^e} \cdots x_d^{(\alpha_d - p^e + 1)/p^e} & \text{if } p^e | (\alpha_i - p^e + 1)\ \forall i, \\ 0 & \text{otherwise.} \end{cases}$$

We have that Φ is a generator of $\operatorname{Hom}_A(A^{1/p^e}, A)$ as an A^{1/p^e}-module [6, p. 22]. The map Φ is often called the *trace map* of A. We point out that, if K is not perfect, Φ depends on γ, but this is usually omitted from the notation.

2.3 | Local cohomology and Castelnuovo–Mumford regularity

For an ideal $I \subseteq A$, we define the *i-th local cohomology of M with support in I* as $H_I^i(M) := H^i(\check{C}^\bullet(\underline{f}; A) \otimes_A M)$, where $\check{C}^\bullet(\underline{f}; A)$ is the Čech complex on a set of generators $\underline{f} = f_1, \dots, f_\ell$ of I. We note that $H_I^i(M)$ does not depend on the choice of generators of I. Moreover, it only depends on the radical of I. We recall that the i-th local cohomology functor $H_I^i(-)$ can also be defined as the i-th right derived functor of $\Gamma_I(-)$, where $\Gamma_I(M) = \{m \in M \mid I^n m = 0$ for some $n \in \mathbb{Z}_{\geqslant 0}\}$. If $I = \mathfrak{m}$ is a maximal ideal and M is finitely generated, then $H_{\mathfrak{m}}^i(M)$ is Artinian.

If $M = \bigoplus_{\frac{n}{p^e} \in \frac{1}{p^e}\mathbb{Z}} M_{\frac{n}{p^e}}$ is a $\frac{1}{p^e}\mathbb{Z}_{\geqslant 0}$-graded R-module, and we let $A_+ = \bigoplus_{n>0} A_n$, then $H_{A_+}^i(M)$ is a $\frac{1}{p^e}\mathbb{Z}$-graded A-module. Moreover, $[H_{A_+}^i(M)]_{\frac{n}{p^e}}$ is a finitely generated A_0-module for every $n \in \mathbb{Z}$, and $H_{A_+}^i(M)_{\frac{n}{p^e}} = 0$ for $n \gg 0$ [7, Theorem 16.1.5]. We define the *a_i-invariant of M* as

$$a_i(M) = \max\left\{ \frac{n}{p^e} \,\middle|\, [H_{A_+}^i(M)]_{\frac{n}{p^e}} \neq 0 \right\}$$

if $H_{A_+}^i(M) \neq 0$, and $a_i(M) = -\infty$ otherwise.

Remark 2.4. Given a finitely generated $\mathbb{Z}$-graded A-module M, we have $a_i(M^{1/p^e}) = a_i(M)/p^e$ for all $i \in \mathbb{Z}_{\geqslant 0}$. In fact, $H_{A_+}^i(M^{1/p^e}) \cong H_{A_+}^i(M)^{1/p^e}$ since the functor $(-)^{1/p^e}$ is exact.

Remark 2.5. If A is an F-split $\mathbb{Z}_{\geqslant 0}$-graded ring, then $a_i(A) \leqslant 0$ for all $i \in \mathbb{Z}$ [54, Lemma 2.3].

Given a finitely generated $\mathbb{Z}$-graded A-module, the *Castelnuovo–Mumford regularity of* M is defined as

$$\operatorname{reg}(M) = \max\{a_i(M) + i \mid i \in \mathbb{Z}_{\geqslant 0}\}.$$

Remark 2.6. If $A = A_0[x_1, \dots, x_r]$ is a polynomial ring over A_0, such that x_i has degree $d_i > 0$ for every $1 \leqslant i \leqslant r$, then $\operatorname{reg}(A) = r - \sum_{i=1}^{r} d_i$.

2.4 | Filtrations and blowup algebras

Let R be a commutative ring. We say that a sequence of ideals $\{I_n\}_{n\in\mathbb{Z}_{\geqslant 0}}$ of R is a *filtration* if $I_0 = R$, $I_{n+1} \subseteq I_n$ for every $n \in \mathbb{Z}_{\geqslant 0}$, and $I_n I_m \subseteq I_{n+m}$ for every $n, m \in \mathbb{Z}_{\geqslant 0}$.

Definition 2.7. Let R be a ring. Consider the following graded algebras associated to a filtration $\mathbb{I} = \{I_n\}_{n\in\mathbb{Z}_{\geqslant 0}}$:

(i) The the *Rees algebra* of $\mathbb{I}$: $\mathscr{R}(\mathbb{I}) = \bigoplus_{n\in\mathbb{Z}_{\geqslant 0}} I_n T^n \subseteq R[T]$, where T is a variable.
(ii) The *associated graded algebra* of $\mathbb{I}$: $\operatorname{gr}(\mathbb{I}) = \bigoplus_{n\in\mathbb{Z}_{\geqslant 0}} I_n/I_{n+1}$.

We generally refer to the above as the blowup algebras associated to the filtration $\mathbb{I}$ [97].

If the Rees algebra is Noetherian, we can compute the dimensions of the blowup algebras. We show this in the next proposition.

Proposition 2.8. *Assume that the ideal I_1 has positive height and that $\mathscr{R}(\mathbb{I})$ is finitely generated as an R-algebra. Then,* $\dim(\mathscr{R}(\mathbb{I})) = \dim(R) + 1$ *and* $\dim(\operatorname{gr}(\mathbb{I})) = \dim(R)$.

Proof. Consider the extended Rees algebra $B := R[\mathbb{I}T, T^{-}1] = \oplus_{n\in\mathbb{Z}} I_n T^n$, where $I_n = R$ for $n \leqslant 0$. Since $\mathscr{R}(\mathbb{I})$ is Noetherian, there exists $\ell \in \mathbb{Z}_{>0}$ such that

$$I_{n+\ell} = I_\ell I_n \text{ for every } n \geqslant \ell \;\; [77, \text{Remark } 2.4.3]. \tag{2.4.1}$$

Thus, B is an integral extension of $R[I_\ell T, T^{-1}] = \oplus_{n\in\mathbb{Z}} I_\ell^n T^n$, and $\mathscr{R}(\mathbb{I})$ is an integral extension of $R[I_\ell T] = \oplus_{n\in\mathbb{Z}} I_\ell^n T^n$, and hence they both have dimension $\dim(R) + 1$ [58, Theorem 2.2.5, Theorem 5.1.4(1)(2)]. Now, T^{-1} is a homogeneous regular element of B, thus $B/(T^{-1}) \cong \operatorname{gr}(\mathbb{I})$ has dimension $\dim(R)$, finishing the proof. □

3 | GENERATORS OF DEFINING EQUATIONS OF *F*-SPLIT BLOWUP ALGEBRAS

This section is devoted to find bounds for the degrees of the defining equations of the algebras introduced in Definition 2.7 when they are F-split. The main results of this section are Theorems 3.3 and 3.4.

Let $A = \oplus_{n\geqslant 0} A_n$ be a $\mathbb{Z}_{\geqslant 0}$-graded Noetherian ring. Given a finitely generated graded A-module $M = \oplus_{n\in\mathbb{Z}} M_n$, we let

$$\beta_A(M) = \inf\{i \mid M = A \cdot (\oplus_{n\leqslant i} M_n)\},$$

that is, the largest degree of a minimal homogeneous generator of M. The following lemma states an upper bound on $\beta_A(M)$ in terms of the Castelnuovo–Mumford regularity. This statement can be found in the literature when A is generated as an algebra in degree one [7, Theorem 16.3.1], or when A_0 is a field [22, Theorem 3.5] [21, Theorem 2.2]. While the same result in our setup may be well known to experts, we could not find a reference in the literature. We include its proof here for the sake of completeness.

Proposition 3.1. *Let $A = \oplus_{n \geqslant 0} A_n$ be a $\mathbb{Z}_{\geqslant 0}$-graded Noetherian ring. Let $d_1, \ldots, d_r > 0$ be the generating degrees of A as an A_0-algebra. Let M be a finitely generated $\mathbb{Z}$-graded A-module. Then, $\beta_A(M) \leqslant \operatorname{reg}(M) + \sum_{i=1}^r (d_i - 1)$.*

Proof. Let $f_1, \ldots, f_r$ be homogeneous generators of A as an A_0-algebra of degree $d_1, \ldots, d_r$, respectively. Without loss of generality, we may assume $1 \leqslant d_1 \leqslant \ldots \leqslant d_r$. Observe $A_+ = \bigoplus_{n>0} A_n = (f_1, \ldots, f_r)$.

We now proceed by induction on $\sum_{i=1}^r d_i \geqslant r$. The base case $d_i = 1$ for all $1 \leqslant i \leqslant r$ is known [7, Theorem 16.3.1]. Let $A' = A[y]$, where $\deg(y) = 1$ and $M' = M \otimes_A A'$. Since y is regular on M', a standard argument via the long exact sequence of local cohomology of

$$0 \to M'(-1) \xrightarrow{y} M' \to M'/yM \cong M \to 0$$

shows $\operatorname{reg}(M) = \operatorname{reg}(M')$, where the regularity of M' is computed with respect to the ideal ${A_+}' = \oplus_{n>0} A'_n$. We observe that $f = f_r - y^{d_r}$ is a homogeneous element of degree d_r, which is regular on M'. The short exact sequence

$$0 \to M'(-d_r) \xrightarrow{f} M' \to M'/fM' \to 0$$

gives $\operatorname{reg}(M'/fM') \leqslant \max\{\operatorname{reg}(M') + d_r - 1, \operatorname{reg}(M')\} = \operatorname{reg}(M') + d_r - 1 \leqslant \operatorname{reg}(M) + d_r - 1$. Note that M'/fM' is an A'/fA'-module and that $A'/fA' \cong A_0[f_1, \ldots, f_{r-1}, y]$. Since $\sum_{i=1}^{r-1} d_i + 1 < \sum_{i=1}^r d_i$, by induction, we have

$$\beta_{A'/fA'}(M'/fM') \leqslant \operatorname{reg}(M'/fM') + \sum_{i=1}^{r-1}(d_i - 1) \leqslant \operatorname{reg}(M) + \sum_{i=1}^{r}(d_i - 1).$$

Let N be the A'-submodule of M' generated by elements of degree at most $\operatorname{reg}(M) + \sum_{i=1}^r (d_i - 1)$. We have just shown that $M' = N + fM'$, and therefore $M' = N + \mathfrak{m}'M'$, where $\mathfrak{m}' = \mathfrak{m}_0 + {A_+}'$. Thus, from the graded Nakayama's lemma, it follows that $M' = N$. In particular, $\beta_{A'}(M') \leqslant \operatorname{reg}(M) + \sum_{i=1}^r (d_i - 1)$. Since $\beta_{A'}(M') = \beta_A(M)$, the proof is complete. □

We need one more lemma before stating the main result of this section.

Lemma 3.2. *Let $A = \oplus_{n \geqslant 0} A_n$ be a Noetherian F-finite and F-split graded ring. Let $d_1, \ldots, d_r$ be the generating degrees of A as an A_0-algebra. Then the defining equations of A over A_0 have degree at most*

$$\dim(A) + \sum_{i=1}^r d_i - \max\{\dim(A), r\}.$$

Proof. Let $f_1, \dots, f_r$ be homogeneous generators of A as an A_0-algebra of degree $d_1, \dots, d_r$, respectively. Let $S = A_0[y_1, \dots, y_r]$ be a polynomial ring over A_0 with $\deg(y_i) = d_i$ for $1 \leqslant i \leqslant r$, and let $\phi : S \to A$ be the graded A_0-algebra homomorphism defined by $\phi(y_i) = f_i$ for $1 \leqslant i \leqslant r$. Let $\mathcal{J} = \operatorname{Ker}(\phi)$.

Set $S_+ = (y_1, \dots, y_r) \subseteq S$ and consider the homogeneous short exact sequence:

$$0 \to \mathcal{J} \to S \to A \to 0.$$

From the long exact sequence of local cohomology modules with support in S_+, we obtain $H^i_{S_+}(A) \cong H^{i+1}_{S_+}(\mathcal{J})$ for $i \leqslant r-2$, and an exact sequence

$$0 \to H^{r-1}_{v}(A) \to H^r_{S_+}(\mathcal{J}) \to H^r_{S_+}(S) \to H^r_{S_+}(A) \to 0.$$

Since $a_i(A) \leqslant 0$ for every $i \in \mathbb{Z}_{\geqslant 0}$ by Remark 2.5, and $a_r(S) = -\sum_{i=1}^r d_i$ by Remark 2.6, we have $a_i(\mathcal{J}) \leqslant 0$ for every $i \in \mathbb{Z}_{\geqslant 0}$. Thus,

$$\operatorname{reg}(\mathcal{J}) = \max\{a_i(\mathcal{J}) + i\} \leqslant \min\{r, \dim(A)\},$$

as $a_i(\mathcal{J}) = -\infty$ for $i > \min\{r, \dim(A)\}$ [7, Theorems 3.3.1 and 6.1.2]. The result now follows by Proposition 3.1, after performing some easy calculations. □

The following is the main theorem of this section, as it provides bounds for the degrees of generators of defining equations of F-split blowup algebras.

Theorem 3.3. *Let R be a Noetherian F-finite and F-split ring of characteristic $p > 0$. Let $\mathbb{I} = \{I_n\}_{n \in \mathbb{Z}_{\geqslant 0}}$ be a filtration such that $\mathcal{R}(\mathbb{I})$ is a finitely generated F-split R-algebra. Let $e_1, \dots, e_\ell$ be the generating degrees of $\mathcal{R}(\mathbb{I})$ as an R-algebra, that is, $\mathcal{R}(\mathbb{I}) = R[I_{e_1}T^{e_1}, \dots, I_{e_\ell}T^{e_\ell}]$, and let $v_1, \dots, v_\ell$ be the number of generators of $I_1, \dots, I_\ell$, respectively. Further assume that I_1 has positive height. The defining equations of $\mathcal{R}(\mathbb{I}) = \bigoplus_{n \geqslant 0} I_n T^n$ over R have degree at most*

$$\dim(R) + 1 + \sum_{i=1}^{\ell} e_i v_i - \max\left\{\dim(R) + 1, \sum_{i=1}^{\ell} v_i\right\}.$$

Moreover, if $\operatorname{gr}(\mathbb{I})$ is F-split, then the defining equations of $\operatorname{gr}(\mathbb{I}) = \bigoplus_{n \geqslant 0} I_n/I_{n+1}$ over R/I_1 have degree at most

$$\dim(R) + \sum_{i=1}^{\ell} e_i v_i - \max\left\{\dim(R), \sum_{i=1}^{\ell} v_i\right\}.$$

Proof. This follows from Lemma 3.2 and Proposition 2.8. □

Theorem 3.4. *Let K be an F-finite field, and R be an F-split graded K-algebra, generated over K by μ elements of degree one. Let $\mathbb{I} = \{I_n\}_{n \in \mathbb{Z}_{\geqslant 0}}$ be a filtration such that $\mathcal{R}(\mathbb{I})$ is a finitely generated F-split R-algebra. Let $e_1, \dots, e_\ell$ be the generating degrees of $\mathcal{R}(\mathbb{I})$ as an R-algebra, that is, $\mathcal{R}(\mathbb{I}) = R[I_{e_1}T^{e_1}, \dots, I_{e_\ell}T^{e_\ell}]$. Set $w_i = \beta_R(I_{e_i})$ for $1 \leqslant i \leqslant \ell$ and let $\nu_1, \dots, \nu_\ell$ be the number of generators of*

$I_1, \dots, I_\ell$, respectively. Further assume that I_1 has positive height. The defining equations of $\mathscr{R}(\mathbb{I})$ over K have total degree at most

$$\dim(R) + 1 + \mu + \sum_{i=1}^{\ell} \nu_i(w_i + e_i) - \max\left\{\dim(R) + 1, \mu + \sum_{i=1}^{\ell} v_i\right\}.$$

Moreover, if $\operatorname{gr}(\mathbb{I})$ is F-split, then the defining equations of $\operatorname{gr}(\mathbb{I})$ over K have total degree at most

$$\dim(R) + \mu + \sum_{i=1}^{\ell} \nu_i(w_i + e_i) - \max\left\{\dim(R), \mu + \sum_{i=1}^{\ell} v_i\right\}.$$

Proof. Both parts of the result follow from Lemma 3.2 and Proposition 2.8. □

4 | *F*-SPLIT FILTRATIONS

Throughout this section, we assume the following setup.

Setup 4.1. Let R be a Noetherian F-finite and F-split ring of characteristic $p > 0$, which is either local or $\mathbb{Z}_{\geqslant 0}$-graded. In the local case, we let $\mathfrak{m}$ denote its unique maximal ideal, and $K = R/\mathfrak{m}$ its residue field. In the graded case, we assume $R = \bigoplus_{n\geqslant 0} R_0$ is a finitely generated R_0-algebra, where $(R_0, \mathfrak{m}_0)$ is a local ring. We let $R_+ = \bigoplus_{n>0} R_n$ and $\mathfrak{m} = \mathfrak{m}_0 + R_+$. We further assume that R is generated in degree one, that is, $R = R_0[R_1]$. In the graded case, every object we consider is homogeneous with respect to the given grading.

4.1 | *F*-split filtrations of ideals

We introduce the main object of study of this paper: *F-split filtrations*. For a related notion in the case of ordinary powers, see [65].

Definition 4.2. Assume Setup 4.1. We say that a sequence of R-ideals $\mathbb{I} = \{I_n\}_{n\in\mathbb{Z}_{\geqslant 0}}$ is an *F-split filtration* if $I_0 = R$, $I_{n+1} \subseteq I_n$ for every $n \in \mathbb{Z}_{\geqslant 0}$, $I_n I_m \subseteq I_{n+m}$ for every $n, m \in \mathbb{Z}_{\geqslant 0}$, and there exists a splitting $\phi : R^{1/p} \to R$ such that $\phi\big((I_{np+1})^{1/p}\big) \subseteq I_{n+1}$ for every $n \in \mathbb{Z}_{\geqslant 0}$.

We now study properties regarding F-splittings, depth, and regularity for ideals appearing in these filtrations (see Theorem 4.7 and Theorem 4.10). In particular, we will show that F-split filtrations yield F-split blowup algebras.

Remark 4.3. Suppose $\phi : R^{1/p} \to R$ is a surjective map such that $\phi\big((I_{np+1})^{1/p}\big) \subseteq I_{n+1}$ for every $n \in \mathbb{Z}_{\geqslant 0}$. Let $g \in R$ such that $\phi(g^{1/p}) = 1$. Then, $\varphi(-) = \phi(g^{1/p}-)$ induces a splitting such that $\varphi\big((I_{np+1})^{1/p}\big) \subseteq I_{n+1}$ for every $n \in \mathbb{Z}_{\geqslant 0}$. Then, it suffices to assume that ϕ is surjective in Definition 4.2.

Remark 4.4. We observe that if $\mathbb{I} = \{I_n\}_{n\in\mathbb{Z}_{\geqslant 0}}$ is an F-split filtration, then R/I_1 is F-split. In fact, by considering $n = 0$ in Definition 4.2, one gets an induced splitting $\phi : (R/I_1)^{1/p} \to R/I_1$. In particular, I_1 is a radical ideal.

Proposition 4.5. *Assume Setup 4.1 and let* $\mathbb{I} = \{I_n\}_{n\in\mathbb{Z}_{\geqslant 0}}$ *be a filtration. The following statements are equivalent:*

(1) $\mathbb{I}$ *is an* F*-split filtration.*
(2) *There exists a splitting* $\phi_e : R^{1/p^e} \to R$ *such that* $\phi_e\big((I_{np^e+s})^{1/p^e}\big) = I_{n+1}$ *for every* $e \in \mathbb{Z}_{>0}$*,* $n \in \mathbb{Z}_{\geqslant 0}$*, and* $s \in \mathbb{Z}$ *such that* $1 \leqslant s \leqslant p^e$*.*
(3) *There exists a splitting* $\phi_e : R^{1/p^e} \to R$ *such that* $\phi_e\big((I_{np^e+1})^{1/p^e}\big) \subseteq I_{n+1}$ *for some* $e \in \mathbb{Z}_{>0}$ *and every* $n \in \mathbb{Z}_{\geqslant 0}$*.*

Proof. We consider the implication (1) ⇒ (2). Let ϕ be as in Definition 4.2. For every $j > 0$, we consider the R-linear map $\varphi_j : R^{1/p^j} \to R^{1/p^{j-1}}$ defined as $\varphi_j(r^{1/p^j}) = (\phi(r^{1/p}))^{1/p^{j-1}}$. We observe that $\varphi_j\big((I_{np^j+s})^{1/p^j}\big) \subseteq \varphi_j\big((I_{np^j+1})^{1/p^j}\big) \subseteq (I_{np^{j-1}+1})^{1/p^{j-1}}$ for every $n \in \mathbb{Z}_{\geqslant 0}$ and $j, s \in \mathbb{Z}_{>0}$. Then, we have

$$\varphi_1\circ\varphi_2\circ\cdots\circ\varphi_e\Big((I_{np^e+s})^{1/p^e}\Big) \subseteq I_{n+1}$$

for every $e > 0$, $n \geqslant 0$, and $s > 0$. Set $\phi_e := \varphi_1\circ\varphi_2\circ\cdots\circ\varphi_e$. It remains to show $I_{n+1} \subseteq \phi_e\big((I_{np^e+s})^{1/p^e}\big)$ for $s \leqslant p^e$. But this inclusion follows by noticing that

$$I_{n+1} \subseteq \phi_e\Big(I_{n+1}R^{1/p^e}\Big) \subseteq \phi_e\Big((I_{np^e+s})^{1/p^e}\Big)$$

for $s \leqslant p^e$.

Since (2) ⇒ (3) is clear, it remains to show the implication (3) ⇒ (1). We consider the natural inclusion $\iota : R^{1/p} \to R^{1/p^e}$ and set $\phi := \phi_e\circ\iota$. We note that, $\iota\big((I_{np+1})^{1/p}\big) \subseteq (I_{np^e+p^{e-1}})^{1/p^e} \subseteq (I_{np^e+1})^{1/p^e}$. As a consequence, we have

$$\phi\Big((I_{np+1})^{1/p}\Big) \subseteq \phi_e\Big((I_{np^e+1})^{1/p^e}\Big) \subseteq I_{n+1}$$

for every $n \in \mathbb{Z}_{\geqslant 0}$, and the result follows. □

For ideals in a regular ring R, we state an effective criterion for F-split filtrations analogous to the classical one by Fedder [33].

Proposition 4.6. *Assume Setup 4.1 with R regular. In the graded case, we further assume that* R_0 *is a field, so that* $\mathfrak{m} = R_+$*. We have that* $\mathbb{I} = \{I_n\}_{n\in\mathbb{Z}_{\geqslant 0}}$ *is an* F*-split filtration if and only*

$$\bigcap_{n\in\mathbb{Z}_{\geqslant 0}} \Big((I_{n+1})^{[p]} :_R I_{np+1}\Big) \not\subseteq \mathfrak{m}^{[p]}.$$

Proof. Since R is regular, we can pick the trace Φ, which is a generator of $\operatorname{Hom}_R(R^{1/p}, R)$ as a free $R^{1/p}$-module described, see Remark 2.3. Then, for $f \in R$ and $\phi := f^{1/p}\cdot\Phi = \Phi(f^{1/p}-)$, we have $\phi\big((I_{np+1})^{1/p}\big) \subseteq I_{n+1}$ for every $n \in \mathbb{Z}_{\geqslant 0}$ if and only if

$$f \in \bigcap_{n\in\mathbb{Z}_{\geqslant 0}} \Big((I_{n+1})^{[p]} :_R I_{np+1}\Big),$$

by Remark 2.3. In addition, ϕ is surjective if and only if $f \notin \mathfrak{m}^{[p]}$, and the result follows. □

4.2 | F-split blowup algebras

In this subsection, we obtain the first significant property for F-split filtrations. In the following theorem, we prove that if $\mathbb{I}$ is an F-split filtration, then the algebras in Definition 2.7 are F-split. This is one of the main motivations for introducing F-split filtrations.

Theorem 4.7. *Assume Setup 4.1. If $\mathbb{I}$ is an F-split filtration, then $\mathscr{R}(\mathbb{I})$ and $\operatorname{gr}(\mathbb{I})$ are F-split.*

Proof. Let ϕ be such that $\mathbb{I}$ is F-split with respect to ϕ, see Definition 4.2. We note that $\mathscr{R}(\mathbb{I})$ is reduced. Then, we can consider the ring of p-th roots $\mathscr{R}(\mathbb{I})^{1/p} = \bigoplus_{n\in\mathbb{Z}_{\geqslant 0}}(I_n)^{1/p}T^{n/p}$. We define $\varphi : \mathscr{R}(\mathbb{I})^{1/p} \to \mathscr{R}(\mathbb{I})$ as the homogeneous homomorphism of $\mathscr{R}(\mathbb{I})$-modules induced by $\varphi(r^{1/p}T^{n/p}) = \phi(r^{1/p})T^{n/p}$ if p divides n, and $\varphi(r^{1/p}Tn/p) = 0$ otherwise. The map φ is well defined because

$$\phi\left((I_{(n+1)p})^{1/p}\right) \subseteq \phi\left((I_{np+1})^{1/p}\right) \subseteq I_{n+1}$$

for every $n \in \mathbb{Z}_{\geqslant 0}$, and it is $\mathscr{R}(\mathbb{I})$-linear since ϕ is R-linear. If $r \in I_n \subseteq (I_{np})^{1/p}$, then $\varphi(rt^{np/p}) = \phi(r)T^n = rT^n$ because ϕ is a splitting. We conclude that φ is a splitting of the inclusion $\mathscr{R}(\mathbb{I}) \to \mathscr{R}(\mathbb{I})^{1/p}$, and hence $\mathscr{R}(\mathbb{I})$ is F-split.

Consider the ideal $\mathscr{J} = \bigoplus_{n\in\mathbb{Z}_{\geqslant 0}} I_{n+1}T^n \subseteq \mathscr{R}(\mathbb{I})$. Since $\phi\left((I_{np+1})^{1/p}\right) \subseteq I_{n+1}$ for every $n \in \mathbb{Z}_{\geqslant 0}$, we obtain $\varphi\left((I_{np+1})^{1/p}T^{np/p}\right) \subseteq I_{n+1}T^n$ for every $n \in \mathbb{Z}_{\geqslant 0}$. Therefore, $\varphi(\mathscr{J}^{1/p}) \subseteq \mathscr{J}$. This induces a splitting $\overline{\varphi} : (\mathscr{R}(\mathbb{I})/\mathscr{J})^{1/p} \to \mathscr{R}(\mathbb{I})/\mathscr{J}$ and then $\mathscr{R}(\mathbb{I})/\mathscr{J} \cong \operatorname{gr}(\mathbb{I})$ is also F-split. □

Remark 4.8. We remark that in order to prove that $\mathscr{R}(\mathbb{I})$ is F-split, we only need a splitting $\phi' : R^{1/p} \to R$ such that $\phi'\left((I_{(n+1)p})^{1/p}\right) \subseteq I_{n+1}$ for every $n \in \mathbb{Z}_{\geqslant 0}$. The stronger requirement in the definition of F-split filtrations is to ensure that $\operatorname{gr}(\mathbb{I})$ is F-split as well.

4.3 | Depth and regularity of F-split filtrations

In this subsection, we study the asymptotic behavior of the depth and Castelnuovo–Mumford regularity of F-split filtrations. We assume Setup 4.1. In the graded case, by depth of a graded R-module we mean its grade with respect to the maximal ideal $\mathfrak{m}$, that is, the length of a maximal regular sequence for M inside $\mathfrak{m}_0 + R_+$. On the other hand, by Castelnuovo–Mumford regularity, we mean the regularity computed with respect to the ideal R_+.

In Theorem 4.10, we show that the sequences $\{\operatorname{depth}(I_n)\}_{n\in\mathbb{Z}_{\geqslant 0}}$ and $\{\frac{\operatorname{reg}(I_n)}{n}\}_{n\in\mathbb{Z}_{\geqslant 0}}$ converge to a limit under mild assumptions. We begin with the following technical result.

Proposition 4.9. *Assume Setup 4.1, and let $\mathbb{I} = \{I_n\}_{n\in\mathbb{Z}_{\geqslant 0}}$ be an F-split filtration. Then,*

(1) $\operatorname{depth}(I_n) \leqslant \operatorname{depth}(I_{\lceil \frac{n}{p^e}\rceil})$ *for every $n, e \in \mathbb{Z}_{\geqslant 0}$;*
(2) *if R is graded, then $a_i(I_n) \geqslant p^e a_i(I_{\lceil \frac{n}{p^e}\rceil})$ for every $n, e \in \mathbb{Z}_{\geqslant 0}$ and $0 \leqslant i \leqslant \dim(R/I_1)$.*

Proof. By Proposition 4.5, the natural map $\iota : I_{n+1} \to (I_{np^e+s})^{1/p^e}$ splits for every $n, e \in \mathbb{Z}_{\geqslant 0}$ and $1 \leqslant s \leqslant p^e$ via a splitting ϕ_e. Therefore, the module $\mathrm{H}^i_{\mathfrak{m}}\left(I_{n+1}\right)$ is a direct summand of

$\mathrm{H}^i_{\mathfrak{m}}\left((I_{np^e+s})^{1/p^e}\right)$ for every $1 \leqslant i \leqslant \dim(R/I_1)$. We note that

$$\mathrm{H}^i_{\mathfrak{m}}\left((I_{np^e+s})^{1/p^e}\right) = \left(\mathrm{H}^i_{\mathfrak{m}}\left(I_{np^e+s}\right)\right)^{1/p^e}.$$

Thus, $\mathrm{H}^i_{\mathfrak{m}}\left(I_{np^e+s}\right) = 0$ implies $\mathrm{H}^i_{\mathfrak{m}}\left((I_{np^e+s})^{1/p^e}\right) = 0$, and hence $\mathrm{H}^i_{\mathfrak{m}}\left(R/I_{n+1}\right) = 0$. Therefore, we have $\operatorname{depth}(I_{n+1}) \geqslant \operatorname{depth}(I_{np^e+s})$, which proves the first part.

We note that $\mathrm{H}^i_{R_+}\left(I_{n+1}\right)$ is also a direct summand of $\mathrm{H}^i_{R_+}\left((I_{np^e+s})^{1/p^e}\right)$ and

$$\mathrm{H}^i_{R_+}\left((I_{np^e+s})^{1/p^e}\right) = \left(\mathrm{H}^i_{R_+}\left(I_{np^e+j}\right)\right)^{1/p^e},$$

thus we obtain

$$a_i(I_{n+1}) \leqslant a_i\left((I_{np^e+s})^{1/p^e}\right) = \frac{1}{p^e} a_i\left(I_{np^e+s}\right),$$

and the second part follows. □

The following is the main result of this section.

Theorem 4.10. *Assume Setup 4.1, and let* $\mathbb{I}$ *be an F-split filtration such that* $\mathscr{R}(\mathbb{I})$ *is Noetherian. Then,*

(1) $\operatorname{depth}(I_n)$ *stabilizes and the stable value is equal to* $\min\{\operatorname{depth}(I_n)\}$,
(2) *if R is graded, then* $\lim\limits_{n\to\infty} \frac{\operatorname{reg}(I_n)}{n}$ *exists. As a consequence, if R is regular, then* $\lim\limits_{n\to\infty} \frac{\operatorname{reg}(R/I_n)}{n}$ *exists.*

Proof. We begin with (1). Since $\mathscr{R}(\mathbb{I})$ is Noetherian, there exists $\ell \in \mathbb{Z}_{>0}$ such that $I_{n+\ell} = I_\ell I_n$ for every $n \geqslant \ell$ [77, Remark 2.4.3]. Hence, for every $j = 0, \ldots, \ell - 1$, there exist $d_j, \ell_j \in \mathbb{Z}_{\geqslant 0}$ such that

$$\operatorname{depth}(I_{(n+1)\ell+j}) = \operatorname{depth}((I_\ell)^n I_{\ell+j}) = d_j,$$

for $n \geqslant \ell_j$ [42, Theorem 1.1].

Let $\delta = \min\{\operatorname{depth}(I_n)\}_{n\in\mathbb{Z}_{\geqslant 0}}$ and fix $s \in \mathbb{Z}_{>0}$ such that $\delta = \operatorname{depth}(I_s)$. Let $q = p^e$ be such that $q > \ell$, and $q(s-1) > (\ell_j + 1)\ell$ for every $j = 0, \ldots, \ell - 1$. From Proposition 4.9, it follows that

$$\operatorname{depth}(I_{q(s-1)+i}) \leqslant \operatorname{depth}(I_s)$$

for every $i = 1, \ldots, q$. By our choice for q, for each $j = 0, \ldots, \ell - 1$, there exist natural numbers $m \geqslant \ell_j + 1$ and $1 \leqslant i \leqslant q$ such that $q(s-1) + i = m\ell + j$. Then,

$$\begin{aligned}\delta = \operatorname{depth}(I_s) &\geqslant \operatorname{depth}(I_{q(s-1)+i}) \\ &= \operatorname{depth}(I_{mt+j}) = d_j.\end{aligned}$$

We conclude that $\delta = d_j$ for every $j = 0, \dots, \ell - 1$ because δ is the minimum depth. Then,

$$\operatorname{depth}(I_n) = \delta, \text{ for } n \gg 0.$$

We proceed to prove (2). Since $\mathscr{R}(\mathbb{I})$ is Noetherian and by Equation (2.4.1), the sequence $\operatorname{reg}(I_n)$ eventually agrees with a linear quasi-polynomial [94, Theorem 3.2]. Then, there exists $w \in \mathbb{Z}_{\geqslant 0}$ and $c_1, \dots, c_w, b_1, \dots, b_w \in \mathbb{Z}_{\geqslant 0}$ such that $\operatorname{reg}(I_n) = c_j n + b_j$ for $n \equiv j \pmod{w}$ and $n \gg 0$. We want to show $c_1 = \cdots = c_w$. Set $\alpha_n = \max\{a_i(I_n)\}$ and notice that $\alpha_n \neq -\infty$ for every $n \in \mathbb{Z}_{\geqslant 0}$. We have

$$\lim_{m\to\infty} \frac{\alpha_{wm+j}}{wm+j} = c_j$$

for every $j = 0, \dots, w-1$. We fix $i, j \in \{1, \dots, w\}$, and $e \in \mathbb{Z}_{\geqslant 0}$ such that $q = p^e > w$. Fix $\varepsilon \in \mathbb{R}_{>0}$ and let $r \in \mathbb{Z}_{\geqslant 0}$ be such that $c_j - \frac{\alpha_{wm+j}}{wm+j} < \varepsilon$ for every $m \geqslant r$. From Proposition 4.9, we obtain

$$\alpha_{wr+j} \leqslant \frac{\alpha_{q^\theta(wr+j-1)+b}}{q^\theta}$$

for every $\theta \in \mathbb{Z}_{\geqslant 0}$ and $b = 1, \dots, q^\theta$. Then,

$$c_j - \varepsilon \leqslant \frac{\alpha_{wr+j}}{wr+j} \leqslant \frac{\alpha_{q^\theta(wr+j-1)+b}}{q^\theta(wr+j)} \leqslant \frac{\alpha_{q^\theta(wr+j-1)+b}}{q^\theta(wr+j-1)+b}.$$

Since this inequality holds for every $b = 1, \dots, q^\theta$ and since $w < q^\theta$, there exist infinitely many pairs θ, b such that $q^\theta(wr+j-1)+b \cong i \pmod{w}$. We conclude that $c_j - \varepsilon \leqslant c_i$ for every ϵ, and then $c_j \leqslant c_i$. Since i, j were chosen arbitrarily, we conclude that $c_1 = \cdots = c_w$. □

Under some extra assumptions, we can say more about the stable value $\lim\limits_{n\to\infty} \operatorname{depth}(I_n)$.

Corollary 4.11. *Assume Setup 4.1, and let $\mathbb{I}$ be an F-split filtration such that $\mathscr{R}(\mathbb{I})$ is a Noetherian Cohen–Macaulay algebra. Then,*

$$\lim_{n\to\infty} \operatorname{depth}(I_n) = \dim(R) - \dim(\mathscr{R}(\mathbb{I})/\mathfrak{m}\mathscr{R}(\mathbb{I})) + 1.$$

Proof. There exists $\ell \in \mathbb{Z}_{>0}$ such that $\mathscr{R}(\ell)$ is generated in degree one as an algebra [77, 2.4.4]. If $\mathscr{R} = \mathscr{R}(\mathbb{I})$ is Cohen–Macaulay, then so is $\mathscr{R}(\ell) = \sum_{n\in\mathbb{Z}_{\geqslant 0}} I_{n\ell} T^n$ as it is a direct summand of $\mathscr{R}$. Therefore,

$$\begin{aligned}
\lim_{n\to\infty} \operatorname{depth}(I_n) &= \dim(\mathscr{R}(\ell)) - \dim(\mathscr{R}(\ell)/\mathfrak{m}\mathscr{R}(\ell)) \quad [42, \text{Theorem 1.1}] \\
&= \dim(R) - \dim(\mathscr{R}(\ell)/\mathfrak{m}\mathscr{R}(\ell)) + 1 \\
&= \dim(R) - \dim(\mathscr{R}/\mathfrak{m}\mathscr{R}) + 1.
\end{aligned}$$

□

5 | SYMBOLIC F-SPLIT IDEALS AND SYMBOLIC POWERS

We now focus on F-split filtrations that are given by symbolic powers. We recall that, given a ring R and an ideal $I \subseteq R$, for $n \in \mathbb{Z}_{\geqslant 0}$ the *n-th symbolic power of I* is defined as $I^{(n)} = I^n R_W$, where W is the complement of the union of the minimal primes of I.

Throughout this section, we assume Setup 4.1 with R regular. In the graded case, we further assume that R_0 is a field, that is, R is standard graded. We recall it here more explicitly for future reference:

Setup 5.1. Let $(R, \mathfrak{m}, K)$ be an F-finite regular ring of characteristic $p > 0$, which is either local, or $R = \bigoplus_{n\geqslant 0} R_n = K[R_1]$ is a standard graded polynomial ring over the field K, with homogeneous maximal ideal $\mathfrak{m} = \bigoplus_{n>0} R_n$. We denote by

$$\mathscr{R}^s(I) = \bigoplus_{n\in\mathbb{Z}_{\geqslant 0}} I^{(n)}T^n \qquad \text{and} \qquad \mathrm{gr}^s(I) = \bigoplus_{n\in\mathbb{Z}_{\geqslant 0}} I^{(n)}/I^{(n+1)}$$

the *symbolic Rees algebra* and *symbolic associated graded algebra* of I, respectively. We also let

$$\mathscr{R}(I) = \bigoplus_{n\in\mathbb{Z}_{\geqslant 0}} I^nT^n \qquad \text{and} \qquad \mathrm{gr}(I) = \bigoplus_{n\in\mathbb{Z}_{\geqslant 0}} I^n/I^{n+1}$$

be the *Rees algebra* and *associated graded algebra* of I, respectively.

The interest in symbolic blowups has significantly increased, even in recent years [37, 79].

Definition 5.2. We say that an ideal I is *symbolic F-split* if $\mathbb{I} = \{I^{(n)}\}_{n\in\mathbb{Z}_{\geqslant 0}}$ is an F-split filtration.

We start by studying equality between ordinary and symbolic powers for symbolic F-split ideals. We use the following remark to study symbolic powers.

Remark 5.3. Assume Setup 5.1. We note that $\mathrm{gr}(I)$ is torsion free over R/I if and only if $I^n = I^{(n)}$ for every $n \in \mathbb{Z}_{\geqslant 0}$. In fact, if $\mathrm{gr}(I)$ is torsion free over R/I, then $\mathrm{Ass}_R(I^n/I^{n+1}) \subseteq \mathrm{Ass}_R(R/I)$ for every $n \in \mathbb{Z}_{\geqslant 0}$. From $0 \to I^n/I^{n+1} \to R/I^{n+1} \to R/I^n \to 0$, we obtain $\mathrm{Ass}_R(R/I^{n+1}) \subseteq \mathrm{Ass}_R(I^n/I^{n+1}) \cup \mathrm{Ass}_R(R/I^n)$. Therefore, proceeding by induction on n, we obtain $\mathrm{Ass}_R(R/I^{n+1}) \subseteq \mathrm{Ass}_R(R/I)$ for every $n \in \mathbb{Z}_{\geqslant 0}$, which implies $I^n = I^{(n)}$ for every $n \in \mathbb{Z}_{\geqslant 0}$. Conversely, if $I^n = I^{(n)}$ for every $n \in \mathbb{Z}_{\geqslant 0}$, then $\mathrm{Ass}_R(I^n/I^{n+1}) \subseteq \mathrm{Ass}_R(R/I^{n+1}) = \mathrm{Ass}_R(R/I)$ for every $n \in \mathbb{Z}_{\geqslant 0}$. This implies $\mathrm{Ass}_R(\mathrm{gr}(I)) \subseteq \mathrm{Ass}_R(R/I)$, that is, $\mathrm{gr}(I)$ is torsion free over R/I.

We can now rephrase a result due to Huneke, Simis, and Vasconcelos as follows:

Lemma 5.4 [57, Corollary 1.10]. *Assume Setup 5.1. Let $I \subseteq R$ be a radical ideal. Then, $I^n = I^{(n)}$ for every $n \in \mathbb{Z}_{\geqslant 0}$ if and only if the associated graded algebra $\mathrm{gr}(I)$ is reduced.*

Proposition 5.5. *Assume Setup 5.1. Let $I \subseteq R$ be a symbolic F-split ideal. Then, $I^n = I^{(n)}$ for every $n \in \mathbb{Z}_{\geqslant 0}$ if and only if $\mathrm{gr}(I)$ is an F-split ring.*

Proof. If $I^n = I^{(n)}$ for every $n \in \mathbb{Z}_{\geqslant 0}$, then $\mathrm{gr}(I) = \mathrm{gr}_I^s(R)$. Hence, $\mathrm{gr}(I)$ is F-split by Theorem 4.7. Conversely, if $\mathrm{gr}(I)$ is F-split, then it is reduced. Therefore, $I^n = I^{(n)}$ for every $n \in \mathbb{Z}_{\geqslant 0}$ by Lemma 5.4. □

For the proof of Theorem 5.7, we need the following well-known lemma. Here, we denote by $\mu(I)$ the minimal number of (homogeneous) generators of I.

Lemma 5.6. *Assume Setup 5.1. Let $I \subseteq R$ be any ideal. If $r \geqslant \mu(I)(p-1)+1$, then $I^r = I^{r-p}I^{[p]}$.*

Proof. Let $u = \mu(I)$ and $f_1, \dots, f_u$ a minimal set of generators of I. Let $\alpha_1, \dots, \alpha_u \in \mathbb{Z}_{\geqslant 0}$ be such that $\alpha_1 + \cdots + \alpha_u = r$, then by assumption, there must exist α_i such that $\alpha_i \geqslant p$. Therefore, $f_1^{\alpha_1} \cdots f_u^{\alpha_u} = f_1^{\alpha_1} \cdots f_i^{\alpha_i - p} f_u^{\alpha_u} \cdot f_i^p \in I^{r-p}I^{[p]}$. This shows $I^r \subseteq I^{r-p}I^{[p]}$. To obtain the other containment, we observe that $I^{[p]} \subseteq I^p$. □

The following result gives a finite test to verify whether all the symbolic and ordinary powers of a symbolic F-split ideal coincide.

Theorem 5.7. *Assume Setup 5.1. Let $I \subseteq R$ be a symbolic F-split ideal. If $I^n = I^{(n)}$ for every $n \leqslant \lceil \frac{\mu(I)(p-1)}{p} \rceil$, then $I^n = I^{(n)}$ for every $n \in \mathbb{Z}_{\geqslant 0}$.*

Proof. By Proposition 5.5, it suffices to show $\mathrm{gr}(I)$ is F-split. Let ϕ be such that I is symbolic F-split with respect to ϕ. Proceeding as in Theorem 4.7, it suffices to prove $\phi\big((I^{np+1})^{1/p}\big) \subseteq I^{n+1}$ for every $n \in \mathbb{Z}_{\geqslant 0}$. By assumption, this inclusion holds for $n < \lceil \frac{\mu(I)(p-1)}{p} \rceil$, as for these values $I^{(n+1)} = I^{n+1}$. We fix $n \geqslant \lceil \frac{\mu(I)(p-1)}{p} \rceil$. Then, $I^{np+1} = I^{(n-1)p+1}I^{[p]}$ by Lemma 5.6. The latter is equivalent to $(I^{np+1})^{1/p} = (I^{(n-1)p+1})^{1/p}I$. Therefore, by induction on n,

$$\phi\Big((I^{np+1})^{1/p}\Big) = \phi\Big((I^{(n-1)p+1})^{1/p}I\Big) = \phi\Big((I^{(n-1)p+1})^{1/p}\Big)I \subseteq I^n I = I^{n+1}. \qquad \square$$

We continue with a version of Fedder's Criterion for symbolic F-split ideals. This improves Proposition 4.6 as it only requires to verify that a finite intersection of colon ideals is not contained in $\mathfrak{m}^{[p]}$. We recall that the *big height* of an ideal I, denoted by bigheight(I), is the largest height of a minimal prime of I.

Theorem 5.8. *Assume Setup 5.1. Let $I \subseteq R$ be a radical ideal and set $H = \mathrm{bigheight}(I)$. Let $\delta = 1$ if $p \leqslant H$ and $\delta = 0$ otherwise. Then, I is symbolic F-split if and only if*

$$\bigcap_{n=0}^{\max\{0, H-1-\delta\}} \Big((I^{(n+1)})^{[p]} :_R I^{(np+1)}\Big) \not\subseteq \mathfrak{m}^{[p]}.$$

Proof. By Proposition 4.6, it suffices to show

$$\bigcap_{n \in \mathbb{Z}_{\geqslant 0}} \Big((I^{(n+1)})^{[p]} :_R I^{(np+1)}\Big) = \bigcap_{n=0}^{\max\{0, H-1-\delta\}} \Big((I^{(n+1)})^{[p]} :_R I^{(np+1)}\Big).$$

We note that if $H = 0$, then $I = 0$ and the result follows. If $H = 1$, then I is a principal ideal. Therefore,

$$(I^{(n+1)})^{[p]} :_R I^{(np+1)} = I^{np+p} :_R I^{np+1} = I^{p-1} = I^{[p]} :_R I$$

and the result follows. Hence, for the rest of the proof, we may assume $H \geqslant 2$. Let J be the ideal $(I^{(H-\delta)})^{[p]} :_R I^{((H-1-\delta)p+1)}$. We claim that $J \subseteq \left(I^{(n+1)}\right)^{[p]} :_R I^{(np+1)}$ for every $n \geqslant H - 1 - \delta$. We proceed by induction on n. The base of induction follows from the definition of J. Suppose $J \subseteq (I^{(n+1)})^{[p]} :_R I^{(np+1)}$, we need to show $JI^{((n+1)p+1)} \subseteq (I^{(n+2)})^{[p]}$, and it suffices to show this containment locally at every prime ideal in $\mathrm{Ass}_R(R/I^{(n+2)^{[p]}}) = \mathrm{Ass}_R(R/I^{(n+2)}) = \mathrm{Ass}_R(R/I)$, where the first equality holds by flatness of Frobenius. Let $Q \in \mathrm{Ass}_R(R/I)$ and set $\widetilde{Q} = QR_Q$. We observe that since $n + 1 \geqslant H - \delta$, we have $(n + 1)p + 1 \geqslant Hp - \delta p + 1 \geqslant H(p - 1) + 1$. Then, Lemma 5.6 implies

$$\left(\widetilde{Q}^{n+2}\right)^{[p]} :_R \widetilde{Q}^{(n+1)p+1} = \widetilde{Q}^{[p]}\left(\widetilde{Q}^{n+1}\right)^{[p]} :_R \widetilde{Q}^{[p]}\widetilde{Q}^{(np+1)} \supseteq \left(\widetilde{Q}^{n+1}\right)^{[p]} :_R \widetilde{Q}^{(np+1)}.$$

Since $JR_Q \subseteq (\widetilde{Q}^{n+1})^{[p]} :_R \widetilde{Q}^{(np+1)}$ by induction hypothesis, the proof of our claim follows. We conclude that

$$\bigcap_{n\in\mathbb{Z}_{\geqslant 0}} \left((I^{(n+1)})^{[p]} :_R I^{(np+1)}\right) = \bigcap_{n=0}^{H-1-\delta} \left((I^{(n+1)})^{[p]} :_R I^{(np+1)}\right),$$

and the result follows. □

Remark 5.9. The correction given by $\delta = 0$ in the case $p > H$ of Theorem 5.8 is needed. Indeed, if we could always use $\delta = 1$, that is, if the condition

$$\bigcap_{n=0}^{\max\{0,H-2\}} \left((I^{(n+1)})^{[p]} :_R I^{(np+1)}\right) \nsubseteq \mathfrak{m}^{[p]}$$

implies that I is symbolic F-split, then every F-split ideal I such that $\mathrm{bigheight}(I) = 2$ would be symbolic F-split. This is not the case, as we show in Example 5.13.

Corollary 5.10. *Assume Setup 5.1. Let $I \subseteq R$ be a radical ideal and set $H = \mathrm{bigheight}(I)$. If $I^{(H(p-1))} \nsubseteq \mathfrak{m}^{[p]}$, then I is symbolic F-split.*

Proof. For every $n \in \mathbb{Z}_{\geqslant 0}$, we have $I^{(H(p-1))}I^{(np+1)} \subseteq I^{(H(p-1)+np+1)} \subseteq (I^{(n+1)})^{[p]}$ [36, Lemma 2.6]. Therefore, $I^{(H(p-1))} \subseteq (I^{(n+1)})^{[p]} :_R I^{(np+1)}$. The result now follows from Theorem 5.8. □

Example 5.11. Let I be a square-free monomial ideal in a polynomial ring $K[x_1, \ldots, x_r]$. We observe that

$$(x_1 \cdots x_r)^{p-1} \in \left(\bigcap_{n=0}^{r} \left((I^{(n+1)})^{[p]} :_R I^{(np+1)}\right)\right) \setminus \mathfrak{m}^{[p]}.$$

Then, by Theorem 5.8, I is symbolic F-split. This can also be proven via monomial valuations (see Example 7.3(2)).

We also obtain the following sufficient condition for $\mathscr{R}^s(I)$ to be F-split.

Proposition 5.12. *Assume Setup 5.1. Let $I \subseteq R$ be a radical ideal and set $H = \operatorname{bigheight}(I)$. Let $\delta' = 1$ if $p \leqslant H-1$ and $\delta' = 0$ otherwise. If*

$$\bigcap_{n=0}^{H-2-\delta'} \left((I^{(n+1)})^{[p]} :_R I^{((n+1)p)} \right) \not\subseteq \mathfrak{m}^{[p]},$$

then $\mathscr{R}^s(I)$ is F-split. In particular, $\mathscr{R}^s(I)$ is F-split if $I^{((H-1)(p-1))} \not\subseteq \mathfrak{m}^{[p]}$.

Proof. We note that $\mathscr{R}^s(I)$ is F-split if there is a splitting $\phi' : R^{1/p} \to R$ for which the inclusion

$$\phi'\left((I^{((n+1)p)})^{1/p} \right) \subseteq I^{(n+1)}$$

holds for every $n \in \mathbb{Z}_{\geqslant 0}$ (see Remark 4.8). Let $\delta' = 1$ if $p \leqslant H-1$ and $\delta' = 0$ otherwise. We can adapt Lemma 4.6 and the proof of Theorem 5.8 to obtain that

$$\bigcap_{n=0}^{H-2-\delta'} \left((I^{(n+1)})^{[p]} :_R I^{((n+1)p)} \right) \not\subseteq \mathfrak{m}^{[p]}$$

implies that $\mathscr{R}^s(I)$ is F-split.

For the last statement, we can proceed as in Corollary 5.10 to obtain

$$I^{((H-1)(p-1))} \subseteq \left((I^{(n+1)})^{[p]} :_R I^{((n+1)p)} \right),$$

whence the conclusion follows. □

Since every symbolic F-split ideal is F-split, one may ask whether these two conditions are equivalent. The following example shows that this is not the case.

Example 5.13. Let $R = K[a,b,c,d]$ be a polynomial ring and char $p \geqslant 3$. Consider the following matrix:

$$A = \begin{bmatrix} a^2 & b & d \\ c & a^2 & b-d \end{bmatrix}.$$

Let $I = I_2(A)$ be the ideal generated by the 2×2 minors of A. The ring R/I is F-split, in fact strongly F-regular [86, Proposition 4.3]. The ideal I is prime of height 2. Moreover, the symbolic and ordinary powers of I coincide [36, Corollary 4.4]. Considering $p = 3$, we verify with Macaulay2 [35] that $(I^2)^{[3]} :_R I^4 \subseteq \mathfrak{m}^{[3]}$. Therefore, I is not symbolic F-split by Theorem 5.8.

The following definition due to Huneke provides a sufficient condition for an ideal to be symbolic F-split.

Definition 5.14 (Huneke). Assume Setup 5.1. Let $I \subseteq R$ be a radical ideal of height h. We say that I is *F-König* if there exists a regular sequence $f_1, \dots, f_h \in I$ such that $R/(f_1, \dots, f_h)$ is F-split.

In particular, if I is an ideal generated by a regular sequence and such that R/I is F-pure, then I is F-König.

Proposition 5.15. *Assume Setup 5.1. If $I \subseteq R$ is equidimensional and F-König, then it is symbolic F-split.*

Proof. Let $h = \mathrm{ht}(I)$ and $f_1, \dots, f_h \in I$ a regular sequence such that $R/(f_1, \dots, f_h)$ is F-split. We consider $J = (f_1, \dots, f_h)$. Since J is F-split, we have $f_1^{p-1} \cdots f_h^{p-1} \in J^{h(p-1)} \setminus \mathfrak{m}^{[p]}$ [33, Proposition 2.1]. The result now follows from Corollary 5.10 since $J^{h(p-1)} \subseteq I^{(h(p-1))}$. □

Example 5.16. Let A and B be two generic matrices of size $n \times n$ with entries in disjoint sets of variables. Let J be the ideal generated by the entries of $AB - BA$ and I the ideal generated by the off-diagonal entries of this matrix. Then if $n = 2$, or 3, the ideals I and J are F-König [62], and hence symbolic F-split.

We now mention and answer a question that was raised by Huneke at the BIRS-CMO workshop on *Ordinary and Symbolic Powers of Ideals* during the summer of 2017 at Casa Matemática Oaxaca, which arose in connection with the Conforti–Cornuéjols conjecture [18].

Question 5.17 (Huneke). Let $Q \subseteq R$ be a prime ideal such that R/Q is F-split, and $Q^{(n)} = Q^n$ for every $n \in \mathbb{Z}_{\geqslant 0}$. Is Q F-König?

The following example shows that the answer to this question is negative.

Example 5.18. Let I and R be as in Example 5.13 with $p \geqslant 3$. Then, I is a prime ideal of height 2. As noted before, $I^{(n)} = I^n$ for every $n \in \mathbb{Z}_{\geqslant 0}$; however, I is not F-König. Indeed, by Proposition 5.15 and its proof, it suffices to show $I^{(2(p-1))} = I^{2(p-1)} \subseteq \mathfrak{m}^{[p]}$. Assume that the variable a has degree 1 and that the variables b, c, and, d have degree 2. Hence, I is generated in degree 4 and then $I^{2(p-1)}$ is generated in degree $8(p-1)$. On the other hand, if $f := a^{n_1} b^{n_2} c^{n_3} d^{n_4} \notin \mathfrak{m}^{[p]}$, we must have $n_i \leqslant p - 1$ for each i. Therefore, such an f has degree at most $7(p-1)$.

6 | SYMBOLIC AND ORDINARY POWERS OF DETERMINANTAL IDEALS

In this section, we prove our main results on symbolic powers of several types of determinantal ideals. A key point in our proofs is the construction of specific polynomials that allow us to directly apply Fedder's Criterion for an ideal to be symbolic F-split, Theorem 5.8; this construction is inspired by the work of Seccia in the context of Knutson ideals [81, 82].

Notation 6.1. Let A be an $r \times s$ matrix, and $i, j, k, \ell \in \mathbb{Z}$ be such that $1 \leqslant i \leqslant k \leqslant r$ and $1 \leqslant j \leqslant \ell \leqslant s$. We denote by $A_{[j,\ell]}^{[i,k]}$ the submatrix of A with row indices $i, \dots, k$ and column indices $j, \dots, \ell$.

In the next subsections, we repeatedly use the following lemma.

Lemma 6.2. *Let R be a $\mathbb{Z}_{\geqslant 0}$-graded polynomial ring over an F-finite field K, and let $\mathfrak{m}$ be the homogeneous maximal ideal of R. Let I be a radical homogeneous ideal with $H = \operatorname{bigheight}(I)$. Assume that there exist a homogeneous polynomial $f \in R$ and a monomial order $<$ such that $\operatorname{in}_<(f)$ is square free.*

(1) *If $f \in I^{(H)}$, then I is symbolic F-split.*
(2) *If $f^{p-1} \in (I^n)^{[p]} : I^{np}$ for every $n \in \mathbb{Z}_{\geqslant 0}$, then the Rees algebra $\mathscr{R}(I)$ is F-split.*

Proof. We first prove (1). The assumption that $\operatorname{in}_<(f)$ is a square-free monomial implies $f^{p-1} \notin \mathfrak{m}^{[p]}$. Since $f^{p-1} \in (I^{(H)})^{p-1} \subseteq I^{(H(p-1))}$, we conclude by Corollary 5.10 that I is symbolic F-split.

In order to prove (2), we let Φ be the trace map (see Remark 2.3), and we consider $\phi = \Phi(f^{p-1}-)$. Because of our assumptions, the map ϕ induces an $\mathscr{R}(I)$-linear map $\Psi : (\mathscr{R}(I))^{1/p} \to \mathscr{R}(I)$. As above, we have $f^{p-1} \notin \mathfrak{m}^{[p]}$, and therefore Ψ is surjective. It follows that $\mathscr{R}(I)$ is F-split. □

6.1 | Ideals of minors of a generic matrix

In this subsection, we use the following setup.

Setup 6.3. Let $r, s \in \mathbb{Z}_{>0}$ be such that $r \leqslant s$. Let $X = (x_{i,j})$ be a generic $r \times s$ matrix of variables, K be an F-finite field of characteristic $p > 0$, $R = K[X]$, and $\mathfrak{m} = (x_{i,j})$. For $t \in \mathbb{Z}_{>0}$ such that $t \leqslant r$, we let $I_t(X)$ be the ideal generated by the $t \times t$ minors of X. We let

$$f_u(X) = \left(\prod_{\ell=u}^{r-1} \det\left(X_{[1,\ell]}^{[r-\ell+1,r]}\right) \det\left(X_{[s-\ell+1,s]}^{[1,\ell]}\right)\right) \cdot \left(\prod_{\ell=1}^{s-r+1} \det\left(X_{[\ell,r+\ell-1]}^{[1,r]}\right)\right),$$

for $u \leqslant r$. We consider the lexicographical monomial order on R induced by

$$x_{1,1} > x_{1,2} > \ldots > x_{1,s} > x_{2,1} > x_{2,2} > \cdots > x_{r,s-1} > x_{r,s}.$$

Remark 6.4. For any $t \in \mathbb{Z}_{\geqslant 0}$, we note that the initial form $\operatorname{in}_<(f_t(X))$ is a square-free monomial.

We begin by showing that generic determinantal ideals are symbolic F-split.

Theorem 6.5. *Assuming Setup 6.3, the ideal $I_t(X)$ is symbolic F-split.*

Proof. Let $h = (s-t+1)(r-t+1) = \operatorname{ht}(I_t(X))$. Let $f = f_t(X)$, and note that

$$f \in \left(\prod_{\ell=t}^{r-1} I_\ell(X)\right)^2 I_r(X)^{s-r+1}$$

$$\subseteq \left(\prod_{\ell=t}^{r-1} I_t(X)^{(\ell-t+1)}\right)^2 \left(I_t(X)^{(r-t+1)}\right)^{s-r+1} \quad [14, \text{Proposition } 10.2]$$

$$\subseteq I_t(X)^{((r-t)(r-t+1))} I_t(X)^{((r-t+1)(s-r+1))}$$
$$\subseteq I_t(X)^{(s-t+1)(r-t+1)} = I_t(X)^{(h)}.$$

The conclusion follows from Remark 6.4 and Lemma 6.2(1). □

From the previous proposition, we obtain the following consequences.

Theorem 6.6. *Assuming Setup 6.3, the limit*

$$\lim_{n\to\infty} \frac{\operatorname{reg}\left(R/I_t(X)^{(n)}\right)}{n}$$

exists and

$$\operatorname{depth}\left(R/I_t(X)^{(n)}\right)$$

stabilizes for $n \gg 0$. *Furthermore, if* $1 \leqslant t \leqslant \min\{m, r\}$, *then*

$$\lim_{n\to\infty} \operatorname{depth}\left(R/I_t(X)^{(n)}\right) = \dim(R) - \dim\left(\mathscr{R}^s(I_t(X))/\mathfrak{m}\mathscr{R}^s(I_t(X))\right) = t^2 - 1.$$

Proof. We know that $\mathscr{R}^s(I_t(X))$ is Noetherian [14, Proposition 10.2, Theorem 10.4]. Hence, the result follows by combining Theorem 6.5 and Theorem 4.10. Since $\mathscr{R}^s(I_t(X))$ is Cohen–Macaulay [9, Corollary 3.3], we have

$$\begin{aligned}
\lim_{n\to\infty} \operatorname{depth}\left(R/I_t(X)^{(n)}\right) &= \dim(R) - \dim\left(\mathscr{R}^s(I_t(X))/\mathfrak{m}\mathscr{R}^s(I_t(X))\right) && \text{by Corollary 4.11,}\\
&= \min\{\operatorname{depth}\left(R/I_t(X)^{(n)}\right)\} && \text{by Theorem 4.10,}\\
&= \operatorname{grade}\left(\mathfrak{m}\operatorname{gr}^s(I_t(X))\right) && \text{[14, Proposition 9.23],}\\
&= t^2 - 1 && \text{[14, Proposition 10.8].}
\end{aligned}$$

□

We now show that $\mathscr{R}^s(I_t(X))$ and $\operatorname{gr}^s(I_t(X))$ are strongly F-regular. This strengthens a result of Bruns and Conca [8] showing that $\mathscr{R}^s(I_t(X))$ is F-rational using techniques from the theory of Sagbi bases [16].

Theorem 6.7. *Assuming Setup 6.3, the algebras* $\mathscr{R}^s(I_t(X))$ *and* $\operatorname{gr}^s(I_t(X))$ *are strongly* F*-regular.*

Proof. We know that $\mathscr{R}^s(I_t(X))$ and $\operatorname{gr}^s(I_t(X))$ are Noetherian [14, Proposition 10.2, Theorem 10.4]. We proceed by induction on t. If $t = 1$, then the result follows because $I_1(X) = \mathfrak{m}$. We now assume the result is true for $(t-1)\times(t-1)$-minors of a generic matrix. If we let $f = f_t(X)$, then $\operatorname{in}_<(f)$ is a square-free monomial, which is not divisible by $x_{r,1}$, because $t \geqslant 2$. Let $g = \frac{\prod_{i,j} x_{i,j}}{x_{r,1}\operatorname{in}_<(f)}$. We note that $\operatorname{in}_<(f^{p-1}g^{p-1}) = \frac{\prod_{i,j} x_{i,j}^{p-1}}{x_{r,1}^{p-1}}$, and as a consequence $f^{p-1}g^{p-1} \notin \mathfrak{m}^{[p]}$. Let $\phi = \Phi\left(f^{(p-1)/p}g^{(p-1)/p}-\right)$, where $\Phi : R^{1/p} \to R$ denotes the trace map introduced in Remark 2.3. We note that $\phi\left(x_{r,1}^{(p-1)/p}\right) = 1$.

We have $f \in I_t(X)^{(\mathrm{ht}(I_t(X))}$ as shown in the proof of Theorem 6.5. It follows that $f^{p-1} \in \left(I_t(X)^{(n+1)}\right)^{[p]} : I_t(X)^{(np+1)}$ for every $n \in \mathbb{Z}_{\geqslant 0}$ [36, Lemma 2.6] (see also proof of Corollary 5.10). As a consequence, ϕ induces maps $\Psi : \mathcal{R}^s(I_t(X))^{1/p} \to \mathcal{R}^s(I_t(X))$ and $\overline{\Psi} : \mathrm{gr}^s(I_t(X))^{1/p} \to \mathrm{gr}^s(I_t(X))$, which satisfy

$$\Psi\left(x_{r,1}^{(p-1)/p}\right) = 1 \quad \text{and} \quad \overline{\Psi}\left(\overline{x_{r,1}}^{(p-1)/p}\right) = 1. \tag{6.1.1}$$

Let $A = K[U]$, where $U = (u_{i,j})_{\substack{1\leqslant i\leqslant r-1,\\ 2\leqslant j\leqslant s}}$ is a generic matrix of size $(r-1)\times(s-1)$, and let

$$S = A[x_{1,1}, \dots, x_{r-1,1}, x_{r,1}, \dots, x_{r,s}].$$

We have an isomorphism $\gamma : R[x_{r,1}^{-1}] \to S[x_{r,1}^{-1}]$ defined by $x_{i,j} \mapsto u_{i,j} + x_{r,j}x_{i,1}x_{r,1}^{-1}$, $x_{i,1} \mapsto x_{i,1}$, and $x_{r,j} \mapsto x_{r,j}$ for $i \leqslant r-1$ and $j \geqslant 2$. Furthermore, we have $\gamma\left(I_t(X)R[x_{r,1}^{-1}]\right) = I_{t-1}(U)S[x_{r,1}^{-1}]$, and then $\gamma(I_t(X)^{(n)}R[x_{r,1}^{-1}]) = I_{t-1}(U)^{(n)}S[x_{r,1}^{-1}]$ for every $n \in \mathbb{Z}_{\geqslant 0}$ [14, Lemma 10.1]. By the induction hypothesis, $\mathcal{R}^s(I_{t-1}(U))$ and $\mathrm{gr}^s(I_{t-1}(U))$ are strongly F-regular. It follows that $\mathcal{R}^s(I_{t-1}(U)) \otimes_A S[x_{r,1}^{-1}]$ and $\mathrm{gr}^s(I_{t-1}(U)) \otimes_A S[x_{r,1}^{-1}]$ are strongly F-regular, because strong F-regularity is preserved by adding variables and localizing. Therefore, thanks to the isomorphism γ, the rings $\mathcal{R}^s(I_t(X)) \otimes_R R[x_{r,1}^{-1}]$ and $\mathrm{gr}^s(I_t(X)) \otimes_R R[x_{r,1}^{-1}]$ are also strongly F-regular. From this and Equation (6.1.1), we conclude that $\mathcal{R}^s(I_t(X))$ and $\mathrm{gr}^s(I_t(X))$ are strongly F-regular [48, Theorem 3.3]. □

We now show that the ordinary Rees algebra of a generic determinantal ideal is F-split. We note that it was already known that $\mathcal{R}(I_t(X))$ is F-rational [8]. However, F-rationality does not imply that the ring is F-split.

Theorem 6.8. *In addition to assuming Setup 6.3, suppose $p > \min\{t, r-t\}$. Then the Rees algebra $\mathcal{R}(I_t(X))$ is F-split.*

Proof. Let $f = f_1(X)$, and note that $f \in I_\ell(X)^{(\mathrm{ht}(I_\ell(X))}$ for every $\ell \leqslant r$, as shown in the proof of Theorem 6.5. It follows that $f^{p-1} \in \left(I_\ell(X))^{(n+1)}\right)^{[p]} : I_\ell(X)^{(np+1)}$ for every $\ell \leqslant r$ and $n \in \mathbb{Z}_{\geqslant 0}$ [36, Lemma 2.6] (cf. proof of Corollary 5.10). Thus,

$$f^{p-1}I_\ell(X)^{((n+1)p)} \subseteq f^{p-1}I_\ell(X)^{(np+1)} \subseteq \left(I_\ell(X)^{(n+1)}\right)^{[p]}$$

for every $\ell \leqslant r$ and $n \in \mathbb{Z}_{\geqslant 0}$. Then,

$$\begin{aligned} f^{p-1}I_t(X)^{np} &= f^{p-1}\left(\bigcap_{\ell=1}^{t} I_\ell(X)^{((t-\ell+1)np)}\right) && \text{[14, Corollary 10.13]} \\ &\subseteq \bigcap_{\ell=1}^{t} f^{p-1}\left(I_\ell(X)^{((t-\ell+1)np)}\right) \\ &\subseteq \bigcap_{\ell=1}^{t} \left(I_\ell(X)^{((t-\ell+1)n)}\right)^{[p]} \end{aligned}$$

$$= \left(\bigcap_{\ell=1}^{t} I_\ell(X)^{((t-\ell+1)n)} \right)^{[p]}$$

$$= (I_t(X)^n)^{[p]} \qquad [14, \text{Corollary 10.13}].$$

The conclusion follows from Remarks 6.4 and Lemma 6.2(2). □

We end this subsection with the following results, which provide bounds for the degrees of the defining equations of symbolic and ordinary Rees algebras and associated graded algebra of determinantal ideals of generic matrices.

Theorem 6.9. *Assume Setup 6.3. Set* $\mu = \binom{r}{t}\binom{s}{t}$.

(1) *Suppose* $\deg(x_{i,j}) = 0$ *for every* i, j, *then the defining equations of* $\mathcal{R}(I_t(X))$ *over* R *have degree at most* $\min\{rs+1, \mu\}$.
(2) *Suppose* $\deg(x_{i,j}) = 1$ *for every* i, j, *then the defining equations of* $\mathcal{R}(I_t(X))$ *over* K *have total degree at most* $rs + \mu(t+1)$.

Proof. The result follows from Theorems 6.8 and 3.3, and Proposition 2.8. □

Theorem 6.10. *Assume Setup 6.3 and that* $t < r$. *For* $j = t, \dots, r$, *set* $\mu_j = \binom{r}{j}\binom{s}{j}$.

(1) *Suppose* $\deg(x_{i,j}) = 0$ *for every* i, j, *then the defining equations of* $\mathcal{R}^s(I_t(X))$ *over* R *have degree at most* $\min\{rs + 1 + \sum_{j=t+1}^{r} \mu_j(j-t), \sum_{j=t}^{r} \mu_j(j-t+1)\}$, *and of* $\operatorname{gr}^s(I_t(X))$ *over* $R/I_t(X)$, *have degree at most* $\min\{rs + \sum_{j=t+1}^{r} \mu_j(j-t), \sum_{j=t}^{r} \mu_j(j-t+1)\}$.
(2) *Suppose* $\deg(x_{i,j}) = 1$ *for every* i, j, *then the defining equations of* $\mathcal{R}^s(I_t(X))$ *and* $\operatorname{gr}^s(I_t(X))$ *over* K *have total degree at most* $rs + \sum_{j=t}^{r} \mu_j(2j-t+1)$.

Proof. By Theorem 6.5 and Theorem 4.7, the algebras $\mathcal{R}^s(I_t(X))$ and $\operatorname{gr}^s(I_t(X))$ are F-split. Both parts of the result now follow from Theorem 3.4, Proposition 2.8, and the equality

$$\mathcal{R}^s(I_t(X)) = R[I_t(X)T, I_{t+1}(X)T^2, \dots, I_r(X)T^{r-t+1}] \qquad [14, \text{Proposition 10.2, Theorem 10.4}].$$ □

6.2 | Ideals of minors of a symmetric matrix

In this subsection, we use the following setup.

Setup 6.11. Let $r \in \mathbb{Z}_{>0}$, and $Y = (y_{i,j})$ be a generic symmetric matrix of size $r \times r$. Let K be an F-finite field of characteristic $p > 0$, $R = K[Y]$, and $\mathfrak{m} = (y_{i,j})$. For $t \in \mathbb{Z}_{>0}$ with $t \leqslant r$, we let $I_t(Y)$ be ideal generated by the $t \times t$ minors of Y. We let

$$f_u(Y) = \prod_{\ell=u}^{r} \det\left(Y^{[1,\ell]}_{[r-\ell+1,r]}\right),$$

for $u \leqslant r$. We consider the lexicographical monomial order on R induced by

$$y_{1,1} > y_{1,2} > \dots > y_{1,r} > y_{2,2} > \dots > y_{r,r}.$$

Remark 6.12. For any $t \in \mathbb{Z}_{\geqslant 0}$, we note that the initial form $\mathrm{in}_<(f_t(Y))$ is a square-free monomial.

We now show that ideals of minors of a generic symmetric matrix is symbolic F-split.

Theorem 6.13. *Assuming Setup 6.11, the ideal $I_t(Y)$ is symbolic F-split. In particular, the rings $\mathscr{R}^s(I_t(Y))$ and $\mathrm{gr}^s(I_t(Y))$ are F-split.*

Proof. Let $h = \frac{(r-t+1)(r-t+2)}{2} = \mathrm{ht}(I_t(Y))$ [60, Corollary 2.4]. Let $f = f_t(Y)$, and note that

$$\begin{aligned} f &\in \prod_{\ell=t}^{r} I_\ell(Y) \\ &\subseteq \prod_{\ell=t}^{r} (I_t(Y))^{(\ell-t+1)} \quad [59, \text{Theorem } 4.4] \\ &\subseteq I_t(Y)^{(h)}. \end{aligned}$$

The first statement now follows from Remark 6.12 and Lemma 6.2(1), and the second statement from Theorem 4.7. □

Lemma 6.14. *Assuming Setup 6.11, the rings $\mathscr{R}^s(I_t(Y))$ and $\mathrm{gr}^s(I_t(Y))$ are Noetherian. Moreover, $\mathscr{R}^s(I_t(Y)) = R[I_t(Y)T, I_{t+1}(Y)T^2, \dots, I_r(Y)T^{r-t+1}]$.*

Proof. We have $I_{t+n-1}(Y) \subseteq I_t(Y)^{(n)}$ for every $n \geqslant 1$ [59, Theorem 4.4]. Moreover, we have $I_t(Y)^{(n)} = \sum I_{t+n_1-1}(Y) \cdots I_{t+n_s-1}(Y)$, where the sum ranges over the integers $n_1, \dots, n_s \geqslant 1$, such that $s \leqslant n$ and $n_1 + \cdots + n_s \geqslant n$ [59, Proposition 4.3]. The conclusion clearly follows. □

We obtain the following homological consequences.

Theorem 6.15. *Assuming Setup 6.11, the limit*

$$\lim_{n\to\infty} \frac{\mathrm{reg}(R/I_t(Y)^{(n)})}{n}$$

exists and

$$\mathrm{depth}(R/I_t(Y)^{(n)})$$

stabilizes for $n \gg 0$.

Proof. Since $\mathscr{R}^s(I_t(Y))$ is Noetherian by Lemma 6.14, the result follows by combining Theorems 6.13 and 4.10. □

We now show that the symbolic Rees algebra of a determinantal ideal of a generic symmetric matrix is strongly F-regular.

Theorem 6.16. *Assuming Setup 6.11, $\mathscr{R}^s(I_t(Y))$ is strongly F-regular.*

Proof. We set $h = \mathrm{ht}(I_t(Y))$. We know that $\mathscr{R}^s(I_t(Y))$ Noetherian by Lemma 6.14.

If $t = r$, then $I_t(Y)$ is principal, and so, $\mathscr{R}^s(I_t(Y)) = R[\det(Y)T]$, which is isomorphic to a polynomial ring over K. Then, $\mathscr{R}^s(I_t(Y))$ is strongly F-regular.

We proceed by induction on t. If $t = 1$, then the result follows because $I_1(Y) = \mathfrak{m}$. We now assume the result is true for $(t-1) \times (t-1)$-minors of a generic symmetric matrix. If we let

$$f = \det\left(Y_{[2,r]}^{[2,r]}\right) \cdot \prod_{\ell=1}^{r-1} \det\left(Y_{[r-\ell+1,r]}^{[1,\ell]}\right),$$

as in the proof of Theorem 6.13, we have that $f \in I_t(Y)^{(h-1)}$. In addition, then $\mathrm{in}_<(f)$ is a square-free monomial, which is not divisible by $y_{1,1}$, because $t \geqslant 2$. We have $f^{p-1} \notin \mathfrak{m}^{[p]}$. Let $\phi = \Phi\left(f^{(p-1)/p} -\right)$, where $\Phi : R^{1/p} \to R$ denotes the trace map introduced in Remark 2.3. We note that $\phi\left(y_{1,1}^{(p-1)/p}\right) = 1$. It follows that $f^{p-1} \in \left(I_t(Y)^{(n+1)}\right)^{[p]} : I_t(Y)^{((n+1)p)}$ for every $n \in \mathbb{Z}_{\geqslant 0}$ by Proposition 5.12.

As a consequence, ϕ induces maps $\Psi : \mathscr{R}^s(I_t(X))^{1/p} \to \mathscr{R}^s(I_t(X))$, which satisfy

$$\Psi\left(y_{1,1}^{(p-1)/p}\right) = 1 \quad \text{and} \quad \overline{\Psi}\left(\overline{y_{1,1}}^{(p-1)/p}\right) = 1. \tag{6.2.1}$$

Let $A = K[U]$, where $U = (u_{i,j})_{\substack{2 \leqslant i \leqslant r-1,\\ 2 \leqslant j \leqslant r}}$ is a generic symmetric matrix of size $(r-1) \times (r-1)$, and let

$$S = A[y_{1,1}, \dots, y_{2,1}, \dots, y_{r,1}].$$

We have an isomorphism $\gamma : R[y_{1,1}^{-1}] \to S[y_{y,1}^{-1}]$ defined by $y_{i,j} \mapsto u_{i,j} + y_{1,j}x_{i,1}y_{1,1}^{-1}$, $y_{i,1} \mapsto y_{i,1}$, and $y_{1,j} \mapsto y_{1,j}$ for $j \geqslant 2$ [69] (see also [60, Lemma 1.1].

Furthermore, we have $\gamma\left(I_t(Y)R[y_{1,1}^{-1}]\right) = I_{t-1}(U)S[y_{1,1}^{-1}]$, and then $\gamma(I_t(Y)^{(n)}R[x_{1,1}^{-1}]) = I_{t-1}(U)^{(n)}S[y_{1,1}^{-1}]$ for every $n \in \mathbb{Z}_{\geqslant 0}$ [14, Lemma 10.1]. By the induction hypothesis, $\mathscr{R}^s(I_{t-1}(U))$ and $\mathrm{gr}^s(I_{t-1}(U))$ are strongly F-regular. It follows that $\mathscr{R}^s(I_{t-1}(U)) \otimes_A S[y_{1,1}^{-1}]$ is strongly F-regular, because strong F-regularity is preserved by adding variables and localizing. Therefore, thanks to the isomorphism γ, the ring $\mathscr{R}^s(I_t(Y)) \otimes_R R[y_{1,1}^{-1}]$ is also strongly F-regular. From this and Equation (6.2.1), we conclude that $\mathscr{R}^s(I_t(Y))$ is strongly F-regular [48, Theorem 3.3]. □

We now show that the ordinary Rees algebra of a generic symmetric determinantal ideal is F-split.

Theorem 6.17. *In addition to Setup 6.11, suppose $p > \min\{t, r-t\}$. Then, the Rees algebra $\mathscr{R}(I_t(Y))$ is F-split.*

Proof. Let $f = f_1(Y)$, and note that $f \in I_\ell(Y)^{(\mathrm{ht}(I_\ell(Y))}$ for every $\ell \leqslant r$, as shown in the proof of Theorem 6.13. It follows that $f^{p-1} \in \left(I_\ell(Y))^{(n+1)}\right)^{[p]} : I_\ell(Y)^{(np+1)}$ for every $\ell \leqslant r$ and $n \in \mathbb{Z}_{\geqslant 0}$ [36, Lemma 2.6] (cf. proof of Corollary 5.10). Thus,

$$f^{p-1}I_\ell(Y)^{((n+1)p)} \subseteq f^{p-1}I_\ell(Y)^{(np+1)} \subseteq \left(I_\ell(Y)^{(n+1)}\right)^{[p]}$$

for $n \in \mathbb{Z}_{\geqslant 0}$. Then,

$$\begin{aligned} f^{p-1}I_t(Y)^{np} &= f^{p-1}\left(\bigcap_{\ell=1}^{t} I_\ell(Y)^{((t-\ell+1)np)}\right) && \text{[59, Theorem 4.4]} \\ &\subseteq \bigcap_{\ell=1}^{t} f^{p-1}\left(I_\ell(Y)^{((t-\ell+1)np)}\right) \\ &\subseteq \bigcap_{\ell=1}^{t} \left(I_\ell(Y)^{((t-\ell+1)n)}\right)^{[p]} \\ &= \left(\bigcap_{\ell=1}^{t} I_\ell(Y)^{((t-\ell+1)n)}\right)^{[p]} \\ &= (I_t(Y)^n)^{[p]} && \text{[59, Theorem 4.4].} \end{aligned}$$

The result follows from Remark 6.12 and Lemma 6.2(2). □

We end with the following results about degrees of defining equations for ordinary Rees and associated graded algebras in the case of generic symmetric matrices.

Theorem 6.18. *Assume Setup 6.11. Set* $\mu = \frac{1}{2}\binom{r}{t}^2$.

(1) *Suppose* $\deg(y_{i,j}) = 0$ *for every* i, j, *then the defining equations of* $\mathscr{R}(I_t(Y)$ *over* R *have degree at most* $\min\{\binom{r+1}{2} + 1, \mu\}$.
(2) *Suppose* $\deg(y_{i,j}) = 1$ *for every* i, j, *then the defining equations of* $\mathscr{R}(I_t(Y))$ *over* K *have total degree at most* $\binom{r+1}{2} + \mu(t+1)$.

Proof. The result follows from Theorems 6.17 and 3.3, and Proposition 2.8. □

Theorem 6.19. *Assume Setup 6.11. For* $j = t, \dots, r$, *set* $\mu_j = \frac{1}{2}\binom{r}{j}^2$.

(1) *Suppose* $\deg(y_{i,j}) = 0$ *for every* i, j, *then the defining equations of* $\mathscr{R}^s(I_t(Y))$ *over* R *have degree at most* $\min\{\binom{r+1}{2} + 1 + \sum_{j=t+1}^{r} \mu_j(j-t), \sum_{j=t}^{r} \mu_j(j-t+1)\}$, *and of* $\mathrm{gr}^s(I_t(Y))$ *over* $R/I_t(Y)$, *have degree at most* $\min\{\binom{r+1}{2} + \sum_{j=t+1}^{r} \mu_j(j-t), \sum_{j=t}^{r} \mu_j(j-t+1)\}$.
(2) *Suppose* $\deg(y_{i,j}) = 1$ *for every* i, j, *then the defining equations of* $\mathscr{R}^s(I_t(Y))$ *and* $\mathrm{gr}^s(I_t(Y))$ *over* K *have total degree at most* $\binom{r+1}{2} + \sum_{j=t}^{r} \mu_j(2j-t+1)$.

Proof. The result follows from Theorem 6.13, Theorem 3.4, Proposition 2.8, and Lemma 6.14. □

6.3 | Ideals of Pfaffians of a skew-symmetric matrix

For the convenience of the reader, we recall the definition of Pfaffians.

Definition 6.20. Let $Z = (z_{i,j})$ be a generic $r \times r$ skew-symmetric matrix, that is, $z_{i,j} = -z_{j,i}$ for every $1 \leqslant i < j \leqslant r$, and $z_{i,i} = 0$ for every $1 \leqslant i \leqslant m$. A minor of the form $\det\left(Z^{[i_1,\dots,i_{2t}]}_{[i_1,\dots,i_{2t}]}\right)$ is the square of a polynomial $\operatorname{pf}\left(Z_{[i_1,\dots,i_{2t}]}\right) \in R = K[Z]$. Such a polynomial is called a $2t$-*Pfaffian* of Z.

In this subsection, we use the following setup.

Setup 6.21. Let $r \in \mathbb{Z}_{>0}$, and $Z = (z_{i,j})$ be a generic skew-symmetric matrix of size $r \times r$. Let K be an F-finite field of characteristic $p > 0$, $R = K[Z]$, and $\mathfrak{m} = (z_{i,j})$. For $t \in \mathbb{Z}_{>0}$ such that $2t \leqslant r$, we let $P_{2t}(Z)$ be the ideal generated by the $2t$-Pfaffians of Z. If r is odd, we set $b = \lfloor r/2 \rfloor$, and we let

$$f_{2u}(Z) = \left(\prod_{\ell=u}^{b-1} \operatorname{pf}\left(Z_{[1,\dots,2\ell]}\right) \operatorname{pf}\left(Z_{[1,\dots,\ell,\ell+2,\dots,2\ell+1]}\right) \operatorname{pf}\left(Z_{[r+1-2\ell,\dots,r]}\right) \operatorname{pf}\left(Z_{[r-2\ell,\dots,r-\ell-1,r-\ell+1,\dots,r]}\right)\right) \cdot \left(\operatorname{pf}\left(Z_{[1,\dots,r-1]}\right) \operatorname{pf}\left(Z_{[2,\dots,r]}\right) \operatorname{pf}\left(Z_{[1,\dots,b,b+2,\dots,r]}\right)\right),$$

for $u \leqslant r/2$. If r is even, we set $b = r/2$, and we let

$$f_{2u}(Z) = \left(\prod_{\ell=u}^{b-1} \operatorname{pf}\left(Z_{[1,\dots,2\ell]}\right) \operatorname{pf}\left(Z_{[1,\dots,\ell,\ell+2,\dots,2\ell+1]}\right) \operatorname{pf}\left(Z_{[r+1-2\ell,\dots,r]}\right) \operatorname{pf}\left(Z_{[r-2\ell\dots,r-\ell-1,r-\ell+1\dots,r]}\right)\right) \operatorname{pf}(Z),$$

for $u \leqslant r/2$. We consider the lexicographical monomial order on R induced by

$$z_{1,r} > z_{1,r-1} > \dots > z_{1,2} > z_{2,r} > \dots > z_{r-1,r}.$$

Remark 6.22. For any $t \in \mathbb{Z}_{\geqslant 0}$, we note that the initial form $\operatorname{in}_<(f_{2t}(Z))$ is a square-free monomial.

In the following result, we show ideals of Pfaffians are symbolic F-split.

Theorem 6.23. *Assuming Setup 6.21, the ideal $P_{2t}(Z)$ is symbolic F-split.*

Proof. We set $h = \frac{(r-2t+1)(r-2t+2)}{2} = \operatorname{ht}(P_{2t}(Z))$ [61, Theorem 2.3]. Let $b = \lfloor r/2 \rfloor$, and $f = f_{2t}(Z)$. If r is odd, we have

$$f \in \left(\prod_{\ell=t}^{b-1} P_{2\ell}(Z)\right)^4 \cdot P_{2b}(Z)^3$$

$$\subseteq \left(\prod_{\ell=t}^{b-1} P_{2t}(Z)^{(\ell-t+1)}\right)^4 \cdot \left(P_{2t}(Z)^{(b-t+1)}\right)^3 \qquad [59, \text{Theorem 4.6}]$$

$$= \left(\prod_{a=1}^{b-t} P_{2t}(Z)^{(a)}\right)^4 \cdot \left(P_{2t}(Z)^{(b-t+1)}\right)^3$$

$$\subseteq P_{2t}(Z)^{(2(b-t)(b-t+1))} \cdot P_{2t}(Z)^{(3(b-t+1))}$$

$$\subseteq P_{2t}(Z)^{(2(b-t)(b-t+1)+3(b-t+1))}$$

$$= P_{2t}(Z)^{(h)}.$$

On the other hand, if r is even, we have

$$f \in \left(\prod_{\ell=t}^{b-1} P_{2\ell}(Z)\right)^4 \cdot P_r(Z)$$

$$\subseteq \left(\prod_{\ell=t}^{b-1} P_{2t}(Z)^{(\ell-t+1)}\right)^4 \cdot P_{2t}(Z)^{(b-t+1)} \qquad [59, \text{Theorem 4.6}]$$

$$= \left(\prod_{a=1}^{b-t} P_{2t}(Z)^{(a)}\right)^4 \cdot P_{2t}(Z)^{(b-t+1)}$$

$$\subseteq P_{2t}(Z)^{(2(b-t)(b-t+1))} \cdot P_{2t}(Z)^{(b-t+1)}$$

$$\subseteq P_{2t}(Z)^{(2(b-t)(b-t+1)+(b-t+1))}$$

$$= P_{2t}(Z)^{(h)}.$$

The conclusion follows from Remarks 6.22 and Lemma 6.2(2). □

The previous result leads to the following homological consequences.

Theorem 6.24. *Assuming Setup 6.21, the limit*

$$\lim_{n\to\infty} \frac{\operatorname{reg}(R/P_{2t}(Z)^{(n)})}{n}$$

exists and

$$\operatorname{depth}(R/P_{2t}(Z)^{(n)})$$

stabilizes for $n \gg 0$, Furthermore,

$$\lim_{n\to\infty} \operatorname{depth}(R/P_{2t}(Z)^{(n)}) = \dim(R) - \dim\left(\mathscr{R}^s(P_{2t}(Z))/\mathfrak{m}\mathscr{R}^s(P_{2t}(Z))\right) = t(2t-1) - 1.$$

Proof. We recall that $\mathscr{R}^s(P_{2t}(Z))$ is Noetherian [1] (see also [2, Section 3]). Hence, the result follows by combining Theorem 6.23 and Theorem 4.10. Since $\mathscr{R}^s(P_{2t}(Z))$ is Cohen–Macaulay [2, Corollary 3.2], we have

$$\begin{aligned}\lim_{n\to\infty} \operatorname{depth}\left(R/P_{2t}(Z)^{(n)}\right) &= \dim(R) - \dim\left(\mathscr{R}^s(P_{2t}(Z))/\mathfrak{m}\mathscr{R}^s(P_{2t}(Z))\right) && \text{by Remark 4.11,}\\ &= \min\left\{\operatorname{depth}\left(R/P_{2t}(Z)^{(n)}\right)\right\} && \text{by Theorem 4.10,}\\ &= \operatorname{grade}\left(\mathfrak{m}\operatorname{gr}^s(P_{2t}(Z))\right) && \text{[14, Proposition 9.23].}\end{aligned}$$

It remains to show $\operatorname{grade}\left(\mathfrak{m}\operatorname{gr}^s(P_{2t}(Z))\right) = t(2t-1)-1$. This computation is already known in the generic case in arbitrary characteristic [14, Proposition 10.8], we adapt the proof for ideals of Pfaffians. We let H be the poset of all the Pfaffians of Z, and consider the partial order induced by

$$\operatorname{pf}\left(Z_{[i_1,\dots,i_{2u}]}\right) \leqslant \operatorname{pf}\left(Z_{[a_1,\dots,a_{2v}]}\right) \iff u \geqslant v \text{ and } i_s \leqslant a_s \text{ for all } 1 \leqslant s \leqslant 2v.$$

We let Ω be the subposet of H consisting of the $2s$-Pfaffians with $s \geqslant t$. We note that Ω is also given by

$$\Omega = \{\delta \in H \mid \delta \leqslant [r-2t+1,\dots,r]\}.$$

Since $\mathscr{R}^s(P_{2t}(Z))$ is Cohen–Macaulay, then so is $\operatorname{gr}^s(P_{2t}(Z))$ [95, Proof of Proposition 2.4]. Then, we have $\operatorname{grade}\left(\mathfrak{m}\operatorname{gr}^s(P_{2t}(Z))\right) = \operatorname{rk}(H) - \operatorname{rk}(\Omega)$ [14, Proof of Proposition 10.8], where the rank of a poset P is defined as

$$\operatorname{rk}(P) = \max\{i \mid \text{ there exists a chain } p_1 < p_2 < \cdots < p_i \text{ of elements of } P\}.$$

We note that every maximal chain of H has length $\operatorname{rk}(H) = \dim(R) = \binom{r}{2}$ [14, Lemma 5.13(d) and Proposition 5.10]. We also note that a maximal chain of Ω can be extended to a maximal chain of H by adjoining a maximal chain of Pfaffians of the submatrix of Z with rows $\{r-t+1,\dots,r\}$ and columns $\{r-t+1,\dots,r\}$. Then, $\operatorname{rk}(H) - \operatorname{rk}(\Omega) = \binom{r}{2} - \left(\binom{r}{2} - \binom{2t}{2} + 1\right) = t(2t-1)-1$, finishing the proof. □

We now show that $\mathscr{R}^s(P_{2t}(Z))$ and $\operatorname{gr}^s(P_{2t}(Z))$ are strongly F-regular.

Theorem 6.25. *Assuming Setup 6.21, the rings $\mathscr{R}^s(P_{2t}(Z))$ and $\operatorname{gr}^s(P_{2t}(Z))$ are strongly F-regular.*

Proof. We know that $\mathscr{R}^s(P_{2t}(Z))$ and $\operatorname{gr}^s(P_{2t}(Z))$ are Noetherian [1] (see also [2, Section 3]). We proceed by induction on t. If $t = 1$, then the result follows because $P_{2t}(Z) = \mathfrak{m}$. We now assume the result is true for the ideal of $(2t-2)$-Pfaffians.

Let $f = f_{2t}(Z)$, and note that $\operatorname{in}_<(f)$ is a square-free monomial, which is not divisible by $z_{1,2}$, because $t \geqslant 2$. Let $g = \frac{\prod_{i<j} z_{i,j}}{z_{1,2}\operatorname{in}_<(f)}$. We note that $\operatorname{in}(f^{p-1}g^{p-1}) = \frac{\prod_{i<j} z_{i,j}^{p-1}}{z_{1,2}^{p-1}}$, and so,

$f^{p-1}g^{p-1} \notin \mathfrak{m}^{[p]}$. Let $\phi = \Phi(f^{(p-1)/p}g^{(p-1)/p}-)$, where $\Phi : R^{1/p} \to R$ denotes the trace map introduced in Remark 2.3. We note $\varphi(z_{1,2}^{(p-1)/p}) = 1$.

We have $f \in P_{2t}(Z)^{(\mathrm{ht}(P_{2t}(Z))}$, as shown in the proof of Theorem 6.23. We therefore have $f^{p-1} \in \left(P_{2t}(Z)^{(n+1)}\right)^{[p]} : P_{2t}(Z)^{(np+1)}$ for every $n \in \mathbb{Z}_{\geqslant 0}$ [36, Lemma 2.6] (see also proof of Corollary 5.10). Thus, ϕ induces maps $\Psi : \mathscr{R}^s(P_{2t}(Z))^{1/p} \to \mathscr{R}^s(P_{2t}(Z))$ and $\overline{\Psi} : \mathrm{gr}^s(P_{2t}(Z))^{1/p} \to \mathrm{gr}^s(P_{2t}(Z))$, which satisfy

$$\Psi\left(z_{1,2}^{(p-1)/p}\right) = 1 \quad \text{and} \quad \overline{\Psi}\left(\overline{z_{1,2}}^{(p-1)/p}\right) = 1. \tag{6.3.1}$$

Let $A = K[U]$, where $U = (u_{i,j})_{3\leqslant i<j\leqslant r}$ be a generic skew-symmetric matrix of size $(r-2)\times(r-2)$, and let $S = A[z_{1,3}, \dots, z_{1,r}, z_{2,3}, \dots, z_{2,r}]$. We have an isomorphism $\gamma : R[z_{1,2}^{-1}] \to S[z_{1,2}^{-1}]$ defined by $z_{i,j} \mapsto u_{i,j} - z_{1,j}z_{2,i}z_{1,2}^{-1} + z_{1,i}z_{2,j}z_{1,2}^{-1}$ for $3 \leqslant i < j \leqslant r$, $z_{1,i} \mapsto z_{1,i}$ for $i \geqslant 2$, and $z_{2,i} \mapsto z_{2,i}$ for $i \geqslant 3$. Furthermore, $\gamma(P_{2t}(Z))R[z_{1,2}^{-1}]) = P_{2t-2}(U)S[z_{1,2}^{-1}]$, and then $\gamma(P_{2t}(Z))^{(n)}R[z_{1,2}^{-1}]) = \Psi(P_{2t-2}(U))^{(n)}S[z_{1,2}^{-1}])$ for every $n \in \mathbb{Z}_{\geqslant 0}$ [61, Lemma 1.2] (see also [14, Lemma 10.1]). By the induction hypothesis, the rings $\mathscr{R}^s(P_{2t-2}(U))$ and $\mathrm{gr}^s(P_{2t-2}(U))$ are strongly F-regular. It follows that $\mathscr{R}^s(P_{2t-2}(U)) \otimes_A S[z_{1,2}^{-1}]$ and $\mathrm{gr}^s(P_{2t-2}(U)) \otimes_A S[z_{1,2}^{-1}]$ are strongly F-regular, because strong F-regularity is preserved by adding variables and localizing. Therefore, thanks to the isomorphism γ, the rings $\mathscr{R}^s(P_{2t}(Z)) \otimes_R R[z_{1,2}^{-1}]$ and $\mathrm{gr}^s(P_{2t}(Z)) \otimes_R R[z_{1,2}^{-1}]$ are also strongly F-regular. From this and Equation (6.3.1), we conclude that $\mathscr{R}^s(P_{2t}(Z))$ and $\mathrm{gr}^s(P_{2t}(Z))$ are strongly F-regular [48, Theorem 3.3]. □

In the following result, we show that ordinary Rees algebras of ideals of Pfaffians are also F-split.

Theorem 6.26. *In addition to assuming Setup 6.21, suppose $p > \min\{2t, r-2t\}$. Then the Rees algebra $\mathscr{R}(P_{2t}(Z))$ is F-split.*

Proof. Let $f = f_2(Z)$, and note $f \in P_{2\ell}(Z)^{\mathrm{ht}(P_{2\ell}(Z))}$, as shown in the proof of Theorem 6.23. It follows that $f^{p-1} \in \left(P_{2\ell}(Z)^{(n+1)}\right)^{[p]} : P_{2\ell}(Z)^{(np+1)}$ for every $\ell \leqslant r/2$ and $n \in \mathbb{Z}_{\geqslant 0}$ [36, Lemma 2.6] (cf. proof of Corollary 5.10). Thus,

$$f^{p-1}P_{2\ell}(Z)^{((n+1)p)} \subseteq f^{p-1}I_\ell(X)^{(np+1)} \subseteq \left(P_{2\ell}(Z)^{(n+1)}\right)^{[p]}$$

for every $\ell \leqslant r/2$ and $n \in \mathbb{Z}_{\geqslant 0}$. We then get

$$\begin{aligned} f^{p-1}P_{2t}(Z)^{np} &= f^{p-1}\left(\bigcap_{\ell=1}^{t} P_{2\ell}(Z)^{((t-\ell+1)np)}\right) && \text{[26, Proposition 2.6]} \\ &\subseteq \bigcap_{\ell=1}^{t} f^{p-1}\left(P_{2\ell}(Z)^{((t-\ell+1)np)}\right) \\ &\subseteq \bigcap_{\ell=1}^{t} \left(P_{2\ell}(Z)^{((t-\ell+1)n)}\right)^{[p]} \end{aligned}$$

$$\subseteq \left(\bigcap_{\ell=1}^{t} P_{2\ell}(Z)^{((t-\ell+1)n)} \right)^{[p]}$$

$$= (P_{2t}(Z)^n)^{[p]} \qquad [26, \text{Proposition 2.6}].$$

The result follows from Remark 6.22 and Lemma 6.2(2). □

As in the previous subsections, we end with the following results about degrees of defining equations for ordinary blowup algebras for ideals of Pfaffians of generic skew-symmetric matrices.

Theorem 6.27. *Assume Setup 6.21. Set* $\mu = \binom{r}{2t}$.

(1) *Suppose* $\deg(z_{i,j}) = 0$ *for every i, j, then the defining equations of* $\mathscr{R}(P_{2t}(Z))$ *over R have degree at most* $\min\{\binom{r}{2} + 1, \mu\}$.
(2) *Suppose* $\deg(z_{i,j}) = 1$ *for every i, j, then the defining equations of* $\mathscr{R}(P_{2t}(Z))$ *over K have total degree at most* $\binom{r}{2} + \mu(t+1)$.

Proof. The result follows from Theorems 6.26 and 3.3, and Proposition 2.8. □

Theorem 6.28. *Assume Setup 6.21. For* $j = t, \dots, \lfloor r \rfloor$, *set* $\mu_j = \binom{r}{2j}$.

(1) *Suppose* $\deg(z_{i,j}) = 0$ *for every i, j, then the defining equations of* $\mathscr{R}^s(P_{2t}(Z))$ *over R have degree at most* $\min\{\binom{r}{2} + 1 + \sum_{j=t+1}^{r} \mu_j(j-t), \sum_{j=t}^{r} \mu_j(j-t+1)\}$, *and of* $\mathrm{gr}^s(P_{2t}(Z))$ *over* $R/P_{2t}(Z)$, *have degree at most* $\min\{\binom{r}{2} + \sum_{j=t+1}^{\lfloor r/2 \rfloor} \mu_j(j-t), \sum_{j=t}^{\lfloor r/2 \rfloor} \mu_j(j-t+1)\}$.
(2) *Suppose* $\deg(z_{i,j}) = 1$ *for every i, j, then the defining equations of* $\mathscr{R}^s(P_{2t}(Z))$ *and* $\mathrm{gr}^s(P_{2t}(Z))$ *over K have total degree at most* $\binom{r}{2} + \sum_{j=t}^{\lfloor r/2 \rfloor} \mu_j(2j-t+1)$.

Proof. By Theorem 6.23 and Theorem 4.7, the algebras $\mathscr{R}^s(P_{2t}(Z))$ and $\mathrm{gr}^s(P_{2t}(Z))$ are F-split. Both parts of the result now follow from Theorem 3.4, Proposition 2.8, and the equality

$$\mathscr{R}^s(P_{2t}(Z)) = R[P_{2t}(Z)T, P_{2t+2}(Z)T^2, \dots, P_{2\lfloor r/2 \rfloor}(Z)T^{\lfloor r/2 \rfloor}] \quad [1] \text{ (see also [2, Section 3])}.$$ □

6.4 | Ideals of minors of a Hankel matrix

We first recall the definition of Hankel matrix.

Definition 6.29. Let $j, c \in \mathbb{Z}_{>0}$, with $j \leqslant c$. Let $w_1, \dots, w_c$ be variables. We denote by W_j^c the $j \times (c+1-j)$ *Hankel matrix*, which has the following entries

$$W_j^c = \begin{pmatrix} w_1 & w_2 & \cdots & w_{c+1-j} \\ w_2 & w_3 & \cdots & \cdots \\ w_3 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ w_j & \cdots & \cdots & w_c \end{pmatrix}.$$

Setup 6.30. Let $j, c \in \mathbb{Z}_{>0}$, with $j \leqslant c$, and W_j^c be the $j \times (c+1-j)$ Hankel matrix. Let K be an F-finite field of characteristic $p > 0$, $R = K[W_j^c]$, and $\mathfrak{m} = (w_1, \dots, w_c)$. For $t \in \mathbb{Z}_{>0}$ with $t \leqslant \min\{j, c+1-j\}$, we denote by $I_t(W_j^c)$ the ideal generated by the minors of size t of W_j^c. If c is odd, we set $m = \frac{c+1}{2}$, and we let

$$f_{\mathrm{odd}}(W_j^c) = \det\left(W_m^c\right) \det\left((W_m^c)_{[1,m-1]}^{[2,m]}\right).$$

If c is even, we set $m = \frac{c}{2}$, and we let

$$f_{\mathrm{even}}(W_j^c) = \det\left((W_m^c)_{[1,m]}^{[1,m]}\right) \det\left((W_m^c)_{[2,m+1]}^{[1,m]}\right).$$

Consider the lexicographical monomial order on R induced by

$$w_1 > w_3 > \dots > w_c > w_2 > w_4 > \dots > w_{c-1}.$$

Remark 6.31. We note that the initial forms $\mathrm{in}_<(f_{\mathrm{odd}}(W_j^c))$ and $\mathrm{in}_<(f_{\mathrm{even}}(W_j^c))$ are square-free monomials.

Remark 6.32. It is well known that $I_t(W_j^c)$ only depends on c and t, that is, $I_t(W_j^c) = I_t(W_t^c)$ for every $t \leqslant \min\{j, c+1-j\}$.

Theorem 6.33. *Assuming Setup 6.30, the ideal $I_t(W_j^c)$ is symbolic F-split for every $t \leqslant \min\{j, c+1-j\}$. In particular, the rings $\mathscr{R}^s(I_t(W_j^c))$ and $\mathrm{gr}^s(I_t(W_j^c))$ are F-split.*

Proof. Let $m = \lfloor \frac{c+1}{2} \rfloor$, and observe that $t \leqslant m$. Moreover, we have $I_t(W_j^c) = I_t(W_m^c)$ by Remark 6.32.

If c is odd, let $f = f_{\mathrm{odd}}(W_j^c)$, and observe $h = \mathrm{ht}(I_t(W_j^c)) = c - 2t + 2 = 2m - 2t + 1$. We then have

$$\begin{aligned} f &\in I_m(W_m^c) I_{m-1}(W_m^c) \\ &\subseteq I_t(W_m^c)^{(m-t+1)} I_t(W_m^c)^{(m-t)} && \text{[15, Theorem 3.16 (a)]} \\ &\subseteq I_t(W_m^c)^{(2m-2t+1)} = I_t(W_m^c)^{(h)}. \end{aligned}$$

If c is even, we let $f = f_{\mathrm{even}}(W_j^c)$, and we observe that $h = \mathrm{ht}(I_t(W_j^c)) = c - 2t + 2 = 2m - 2t + 2$. In this case, we have

$$\begin{aligned} f &\in I_m(W_m^c) I_m(W_m^c) \\ &\subseteq I_t(W_m^c)^{(m-t+1)} I_t(W_m^c)^{(m-t+1)} && \text{[15, Theorem 3.16(a)]} \\ &\subseteq I_t(W_m^c)^{(2m-2t+2)} = I_t(W_m^c)^{(h)}. \end{aligned}$$

In both cases, we have shown that $f \in I_t(W_m^c)^{(h)}$. The first statement now follows from Remark 6.31 and Lemma 6.2(1), and the second statement from Theorem 4.7. □

We obtain the following homological consequences.

Theorem 6.34. *Assuming Setup 6.11, the limit*

$$\lim_{n\to\infty} \frac{\operatorname{reg}(R/I_t(W_j^c)^{(n)})}{n}$$

exists and

$$\operatorname{depth}(R/I_t(W_j^c)^{(n)})$$

stabilizes for $n \gg 0$.

Proof. Since $\mathscr{R}^s\left(I_t(W_j^c)\right)$ is Noetherian [15, Theorem 4.1], the result follows by combining Theorems 6.33 and 4.10. □

We now show that ordinary Rees algebras of a determinantal ideals of Hankel matrices are F-split.

Theorem 6.35. *Assume Setup 6.30. Then, the Rees algebra $\mathscr{R}(I_t(W_j^c))$ is F-split.*

Proof. Let $m = \lfloor \frac{c+1}{2} \rfloor$, and observe that $t \leqslant m$. We have $I_\ell(W_j^c) = I_\ell(W_m^c)$ for every $\ell \leqslant m$ by Remark 6.32. If c is odd, we set $f = f_{\text{odd}}(W_j^c)$, otherwise we set $f = f_{\text{even}}(W_j^c)$. From the proof of Theorem 6.33, we see that $f \in I_\ell(W_m^c)^{(\operatorname{ht}(I_\ell(W_m^c)))}$ for every $\ell \leqslant m$. It follows that $f^{p-1} \in \left(I_\ell(W_m^c)^{(n+1)}\right)^{[p]} : \left(I_\ell(W_m^c)\right)^{(np+1)}$ for every $\ell \leqslant m$ and $n \in \mathbb{Z}_{\geqslant 0}$ [36, Lemma 2.6] (cf. proof of Corollary 5.10). Thus,

$$f^{p-1} I_\ell(W_m^c)^{((n+1)p)} \subseteq f^{p-1} I_\ell(W_m^c)^{(np+1)} \subseteq \left(I_\ell(W_m^c)^{(n+1)}\right)^{[p]}$$

for all $\ell \leqslant m$ and $n \in \mathbb{Z}_{\geqslant 0}$. Then,

$$\begin{aligned}
f^{p-1} I_t(W_m^c)^{np} &= f^{p-1}\left(\bigcap_{\ell=1}^{t} I_\ell(W_m^c)^{(np(t+1-\ell))}\right) && \text{[15, Theorem 3.16 (a)]}\\
&\subseteq \bigcap_{\ell=1}^{t} \left(f^{p-1} I_\ell(W_m^c)^{(np(t+1-\ell))}\right)\\
&\subseteq \bigcap_{\ell=1}^{t} \left(I_\ell(W_m^c)^{(n(t+1-\ell))}\right)^{[p]}\\
&= \left(\bigcap_{\ell=1}^{t} I_\ell(W_m^c)^{(n(t+1-\ell))}\right)^{[p]}\\
&= \left(I_t(W_m^c)^n\right)^{[p]} && \text{[15, Theorem 3.16(a)].}
\end{aligned}$$

The conclusion follows from Remarks 6.31 and Lemma 6.2(2). □

Finally, we prove the following results about degrees of defining equations for ordinary Rees and associated graded algebra for ideals of minors of generic Hankel matrices.

Theorem 6.36. *Assume Setup 6.30. Set* $\mu = \binom{c+1-t}{t}$.

(1) *Suppose* $\deg(w_i) = 0$ *for every i, then the defining equations of* $\mathscr{R}(I_t(W_j^c))$ *over R have degree at most* $\min\{c, \mu\}$.
(2) *Suppose* $\deg(w_i) = 1$ *for every i, then the defining equations of* $\mathscr{R}(I_t(W_j^c))$ *over K have total degree at most* $c + \mu(t+1)$.

Proof. The result follows from Theorems 6.35 and 3.3, and Proposition 2.8. □

Theorem 6.37. *Assume Setup 6.30. For* $j = t, \dots, m$, *set* $\mu_j = \binom{c+1-j}{j}$.

(1) *Suppose* $\deg(w_i) = 0$ *for every i. The defining equations of* $\mathscr{R}^s(I_t(W_j^c))$ *over R have degree at most* $\min\{c + 1 + \sum_{j=t+1}^m \mu_j(j-t), \sum_{j=t}^m \mu_j(j-t+1)\}$, *and the defining equations of* $\mathrm{gr}^s(I_t(W_j^c))$ *over* $R/I_t(W_j^c)$ *have degree at most* $\min\{c + \sum_{j=t+1}^m \mu_j(j-t), \sum_{j=t}^m \mu_j(j-t+1)\}$.
(2) *Suppose* $\deg(w_i) = 1$ *for every i, then the defining equations of* $\mathscr{R}^s(I_t(W_j^c))$ *and* $\mathrm{gr}^s(I_t(W_j^c))$ *over K have total degree at most* $c + \sum_{j=t}^m \mu_j(2j-t+1)$.

Proof. The result follows from Theorem 6.33, Theorem 3.4, Proposition 2.8, and the equality

$$\mathscr{R}^s\left(I_t(W_j^c)\right) = R[I_t(W_j^c)T, I_{t+1}(W_j^c)T^2, \dots, I_m(W_j^c)T^{m-t+1}] \quad \text{[15, Proposition 4.1]}.$$ □

6.5 | Binomial edge ideals

We now give another example of symbolic F-split ideals, the binomial edge ideals, which are generated by minors of certain matrices related to graphs.

Definition 6.38 [43, 74]. Let $G = (V(G), E(G))$ be a simple graph such that $V(G) = [n] = \{1, 2, \dots, n\}$. Let K be a field and $S = K[x_1, \dots, x_n, y_1, \dots, y_n]$ the ring of polynomials in $2n$ variables. The *binomial edge ideal*, $\mathcal{J}_G$, of G is defined by

$$\mathcal{J}_G = \left(x_i y_j - x_j y_i \mid \text{ for } \{i, j\} \in E(G)\right).$$

Definition 6.39 [43]. A graph G on $[n]$ is *closed* if G has a labeling of the vertices such that for all edges $\{i, j\}$ and $\{k, l\}$ with $i < j$ and $k < l$, one has $\{j, l\} \in E(G)$ if $i = k$, and $\{i, k\} \in E(G)$ if $j = l$.

The binomial ideals of closed graphs class of graphs can be characterized via Gröbner bases for binomial edge ideals [43]. This class of binomial edge ideals has been studied in several works [19, 31, 32, 43]. For example, it is known that, for closed graphs, a binomial edge ideal is equidimensional if and only if it is Cohen–Macaulay [32, Theorem 3.1]. Since their initial ideals correspond to a bipartite graphs, we have that the ordinary and symbolic powers of closed binomial edge ideals coincide [31, Corollary 3.4]. This follows from the analogous result for monomial edge ideals of bipartite graphs [85, Theorem 5.9].

Proposition 6.40. *Let G be a closed connected graph such that $\mathcal{J}_G$ is equidimensional. Then, $\mathcal{J}_G$ is symbolic F-split. In particular, the rings $\mathscr{R}^s(\mathcal{J}_G)$ and $\mathrm{gr}^s(\mathcal{J}_G)$ are F-split.*

Proof. Since G is connected and $S/\mathcal{J}_G$ is equidimensional, we have bigheight$(\mathcal{J}_G) = n-1$ ([32, Theorem 3.1], [43, Corollary 3.4]). We also have $\mathcal{J}_G^{(n)} = \mathcal{J}_G^n$ [31, Corollary 3.4]. Now, since a closed graph is a *proper interval graph*, it contains a Hamiltonian graph [4, 19]. We assume with out loss of generality that this path is given by $1, \dots, n$, in this order. We set $f = \prod_{i=1}^{n-1}(x_i y_{i+1} - x_{i+1} y_i)$. Then, $f^{p-1} \in \mathcal{J}^{((n-1)(p-1))} \setminus (x_1^p, \dots, x_n^p, y_1^p, \dots, y_n^p)$. Therefore, $\mathcal{J}_G$ is symbolic F-split by Corollary 5.10. The second statement follows from Theorem 4.7. □

7 | EXAMPLES OF MONOMIAL *F*-SPLIT FILTRATIONS

In this section, we present several classes of filtrations of monomial ideals that are F-split. The list of examples include symbolic powers and rational powers of squarefree monomial ideals, and initial ideals of symbolic and ordinary powers of determinantal ideals of generic and Hankel matrices of variables, and of Pfaffians of generic skew-symmetric matrices.

Throughout this section, we assume the following setup.

Setup 7.1. Let R be a standard graded polynomial ring $R = K[x_1, \dots, x_d]$ over an F-finite field K of characteristic $p > 0$. For a vector $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d_{\geqslant 0}$, we set $\mathbf{x}^{\mathbf{n}} = x_1^{n_1} \cdots x_d^{n_d}$.

7.1 | *F*-split filtrations obtained from monomial valuations

Assuming Setup 7.1, let $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{Z}^d_{\geqslant 0}$ and consider the following function on the set of monomials in R :

$$v(\mathbf{x}^{\mathbf{n}}) = \mathbf{n} \cdot \mathbf{a}.$$

We extend v to the entire R by setting

$$v(f) = v\Big(\sum c_i \mathbf{x}^{\mathbf{n}_i}\Big) := \min\{v(\mathbf{x}^{\mathbf{n}_i})\}$$

for a polynomial $f = \sum c_i \mathbf{x}^{\mathbf{n}_i} \in R$ with $0 \neq c_i \in K$. Such a function is called a *monomial valuation* of R [58, Definition 6.1.4].

The following is the main result of this subsection.

Theorem 7.2. *Assuming Setup 7.1, let $v_1, \dots, v_r$ be monomial valuations of R. For each $n \in \mathbb{Z}_{\geqslant 0}$, we set*

$$I_n = \{f \in R \mid v_i(f) \geqslant n, \textit{ for every } 1 \leqslant i \leqslant r\}.$$

(1) *The sequence* $\mathbb{I} = \{I_n\}_{n\in\mathbb{Z}_{\geqslant 0}}$ *is an* F*-split filtration of monomial ideals.*
(2) $\mathscr{R}(\mathbb{I})$ *is Noetherian and strongly* F*-regular.*

Proof. We begin with the proof of (1). We note that each I_n is a monomial ideal by the definition of valuation [58, Definition 6.1.1]. Let $\rho : K^{1/p} \to K$ be a splitting. Let $\phi : R^{1/p} \to R$ be the K-linear map defined on the monomials of R as:

$$\phi\left((c\mathbf{x}^{\mathbf{n}})^{1/p}\right) = \phi\left(c^{1/p}\left(x_1^{n_1}\cdots x_d^{n_d}\right)^{1/p}\right)$$
$$= \begin{cases} \rho(c^{1/p})x_1^{n_1/p}\cdots x_d^{n_d/p}, & \text{if } n_1 \equiv \cdots \equiv n_d \equiv 0 \pmod p,\\ 0 & \text{otherwise.}\end{cases}$$

In particular, $\phi(c^{1/p}(\mathbf{x}^{\mathbf{n}})^{1/p}) = \phi(c^{1/p})\phi((\mathbf{x}^{\mathbf{n}})^{1/p})$. If $\widetilde{c}, c \in K$ and $\mathbf{x}^{\widetilde{\mathbf{n}}}, \mathbf{x}^{\mathbf{n}} \in R$ are monomial, then

$$\phi((\widetilde{c}\mathbf{x}^{\widetilde{\mathbf{n}}})c^{1/p}\mathbf{x}^{\mathbf{n}/p}) = \phi(\widetilde{c}c^{1/p})\phi(\mathbf{x}^{\widetilde{\mathbf{n}}}\mathbf{x}^{\mathbf{n}/p}) = \widetilde{c}\phi(c^{1/p})\mathbf{x}^{\widetilde{\mathbf{n}}}\phi(\mathbf{x}^{\mathbf{n}/p}) = \widetilde{c}\mathbf{x}^{\widetilde{\mathbf{n}}}\phi(c^{1/p}\mathbf{x}^{\mathbf{n}/p}).$$

Then,

$$\phi\left(f\left(g^{1/p}\right)\right) = f\phi\left(g^{1/p}\right) \quad \text{for every} \quad f, g \in R. \tag{7.1.1}$$

We have that ϕ is an R-homomorphism and thus a splitting of the natural inclusion $R \hookrightarrow R^{1/p}$.

Now, let $n \in \mathbb{Z}_{\geqslant 0}$ be arbitrary and $\mathbf{x}^{\mathbf{n}} = x_1^{n_1}\cdots x_d^{n_d} \in I_{np+1}$ be such that $\phi((\mathbf{x}^{\mathbf{n}})^{1/p}) \neq 0$. Therefore, $p | n_i$ for every $i = 1, \dots, d$ and then

$$v_i\left(\phi((\mathbf{x}^{\mathbf{n}})^{1/p})\right) = \sum_{j=1}^{d} \frac{n_j}{p} v_i(x_j) \geqslant \left\lceil \frac{np+1}{p} \right\rceil = n+1,$$

for every $1 \leqslant i \leqslant r$. It follows that $\phi((\mathbf{x}^{\mathbf{n}})^{1/p}) \in I_{n+1}$. Therefore, $\phi((I_{np+1})^{1/p}) \in I_{n+1}$ for every $n \in \mathbb{Z}_{\geqslant 0}$, which implies $\mathbb{I}$ is an F-split filtration.

We continue with the proof of (2). Let $\mathbf{a}_i = (v_i(x_1), \dots, v_i(x_d)) \in \mathbb{Z}^d_{\geqslant 0}$ and let $\mathscr{M}_i$ be the *affine semigroup*

$$\mathscr{M}_i = \left\{(\mathbf{n}, n) \in \mathbb{Z}^{d+1}_{\geqslant 0} \mid (\mathbf{a}_i, -1)\cdot(\mathbf{n}, n) \geqslant 0\right\} \subseteq \mathbb{Z}^{d+1}_{\geqslant 0}$$

for every $1 \leqslant i \leqslant r$. Then, $\mathscr{M} = \mathscr{M}_1 \cap \dots \cap \mathscr{M}_r$ is a finitely generated affine semigroup [44, Theorem 1.1 and Corollary 1.2]. We note that $\mathscr{R}(\mathbb{I}) = K[\mathscr{M}]$, and so, $\mathscr{R}(\mathbb{I})$ is a finitely generated K-algebra.

Since $\mathscr{R}(\mathbb{I}) = K[\mathscr{M}]$ is F-split regardless of the characteristic of the field by part (1), we have that $\mathscr{M}$ is a normal monoid [13, Corollary 6.3]. As a consequence, $\mathscr{R}(\mathbb{I})$ is a strongly F-regular ring [47, Theorem 1], finishing the proof. □

In the following example, we include several well-studied filtrations of monomial ideals covered by Theorem 7.2.

Example 7.3. Some examples of F-split filtrations of monomial ideals.

(1) (*Rational powers of monomial ideals*) Let I be a monomial ideal and $u_1, \dots, u_r$ its *Rees valuations*, which in this setting are also monomial valuations (see [58, Proposition 10.3.4]). For each i, set $u_i(I) = \min\{u_i(f) \mid f \in I\}$ and let $u = \mathrm{mcm}(u_1(I), \dots, u_r(I))$. For each $1 \leqslant i \leqslant, r$, consider the monomial valuation $v_i = \frac{u}{u_i(I)} u_i$. Then, the monomial ideal

$$I_n = \{f \in R \mid v_i(f) \geqslant n, \text{ for every } 1 \leqslant i \leqslant r\}$$

is the $\frac{n}{u}$-*rational power* of I (see [58, Proposition 10.5.5], [66]). Therefore, by Theorem 7.2(1), $\mathbb{I} = \{I_n\}_{n \in \mathbb{Z}_{\geqslant 0}}$ is an F-split filtration. Thus, $\mathscr{R}(\mathbb{I})$ is Noetherian and strongly F-regular by Theorem 7.2(2) and $\mathrm{gr}(\mathbb{I})$ are F-split by Theorem 4.7. Since these algebras are Noetherian, the conclusions of Theorem 4.10 hold for $\mathbb{I}$.

The sequence of *integral closure powers* $\{\overline{I^n}\}_{n \in \mathbb{Z}_{\geqslant 0}}$ is a subsequence of the rational powers. Indeed, one has $\overline{I^n} = I_{nu}$ [58, Proposition 10.5.2 (5)]. Thus, the conclusions of Theorem 4.10 hold for $\{\overline{I^n}\}_{n \in \mathbb{Z}_{\geqslant 0}}$ and, since direct summands of strongly F-regular rings are strongly F-regular, the *normal Rees algebra* $\overline{\mathscr{R}(I)} = \bigoplus_{n \in \mathbb{Z}_{\geqslant 0}} \overline{I^n} T^n$ is also strongly F-regular.

(2) (*Symbolic powers of squarefree monomial ideals*) Let I be a square-free monomial ideal and $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be its minimal primes. The functions $v_i(f) = \max\{n \mid f \in \mathfrak{p}_i^n\}$ are monomial valuations, therefore, the *symbolic powers* of I

$$I^{(n)} = \{g \in R \mid v_i(g) \geqslant n, \text{ for every } 1 \leqslant i \leqslant r\}$$

form an F-split filtration.

7.2 | F-split filtrations obtained from initial ideals

Setup 7.4. Assume Setup 7.1 and suppose that R is equipped with a monomial order $<$. If $\mathbb{I}$ is a filtration, then $\mathrm{in}_<(\mathbb{I}) = \{\mathrm{in}(I_n)\}_{n \in \mathbb{Z}_{\geqslant 0}}$ is also a filtration.

In the following proposition, we provide sufficient conditions for certain filtrations and algebras obtained from initial ideals to be F-split.

Proposition 7.5. *Assuming Setup 7.4, let $I \subseteq R$ be a homogeneous equidimensional radical ideal of height h such that $\mathrm{in}_<(I)$ is radical.*

(1) *If there exists $f \in I^{(h)}$ such that $\mathrm{in}_<(f)$ is square free, then the filtration $\{\mathrm{in}_<(I^{(n)})\}_{n \in \mathbb{Z}_{\geqslant 0}}$ is F-split.*

(2) *If there exists $f \in \bigcap_{n \in \mathbb{Z}_{\geqslant 0}} (I^n)^{[p]} : I^{np}$ such that $\mathrm{in}_<(f)$ is square free, then $\mathscr{R}(\{\mathrm{in}_<(I^n)\})$ is F-split.*

Proof. We begin with (1). Since $f \in I^{(h)}$, we have $f^{p-1} \in \left(I^{(n+1)}\right)^{[p]} : I^{(np+1)}$ for every $n \in \mathbb{Z}_{\geqslant 0}$ [36, Lemma 2.6] (cf. proof of Corollary 5.10). Thus,

$$\mathrm{in}_< \left(f^{p-1}\right) \mathrm{in}_< \left(I^{(np+1))}\right) = \mathrm{in}_< \left(f^{p-1} I^{(np+1)}\right)$$

$$\subseteq \operatorname{in}_<\left(\left(I^{(n+1)}\right)^{[p]}\right)$$

$$= \operatorname{in}_<\left(I^{(n+1)}\right)^{[p]}.$$

Then, $\operatorname{in}_<(f^{p-1}) \notin \mathfrak{m}^{[p]}$ and $\operatorname{in}_<(f^{p-1}) \in \left(\operatorname{in}_<(I^{(n+1)})\right)^{[p]} : \operatorname{in}_<(I^{(np+1)})$ for every $n \in \mathbb{Z}_{\geqslant 0}$. It follows that $\{\operatorname{in}_<(I^{(n)})\}_{n\in\mathbb{Z}_{\geqslant 0}}$ is an F-split filtration by Proposition 4.6.

We continue with (2). The assumptions of this part guarantee that

$$\operatorname{in}_<\left(f^{p-1}\right)\operatorname{in}_<\left(I^{np}\right) = \operatorname{in}_<\left(f^{p-1}I^{np}\right)$$

$$\subseteq \operatorname{in}_<\left(\left(I^{(n)}\right)^{[p]}\right)$$

$$= \operatorname{in}_<\left(I^{n}\right)^{[p]}.$$

Then, $\operatorname{in}_<(f^{p-1}) \notin \mathfrak{m}^{[p]}$ and $\operatorname{in}_<(f^{p-1}) \in \left(\operatorname{in}_<(I^{n})\right)^{[p]} : \operatorname{in}_<(I^{np})$ for every $n \in \mathbb{Z}_{\geqslant 0}$. We conclude that $\mathscr{R}(\{\operatorname{in}_<(I^n)\})$ is F-split proceeding as in the proof of Lemma 6.2(2). □

Our next goal is to prove a technical result, Theorem 7.9, which is crucially used in the rest of this section. First we need two lemmas.

Lemma 7.6. *Assume Setup 7.4 with K a perfect field. Let $P \subseteq R$ be a homogeneous prime ideal such that $\operatorname{in}_<(P)$ is radical, and let $Q = (x_1, \dots, x_h)$. Suppose that $\operatorname{in}_<(P) \subseteq Q$. Let $\{x^{\mathbf{n}_1}T^{b_1}, \dots, x^{\mathbf{n}_m}T^{b_m}\}$ be a set of generators of $S = \bigoplus_{n\geqslant 0}\operatorname{in}_<(P^{(n)})T^n$ as an R-algebra. Write $\mathbf{n}_i = (n_{i,1} \dots, n_{i,d})$ for each $1 \leqslant i \leqslant m$ and suppose $p \geqslant \max\{n_{i,j}, b_i\}_{1\leqslant i\leqslant m,\, 1\leqslant j\leqslant d}$. Then, $\operatorname{in}_<(P^{(n+1)}) \subseteq Q\operatorname{in}_<(P^{(n)})$ for every $n \in \mathbb{Z}_{\geqslant 0}$.*

Proof. Set $\mathscr{J} := \bigoplus_{n\geqslant 0}\operatorname{in}_<(P^{(n+1)})T^n \subseteq S$. We have that $\mathscr{A} := \{x^{\mathbf{n}_1}T^{b_1-1}, \dots, x^{\mathbf{n}_m}T^{b_m-1}\}$ generates $\mathscr{J}$ as an S-ideal. Fix $x^{\mathbf{n}_i}T^{b_i-1} \in \mathscr{A}$. We claim that there exists $1 \leqslant j \leqslant h$ such that $n_{i,j} \geqslant 1$. If not, we would get that $x^{\mathbf{n}_i}T^{b_i-1}R_Q = R_Q$, and therefore $\operatorname{in}_<(P^{(b_i)}) \nsubseteq Q$. However, this contradicts the assumption $\operatorname{in}_<(P) \subseteq Q$. It follows from this claim and by the assumption on p that $\partial_j(x^{\mathbf{n}_i}T^{b_i-1}) \neq 0$. Let $g \in P^{(b_i)}$ be such that $\operatorname{in}_<(g) = x^{\mathbf{n}_i}$, and set $\partial_j = \frac{\partial}{\partial x_j}$. We have

$$0 \neq \partial_j(x^{\mathbf{n}_i}T^{b_i-1}) = \partial_j(\operatorname{in}_<(g)T^{b_i-1}) = \operatorname{in}_<(\partial_j(g))T^{b_i-1} \in \operatorname{in}_<(P^{(b_i-1)})T^{b_i-1}$$

by the characterization of symbolic powers in terms of differential operators [23, 27, 73, 98]. Hence,

$$x^{\mathbf{n}_i}T^{b_i-1} \in x_j\operatorname{in}_<\left(P^{(b_i-1)}\right)T^{b_i-1} \subseteq Q\operatorname{in}_<\left(P^{(b_i-1)}\right)T^{b_i-1} \subseteq Q\cdot S.$$

We have $\mathscr{J} \subseteq Q\cdot S$, and therefore $\operatorname{in}_<(P^{(n+1)}) \subseteq Q\operatorname{in}_<(P^{(n)})$ for every $n \in \mathbb{Z}_{\geqslant 0}$. □

Lemma 7.7. *Let $(S, \mathfrak{n})$ be a universally catenary local domain. Let $\mathbb{I}$ be a filtration of nonzero S-ideals, such that the Rees algebra $\mathscr{R}(\mathbb{I})$ is finitely generated as S-algebra. Then, $\operatorname{gr}(\mathbb{I})$ is equidimensional of dimension $\dim(S)$.*

Proof. Consider the *extended Rees algebra* $A := S[\mathbb{I}T, T^{-1}] = \bigoplus_{n\in\mathbb{Z}} I_n T^n$ where $I_n = S$ for $n \leqslant 0$. Thus, there exists $\ell \in \mathbb{Z}_{>0}$ such that A is an integral extension of $S[I_\ell T, T^{-1}] = \bigoplus_{n\in\mathbb{Z}} I_\ell^n T^n$; see Equation (2.4.1). Therefore, A has dimension $\dim(S) + 1$ [58, Theorem 2.2.5, Theorem 5.1.4(1)]. Now, T^{-1} is a homogeneous nonzero element of A and then every minimal prime of (t^{-1}) has height one. We also have that $A/(T^{-1}) \cong \operatorname{gr}(\mathbb{I})$. The result now follows by noticing that A is a catenary graded domain, which has a unique homogeneous maximal ideal $\oplus_{n<0} I_n t^n \oplus \mathfrak{n} \oplus_{n>0} I_n t^n$, and thus for any homogeneous ideal $\mathcal{J} \subseteq A$, one has $\dim(A/\mathcal{J}) + \operatorname{ht}(\mathcal{J}) = \dim(A)$. □

Remark 7.8. Assuming Setup 7.4 with K algebraically closed, Sullivant proved that for all $n \in \mathbb{Z}_{\geqslant 0}$ and all radical ideals I such that $\operatorname{in}_<(I)$ is radical, one has $\operatorname{in}_<(I^{(n)}) \subseteq \operatorname{in}_<(I)^{(n)}$ [92]. We point out that his proof works, more generally, if K is just any perfect field.

We are now ready to present the technical theorem (cf. [57, Theorem 1.2]).

Theorem 7.9. *Assume Setup 7.4 with K a perfect field. Let $P \subseteq R$ be a homogeneous prime ideal such that $\operatorname{in}_<(P)$ is radical. Let $S = \bigoplus_{n\geqslant 0} \operatorname{in}_<(P^{(n)})T^n$, $\mathcal{J} = \bigoplus_{n\geqslant 0} \operatorname{in}_<(P^{(n+1)})T^n \subseteq S$, and $G = S/\mathcal{J}$. Assume that S is Noetherian and let $b_1, \dots, b_m$ be the generating degrees of S and an R-algebra. Assume $p > \operatorname{lcm}(b_1, \dots, b_m)$ and that G is reduced. Then, there is a one-to-one correspondence between primes of R minimal over $\operatorname{in}_<(P)$ and minimal primes of G.*

More specifically, if $\mathfrak{q} \in \operatorname{Spec}(R)$ is minimal over $\operatorname{in}_<(P)$, then $Q = \ker(G \to G \otimes_R R_q)$ is a minimal prime of G such that $Q \cap R = q$, and every minimal prime of G is of this form.

Proof. For all $n \geqslant 1$, set $J_n = \operatorname{in}_<(P^{(n)})$. Without loss of generality, we may assume that K is infinite. For all $n \geqslant 1$, we have $J_1^n \subseteq J_n$, and by Remark 7.8, we also have $J_n \subseteq J_1^{(n)}$. Let $\mathfrak{q} \in \operatorname{Spec}(R)$ be minimal over J_1. After localizing at $\mathfrak{q}$, the above inclusions all become equalities. Moreover, since J_1 is radical, we have $J_1 R_\mathfrak{q} = \mathfrak{q}R_\mathfrak{q}$, and therefore $J_n R_\mathfrak{q} = (\mathfrak{q}R_\mathfrak{q})^n$. It follows that $G \otimes_R R_\mathfrak{q} \cong \operatorname{gr}_{\mathfrak{q}R_\mathfrak{q}}(R_\mathfrak{q})$, and since $R_\mathfrak{q}$ is regular, this associated graded ring is a domain. If we let $Q = \ker(G \to G \otimes_R R_\mathfrak{q})$, then G/Q is a subring of a domain, hence a domain itself. It follows that Q is a prime ideal of G, and it is easy to see that $Q \cap R = \ker(R/J_1 \to R_\mathfrak{q}/\mathfrak{q}R_\mathfrak{q}) = \mathfrak{q}$. Finally, the map $G \to G \otimes_R R_\mathfrak{q}$ becomes an isomorphism when localized at Q. Therefore, $QG_Q = 0$, that is, Q is a minimal prime of G.

Now let $\overline{Q}$ be a minimal prime of G. Let Q be a lift of $\overline{Q}$ to S, so that Q is a prime of S, which is minimal over $\mathcal{J}$. Let $q = Q \cap R$, which is a monomial ideal. Hence, $\mathfrak{q}$ generated by variables, say $x_1, \dots, x_h$. Consider the multiplicative system $W = K[x_{h+1}, \dots, x_d] \setminus \{0\} \subseteq R$, let $K' = K(x_{h+1}, \dots, x_d) = W^{-1}K[x_{h+1}, \dots, x_d]$ and $R' = W^{-1}R = K'[x_1, \dots, x_h]$. In addition, $W^{-1}q = W^{-1}Q' \cap W^{-1}R = (x_1, \dots, x_h)R'$. We replace K by K', and may assume $Q \cap R = \mathfrak{m} = (x_1, \dots, x_h)$. By Lemma 7.7, we have $\dim(S/Q) = \dim(G) = d$. We want to show that $\mathfrak{m}$ is minimal over J_1.

First, we want to show that, in our current setup, the monomial ideal J_1 is generated by variables. If not, after possibly relabeling the indeterminates, we may find integers $1 \leqslant \ell_1 \leqslant \ell_2 \leqslant d$, a square-free monomial ideal $A \subseteq (x_1, \dots, x_{\ell_1})^2$, and an ideal $L = (x_{\ell_1+1}, \dots, x_{\ell_2})$ generated by variables such that $J_1 = A + L$.

By Lemma 7.6, we have that $J_{n+1} \subseteq \mathfrak{m}J_n$ for all $n \geqslant 0$, and thus $\mathcal{J} \subseteq \mathfrak{m}S \subseteq Q$. Since $G = S/\mathcal{J}$ is reduced, and Q is a minimal prime of $\mathcal{J}$, we have $\mathcal{J}S_Q = QS_Q$, and thus $\mathcal{J}S_Q = \mathfrak{m}S_Q$. In particular, there exist $n \geqslant 0$ and $fT^n \in S \setminus Q$ such that $fT^n(\mathfrak{m}S) \subseteq \mathcal{J}$. Thus, $f \in J_n$ is an element such that $f\mathfrak{m} \subseteq J_{n+1}$, and $fT^n \notin Q$. In our assumptions, if $s = \operatorname{lcm}(b_1, \dots, b_m)$, then we can find homogeneous elements $a_1T^s, \dots, a_dT^s$, which form a full system of parameters for the finitely generated graded K-algebra $S/\mathfrak{m}S$. In particular, notice that $a_iT^s \notin Q$ for all i, since Q is a minimal prime of $\mathfrak{m}S$.

First, we claim that $a_i \notin \mathfrak{m}^{s+1}$ for all $i = 1, \dots, d$, that is, each a_i has degree at most s. By way of contradiction, assume $a_i \in \mathfrak{m}^{s+1}$ for some i. We have

$$a_iT^s(fT^n)^{s+1} \subseteq (f\mathfrak{m})^{s+1}T^{(n+1)(s+1)-1} \subseteq J_{(n+1)(s+1)}T^{(n+1)(s+1)-1} \subseteq \mathcal{J} \subseteq Q,$$

which contradicts the fact that a_iT^s and fT^n do not belong to Q.

We now claim that $J_1^{(s)} = (A+L)^{(s)} \subseteq (x_1, \dots, x_{\ell_1})^{s+1} + L$. Let $R_1 = K[x_1, \dots, x_{\ell_1}]$, $\mathfrak{m}_1 = (x_1, \dots, x_{\ell_1})R_1$ and $B = A \cap R_1$. Since we have $(A+L)^{(s)} \subseteq A^{(s)} + L$, it suffices to show $B^{(s)} \subseteq \mathfrak{m}_1^{s+1}$ in R_1. Since $p = \operatorname{char}(K) > s$, every K-linear differential operator ∂ of R' of order at most s can be written as $\partial_{x_i} \circ \partial'$ for some K-linear differential operator ∂' of order at most $s-1$, and some $1 \leqslant i \leqslant \ell_1$. We have

$$\partial(B^{(s)}) = \partial_{x_i}(\partial'(B^{(s)})) \subseteq \partial_{x_i}(B) \subseteq \partial_{x_i}(\mathfrak{m}_1^2) \subseteq \mathfrak{m}_1 \text{ (for instance, [23, Proposition 2.14]).}$$

Therefore, we have $B^{(s)} \subseteq \mathfrak{m}_1^{(s+1)} = \mathfrak{m}_1^{s+1}$ [23, Proposition 2.14]. At this point, we have shown that $(a_1, \dots, a_d) \subseteq J_s \subseteq J_1^{(s)} \subseteq (x_1, \dots, x_{\ell_1})^{s+1} + L$. Since each a_i is homogeneous, and has degree at most s, we must have $(a_1, \dots, a_d) \subseteq L$.

Finally, let $I = (a_1, \dots, a_d)$; we claim that $\sqrt{I} = J_1$. Once we have shown this, we have $J_1 = \sqrt{I} \subseteq L \subseteq J_1$, which implies $J_1 = L$ is generated by variables. Since $a_1T^s, \dots, a_dT^s$ are a full system of parameters for $S/\mathfrak{m}S$, we can find an integer $N \gg 0$ such that $(J_s^N T^{Ns})S/\mathfrak{m}S \subseteq (a_1T^s, \dots, a_dT^s)S/\mathfrak{m}S$, so that $J_sT^{Ns} \subseteq (a_1T^s, \dots, a_dT^s)J_{Ns-s}T^{Ns} + (\mathfrak{m}S)_{Ns}T^{Ns}$. In particular, we have a containment $J_s^N \subseteq (a_1, \dots, a_d) + \mathfrak{m}J_{Ns}$. Because S is generated in degree at most s, we have $J_{Ns} = J_s^N$, and we conclude that $J_{Ns} \subseteq (a_1, \dots, a_d) + \mathfrak{m}J_{Ns}$. It follows from graded Nakayama's Lemma that $J_{Ns} \subseteq (a_1, \dots, a_d)$, and since $J_1^{Ns} \subseteq J_{Ns}$, we conclude that $J_1 \subseteq \sqrt{I}$. Since $I \subseteq J_1$, the other inclusion is trivial.

To conclude the proof, observe that since J_1 is generated by variables, we have $J_1^n = J_1^{(n)}$ for all n. It follows that $G = \bigoplus_{n \geqslant 0} J_n/J_{n+1} = \bigoplus_{n \geqslant 0} J_1^n/J_1^{n+1} = \operatorname{gr}_{J_1}(R)$ is reduced. Then, $\mathfrak{m}$ is a minimal prime over J_1, by the one-to-one correspondence between minimal primes of G and R/J_1 already established in this case [57, Theorem 1.2]. □

From the previous theorem, we obtain the following useful corollary.

Corollary 7.10. *Assume Setup 7.4 with K a perfect field. Let $P \subseteq R$ be a homogeneous prime ideal such that $\operatorname{in}_<(P)$ is radical. Let $S = \bigoplus_{n \geqslant 0} \operatorname{in}_<(P^{(n)})T^n$, $\mathcal{J} = \bigoplus_{n \geqslant 0} \operatorname{in}_<(P^{(n+1)})T^n \subseteq S$, and $G = S/\mathcal{J}$. Assume S is Noetherian and let $b_1, \dots, b_m$ be the generating degrees of S and an R-algebra. Assume $p > \operatorname{lcm}(b_1, \dots, b_m)$ and that G is reduced. Then, $\operatorname{in}_<(P^{(n)}) = \operatorname{in}_<(P)^{(n)}$ for all $n \geqslant 1$.*

Proof. For all $n \geqslant 1$, set $J_n = \operatorname{in}_<(P^{(n)})$. By Theorem 7.9, we have that $G = \bigoplus_{n \geqslant 0} J_n/J_{n+1}$ is a torsion-free R/J_1-module. The same argument used in Remark 5.3 shows $\operatorname{Ass}_R(R/J_n) \subseteq \operatorname{Min}(J_1)$ for all $n \geqslant 1$. Thus, since J_n contains J_1^n, it must in fact contain $J_1^{(n)}$. Finally, because the containment $J_n \subseteq J_1^{(n)}$ always holds, we obtained the desired equality. □

The following observation shows how close is the equality $\mathrm{in}_<(P^{(n)}) = \mathrm{in}_<(P)^{(n)}$, for every $n \in \mathbb{Z}_{\geqslant 0}$, to I being symbolic F-split.

Remark 7.11. Assume Setup 7.4 with K a perfect field. Let $P \subseteq R$ be a homogeneous prime ideal such that $\mathrm{in}_<(P)$ is radical and $\mathrm{in}_<(P^{(n)}) = \mathrm{in}_<(P)^{(n)}$ for all $n \geqslant 1$. Then, $x_1 \cdots x_d \in \mathrm{in}_<(P)^{(h)} = \mathrm{in}_<(P^{(h)})$, where $h = \mathrm{height}(P)$. Let $f \in P^{(h)}$ be a homogeneous polynomial such that $\mathrm{in}_<(f) = x_1 \cdots x_d$. Thus, $f \notin \mathfrak{m}^{[p]}$, and so, P is symbolic F-split by Corollary 5.10.

7.3 | Results in characteristic zero

Let $J \subseteq R = \mathbb{Q}[x_1, \dots, x_d]$, and let $A = \mathbb{Z}[x_1, \dots, x_d]$. Let $J_A = J \cap A$, and let $f_1, \dots, f_s \in A$ be generators of J_A. Note that $(f_1, \dots, f_s)R = J$. For every prime $p \in \mathbb{Z}$, we let $J(p) = J_A \cdot A(p)$, where $A(p) = \mathbb{Z}/(p)[x_1, \dots, x_d]$. Note that, if A/J_A is flat over A, then $J_A \otimes_{\mathbb{Z}} \mathbb{Z}/(p)$ can be identified with the ideal $J(p)$ of $A(p) \cong A \otimes_{\mathbb{Z}} \mathbb{Z}/(p)$.

Lemma 7.12. *Given an integer $n \in \mathbb{Z}_{\geqslant 0}$ and an ideal $J \subseteq R = \mathbb{Q}[x_1, \dots, x_d]$, we have $(J(p))^{(n)} = ((J_A)^{(n)})(p)$ for all primes $p \gg 0$.*

Proof. Consider a minimal primary decomposition $(J_A)^n = I_1 \cap \dots \cap I_s$ in A. We collect the primary components and write $(J_A)^n = (J_A)^{(n)} \cap I$, where $(J_A)^{(n)} = I_1 \cap \dots \cap I_t$ is the n-th symbolic power of J_A in A. Let $P_j = \sqrt{I_j}$. By generic freeness [53, Lemma 8.1], there exists an element $a \in \mathbb{Z}$ such that all the modules $(A/I_j)_a$, $(A/J_A^n)_a$, $(A/J_A^{(n)})_a$, and $(A/J_A)_a$ are free over $\mathbb{Z}_a$. Since we are seeking to get the equality $(J(p))^{(n)} = ((J_A)^{(n)})(p)$ only for $p \gg 0$, without loss of generality, we directly assume that all the above modules are free over $\mathbb{Z}$. By flatness, we have $J_A^n \otimes_{\mathbb{Z}} \mathbb{Z}/(p) \cong J_A^n(p) = (J(p))^n = I_1(p) \cap \dots \cap I_s(p)$ as ideals of $A(p)$ and, in particular, $((J_A)^{(n)})(p) = I_1(p) \cap \dots I_t(p)$. It is left to show $I_1(p) \cap \dots \cap I_t(p) = (J(p))^{(n)}$. Since $(J_A)^{(n)} \otimes_{\mathbb{Z}} \mathbb{Q} \cong J^{(n)}$, we have that for $p \gg 0$ there is no associated prime of $I_1(p) \cap \dots \cap I_t(p)$, which is embedded [52, Theorem 2.3.9], and the desired equality follows. □

Remark 7.13. Assume that $<$ is a monomial order on $R = \mathbb{Q}[x_1, \dots, x_d]$, and let $J \subseteq R$ be an ideal. We have that $\mathrm{in}_<(J_A)(p) = \mathrm{in}_<(J(p))$ for all $p \gg 0$ [81, Lemma 2.3]. Moreover, any minimal monomial generating set of $\mathrm{in}_<(J)$ is a minimal monomial generating set of $\mathrm{in}_<(J(p))$ for $p \gg 0$.

Remark 7.14. If $I \subseteq J$ are two ideals of $R = \mathbb{Q}[x_1, \dots, x_d]$ such that $I(p) = J(p)$ for all $p \gg 0$, then $I = J$. In fact, after localizing at a nonzero element $a \in \mathbb{Z}$, we may assume that $(J_A/I_A)_a$ is a free $\mathbb{Z}_a$-module, by generic freeness. Our assumptions guarantee that there is a sufficiently large prime integer p such that $(J_A/I_A)_a \otimes_{\mathbb{Z}_a} \mathbb{Z}_a/(p) \cong J_A/I_A \otimes_{\mathbb{Z}} \mathbb{Z}/(p) \cong J(p)/I(p) = 0$, and since $(J_A/I_A)_a$ is free over $\mathbb{Z}_a$, this implies $(I_A)_a = (J_A)_a$. In particular, $I = J$.

Theorem 7.15. *Let $B = \mathbb{Q}[x_1, \dots, x_d]$ be equipped with a monomial order $<$. Let $f_1, \dots, f_s \in A = \mathbb{Z}[x_1, \dots, x_d]$ be homogeneous elements, and $Q = (f_1, \dots, f_s)B$. Assume that Q is prime, and that $\mathrm{in}_<(Q)$ is radical. For a prime integer p, we let $S(p) = \bigoplus_{n \geqslant 0} \mathrm{in}_<(Q(p)^{(n)})T^n$, $\mathcal{J}(p) = \bigoplus_{n \geqslant 0} \mathrm{in}_<(Q(p)^{(n+1)})T^n$, and $G(p) = S(p)/\mathcal{J}(p)$. Assume that $S(p)$ is Noetherian and that $G(p)$*

is reduced for all $p \gg 0$. If K is any field of characteristic zero, $R = K[x_1, \dots, x_d]$ is equipped with the same monomial order $<$ as B, and $P = QR$, then $\mathrm{in}_<(P^{(n)}) = \mathrm{in}_<(P)^{(n)}$ for all $n \geqslant 1$.

Proof. First we prove the statement for $K = \mathbb{Q}$, that is, for $R = B$ and $P = Q$. Let $J = Q_A$ and fix $n \in \mathbb{Z}_{\geqslant 0}$; by Remarks 7.8 and 7.14, we only have to show $\mathrm{in}_<(J)^{(n)}(p) = \mathrm{in}_<(J^{(n)})(p)$ for all $p \gg 0$. By Lemma 7.12 and Remark 7.13, for $p \gg 0$, we have

$$\mathrm{in}_<(J)^{(n)}(p) = (\mathrm{in}_<(J)(p))^{(n)} = \mathrm{in}_<(Q(p))^{(n)}$$

and

$$\mathrm{in}_<\left(J^{(n)}\right)(p) = \mathrm{in}_<\left(J^{(n)}(p)\right) = \mathrm{in}_<\left(Q(p)^{(n)}\right).$$

Moreover, by Remark 7.13, we have that $\mathrm{in}_<(Q(p))$ is square free for all $p \gg 0$, given that $\mathrm{in}_<(Q)$ is square free by assumption. Since $S(p)$ is finitely generated and $G(p)$ is reduced for all $p \gg 0$, we conclude by Corollary 7.10 that $\mathrm{in}_<(Q(p))^{(n)} = \mathrm{in}_<(Q(p)^{(n)})$ for all $p \gg 0$, and the proof is complete in this case.

Now let K be any field of characteristic zero and fix $n \in \mathbb{Z}_{\geqslant 0}$. Since Buchberger's algorithm is stable under base extensions, we have $\mathrm{in}_<(I)R = \mathrm{in}_<(IR)$ for any ideal $I \subseteq B$. Moreover, as the natural inclusion $B \hookrightarrow R \cong B \otimes_{\mathbb{Q}} K$ is flat and $\mathbb{Q} \to K$ is separable, we have $I^{(n)}R = (IR)^{(n)}$ for any radical ideal $I \subseteq B$. By what we have already shown, we finally get

$$\mathrm{in}_<\left(P^{(n)}\right) = \mathrm{in}_<\left((QR)^{(n)}\right) = \mathrm{in}_<\left(Q^{(n)}\right)R = \left(\mathrm{in}_<(Q)^{(n)}\right)R = \mathrm{in}_<(QR)^{(n)} = \mathrm{in}_<(P)^{(n)}. \qquad \square$$

7.4 | Main results of this section

We are ready to present the main results of this section in the context of determinantal ideals. In the generic case, the equality between initial ideals of symbolic powers and symbolic powers of initial ideals was proved by Bruns and Conca [10, Lemma 7.2]. The methods developed in this paper allow us to recover this result.

Theorem 7.16. *Assume Setup 6.3 and $p > \min\{t, r-t\}$. Then, the filtration $\{\mathrm{in}_<(I_t(X)^{(n)})\}_{n \in \mathbb{Z}_{\geqslant 0}}$ and the algebra $\mathscr{R}(\{\mathrm{in}_<(I_t(X)^n)\})$ are F-split. Moreover,*

$$\mathrm{in}_<(I_t(X)^{(n)}) = \mathrm{in}_<(I_t(X))^{(n)}$$

for every $n \in \mathbb{Z}_{\geqslant 0}$.

Proof. We consider $f_u(X)$ as in Notation 6.3. We have that $\mathrm{in}_<(f_1(X))$ is square free and $f_1(X) \in \bigcap_{n \in \mathbb{Z}_{\geqslant 0}} (I_t(X)^n)^{[p]} : I_t(X)^{np}$ by the proof of Theorem 6.8. Then, $\mathscr{R}(\{\mathrm{in}_<(I_t(X)^n)\})$ is F-split by Proposition 7.5(2). We also have $f_t(X) \in I_t(X)^{(\mathrm{ht}(I_t(X)))}$ by the proof of Theorem 6.5. Then, $\{\mathrm{in}_<(I_t(X)^{(n)})\}_{n \in \mathbb{Z}_{\geqslant 0}}$ is an F-split filtration by Proposition 7.5 (1). Thus,

$$G := \bigoplus_{n \in \mathbb{Z}_{\geqslant 0}} \frac{\mathrm{in}_<(I_t(X)^{(n)})}{\mathrm{in}_<(I_t(X)^{(n+1)})}$$

is an F-split ring by Theorem 4.7, and so, it is reduced. Since $\mathscr{R}(\{\mathrm{in}_<(I_t(X)^{(n)})\})$ is a finitely generated algebra [10, Lemma 7.1.], we conclude that

$$\mathrm{in}_<(I_t(X)^{(n)}) = \mathrm{in}_<(I_t(X))^{(n)}$$

for every $n \in \mathbb{Z}_{\geqslant 0}$ by Corollary 7.10. □

Corollary 7.17. *Let K be a field of characteristic zero, X be a generic $r \times s$ matrix of variables, and $R = K[X]$. For every $t \leqslant \min\{r, s\}$ and every $n \in \mathbb{Z}_{\geqslant 0}$, we have*

$$\mathrm{in}_<(I_t(X)^{(n)}) = \mathrm{in}_<(I_t(X))^{(n)}.$$

Proof. This is immediate consequence of Theorems 7.15 and 7.16. □

Finally, we now turn our attention to the case of Pfaffians, which, to the best of our knowledge, was not previously known.

Theorem 7.18. *Assume Setup 6.21 and $p > \min\{2t, r-2t\}$. Then, the filtration $\{\mathrm{in}_<(P_{2t}(Z)^{(n)})\}_{n \in \mathbb{Z}_{\geqslant 0}}$ and the algebra $\mathscr{R}(\{\mathrm{in}_<(P_{2t}(Z)^n)\})$ are F-split. Moreover,*

$$\mathrm{in}_<\left(P_{2t}(Z)^{(n)}\right) = \mathrm{in}_<\left(P_{2t}(Z)\right)^{(n)}$$

for every $n \in \mathbb{Z}_{\geqslant 0}$.

Proof. We consider $f_{2u}(Z)$ as in Notation 6.21. We have that $\mathrm{in}_<(f_2)$ is a square-free monomial and $f_2(Z) \in \bigcap_{n \in \mathbb{Z}_{\geqslant 0}} (P_{2t}(Z)^n)^{[p]} : P_{2t}(Z)^{np}$ by the proof of Theorem 6.26. Then, $\mathscr{R}(\{\mathrm{in}_<(P_{2t}(Z)^n)\})$ is F-split by Proposition 7.5(2). We also have $f_{2t}(Z) \in P_{2t}(Z)^{(\mathrm{ht}(P_{2t}(Z)))}$ by the proof of Theorem 6.23, therefore $\{\mathrm{in}_<(P_{2t}(Z)^{(n)})\}_{n \in \mathbb{Z}_{\geqslant 0}}$ is an F-split filtration by Proposition 7.5(1). In particular,

$$G = \bigoplus_{n \in \mathbb{Z}_{\geqslant 0}} \frac{\mathrm{in}_<(P_{2t}(Z)^{(n)})}{\mathrm{in}_<(P_{2t}(Z)^{(n+1)})}$$

is a F-split by Theorem 4.7, and so, it is reduced. As $\mathscr{R}(\{\mathrm{in}_<(P_{2t}(Z)^{(n)})\})$ is a finitely generated algebra [2, Proof of Proposition 3.1], we conclude that

$$\mathrm{in}_<(P_{2t}(Z)^{(n)}) = \mathrm{in}_<(P_{2t}(Z))^{(n)}$$

for every $n \in \mathbb{Z}_{\geqslant 0}$ by Corollary 7.10. □

Corollary 7.19. *Let K be a field of characteristic zero, Z be a generic $r \times r$ skew-symmetric matrix, and $R = K[Z]$. For every $t \leqslant \lfloor \frac{r}{2} \rfloor$ and every $n \in \mathbb{Z}_{\geqslant 0}$, we have*

$$\mathrm{in}_<(P_{2t}(Z)^{(n)}) = \mathrm{in}_<(P_{2t}(Z))^{(n)}.$$

Proof. This is an immediate consequence of Theorems 7.15 and 7.18. □

The case of Hankel matrix was also known, and it is due to Conca [15, Lemma 3.5 and Theorem 3.8]. We recover it here.

Theorem 7.20. *Assume Setup 6.30 and $p > \min\{t, r - t\}$. Then, the filtration $\{\text{in}_< \left(I_t\left(W_j^c\right)^{(n)}\right)\}_{n\in\mathbb{Z}_{\geqslant 0}}$ and the algebra $\mathscr{R}(\{\text{in}_<(I_t(W_j^c)^n)\})$ are F-split. Moreover,*

$$\text{in}_< \left(I_t(W_j^c)^{(n)}\right) = \text{in}_< \left(I_t(W_j^c)\right)^{(n)}$$

for every $n \in \mathbb{Z}_{\geqslant 0}$.

Proof. We set $f = f_{\text{odd}}(W_j^c)$ if d is odd and $f = f_{\text{even}}(W_j^c)$ if d is even as in Notation 6.30, and $W = W_j^c$. We have that $\text{in}_<(f)$ is square free and $f \in \bigcap_{n\in\mathbb{Z}_{\geqslant 0}} (I_t(W)^n)^{[p]} : I_t(W)^{np}$ by the proof of Theorem 6.35. It follows that $\mathscr{R}(\{\text{in}_<(I_t(W)^n)\})$ is an F-split ring by Theorem 4.7, and so, it is reduced. We also have that $f \in I_t(W)^{(\text{ht}(I_t(W)))}$ by the proof of Theorem 6.33, and therefore $\{\text{in}_<(I_t(W)^{(n)})\}_{n\in\mathbb{Z}_{\geqslant 0}}$ is an F-split filtration by Proposition 7.5(1). We also have

$$G = \bigoplus_{n\in\mathbb{Z}_{\geqslant 0}} \frac{\text{in}_<(I_t(W)^{(n)})}{\text{in}_<(I_t(W)^{(n+1)})}$$

is an F-split, and so, it is reduced. Since $\mathscr{R}(\{I_t(W)^{(n)}\})$ is a finitely generated algebra [15, Theorem 4.1], we conclude that

$$\text{in}_<(I_t(W)^{(n)}) = \text{in}_<(I_t(W))^{(n)}$$

for every $n \in \mathbb{Z}_{\geqslant 0}$ by Corollary 7.10. □

Corollary 7.21. *Let K be a field of characteristic zero, W_j^c be a $j \times c + 1 - j$ Hankel matrix of variables, and $R = K[W_j^c]$. For every $t \leqslant \min\{j, c + 1 - j\}$ and every $n \in \mathbb{Z}_{\geqslant 0}$, we have*

$$\text{in}_<(I_t(W_j^c)^{(n)}) = \text{in}_<(I_t(W_j^c))^{(n)}.$$

Proof. This is an immediate consequence of Theorems 7.15 and 7.20. □

Remark 7.22. In the case of minors of a generic symmetric matrix Y, it is not known whether the algebra $\mathscr{R}(\{\text{in}_<(I_t(Y)^{(n)})\})$ is finitely generated. For this reason, we cannot use the same strategy used above for the other three types of determinantal ideals.

Remark 7.23. If R is a standard graded polynomial ring over a field K and $I \subseteq R$ is a homogeneous ideal, we denote by $\alpha(I)$ the smallest degree of a minimal generator of I. Let $<$ be a monomial order on R. We note that $\alpha(I) = \alpha(\text{in}_<(I))$. In particular, if I and $\text{in}_<(I)$ are radical and $\text{in}_<(I^{(n)}) = \text{in}_<(I)^{(n)}$ for every n, then their Waldschmidt constants coincide. Specifically,

$$\widehat{\alpha}(I) = \lim_{n\to\infty} \frac{\alpha\left(I^{(n)}\right)}{n} = \lim_{n\to\infty} \frac{\alpha(\text{in}_<(I)^{(n)})}{n} = \widehat{\alpha}(\text{in}_<(I)).$$

In particular, Theorems 7.16, 7.18, and 7.20 allow us to compute the Waldschmidt constant of certain determinantal rings via their initial ideals, for which a formula has already been proved [5]. We point out that one can also compute directly that $\widehat{\alpha}(I_t(X)) = \frac{r}{r-t+1}$ [10, Lemma 7.1.], $\widehat{\alpha}(P_{2t}(Z)) = \frac{\lfloor r/2 \rfloor}{\lfloor r/2 \rfloor - t+1}$ [2, Proof of Proposition 3.1], and $\widehat{\alpha}(I_t(W_j^c)) = \frac{\lfloor (c+1)/2 \rfloor}{\lfloor (c+1)/2 \rfloor - t+1}$ [15, Theorem 4.1].

Remark 7.24. If R is a polynomial ring, $<$ is a monomial order, and $I \subseteq R$ is a homogeneous ideal, then $I^{(a)} \subseteq I^b$ implies $\text{in}_<(I^{(a)}) \subseteq \text{in}_<(I^b)$. We recall that the resurgence of I is defined by $\rho(I) = \sup\{\frac{a}{b} \mid I^{(a)} \nsubseteq I^b\}$. If $\text{in}_<(I)$ are radical, $\text{in}_<(I^{(n)}) = \text{in}_<(I)^{(n)}$, and $\text{in}_<(I^n) = \text{in}_<(I)^n$ for every n, then $\rho(\text{in}_<(I)) \leqslant \rho(I)$. In particular, this case occurs for ideals of minors of Hankel matrices (see Theorem 7.20 and [15, Theorem 3.16(b)]).

Hoa and Trung showed that the limit above exists for square-free monomial ideals. In fact, they showed a stronger version for the a-invariants [46, Theorems 4.7 and 4.9].

Corollary 7.25. *Assume Setup 6.30.*

(1) *Assume Setup 6.3. Then* $\lim_{n\to\infty} \frac{\text{reg}\left(K[X]/\,\text{in}_<\left(I_t(X)^{(n)}\right)\right)}{n}$ *exists. Moreover, for* $n \gg 0$, *we have that* $\text{depth}\left(K[X]/\,\text{in}_<\left(I_t(X)^{(n)}\right)\right)$ *stabilizes.*

(2) *Assume Setup 6.21. Then* $\lim_{n\to\infty} \frac{\text{reg}\left(K[Z]/\,\text{in}_<\left(P_{2t}(Z)^{(n)}\right)\right)}{n}$ *exists. Moreover, for* $n \gg 0$, *we have that* $\text{depth}\left(K[Z]/\,\text{in}_<\left(P_{2t}(Z)^{(n)}\right)\right)$ *stabilizes.*

(3) *Assume Setup 6.30. Then* $\lim_{n\to\infty} \frac{\text{reg}\left(K[W_j^c]/\,\text{in}_<\left(I_t(W_j^c)^{(n)}\right)\right)}{n}$ *exists. Moreover, for* $n \gg 0$, *we have that* $\text{depth}\left(K[W_j^c]/\,\text{in}_<\left(I_t(W_j^c)^{(n)}\right)\right)$ *stabilizes.*

Proof. If J is a square-free monomial ideal in a polynomial ring R, it is already known that $\lim_{n\to\infty} \frac{\text{reg}(R/J^{(n)})}{n}$ exists, and that $\text{depth}(R/J^{(n)})$ stabilizes [46]. Then, the result follows from Theorems 7.16, 7.18, and 7.20. □

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