POISSONIAN PAIR CORRELATION FOR αn^θ

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ABSTRACT. We show that sequences of the form αn^θ (mod 1) with $\alpha > 0$ and $0 < \theta < \frac{43}{117} = \frac{1}{3} + 0.0341\ldots$ have Poissonian pair correlation. This improves upon the previous result by Lutsko, Sourmelidis, and Technau, where this was established for $\alpha > 0$ and $0 < \theta < \frac{14}{41} = \frac{1}{3} + 0.0081\ldots$.

We reduce the problem of establishing Poissonian pair correlation to a counting problem using a form of amplification and the Bombieri-Iwaniec double large sieve. The counting problem is then resolved non-optimally by appealing to the bounds of Robert-Sargos and (Fouvry-Iwaniec-)Cao-Zhai. The exponent $\theta = \frac{2}{5}$ is the limit of our approach.

1. Introduction

A real-valued sequence $(x_n)_{n\geqslant 1}$ is *equidistributed mod 1* if for any $0 < a < b < 1$,

$$\lim_{N\to+\infty} \frac{1}{N}\#\big\{n \leqslant N : \{x_n\} \in [a, b]\big\} = b - a.$$

Equidistribution is one of the most basic properties of sequences that indicates their pseudorandom behavior. Weyl [23] showed that equidistribution modulo 1 of a sequence x_n is equivalent to showing cancellation in exponential sums

$$\sum_{n<N} e(kx_n) = o(N),$$

for every fixed integer $k \neq 0$. Subsequently, Weyl established that αn^d is equidistributed for all irrational α and $d \geqslant 1$ integer. Fejer and Csillag ([11, Corollary 2.1], [5]) established the same property when $d > 0$ is not an integer and α is non-zero.

Equidistribution is a "large-scale" property: it only provides information about the number of points inside intervals of size $\asymp 1$. A finer question is to ask about "small-scale" properties, that is distribution properties at the scale $\asymp \frac{1}{N}$. The most common statistic at this scale is the *gap distribution* describing the spacings $N(y_{k+1}-y_k)$ where $y_1 < y_2 < \ldots < y_N$ is an ordering of $x_1, \ldots, x_N \in [0, 1]$.

It is natural to expect that the sequence αn^θ with $\theta \notin \mathbb{Z}$ and $\alpha \neq 0$ is distributed like a random set of points modulo 1. This leads one to the following conjecture.

Conjecture 1. *Let $\alpha \neq 0$ and $\theta > 0$ with $\theta \neq \frac{1}{2}$ and $\theta \notin \mathbb{Z}$. Let $y_1^{(N)} < \ldots < y_N^{(N)}$ denote an ordering of the points $\alpha n^\theta \pmod 1$ with $n < N$. Then,*

$$\lim_{N\to\infty} \frac{1}{N} \# \Big\{ i < N : N(y_{i+1}^{(N)} - y_i^{(N)}) \in (a,b) \Big\} \to \int_a^b e^{-t} dt.$$

We then say that the gap distribution is Poissonian.

We exclude the case $\theta = \frac{1}{2}$ because Elkies-McMullen [7] have shown that the gap distribution *is not* Poissonian in this case. This is also the only case in which the gap distribution is known to exist. A weaker statistic than the gap distribution is the pair correlation. The belief that the sequence αn^θ behaves randomly leads to the following conjecture for the pair correlation.

Conjecture 2. *Let $\alpha \neq 0$ and $\theta > 0$ with $\theta \neq \frac{1}{2}$ and $\theta \notin \mathbb{Z}$. Then, for any $s > 0$,*

$$R_2\Big([-s,s], \alpha n^\theta, N\Big) := \frac{1}{N} \cdot \# \Big\{ 1 \leqslant n \neq m \leqslant N : N \| \alpha n^\theta - \alpha m^\theta \| \leqslant s \Big\} \to 2s$$

as $N \to \infty$ and where $\|x\|$ denotes the distance of x from the nearest integer. Whenever this holds we say that the sequence αn^θ has Poissonian Pair Correlation (PPC).

One can think of the pair correlation as a second moment of the gap distribution. In particular if all higher m-correlations exist and are Poisson then Conjecture 1 would follow (see [12, Appendix A]).

We now briefly summarize the state of the art regarding Conjecture 2. There are broadly two types of results: metric results that allow for averaging in α or θ, or deterministic results valid for specific α and θ. As far as metric results go, PPC has been shown for sequences of the form αn^d with integer $d \geqslant 2$ [9, 15, 18, 19] for almost all α. Moreover, Heath-Brown provided an algorithm for constructing a dense set of such α's in [9]. The case $d = 2$ is of particular interest due to its connection to quantum chaos (see [3, 17]). For sequences of the form αn^θ with non-integer fixed $\theta > 0$ PPC is known for almost all α following Aistleitner, El-Baz, and Munsch [1], as well as Rudnick and Technau [20]. While in [22] Technau and Yesha show that the sequence n^θ has PPC for almost all $\theta > 7$.

Deterministic results valid for specific θ and α are harder to come by. For $\theta = \frac{1}{2}$, El-Baz, Marklof, and Vinogradov [6] showed that the sequence $(\sqrt{n})_{n \neq \square}$ admits PPC. Finally, Lutsko, Sourmelidis, and Technau [14] recently verified PPC for all sequences of the form αn^θ, where $\alpha > 0$ and $0 < \theta < \frac{14}{41} = \frac{1}{3} + 0.008\ldots$. Going beyond $\frac{1}{3}$ requires non-trivial bounds for certain exponential sums. The goal of this work is to extend the range of θ by deploying heavy exponential sums machinery.

Theorem 1.1. *The sequence αn^θ has a Poissonian pair correlation for all $\alpha > 0$ and $0 < \theta < \frac{43}{117} = \frac{1}{3} + 0.03418\ldots$*

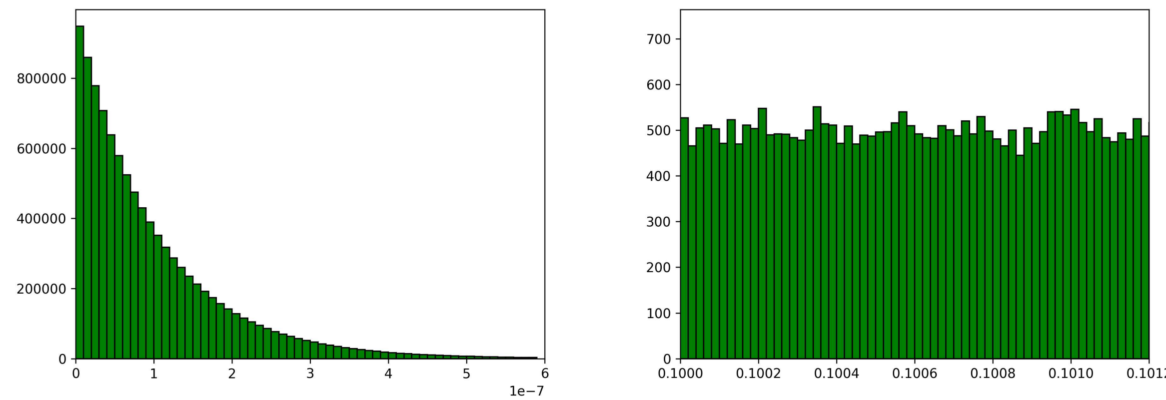

Figure 1. The distribution of gaps and spacings for the sequence $n^{1/3}$, with a step size $\frac{0.1}{N}$, $N = 10^7$ for gaps, and $N = 5000$ for spacings.

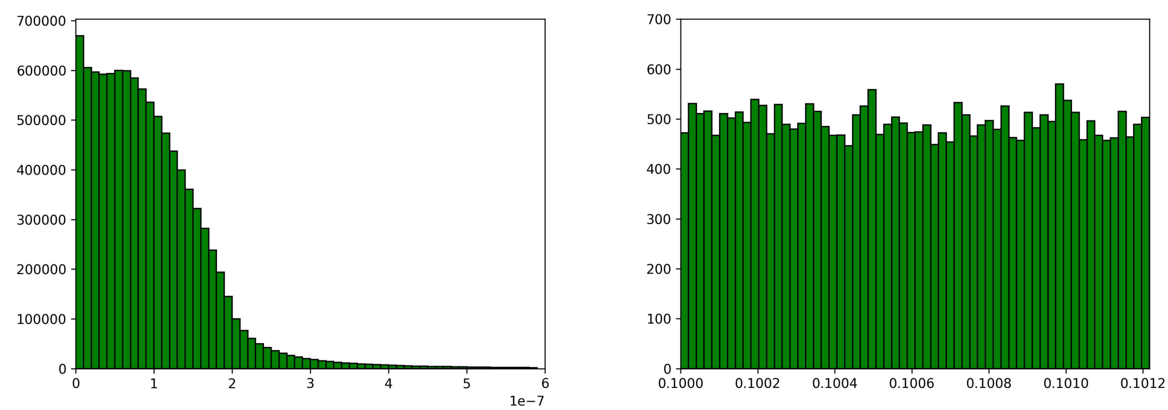

Figure 2. The distribution of gaps and spacings for the sequence $n^{1/2}$, with a step size $\frac{0.1}{N}$, $N = 10^7$ for gaps, and $N = 5000$, $n \neq \square$ for spacings. Elkies and McMullen [7] demonstrated that the gap distribution in this case is not Poissonian, whereas El-Baz, Marklof, and Vinogradov [6] showed that the pair correlation function is Poissonian.

1.1. **Reduction to exponential sums and sketch of the proof.** We define the integral version of the pair correlation function in the standard way:

$$\tilde{R}_2\big(f, \{x_n\}, N\big) := \frac{1}{N} \sum_{k \in \mathbb{Z}} \sum_{1 \leqslant i \neq j \leqslant N} f\big(N(x_i - x_j + k)\big). \tag{1.1}$$

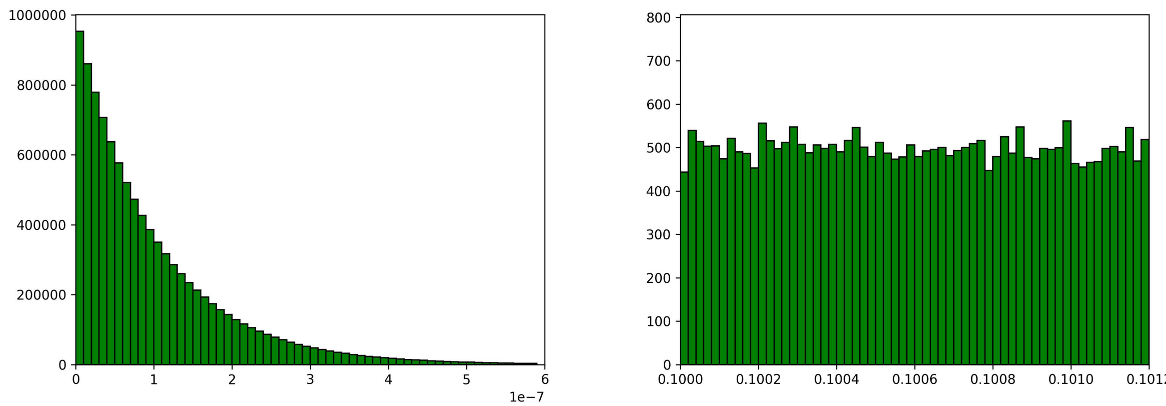

FIGURE 3. The distribution of gaps and spacings for the sequence $n^{2/3}$, with a step size $\frac{0.1}{N}$, $N = 10^7$ for gaps, and $N = 5000$ for spacings.

The condition $\|x_i - x_j\| \leqslant \frac{s}{N}$ is equivalent to $N(x_i - x_j + k) \in [-s, s]$ for some $k \in \mathbb{Z}$. When N becomes large compared to s, there is only one such k. Then

$$\lim_{N\to+\infty} R_2\big([-s,s], \{x_n\}, N\big) = \lim_{N\to+\infty} \tilde{R}_2\big(\mathbb{1}_{[-s,s]}, \{x_n\}, N\big).$$

Smoothing $\mathbb{1}_{[-s,s]}$ and expanding into Fourier series reduces the problem of PPC for αn^θ to a problem about exponential sums. We state this below.

Proposition 1.1. *Let $\alpha > 0$ and $\theta > 0$ be given. Suppose that for every even smooth compactly supported function f, every $\varepsilon > 0$,*

$$\lim_{N\to\infty} \frac{2}{N^2} \sum_{0<|k|\leqslant N^{1+\varepsilon}} \widehat{f}\Big(\frac{k}{N}\Big) \Big| \sum_{1\leqslant y\leqslant N} e(\alpha k y^\theta)\Big|^2 = f(0).$$

Then the sequence αn^θ has Poissonian Pair Correlation (PPC).

Thus Conjecture 2 amounts to showing that certain exponential sums have square-root cancellation on average.

Since the growth rate of the phase $\alpha k y^\theta$ is $\ll \alpha N^{1+\theta+\varepsilon}$, its first derivative is much smaller than N. Therefore, a natural first step is to apply the Poisson summation formula to each of the three sums over k, y_1, and y_2 as it will shorten each of these sums to length N^θ. This step is formalized in the following proposition.

Proposition 1.2. *Let $\alpha, \theta, \varepsilon$ be real numbers such that $\alpha > 0$, $\varepsilon > 0$, $3\varepsilon < \theta < 1-3\varepsilon$, and for $K, Y_1, Y_2 > 0$ define the subset $\mathcal{N}(K, Y_1, Y_2)$ as follows:*

$$\begin{gathered}\mathcal{N}(K,Y_1,Y_2) := \big\{\eta : \eta = m_1^{-\theta/(1-\theta)} - m_2^{-\theta/(1-\theta)}, \ \textit{where} \\ \alpha\theta K(2Y_1)^{\theta-1} \leqslant m_1 < 2\alpha\theta K Y_1^{\theta-1}; \\ \alpha\theta K(2Y_2)^{\theta-1} \leqslant m_2 < 2\alpha\theta K Y_2^{\theta-1}; \\ m_1 < m_2\big\}.\end{gathered}$$

Then we have

$$\tag{1.2} \frac{2}{N^2}\sum_{k\leqslant N^{1+\varepsilon}} \hat{f}\big(\frac{k}{N}\big)\Big|\sum_{1\leqslant y\leqslant N} e\big(\alpha k y^\theta\big)\Big|^2 = \\ f(0) + O\Big(N^{-\theta/2+\varepsilon/2}\log N + N^{-1/4+\theta/4+3\varepsilon/4}\log N + S(N)\Big),$$

where

$$\tag{1.3} S(N) := \frac{1}{N^2}\sum_{\substack{K\leqslant N^{1+\varepsilon}\\ \textit{dyadic}}}\sum_{\substack{Y_1,Y_2\leqslant N\\ \textit{dyadic}}} K^{(2-\theta)/(1-\theta)-1}\max_{K\leqslant \tilde{K}\leqslant 2K}\big|\tilde{S}(K,\tilde{K},Y_1,Y_2)\big|,$$

$$\tilde{S}(K,\tilde{K},Y_1,Y_2) := \\ \sum_{\eta\in\mathcal{N}(K,Y_1,Y_2)}\Big(\frac{1}{m_1m_2}\Big)^{(2-\theta)/(2-2\theta)}\sum_{L\leqslant \ell<\tilde{L}}\frac{c_1}{\sqrt{\eta}}\Big(\frac{\ell}{\eta}\Big)^{1/(2\theta)-1} e\Big(c_2\ell^{1/\theta}\eta^{1-1/\theta}\Big),$$

$$L = \frac{c_3\eta}{1-\theta}K_1^{\theta/(1-\theta)}, \qquad \tilde{L} = \frac{c_3\eta}{1-\theta}K_2^{\theta/(1-\theta)}, \qquad c_3 = (\alpha\theta)^{1/(1-\theta)}\Big(\frac{1}{\theta}-1\Big),$$
$$c_1 = \frac{1-\theta}{\sqrt{c_3\theta}}\Big(\frac{1-\theta}{c_3}\Big)^{1/(2\theta)-1}, \qquad c_2 = -\theta\Big(\frac{1-\theta}{c_3}\Big)^{1/\theta-1},$$

$$\tag{1.4} K_1 := \max\Big(K, \frac{m_1}{2\alpha\theta Y_1^{\theta-1}}, \frac{m_2}{2\alpha\theta Y_2^{\theta-1}}\Big),$$

$$\tag{1.5} K_2 := \min\Big(\tilde{K}, \frac{m_1}{\alpha\theta(2Y_1)^{\theta-1}}, \frac{m_2}{\alpha\theta(2Y_2)^{\theta-1}}\Big).$$

The implied constant in the error term in (1.2) depends on f, α, θ and ε.

A similar transform was used in [14]. An application of Proposition 1.2, combined with a trivial upper bound for $|\tilde{S}(K,\tilde{K},Y_1,Y_2)|$ on the right side of (1.3) yields Theorem 1.1 in the range $\varepsilon < \theta < \frac{1}{3}-\varepsilon$. Thus, going beyond $\frac{1}{3}$ requires a non-trivial bound for $S(N)$. We note incidentally that such non-trivial bounds are not always possible, for example, we have $S(N) \gg 1$ for $\alpha = 1$ and $\theta \in \{\frac{1}{2}, 1, 2, 3, \ldots\}$.

To a first approximation,

$$S(N) \approx N^{-1/2-3\theta/2} \sum_{\eta \in \mathcal{N}} \sum_{\ell \sim N^\theta} e\Big(\ell^{1/\theta} \eta^{1-1/\theta}\Big)$$

where

$$\mathcal{N} := \Big\{ m_1, m_2 \sim N^\theta : m_1^{-\theta/(1-\theta)} - m_2^{-\theta/(1-\theta)} \Big\}.$$

For the purpose of the sketch we can also assume that elements of $\mathcal{N}$ are in "generic position" and thus of size $\asymp N^{-\theta^2/(1-\theta)}$. Given $\eta \in \mathcal{N}$ we let $\widetilde{\eta} := \eta N^{\theta^2/(1-\theta)}$ so that for generic $\eta \in \mathcal{N}$ we have $|\widetilde{\eta}| \asymp 1$. Likewise for $\ell \sim N^\theta$ we denote by $\widetilde{\ell} := \ell / N^\theta$ so that $|\widetilde{\ell}| \asymp 1$. To bound $S(N)$ we apply Holder's inequality.

$$S(N) \ll N^{-1/2-3\theta/2} \Big(\# \mathcal{N} \Big)^{1-1/k} \cdot \Big(\sum_{\eta \in \mathcal{N}} \Big| \sum_{\ell \sim N^\theta} e\Big(\ell^{1/\theta} \eta^{1-1/\theta}\Big) \Big|^k \Big)^{1/k}.$$

We notice that the phase is of size $X := N^{1+\theta}$. Using the double large sieve we have the bounds,

$$\begin{aligned} &\sum_{\eta \in \mathcal{N}} \Big| \sum_{\ell \sim N^\theta} e\Big(\ell^{1/\theta} \eta^{1-1/\theta}\Big) \Big|^k \\ &\qquad \ll X^{1/2} \cdot \Big(\sum_{\eta_1, \eta_2 \in \mathcal{N}} \mathbf{1}\Big(\Big| \widetilde{\eta}_1^{1-1/\theta} - \widetilde{\eta}_2^{1-1/\theta} \Big| \leqslant \frac{1}{X} \Big) \Big)^{1/2} \cdot \\ &\qquad\qquad \Big(\sum_{\ell_1, \ldots, \ell_{2k} \sim N^\theta} \mathbf{1}\Big(\Big| \widetilde{\ell}_1 + \ldots - \widetilde{\ell}_{2k} \Big| \leqslant \frac{1}{X} \Big) \Big)^{1/2}. \end{aligned}$$

In the best case scenario the probability of each event inside the indicator function is $1/X$. Taking care to exclude the diagonal, this gives in the best case scenario the bound,

$$\sum_{\eta \in \mathcal{N}} \Big| \sum_{\ell \sim N^\theta} e\Big(\ell^{1/\theta} \eta^{1-1/\theta}\Big) \Big|^k \ll N^{(1+\theta)/2} \Big(N^{2\theta} + \frac{N^{4\theta}}{N^{1+\theta}} \Big)^{1/2} \cdot \Big(N^{k\theta} + \frac{N^{2k\theta}}{N^{1+\theta}} \Big)^{1/2}.$$

If θ_k is the optimal exponent that such an approach can yield, then we find $\theta_2 = \frac{1}{3}$, $\theta_3 = \theta_4 = \frac{2}{5}$, $\theta_5 = \frac{5}{13}$, $\theta_6 = \frac{3}{8}$ followed by smaller exponents. Thus the most beneficial choice of exponent is $k = 4$. Unfortunately we are not able to directly resolve the problem of counting eight-tuples of $(\ell_1, \ldots, \ell_8)$ all of size $\approx N^\theta$ such that,

$$|\ell_1^{1/\theta} + \ldots - \ell_8^{1/\theta}| \leqslant N^{-\theta}.$$

Instead we perturb the problem a little and apply van der Corput differencing twice, to get that,

$$\sum_{\eta\in\mathcal{N}}\Big|\sum_{\ell\sim N^{\theta}} e(\ell^{1/\theta}\eta^{1-1/\theta})\Big|^4 \ll$$
$$\frac{N^{4\theta}\#\mathcal{N}}{H_1^2}+\frac{N^{4\theta}\#\mathcal{N}}{H_2}+\frac{N^{3\theta}}{H_1H_2}\sum_{\eta\in\mathcal{N}}\sum_{\substack{0<|h_1|\leqslant H_1\\ 0<|h_2|\leqslant H_2}}\gamma(h_1,h_2)\sum_{\ell\sim N^{\theta}} e\Big(\eta^{1-1/\theta}\ t(\ell,h_1,h_2)\Big)$$

where

$$t(\ell,h_1,h_2):=((\ell+h_1+h_2)^{1/\theta}-(\ell+h_2)^{1/\theta})-((\ell+h_1)^{1/\theta}-\ell^{1/\theta}).$$

We now apply the double large sieve, and this leads to the bound

$$\sum_{\eta\in\mathcal{N}}\sum_{\substack{0<|h_1|\leqslant H_1\\ 0<|h_2|\leqslant H_2}}\gamma(h_1,h_2)\sum_{\ell\sim N^{\theta}} e\Big(\eta^{1-1/\theta}t(\ell,h_1,h_2)\Big)\ll X^{1/2}\mathcal{B}_1^{1/2}\mathcal{B}_2^{1/2}$$

where $X:=N^{1-\theta}H_1H_2$ and

$$\mathcal{B}_1:=\sum_{\eta_1,\eta_2\in\mathcal{N}}\mathbf{1}\Big(|\widetilde{\eta}_1^{1-1/\theta}-\widetilde{\eta}_2^{1-1/\theta}|\leqslant\frac{1}{X}\Big)$$

and

$$\mathcal{B}_2:=\sum_{\substack{0<|h_1|,|h_1'|\leqslant H_1\\ 0<|h_2|,|h_2'|\leqslant H_2}}\sum_{\ell,\ell'\sim N^{\theta}}\mathbf{1}\Big(\Big|\widetilde{t}(\ell,h_1,h_2)-\widetilde{t}(\ell',h_1',h_2')\Big|\leqslant\frac{1}{X}\Big)$$

where as before $\widetilde{t}(\ell,h_1,h_2)$ denotes a suitably normalized version of $t(\ell,h_1,h_2)$ so that $|\widetilde{t}(\ell,h_1,h_2)|\asymp 1$. We bound $\mathcal{B}_1$ using a Taylor expansion and the bound of Robert-Sargos [16], which gives an optimal bound for the number of tuples (m_1,m_2,m_3,m_4) of size M such that

$$|m_1^{\alpha}+m_2^{\alpha}-m_3^{\alpha}-m_4^{\alpha}|\leqslant\delta.$$

We then bound $\mathcal{B}_2$ using results of Cao and Zhai [4]. Such bounds for "shifted variables" go back to the work of Fouvry-Iwaniec [8] with subsequent improvements by Liu [13] and Sargos-Wu [21]. Due to the complicated nature of the bound of Cao-Zhai (see Lemma 4.4) we refrain from stating the exact bounds here. It suffices to say that after inputting all these bounds we find an appropriate choice of H_1 and H_2 that yields the following main technical proposition:

Proposition 1.3. *Let $\alpha,\varepsilon,\mathcal{N}$ and $S(N)$ be as in Proposition 1.2, and let θ be such that $3\varepsilon<\theta<\frac{43}{117}-5\varepsilon$. Then we have*

$$S(N)=o(1)\qquad\text{when } N\to+\infty,$$

where the implied constant depends on α, θ and ε.

Theorem 1.1 clearly follows from Propositions 1.1, 1.2 and 1.3. We prove Proposition 1.1 in Section 2. We then prove Proposition 1.2 in Section 3, and Proposition 1.3 in Section 4.

1.2. **Notation.** We use the standard notation for the real character $e(x) := e^{2\pi i x}$.

The relations $f(x) \ll g(x)$ and $f(x) = O(g(x))$ mean $|f(x)| \leqslant Cg(x)$ for a fixed number $C > 0$ and all large enough x. The symbols $\ll_\varepsilon$ and O_ε mean that the constant C might depend on the parameter ε. The relation $f(x) = o(g(x))$ for $g(x) > 0$ means $f(x)/g(x) \to 0$ as $x \to +\infty$. The relation $f(x) \asymp g(x)$ means $f(x) \ll g(x) \ll f(x)$. Finally, the relation $f(x) \sim g(x)$ means $f(x)/g(x) \to 1$ as $x \to +\infty$.

For brevity, the limits in the sums over dyadic intervals, such as $\sum_{K,Y_1,Y_2,R}$, are often not indicated.

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2. Reduction to exponential sums

We prove in this section Proposition 1.1. There is a sequence of $C_c^\infty(\mathbb{R})$ functions f_j^+ and f_j^-, such that $f_j^- \leqslant \mathbb{1}_{[-s,s]} \leqslant f_j^+$ for all j,

$$\int_{\mathbb{R}} \big(f_j^+(x) - f_j^-(x)\big) dx \to 0$$

when $j \to +\infty$, and $\hat{f}_j^{\pm}(x) \ll x^{-A}$ for arbitrarily large $A > 0$. Then, for the proof of Theorem 1.1, it is enough to show that for any $f \in C_c^\infty(R)$, one has

$$\tilde{R}_2\big(f, \{\alpha n^\theta\}, N\big) \to \int_{\mathbb{R}} f(x) dx$$

when $N \to +\infty$. Applying Poisson summation to the sum over $k \in \mathbb{Z}$, we obtain

$$\tilde{R}_2\big(f, \{\alpha n^\theta\}, N\big) = \frac{1}{N^2} \sum_{k \in \mathbb{Z}} \hat{f}\Big(\frac{k}{N}\Big) \sum_{1 \leqslant y_1 \neq y_2 \leqslant N} e\big(\alpha k (y_1^\theta - y_2^\theta)\big).$$

Moving away the term $k = 0$, we find

$$\tilde{R}_2\big(f, \{\alpha n^\theta\}, N\big) = \frac{1}{N^2} \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} \hat{f}\Big(\frac{k}{N}\Big) \sum_{1 \leqslant y_1 \neq y_2 \leqslant N} e\big(\alpha k (y_1^\theta - y_2^\theta)\big) + \hat{f}(0) + O\Big(\frac{\hat{f}(0)}{N}\Big).$$

For convenience we add and subtract the diagonal term $y_1 = y_2$:

$$\tilde{R}_2\big(f, \{\alpha n^\theta\}, N\big) =$$
$$\frac{1}{N^2}\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \hat{f}\Big(\frac{k}{N}\Big)\Big|\sum_{1\leqslant y\leqslant N} e\big(\alpha k y^\theta\big)\Big|^2 - \frac{1}{N^2}\sum_{k\in\mathbb{Z}}\sum_{1\leqslant y\leqslant N}\hat{f}\Big(\frac{k}{N}\Big) + \hat{f}(0) + O\Big(\frac{1}{N}\Big).$$

Note that the term $k = 0$ is absorbed by the error $O(1/N)$. By Poisson summation,

$$\frac{1}{N^2}\sum_{k\in\mathbb{Z}}\sum_{1\leqslant y\leqslant N}\hat{f}\Big(\frac{k}{N}\Big) = \frac{1}{N}\sum_{k\in\mathbb{Z}}\hat{f}\Big(\frac{k}{N}\Big) = f(0).$$

Thus,

$$\tilde{R}_2\big(f, \{\alpha n^\theta\}, N\big) = \frac{1}{N^2}\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \hat{f}\Big(\frac{k}{N}\Big)\Big|\sum_{1\leqslant y\leqslant N} e\big(\alpha k y^\theta\big)\Big|^2 - f(0) + \hat{f}(0) + O\Big(\frac{1}{N}\Big).$$

Since $\hat{f}(0) = \int_{\mathbb{R}} f(x)dx$, the problem reduces to showing that

$$\frac{1}{N^2}\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \hat{f}\Big(\frac{k}{N}\Big)\Big|\sum_{1\leqslant y\leqslant N} e\big(\alpha k y^\theta\big)\Big|^2 = f(0) + o(1).$$

Without loss of generality, we can assume that f is even. Then the last formula is equivalent to

$$\frac{2}{N^2}\sum_{k=1}^{+\infty} \hat{f}\Big(\frac{k}{N}\Big)\Big|\sum_{1\leqslant y\leqslant N} e\big(\alpha k y^\theta\big)\Big|^2 = f(0) + o(1).$$

Due to the fast decay of $\hat{f}(x)$, the summation over k can be restricted to $k \leqslant N^{1+\varepsilon}$ for arbitrary small $\varepsilon > 0$. Thus for the proof of Theorem 1.1 it is enough to show that

$$\frac{2}{N^2}\sum_{k\leqslant N^{1+\varepsilon}} \hat{f}\Big(\frac{k}{N}\Big)\Big|\sum_{1\leqslant y\leqslant N} e\big(\alpha k y^\theta\big)\Big|^2 = f(0) + o_\varepsilon(1) \tag{2.1}$$

for any $\theta < \frac{43}{117} - 5\varepsilon$ (say).

3. Proof of Proposition 1.2

3.1. **Poisson summation on y.** We use the following form of the Poisson summation formula (see [10, Theorem 8.16]):

Lemma 3.1. *Let $f(x)$ be a real function on $[a, b]$ such that*

$$\Lambda \leqslant f''(x) \leqslant c\Lambda, \qquad |f^{(3)}(x)| \leqslant c\Lambda(b-a)^{-1}, \qquad |f^{(4)}(x)| \leqslant c\Lambda(b-a)^{-2}$$

for some $\Lambda > 0$ *and* $c \geqslant 1$*. Then*

$$\sum_{a<n<b} e\big(f(n)\big) = \sum_{\alpha<m<\beta} \frac{1}{\sqrt{f''(x_m)}} e\Big(f(x_m) - mx_m + \frac{1}{8}\Big) + R_f(a,b),$$

where $\alpha = f'(a), \beta = f'(b),\ f'(x_m) = m,$

$$R_f(a,b) \ll \Lambda^{-1/2} + c^2 \log(\beta - \alpha + 1).$$

When Y_1 and Y_2 are much different in size we estimate their contribution using van der Corput's theorem (see [10, Theorem 8.20]):

Lemma 3.2. *Let* $b - a \geqslant 1$*,* $f(x)$ *be a real function on* $[a,b]$*, and* $\nu \geqslant 2$ *such that* $\Lambda \leqslant f^{(\nu)}(x) \leqslant c\Lambda$*, where* $\Lambda > 0$ *and* $c \geqslant 1$*. Then*

$$\sum_{a<n<b} e\big(f(n)\big) \ll \Lambda^{\kappa}(b-a) + \Lambda^{-\kappa}(b-a)^{1-2^{2-\nu}}$$

where $\kappa = (2^\nu - 2)^{-1}$ *and the implied constant is absolute.*

Let us split the sum over $y \leqslant N$ in (2.1) to dyadic intervals $(Y, 2Y]$, so that $Y \leqslant N/2$. Then, applying Lemma 3.1 with $f(y) = -\alpha k y^\theta$, substituting $m \to -m$, and taking the conjugation, we get

$$\sum_{Y \leqslant y < 2Y} e\big(\alpha k y^\theta\big) = T_k(Y) + R_k(Y),$$

where

$$T_k(Y) = \sum_{\alpha\theta k(2Y)^{\theta-1} \leqslant m < \alpha\theta k Y^{\theta-1}} \frac{c_4}{\sqrt{k}} \Big(\frac{k}{m}\Big)^{(2-\theta)/(2-2\theta)} e\Big(c_3 k^{1/(1-\theta)} m^{-\theta/(1-\theta)} - \frac{1}{8}\Big),$$

$$c_4 = \frac{1}{\sqrt{\alpha\theta(1-\theta)}} \big(\alpha\theta\big)^{(2-\theta)/(2-2\theta)}, \qquad c_3 = \big(\alpha\theta\big)^{1/(1-\theta)} \Big(\frac{1}{\theta} - 1\Big),$$

$$R_k(Y) \ll \frac{Y^{1-\theta/2}}{\sqrt{k}} + \max\big(1, \log(kY^{\theta-1})\big). \tag{3.1}$$

Similarly, split the sum over $k \leqslant N^{1+\varepsilon}$ in (2.1) to dyadic intervals $(K, 2K]$:

$$\frac{2}{N^2} \sum_K \sum_{K \leqslant k < 2K} \hat{f}\big(\frac{k}{N}\big) \Big| \sum_Y \big(T_k(Y) + R_k(Y)\big)\Big|^2 =: \frac{2}{N^2} \sum_K \Big(S_{1,1}(K) + S_{1,2}(K) + S_{2,1}(K) + S_{2,2}(K)\Big),$$

where the sums $S_{1,1}, S_{1,2}, S_{2,1}, S_{2,2}$ correspond respectively to $T_k(Y_1)\overline{T_k(Y_2)}$, $T_k(Y_1)\overline{R_k(Y_2)}$, $R_k(Y_1)\overline{T_k(Y_2)}$, $R_k(Y_1)\overline{R_k(Y_2)}$.

We start by estimating the contribution from $S_{2,2}$. Applying trivially $\hat{f}\,(k/N) \ll 1$ and (3.1), we find

$$S_{2,2}(K) = \sum_{K\leqslant k<2K} \hat{f}\Big(\frac{k}{N}\Big)\sum_{Y_1,Y_2} R_k(Y_1)\overline{R_k(Y_2)} \ll K(\log N)^2\Big(\Big(\frac{N^{1-\theta/2}}{\sqrt{K}}\Big)^2 + \big(\log K\big)^2\Big).$$

Hence,

$$\frac{2}{N^2}\sum_K S_{2,2}(K) \ll N^{-\theta}(\log N)^3 + N^{-1+\varepsilon}(\log N)^5, \tag{3.2}$$

which is negligible.

The sums $S_{1,2}$ and $S_{2,1}$ are similar, so we only consider $S_{1,2}$:

$$S_{1,2}(K) = \sum_{K\leqslant k<2K} \hat{f}\Big(\frac{k}{N}\Big)\sum_{Y_1,Y_2} T_k(Y_1)\overline{R_k(Y_2)}.$$

Applying Cauchy inequality to the sum over $K \leqslant k < 2K$, we obtain

$$S_{1,2}(K) \leqslant \sum_{Y_1,Y_2}\Big(\sum_{K\leqslant k<2K}\big|\hat{f}\Big(\frac{k}{N}\Big)\big|^2|R_k(Y_2)|^2\frac{k^{(2-\theta)/(1-\theta)}}{k}\Big)^{1/2}.$$

$$\Big(\sum_{K\leqslant k<2K}\Big|\sum_{\alpha\theta k(2Y_1)^{\theta-1}\leqslant m<\alpha\theta k Y_1^{\theta-1}}\Big(\frac{c_4}{m}\Big)^{(2-\theta)/(2-2\theta)} e\Big(c_3 k^{1/(1-\theta)} m^{-\theta/(1-\theta)} - \frac{1}{8}\Big)\Big|^2\Big)^{1/2}.$$

Hence,

$$S_{1,2}(K) \ll \sum_{Y_1,Y_2}\Big(\sum_{K\leqslant k<2K}|R_k(Y_2)|^2 k^{1/(1-\theta)}\Big)^{1/2}\big|U_{1,2}(K)\big|^{1/2}, \tag{3.3}$$

where

$$U_{1,2}(K) := \sum_{K\leqslant k<2K}\ \sum_{\substack{m_1,m_2\in\\ [\alpha\theta k(2Y_1)^{\theta-1},\alpha\theta k Y_1^{\theta-1})}}\Big(\frac{c_4^2}{m_1 m_2}\Big)^{(2-\theta)/(2-2\theta)} e\Big(c_3 k^{1/(1-\theta)}\eta(m_1,m_2)\Big),$$

where $\eta(m_1, m_2) = m_1^{-\theta/(1-\theta)} - m_2^{-\theta/(1-\theta)}$. Changing the order of summation we obtain

$$U_{1,2}(K) = \sum_{\substack{m_1,m_2\in\\ [\alpha\theta K(2Y_1)^{\theta-1},2\alpha\theta K Y_1^{\theta-1})}}\Big(\frac{c_4^2}{m_1 m_2}\Big)^{(2-\theta)/(2-2\theta)}\sum_{K\leqslant k<2K}{}' e\Big(c_3 k^{1/(1-\theta)}\eta(m_1,m_2)\Big),$$

where $\sum'$ denotes the sum with the additional restrictions

$$m_1, m_2 \in [\alpha\theta k(2Y_1)^{\theta-1}, \alpha\theta k Y_1^{\theta-1}].$$

Next, substitute

$$m_1 := m, \qquad m_2 := m + r.$$

Without loss of generality, let us assume that $m_1 \leqslant m_2$, which implies that $r \geqslant 0$. Next, we can split the sum $U_{1,2}$ into its diagonal ($r = 0$) and non-diagonal ($r > 0$) terms. Furthermore, we can split the non-diagonal sum into dyadic intervals $r \in (R, 2R]$, where R satisfies $R \leqslant (2 - 2^{\theta-1})\alpha\theta K Y_1^{\theta-1}$. We get

$$U_{1,2}(K) = D_{1,2}(K) + \sum_{R} \sum_{R \leqslant r < 2R} \sum_{\substack{m:\ m, m+r \in \\ [\alpha\theta K(2Y_1)^{\theta-1}, 2\alpha\theta K Y_1^{\theta-1})}} \Big(\frac{c_4^2}{m(m+r)}\Big)^{(2-\theta)/(2-2\theta)} \sum_{K \leqslant k < 2K}{}^{'} e\Big(c_3 k^{1/(1-\theta)}\eta\Big), \tag{3.4}$$

where $\eta = \eta(m, m+r) = m^{-\theta/(1-\theta)} - (m+r)^{-\theta/(1-\theta)} \sim r m^{-1/(1-\theta)}$. Note that the sum over k can be empty, in which case the necessary bound follows immediately.

Remark 3.1. *From now on, we assume that $\alpha\theta K Y_1^{\theta-1} \geqslant 10$. In this case, the sum over m contains more than one term. Otherwise, we trivially have $U_{1,2}(K) \ll K$, which, by (3.3), implies*

$$S_{1,2}(K) \ll \sum_{\substack{Y_1, Y_2 \\ Y_1 \gg K^{1/(1-\theta)}}} \bigg(\Big(\frac{Y_2^{1-\theta/2}}{\sqrt{K}}\Big)^2 K^{1/(1-\theta)}\bigg)^{1/2} K^{1/2} \ll K^{1/(2-2\theta)} N^{1-\theta/2} \log N,$$

which contributes to the right hand side of (1.2) no more than

$$\frac{2}{N^2} \sum_{K \ll N^{1-\theta}} K^{1/(2-2\theta)} N^{1-\theta/2} \log N \ll N^{-1/2-\theta/2} \log N,$$

which is negligible.

For the diagonal term $D_{1,2}(K)$, we have

$$D_{1,2}(K) \leqslant \sum_{K \leqslant k < 2K} \sum_{\substack{m \in \\ [\alpha\theta K(2Y_1)^{\theta-1}, 2\alpha\theta K Y_1^{\theta-1})}} \Big(\frac{c_4}{m}\Big)^{(2-\theta)/(1-\theta)} \ll K^{-\theta/(1-\theta)} Y_1. \tag{3.5}$$

Next, we treat the non-diagonal part of $U_{1,2}$. We apply the first part of Lemma 3.2 to the inner sum over k (assuming it is non-empty) in (3.4) with $\Lambda = R Y_1 K^{-2}$ and $\nu = 2$. We get

$$\sum_{K \leqslant k < 2K}{}^{'} e\big(c_3 k^{1/(1-\theta)}\eta\big) \ll K\big(R Y_1 K^{-2}\big)^{1/2} + \big(R Y_1 K^{-2}\big)^{-1/2} = (R Y_1)^{1/2} + K (R Y_1)^{-1/2}.$$

Thus, combining this bound with (3.4) and (3.5), we find

$$U_{1,2}(K) \ll K^{-\theta/(1-\theta)} Y_1 + \sum_R \Big(RKY_1^{\theta-1} (KY_1^{\theta-1})^{-(2-\theta)/(1-\theta)} \big[(RY_1)^{1/2} + K(RY_1)^{-1/2} \big] \Big) \ll K^{-\theta/(1-\theta)} Y_1 + K^{1/2-\theta/(1-\theta)} Y_1^{3\theta/2} + K^{-1/2-\theta/(1-\theta)} Y_1^{\theta/2} \ll K^{-\theta/(1-\theta)} Y_1 + K^{1/2-\theta/(1-\theta)} Y_1^{3\theta/2}.$$

Hence, from (3.3),

$$S_{1,2}(K) \ll \sum_{Y_1,Y_2} \Big(\sum_{K \leqslant k < 2K} \max\Big(1, \big(\log(kY_2^{\theta-1})\big)^2, \frac{Y_2^{2-\theta}}{k}\Big) k^{1/(1-\theta)} \Big)^{1/2} \cdot \big(K^{-\theta/(1-\theta)} Y_1 + K^{1/2-\theta/(1-\theta)} Y_1^{3\theta/2} \big)^{1/2} \ll \sum_{Y_1,Y_2} \big(K^{1/2} \log N + Y_2^{1-\theta/2} \big) \big((KY_1)^{1/2} + (KY_1^\theta)^{3/4} \big).$$

Here we used the inequality $\max(1, (\log kY_2^{\theta-1})^2) \leqslant (\log N)^2$. Finally,

$$S_{1,2}(K) \ll N^{1-\theta/2} \log N \big((KN)^{1/2} + (KN^\theta)^{3/4} \big),$$

and so

$$\frac{2}{N^2} \sum_K S_{1,2}(K) \ll \frac{\log N}{N^2} \big(N^{2-\theta/2+\varepsilon/2} + N^{7/4+\theta/4+3\varepsilon/4} \big), \tag{3.6}$$

which is $o(1)$ if $\varepsilon < \theta < 1 - 3\varepsilon$. The contribution from $S_{2,1}(K)$ is estimated similarly.

It remains only to evaluate the contribution from $S_{1,1}(K)$:

$$S_{1,1}(K) = \sum_{Y_1,Y_2} \sum_{K \leqslant k < 2K} \hat{f}\Big(\frac{k}{N}\Big) T_k(Y_1) \overline{T_k(Y_2)} =: D_{1,1}(K) + E_{1,1}(K),$$

where $D_{1,1}(K)$ is the diagonal term ($Y_1 = Y_2$, $m_1 = m_2$),

$$D_{1,1}(K) = \sum_Y \sum_{K \leqslant k < 2K} \hat{f}\Big(\frac{k}{N}\Big) \frac{1}{\alpha\theta(1-\theta)k} \big(\alpha k \theta\big)^{(2-\theta)/(1-\theta)} \sum_{\alpha\theta k(2Y)^{\theta-1} \leqslant m < \alpha\theta k Y^{\theta-1}} m^{-(2-\theta)/(1-\theta)}.$$

By [2, Theorem 3.2], the inner sum is

$$\frac{\big(\alpha\theta k Y^{\theta-1}\big)^{1-(2-\theta)/(1-\theta)} - \big(2^{\theta-1}\alpha\theta k Y^{\theta-1}\big)^{1-(2-\theta)/(1-\theta)}}{1-(2-\theta)/(1-\theta)} + O\big((KY)^{-(2-\theta)/(1-\theta)}\big).$$

Simplifying the coefficients and using $\sum_Y Y = N + O(1)$, we obtain

$$S_{1,1}(K) = \sum_{Y} \sum_{K \leqslant k < 2K} \hat{f}\Big(\frac{k}{N}\Big) Y + O\Big(\sum_{Y} \sum_{K \leqslant k < 2K} k^{1/(1-\theta)} (KY)^{-(2-\theta)/(1-\theta)}\Big) + E_{1,1}(K) = N \sum_{K \leqslant k < 2K} \hat{f}\Big(\frac{k}{N}\Big) + O\big(K\big) + O\big(\log N\big) + E_{1,1}(K).$$

Finally,

$$\frac{2}{N^2} \sum_{K} S_{1,1}(K) = \frac{2}{N} \sum_{k \leqslant N^{1+\varepsilon}} \hat{f}\Big(\frac{k}{N}\Big) + O\Big(\frac{1}{N}\Big) + O\Big(\frac{(\log N)^2}{N^2}\Big) + \frac{2}{N^2} \sum_{K} E_{1,1}(K) = \frac{1}{N} \sum_{k \in \mathbb{Z}} \hat{f}\Big(\frac{k}{N}\Big) + O_\varepsilon\Big(\frac{1}{N}\Big) + O\Big(\frac{(\log N)^2}{N^2}\Big) + \frac{2}{N^2} \sum_{K} E_{1,1}(K) = f(0) + O\Big(\frac{(\log N)^2}{N^2}\Big) + \frac{2}{N^2} \sum_{K} E_{1,1}(K). \tag{3.7}$$

To complete the proof of Proposition 1.2, we need to show that the sum with $E_{1,1}(K)$ is bounded by $S(N)$ given in (1.3).

3.2. **Poisson summation on k.** We have

$$E_{1,1}(K) = \sum_{Y_1, Y_2} \sum_{K \leqslant k < 2K} \hat{f}\Big(\frac{k}{N}\Big) \sum_{\substack{m_1 \neq m_2 \\ \alpha\theta k (2Y_i)^{\theta-1} \leqslant m_i < \alpha\theta k Y_i^{\theta-1}}} \Big(\frac{c_4}{\sqrt{k}}\Big)^2 \Big(\frac{k^2}{m_1 m_2}\Big)^{(2-\theta)/(2-2\theta)} \cdot e\Big(c_3 k^{1/(1-\theta)} \eta(m_1, m_2)\Big).$$

We first remove the factor $\hat{f}(k/N)$ by partial summation:

Lemma 3.3. *Let $\{a_k\}$ and $\{b_k\}$ be sequences of complex numbers, and $c > 1$ be a constant. If $T > 0$ is such that $|b_k - b_{k+1}| \leqslant T/k$, then*

$$\Big| \sum_{A < k < B} a_k b_k \Big| \leqslant \Big(\max_{A \leqslant k \leqslant cA} |b_k| + O(T) \Big) \max_{A \leqslant \tilde{A} \leqslant cA} \Big| \sum_{A \leqslant k < \tilde{A}} a_k \Big|$$

for any positive integers A and B satisfying $A \leqslant B \leqslant cA$.

Now we apply Lemma 3.3 to $E_{1,1}(K)$ with $b_k = \hat{f}(k/N)$ and

$$a_k = \sum_{\substack{m_1 \neq m_2 \\ \alpha\theta k (2Y_i)^{\theta-1} \leqslant m_i < \alpha\theta k Y_i^{\theta-1}}} \Big(\frac{c_4}{\sqrt{k}}\Big)^2 \Big(\frac{k^2}{m_1 m_2}\Big)^{(2-\theta)/(2-2\theta)} e\Big(c_3 k^{1/(1-\theta)} \eta(m_1, m_2)\Big).$$

Since clearly $|\hat{f}(k/N) - \hat{f}((k+1)/N)| \ll 1/N$, we can choose $T \asymp 1$. Thus,

$$E_{1,1}(K) \ll \sum_{Y_1,Y_2} \max_{K\leqslant \tilde{K}\leqslant 2K} \Big| \sum_{K\leqslant k<\tilde{K}} c_4^2 k^{(2-\theta)/(1-\theta)-1} \cdot \sum_{\substack{m_1\neq m_2 \\ \alpha\theta k(2Y_i)^{\theta-1}\leqslant m_i<\alpha\theta k Y_i^{\theta-1}}} \Big(\frac{1}{m_1 m_2}\Big)^{(2-\theta)/(2-2\theta)} e\Big(c_3 k^{1/(1-\theta)}\eta(m_1,m_2)\Big)\Big|.$$

Next, we remove the factor $c_4^2 k^{(2-\theta)/(1-\theta)-1}$ from the sum over k by the second application of Lemma 3.3 with

$$a_k = \sum_{\substack{m_1\neq m_2 \\ \alpha\theta k(2Y_i)^{\theta-1}\leqslant m_i<\alpha\theta k Y_i^{\theta-1}}} \Big(\frac{1}{m_1 m_2}\Big)^{(2-\theta)/(2-2\theta)} e\Big(c_3 k^{1/(1-\theta)}\eta(m_1,m_2)\Big),$$

$$b_k = c_4^2 k^{(2-\theta)/(1-\theta)-1}, \qquad T = cK^{(2-\theta)/(1-\theta)-1}$$

with some constant $c > 0$. Note that

$$\big|b_k - b_{k+1}\big| \ll \frac{1}{k} k^{(2-\theta)/(1-\theta)-1}.$$

Thus,

$$E_{1,1}(K) \ll K^{(2-\theta)/(1-\theta)-1} \sum_{Y_1,Y_2} \max_{K\leqslant \tilde{K}\leqslant 2K} \Big| \sum_{K\leqslant k<\tilde{K}} \sum_{\substack{m_1\neq m_2 \\ \alpha\theta k(2Y_i)^{\theta-1}\leqslant m_i<\alpha\theta k Y_i^{\theta-1}}} \Big(\frac{1}{m_1 m_2}\Big)^{(2-\theta)/(2-2\theta)} e\Big(c_3 k^{1/(1-\theta)}\eta(m_1,m_2)\Big)\Big|.$$

Changing the order of summation, we obtain

$$E_{1,1}(K) \ll K^{(2-\theta)/(1-\theta)-1} \sum_{Y_1,Y_2} \max_{K\leqslant \tilde{K}\leqslant 2K} \Big| \sum_{\substack{m_1\neq m_2 \\ \alpha\theta K(2Y_i)^{\theta-1}\leqslant m_i<2\alpha\theta K Y_i^{\theta-1}}} \Big(\frac{1}{m_1 m_2}\Big)^{(2-\theta)/(2-2\theta)} \sum_{K_1\leqslant k\leqslant K_2} e\Big(c_3 k^{1/(1-\theta)}\eta(m_1,m_2)\Big)\Big|, \tag{3.8}$$

where K_1 and K_2 were defined in (1.4) and (1.5).

Due to the symmetry between m_1 and m_2, we can add the restriction $m_1 < m_2$ to the corresponding sums in (3.8). Then, applying Lemma 3.1 to the sum over k

in (3.8), we get

$$E_{1,1}(K) \ll K^{(2-\theta)/(1-\theta)-1} \sum_{Y_1,Y_2} \max_{K \leqslant \tilde{K} \leqslant 2K} \Big| \sum_{\eta \in \mathcal{N}(K,Y_1,Y_2)} \Big(\frac{1}{m_1 m_2}\Big)^{(2-\theta)/(2-2\theta)} \cdot \sum_{L \leqslant \ell < \tilde{L}} \frac{c_1}{\sqrt{\eta}} \Big(\frac{\ell}{\eta}\Big)^{1/(2\theta)-1} e\Big(c_2 \ell^{1/\theta} \eta^{1-1/\theta} + \frac{1}{8}\Big)\Big|, \tag{3.9}$$

where $L, \tilde{L}, c_1, c_2$ were defined in Proposition 1.2. The factor $e(1/8)$ does not affect the absolute value of the inner double sum and, thus, can be omitted. We complete the proof by combining together the estimates (3.2), (3.6), (3.7) and (3.9).

4. Proof of Proposition 1.3

4.1. **Van der Corput differencing.** Let us denote the sum over $\eta \in \mathcal{N}(K, Y_1, Y_2)$ in (3.9) by $\tilde{S}(K, \tilde{K}, Y_1, Y_2)$, as in Proposition 1.2. Set the notation $m_2 =: m$, $m_1 =: m - r$. We then consider the sum over r. It has $\asymp KY_1^{\theta-1}$ terms with the restrictions $r \ll KY_2^{\theta-1}$ on the one side and (when $Y_1 > 2Y_2$) $r \gg KY_2^{\theta-1}$ on the other side. We split it to the dyadic intervals $(R, 2R]$ as follows:

$$\big|\tilde{S}(K, \tilde{K}, Y_1, Y_2)\big| = \sum_{\substack{R \ll KY_2^{\theta-1} \\ \text{dyadic}}} \big|W(K, \tilde{K}, Y_1, Y_2, R)\big|,$$

where

$$W(K, \tilde{K}, Y_1, Y_2, R) := \sum_{\eta \in \mathcal{N}_R(K,Y_1,Y_2)} \Big(\frac{1}{(m-r)m}\Big)^{(2-\theta)/(2-2\theta)} \sum_{L \leqslant \ell < \tilde{L}} \frac{c_1}{\sqrt{\eta}} \Big(\frac{\ell}{\eta}\Big)^{1/(2\theta)-1} e\Big(c_2 \ell^{1/\theta} \eta^{1-1/\theta}\Big)$$

with

$$\begin{aligned} \mathcal{N}_R(K, Y_1, Y_2) := \big\{\eta : \eta = m_1^{-\theta/(1-\theta)} - m_2^{-\theta/(1-\theta)}, \text{ where} \\ \alpha\theta K(2Y_1)^{\theta-1} \leqslant m_1 < 2\alpha\theta K Y_1^{\theta-1}; \\ \alpha\theta K(2Y_2)^{\theta-1} \leqslant m_2 < 2\alpha\theta K Y_2^{\theta-1}; \\ m_1 < m_2,\ m_2 - m_1 \in [R, 2R)\big\}. \end{aligned}$$

Applying Cauchy inequality, we obtain

$$\big|W(K, \tilde{K}, Y_1, Y_2, R)\big|^2 \leqslant \Big(\sum_{\eta \in \mathcal{N}_R(K,Y_1,Y_2)} \Big(\frac{1}{(m-r)m}\Big)^{(2-\theta)/(1-\theta)} \Big(\frac{1}{\eta}\Big)^{1/\theta-1}\Big) \cdot \Big(\sum_{\eta \in \mathcal{N}_R(K,Y_1,Y_2)} \Big| \sum_{L \leqslant \ell < \tilde{L}} \ell^{1/(2\theta)-1} e\Big(c_2 \ell^{1/\theta} \eta^{1-1/\theta}\Big)\Big|^2\Big).$$

For the number of terms in $\mathcal{N}_R(K, Y_1, Y_2)$ we clearly have

$$\#\mathcal{N}_R(K, Y_1, Y_2) \ll \begin{cases} KY_1^{\theta-1}KY_2^{\theta-1}, & \text{if } Y_1 > 2Y_2; \\ RKY_2^{\theta-1}, & \text{if } Y_2/2 \leqslant Y_1 \leqslant 2Y_2. \end{cases} \tag{4.1}$$

In fact, $\#\mathcal{N}_R(K, Y_1, Y_2) = 0$ when Y_1 is much larger than Y_2, and, at the same time, R is small compared to $KY_2^{\theta-1}$. Note also that $\#\mathcal{N}_R(K, Y_1, Y_2) = 0$ when $Y_1 \leqslant Y_2/4$ by the definition. Next, recall that

$$L \asymp \eta K^{\theta/(1-\theta)}. \tag{4.2}$$

For $\eta \in \mathcal{N}_R(K, Y_1, Y_2)$ one has $\eta \asymp Z$, where

$$Z = Z(K, Y_1, Y_2, R) := \begin{cases} Y_1^\theta K^{-\theta/(1-\theta)}, & \text{if } Y_1 > 2Y_2; \\ RY_2K^{-1/(1-\theta)}, & \text{if } Y_2 \leqslant Y_1 \leqslant 2Y_2. \end{cases} \tag{4.3}$$

Then, applying partial summation to the sum over ℓ (with $a_\ell = e(c_2\ell^{1/\theta}\eta^{1-1/\theta})$, $b_\ell = \ell^{1/(2\theta)-1}$, $T = L^{1/(2\theta)}$), we find

$$\begin{aligned} &\left|W(K, \tilde{K}, Y_1, Y_2, R)\right|^2 \ll \\ &\qquad \left(\#\mathcal{N}_R(K, Y_1, Y_2)\left(\frac{1}{KY_1^{\theta-1}KY_2^{\theta-1}}\right)^{(2-\theta)/(1-\theta)}\left(\frac{1}{Z}\right)^{1/\theta-1}\right) \cdot \\ &\qquad \left(ZK^{\theta/(1-\theta)}\right)^{1/\theta-2}\Bigg(\sum_{\substack{\eta\in\mathcal{N}(K,Y_1,Y_2)\\ \eta\asymp Z}}\Bigg|\sum_{L\leqslant\ell<L_1} e\left(c_2\ell^{1/\theta}\eta^{1-1/\theta}\right)\Bigg|^2\Bigg) = \\ &\#\mathcal{N}_R(K, Y_1, Y_2)K^{-3/(1-\theta)}(Y_1Y_2)^{2-\theta}Z^{-1}\Bigg(\sum_{\substack{\eta\in\mathcal{N}(K,Y_1,Y_2)\\ \eta\asymp Z}}\Bigg|\sum_{L\leqslant\ell<L_1} e\left(c_2\ell^{1/\theta}\eta^{1-1/\theta}\right)\Bigg|^2\Bigg). \end{aligned}$$

Then

$$\begin{aligned} &\left|\tilde{S}(K, \tilde{K}, Y_1, Y_2)\right| \ll \\ &\quad \left(K^{-3/(1-\theta)}(Y_1Y_2)^{2-\theta}\right)^{1/2}\sum_{\substack{R\ll KY_2^{\theta-1}\\ \text{dyadic}}}\left(\frac{\#\mathcal{N}_R(K, Y_1, Y_2)}{Z}\right)^{1/2}U(K, \tilde{K}, Y_1, Y_2, R)^{1/2}, \end{aligned} \tag{4.4}$$

where

$$U(K, \tilde{K}, Y_1, Y_2, R) := \sum_{\eta\in\mathcal{N}_R(K,Y_1,Y_2)}\Bigg|\sum_{L\leqslant\ell<L_1} e\left(c_2\ell^{1/\theta}\eta^{1-1/\theta}\right)\Bigg|^2. \tag{4.5}$$

To estimate $U(K, \tilde{K}, Y_1, Y_2, R)$, we first apply the van der Corput inequality (see [10, Lemma 8.17]):

Lemma 4.1. *For any complex numbers z_n we have*

$$\Big|\sum_{a<n<b} z_n\Big|^2 \leqslant \Big(1+\frac{b-a}{H}\Big)\sum_{|h|<H}\Big(1-\frac{|h|}{H}\Big)\sum_{a<n,n+h<b} z_{n+h}\overline{z_n},$$

where H is any positive integer.

Applying this lemma to the sum over ℓ in (4.5) with $H = H_1 = L^{\kappa_1}$ for some $0 < \kappa_1 < 1$, which will be chosen later, we obtain

$$\big|U(K,\tilde{K},Y_1,Y_2,R)\big| \leqslant \sum_{\eta\in\mathcal{N}_R(K,Y_1,Y_2)}\Big(1+\frac{L_1-L}{H_1}\Big)\sum_{|h_1|<H_1}\Big(1-\frac{|h_1|}{H_1}\Big)\sum_{L\leqslant \ell<L_1-h_1} e\Big(c_2\eta^{1-1/\theta}t(\ell,h_1)\Big),$$

where

$$t(\ell,h_1) := (\ell+h_1)^{1/\theta}-\ell^{1/\theta}.$$

Then

$$\big|U(K,\tilde{K},Y_1,Y_2,R)\big| \ll \sum_{\eta\in\mathcal{N}_R(K,Y_1,Y_2)}\Big(\frac{L^2}{H_1}+\frac{L}{H_1}\sum_{0<|h_1|<H_1}\Big|\sum_{L\leqslant \ell<L_1-h_1} e\Big(c_2\eta^{1-1/\theta}t(\ell,h_1)\Big)\Big|\Big) \ll$$
$$\#\mathcal{N}_R(K,Y_1,Y_2)\frac{L^2}{H_1}+\frac{L}{H_1}\sum_{\eta\in\mathcal{N}_R(K,Y_1,Y_2)}\sum_{0<|h_1|<H_1}\Big|\sum_{L\leqslant \ell<L_1-h_1} e\Big(c_2\eta^{1-1/\theta}t(\ell,h_1)\Big)\Big|.$$

Applying Cauchy inequality again, we find

$$\big|U(K,\tilde{K},Y_1,Y_2,R)\big| \leqslant \#\mathcal{N}_R(K,Y_1,Y_2)\frac{L^2}{H_1}+\frac{L}{H_1}\Big(\sum_{\eta\in\mathcal{N}_R(K,Y_1,Y_2)}\sum_{0<|h_1|<H_1}1\Big)^{1/2}\cdot$$
$$\Big(\sum_{\eta\in\mathcal{N}_R(K,Y_1,Y_2)}\sum_{0<|h_1|<H_1}\Big|\sum_{L\leqslant \ell<L_1-h_1} e\Big(c_2\eta^{1-1/\theta}t(\ell,h_1)\Big)\Big|^2\Big)^{1/2}.$$

Next, we apply Lemma 4.1 for the second time with $H = H_2 = L^{\kappa_2}$ for some $0 < \kappa_2 < 1$, which will be chosen later:

$$\big|U(K,\tilde{K},Y_1,Y_2,R)\big| \ll \#\mathcal{N}_R(K,Y_1,Y_2)\frac{L^2}{H_1}+\frac{L}{H_1}\big(\#\mathcal{N}_R(K,Y_1,Y_2)H_1\big)^{1/2}\cdot$$
$$\Big(\sum_{\eta\in\mathcal{N}_R(K,Y_1,Y_2)}\sum_{0\leqslant|h_1|<H_1}\Big(1+\frac{L_1-h_1-L}{H_2}\Big)\cdot$$
$$\sum_{0<|h_2|<H_2}\gamma(h_2)\sum_{L\leqslant \ell<L_1-h_1-h_2} e\Big(c_2\eta^{1-1/\theta}t(\ell,h_1,h_2)\Big)\Big)^{1/2},$$

where $\gamma(h_2) := 1 - |h_2|/H_2$,

$$t(\ell, h_1, h_2) := \big((\ell + h_2 + h_1)^{1/\theta} - (\ell + h_2)^{1/\theta}\big) - \big((\ell + h_1)^{1/\theta} - \ell^{1/\theta}\big).$$

Hence,

$$\begin{aligned}
&\big|U(K, \tilde{K}, Y_1, Y_2, R)\big| \ll \\
&\quad \#\mathcal{N}_R(K, Y_1, Y_2)\frac{L^2}{H_1} + \frac{L}{H_1}\big(\#\mathcal{N}_R(K, Y_1, Y_2)H_1\big)^{1/2}\Big(\#\mathcal{N}_R(K, Y_1, Y_2)H_1\frac{L^2}{H_2}\Big)^{1/2} + \\
&\qquad \frac{L}{H_1}\big(\#\mathcal{N}_R(K, Y_1, Y_2)H_1\big)^{1/2}\Big(\frac{L}{H_2}\Big)^{1/2} \cdot \\
&\qquad \Big(\Big|\sum_{\substack{\eta, h_1, h_2 \\ \eta \in \mathcal{N}_R(K, Y_1, Y_2) \\ 0 < |h_i| < H_i}} \gamma(h_2) \sum_{L \leqslant \ell < L_1 - h_1 - h_2} e\Big(c_2 \eta^{1-1/\theta} t(\ell, h_1, h_2)\Big)\Big|\Big)^{1/2}.
\end{aligned}$$

We rewrite the last inequality as

$$\begin{aligned}
\big|U(K, \tilde{K}, Y_1, Y_2, R)\big| &\leqslant A_1(R) + A_2(R) + \\
& A_3(R)\Big(\Big|\sum_{\substack{\eta, h_1, h_2 \\ \eta \in \mathcal{N}_R(K, Y_1, Y_2) \\ 0 < |h_i| < H_i}} \gamma(h_2) \sum_{L \leqslant \ell < L_1 - h_1 - h_2} e\Big(c_2 \eta^{1-1/\theta} t(\ell, h_1, h_2)\Big)\Big|\Big)^{1/2},
\end{aligned} \tag{4.6}$$

where

$$A_1(R) := \#\mathcal{N}_R(K, Y_1, Y_2)\frac{L^2}{H_1}, \qquad A_2(R) := \#\mathcal{N}_R(K, Y_1, Y_2)\frac{L^2}{H_2^{1/2}},$$

$$A_3(R) := \frac{L^{3/2}(\#\mathcal{N}_R(K, Y_1, Y_2))^{1/2}}{(H_1 H_2)^{1/2}}.$$

Substituting the bound from (4.6) into (4.4), we get

$$\begin{aligned}
&\big|\tilde{S}(K, \tilde{K}, Y_1, Y_2)\big| \ll \\
&\quad \big(K^{-3/(1-\theta)}(Y_1 Y_2)^{2-\theta}\big)^{1/2} \sum_R \Big(\frac{\#\mathcal{N}_R(K, Y_1, Y_2)}{Z}\Big)^{1/2}\Big(|A_1(R)|^{1/2} + \\
&|A_2(R)|^{1/2} + |A_3(R)|^{1/2}\Big(\Big|\sum_{\substack{\eta, h_1, h_2 \\ \eta \in \mathcal{N}_R(K, Y_1, Y_2) \\ 0 < |h_i| < H_i}} \gamma(h_2) \sum_{L \leqslant \ell < L_1 - h_1 - h_2} e\Big(c_2 \eta^{1-1/\theta} t(\ell, h_1, h_2)\Big)\Big|\Big)^{1/4}\Big).
\end{aligned}$$

Then we have

$$\tilde{S}(K,\tilde{K},Y_1,Y_2) \ll \sum_{R} \Big(B_1(R) + B_2(R) + \\ B_3(R) \Big(\Big| \sum_{\substack{\eta,h_1,h_2 \\ \eta\in\mathcal{N}_R(K,Y_1,Y_2) \\ 0<|h_i|<H_i}} \gamma(h_2) \sum_{L\leqslant \ell < L_1-h_1-h_2} e\Big(c_2\eta^{1-1/\theta} t(\ell,h_1,h_2)\Big)\Big|\Big)^{1/4}\Big),$$

where

$$B_1(R) := \Big(K^{-3/(1-\theta)}(Y_1Y_2)^{2-\theta}\frac{\#\mathcal{N}_R^2(K,Y_1,Y_2)L^2}{ZH_1}\Big)^{1/2}, \tag{4.7}$$

$$B_2(R) := \Big(K^{-3/(1-\theta)}(Y_1Y_2)^{2-\theta}\frac{\#\mathcal{N}_R^2(K,Y_1,Y_2)L^2}{ZH_2^{1/2}}\Big)^{1/2}, \tag{4.8}$$

$$B_3(R) := \Big(K^{-3/(1-\theta)}(Y_1Y_2)^{2-\theta}\frac{\#\mathcal{N}_R^2(K,Y_1,Y_2)L^{3/2}}{Z(H_1H_2)^{1/2}}\Big)^{1/2}. \tag{4.9}$$

Upon inserting it into (1.3), we obtain

$$S(N) \ll \frac{1}{N^2}\sum_{K,Y_1,Y_2} K^{(2-\theta)/(1-\theta)-1} \max_{K\leqslant \tilde{K}\leqslant 2K} \sum_{R} \Big(B_1(R) + B_2(R) + \\ B_3(R) \Big(\Big| \sum_{\substack{\eta,h_1,h_2 \\ \eta\in\mathcal{N}_R(K,Y_1,Y_2) \\ 0<|h_i|<H_i}} \gamma(h_2) \sum_{L\leqslant \ell < L_1-h_1-h_2} e\Big(c_2\eta^{1-1/\theta} t(\ell,h_1,h_2)\Big)\Big|\Big)^{1/4}\Big) =: \\ S_{B_1}(N) + S_{B_2}(N) + S_{B_3}(N), \tag{4.10}$$

where the meaning of the notation $S_{B_i}(N)$ is clear.

Let us first estimate $S_{B_1}(N)$ and $S_{B_2}(N)$.

Case 1. $Y_2/2 \leqslant Y_1 \leqslant 2Y_2$. Combining (4.1), (4.2), (4.3), and (4.7), we get

$$\frac{1}{N^2} \sum_{\substack{K,Y_1,Y_2 \\ Y_2/2 \leqslant Y_1 \leqslant 2Y_2}} K^{(2-\theta)/(1-\theta)-1} \sum_R B_1(R) \ll$$

$$\frac{1}{N^2} \sum_{\substack{K,Y_1,Y_2 \\ Y_2/2 \leqslant Y_1 \leqslant 2Y_2}} K^{(2-\theta)/(1-\theta)-1} K^{-3/(2(1-\theta))} (Y_1Y_2)^{1-\theta/2}.$$

$$\sum_R \frac{RKY_2^{\theta-1}RY_2K^{-1}}{\left(RY_2K^{-1/(1-\theta)}\right)^{1/2}\left(RY_2K^{-1}\right)^{\kappa_1/2}} \ll \frac{1}{N^2} \sum_{\substack{K,Y_1,Y_2 \\ Y_2/2 \leqslant Y_1 \leqslant 2Y_2}} K^{\kappa_1/2} Y_1^{1-\theta/2} Y_2^{1/2+\theta/2-\kappa_1/2}.$$

$$\sum_R R^{3/2-\kappa_1/2} \ll \frac{1}{N^2} \sum_{\substack{K,Y_1,Y_2 \\ Y_2/2 \leqslant Y_1 \leqslant 2Y_2}} K^{3/2} Y_1^{1-\theta/2} Y_2^{-1+2\theta-\kappa_1\theta/2} \ll N^{-1/2+3\theta/2-\kappa_1\theta/2+3\varepsilon/2}.$$

Similarly,

$$\frac{1}{N^2} \sum_{\substack{K,Y_1,Y_2 \\ Y_2/2 \leqslant Y_1 \leqslant 2Y_2}} K^{(2-\theta)/(1-\theta)-1} \sum_R B_2(R) \ll$$

$$\frac{1}{N^2} \sum_{\substack{K,Y_1,Y_2 \\ Y_2/2 \leqslant Y_1 \leqslant 2Y_2}} K^{(2-\theta)/(1-\theta)-1} K^{-3/(2(1-\theta))} (Y_1Y_2)^{1-\theta/2}.$$

$$\sum_R \frac{RKY_2^{\theta-1}RY_2K^{-1}}{\left(RY_2K^{-1/(1-\theta)}\right)^{1/2}\left(RY_2K^{-1}\right)^{\kappa_2/4}} \ll \frac{1}{N^2} \sum_{\substack{K,Y_1,Y_2 \\ Y_2/2 \leqslant Y_1 \leqslant 2Y_2}} K^{\kappa_2/4} Y_1^{1-\theta/2} Y_2^{1/2-\kappa_2/4+\theta/2}.$$

$$\sum_R R^{3/2-\kappa_2/4} \ll \frac{1}{N^2} \sum_{\substack{K,Y_1,Y_2 \\ Y_2/2 \leqslant Y_1 \leqslant 2Y_2}} K^{3/2} Y_1^{1-\theta/2} Y_2^{-1+2\theta-\kappa_2\theta/4} \ll N^{-1/2+3\theta/2-\kappa_2/4+3\varepsilon/2}.$$

Case 2. $Y_1 > 2Y_2$. We have

$$\frac{1}{N^2} \sum_{\substack{K,Y_1,Y_2 \\ Y_1 > 2Y_2}} K^{(2-\theta)/(1-\theta)-1} \sum_R B_1(R) \ll$$

$$\frac{1}{N^2} \sum_{\substack{K,Y_1,Y_2 \\ Y_1 > 2Y_2}} K^{(2-\theta)/(1-\theta)-1} \sum_R K^{-3/(2(1-\theta))} (Y_1Y_2)^{1-\theta/2} \frac{K^2 (Y_1Y_2)^{\theta-1} Y_1^{\theta}}{Y_1^{\theta/2} K^{-\theta/(2(1-\theta))} Y_1^{\kappa_1\theta/2}} \ll$$

$$\frac{\log N}{N^2} \sum_{\substack{K,Y_1,Y_2 \\ Y_1 > 2Y_2}} K^{3/2} Y_1^{\theta-\kappa_1\theta/2} Y_2^{\theta/2} \ll N^{-1/2+3\theta/2-\kappa_1\theta/2+3\varepsilon/2} \log N.$$

Similarly,

$$\frac{1}{N^2}\sum_{\substack{K,Y_1,Y_2\\Y_1>2Y_2}} K^{(2-\theta)/(1-\theta)-1}\sum_R B_2(R) \ll$$

$$\frac{1}{N^2}\sum_{\substack{K,Y_1,Y_2\\Y_1>2Y_2}} K^{(2-\theta)/(1-\theta)-1}\sum_R K^{-3/(2(1-\theta))}(Y_1Y_2)^{1-\theta/2}\frac{K^2(Y_1Y_2)^{\theta-1}Y_1^{\theta}}{Y_1^{\theta/2}K^{-\theta/(2(1-\theta))}Y_1^{\kappa_2\theta/4}} \ll$$

$$\frac{\log N}{N^2}\sum_{\substack{K,Y_1,Y_2\\Y_1>2Y_2}} K^{3/2}Y_1^{\theta/2}Y_2^{\theta-\kappa_2\theta/4} \ll N^{-1/2+3\theta/2-\kappa_2\theta/4+3\varepsilon/2}\log N.$$

Thus,

$$S_{B_1}(N) \ll N^{-1/2+3\theta/2-\kappa_1\theta/2+3\varepsilon/2}\log N, \tag{4.11}$$

$$S_{B_2}(N) \ll N^{-1/2+3\theta/2-\kappa_2\theta/4+3\varepsilon/2}\log N. \tag{4.12}$$

In the remainder of the paper, we will estimate $S_{B_3}(N)$.

4.2. **Double large sieve.** We handle the fivefold exponential sum over η, ℓ, h_1, h_2 by combining the techniques of Bombieri-Iwaniec, Fourvy-Iwaniec-Cao-Zhai, and Robert-Sargos. Specifically, we use the double large sieve (as described in [16, Lemma 8]) to reduce the sum to two spacing inequalities.

Lemma 4.2. *Define*

$$S = \sum_{k\leqslant K}\sum_{\ell\leqslant L} a(k)b(\ell)e\big(Xu(k)v(\ell)\big),$$

$$\mathcal{B}_1 = \sum_{\substack{1\leqslant k_1,k_2\leqslant K\\|u(k_1)-u(k_2)|\leqslant X^{-1}}} 1, \qquad \mathcal{B}_2 = \sum_{\substack{1\leqslant \ell_1,\ell_2\leqslant L\\|v(\ell_1)-v(\ell_2)|\leqslant X^{-1}}} 1,$$

where $|a(k)|\leqslant 1$, $|b(\ell)|\leqslant 1$ *are complex valued functions,* $|u(k)|\leqslant 1$, $|v(\ell)|\leqslant 1$ *are real valued functions. Then*

$$S \ll X^{1/2}\mathcal{B}_1^{1/2}\mathcal{B}_2^{1/2}.$$

For convenience, we change the range of the summation over h_1 and h_2 in $S_{B_3}(N)$ in (4.10) from $|h_i| < H_i$ to dyadic intervals $h_i \asymp \tilde{H}_i$, where $\tilde{H}_i$ denotes the size of the dyadic interval. There exist $\tilde H_1 \leqslant H_1$ and $\tilde H_2 \leqslant H_2$ such that

$$S_{B_3}(N) \ll \frac{\log N}{N^2} \sum_{K,Y_1,Y_2} K^{(2-\theta)/(1-\theta)-1} \max_{K\leqslant \tilde K \leqslant 2K} \sum_R B_3(R) \cdot \Bigg(\Bigg| \sum_{\substack{\eta,h_1,h_2\\ \eta\in\mathcal{N}_R(K,Y_1,Y_2)\\ h_i \asymp \tilde H_i}} \gamma(h_2) \sum_{L\leqslant \ell < L_1-h_1-h_2} e\Big(c_2 \eta^{1-1/\theta} t(\ell,h_1,h_2)\Big)\Bigg|\Bigg)^{1/4}.$$

Next, we apply Lemma 4.2 to the inner sum over η, h_1, h_2, ℓ in (4.10) with $a \equiv 1$, $b = \gamma(h_2)$, $X = Z^{1-1/\theta} L^{1/\theta - 2} \tilde H_1 \tilde H_2$,

$$\mathcal{B}_1 = \sum_{\substack{\eta_1,\eta_2\in\mathcal{N}_R(K,Y_1,Y_2)\\ |\eta_1^\Theta - \eta_2^\Theta| \leqslant Z^\Theta X^{-1}}} 1, \qquad \Theta := 1 - \frac{1}{\theta},$$

$$\mathcal{B}_2 = \sum_{\substack{L\leqslant \ell,\tilde\ell < L_1-h_1-h_2\\ h_1\asymp \tilde H_1,\ h_2\asymp\tilde H_2\\ |t(\ell,h_1,h_2)-t(\tilde\ell,\tilde h_1,\tilde h_2)|\leqslant TX^{-1}}} 1, \qquad T = L^{1/\theta-2}\tilde H_1 \tilde H_2.$$

Then, using (4.9), we find

$$S_{B_3}(N) \ll \frac{\log N}{N^2} \sum_{K,Y_1,Y_2} K^{(2-\theta)/(1-\theta)-1} \max_{K\leqslant\tilde K\leqslant 2K} \sum_R B_3(R) X^{1/8} \mathcal{B}_1^{1/8} \mathcal{B}_2^{1/8} \ll \frac{1}{N^2} \sum_{K,Y_1,Y_2} K^{(2-\theta)/(1-\theta)-1} \max_{K\leqslant\tilde K\leqslant 2K} \sum_R K^{-3/(2(1-\theta))} (Y_1Y_2)^{1-\theta/2} \cdot \frac{\big(\#\mathcal{N}_R(K,Y_1,Y_2)L\big)^{3/4}}{Z^{1/2}(H_1H_2)^{1/4}} X^{1/8}\mathcal{B}_1^{1/8}\mathcal{B}_2^{1/8}. \tag{4.13}$$

We estimate $\mathcal{B}_1$ and $\mathcal{B}_2$ separately. First, we replace $\mathcal{B}_1$ by a larger quantity. We drop the restriction $r \in [R, 2R)$ and set

$$\eta_1 := m_1^{-\theta/(1-\theta)} - m_2^{-\theta/(1-\theta)}, \qquad \eta_2 := m_3^{-\theta/(1-\theta)} - m_4^{-\theta/(1-\theta)},$$

where $m_1, m_3 \asymp KY_1^{\theta-1}$ and $m_2, m_4 \asymp KY_2^{\theta-1}$. Applying the first-order Taylor approximation to

$$\Big(\big(m_3^{-\theta/(1-\theta)} - m_4^{-\theta/(1-\theta)}\big)^\Theta + O(Z^\Theta X^{-1})\Big)^{1/\Theta},$$

we get

$$m_1^{-\theta/(1-\theta)} - m_2^{-\theta/(1-\theta)} = m_3^{-\theta/(1-\theta)} - m_4^{-\theta/(1-\theta)} + O\big(ZX^{-1}\big).$$

Then

$$\mathcal{B}_1 \leqslant \sum_{\substack{\alpha\theta K(2Y_1)^{\theta-1} \leqslant m_1, m_3 < 2\alpha\theta K Y_1^{\theta-1} \\ \alpha\theta K(2Y_2)^{\theta-1} \leqslant m_2, m_4 < 2\alpha\theta K Y_2^{\theta-1} \\ |m_1^\beta - m_2^\beta - m_3^\beta + m_4^\beta| \leqslant cZX^{-1}}} 1 \leqslant \sum_{\substack{1 \leqslant m_1, \dots, m_4 < 4\alpha\theta K Y_2^{\theta-1} \\ |m_1^\beta - m_2^\beta - m_3^\beta + m_4^\beta| \leqslant cZX^{-1}}} 1,$$

where $\beta := -\theta/(1-\theta)$ and $c > 0$ is the appropriate constant. To estimate this quantity, we use a slight variation of Robert and Sargos's inequality (see [16, Theorem 2]):

Lemma 4.3. *Let $M \geqslant 2$ be an integer, $a_2 > a_1 \geqslant 0$, $\delta > 0$ and $\alpha \neq 0, 1$ be real numbers, $\mathcal{N}(M, \delta)$ be the number of quadruplets (m_1, m_2, m_3, m_4) of integers, $a_1 M \leqslant m_i < a_2 M$, $i = 1, \dots, 4$, satisfying the inequality*

$$|m_1^\alpha + m_2^\alpha - m_3^\alpha - m_4^\alpha| \leqslant \delta M^\alpha.$$

Then

$$\mathcal{N}(M, \delta) \ll_\varepsilon M^{2+\varepsilon} + \delta M^{4+\varepsilon}.$$

Thus,

$$\mathcal{B}_1 \ll_{\varepsilon_1} (KY_2^{\theta-1})^{2+\varepsilon_1} + \big((KY_2^{\theta-1})^{\theta/(1-\theta)} ZX^{-1}\big)(KY_2^{\theta-1})^{4+\varepsilon_1} \tag{4.14}$$

with some $\varepsilon_1 > 0$.

We estimate $\mathcal{B}_2$ using the result of Cao and Zhai (see [4, Theorems 1 and 2]):

Lemma 4.4. *Let M, H_1, H_2 be integers $\geqslant 10$ such that $H_1 \leqslant H_2 \leqslant M^{2/3-\varepsilon}$, $\Delta > 0$, $\alpha \neq 0, 1, 2, 3$ be real, and let $T := M^{\alpha-2} H_1 H_2$. Furthermore, let $\mathcal{F}(M, H_1, H_2, \Delta)$ be the number of sextuplets $(m, \tilde{m}, h_1, \tilde{h}_1, h_2, \tilde{h}_2)$ with $M \leqslant m, \tilde{m} < 2M$, $h_1, \tilde{h}_1 \asymp H_1$, and $h_2, \tilde{h}_2 \asymp H_2$, satisfying*

$$\big|t(m, h_1, h_2) - t(\tilde{m}, \tilde{h}_1, \tilde{h}_2)\big| \leqslant \Delta T.$$

Then

$$\mathcal{F}(M, H_1, H_2, \Delta) M^{-2\varepsilon} \ll M H_1 H_2 + \Delta (M H_1 H_2)^2 + M^{-2} H_1^2 H_2^6 + H_1^2 H_2^{8/3}.$$

Alternatively, if H_1 is much smaller than H_2, namely one has $H_1 M^\varepsilon < H_2 \leqslant M^{1-\varepsilon}$, and $H_1 H_2 \leqslant M^{3/2-\varepsilon}$, then one also has the inequality

$$\mathcal{F}(M, H_1, H_2, \Delta) M^{-2\varepsilon} \ll M H_1 H_2 + \Delta (M H_1 H_2)^2 + (M H_1^7 H_2^9)^{1/4} + M^{-2} H_1^4 H_2^4 + (\Delta M^4 H_1^{15} H_2^{17})^{1/8} + (\Delta M^4 H_1^3 H_2)^{1/2} + (H_1^{13} H_2^{15})^{1/6} + (\Delta M^2 H_1^8 H_2^{10})^{1/4} + (M^{-1} H_1^5 H_2^6)^{1/2}.$$

We first assume that $\tilde{H}_1$ and $\tilde{H}_2$ are separated at least by the factor of L^{ε_1}. Precisely, we have either $\tilde{H}_1 L^{\varepsilon_1} \ll \tilde{H}_2$ or $\tilde{H}_2 L^{\varepsilon_1} \ll \tilde{H}_1$, where ε_1 will be chosen later. The case $\tilde{H}_1 \approx \tilde{H}_2$ will be considered at the end of the paper. Applying the second part of Lemma 4.4 to $\mathcal{B}_2$ we obtain

$$\mathcal{B}_2 \ll_{\varepsilon_1} L^{2\varepsilon_1}\Big(L\tilde{H}_1\tilde{H}_2+X^{-1}(L\tilde{H}_1\tilde{H}_2)^2+(L\tilde{H}_1^7\tilde{H}_2^9)^{1/4}+L^{-2}\tilde{H}_1^4\tilde{H}_2^4+(X^{-1}L^4\tilde{H}_1^{15}\tilde{H}_2^{17})^{1/8}+ (X^{-1}L^4\tilde{H}_1^3\tilde{H}_2)^{1/2}+(\tilde{H}_1^{13}\tilde{H}_2^{15})^{1/6}+(X^{-1}L^2\tilde{H}_1^8\tilde{H}_2^{10})^{1/4}+(L^{-1}\tilde{H}_1^5\tilde{H}_2^6)^{1/2}\Big).$$

Here we make the same choice of ε_1.

Next, note that the expression $X^{1/8}\mathcal{B}_1^{1/8}\mathcal{B}_2^{1/8}$ is maximized when $\tilde{H}_1$ and $\tilde{H}_2$ take their maximum values, that is, when $\tilde{H}_1 = H_1$ and $\tilde{H}_2 = H_2$. Therefore,

$$S_{B_3}(N) \ll \frac{N^{\varepsilon+\varepsilon_1}}{N^2}\sum_{K,Y_1,Y_2} K^{(2-\theta)/(1-\theta)-1}\max_{K\leqslant\tilde{K}\leqslant 2K}\sum_R K^{-3/(2(1-\theta))}(Y_1Y_2)^{1-\theta/2}\cdot \frac{\big(\#\mathcal{N}_R(K,Y_1,Y_2)L\big)^{1/4}}{Z^{1/2}(H_1H_2)^{1/4}}\big(Z^{1-1/\theta}L^{1/\theta-2}H_1H_2\big)^{1/8}\cdot \Big((KY_2^{\theta-1})^2+\big((KY_2^{\theta-1})^{\theta/(1-\theta)}ZX^{-1}\big)(KY_2^{\theta-1})^4\Big)^{1/8}\cdot \Big(LH_1H_2+X^{-1}(LH_1H_2)^2+(LH_1^7H_2^9)^{1/4}+L^{-2}H_1^4H_2^4+(X^{-1}L^4H_1^{15}H_2^{17})^{1/8}+ (X^{-1}L^4H_1^3H_2)^{1/2}+(H_1^{13}H_2^{15})^{1/6}+(X^{-1}L^2H_1^8H_2^{10})^{1/4}+(L^{-1}H_1^5H_2^6)^{1/2}\Big)^{1/8}.$$

Next, we consider two cases once again.

Case 1. $Y_2/2 \leqslant Y_1 \leqslant 2Y_2$.

From (4.1), (4.2), and (4.3) we have

$$\#\mathcal{N}_R(K,Y_1,Y_2) \ll RKY_2^{\theta-1}, \qquad Z \asymp RY_2K^{-1/(1-\theta)}, \qquad L \asymp ZK^{\theta/(1-\theta)} \asymp RY_2K^{-1}, \quad X = Z^{1-1/\theta}L^{1/\theta-2}H_1H_2 = K^2Y_2^{-1}R^{-1}H_1H_2.$$

Then

$$
\begin{aligned}
(4.15)\quad & S_{B_3}^{(1)}(N) := S_{B_3}(N;\ Y_2/2 \leqslant Y_1 \leqslant 2Y_2) \ll \\
& \frac{N^{\varepsilon+\varepsilon_1}}{N^2} \sum_{\substack{K, Y_1, Y_2 \\ Y_2/2 \leqslant Y_1 \leqslant 2Y_2}} Y_1^{1-\theta/2} Y_2^{1/2+\theta/4} \sum_R \frac{R}{(H_1 H_2)^{1/4}} \cdot \\
& \Big(K^2 Y_2^{-1} R^{-1} H_1 H_2\Big)^{1/8} \Big((KY_2^{\theta-1})^2 + \frac{(KY_2^{\theta-1})^4}{K^3 Y_2^{\theta-2} R^{-2} H_1 H_2}\Big)^{1/8} \cdot \\
& \Big(RY_2K^{-1}H_1H_2 + \frac{(RY_2K^{-1}H_1H_2)^2}{K^2Y_2^{-1}R^{-1}H_1H_2} + (RY_2K^{-1}H_1^7H_2^9)^{1/4} + (RY_2K^{-1})^{-2}H_1^4H_2^4 + \\
& \Big(\frac{(RY_2K^{-1})^4H_1^{15}H_2^{17}}{K^2Y_2^{-1}R^{-1}H_1H_2}\Big)^{1/8} + \Big(\frac{(RY_2K^{-1})^4H_1^3H_2}{K^2Y_2^{-1}R^{-1}H_1H_2}\Big)^{1/2} + (H_1^{13}H_2^{15})^{1/6} + \\
& \Big(\frac{(RY_2K^{-1})^2H_1^8H_2^{10}}{K^2Y_2^{-1}R^{-1}H_1H_2}\Big)^{1/4} + ((RY_2K^{-1})^{-1}H_1^5H_2^6)^{1/2}\Big)^{1/8}.
\end{aligned}
$$

The most contribution to the sum over R comes from R maximal, namely $R \asymp KY_2^{\theta-1}$. It corresponds to $Z \asymp Y_2^{\theta} K^{-\theta/(1-\theta)}$, $X \asymp KY_2^{-\theta}H_1H_2$, $L \asymp Y_2^{\theta}$, $H_1 \asymp Y_2^{\kappa_1\theta}$, $H_2 \asymp Y_2^{\kappa_2\theta}$. Then, we get

$$
\begin{aligned}
S_{B_3}^{(1)}(N) \ll & \frac{N^{\varepsilon+\varepsilon_1}}{N^2} \sum_{\substack{K, Y_1, Y_2 \\ Y_2/2 \leqslant Y_1 \leqslant 2Y_2}} \frac{KY_2^{1/2+3\theta/4}}{(H_1H_2)^{1/4}} \cdot \\
& \Big(KY_2^{-\theta}H_1H_2\Big)^{1/8} \Big((KY_2^{\theta-1})^2 + \frac{(KY_2^{\theta-1})^4}{KY_2^{-\theta}H_1H_2}\Big)^{1/8} \cdot \\
& \Big(Y_2^{\theta}H_1H_2 + \frac{(Y_2^{\theta}H_1H_2)^2}{KY_2^{-\theta}H_1H_2} + (Y_2^{\theta}H_1^7H_2^9)^{1/4} + Y_2^{-2\theta}H_1^4H_2^4 + \\
& \Big(\frac{Y_2^{4\theta}H_1^{15}H_2^{17}}{KY_2^{-\theta}H_1H_2}\Big)^{1/8} + \Big(\frac{Y_2^{4\theta}H_1^3H_2}{KY_2^{-\theta}H_1H_2}\Big)^{1/2} + (H_1^{13}H_2^{15})^{1/6} + \\
& \Big(\frac{Y_2^{2\theta}H_1^8H_2^{10}}{KY_2^{-\theta}H_1H_2}\Big)^{1/4} + (Y_2^{-\theta}H_1^5H_2^6)^{1/2}\Big)^{1/8}.
\end{aligned}
$$

The maximum contribution to the sum over K, Y_1, Y_2 comes from Y_2 (which is $\asymp Y_1$) and K maximal, namely $Y_1, Y_2 \asymp N$, $K \asymp N^{1+\varepsilon}$. Thus,

$$S_{B_3}^{(1)}(N) \ll N^{3\varepsilon+\varepsilon_1} \frac{N^{-1/2+3\theta/4}}{(H_1H_2)^{1/4}} \big(N^{1-\theta}H_1H_2\big)^{1/8} \Big(N^{2\theta} + \frac{N^{4\theta}}{N^{1-\theta}H_1H_2}\Big)^{1/8} \cdot \Big(N^\theta H_1 H_2 + \frac{\big(N^\theta H_1 H_2\big)^2}{N^{1-\theta}H_1H_2} + (N^\theta H_1^7 H_2^9)^{1/4} + N^{-2\theta} H_1^4 H_2^4 + (N^{5\theta-1} H_1^{14} H_2^{16})^{1/8} + (N^{5\theta-1} H_1^2)^{1/2} + (H_1^{13} H_2^{15})^{1/6} + (N^{3\theta-1} H_1^7 H_2^9)^{1/4} + (N^{-\theta} H_1^5 H_2^6)^{1/2}\Big)^{1/8}. \tag{4.16}$$

Case 2. $Y_1 > 2Y_2$.

From (4.1), (4.2), and (4.3) we have

$$\#\mathcal{N}_R(K, Y_1, Y_2) \ll KY_1^{\theta-1}KY_2^{\theta-1}, \qquad Z = Y_1^\theta K^{-\theta/(1-\theta)}, \qquad L \asymp ZK^{\theta/(1-\theta)} = Y_1^\theta, \\ X \asymp KY_1^{-\theta} H_1 H_2.$$

Simplifying the right hand side in (4.13), we obtain

$$S_{B_3}^{(2)}(N) := S_{B_3}\big(N;\ Y_1 > 2Y_2\big) \ll \frac{\log N}{N^2} \sum_{\substack{K, Y_1, Y_2 \\ Y_1 > 2Y_2}} \frac{KY_1^{1/4+\theta/2} Y_2^{1/4+\theta/4}}{(H_1H_2)^{1/4}} \big(KY_1^{-\theta} H_1 H_2\big)^{1/8} \mathcal{B}_1^{1/8} \mathcal{B}_2^{1/8},$$

where $(\log N)$-factor comes from the sum over R.

Here we need to consider two more cases. First, we assume that the first term in the bound (4.14) for $\mathcal{B}_1$ is dominating:

Case 2.1. $(KY_2^{\theta-1})^{2+\varepsilon_1} \gg \big((KY_2^{\theta-1})^{\theta/(1-\theta)} ZX^{-1}\big)(KY_2^{\theta-1})^{4+\varepsilon_1}$.

This condition can be rewritten as $Y_2^{2-\theta} \gg KY_1^{2\theta}(H_1H_2)^{-1}$. We have

$$(4.17)\quad S_{B_3}^{(2.1)}(N) := S_{B_3}\big(N;\ Y_1 > 2Y_2;\ Y_2^{2-\theta} \gg KY_1^{2\theta}(H_1H_2)^{-1}\big) \ll \frac{N^{\varepsilon+\varepsilon_1}}{N^2} \sum_{\substack{K,Y_1,Y_2 \\ Y_1>2Y_2 \\ Y_2^{2-\theta}\gg KY_1^{2\theta}(H_1H_2)^{-1}}} \frac{KY_1^{1/4+\theta/2}Y_2^{1/4+\theta/4}}{(H_1H_2)^{1/4}}\big(KY_1^{-\theta}H_1H_2\big)^{1/8}\cdot$$
$$\big(KY_2^{\theta-1}\big)^{1/4}\Big(Y_1^{\theta}H_1H_2 + \frac{(Y_1^{\theta}H_1H_2)^2}{KY_1^{-\theta}H_1H_2} + (Y_1^{\theta}H_1^7H_2^9)^{1/4} + Y_1^{-2\theta}H_1^4H_2^4 + \Big(\frac{Y_1^{4\theta}H_1^{15}H_2^{17}}{KY_1^{-\theta}H_1H_2}\Big)^{1/8} + \Big(\frac{Y_1^{4\theta}H_1^{3}H_2}{KY_1^{-\theta}H_1H_2}\Big)^{1/2} + (H_1^{13}H_2^{15})^{1/6} + \Big(\frac{Y_1^{2\theta}H_1^{8}H_2^{10}}{KY_1^{-\theta}H_1H_2}\Big)^{1/4} + (Y_1^{-\theta}H_1^5H_2^6)^{1/2}\Big)^{1/8}.$$

Recall that $H_1H_2 = Y_1^{(\kappa_1+\kappa_2)\theta}$ with $\kappa_1+\kappa_2 \leqslant \frac{3}{2}-\varepsilon_1$. The most contribution in the last upper bound for $S_{B_3}^{(2.1)}(N)$ comes from K, Y_1, Y_2 maximal. We get

$$(4.18)\quad S_{B_3}^{(2.1)}(N) \ll N^{3\varepsilon+\varepsilon_1}\frac{N^{-1/2+3\theta/4}}{(H_1H_2)^{1/4}}\big(N^{1-\theta}H_1H_2\big)^{1/8}N^{\theta/4}\cdot$$
$$\Big(N^{\theta}H_1H_2 + \frac{\big(N^{\theta}H_1H_2\big)^2}{N^{1-\theta}H_1H_2} + (N^{\theta}H_1^7H_2^9)^{1/4} + N^{-2\theta}H_1^4H_2^4 + (N^{5\theta-1}H_1^{14}H_2^{16})^{1/8} + (N^{5\theta-1}H_1^{2})^{1/2} + (H_1^{13}H_2^{15})^{1/6} + (N^{3\theta-1}H_1^{7}H_2^{9})^{1/4} + (N^{-\theta}H_1^5H_2^6)^{1/2}\Big)^{1/8}.$$

Case 2.2. $(KY_2^{\theta-1})^{2+\varepsilon_1} \ll \big((KY_2^{\theta-1})^{\theta/(1-\theta)}ZX^{-1}\big)(KY_2^{\theta-1})^{4+\varepsilon_1}$.

Similarly, this condition can be rewritten as $Y_2^{2-\theta} \ll KY_1^{2\theta}(H_1H_2)^{-1}$. Since the second term in the bound for $\mathcal{B}_1$ (4.14) dominates in this case, we obtain:

$$
\begin{aligned}
(4.19)\quad S_{B_3}^{(2.2)}(N) &:= S_{B_3}\big(N;\ Y_1 > 2Y_2;\ Y_2^{2-\theta} \ll KY_1^{2\theta}(H_1H_2)^{-1}\big) \ll \\
&\frac{N^{\varepsilon+\varepsilon_1}}{N^2} \sum_{\substack{K,Y_1,Y_2 \\ Y_1>2Y_2 \\ Y_2^{2-\theta} \ll KY_1^{2\theta}(H_1H_2)^{-1}}} \frac{KY_1^{1/4+\theta/2}Y_2^{1/4+\theta/4}}{(H_1H_2)^{1/4}} \big(KY_1^{-\theta}H_1H_2\big)^{1/8} \cdot \\
&\Big(\frac{(KY_2^{\theta-1})^4}{KY_1^{-2\theta}Y_2^{\theta}H_1H_2}\Big)^{1/8} \Big(Y_1^\theta H_1H_2 + \frac{(Y_1^\theta H_1H_2)^2}{KY_1^{-\theta}H_1H_2} + (Y_1^\theta H_1^7H_2^9)^{1/4} + Y_1^{-2\theta}H_1^4H_2^4 + \\
&\Big(\frac{Y_1^{4\theta}H_1^{15}H_2^{17}}{KY_1^{-\theta}H_1H_2}\Big)^{1/8} + \Big(\frac{Y_1^{4\theta}H_1^3H_2}{KY_1^{-\theta}H_1H_2}\Big)^{1/2} + (H_1^{13}H_2^{15})^{1/6} + \\
&\Big(\frac{Y_1^{2\theta}H_1^8H_2^{10}}{KY_1^{-\theta}H_1H_2}\Big)^{1/4} + (Y_1^{-\theta}H_1^5H_2^6)^{1/2}\Big)^{1/8}.
\end{aligned}
$$

The maximum contribution to this expression comes from K, Y_1 maximal and $Y_2 \asymp 1$. We get

$$
\begin{aligned}
(4.20)\quad S_{B_3}^{(2.2)}(N) &\ll N^{3\varepsilon+\varepsilon_1}\frac{N^{-1/4+5\theta/8}}{(H_1H_2)^{1/4}}\Big(N^\theta H_1H_2 + N^{3\theta-1}H_1H_2 + (N^\theta H_1^7H_2^9)^{1/4} + \\
&N^{-2\theta}H_1^4H_2^4 + (N^{5\theta-1}H_1^{14}H_2^{16})^{1/8} + (N^{5\theta-1}H_1^2)^{1/2} + (H_1^{13}H_2^{15})^{1/6} + \\
&(N^{3\theta-1}H_1^7H_2^9)^{1/4} + (N^{-\theta}H_1^5H_2^6)^{1/2}\Big)^{1/8}.
\end{aligned}
$$

Combining (4.16) and (4.18), we find

$$
\begin{aligned}
(4.21)\quad S_{B_3}^{(1)}(N) + S_{B_3}^{(2.1)}(N) &\ll \\
&N^{3\varepsilon+\varepsilon_1}\frac{N^{-1/2+3\theta/4}}{(H_1H_2)^{1/4}}\big(N^{1-\theta}H_1H_2\big)^{1/8}\Big(N^{2\theta} + \frac{N^{4\theta}}{N^{1-\theta}H_1H_2}\Big)^{1/8} \cdot \\
&\Big(N^\theta H_1H_2 + N^{3\theta-1}H_1H_2 + (N^\theta H_1^7H_2^9)^{1/4} + N^{-2\theta}H_1^4H_2^4 + (N^{5\theta-1}H_1^{14}H_2^{16})^{1/8} + \\
&(N^{5\theta-1}H_1^2)^{1/2} + (H_1^{13}H_2^{15})^{1/6} + (N^{3\theta-1}H_1^7H_2^9)^{1/4} + (N^{-\theta}H_1^5H_2^6)^{1/2}\Big)^{1/8}.
\end{aligned}
$$

We can drop the term $N^{4\theta}/(N^{1-\theta}H_1H_2)$ in the second factor by assuming that $H_1H_2 \gg N^{3\theta-1}$. Choose $\varepsilon_1 < \varepsilon$. Then, from (4.10) and (4.21),

$$
S(N) \ll S_{B_1}(N) + S_{B_2}(N) + N^{4\varepsilon}\sum_{i=1}^{9} S_i + S_{B_3}^{(2.2)}(N),
$$

where S_i correspond to nine terms in the bound in (4.21). We have

$$S_1 = N^{-1/2+3\theta/4+1/8+\theta/4}, \qquad S_2 = N^{-1/2+5\theta/4}.$$

Then $S_1, S_2 = o(N^{-4\varepsilon})$ if $\theta < \frac{3}{8} - 4\varepsilon = 0.375 - 4\varepsilon$. Similarly,

$$S_3 = N^{-3/8+29\theta/32} H_1^{3/32} H_2^{5/32}, \qquad S_4 = N^{-3/8+5\theta/8} H_1^{3/8} H_2^{3/8},$$
$$S_5 = N^{-25/64+61\theta/64} H_1^{6/64} H_2^{8/64}, \qquad S_6 = N^{-7/16+19\theta/16} H_2^{-1/8},$$
$$S_7 = N^{-3/8+7\theta/8} H_1^{13/48} H_2^{15/48}, \qquad S_8 = N^{-13/32+31\theta/32} H_1^{3/32} H_2^{5/32},$$
$$S_9 = N^{-3/8+13\theta/16} H_1^{3/16} H_2^{4/16}.$$

By combining the expressions with the bounds (4.11) and (4.12) for $S_{B_1}(N)$ and $S_{B_2}(N)$, we obtain that $S(N) = o(1)$ when

$$\theta < \min\Big(\frac{1}{3-\kappa_1}, \frac{2}{6-\kappa_2}, \frac{12}{29+3\kappa_1+5\kappa_2}, \frac{3}{5+3\kappa_1+3\kappa_2}, \frac{25}{61+6\kappa_1+8\kappa_2}, \frac{7}{19-2\kappa_2}, \frac{18}{42+7\kappa_1+9\kappa_2}, \frac{13}{31+3\kappa_1+5\kappa_2}, \frac{6}{13+3\kappa_1+4\kappa_2}\Big) - 5\varepsilon,$$

which is maximized at $(\kappa_1, \kappa_2) = (\frac{12}{43}, \frac{24}{43})$. Therefore, we have $\theta < \frac{43}{117} - 5\varepsilon$, with the largest term being S_7. From (4.20) we find

$$S_{B_3}^{(2.2)}(N) \ll N^{4\varepsilon}\Big(N^{-1/4+3\theta/4-(\kappa_1+\kappa_2)\theta/8} + N^{-3/8+\theta-(\kappa_1+\kappa_2)\theta/8} + N^{-1/4+21\theta/32-\kappa_1\theta/32+\kappa_2\theta/32} + N^{-1/4+3\theta/8+(\kappa_1+\kappa_2)\theta/4} + N^{-17/64+45\theta/64-16\kappa_1\theta/64} + N^{-5/16+15\theta/16-2\kappa_1\theta/16-4\kappa_2\theta/16} + N^{-1/4+5\theta/8+\kappa_1\theta/48+3\kappa_2\theta/48} + N^{-9/32+23\theta/32-\kappa_1\theta/32+\kappa_2\theta/32} + N^{-1/4+9\theta/16+\kappa_1\theta/16+2\kappa_2\theta/16}\Big),$$

which is $o(1)$ when

$$\theta < \min\Big(\frac{2}{6-\kappa_1-\kappa_2}, \frac{3}{8-\kappa_1-\kappa_2}, \frac{8}{21-\kappa_1+\kappa_2}, \frac{2}{3+2\kappa_1+2\kappa_2}, \frac{17}{45-16\kappa_1}, \frac{5}{15-2\kappa_1-4\kappa_2}, \frac{12}{30+\kappa_1+3\kappa_2}, \frac{9}{23-\kappa_1+\kappa_2}, \frac{4}{9+\kappa_1+2\kappa_2}\Big) - 5\varepsilon.$$

It covers the interval $\theta \in [0, \frac{43}{117})$ when $(\kappa_1, \kappa_2) = (\frac{12}{43}, \frac{24}{43})$. Note that with this choice of (κ_1, κ_2) the condition $H_1 H_2 \gg N^{3\theta-1}$ is also satisfied.

4.3. **Case $\tilde{H}_1 \approx \tilde{H}_2$.** We assume that $L^{-\varepsilon_1}\tilde{H}_1 \ll \tilde{H}_2 \ll L^{\varepsilon_1}\tilde{H}_1$. We estimate $S_{B_3}(N)$ using the first part of Lemma 4.4:

$$\mathcal{B}_2 \ll_{\varepsilon_1} L^{2\varepsilon_1}\Big(L\tilde{H}_1\tilde{H}_2 + X^{-1}(L\tilde{H}_1\tilde{H}_2)^2 + L^{-2}\tilde{H}_1^2\tilde{H}_2^6 + \tilde{H}_1^2\tilde{H}_2^{8/3}\Big).$$

We can similarly conclude that the expression $X^{1/8}\mathcal{B}_1^{1/8}\mathcal{B}_2^{1/8}$ is maximized when $\tilde{H}_1$ and $\tilde{H}_2$ are both maximal, which occurs when $\tilde{H}_1 = H_1$ and $\tilde{H}_2 = L^{\varepsilon_1} H_1$. Then

$$S_{B_3}(N) \ll \frac{N^{\varepsilon+\varepsilon_1}}{N^2} \sum_{K,Y_1,Y_2} K^{(2-\theta)/(1-\theta)-1} \max_{K \leqslant \tilde{K} \leqslant 2K} \sum_R K^{-3/(2(1-\theta))}(Y_1Y_2)^{1-\theta/2} \cdot \frac{\big(\#\mathcal{N}_R(K,Y_1,Y_2)L\big)^{1/4}}{Z^{1/2}(H_1H_2)^{1/4}} \big(Z^{1-1/\theta}L^{1/\theta-2}H_1^2\big)^{1/8} \cdot \Big((KY_2^{\theta-1})^2 + \big((KY_2^{\theta-1})^{\theta/(1-\theta)} Z^{1/\theta} L^{2-1\theta} H_1^{-2}\big)(KY_2^{\theta-1})^4\Big)^{1/8} \cdot \Big(LH_1^2 + \frac{(LH_1^2)^2}{Z^{1-1/\theta}L^{1/\theta-2}H_1^2} + L^{-2}H_1^8 + H_1^{14/3}\Big)^{1/8}.$$

Case 1. $Y_2/2 \leqslant Y_1 \leqslant 2Y_2$.

Similarly to (4.15), we have

$$S_{B_3}^{(1)}(N) \ll \frac{N^{\varepsilon+\varepsilon_1}}{N^2} \sum_{\substack{K,Y_1,Y_2 \\ Y_2/2 \leqslant Y_1 \leqslant 2Y_2}} Y_1^{1-\theta/2} Y_2^{1/2+\theta/4} \sum_R \frac{R}{(H_1H_2)^{1/4}} \cdot \big(K^2Y_2^{-1}R^{-1}H_1^2\big)^{1/8}\Big((KY_2^{\theta-1})^2 + \frac{(KY_2^{\theta-1})^4}{K^3Y_2^{\theta-2}R^{-2}H_1^2}\Big)^{1/8} \cdot \Big(RY_2K^{-1}H_1^2 + \frac{\big(RY_2K^{-1}H_1^2\big)^2}{K^2Y_2^{-1}R^{-1}H_1^2} + \frac{H_1^8}{\big(RY_2K^{-1}\big)^2} + H_1^{14/3}\Big)^{1/8}.$$

Again, the most contribution to the sum over R comes from R maximal, so we get

$$S_{B_3}^{(1)}(N) \ll \frac{N^{\varepsilon+\varepsilon_1}}{N^2} \sum_{\substack{K,Y_1,Y_2 \\ Y_2/2 \leqslant Y_1 \leqslant 2Y_2}} \frac{KY_2^{1/2+3\theta/4}}{(H_1H_2)^{1/4}} \cdot \big(KY_2^{-\theta}H_1^2\big)^{1/8}\Big((KY_2^{\theta-1})^2 + \frac{(KY_2^{\theta-1})^4}{KY_2^{-\theta}H_1^2}\Big)^{1/8} \cdot \Big(Y_2^\theta H_1^2 + \frac{(Y_2^\theta H_1^2)^2}{KY_2^{-\theta}H_1^2} + \frac{H_1^8}{Y_2^{2\theta}} + H_1^{14/3}\Big)^{1/8}.$$

Next, we take maximum values of K, Y_1, Y_2, yielding

$$S_{B_3}^{(1)}(N) \ll N^{3\varepsilon+\varepsilon_1} \frac{N^{-1/2+3\theta/4}}{(H_1H_2)^{1/4}} \big(N^{1-\theta}H_1^2\big)^{1/8} \Big(N^{2\theta} + \frac{N^{4\theta}}{N^{1-\theta}H_1^2}\Big)^{1/8} \cdot \Big(N^\theta H_1^2 + \frac{\big(N^\theta H_1^2\big)^2}{N^{1-\theta}H_1^2} + \frac{H_1^8}{N^{2\theta}} + H_1^{14/3}\Big)^{1/8}. \tag{4.22}$$

Case 2. $Y_1 > 2Y_2$.

We have

$$S_{B_3}^{(2)}(N) \ll \frac{\log N}{N^2} \sum_{\substack{K, Y_1, Y_2 \\ Y_1 > 2Y_2}} \frac{KY_1^{1/4+\theta/2}Y_2^{1/4+\theta/4}}{(H_1H_2)^{1/4}} \big(KY_1^{-\theta}H_1^2\big)^{1/8} \mathcal{B}_1^{1/8}\mathcal{B}_2^{1/8}.$$

Case 2.1. $(KY_2^{\theta-1})^{2+\varepsilon_1} \gg \Big((KY_2^{\theta-1})^{\theta/(1-\theta)}ZX^{-1}\Big)(KY_2^{\theta-1})^{4+\varepsilon_1}$.

Note that now $X \asymp KY_1^{-\theta}H_1^2L^{O(\varepsilon_1)}$. Similarly to (4.17), we get

$$S_{B_3}^{(2.1)}(N) \ll \frac{N^{\varepsilon+\varepsilon_1}}{N^2} \sum_{\substack{K, Y_1, Y_2 \\ Y_1 > 2Y_2 \\ Y_2^{2-\theta} \gg KY_1^{2\theta}(H_1^2)^{-1}N^{O(\varepsilon_1)}}} \frac{KY_1^{1/4+\theta/2}Y_2^{1/4+\theta/4}}{(H_1H_2)^{1/4}} \cdot \big(KY_1^{-\theta}H_1^2\big)^{1/8}(KY_2^{\theta-1})^{1/4}\Big(Y_1^\theta H_1^2 + \frac{(Y_1^\theta H_1^2)^2}{KY_1^{-\theta}H_1^2} + \frac{H_1^8}{Y_1^{2\theta}} + H_1^{14/3}\Big)^{1/8}.$$

Taking the maximal K, Y_1, Y_2, we obtain

$$S_{B_3}^{(2.1)}(N) \ll N^{3\varepsilon+\varepsilon_1} \frac{N^{-1/2+3\theta/4}}{(H_1H_2)^{1/4}} \big(N^{1-\theta}H_1^2\big)^{1/8} N^{\theta/4} \cdot \Big(N^\theta H_1^2 + \frac{(N^\theta H_1^2)^2}{N^{1-\theta}H_1^2} + \frac{H_1^8}{N^{2\theta}} + H_1^{14/3}\Big)^{1/8}. \tag{4.23}$$

From (4.22) and (4.23), we find

$$S_{B_3}^{(1)}(N) + S_{B_3}^{(2.1)}(N) \ll N^{3\varepsilon+\varepsilon_1} \frac{N^{-1/2+3\theta/4}}{(H_1H_2)^{1/4}} \big(N^{1-\theta}H_1^2\big)^{1/8} \Big(N^{2\theta} + \frac{N^{4\theta}}{N^{1-\theta}H_1^2}\Big)^{1/8} \cdot \Big(N^\theta H_1^2 + \frac{\big(N^\theta H_1^2\big)^2}{N^{1-\theta}H_1^2} + \frac{H_1^8}{N^{2\theta}} + H_1^{14/3}\Big)^{1/8}.$$

With our choice of (κ_1,κ_2) we have $N^{2\theta} > N^{4\theta}/(N^{1-\theta}H_1^2)$. Thus,

$$S_{B_3}^{(1)}(N) + S_{B_3}^{(2.1)}(N) \ll N^{3\varepsilon+\varepsilon_1}\frac{N^{-3/8+7\theta/8}}{H_2^{1/4}}\cdot$$
$$\Big(N^\theta H_1^2 + \frac{\big(N^\theta H_1^2\big)^2}{N^{1-\theta}H_1^2} + \frac{H_1^8}{N^{2\theta}} + H_1^{14/3}\Big)^{1/8} = N^{4\varepsilon}\Big(N^{-3/8+\theta+\kappa_1\theta/4-\kappa_2\theta/4} +$$
$$N^{-1/2+5\theta/4+\kappa_1\theta/4-\kappa_2\theta/4} + N^{-3/8+5\theta/8+\kappa_1\theta-\kappa_2\theta/4} + N^{-3/8+7\theta/8+7\kappa_1\theta/12-\kappa_2\theta/4}\Big),$$

which is $o(1)$ when

$$\theta < \min\Big(\frac{3}{8+2\kappa_1-2\kappa_2},\ \frac{2}{5+\kappa_1-\kappa_2},\ \frac{3}{5+8\kappa_1-2\kappa_2},\ \frac{9}{21+14\kappa_1-6\kappa_2}\Big) - 5\varepsilon,$$

which covers the range $[0,\frac{43}{117})$ when $(\kappa_1,\kappa_2) = (\frac{12}{43},\frac{24}{43})$.

Case 2.2. $(KY_2^{\theta-1})^{2+\varepsilon_1} \ll \big((KY_2^{\theta-1})^{\theta/(1-\theta)}ZX^{-1}\big)(KY_2^{\theta-1})^{4+\varepsilon_1}$.

Similarly to (4.19), we find

$$S_{B_3}^{(2.2)}(N) \ll \frac{N^{\varepsilon+\varepsilon_1}}{N^2}\sum_{\substack{K,Y_1,Y_2\\ Y_1>2Y_2\\ Y_2^{2-\theta}\ll KY_1^{2\theta}(H_1^2)^{-2}N^{O(\varepsilon_1)}}}\frac{KY_1^{1/4+\theta/2}Y_2^{1/4+\theta/4}}{(H_1H_2)^{1/4}}\big(KY_1^{-\theta}H_1^2\big)^{1/8}\cdot$$
$$\Big(\frac{(KY_2^{\theta-1})^4}{KY_1^{-2\theta}Y_2^\theta H_1^2}\Big)^{1/8}\Big(Y_1^\theta H_1^2 + \frac{(Y_1^\theta H_1^2)^2}{KY_1^{-\theta}H_1^2} + \frac{H_1^8}{Y_1^{2\theta}} + H_1^{14/3}\Big)^{1/8}.$$

The maximum contribution comes from $K \asymp N^{1+\varepsilon}$, $Y_1 \asymp N$, $Y_2 \asymp 1$:

$$S_{B_3}^{(2.2)}(N) \ll N^{3\varepsilon+\varepsilon_1}\frac{N^{-1/4+5\theta/8}}{(H_1H_2)^{1/4}}\Big(N^\theta H_1^2 + N^{3\theta-1}H_1^2 + \frac{H_1^8}{N^{2\theta}} + H_1^{14/3}\Big)^{1/8} \ll$$
$$N^{4\varepsilon}\Big(N^{-1/4+3\theta/4-\kappa_2\theta/4} + N^{-3/8+\theta-\kappa_2\theta/4} +$$
$$N^{-1/4+3\theta/8+3\kappa_1\theta/4-\kappa_2\theta/4} + N^{-1/4+5\theta/8+\kappa_1\theta/3-\kappa_2\theta/4}\Big),$$

which is $o(1)$ when

$$\theta < \min\Big(\frac{1}{3-\kappa_2},\ \frac{3}{8-2\kappa_2},\ \frac{2}{3+6\kappa_1-2\kappa_2},\ \frac{6}{15+8\kappa_1-6\kappa_2}\Big) - 5\varepsilon,$$

which covers the range $[0,\frac{43}{117})$ when $(\kappa_1,\kappa_2) = (\frac{12}{43},\frac{24}{43})$.

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