BIAS IN CUBIC GAUSS SUMS: PATTERSON'S CONJECTURE

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ABSTRACT. Let W be a smooth test function with compact support in $(0, \infty)$. Conditional on the Generalized Riemann Hypothesis for Hecke L-functions over $\mathbb{Q}(\omega)$, we prove that

$$\sum_{p \equiv 1 \pmod 3} \frac{1}{2\sqrt{p}} \cdot \Big(\sum_{x \pmod p} e^{2\pi i x^3 / p} \Big) W\Big(\frac{p}{X}\Big) \sim \frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})} \int_0^\infty W(x) x^{-1/6} dx \cdot \frac{X^{5/6}}{\log X},$$

as $X \to \infty$ and p runs over primes. This explains a well-known numerical bias in the distribution of cubic Gauss sums first observed by Kummer in 1846 and confirms (conditionally on the Generalized Riemann Hypothesis) a conjecture of Patterson [Pat78b] from 1978.

There are two important byproducts of our proof. The first is an explicit level aspect Voronoi summation formula for cubic Gauss sums, extending computations of Patterson and Yoshimoto. Secondly, we show that Heath-Brown's cubic large sieve is sharp up to factors of $X^{o(1)}$ under the Generalized Riemann Hypothesis. This disproves the popular belief that the cubic large sieve can be improved.

An important ingredient in our proof is a dispersion estimate for cubic Gauss sums. It can be interpreted as a cubic large sieve with correction by a non-trivial asymptotic main term. This estimate relies on the Generalized Riemann Hypothesis, and is one of the fundamental reasons why our result is conditional.

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1. Introduction

1.1. Exponential sums over primes. Kummer [Kum75, Paper 16,17] studied the distribution of the cubic exponential sums

$$S_p = \sum_{n=1}^{p} e\left(\frac{n^3}{p}\right), \quad e(x) := e^{2\pi i x},$$

with $p \equiv 1 \pmod 3$ prime. The bound $|S_p| \leqslant 2\sqrt{p}$ is well-known, and we can consequently write

$$\frac{S_p}{2\sqrt{p}} = \cos(2\pi\theta_p), \quad \theta_p \in [0,1]. \tag{1.1}$$

This specifies the value of $\theta_p - \frac{1}{2}$ up to sign. This sign ambiguity can be resolved by noticing that (1.1) is the real part of an explicit root of unity defined in (1.7). To probe whether θ_p is equidistributed, Kummer computed the frequency with which $\cos(2\pi\theta_p)$ lay in the intervals $I_1 = [\frac{1}{2}, 1]$, $I_2 = [-\frac{1}{2}, \frac{1}{2}]$ and $I_3 = [-1, -\frac{1}{2}]$, for $p \leqslant 500$. Kummer observed that $\cos(2\pi\theta_p)$ tended to lay more frequently in I_1 than in I_2 or I_3 (the ratio he observed was $3 : 2 : 1$ respectively). If this bias persisted, then the angles θ_p *would not be* uniformly distributed. Subsequent calculations by von Neumann-Goldstine [vNG53], Lehmer [Leh56] and Cassels [Cas69] cast doubt on the persistence of this observation and suggested that $\cos(2\pi\theta_p)$ lay equally frequently in I_1, I_2 and I_3, and that θ_p was uniformly distributed. In light of the new numerical evidence, Patterson [Pat78b] enunciated a corrected conjecture. This conjecture explained the bias observed by Kummer, and was consistent with the numerical data.

Conjecture 1 (Patterson, 1978). As $X \to \infty$,

$$\sum_{\substack{p \leqslant X \\ p \equiv 1 \pmod 3}} \frac{S_p}{2\sqrt{p}} \sim \frac{2(2\pi)^{2/3}}{5\Gamma(\frac{2}{3})} \cdot \frac{X^{5/6}}{\log X},$$

where p runs through primes.

Patterson obtained this conjecture by developing Kubota's theory of metaplectic forms [Kub69,Kub71], and by appealing to a heuristic form of the circle method [Pat78b]. Unfortunately, even under the assumption of the Generalized Riemann Hypothesis, Patterson's heuristic fell short of a proof. This was due to insufficient bounds for the minor arcs. Subsequently, in 1979, Heath-Brown and Patterson [HBP79] established that θ_p is uniformly distributed in $[0, 1]$ as p varies among primes congruent to 1 modulo 3. This decisively disproved Kummer's guess. A nice summary up to this point can be found in a standard text of Davenport [Dav00, Chap. 3]. Some 20 years later, in 2000, Heath-Brown [HB00] sharpened his earlier result with Patterson and obtained unconditionally the nearly tight upper bound

$$\sum_{\substack{p \leqslant X \\ p \equiv 1 \pmod 3}} \frac{S_p}{2\sqrt{p}} \ll_\varepsilon X^{5/6+\varepsilon}, \tag{1.2}$$

for any given $\varepsilon > 0$. Heath-Brown [HB00, pg. 99] also stated a refined form of Patterson's conjecture that features an error term capturing square root cancellation[1].

In this paper we confirm Patterson's conjecture, conditionally on the assumption of the Generalized Riemann Hypothesis. This can be also viewed as a conditional sharpening of (1.2). We will explain in a later part of the introduction why the assumption of the Riemann Hypothesis (or similar unproven hypothesis) appears to be necessary at this point.

Theorem 1.1. *Assume the Generalized Riemann Hypothesis for Hecke L-functions over* $\mathbb{Q}(\omega)$. *Let* W *be a smooth function that is compactly supported in* $(0, \infty)$. *Then as* $X \to \infty$ *we have*

$$\sum_{\substack{p \equiv 1 \pmod{3}}} \frac{S_p}{2\sqrt{p}} \cdot W\left(\frac{p}{X}\right) \sim \frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})} \int_0^\infty W(x) x^{-1/6} dx \cdot \frac{X^{5/6}}{\log X},$$

where p *runs through primes.*

Notice that the constant that we get is consistent with Patterson's [Pat78b] prediction: if $W(x) \to \mathbf{1}_{[0,1]}(x)$ then,

$$\frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})} \int_0^\infty W(x) x^{-1/6} dx \to \frac{2(2\pi)^{2/3}}{5\Gamma(\frac{2}{3})}.$$

Theorem 1.1 shows that the angles θ_p cannot be equidistributed with square-root cancellation in the error term. We make this precise in Theorem 1.2.

Theorem 1.2. *Assume the Generalized Riemann Hypothesis for Hecke L-functions over* $\mathbb{Q}(\omega)$. *Let* f *be a smooth* 1*-periodic function and* W *be a smooth function that is compactly supported in* $(0, \infty)$. *Then we have*

$$\sum_{\substack{p \equiv 1 \pmod{3}}} f(\theta_p) W\left(\frac{p}{X}\right) = \int_0^1 f(x) dx \sum_{\substack{p \equiv 1 \pmod{3}}} W\left(\frac{p}{X}\right) \tag{1.3}$$

$$+ 2 \int_0^1 f(x) \cos(2\pi x) dx \cdot \frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})} \int_0^\infty W(x) x^{-1/6} dx \cdot \frac{X^{5/6}}{\log X} + o\left(\frac{X^{5/6}}{\log X}\right),$$

as $X \to \infty$.

It is unlikely that (1.3) can be established unconditionally given the current state of knowledge. For instance, with the choice $f(x) = e(3\ell x)$, $\ell \neq 0$, (1.3) implies a zero-free strip for certain L-functions associated to Größencharaktern.

Before proceeding to a high level sketch of the ideas in the paper, we remark that it is possible to slightly sharpen the rate of convergence in Theorems 1.1 and 1.2. One can save roughly one power of $\log X$.

1.2. Gauss sums over Eisenstein integers. Let $\omega = e^{2\pi i/3}$ and let $\mathbb{Z}[\omega]$ denote the ring of Eisenstein integers (in $\mathbb{Q}(\omega)$). It is well known that any non-zero element of $\mathbb{Z}[\omega]$ can be uniquely written as $\zeta \lambda^i c$ with $\zeta \in \{\pm 1, \pm \omega, \pm \omega^2\}$ a unit, $\lambda = \sqrt{-3} = 1 + 2\omega$ the unique ramified prime in $\mathbb{Z}[\omega]$, $i \in \mathbb{Z}_{\geq 0}$, and $c \in \mathbb{Z}[\omega]$ satisfying $c \equiv 1 \pmod{3}$.

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[1]The constant in Patterson's conjecture appearing in [HB00] is mistated due to a misprint

Furthermore, we have a cubic symbol $\left(\frac{a}{\varpi}\right)_3$ defined for $a \equiv 1 \pmod{3}$ and $\varpi \equiv 1 \pmod{3}$ prime. This is defined by

$$\left(\frac{a}{\varpi}\right)_3 \equiv a^{(N(\varpi)-1)/3} \pmod{\varpi},$$

and the condition that it takes values in $\{1, \omega, \omega^2\}$. It is clearly multiplicative in a and can be extended to a multiplicative function in ϖ by setting

$$\left(\frac{a}{b}\right)_3 = \prod_i \left(\frac{a}{\varpi_i}\right)$$

for any $b \equiv 1 \pmod{3}$ with $b = \prod_i \varpi_i$ and ϖ_i primes. The cubic symbol obeys cubic reciprocity: given $a, b \equiv 1 \pmod{3}$ we have

$$\left(\frac{a}{b}\right)_3 = \left(\frac{b}{a}\right)_3. \tag{1.4}$$

We also have supplementary laws for units and the ramified prime. Given $d \equiv 1 + \alpha_2 \lambda^2 + \alpha_3 \lambda^3 \pmod{9}$ with $\alpha_2, \alpha_3 \in \{-1, 0, 1\}$, then

$$\left(\frac{\omega}{d}\right)_3 = \omega^{\alpha_2} \quad \text{and} \quad \left(\frac{\lambda}{d}\right)_3 = \omega^{-\alpha_3}. \tag{1.5}$$

The cubic exponential sums S_p are intimately connected to cubic Gauss sums over Eisenstein integers. For any rational prime $p \equiv 1 \pmod{3}$, we can write $p = \varpi \overline{\varpi}$ with ϖ prime in $\mathbb{Z}[\omega]$. Then

$$\frac{S_p}{2\sqrt{p}} = \operatorname{Re} \tilde{g}(\varpi), \tag{1.6}$$

where the normalised Gauss sum is given by

$$\tilde{g}(c) = \frac{1}{|c|} \sum_{x \pmod{c}} \left(\frac{x}{c}\right)_3 \check{e}\left(\frac{x}{c}\right), \quad \check{e}(z) = e^{2\pi i(z+\overline{z})}, \tag{1.7}$$

for any $c \in \mathbb{Z}[\omega]$ with $c \equiv 1 \pmod{3}$. Here $|\cdot|$ denotes the Euclidean distance of c from the origin. We write $g(c)$ for the unnormalized Gauss sum, namely $g(c) := |c|\tilde{g}(c)$. We also note that

$$\tilde{g}(c)^3 = \mu(c)\frac{c}{|c|}. \tag{1.8}$$

Thus $\tilde{g}(c)$ is a cube root of $\mu(c)c/|c|$ (see [Has50, pp. 443–445]). However, given a prime $\varpi \equiv 1 \pmod{3}$, there is no known formula efficiently predicting which cube root $\tilde{g}(\varpi)$ corresponds to[2].

Formula (1.6) shows that Patterson's conjecture is equivalent to the statement

$$\sum_{\substack{N(\varpi) \leqslant X \\ \varpi \text{ prime} \\ \varpi \equiv 1 \pmod{3}}} \tilde{g}(\varpi) \sim \frac{2(2\pi)^{2/3}}{5\Gamma(\frac{2}{3})} \cdot \frac{X^{5/6}}{\log X}.$$

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[2]We note that the work of Matthews [Mat79] gives an explicit formula expressing $\tilde{g}(\varpi)$ as a product of the Weierstrass $\wp$-functions evaluated at $(N(\pi) - 1)/3$ values. Despite the beauty of Matthews' formula, it is not computationally efficient.

From the point of view of Weyl's equidistribution criterion it is also natural to ask about the behavior of $\tilde{g}(\varpi)^k$ with $k \in \mathbb{Z}$. Patterson enunciated in [Pat78b] a complementary conjecture. It states that for all $k \notin \{0, 1, -1\}$ we have,

$$\sum_{\substack{N(\varpi) \leqslant X \\ \varpi \text{ prime} \\ \varpi \equiv 1 \pmod 3}} \tilde{g}(\varpi)^k = o\Big(\frac{X^{5/6}}{\log X}\Big) \tag{1.9}$$

as $X \to \infty$. We conditionally establish a version of this conjecture with wide uniformity in k.

Theorem 1.3. *Assume the Generalized Riemann Hypothesis for Hecke L-functions over $\mathbb{Q}(\omega)$. Let W be a smooth function compactly supported in $(0, \infty)$. Then as $X \to \infty$ we have*

$$\sum_{\substack{\varpi \equiv 1 \pmod 3 \\ \varpi \text{ prime}}} \tilde{g}(\varpi)^k \cdot W\Big(\frac{N(\varpi)}{X}\Big) = o\Big(\frac{X^{5/6}}{\log X}\Big),$$

uniformly in $1 < |k| \leqslant X^{1/100}$

Theorem 1.2 is a nearly immediate consequence of Theorem 1.1 and Theorem 1.3. Notice that in the case $k \equiv 0 \pmod 3$ ($k \neq 0$), Theorem 1.3 unambiguously requires a zero-free strip for L-functions associated to Größencharaktern.

1.3. Cubic Gauss sums and automorphic forms. Developing Kubota's theory [Kub69, Kub71], Patterson [Pat77] established a functional equation for a Dirichlet series of the form

$$\sum_{c \equiv 1 \pmod 3} \frac{\tilde{g}(\mu, c)}{N(c)^s},$$

where

$$\tilde{g}(\mu, c) := \frac{1}{|c|} \sum_{x \pmod c} \Big(\frac{x}{c}\Big)_3 \check{e}\Big(\frac{\mu x}{c}\Big).$$

Subsequently, Yoshimoto [Yos87] followed Patterson's approach to obtain a functional equation for the Dirichlet series

$$\sum_{c \equiv 1 \pmod 3} \frac{\tilde{g}(c)\psi(c)}{N(c)^s},$$

where ψ is a primitive Dirichlet character such that ψ^3 is not principal. Yoshimoto specifically excludes the case when ψ^3 is principal to prevent the (Kubota) multiplier from interfering with ψ. We develop both of these computations further, obtaining a functional equation for the Dirichlet series

$$\sum_{c \equiv 1 \pmod 3} \frac{\tilde{g}(c)\phi(c)}{N(c)^s}$$

and ϕ a periodic function modulo r with $r \equiv 1 \pmod 3$. We specialise our computation to the case when ϕ is the conjugate of a cubic character to modulus r. The result for this specific choice of ϕ could have been obtained more directly by combining [Pat77, Theorem

6.1] and [HB00, Lemma 4.1]. However, we found it advantageous to develop a more general approach. First, we believe that the result will be useful in later works. Second, this more general approach improved our understanding of (and confidence in) the formula. Third, our functional equations explicate the root number. These formulas are too lengthy to be introduced here. We refer the reader to Section 5 where they are stated in detail.

As in earlier works [HB00, HBP79], this Voronoi formula is used to understand the so-called Type-I sums

$$\sum_{c,r\equiv 1 \pmod 3} \tilde{g}(cr)\alpha_r V\Big(\frac{N(r)}{R}\Big)V\Big(\frac{N(c)}{C}\Big), \quad CR = X, \tag{1.10}$$

with α_r arbitrary coefficients bounded in absolute value by 1. A sharp bound for (1.10) in the range $C > R^2$ was established in [HB00]. In the proof of Theorem 1.1 we need an asymptotic slightly past this range (with an error term $\ll X^{5/6-\varepsilon}$). In Section 8 we use the Generalized Riemann Hypothesis (to cancel out the contribution of cubes) to obtain adequate pointwise (for a single value of r) Type-I information as long as $C > N(r)^{1+\varepsilon}$, for any given $\varepsilon > 0$. We also give alternative estimates in Section 11 that use the averaging over r in a non-trivial way: we obtain adequate Type-I information on average in the range $C > R^{2-\varepsilon}$, under the Generalized Riemann Hypothesis. For the interested reader we note that there are two ways of bypassing the Riemann Hypothesis in this case. One is to assume that the sequence α_r has a bilinear structure. The second would be to obtain "subconvex" bounds in the r aspect for the Dirichlet series $\sum_{c\equiv 1 \pmod 3} \tilde{g}(cr)N(c)^{-s}$. Since a more significant bottleneck appears elsewhere we have not endeavoured to make these results unconditional.

1.4. Cubic Gauss sums and the cubic large sieve. In order to obtain the bound

$$\sum_{\substack{N(\varpi)\leqslant X \\ \varpi \text{ prime} \\ \varpi\equiv 1 \pmod 3}} \tilde{g}(\varpi) \ll_\varepsilon X^{5/6+\varepsilon},$$

Heath-Brown develops in [HB00, Theorem 2] the so-called "cubic large sieve". The cubic-large sieve states that for any sequence β_b supported on squarefree $b \in \mathbb{Z}[\omega]$,

$$\sum_{\substack{N(a)\leqslant A \\ a\equiv 1 \pmod 3}} \mu^2(a)\Big| \sum_{\substack{N(b)\leqslant B \\ b\equiv 1 \pmod 3}} \beta_b\Big(\frac{b}{a}\Big)_3\Big|^2 \ll_\varepsilon (AB)^\varepsilon\cdot(A+B+(AB)^{2/3}) \sum_{\substack{N(b)\leqslant B \\ b\equiv 1 \pmod 3}} |\beta_b|^2. \tag{1.11}$$

Immediately after stating (1.11) in [HB00], Heath-Brown writes:

> "It seems possible that the term $(AB)^{2/3}$ could be removed with further effort, and the bound would then be essentially best possible. However, the above suffices for our purposes. It should be noted that if the variables are not restricted to be squarefree, a result as sharp as Theorem 2 would be impossible. The proof of Theorem 2 is modelled on the corresponding argument for sums (over $\mathbb{Z}$) containing the quadratic residue symbol, due to the author [3] (local cit. [HB95]). The latter is distinctly unpleasant, but fortunately some of the difficulties may be reduced in our situation by the introduction of the term $(AB)^{2/3}$ in Theorem 2. "

This assertion that the term $(AB)^{2/3}$ can be removed is then frequently repeated in subsequent literature. For example, in [BGL14] it is asserted that

> "As in [12] (local cit. [HB00]), the term $(AB)^{2/3}$ is not optimal and can most likely be replaced with $(AB)^{1/2}$."

To our great surprise we found that the term $(AB)^{2/3}$ in Heath-Brown's cubic large sieve *can't* be removed. We state our optimality result in terms of operator norms. For $A, B \geqslant 10$, and $(\beta_b)_{b \in \mathbb{Z}[\omega]}$ an arbitrary sequence of complex numbers with support contained in the set of squarefree elements of $\mathbb{Z}[\omega]$, let

$$\Sigma(A, B, \boldsymbol{\beta}) := \sum_{\substack{A < N(a) \leqslant 2A \\ a \equiv 1 \pmod 3}} \mu^2(a) \Big| \sum_{\substack{B < N(b) \leqslant 2B \\ b \equiv 1 \pmod 3}} \beta_b \Big(\frac{b}{a}\Big)_3 \Big|^2.$$

For $A, B \geqslant 1$, consider the operator norm

$$\mathcal{B}(A, B) := \sup_{\boldsymbol{\beta}} \Big\{ \Sigma(A, B, \boldsymbol{\beta}) : \sum_b |\beta_b|^2 = 1 \Big\}. \tag{1.12}$$

Theorem 1.4. *Assume the Generalized Riemann Hypothesis for Hecke L-functions over $\mathbb{Q}(\omega)$. Let $A, B \geqslant 10$, then for any $\varepsilon > 0$ we have*

$$\begin{cases} (AB)^{2/3} \ll_\varepsilon \mathcal{B}(A, B) \ll_\varepsilon (AB)^{2/3+\varepsilon} & \text{for } A \in [\sqrt{B}, B^2] \backslash [B^{1-\varepsilon}, B^{1+2\varepsilon}], \\ (AB)^{2/3-3\varepsilon} \ll_\varepsilon \mathcal{B}(A, B) \ll_\varepsilon (AB)^{2/3+\varepsilon} & \text{for } A \in [B^{1-\varepsilon}, B^{1+2\varepsilon}]. \end{cases}$$

Remark 1.1. Only the lower bounds in Theorem 1.4 are conditional on the Generalized Riemann Hypothesis for Hecke L-functions over $\mathbb{Q}(\omega)$. The upper bounds follow unconditionally from Heath-Brown's cubic large sieve, and are included for the sake of comparison with the lower bounds.

One example establishing optimality in the range $A = B^{1-\varepsilon}$ (for any given small $\varepsilon > 0$) is $\beta_b = \overline{\widetilde{g}(b)}$. This follows from applying our Voronoi summation formula in Proposition 8.1, and then subsequently using the non-trivial main term that arises when summing cubic Gauss sums over all elements of $\mathbb{Z}[\omega]$ (see Section 9 for details). This is far from the only obstruction. Any sequence of the form $\beta_b \overline{\widetilde{g}(b)}$ with β_b non-negative and not correlated with cubic symbols would provide a counterexample.

To address this limitation of the cubic large sieve we introduce a correction term. This allows us to beat the exponent $(AB)^{2/3}$, albeit only for sequences that have substantial cancellations against all non-principal cubic characters. We show that there exists a small fixed $\delta > 0$ such that for any sequence $\boldsymbol{\beta}$ on $\mathbb{Z}[\omega]$ satisfying

- $|\beta_b| \leqslant 1$ for all $b \in \mathbb{Z}[\omega]$;
- β_b supported on square-free w-rough integers (i.e all prime factors of $b \in \mathbb{Z}[\omega]$ have norm $> w$);
- β_b supported on $b \equiv 1 \pmod 3$ with $N(b) \asymp B$;

then

$$\sum_{\substack{a\in\mathbb{Z}[\omega]\\a\equiv 1\pmod 3}} \mu^2(a)V\Big(\frac{N(a)}{A}\Big)\Big|\sum_{b\in\mathbb{Z}[\omega]}\beta_b\tilde{g}(b)\overline{\Big(\frac{b}{a}\Big)_3} - \frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})}\frac{\overline{\tilde{g}(a)}}{N(a)^{1/6}}\sum_{\substack{b\in\mathbb{Z}[\omega]\\(b,a)=1}}\frac{\beta_b}{N(b)^{1/6}}\Big|^2$$

$$\approx \frac{A}{B}\sum_{\substack{h\in\mathbb{Z}[\omega]\\0<|h|\leqslant B^2/A\\h\neq\boxdot}}\Big|\sum_b\beta_b\Big(\frac{b}{h}\Big)_3\Big|^2 + O\Big(\frac{(AB)^{2/3}\cdot B}{w} + (AB)^{2/3-\delta}\cdot B\Big), \quad (1.13)$$

where $\boxdot$ denotes an integer of the form k^3 with $k\in\mathbb{Z}[\omega]$. In particular, if the sequence β_b exhibits square root cancellations against all non-trivial cubic characters and $w > (AB)^\varepsilon$, then (1.13) is $\ll (AB)^{o(1)}(AB + B^2 + (AB)^{2/3-\varepsilon}\cdot B)$. This suggests that in order to beat the cubic large sieve, the correction term alone is not enough; we really need to know additional information about the sequence β_b. It is tempting to try to use Dirichlet polynomial techniques to bound (1.13). However, the *optimal* term $(AB)^{2/3}$ in the cubic large sieve adds substantial technical challenges preventing us from being able to use these techniques. Precise versions of (1.13) are given in Propositions 9.1 and 9.2.

Using the above estimates we are able to show in Section 10 that for a broad class of sequences we have

$$\sum_{a,b\equiv 1\pmod 3}\alpha_a\beta_b\tilde{g}(ab) \sim \frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})}\sum_{a,b\equiv 1\pmod 3}\frac{\alpha_a\beta_b\mu^2(ab)}{N(ab)^{1/6}}. \quad (1.14)$$

Note that $\mu^2(ab)$ can be inserted at will since b is supported on w-rough integers and $w > (\log X)^{10}$ is reasonably large. It is perhaps appropriate to call (1.14) a *dispersion estimate*. Compared to the usual dispersion estimates we use the assumption of the Generalized Riemann Hypothesis instead of the usual Siegel-Walfisz assumption, and the condition $ab \equiv 1 \pmod q$ is replaced by the term $\tilde{g}(ab)$.

The estimate (1.14) will be indispensible in estimating so-called Type-II sums, which we discuss in the next section. Our example suggests that the GL_3-spectral large sieve recently established by Young [You21] might also be optimal. In the same vein, it is also interesting to note that Iwaniec and Li found unexpected main terms appearing in the spectral large sieve for $\Gamma_1(q)$ [IL07]. Other versions of the cubic large sieve have been established by Baier and Young [BY10] in their work on the first moment of Dirichlet L-functions (over $\mathbb{Q}$) twisted by cubic characters.

1.5. The overall strategy of the proof. Having explained above the main ingredients in our proof we will now explain how they are combined in Sections 13 and 14. It will be useful to compare the argument with [HB00]. In order to establish the bound

$$\sum_{N(\varpi)\leqslant X}\tilde{g}(\varpi) \ll_\varepsilon X^{5/6+\varepsilon} \quad (1.15)$$

in [HB00], Heath-Brown needs to address two types of sums,

$$\sum_{a,b\equiv 1 \pmod 3} \alpha_a \tilde{g}(ab) V\Big(\frac{N(a)}{A}\Big) V\Big(\frac{N(b)}{B}\Big), \qquad AB = X; \qquad (1.16)$$

$$\sum_{a,b\equiv 1 \pmod 3} \alpha_a \beta_b \tilde{g}(ab) V\Big(\frac{N(a)}{A}\Big) V\Big(\frac{N(b)}{B}\Big), \qquad AB = X. \qquad (1.17)$$

The first sum is known as a Type-I sum and the second sum as a Type-II sum. If we are aiming for a bound of the form $X^{5/6+\varepsilon}$ for the sum over primes then we need to bound the Type-II sums by $\ll X^{5/6+\varepsilon}$ in the range $A > X^{1/3}$ (since we will apply Cauchy-Schwarz on the a-sum and thus we can hope for at most a saving of $\sqrt{A}$ in the most favourable scenario). We also need to bound the Type-I sums by $\ll X^{5/6+\varepsilon}$ in the range $A \leqslant X^{1/3}$. The main point is that if asymptotic estimates are not sought, then proving bounds with an X^ε-loss is sufficient.

If we aim to refine Heath-Brown's bound (1.15) to an asymptotic then first we need to refine the Type-I estimate to an asymptotic. This can be done simply by a careful derivation of Voronoi summation. We also need to push the range slightly past $A > X^{1/3}$, but this does not present us with any significant difficulties under the Generalized Riemann Hypothesis (other than the tedium of the computations).

Second, we need to refine Type-II estimates to an asymptotic; this is significantly more tricky. For this it is necessary to use our version of the cubic large sieve with the correction term. Since the error term needs to be smaller than $X^{5/6}$ we now need to take $A > X^{1/3+\varepsilon}$ in the Type-II sums. This however creates a problem since the ranges in which we can handle Type-I and Type-II sums are not enough to obtain primes. In fact we now need to also bound the contribution of so-called Type-III sums of the form

$$\sum_{a,b,c\equiv 1 \pmod 3} \tilde{g}(abc) V\Big(\frac{N(a)}{A}\Big) V\Big(\frac{N(b)}{B}\Big) V\Big(\frac{N(c)}{C}\Big), \qquad ABC = X.$$

with $A, B, C = X^{1/3+O(\varepsilon)}$ and a, b, c supported on primes. Bounding these sums with a power-saving presents a real challenge that we do not know how to solve. The main problem arises when $A = B = C = X^{1/3}$. In that regime, executing Voronoi summation on any single variable produces an essentially self-dual situation. Furthermore, the only admissible way of applying Cauchy-Schwarz is by grouping two variables together, and this then leads to a very long off-diagonal that appears even more difficult to handle.

Instead, we use the Generalized Riemann Hypothesis in Section 7 to refine the large sieve bound to a bound that is tight up to constant factors and holds with wider uniformity than the Type-II bound stated above. As a consequence we can show that the overall contribution of these Type-III sums is only ε times the expected main term. This strategy was previously used by Heath-Brown in his work on primes in short intervals [HB88]. Our work is the first to execute this idea in the context of an oscillating sequence with a main term of density $X^{-\delta}$.

Finally we note that our Type-II bound (as stated) is not able to handle the narrow range $X^{1/2-\varepsilon} \leqslant A, B \leqslant X^{1/2+\varepsilon}$. So instead in this range we use the same kind of ideas that we used to handle Type-III sums. This is not necessary and we could have obtained

a power-saving in this range with a little bit more work. However, this wouldn't have made a significant difference so we refrained from this additional work.

2. Basic facts about $\mathbb{Q}(\omega)$

Let $\mathbb{Q}(\omega)$ be the Eisenstein quadratic number field, where ω is identified with $e^{2\pi i/3} \in \mathbb{C}$. It has ring of integers $\mathbb{Z}[\omega]$, discriminant -3, and class number 1. It also has six units $\{\pm 1, \pm\omega, \pm\omega^2\}$ and one ramified prime $\lambda := 1 + 2\omega = \sqrt{-3}$ dividing 3. Let $N(x) := N_{\mathbb{Q}(\omega)/\mathbb{Q}}(x) = |x|^2$ denote the norm form on $\mathbb{Q}(\omega)/\mathbb{Q}$. Each ideal $0 \neq \mathfrak{c} \trianglelefteq \mathbb{Z}[\omega]$ is principal. If $(\mathfrak{c}, 3) = 1$, then $\mathfrak{c}$ has a unique generator $\mathfrak{c} = (c)$ that satisfies $c \equiv 1 \pmod 3$.

Whenever we write $d|c$ with $c \equiv 1 \pmod 3$, it is our convention that $d \equiv 1 \pmod 3$. If $p \equiv 1 \pmod 3$ is a rational prime, then $p = \varpi\overline{\varpi}$ in $\mathbb{Z}[\omega]$ with $N(\varpi) = p$ and ϖ a prime in $\mathbb{Z}[\omega]$. If $p \equiv 2 \pmod 3$ is a rational prime, then $p = \varpi$ is inert in $\mathbb{Z}[\omega]$ and $N(\varpi) = p^2$. Define

$$\check{e}(z) := e^{2\pi i \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(z)} = e^{2\pi i(z+\overline{z})}, \quad z \in \mathbb{C}.$$

For $c \in \mathbb{Z}[\omega]$ with $c \equiv 1 \pmod 3$, the cubic Gauss sum is defined by

$$g(c) := \sum_{d \pmod c} \left(\frac{d}{c}\right)_3 \check{e}\left(\frac{d}{c}\right). \tag{2.1}$$

We have the formula [Has50, pp. 443–445]

$$g(c)^3 = \mu(c)c^2\overline{c}, \tag{2.2}$$

where μ denotes the Möbius function on $\mathbb{Z}[\omega]$. Observe that (2.2) implies that $g(c)$ is supported on squarefree moduli. We write

$$\tilde{g}(c) := \frac{g(c)}{|c|},$$

for the normalised cubic Gauss sum. Note that $|\tilde{g}(c)| = \mu^2(c)$ for all $c \in \mathbb{Z}[\omega]$.

An important property of $g(c)$ is twisted multiplicativity [Has50, pp. 443–445]. It states that

$$\tilde{g}(ab) = \overline{\left(\frac{a}{b}\right)_3} \tilde{g}(a)\tilde{g}(b) \quad \text{for} \quad a, b \in \mathbb{Z}[\omega] \quad \text{satisfying} \quad (a, b) = 1. \tag{2.3}$$

Both sides of (2.3) are zero when $(a, b) \neq 1$, and so (2.3) can be trivially extended to all $a, b \in \mathbb{Z}[\omega]$. Recall that cubic reciprocity and a supplement to cubic reciprocity for the units/ramified prime are is given in (1.4) and (1.5) respectively.

3. Notational conventions and definitions

Throughout the paper, ϖ will denote a general prime in $\mathbb{Z}[\omega]$ satisfying $(\varpi, 3) = 1$, and $\lambda := \sqrt{-3}$ is the unique ramified prime. We also denote by $\boxdot$ an element of the form b^3 with $b \in \mathbb{Z}[\omega]$. For $z \geqslant 3$, let

$$\mathscr{P}(z) := \prod_{\substack{N(\varpi) \leqslant z \\ \varpi \equiv 1 \pmod{3} \\ \varpi \text{ prime}}} \varpi. \tag{3.1}$$

For a given $w \geqslant 1$, we say that $a \in \mathbb{Z}[\omega]$ with $a \equiv 1 \pmod{3}$ is w-rough if and only if $(a, \mathscr{P}(w)) = 1$.

Unless otherwise specified, it should be clear from context whether $\overline{q}$ means modular inverse (with respect to an appropriate modulus) or complex conjugation.

Many estimates in this paper hold for a large class of sequences given in Definition 3.1 below.

Definition 3.1. Given $\eta > 0$, $A \geqslant 1$, and $w \geqslant 1$, let $\mathcal{C}_\eta(A, w)$ denote the set of sequences $\boldsymbol{\alpha} := (\alpha_a)$ such that

(1) $|\alpha_a| \leqslant 1$ for all $a \in \mathbb{Z}[\omega]$;
(2) α_a is supported on squarefree w-rough $a \in \mathbb{Z}[\omega]$ with $a \equiv 1 \pmod{3}$;
(3) $\alpha_a = 0$ unless $N(a) \asymp A$;
(4) For any $\varepsilon > 0$, $t \in \mathbb{R}$, $\ell \in \mathbb{Z}$ and $k, u \in \mathbb{Z}[\omega]$ with $k, u \equiv 1 \pmod{3}$, we have

$$\sum_{\substack{a \in \mathbb{Z}[\omega] \\ a \equiv 1 \pmod{3} \\ u \mid a}} \alpha_a \left(\frac{a}{|a|}\right)^\ell N(a)^{it} \left(\frac{k}{a}\right)_3 \ll_\varepsilon (1 + |\ell|)^\varepsilon N(k)^\varepsilon (1 + |t|)^\varepsilon \left(\frac{A}{N(u)}\right)^{1/2 + \eta + \varepsilon}, \tag{3.2}$$

provided that $\ell \neq 0$, or if $\ell = 0$, then provided that $k \neq \boxdot$.

The Generalized Riemann Hypothesis is used to show that axiom (3.2) above holds for sequences of interest to us (i.e. smoothed indicator functions on the set of w-rough integers in $\mathbb{Z}[\omega]$.) See Section 6 for details.

Where important, the dependence of implied constants on auxiliary parameters will be indicated in subscripts i.e. $O_{\varepsilon,\xi,A,\ldots}$, $\ll_{\varepsilon,\xi,A,\ldots}$ and $\gg_{\varepsilon,\xi,A,\ldots}$. It will be crucial to give the implied constants of certain error terms in the proofs of Theorem 1.1 and Theorem 1.3 explicitly in terms of some of the auxiliary parameters. Such terms are clearly indicated.

4. Poisson summation formula

We will need a minor variant of the number field Poisson summation formula.

Lemma 4.1. *Let* $V : \mathbb{R}^2 \to \mathbb{C}$ *be a smooth Schwartz function. By an abuse of notation, set* $V(x + iy) := V(x, y)$. *Then we have*

$$\sum_{m \in \mathbb{Z}[\omega]} V(m) = \frac{2}{\sqrt{3}} \sum_{k \in \mathbb{Z}[\omega]} \int_{\mathbb{R}^2} V(x, y) \check{e}\left(\frac{k(x + iy)}{\lambda}\right) dx dy. \tag{4.1}$$

Proof. Let $\Lambda := \mathbb{Z}[\omega]$, viewed as a discrete lattice embedded in $\mathbb{C}$. We identify $x + iy$ with $(x, y) \in \mathbb{R}^2$. Poisson summation gives

$$\sum_{\mathbf{x} \in \Lambda} V(\mathbf{x}) = \frac{1}{\text{covol}(\Lambda)} \sum_{\mathbf{x} \in \Lambda^\star} \widehat{V}(\mathbf{x}),$$

where $\Lambda^\star = \lambda^{-1}\mathbb{Z}[\omega]$ is the dual lattice to Λ,

$$\widehat{V}(\mathbf{x}) := \int_{\mathbb{R}^2} V(u, v) e(2(xu + vy)) du dv,$$

and

$$\text{covol}(\Lambda) = \frac{\sqrt{3}}{2}.$$

Observe that

$$\text{Re}[(x + iy)(u - iv)] = xu + yv.$$

Thus Poisson summation for $\mathbb{Z}[\omega]$ is given by

$$\sum_{m \in \mathbb{Z}[\omega]} V(m) = \frac{2}{\sqrt{3}} \sum_{k \in \lambda^{-1}\mathbb{Z}[\omega]} \int_{\mathbb{R}^2} V(x, y) e(2\text{Re}(\overline{k}(x + iy))) dx dy. \tag{4.2}$$

We can replace $\overline{k}$ by k in (4.2), since $\lambda^{-1}\mathbb{Z}[\omega]$ is closed under conjugation. Thus (4.1) holds, as required. ∎

Lemma 4.2. *Let* $0 \neq q \in \mathbb{Z}[\omega]$, $\psi : \mathbb{Z}[\omega] \to \mathbb{C}$ *be a q-periodic function, and* $V : \mathbb{R}^2 \to \mathbb{C}$ *be a smooth Schwartz function. Then*

$$\sum_{m \in \mathbb{Z}[\omega]} \psi(m) V(m) = \frac{2}{\sqrt{3}N(q)} \sum_{k \in \mathbb{Z}[\omega]} \dot{\psi}(k) \dot{V}\left(\frac{k}{q}\right),$$

where

$$\dot{\psi}(k) := \sum_{t \pmod q} \psi(t) \breve{e}\left(-\frac{tk}{q\lambda}\right),$$

and

$$\dot{V}(u) := \int_{\mathbb{R}^2} V(x, y) \breve{e}\left(\frac{u(x + iy)}{\lambda}\right) dx dy.$$

Remark 4.1. For each $t \in \mathbb{Z}[\omega]$, note that the additive character

$$k \mapsto \breve{e}\left(-\frac{tk}{3\lambda q}\right)$$

has minimal period $3q$ (not $3\lambda q$).

Proof. We have

$$\sum_{m \in \mathbb{Z}[\omega]} \psi(m) V(m) = \sum_{t \pmod q} \psi(t) \sum_{m \in \mathbb{Z}[\omega]} V(mq + t)$$

Application of Lemma 4.1 to the summation over m gives

$$\sum_{m \in \mathbb{Z}[\omega]} V(mq + t) = \frac{2}{\sqrt{3}} \sum_{k \in \mathbb{Z}[\omega]} \int_{\mathbb{R}^2} V((x + iy)q + t) \breve{e}\left(\frac{k(x + iy)}{\lambda}\right) dx dy.$$

A linear change of variable then shows that

$$\sum_{m \in \mathbb{Z}[\omega]} V(mq + t) = \breve{e}\Big(-\frac{kt}{q\lambda}\Big)\frac{2}{\sqrt{3}N(q)}\sum_{k \in \mathbb{Z}[\omega]}\int_{\mathbb{R}^2} V(x, y)\breve{e}\Big(\frac{k(x+iy)}{q\lambda}\Big)dxdy. \qquad (4.3)$$

The result follows upon summing both sides of (4.3) over $t \pmod q$ with the q-periodic weights $\psi(t)$. ∎

We will specialise to the case where the test function is radially symmetric.

Lemma 4.3. *Let $q \in \mathbb{Z}[\omega]$ with $q \equiv 1 \pmod 3$, $\psi : \mathbb{Z}[\omega] \to \mathbb{C}$ be a q-periodic function, and $V : \mathbb{R} \to \mathbb{C}$ be a smooth Schwartz function. Then for any $M > 0$ we have*

$$\sum_{\substack{m \in \mathbb{Z}[\omega] \\ m \equiv 1 \pmod 3}} \psi(m)V\Big(\frac{N(m)}{M}\Big) = \frac{4\pi M}{9\sqrt{3}N(q)}\sum_{k \in \mathbb{Z}[\omega]} \ddot{\psi}(k)\ddot{V}\Big(\frac{k\sqrt{M}}{q}\Big),$$

where

$$\ddot{\psi}(k) := \breve{e}\Big(-\frac{k}{3\lambda}\Big)\sum_{x \pmod q} \psi(3\lambda x)\breve{e}\Big(-\frac{kx}{q}\Big),$$

and $\ddot{V} : \mathbb{C} \to \mathbb{C}$ is defined by

$$\ddot{V}(u) := \int_0^\infty rV(r^2)J_0\Big(\frac{4\pi r|u|}{3\sqrt{3}}\Big)dr. \qquad (4.4)$$

Proof. Application of Lemma 4.2 gives

$$\sum_{\substack{m \in \mathbb{Z}[\omega] \\ m \equiv 1 \pmod 3}} \psi(m)V\Big(\frac{N(m)}{M}\Big) = \frac{1}{N(3q)}\frac{2}{\sqrt{3}}\sum_{k \in \mathbb{Z}[\omega]} \dot{\psi}(k)\int_{\mathbb{R}^2} V\Big(\frac{x^2+y^2}{M}\Big)\breve{e}\Big(\frac{k(x+iy)}{3\lambda q}\Big)dxdy,$$

where

$$\dot{\psi}(k) = \sum_{\substack{t \pmod{3q} \\ t \equiv 1 \pmod 3}} \psi(t)\breve{e}\Big(-\frac{tk}{3\lambda q}\Big). \qquad (4.5)$$

We first simplify the integral. A change of variable gives

$$\int_{\mathbb{R}^2} V\Big(\frac{x^2+y^2}{M}\Big)\breve{e}\Big(\frac{k(x+iy)}{3\lambda q}\Big)dxdy = M\int_{\mathbb{R}^2} V(x^2+y^2)\breve{e}\Big(\frac{k(x+iy)\sqrt{M}}{3\lambda q}\Big)dxdy. \qquad (4.6)$$

We change $x + iy$ to polar coordinates via the substitution $x + iy = re^{i\vartheta}$. Let $\kappa \in [-\pi, \pi)$ be a fixed angle (depending on k and q) such that

$$e^{-i\kappa} = \begin{cases} \frac{k}{|k|}\frac{|\lambda q|}{\lambda q} & \text{if} \quad k \neq 0 \\ 1 & \text{if} \quad k = 0. \end{cases}$$

Then (4.6) becomes

$$M\int_0^\infty rV(r^2)\int_0^{2\pi} \breve{e}\Big(\frac{re^{i\vartheta - i\kappa}|k|\sqrt{M}}{3\sqrt{3}|q|}\Big)d\vartheta dr. \qquad (4.7)$$

We eliminate the translation by κ by a linear change of variable in ϑ and the fact that the integrand is periodic with period 2π. Therefore (4.7) becomes

$$M \int_0^\infty rV(r^2) \int_0^{2\pi} \exp\Big(\frac{4\pi i r \cos(\vartheta)|k|\sqrt{M}}{3\sqrt{3}|q|}\Big) d\vartheta dr.$$

Observe that by [DLMF, (10.9.2)] and the fact that $J_0(x)$ is real-valued we obtain

$$J_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{ix\cos(\vartheta)} d\vartheta.$$

Thus (4.7) is equal to

$$2\pi M \int_0^\infty rV(r^2) J_0\Big(\frac{4\pi r|k|\sqrt{M}}{3\sqrt{3}|q|}\Big) dr.$$

It remains to compute $\dot{\psi}(k)$. The Chinese Remainder theorem guarantees that we can write any $t \pmod{3q}$ as $t = aq + 3b$ with a a representative of a residue class $\pmod 3$ and b a representative of a residue class $\pmod q$. Necessarily $a \equiv \overline{q} \pmod 3$. Thus (4.5) becomes

$$\dot{\psi}(k) := \Big(\sum_{\substack{a \pmod 3 \\ a \equiv \overline{q} \pmod 3}} \check{e}\Big(-\frac{ak}{3\lambda}\Big)\Big)\Big(\sum_{b \pmod q} \psi(3b)\check{e}\Big(-\frac{bk}{\lambda q}\Big)\Big)$$

$$= \check{e}\Big(-\frac{k}{3\lambda}\Big) \sum_{b \pmod q} \psi(3\lambda b)\check{e}\Big(-\frac{bk}{q}\Big),$$

where the displays followed from the fact that $q \equiv 1 \pmod 3$, ψ is periodic modulo q with $(q,3) = 1$, and Remark 4.1. ∎

We now state the final version of the Poisson summation formula needed for this paper.

Corollary 4.1. *Let $n_1, n_2 \in \mathbb{Z}[\omega]$ be squarefree and satisfy $n_1 \equiv n_2 \equiv 1 \pmod 3$. Let $d := (n_1, n_2)$ and $V : \mathbb{R} \to \mathbb{C}$ be a smooth Schwartz function. Then for any $M > 0$ we have*

$$\sum_{\substack{m \in \mathbb{Z}[\omega] \\ m \equiv 1 \pmod 3}} \Big(\frac{m}{n_1}\Big)_3 \overline{\Big(\frac{m}{n_2}\Big)_3} V\Big(\frac{N(m)}{M}\Big) = \frac{4\pi \big(\frac{d}{n_1/d}\big)_3 \overline{\big(\frac{d}{n_2/d}\big)_3} M g(n_1/d)\overline{g(n_2/d)}}{9\sqrt{3}N\big(n_1 n_2/d\big)}$$

$$\times \sum_{k \in \mathbb{Z}[\omega]} \widetilde{c}_d(k)\overline{\Big(\frac{k}{n_1/d}\Big)_3}\Big(\frac{k}{n_2/d}\Big)_3 \ddot{V}\Big(\frac{kd\sqrt{M}}{n_1 n_2}\Big),$$

where

$$\widetilde{c}_d(k) := \check{e}\Big(-\frac{k}{3\lambda}\Big) \sum_{\substack{x \pmod d \\ (x,d)=1}} \check{e}\Big(-\frac{kx}{d}\Big), \tag{4.8}$$

and $\ddot{V} : [0,\infty) \to \mathbb{C}$ is given by

$$\ddot{V}(u) := \int_0^\infty rV(r^2) J_0\Big(\frac{4\pi r|u|}{3\sqrt{3}}\Big) dr.$$

Proof. We apply Lemma 4.3 to

$$\psi_{n_1,n_2}(m) := \left(\frac{m}{n_1}\right)_3 \overline{\left(\frac{m}{n_2}\right)_3} = \left(\frac{m}{n_1/d}\right)_3 \overline{\left(\frac{m}{n_2/d}\right)_3} \mathbf{1}_d(m), \quad m \in \mathbb{Z}[\omega],$$

where $\mathbf{1}_d$ denotes the principal character modulo d. Observe that ψ_{n_1,n_2} is n_1n_2/d periodic. All that remains to do is to compute $\ddot{\psi}_{n_1,n_2}$. We have

$$\ddot{\psi}_{n_1,n_2}(k) = \check{e}\left(-\frac{k}{3\lambda}\right) \sum_{\substack{x \pmod{n_1n_2/d} \\ (x,d)=1}} \left(\frac{3\lambda x}{n_1/d}\right)_3 \overline{\left(\frac{3\lambda x}{n_2/d}\right)_3} \check{e}\left(-\frac{kx}{(n_1n_2/d)}\right). \qquad (4.9)$$

Observe that $3\lambda = (-\lambda)^3$ and $(n_1n_2/d, 3) = 1$, so (4.9) becomes

$$\ddot{\psi}_{n_1,n_2}(k) = \check{e}\left(-\frac{k}{3\lambda}\right) \sum_{\substack{x \pmod{n_1n_2/d} \\ (x,d)=1}} \left(\frac{x}{n_1/d}\right)_3 \overline{\left(\frac{x}{n_2/d}\right)_3} \check{e}\left(-\frac{kx}{(n_1n_2/d)}\right).$$

Since n_1 and n_2 are squarefree we have $(n_1n_2/d^2, d) = 1$. We use the Chinese remainder theorem to write $x = a(n_1n_2/d^2) + bd$. We find that

$$\ddot{\psi}_{n_1,n_2}(k) = \check{e}\left(-\frac{k}{3\lambda}\right) \sum_{\substack{x \pmod{n_1n_2/d} \\ (x,d)=1}} \left(\frac{x}{n_1/d}\right)_3 \overline{\left(\frac{x}{n_2/d}\right)_3} \check{e}\left(-\frac{kx}{(n_1n_2/d)}\right)$$

$$= \left(\check{e}\left(-\frac{k}{3\lambda}\right) \sum_{\substack{a \pmod{d} \\ (a,d)=1}} \check{e}\left(-\frac{ak}{d}\right)\right) \left(\sum_{b \pmod{n_1n_2/d^2}} \left(\frac{bd}{n_1/d}\right)_3 \overline{\left(\frac{bd}{n_2/d}\right)_3} \check{e}\left(-\frac{kb}{(n_1n_2/d^2)}\right)\right).$$

Observe that $(n_1/d, n_2/d) = 1$. To evaluate the sum over b we use the Chinese remainder theorem again. Writing $b = t(n_1/d) + u(n_2/d)$ gives

$$\ddot{\psi}_{n_1,n_2}(k) = \tilde{c}_d(k)\left(\sum_{t \pmod{n_2/d}} \overline{\left(\frac{tn_1}{n_2/d}\right)_3} \check{e}\left(-\frac{tk}{n_2/d}\right)\right) \left(\sum_{u \pmod{n_1/d}} \left(\frac{un_2}{n_1/d}\right)_3 \check{e}\left(-\frac{uk}{n_1/d}\right)\right)$$

$$= \tilde{c}_d(k)\left(\frac{k}{n_2/d}\right)_3 \overline{\left(\frac{k}{n_1/d}\right)_3} \left(\frac{n_2}{n_1/d}\right)_3 \overline{\left(\frac{n_1}{n_2/d}\right)_3} \cdot g(n_1/d)\overline{g(n_2/d)},$$

where the last display follows from the primitivity of characters $\left(\frac{\cdot}{n_1/d}\right)_3$ and $\overline{\left(\frac{\cdot}{n_2/d}\right)_3}$. Finally,

$$\left(\frac{n_2}{n_1/d}\right)_3 \overline{\left(\frac{n_1}{n_2/d}\right)_3} = \left(\frac{n_2/d}{n_1/d}\right)_3 \left(\frac{d}{n_1/d}\right)_3 \overline{\left(\frac{n_1/d}{n_2/d}\right)_3} \overline{\left(\frac{d}{n_2/d}\right)_3} = \left(\frac{d}{n_1/d}\right)_3 \overline{\left(\frac{d}{n_2/d}\right)_3},$$

where the last equality follows from cubic reciprocity. This completes the proof. ∎

We close this section with standard estimate for $\ddot{V}$.

Lemma 4.4. *Let $V : \mathbb{R} \to \mathbb{C}$ be a smooth compactly supported function. Then for any integer $k \geq 0$,*

$$|\ddot{V}(u)| \ll_{k,V} (1 + |u|)^{-k}, \quad u \in \mathbb{C}. \qquad (4.10)$$

Proof. Integrating (4.4) by parts $k \in \mathbb{Z}_{\geqslant 0}$ times using [DLMF, (10.22.1)] gives

$$\ddot{V}(u) = (-1)^k \left(\frac{3\sqrt{3}}{2\pi}\right)^k \frac{1}{|u|^k} \int_0^\infty V^{(k)}(r^2) \cdot r^{k+1} J_k\left(\frac{4\pi r |u|}{3\sqrt{3}}\right) dr. \tag{4.11}$$

The claim immediately follows. ■

5. Voronoi summation in the level aspect

The Fourier coefficients of the cubic theta function essentially sample cubic Gauss sums. Naturally, automorphy of the theta function is a key input in the proof of our level aspect Voronoi summation formula.

5.1. Geometry, groups, and the cubic theta function at cusps. Let $\mathbb{H}^3$ denote the hyperbolic 3-space $\mathbb{C} \times \mathbb{R}^+$. We embed $\mathbb{H}^3$ in the Hamilton quaternions by identifying $i = \sqrt{-1}$ with $\hat{i}$ and $w = (z, v) = (x + iy, v) \in \mathbb{H}^3$ with $x + y\hat{i} + v\hat{k}$, where $1, \hat{i}, \hat{j}, \hat{k}$ denote the unit quaternions. In terms of quaternion arithmetic, the group action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{H}^3$ is given by

$$\gamma w = (aw + b)(cw + d)^{-1}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}) \quad \text{and} \quad w \in \mathbb{H}^3.$$

In terms of coordinates,

$$\gamma w = \left(\frac{(az + b)\overline{(cz + d)} + a\bar{c}v^2}{|cz + d|^2 + |c|^2 v^2}, \frac{v}{|cz + d|^2 + |c|^2 v^2}\right), \quad w = (z, v) \in \mathbb{H}^3. \tag{5.1}$$

Let $\Gamma := \mathrm{SL}_2(\mathbb{Z}[\omega])$. It is a standard fact that Γ is generated by the elements

$$P := \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad E := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let $A \in \mathbb{Z}[\omega]$ satisfy $A \equiv 0 \pmod 3$, and

$$\Gamma_1(A) := \{\gamma \in \Gamma : \gamma \equiv I \pmod A\}.$$

Observe that $\Gamma_1(A)$ is a normal subgroup of Γ since it is the kernel of the reduction modulo A map. Let

$$\Gamma_2 := \langle \mathrm{SL}_2(\mathbb{Z}), \Gamma_1(3) \rangle = \mathrm{SL}_2(\mathbb{Z})\Gamma_1(3) = \Gamma_1(3)\,\mathrm{SL}_2(\mathbb{Z}), \tag{5.2}$$

where the last two equalities follow because $\Gamma_1(3)$ is normal in Γ. We also have $[\Gamma : \Gamma_2] = 27$ (see [Pat77, §2] for the calculation). We also observe the trivial (but useful) fact that for any $g, g' \in \Gamma$ we have that

$$\Gamma_1(3)g = \Gamma_1(3)g' \iff g \equiv g' \pmod 3.$$

Furthermore, for any $g, g' \in \Gamma$ we have that

$$g \equiv g' \pmod 3 \implies \Gamma_2 g = \Gamma_2 g'. \tag{5.3}$$

More properties of these groups can be found in [Pat77, §2].

Let

$$\chi : \Gamma_1(3) \to \{1, \omega, \omega^2\}$$

be the famous cubic Kubota character [Kub66, Kub69], given by

$$\chi(\gamma) := \begin{cases} \left(\frac{c}{a}\right)_3 & \text{if} \quad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(3) \quad \text{and} \quad c \neq 0, \\ 1 & \text{otherwise.} \end{cases} \tag{5.4}$$

It was shown by Patterson [Pat77, §2] that χ extends to a well-defined homomorphism

$$\chi : \Gamma_2 \to \{1, \omega, \omega^2\},$$

when one defines $\chi|_{\text{SL}_2(\mathbb{Z})} \equiv 1$.

Remark 5.1. There is a useful alternative expression for $\chi(\gamma)$ to the one given in (5.4). For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(3)$, we have the determinant equation $ad - bc = 1$ with $1 + bc \equiv 1 \pmod 9$. Given $0 \neq c \in \mathbb{Z}[\omega]$ we write $c = \pm \lambda^k \omega^j c'$ where $k \geqslant 2$, $0 \leqslant j \leqslant 2$ are integers, and $c' \in \mathbb{Z}[\omega]$ is such that $c' \equiv 1 \pmod 3$. Thus

$$\left(\frac{c}{ad}\right)_3 = \left(\frac{c}{1+bc}\right)_3 = \left(\frac{\lambda}{1+bc}\right)_3^k \left(\frac{\omega}{1+bc}\right)_3^j \left(\frac{c'}{1+bc}\right)_3 = 1,$$

where the last equality follows from $1 + bc \equiv 1 \pmod 9$, (1.4), and (1.5). Hence

$$\left(\frac{c}{a}\right)_3 = \overline{\left(\frac{c}{d}\right)_3} = \left(\frac{b}{d}\right)_3,$$

and we obtain

$$\chi(\gamma) = \begin{cases} \left(\frac{b}{d}\right)_3 & \text{if} \quad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(3) \quad \text{and} \quad c \neq 0, \\ 1 & \text{otherwise.} \end{cases} \tag{5.5}$$

Let $\theta(w)$ denote the cubic metaplectic theta function of Kubota on $\mathbb{H}^3$. It is automorphic on Γ_2 with multiplier χ. It has Fourier expansion (at ∞) given by

$$\theta(w) = \sigma v^{2/3} + \sum_{\mu \in \lambda^{-3}\mathbb{Z}[\omega]} \tau(\mu) v K_{\frac{1}{3}}(4\pi|\mu|v)\breve{e}(\mu z), \quad w \in \mathbb{H}^3,$$

where

$$\sigma := 3^{5/2}/2, \tag{5.6}$$

and the other Fourier coefficients were computed by Patterson [Pat77, Theorem 8.1]. They are

$$\tau(\mu) = \begin{cases} \overline{g(\lambda^2, c)}\left|\frac{d}{c}\right|3^{n/2+2} & \text{if} \quad \mu = \pm\lambda^{3n-4}cd^3, \quad n \geqslant 1 \\ e^{-\frac{2\pi i}{9}}\overline{g(\omega\lambda^2, c)}\left|\frac{d}{c}\right|3^{n/2+2} & \text{if} \quad \mu = \pm\omega\lambda^{3n-4}cd^3, \quad n \geqslant 1 \\ e^{\frac{2\pi i}{9}}\overline{g(\omega^2\lambda^2, c)}\left|\frac{d}{c}\right|3^{n/2+2} & \text{if} \quad \mu = \pm\omega^2\lambda^{3n-4}cd^3, \quad n \geqslant 1 \\ \overline{g(1, c)}\left|\frac{d}{c}\right|3^{n/2+5/2} & \text{if} \quad \mu = \pm\lambda^{3n-3}cd^3, \quad n \geqslant 0 \\ 0 & \text{otherwise,} \end{cases} \tag{5.7}$$

where

$$c, d \in \mathbb{Z}[\omega], \quad c, d \equiv 1 \pmod 3, \quad \text{and} \quad \mu^2(c) = 1. \tag{5.8}$$

It follows from (5.7) that $\tau(\cdot)$ is an even function.

Implicit in [Pat77, §7 and §8] is a careful study of $\theta(w)$ at various cusps of Γ_2. We extract the information that will be of use to us. Let $\{\gamma_j : j = 1, \ldots 27\}$ be the complete

set of inequivalent representatives for $\Gamma_2\backslash\Gamma$ given in [Pat77, Table II pg. 129]. Particular coset representatives γ_j of $\Gamma_2\backslash\Gamma$ of importance to us are

$$\gamma_1 = I, \gamma_2 = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}, \gamma_3 = \begin{pmatrix} 1 & -\omega \\ 0 & 1 \end{pmatrix}, \gamma_{10} = \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix}, \text{ and } \gamma_{19} = \begin{pmatrix} 1 & 0 \\ \omega^2 & 1 \end{pmatrix}.$$

For each $j = 1, 2, \ldots, 27$, let

$$F_j(w) := \theta(\gamma_j(w)), \quad w \in \mathbb{H}^3. \tag{5.9}$$

If $g \in \Gamma$, then

$$\gamma_j g = g_j(g)\gamma_{k_j(g)}, \quad \text{for some} \quad g_j(g) \in \Gamma_2 \quad \text{and} \quad 1 \leqslant k_j(g) \leqslant 27. \tag{5.10}$$

Thus

$$F_j(g(w)) = \chi(g_j(g))F_{k_j(g)}(w) \quad \text{for all} \quad w \in \mathbb{H}^3. \tag{5.11}$$

Each F_j is automorphic on $\Gamma_1(9)$ with multiplier χ by [Pat78a, Lemma 2.1]. Following Patterson, we define

$$F_j^\star(w) := \sum_\mu d_j(\mu)vK_{1/3}(4\pi|\mu|v)\breve{e}(\mu z), \quad w \in \mathbb{H}^3, \tag{5.12}$$

where the $d_j(\mu)$ have support contained in $\lambda^{-4}\mathbb{Z}[\omega]\backslash\{0\}$, and have expressions in terms of $\tau(\mu)$, $\tau_1(\mu)$ [Pat77, (8.8)] and $\tau_2(\mu)$ [Pat77, (8.9)], see Appendix A. For the reader's convenience we state formulae for $\tau_1(\mu)$ and $\tau_2(\mu)$ here. They are given by

$$\tau_1(\mu) = \begin{cases} 9\omega\overline{g(\lambda^2, c)}\left|\frac{d}{c}\right| & \text{if} \quad \mu = \lambda^{-4}cd^3 \\ 9e^{-\frac{2\pi i}{9}}\omega^2\overline{g(\omega\lambda^2, c)}\left|\frac{d}{c}\right| & \text{if} \quad \mu = \omega\lambda^{-4}cd^3 \\ 9e^{\frac{2\pi i}{9}}\overline{g(\omega^2\lambda^2, c)}\left|\frac{d}{c}\right| & \text{if} \quad \mu = \omega^2\lambda^{-4}cd^3 \\ 0 & \text{otherwise,} \end{cases} \tag{5.13}$$

and

$$\tau_2(\mu) = \begin{cases} 9\omega^2\overline{g(\lambda^2, c)}\left|\frac{d}{c}\right| & \text{if} \quad \mu = -\lambda^{-4}cd^3 \\ 9e^{-\frac{2\pi i}{9}}\overline{g(\omega\lambda^2, c)}\left|\frac{d}{c}\right| & \text{if} \quad \mu = -\lambda^{-4}\omega cd^3 \\ 9\omega e^{\frac{2\pi i}{9}}\overline{g(\omega^2\lambda^2, c)}\left|\frac{d}{c}\right| & \text{if} \quad \mu = -\lambda^{-4}\omega^2cd^3 \\ 0 & \text{otherwise,} \end{cases} \tag{5.14}$$

where c and d are as in (5.8). The formulas for the $d_j(\mu)$ are given in [Pat77, Table III pg. 151]. We have also included them in Appendix A. We have the Fourier expansions (at ∞) [Pat77, pg. 148],

$$F_j(w) = \begin{cases} \sigma v(w)^{2/3} + F_j^\star(w) & \text{if} \quad 1 \leqslant j \leqslant 9, \\ F_j^\star(w) & \text{if} \quad 10 \leqslant j \leqslant 27 \end{cases}, \quad w \in \mathbb{H}^3. \tag{5.15}$$

To understand the maps $j \mapsto g_j(\cdot)$, $j \mapsto k_j(\cdot)$ and $j \mapsto \chi(g_j(\cdot))$ occurring in (5.10) and (5.11), it suffices to compute them on the generators of $\Gamma = \mathrm{SL}_2(\mathbb{Z}[\omega])$: P, T, and E. The values of $k_j(E)$ appear in [Pat77, Table III]. We have included the k_j values on all three generators in Appendix A.

5.2. Conjugation and coefficient sieving. It is more convenient for us to work with the $\overline{F_j}(w)$. It follows from (5.15) that they each have Fourier expansion (at ∞) given by

$$\overline{F_j}(w) = \begin{cases} \sigma v^{2/3} + \sum_\mu \overline{d_j(-\mu)} v K_{\frac{1}{3}}(4\pi|\mu|v)\check{e}(\mu z) & \text{if} \quad 1 \leqslant j \leqslant 9 \\ \sum_\mu \overline{d_j(-\mu)} v K_{\frac{1}{3}}(4\pi|\mu|v)\check{e}(\mu z) & \text{if} \quad 10 \leqslant j \leqslant 27 \end{cases}, \quad w \in \mathbb{H}^3, \quad (5.16)$$

since $K_{1/3}(x) \in \mathbb{R}$ for $x > 0$.

The Fourier coefficients of $\overline{F_1}(w)$ are given by

$$\overline{d_1(-\mu)} = \overline{\tau(-\mu)} = \overline{\tau(\mu)},$$

where the last equality follows from the fact that $\tau(\cdot)$ is even. Let

$$S := \{\lambda^{-3}cd^3 \in \mathbb{Q}(\omega) : c, d \in \mathbb{Z}[\omega], \quad c, d \equiv 1 \pmod{3} \quad \text{and} \quad \mu^2(c) = 1\}, \quad (5.17)$$

and

$$\overline{F_1}(w)_S := \sum_{\mu \in S} \overline{\tau(\mu)} v K_{\frac{1}{3}}(4\pi|\mu|v)\check{e}(\mu z), \quad w \in \mathbb{H}^3.$$

Remark 5.2. We study the function $\overline{F_1}(w)_S$ because its Fourier coefficients are supported on S and the $\mu = \lambda^{-3}cd^3$th coefficient is $\widetilde{g}(c)|d|$ up to an absolute constant (cf. (5.7)). These are the coefficients (up to twisting) we want to appear in our Voronoi formula (cf. Proposition 5.3).

Lemma 5.1. *Let $\overline{F_1}(w)_S$ be as above. Then*

$$\overline{F_1}(w)_S = \frac{1}{3}\Big(\overline{F_1}(w) + \omega\overline{F_2}(w) + \omega^2\overline{F_3}(w)\Big),$$

and $\overline{F_1}(w)_S$ is automorphic under $\Gamma_1(9)$ with multiplier $\overline{\chi}$.

Proof. Following [Pat78a, Theorem 5.2], we detect $\mu \in S$ additively. From (5.7), we have

$$\{\mu \in \lambda^{-3}\mathbb{Z}[\omega] : \check{e}(\omega\mu) = \omega^2 \quad \text{and} \quad \tau(\mu) \neq 0\} = S.$$

Thus

$$\overline{F_1}(w)_S = \frac{1}{3}\sigma v^{2/3}(1 + \omega + \omega^2)$$

$$+ \frac{1}{3} \sum_{\mu \in \lambda^{-3}\mathbb{Z}[\omega]} \overline{\tau(\mu)} v K_{\frac{1}{3}}(4\pi|\mu|v)\check{e}(\mu z)\big(1 + \omega\check{e}(\omega\mu) + \omega^2\check{e}(2\omega\mu)\big)$$

$$= \frac{1}{3}\Big(\overline{F_1}(w) + \omega\overline{F_2}(w) + \omega^2\overline{F_3}(w)\Big), \quad (5.18)$$

where the last term in (5.18) was obtained by writing $\left(\begin{smallmatrix} 1 & 2\omega \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 3\omega \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -\omega \\ 0 & 1 \end{smallmatrix}\right)$ and using automorphy of $\overline{F_1}(w) = \overline{\theta}(w)$ on Γ_2 with multiplier $\overline{\chi}$. This proves the first claim. Each $\overline{F_j}$ is automorphic on $\Gamma_1(9)$ with multiplier $\overline{\chi}$, and so the second claim follows. ∎

5.3. Twists. Let $r \in \mathbb{Z}[\omega]$ with $r \equiv 1 \pmod 3$, and ψ be a function on $\mathbb{Z}[\omega]$ that is periodic modulo r. In view of (5.12), the ψ-twist of $\overline{F_j^\star}(w)$ is defined by

$$\overline{F_j^\star}(w; \psi) := \sum_\mu \overline{d_j(-\mu)} \psi(\lambda^4 \mu) v K_{\frac{1}{3}}(4\pi|\mu|v) \breve{e}(\mu z), \quad w \in \mathbb{H}^3. \tag{5.19}$$

In view of (5.16), the ψ-twist of $\overline{F_j}(w)$ is

$$\overline{F_j}(w; \psi) := \begin{cases} \sigma\psi(0)v^{2/3} + \overline{F_j^\star}(w; \psi) & 1 \leqslant j \leqslant 9 \\ \overline{F_j^\star}(w; \psi) & 10 \leqslant j \leqslant 27 \end{cases}, \quad w \in \mathbb{H}^3. \tag{5.20}$$

Remark 5.3. The Fourier coefficients of all the $\overline{F_j}$ have support contained in $\lambda^{-4}\mathbb{Z}[\omega]$. This explains why we define a general twist by $\psi(\lambda^4(\cdot))$ in (5.20). In the special case $\overline{F_1} = \overline{\theta}$, (5.7) tells us that the Fourier coefficients have support contained in $\lambda^{-3}\mathbb{Z}[\omega]$. Thus our twisting definition produces an extraneous $\psi(\lambda)$ factor in this case. This will be immaterial in our final results.

Define the Fourier transform

$$\widehat{\psi}(u) := \sum_{x \pmod r} \psi(x)\breve{e}\Big(\frac{ux}{r}\Big), \quad u \in \mathbb{Z}[\omega]. \tag{5.21}$$

Fourier inversion tells us that

$$\psi(x) = \frac{1}{N(r)} \sum_{u \pmod r} \widehat{\psi}(u)\breve{e}\Big(-\frac{ux}{r}\Big), \quad x \in \mathbb{Z}[\omega].$$

We also define the following non-Archimedean analogue of a Bessel $K_{1/3}$–transform,

$$\widetilde{\psi}(u) := \sum_{x \pmod r} \psi(x)S_{1/3}(x, u; r), \quad u \in \mathbb{Z}[\omega], \tag{5.22}$$

where

$$S_{1/3}(x, u; r) := \Big(\frac{\lambda}{r}\Big)_3 \sum_{\substack{d \pmod r \\ (d,r)=1 \\ (\lambda^4 d)(\lambda^4 a) \equiv 1 \pmod r}} \Big(\frac{a}{r}\Big)_3 \breve{e}\Big(\frac{xd + ua}{r}\Big), \tag{5.23}$$

is the cubic Kloosterman sum. Note that it is convenient for us to have the $\lambda^4 = 9$ factors built into the congruence in the cubic Kloosterman sum. These factors naturally appear when we use the automorphy of the $\overline{F_j}$ (with multiplier $\overline{\chi}$) on the group $\Gamma_1(\lambda^4)$ in the proof of the following result.

To isolate twists of the cubic Gauss sums, we need to analyse

$$\overline{F_1}(w; \psi)_S := \sum_{\mu \in S} \overline{\tau(\mu)} \psi(\lambda^4 \mu) v K_{\frac{1}{3}}(4\pi|\mu|v) \breve{e}(\mu z), \quad w \in \mathbb{H}^3. \tag{5.24}$$

Lemma 5.2. *Suppose $r \in \mathbb{Z}[\omega]$ with $r \equiv 1 \pmod 3$, and ψ is a sequence on $\mathbb{Z}[\omega]$ that is periodic modulo r. Suppose that $\widehat{\psi}$ is supported only on residue classes coprime to r. For $w = (z, v) \in \mathbb{H}^3$, we have*

$$\overline{F_1}(w; \psi)_S = \frac{1}{3N(r)}\Big\{\overline{F_1} + \omega\overline{F_{19}^\star} + \omega^2\overline{F_{10}^\star}\Big\}\Big(\Big(-\frac{\overline{z}}{r^2(|z|^2 + v^2)}, \frac{v}{|r|^2(|z|^2 + v^2)}\Big); \widetilde{\psi}\Big). \tag{5.25}$$

Proof. Fourier inversion and Lemma 5.1 imply that

$$\overline{F_1}(w;\psi)_S = \frac{1}{N(r)} \sum_{\substack{d \pmod{r} \\ (d,r)=1}} \widehat{\psi}(d)\overline{F_1}\left(z - \frac{\lambda^4 d}{r}, v\right)_S$$

$$= \frac{1}{3N(r)} \sum_{\substack{d \pmod{r} \\ (d,r)=1}} \widehat{\psi}(d)\left(\overline{F_1}\left(z - \frac{\lambda^4 d}{r}, v\right) + \omega\overline{F_2}\left(z - \frac{\lambda^4 d}{r}, v\right) + \omega^2\overline{F_3}\left(z - \frac{\lambda^4 d}{r}, v\right)\right).$$

$$(5.26)$$

Given our $r \equiv 1 \pmod 3$, and each d in (5.26), we have $(r, \lambda^4 d) = 1$. Thus there exists

$$\begin{pmatrix} r & -\lambda^4 a \\ \lambda^4 d & b \end{pmatrix} \in \Gamma_1(3),$$

and hence there exists

$$\gamma := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} r & -\lambda^4 a \\ \lambda^4 d & b \end{pmatrix} = \begin{pmatrix} \lambda^4 d & b \\ -r & \lambda^4 a \end{pmatrix} \in \Gamma_2. \tag{5.27}$$

Note that we also used (5.2) implicitly in the above display.

A direct computation shows that

$$\gamma\left(\frac{\lambda^4 a}{r} - \frac{\overline{z}}{r^2(|z|^2 + v^2)}, \frac{v}{|r|^2(|z|^2 + v^2)}\right) = \left(z - \frac{\lambda^4 d}{r}, v\right). \tag{5.28}$$

We now carefully factorise the γ in (5.27) as a word in P, E and T so that (5.28) and automorphy can be used in (5.26). For each $\kappa = m + n\omega \in \mathbb{Z}[\omega]$, $m, n \in \mathbb{Z}$, let

$$A(\kappa) := PT^{-m}PT^{-m+n}P = \begin{pmatrix} 1 & \kappa \\ 0 & 1 \end{pmatrix}.$$

For each $r, b \in \mathbb{Z}[\omega]$ occurring in (5.27), let

$$W(r, b) := E^3 A(r)EA(b)EA(r) = \begin{pmatrix} b & -1 + br \\ 1 - br & 2r - br^2 \end{pmatrix}.$$

Then

$$W(r, b)E\gamma = \begin{pmatrix} -9d + br + 9bdr & -b - 9ab + b^2 r \\ r + 18dr - br^2 - 9bdr^2 & -9a + 2br + 9abr - b^2 r^2 \end{pmatrix} =: \widetilde{\gamma} \in \Gamma_1(9). \tag{5.29}$$

Equivalently,

$$\gamma = E^3 W(r, b)^{-1}\widetilde{\gamma}. \tag{5.30}$$

To obtain (5.31) immediately below we use (5.28), (5.30), (5.10), (5.11), and the fact that each F_j is automorphic on $\Gamma_1(9)$ with multiplier χ. For each $j = 1, 2, 3$,

$$\overline{F_j}\left(z - \frac{\lambda^4 d}{r}, v\right) = \overline{\chi}(g_j(E^3 W(r, b)^{-1})) \cdot \overline{\chi}(\widetilde{\gamma})$$

$$\times \overline{F_{k_j(E^3 W(r,b)^{-1})}}\left(\frac{\lambda^4 a}{r} - \frac{\overline{z}}{r^2(|z|^2 + v^2)}, \frac{v}{|r|^2(|z|^2 + v^2)}\right). \tag{5.31}$$

We claim that

$$k_1\big(E^3 W(r,b)^{-1}\big) = 1; \tag{5.32}$$

$$k_2\big(E^3 W(r,b)^{-1}\big) = 19; \tag{5.33}$$

$$k_3\big(E^3 W(r,b)^{-1}\big) = 10. \tag{5.34}$$

We will use (5.3) and the computations below to establish (5.32)–(5.34). Observe that

$$\gamma_1 E^3 W(r,b)^{-1} \gamma_1^{-1} = \begin{pmatrix} -1 + br & b \\ -2r + br^2 & -1 + br \end{pmatrix}$$

$$\equiv E^3 \pmod 3. \tag{5.35}$$

Thus $\gamma_1 E^3 W(r,b)^{-1} \gamma_1^{-1} \in \Gamma_2$, and (5.32) follows. Note that (5.33) (resp. (5.34)) follow similarly from

$$\gamma_2 E^3 W(r,b)^{-1} \gamma_{19}^{-1} = \begin{pmatrix} (\omega + 1)b - 2\omega r + \omega b r^2 & -\omega + b + \omega b r \\ -(\omega + 1) - 2r + (\omega + 1)br + br^2 & -1 + br \end{pmatrix}$$

$$\equiv T^{-1} E^3 \pmod 3, \tag{5.36}$$

(resp.)

$$\gamma_3 E^3 W(r,b)^{-1} \gamma_{10}^{-1} = \begin{pmatrix} \omega - \omega b + 2\omega r - \omega b r - \omega b r^2 & \omega + b - \omega b r \\ \omega - 2r - \omega b r + br^2 & -1 + br \end{pmatrix}$$

$$\equiv E^3 \pmod 3. \tag{5.37}$$

We now compute the automorphy factor $\overline{\chi}(g_j(E^3 W(r,b)^{-1})) \cdot \overline{\chi}(\widetilde{\gamma})$ for each $j = 1, 2, 3$ in (5.31). The displays (5.35)–(5.37) prove that

$$g_1(E^3 W(r,b)^{-1}) = E^3 W(r,b)^{-1};$$

$$g_2(E^3 W(r,b)^{-1}) = \gamma_2 E^3 W(r,b)^{-1} \gamma_{19}^{-1};$$

$$g_3(E^3 W(r,b)^{-1}) = \gamma_3 E^3 W(r,b)^{-1} \gamma_{10}^{-1},$$

respectively. In order to use either (5.4) or (5.5) to compute $\overline{\chi}(g_j(E^3 W(r,b)^{-1}))$ we note that $\chi|_{\mathrm{SL}_2(\mathbb{Z})} = 1$ and $E g_1(E^3 W(r,b)^{-1}), E T g_2(E^3 W(r,b)^{-1}), E g_3(E^3 W(r,b)^{-1}) \in \Gamma_1(3)$ (cf. (5.35)–(5.37)). We also repeatedly use the determinant equation $81ad + br = 1$ from (5.27). Now,

$$\overline{\chi}(g_1(E^3 W(r,b)^{-1})) = \overline{\chi}(E g_1(E^3 W(r,b)^{-1}))$$

$$= \begin{cases} \overline{\left(\frac{1-br}{b}\right)_3} & \text{if} \quad -1 + br \neq 0 \\ 1 & \text{otherwise} \end{cases}$$

$$= 1. \tag{5.38}$$

Similar computations yield

$$\overline{\chi}(g_3(E^3 W(r,b)^{-1})) = \overline{\chi}(E g_3(E^3 W(r,b)^{-1}))$$

$$= \begin{cases} \overline{\left(\frac{1-br}{\omega+b-\omega br}\right)_3} & \text{if} \quad \omega(1 - b + 2r - br - br^2) \neq 0 \\ 1 & \text{otherwise} \end{cases}$$

$$= 1, \tag{5.39}$$

and

$$\overline{\chi}(g_2(E^3 W(r,b)^{-1}))$$
$$= \overline{\chi}(ET g_2(E^3 W(r,b)^{-1}))$$
$$= \begin{cases} \overline{\left(\frac{1-br}{-\omega+b+\omega br+br-1}\right)_3} & \text{if} \quad (\omega+1)b - 2\omega r + \omega br^2 \\ & \qquad -(\omega+1) - 2r + (\omega+1)br + br^2 \neq 0 \\ 1 & \text{otherwise} \end{cases}$$
$$= 1. \tag{5.40}$$

We also have

$$\overline{\chi}(\widetilde{\gamma}) = \overline{\left(\frac{r + 18dr - br^2 - 9bdr^2}{-9d + br + 9bdr}\right)_3} = \overline{\left(\frac{81adr + 18dr - 9bdr^2}{-9d + br + 9bdr}\right)_3}$$
$$= \overline{\left(\frac{9d}{-9d + br + 9bdr}\right)_3} \cdot \overline{\left(\frac{9ar + 2r - br^2}{-9d + br + 9bdr}\right)_3}$$
$$= \overline{\left(\frac{9d}{1}\right)_3} \cdot \overline{\left(\frac{-9d + br + 9bdr}{9ar + 2r - br^2}\right)_3} \qquad \text{(by cubic reciprocity)}$$
$$= \overline{\left(\frac{-9d + br + 9bdr}{r}\right)} \cdot \overline{\left(\frac{-9d + br + 9bdr}{9a + 2 - br}\right)_3}$$
$$= \overline{\left(\frac{9d}{r}\right)_3} \cdot \overline{\left(\frac{1 - 81ad - 729ad^2}{9a + 1 + 81ad}\right)_3} = \overline{\left(\frac{9a}{r}\right)_3} \cdot \overline{\left(\frac{9d+1}{9a + 1 + 81ad}\right)_3}$$
$$= \overline{\left(\frac{9a}{r}\right)_3} \cdot \overline{\left(\frac{9d+1}{1}\right)_3} = \overline{\left(\frac{\lambda^4 a}{r}\right)_3}. \tag{5.41}$$

We combine (5.31)–(5.41) in (5.26). Note that $\overline{F_{19}} = \overline{F_{19}^\star}$ and $\overline{F_{10}} = \overline{F_{10}^\star}$ by (5.15). We then use the Fourier expansions (5.16) to open $\overline{F_1}$, $\overline{F_{19}^\star}$ and $\overline{F_{10}^\star}$, and assembling the sum over d (equivalently a) shows that

$$\overline{F}_1(w;\psi)_S = \frac{1}{3N(r)}\left(\overline{F_1} + \omega\overline{F_{19}^\star} + \omega^2\overline{F_{10}^\star}\right)\left(-\frac{\overline{z}}{r^2(|z|^2 + v^2)}, \frac{v}{|r|^2(|z|^2 + v^2)}; \Psi\right),$$

where

$$\Psi(u) := \sum_{\substack{d \pmod r \\ (d,r)=1 \\ (\lambda^4 d)(\lambda^4 a) \equiv 1 \pmod r}} \widehat{\psi}(d)\left(\frac{\lambda^4 a}{r}\right)_3 \check{e}\left(\frac{au}{r}\right), \quad u \in \mathbb{Z}[\omega]. \tag{5.42}$$

After opening $\widehat{\psi}(d)$ using the definition (5.21), and interchanging the order of summation, we readily see that $\Psi(u) = \widetilde{\psi}(u)$ for all $u \in \mathbb{Z}[\omega]$. ∎

For the coming lemma it will be instructive to open the definition of $\check{e}(\cdot)$,

$$\check{e}(\mu z) = e(\mu z + \overline{\mu z}), \quad z \in \mathbb{C}.$$

For $\ell \in \mathbb{Z}\backslash\{0\}$ and $1 \leqslant j \leqslant 27$, let

$$\overline{F_j}(w;\psi,\ell) = \overline{F_j^\star}(w;\psi,\ell) := \begin{cases} \sum_\mu \overline{d_j(-\mu)}\psi(\lambda^4\mu)\mu^\ell v K_{\frac{1}{3}}(4\pi|\mu|v)e(\mu z + \overline{\mu z}), & \text{if } \ell > 0 \\ \sum_\mu \overline{d_j(-\mu)}\psi(\lambda^4\mu)\overline{\mu}^{|\ell|} v K_{\frac{1}{3}}(4\pi|\mu|v)e(\mu z + \overline{\mu z}), & \text{if } \ell < 0 \end{cases}.$$

For $\ell = 0$, $\overline{F_j}(w;\psi,0) := \overline{F_j}(w;\psi)$ from before. We have

$$\overline{F_j}(w;\psi,\ell) = \frac{1}{(2\pi i)^{|\ell|}} \begin{cases} \left(\frac{\partial}{\partial z}\right)^\ell \overline{F_j}(w;\psi), & \text{if } \ell > 0 \\ \left(\frac{\partial}{\partial \overline{z}}\right)^{|\ell|} \overline{F_j}(w;\psi), & \text{if } \ell < 0, \end{cases} \qquad w = (z,v) \in \mathbb{H}^3. \qquad (5.43)$$

We apply differential operators in the proof of the next lemma. Thus we remind the reader that $\overline{F_j}(w;\psi,\ell) = \overline{G_j}(z,\overline{z},v;\psi,\ell)$ is a function of $z,\overline{z}$ and v, although the $\overline{F_j}$ notation suppresses this.

With these observations in mind we deduce the following Corollary.

Corollary 5.1. *Suppose $r \in \mathbb{Z}[\omega]$ with $r \equiv 1 \pmod 3$, and ψ is a sequence on $\mathbb{Z}[\omega]$ that is periodic modulo r. Suppose that $\widehat{\psi}$ is supported only on residue classes coprime to r. By abuse of notation, write $F_j((0,v);\psi,\ell)$ as $F_j(v;\psi,\ell)$ for all $v > 0$, $1 \leqslant j \leqslant 27$ and $\ell \in \mathbb{Z}$. Then we have*

$$\overline{F_1}(v;\psi,\ell)_S = \frac{(-1)^\ell}{3N(r)^{1+|\ell|}v^{2|\ell|}}\left(\frac{\overline{r}}{r}\right)^{-\ell}\left(\delta_{\ell\neq 0}\overline{F_1^\star} + \delta_{\ell=0}\overline{F_1} + \omega\overline{F_{19}^\star} + \omega^2\overline{F_{10}^\star}\right)\left(\frac{1}{|r|^2 v};\widetilde{\psi},-\ell\right),$$

or equivalently,

$$\left(\overline{F_1^\star}\delta_{\ell\neq 0} + \overline{F_1}\delta_{\ell=0} + \omega\overline{F_{19}^\star} + \omega^2\overline{F_{10}^\star}\right)(v;\widetilde{\psi},-\ell) = \frac{3\cdot(-1)^\ell}{N(r)^{|\ell|-1}v^{2|\ell|}}\left(\frac{\overline{r}}{r}\right)^\ell \overline{F_1}\left(\frac{1}{|r|^2 v};\psi,\ell\right).$$

Proof. Setting $z = 0$ in (5.25) gives the claim when $\ell = 0$. If $\ell > 0$, we write $|z|^2 = z\overline{z}$ and apply the operator $\frac{1}{(2\pi i)^\ell}\left(\frac{\partial}{\partial z}\right)^\ell\big|_{z=0}$ to both sides of (5.25). A computation with the chain rule yields

$$\overline{F_1}((0,v);\psi,\ell)_S = \frac{(-1)^\ell}{3N(r)^{1+\ell}v^{2\ell}}\left(\frac{\overline{r}}{r}\right)^{-\ell}\cdot\left(\overline{F_1^\star} + \omega\overline{F_{19}^\star} + \omega^2\overline{F_{10}^\star}\right)\left(\left(0,\frac{1}{|r|^2 v}\right);\widetilde{\psi},-\ell\right).$$

When $\ell < 0$ a similar argument with the operator $\frac{1}{(2\pi i)^{|\ell|}}\left(\frac{\partial}{\partial \overline{z}}\right)^{|\ell|}\big|_{z=0}$ yields the analogous result. ∎

5.4. Poles and Dirichlet series. Let ψ be as in Corollary 5.1. For $\text{Re}(s) > 1$, $\ell \in \mathbb{Z}$ and $1 \leqslant j \leqslant 27$, consider the family of Dirichlet series

$$\mathcal{D}(s,\overline{F_j^\star};\psi,\ell) := \sum_\mu \frac{\overline{d_j(-\mu)}\psi(\lambda^4\mu)\left(\frac{\mu}{|\mu|}\right)^\ell}{N(\mu)^s};$$

$$\mathcal{D}(s,\overline{F_1};\psi,\ell)_S := \sum_{\mu\in S} \frac{\overline{\tau(\mu)}\psi(\lambda^4\mu)\left(\frac{\mu}{|\mu|}\right)^\ell}{N(\mu)^s}. \qquad (5.44)$$

For $\mathrm{Re}(s) > 1$, we introduce the integral transforms

$$\Lambda(s, \overline{F_j^\star}; \psi, \ell) := \int_0^\infty \overline{F_j^\star}(v; \psi, \ell) v^{2s+|\ell|-2} dv;$$

$$\Lambda(s, \overline{F_1}; \psi, \ell)_S := \int_0^\infty \overline{F_1}(v; \psi, \ell)_S v^{2s+|\ell|-2} dv,$$

where by abuse of notation we wrote $F_i(v; \psi, \ell) = F_i((0, v); \psi, \ell)$. In the case $\ell = 0$ we will omit the index ℓ from the notation.

Lemma 5.3. *For* $\mathrm{Re}(s) > 1$ *we have*

$$\Lambda(s, \overline{F_j^\star}; \psi, \ell) = \frac{1}{4}(2\pi)^{-2s-|\ell|}\Gamma\left(s + \frac{|\ell|}{2} - \frac{1}{6}\right)\Gamma\left(s + \frac{|\ell|}{2} + \frac{1}{6}\right)\mathcal{D}(s, \overline{F_j^\star}; \psi, \ell);$$

$$\Lambda(s, \overline{F_1}; \psi, \ell)_S = \frac{1}{4}(2\pi)^{-2s-|\ell|}\Gamma\left(s + \frac{|\ell|}{2} - \frac{1}{6}\right)\Gamma\left(s + \frac{|\ell|}{2} + \frac{1}{6}\right)\mathcal{D}(s, \overline{F_1}; \psi, \ell)_S.$$

Proof. The proofs of both identities are virtually identical, so we prove the latter, with $\ell < 0$. For $\mathrm{Re}(s) > 1$ we have

$$\Lambda(s, \overline{F_1}; \psi, \ell)_S = \int_0^\infty \sum_{\mu \in S} \overline{\tau(\mu)}\psi(\lambda^4\mu)\overline{\mu}^{|\ell|}K_{\frac{1}{3}}(4\pi|\mu|v)v^{2s+|\ell|-1}dv$$

$$= \frac{1}{(4\pi)^{2s+|\ell|}}\sum_{\mu \in S}\frac{\overline{\tau(\mu)}\psi(\lambda^4\mu)\left(\frac{\overline{\mu}}{|\mu|}\right)^{|\ell|}}{N(\mu)^s}\int_0^\infty K_{\frac{1}{3}}(T)T^{2s+|\ell|-1}dT$$

$$= \frac{1}{4}(2\pi)^{-2s-|\ell|}\Gamma\left(s + \frac{|\ell|}{2} - \frac{1}{6}\right)\Gamma\left(s + \frac{|\ell|}{2} + \frac{1}{6}\right)\sum_{\mu \in S}\frac{\overline{\tau(\mu)}\psi(\lambda^4\mu)\left(\frac{\mu}{|\mu|}\right)^\ell}{N(\mu)^s}.$$

$$(5.45)$$

The interchange of summation and integration above for $\mathrm{Re}(s) > 1$ is justified by absolute convergence (cf. [DLMF, (10.25.3),(10.30.2)]). Furthermore, (5.45) follows from [DLMF, (10.43.19)]. ∎

Proposition 5.1. *The completed Dirichlet series* $\Lambda(\overline{F_1}, s; \psi)_S$ *admits a meromorphic continuation to the whole complex plane* $\mathbb{C}$. *It has a unique pole (that is simple) at* $s = 5/6$, *with residue*

$$\underset{s=\frac{5}{6}}{\mathrm{Res}}\, \Lambda(s, \overline{F_1}; \psi)_S = \frac{\sigma\widetilde{\psi}(0)}{6N(r)^{5/3}},$$

$$(5.46)$$

where $\sigma = 3^{5/2}/2$ *is as in (5.6). In particular,*

$$\underset{s=\frac{5}{6}}{\mathrm{Res}}\, \mathcal{D}(s, \overline{F_1}; \psi)_S = \frac{2(2\pi)^{5/3}\sigma\widetilde{\psi}(0)}{3\Gamma(\frac{2}{3})N(r)^{5/3}}.$$

$$(5.47)$$

For $\ell \neq 0$ *the Dirichlet series* $\Lambda(\overline{F_1}, s; \psi, \ell)_S$ *is entire. Moreover, for all* $\ell \in \mathbb{Z}$ *we have the functional equation*

$$3(-1)^\ell N(r)^{2s}\left(\frac{\overline{r}}{r}\right)^\ell \Lambda(s, \overline{F_1}; \psi, \ell)_S = \Lambda(1 - s, \overline{F_1^\star} + \omega\overline{F_{19}^\star} + \omega^2\overline{F_{10}^\star}; \widetilde{\psi}, -\ell).$$

$$(5.48)$$

This functional equation also determines the poles of $\Lambda(s, \overline{F_1^\star} + \omega\overline{F_{19}^\star} + \omega^2\overline{F_{10}^\star}; \widetilde{\psi})$.

Proof. For $\mathrm{Re}\, s > 1$ we have

$$\Lambda(s, \overline{F_1}; \psi, \ell)_S = \int_0^{N(r)^{-1}} \overline{F_1}(v; \psi, \ell)_S v^{2s+|\ell|-2} dv + \int_{N(r)^{-1}}^{\infty} \overline{F_1}(v; \psi, \ell)_S v^{2s+|\ell|-2} dv. \quad (5.49)$$

When $\ell = 0$ observe that $\overline{F_1}(v; \psi, \ell)_S$ has exponential decay at ∞ by (5.17) and (5.24). The same claim holds when $\ell \neq 0$ using (5.43) (termwise differentiation of the Fourier series) and the same reasoning as before. Thus the second integral in (5.49) has analytic continuation to an entire function.

Let

$$\overline{G}(w; \widetilde{\psi}, \ell) := \begin{cases} (\overline{F_1} + \omega \overline{F_{19}^\star} + \omega^2 \overline{F_{10}^\star})(w; \widetilde{\psi}, \ell) & \text{if } \ell = 0 \\ (\overline{F_1^\star} + \omega \overline{F_{19}^\star} + \omega^2 \overline{F_{10}^\star})(w; \widetilde{\psi}, \ell) & \text{if } \ell \neq 0 \end{cases}, \quad w \in \mathbb{H}^3. \quad (5.50)$$

Using (5.50) and Corollary 5.1 gives

$$\int_0^{N(r)^{-1}} \overline{F_1}(v; \psi, \ell)_S v^{2s+|\ell|-2} dv$$

$$= \frac{(-1)^\ell}{3N(r)^{1+|\ell|}} \left(\frac{\overline{r}}{r}\right)^{-\ell} \int_0^{N(r)^{-1}} \overline{G}\left(\frac{1}{v|r|^2}; \widetilde{\psi}, -\ell\right) v^{2s-|\ell|-2} dv \quad (5.51)$$

$$= \frac{\sigma \widetilde{\psi}(0) N(r)^{-2s}}{6} \cdot \frac{\delta_{\ell=0}}{s - \frac{5}{6}}$$

$$+ \frac{(-1)^\ell N(r)^{-2s}}{3} \left(\frac{\overline{r}}{r}\right)^{-\ell} \int_1^{\infty} \left(\overline{F_1^\star} + \omega \overline{F_{19}^\star} + \omega^2 \overline{F_{10}^\star}\right)(v; \widetilde{\psi}, -\ell) v^{|\ell|-2s} dv. \quad (5.52)$$

When $\ell = 0$ observe that $(\overline{F_1^\star} + \omega \overline{F_{19}^\star} + \omega^2 \overline{F_{10}^\star})(\cdot, \widetilde{\psi}, -\ell)$ has exponential decay at ∞ by (5.19), Appendix A (the expressions for $d_j(\mu)$), (5.7), (5.13), and (5.14). The same claim holds when $\ell \neq 0$ using (5.43) (termwise differentiation of the Fourier series) and the same reasoning as before. Thus the integral in (5.52) has analytic continuation to an entire function. This gives the meromorphicity and entirety claims in the Proposition, as well as (5.46). Observe that (5.47) follows from (5.46) and Lemma 5.3.

We now prove the functional equation (5.48). From (5.49) and (5.51) we found that

$$\Lambda(s, \overline{F_1}; \psi, \ell)_S = \frac{(-1)^\ell N(r)^{-2s}}{3} \left(\frac{\overline{r}}{r}\right)^{-\ell} \int_1^{\infty} \overline{G}(v; \widetilde{\psi}, -\ell) v^{|\ell|-2s} dv + \int_{N(r)^{-1}}^{\infty} \overline{F_1}(v; \psi, \ell)_S v^{2s+|\ell|-2} dv. \quad (5.53)$$

We now repeat a similar argument, but instead start with

$$\overline{G^\star}(w; \widetilde{\psi}, \ell) = (\overline{F_1^\star} + \omega \overline{F_{19}^\star} + \omega^2 \overline{F_{10}^\star})(w; \widetilde{\psi}, \ell), \quad \text{for all} \quad w \in \mathbb{H}^3 \quad \text{and} \quad \ell \in \mathbb{Z}. \quad (5.54)$$

For $\mathrm{Re}\, s > 1$ we have

$$\Lambda(s, \overline{G^\star}; \widetilde{\psi}, -\ell) = \int_0^1 \overline{G^\star}(v; \widetilde{\psi}, -\ell) v^{2s+|\ell|-2} dv + \int_1^{\infty} \overline{G^\star}(v; \widetilde{\psi}, -\ell) v^{2s+|\ell|-2} dv. \quad (5.55)$$

For $\mathrm{Re}\, s > 1$ we have

$$\int_0^1 \overline{G^\star}(v; \widetilde{\psi}, -\ell) v^{2s+|\ell|-2} dv = \int_0^1 \overline{G}(v; \widetilde{\psi}, -\ell) v^{2s+|\ell|-2} dv - \delta_{\ell=0} \cdot \frac{3\sigma \widetilde{\psi}(0)}{6s - 1}. \quad (5.56)$$

Then (5.56) holds for all $s \in \mathbb{C}$ by meromorphic continuation. Similarly, for $\mathrm{Re}\, s < -1$, we have

$$\int_1^\infty \overline{G^\star}(v; \widetilde{\psi}, -\ell) v^{2s+|\ell|-2} dv = \int_1^\infty \overline{G}(v; \widetilde{\psi}, -\ell) v^{2s+|\ell|-2} dv + \delta_{\ell=0} \frac{3\sigma \widetilde{\psi}(0)}{6s - 1}. \qquad (5.57)$$

Then (5.57) holds for all $s \in \mathbb{C}$ by meromorphic continuation. Insertion of (5.56) and (5.57) into (5.55) gives

$$\Lambda(s, \overline{G^\star}; \widetilde{\psi}, -\ell) = \int_0^1 \overline{G}(v; \widetilde{\psi}, -\ell) v^{2s+|\ell|-2} dv + \int_1^\infty \overline{G}(v; \widetilde{\psi}, -\ell) v^{2s+|\ell|-2} dv, \qquad (5.58)$$

where both integrals are to be interpreted as the meromorphic continuations of the original integrals. Using Corollary 5.1 we obtain

$$\int_0^1 \overline{G}(v; \widetilde{\psi}, -\ell) v^{2s+|\ell|-2} dv = 3(-1)^\ell N(r)^{1-|\ell|} \left(\frac{\overline{r}}{r}\right)^\ell \int_0^1 \overline{F_1}\left(\frac{1}{|r|^2 v}; \psi, \ell\right)_S v^{2s-|\ell|-2} dv$$

$$= 3(-1)^\ell N(r)^{2-2s} \left(\frac{\overline{r}}{r}\right)^\ell \int_{N(r)^{-1}}^\infty \overline{F_1}(v; \psi, \ell)_S v^{|\ell|-2s} dv. \qquad (5.59)$$

Substitution of (5.59) into (5.58) gives

$$\Lambda(s, \overline{G^\star}; \widetilde{\psi}, -\ell) = 3(-1)^\ell N(r)^{2-2s} \left(\frac{\overline{r}}{r}\right)^\ell \int_{N(r)^{-1}}^\infty \overline{F_1}(v; \psi, \ell)_S v^{|\ell|-2s} dv + \int_1^\infty \overline{G}(v; \widetilde{\psi}, -\ell) v^{2s+|\ell|-2} dv.$$

Equivalently,

$$\Lambda(1-s, \overline{G^\star}; \widetilde{\psi}, -\ell) = 3(-1)^\ell N(r)^{2s} \left(\frac{\overline{r}}{r}\right)^\ell \int_{N(r)^{-1}}^\infty \overline{F_1}(v; \psi, \ell)_S v^{2s+|\ell|-2} dv + \int_1^\infty \overline{G}(v; \widetilde{\psi}, -\ell) v^{|\ell|-2s} dv.$$

$$(5.60)$$

After combining (5.53) and (5.60) we obtain

$$3(-1)^\ell N(r)^{2s} \left(\frac{\overline{r}}{r}\right)^\ell \Lambda(s, \overline{F_1}; \psi, \ell)_S = \Lambda(1 - s, \overline{G^\star}; \widetilde{\psi}, -\ell),$$

as required. ∎

5.5. Sieving for $\widetilde{g}(c)$. Let $r \in \mathbb{Z}[\omega]$ satisfy $r \equiv 1 \pmod 3$, ψ be a primitive character to modulus r, and $\ell \in \mathbb{Z}$. Then let

$$\zeta_{\mathbb{Q}(\omega)}(s; \psi, \ell) := \sum_{d \equiv 1 \pmod 3} \frac{\psi(d) \left(\frac{d}{|d|}\right)^\ell}{N(d)^s}, \quad \mathrm{Re}(s) > 1.$$

In the case $\ell = 0$ we omit ℓ from the notation and write $\zeta_{\mathbb{Q}(\omega)}(s; \psi)$. We denote the principal character modulo r by $\mathbf{1}_r$. Any $J \trianglelefteq \mathbb{Z}[\omega]$ with $(J, 3) = 1$ has a unique generator $d \equiv 1 \pmod 3$. Thus when $\ell = 0$ and $\psi = \mathbf{1}_r$, the above L-function coincides with the Dedekind ζ-function of $\mathbb{Q}(\omega)$, except at the local factors of primes dividing (λr). Note that $\zeta_{\mathbb{Q}(\omega)}(s; \psi, \ell)$ has standard meromorphic continuation to all of $\mathbb{C}$; the only case when this function is not holomorphic is when $\psi = \mathbf{1}_r$ is the principal character and $\ell = 0$, in that case there is a unique simple pole at $s = 1$. Standard functional equations for these L-functions can be found in [Miy89, §3.3].

Lemma 5.4. *Let $r \in \mathbb{Z}[\omega]$ with $r \equiv 1 \pmod 3$, and ψ be a primitive cubic Dirichlet character on $\mathbb{Z}[\omega]$ to modulus r. For $\ell \in \mathbb{Z}$ and $\operatorname{Re} s > 1$ we have*

$$(-1)^\ell i^{-\ell} \sum_{c \equiv 1 \pmod 3} \frac{\tilde{g}(c)\psi(\lambda c)\left(\frac{c}{|c|}\right)^\ell}{N(c)^s} = \frac{3^{-5/2-3s} \cdot \mathcal{D}(s,\overline{F_1};\psi,\ell)_S}{\zeta_{\mathbb{Q}(\omega)}(3s - \frac{1}{2}; \mathbf{1}_r, 3\ell)}, \qquad (5.61)$$

where $\mathcal{D}(s,\overline{F_1};\psi,\ell)_S$ is as in (5.44).

Remark 5.4. Note that we have abused notation in the results and proofs that follow: $\mu \in \mathbb{Q}(\omega)$ is used to index Fourier coefficients of various automorphic forms, and $\mu(\cdot)$ denotes the Möbius function on $\mathbb{Z}[\omega]$. Meanings should be clear from context.

Proof. The Dirichlet coefficients on the right side on (5.61) have support contained in $\mathbb{Z}[\omega]\backslash\{0\}$ since $3^{-3s} = N(\lambda^3)^{-s}$. The νth Dirichlet coefficient on the right side of (5.61) is given by

$$\frac{1}{3^{5/2}}\left(\frac{\nu/\lambda^3}{|\nu/\lambda^3|}\right)^\ell \sum_{\substack{\mu \in S \\ d \equiv 1 \pmod 3 \\ (d,r)=1 \\ \nu = \lambda^3 \mu d^3}} \overline{\tau(\mu)}\psi(\lambda^4\mu)\mu(d)|d|. \qquad (5.62)$$

Recall the definition of S in (5.17). If $\nu \notin \lambda^3 S$, then (5.62) is zero. Therefore we can assume that $\nu \in \lambda^3 S$. If $\mu \in S$ and $\tau(\mu) \neq 0$, then by (5.7) we must have

$$\overline{\tau(\mu)} = 3^{5/2}\tilde{g}(e)|f|,$$

where

$$\mu = \lambda^{-3}ef^3 \quad \text{for some} \quad e,f \equiv 1 \pmod 3 \quad \text{and} \quad \mu^2(e) = 1.$$

Thus (5.62) is equal to

$$(-1)^\ell i^{-\ell}\left(\frac{\nu}{|\nu|}\right)^\ell \sum_{\substack{e,f,d \equiv 1 \pmod 3 \\ (d,r)=1 \\ \nu = e(df)^3}} \tilde{g}(e)\psi(\lambda ef^3)\mu(d)|d||f|$$

$$= (-1)^\ell i^{-\ell}\left(\frac{\nu}{|\nu|}\right)^\ell \sum_{\substack{e,f,d \equiv 1 \pmod 3 \\ (df,r)=1 \\ \nu = e(df)^3}} \tilde{g}(e)\psi(\lambda e)\mu(d)|d||f|, \qquad (5.63)$$

where the last display follows from the assumption that ψ is a primitive cubic character to modulus r. Note that it is redundant to have $\mu^2(e) = 1$ in (5.63) because this condition is automatically captured by (2.2). Möbius inversion then tells us that the right side of (5.63) is equal to

$$(-1)^\ell i^{-\ell}\left(\frac{\nu}{|\nu|}\right)^\ell \sum_{\substack{e,u \equiv 1 \pmod 3 \\ (u,r)=1 \\ \nu = eu^3}} \tilde{g}(e)\psi(\lambda e)|u|\left(\sum_{\substack{d|u \\ d \equiv 1 \pmod 3}} \mu(d)\right)$$

$$= (-1)^\ell i^{-\ell}\left(\frac{e}{|e|}\right)^\ell \tilde{g}(e)\psi(\lambda e),$$

as required.   ∎

The following lemma records the standard evaluation of Ramanujan sums over $\mathbb{Z}[\omega]$. The lemma follows from the evaluation

$$c_\varpi(u) = \begin{cases} \varphi(\varpi) & \text{if } u \equiv 0 \pmod{\varpi} \\ -1 & \text{otherwise} \end{cases},$$

$$\text{for } \varpi \equiv 1 \pmod 3 \text{ prime and } u \in \mathbb{Z}[\omega],$$

and the multiplicativity of Ramanujan sums i.e.

$$c_{mn}(u) = c_m(u)c_n(u) \quad \text{for} \quad u, m, n \in \mathbb{Z}[\omega] \quad \text{with} \quad m, n \equiv 1 \pmod 3$$
$$\text{and} \quad (m, n) = 1.$$

Lemma 5.5. *Let $m \in \mathbb{Z}[\omega]$ be squarefree and satisfy $m \equiv 1 \pmod 3$. Then for $u \in \mathbb{Z}[\omega]$,*

$$c_m(u) := \sum_{\substack{x \pmod m \\ (x,m)=1}} \check{e}\left(\frac{ux}{m}\right) = \mu\left(\frac{m}{(m,u)}\right) \frac{\varphi(m)}{\varphi\left(\frac{m}{(m,u)}\right)},$$

where $\varphi(\cdot)$ is the Euler phi function on $\mathbb{Z}[\omega]$.

Let $r \in \mathbb{Z}[\omega]$ be squarefree and satisfy $r \equiv 1 \pmod 3$. We are now able to study the the analytic properties of the Dirichlet series

$$\sum_{c \equiv 1 \pmod 3} \frac{\tilde{g}(c)\overline{\left(\frac{c}{r}\right)_3}\left(\frac{c}{|c|}\right)^\ell}{N(c)^s}, \quad \operatorname{Re}(s) > 1. \tag{5.64}$$

The following result records a level aspect (i.e. in the conductor of the cubic twist in (5.64)) functional equation for (5.64) that generalises [Pat77, Theorem 6.1]. It explicates the root number and level, and is crucial to our paper. Yoshimoto [Yos87] established level aspect analogues of [Pat77, Theorem 6.1] for twists of Gauss sums by arbitrary non-cubic Dirichlet characters. Clearly, Yoshimoto's results do not cover the case we need.

Remark 5.5. The functional equation in [Pat77, Theorem 6.1] could be potentially used to obtain a functional equation for (5.64) similar to the one presented below in Proposition 5.2 (at least when $(r, c) = 1$). Patterson exploited the fact that (5.64) (and its variants) are the Fourier coefficients of cubic metaplectic Eisenstein series attached to the essential cusps of $\Gamma_1(3)$ with respect to χ. The drawback of the functional equation in [Pat77, Theorem 6.1] is that the dual side of is a linear combination of variants of Dirichlet series of twisted cubic Gauss sums that involve the ramified prime, and this obscures the root number. We found it advantageous to work directly with the automorphy of the cubic theta function (this is a more advanced starting point since the purpose of [Pat77] was to compute the Fourier coefficients of the cubic theta function).

Proposition 5.2. *Let $r \in \mathbb{Z}[\omega]$ be squarefree and satisfy $r \equiv 1 \pmod 3$, and $\psi := \overline{\left(\frac{\cdot}{r}\right)_3}$. Let $\ell \in \mathbb{Z}$. Then the Dirichlet series*

$$\mathcal{R}(s; \psi, \ell) := \sum_{c \equiv 1 \pmod 3} \frac{\tilde{g}(c)\overline{\left(\frac{c}{r}\right)_3}\left(\frac{c}{|c|}\right)^\ell}{N(c)^s}, \quad \operatorname{Re}(s) > 1, \tag{5.65}$$

admits meromorphic continuation to all of $\mathbb{C}$. *If* $\ell = 0$, *the Dirichlet series*

$$\zeta_{\mathbb{Q}(\omega)}(3s - \tfrac{1}{2}; \mathbf{1}_r, 3\ell)\mathcal{R}(s; \psi, \ell) \tag{5.66}$$

has a unique pole located at $s = \tfrac{5}{6}$, *and it is simple. If* $\ell \neq 0$ *the Dirichlet series* (5.66) *defines an entire function. We have*

$$Res_{s=\frac{5}{6}}\left(\zeta_{\mathbb{Q}(\omega)}(3s - \tfrac{1}{2}; \mathbf{1}_r)\mathcal{R}(s; \psi)\right) = \frac{(2\pi)^{5/3}\overline{g(r)}\varphi(r)}{\Gamma(\frac{2}{3})3^{7/2}N(r)^{5/3}},$$

and for $\ell \in \mathbb{Z}$ *we have the functional equation*

$$\zeta_{\mathbb{Q}(\omega)}(3s - \tfrac{1}{2}; \mathbf{1}_r, 3\ell)\mathcal{R}(s; \psi, \ell)$$
$$= \overline{g(r)}\left(\frac{\overline{r}}{r}\right)^{-\ell} \cdot \frac{i^\ell (2\pi)^{4s-2}}{3^{7/2}N(r)^{2s}} \frac{\Gamma(\frac{5}{6} + \frac{|\ell|}{2} - s)\Gamma(\frac{7}{6} + \frac{|\ell|}{2} - s)}{\Gamma(s + \frac{|\ell|}{2} - \frac{1}{6})\Gamma(s + \frac{|\ell|}{2} + \frac{1}{6})}$$
$$\times \zeta_{\mathbb{Q}(\omega)}(\tfrac{5}{2} - 3s; \mathbf{1}_r, -3\ell)\mathcal{R}_r^\dagger(1 - s; -\ell),$$

where

$$\mathcal{R}_r^\dagger(s; \ell) := (-1)^\ell i^{-\ell} \sum_{\nu \in \lambda^{-1}\mathbb{Z}[\omega]} \frac{a^\dagger(\nu)b_r^\dagger(\nu)\left(\frac{\nu}{|\nu|}\right)^\ell}{N(\nu)^s}, \quad \mathrm{Re}(s) > 1, \quad \ell \in \mathbb{Z},$$

for some Dirichlet coefficients $a^\dagger(\nu)$. *The coefficients* $a^\dagger(\nu)$ *have support contained in the set*

$$Q^\dagger := \big\{\nu = \lambda^L \zeta h w h'^3 : L \in \mathbb{Z}_{\geqslant -1}, \quad \zeta \in \{\pm 1, \pm \omega, \pm \omega^2\},$$
$$h, h', w \equiv 1 \pmod{3}, \quad h, h' \mid r^\infty, \quad (w, r) = 1 \quad and \quad \mu^2(hw) = 1\big\},$$

and for $\nu \in Q^\dagger$,

$$a^\dagger(\nu) = \begin{cases} \overline{\tau(-\lambda^{L-3}\nu)} & if \quad L \geqslant 0 \\ \omega^2\overline{\tau_1(-\lambda^{L-3}\omega^2\nu)}\check{e}(\lambda^{L-3}\nu) & if \quad L = -1, \quad \zeta \in \{-1, -\omega, -\omega^2\}, \\ \omega\overline{\tau_2(-\lambda^{L-3}\omega\nu)}\check{e}(\lambda^{L-3}\nu) & if \quad L = -1, \quad \zeta \in \{1, \omega, \omega^2\} \end{cases} \tag{5.67}$$

and

$$b_r^\dagger(\nu) = \mu\left(\frac{r}{(\lambda\nu, r)}\right)\frac{\varphi(r)}{\varphi\left(\frac{r}{(\lambda\nu,r)}\right)}. \tag{5.68}$$

Remark 5.6. Recall that $\tau(\cdot), \tau_1(\cdot)$ and $\tau_2(\cdot)$ are given in (5.7), (5.13) and (5.14) respectively.

Proof. Meromorphic continuation of $\zeta_{\mathbb{Q}(\omega)}(3s - \tfrac{1}{2}; \mathbf{1}_r, 3\ell)\mathcal{R}(s; \psi, \ell)$ to all of $\mathbb{C}$ follows from Lemma 5.3, Proposition 5.1 and Lemma 5.4. If $\ell \neq 0$, then it is entire. If $\ell = 0$, then it has a unique simple pole at $s = 5/6$ with residue

$$\mathrm{Res}_{s=\frac{5}{6}}\left(\zeta_{\mathbb{Q}(\omega)}(3s - \tfrac{1}{2}; \mathbf{1}_r)\mathcal{R}(s; \psi)\right) = \left(\frac{\lambda}{r}\right)_3 \frac{(2\pi)^{5/3}\widetilde{\psi}(0)}{3^{7/2}\Gamma(\frac{2}{3})N(r)^{5/3}}.$$

We now evaluate $\widetilde{\psi}(u)$ (defined in (5.22)). Recall from (5.42) and the argument following it that we have

$$\widetilde{\psi}(u) = \left(\frac{\lambda}{r}\right)_3 \sum_{\substack{d \pmod r \\ (d,r)=1 \\ (\lambda^4 d)(\lambda^4 a) \equiv 1 \pmod r}} \widehat{\psi}(d)\left(\frac{a}{r}\right)_3 \breve{e}\left(\frac{au}{r}\right), \quad u \in \mathbb{Z}[\omega]. \tag{5.69}$$

Moreover, using the definition (5.21) and the fact that ψ is primitive gives us

$$\widehat{\psi}(d) = \left(\frac{d}{r}\right)_3 \overline{g(r)}.$$

We have $ad \equiv \overline{\lambda^8} \pmod r$ in (5.69). Therefore

$$\widetilde{\psi}(u) = \left(\frac{\lambda^2}{r}\right)_3 \overline{g(r)}c_r(u), \tag{5.70}$$

where $c_r(\cdot)$ denotes the usual Ramanujan sum. In particular,

$$\widetilde{\psi}(0) = \left(\frac{\lambda^2}{r}\right)_3 \overline{g(r)}\varphi(r).$$

Lemma 5.4 tells us that

$$\zeta_{\mathbb{Q}(\omega)}(3s - \tfrac{1}{2}; \mathbf{1}_r, 3\ell)\mathcal{R}(s; \psi, \ell) = (-1)^\ell i^\ell \left(\frac{\lambda}{r}\right)_3 3^{-5/2-3s}\mathcal{D}(s, \overline{F_1}; \psi, \ell)_S.$$

Thus Lemma 5.3 and Proposition 5.1 imply that

$$\zeta_{\mathbb{Q}(\omega)}(3s - \tfrac{1}{2}; \mathbf{1}_r, 3\ell)\mathcal{R}(s; \psi, \ell)$$
$$= \left(\frac{\overline{r}}{r}\right)^{-\ell} \frac{i^\ell (2\pi)^{4s-2}}{3^{7/2}N(r)^{2s}} \frac{\Gamma(\tfrac{5}{6} + \tfrac{|\ell|}{2} - s)\Gamma(\tfrac{7}{6} + \tfrac{|\ell|}{2} - s)}{\Gamma(s + \tfrac{|\ell|}{2} - \tfrac{1}{6})\Gamma(s + \tfrac{|\ell|}{2} + \tfrac{1}{6})}\zeta_{\mathbb{Q}(\omega)}\left(\tfrac{5}{2} - 3s; \mathbf{1}_r, -3\ell\right)$$
$$\times \left(\frac{\lambda}{r}\right)_3 3^{-3s}\frac{\mathcal{D}(1-s, \overline{G^\star}; \widetilde{\psi}, -\ell)}{\zeta_{\mathbb{Q}(\omega)}(\tfrac{5}{2} - 3s; \mathbf{1}_r, -3\ell)}, \tag{5.71}$$

where $\overline{G^\star}(w; \widetilde{\psi}, \ell)$ is as in (5.54). Observe that (5.70) gives

$$\left(\frac{\lambda}{r}\right)_3 3^{-3s}\frac{\mathcal{D}(s, \overline{G^\star}; \widetilde{\psi}, -\ell)}{\zeta_{\mathbb{Q}(\omega)}(3s - \tfrac{1}{2}; \mathbf{1}_r, -3\ell)} = \overline{g(r)}\mathcal{R}_r^\dagger(s, -\ell),$$

where

$$\mathcal{R}_r^\dagger(s; \ell) := 3^{-3s}\frac{\mathcal{D}(s, \overline{G^\star}; c_r(\cdot), \ell)}{\zeta_{\mathbb{Q}(\omega)}(3s - \tfrac{1}{2}; \mathbf{1}_r, 3\ell)}, \quad \ell \in \mathbb{Z}. \tag{5.72}$$

We now analyse the Dirichlet coefficients of $\mathcal{R}_r^\dagger(s; -\ell)$. Let $a_{-\ell}^\dagger(\nu)$ and $a_{-\ell}^\star(\nu)$ be the Fourier coefficients of $\mathcal{R}_r^\dagger(s; -\ell)$ and $\mathcal{D}(s, \overline{G^\star}; -\ell)$ respectively. Using the definition of $G^\star(w; \widetilde{\psi}, \ell)$ in (5.54) and Appendix A for the closed form expressions for $d_1(\mu), d_{19}(\mu)$,

and $d_{10}(\mu)$ gives

$$a^{\star}_{-\ell}(\mu) = \left(\frac{\mu}{|\mu|}\right)^{-\ell}\left(\overline{d_1(-\mu)} + \omega\overline{d_{19}(-\mu)} + \omega^2\overline{d_{10}(-\mu)}\right)$$

$$= \left(\frac{\mu}{|\mu|}\right)^{-\ell}\left(\overline{\tau(-\mu)} + \omega^2\overline{\tau_1(-\omega^2\mu)}\check{e}(\mu) + \omega\overline{\tau_2(-\omega\mu)}\check{e}(\mu)\right). \qquad (5.73)$$

Consultation with (5.7), (5.13) and (5.14) shows that the $a^{\star}(\mu)$ have support contained in the set

$$U := \left\{\mu = \lambda^k\zeta cj^3 : k \in \mathbb{Z}_{\geqslant -4}, \quad \zeta \in \{\pm 1, \pm\omega, \pm\omega^2\}, \quad c, j \equiv 1 \pmod{3} \quad \text{and} \quad \mu^2(c) = 1\right\}.$$

Each of the three terms in (5.73) have disjoint support. In particular,

$$a^{\star}_{-\ell}(\mu) = \left(\frac{\mu}{|\mu|}\right)^{-\ell} \cdot \begin{cases} \overline{\tau(-\mu)} & \text{if } k \geqslant -3 \\ \omega^2\overline{\tau_1(-\omega^2\mu)}\check{e}(\mu) & \text{if } k = -4, \quad \zeta \in \{-1, -\omega, -\omega^2\} \\ \overline{\omega\tau_2(-\omega\mu)}\check{e}(\mu) & \text{if } k = -4, \quad \zeta \in \{1, \omega, \omega^2\} \end{cases}$$

$$=: \left(\frac{\mu}{|\mu|}\right)^{-\ell}a^{\star}(\mu). \qquad (5.74)$$

Observe that (5.72) and (5.73) imply that the coefficients $a^{\dagger}(\cdot)$ have support contained in $\lambda^3 U \subset \lambda^{-1}\mathbb{Z}[\omega]\backslash\{0\}$. Then

$$a^{\dagger}_{-\ell}(\nu) = \left(\frac{\nu/\lambda^3}{|\nu/\lambda^3|}\right)^{-\ell} \sum_{\substack{\nu = \lambda^3\mu d^3 \\ \mu \in U \\ (d,r)=1 \\ d \equiv 1 \pmod{3}}} a^{\star}(\mu)c_r(\lambda^4\mu)\mu(d)|d|$$

$$= (-1)^{\ell}i^{\ell}\left(\frac{\nu}{|\nu|}\right)^{-\ell} \sum_{\substack{\nu = \lambda^3\mu d^3 \\ \mu \in U \\ (d,r)=1 \\ d \equiv 1 \pmod{3}}} a^{\star}(\mu)c_r(\lambda^4\mu)\mu(d)|d|.$$

Evaluation of the Ramanujan sum using Lemma 5.5 gives

$$a^{\dagger}_{-\ell}(\nu) = (-1)^{\ell}i^{\ell}\left(\frac{\nu}{|\nu|}\right)^{-\ell} \sum_{\substack{\nu = \lambda^3\mu d^3 \\ \mu \in U \\ (d,r)=1 \\ d \equiv 1 \pmod{3}}} a^{\star}(\mu)\mu\left(\frac{r}{(\lambda^4\mu, r)}\right)\frac{\varphi(r)}{\varphi\left(\frac{r}{(\lambda^4\mu,r)}\right)}\mu(d)|d|. \qquad (5.75)$$

To continue the evaluation of $a^{\dagger}(\nu)$ in (5.75), we write each $\mu \in U$ occurring on the right side uniquely as

$$\lambda^4\mu = \lambda^{k+4}\zeta hw(h'w')^3 \quad \text{with} \quad \zeta \in \{\pm 1, \pm\omega, \pm\omega^2\},$$

$$h, h' \mid r^{\infty}, \quad (ww', r) = 1, \quad h, h', w, w' \equiv 1 \pmod{3} \quad \text{and} \quad \mu^2(hw) = 1.$$

Then

$$a^\dagger_{-\ell}(\nu) = (-1)^\ell i^\ell \left(\frac{\nu}{|\nu|}\right)^{-\ell} \sum_{\substack{\nu = \lambda^{k+3}\zeta hw(h'w'd)^3 \\ h,h',w,w',d\equiv 1 \pmod 3 \\ \mu^2(hw)=1 \\ h,h'|r^\infty \\ (dww',r)=1}} a^\star\left(\lambda^k \zeta hw(h'w')^3\right) \mu\left(\frac{r}{(hh'^3,r)}\right) \frac{\varphi(r)}{\varphi\left(\frac{r}{(hh'^3,r)}\right)} \mu(d)|d|.$$

$$(5.76)$$

Furthermore, (5.74) tells us that

$$a^\star\left(\lambda^k \zeta hw(h'w')^3\right)$$
$$= \begin{cases} \overline{\tau\left(-\lambda^k \zeta hw(h'w')^3\right)} & \text{if } k \geqslant -3 \\ \omega^2 \overline{\tau_1\left(-\lambda^k \omega^2 \zeta hw(h'w')^3\right)} \breve{e}\left(\lambda^k \zeta hw(h'w')^3\right) & \text{if } k = -4, \quad \zeta \in \{-1, -\omega, -\omega^2\} \\ \omega \tau_2\left(-\lambda^k \omega \zeta hw(h'w')^3\right) \breve{e}\left(\lambda^k \zeta hw(h'w')^3\right) & \text{if } k = -4, \quad \zeta \in \{1, \omega, \omega^2\} \end{cases}.$$

Further consultation with (5.7), (5.13) and (5.14) shows that

$$a^\star\left(\lambda^k \zeta hw(h'w')^3\right) = b^\star_{k,\zeta}(hw) \frac{|h'w'|}{|hw|}, \tag{5.77}$$

for some sequence of coefficients $b^\star_{k,\zeta}(\cdot)$ on squarefree elements of $\mathbb{Z}[\omega]$ that are congruent to 1 (the sequence depends only on k and ζ). Using (5.77) in (5.76), we obtain

$$a^\dagger_{-\ell}(\nu) = (-1)^\ell i^\ell \left(\frac{\nu}{|\nu|}\right)^{-\ell} \sum_{\substack{\nu = \lambda^{k+3}\zeta hw(h'u)^3 \\ h,h',w,u\equiv 1 \pmod 3 \\ \mu^2(hw)=1 \\ h,h'|r^\infty \\ (uw,r)=1}} \frac{b^\star_{k,\zeta}(hw)}{|hw|} |h'u| \mu\left(\frac{r}{(hh'^3,r)}\right) \frac{\varphi(r)}{\varphi\left(\frac{r}{(hh'^3,r)}\right)} \tag{5.78}$$

$$\times \left(\sum_{\substack{d|u \\ d\equiv 1 \pmod 3}} \mu(d)\right).$$

Möbius inversion tells us that $u = 1$ in (5.78). Subsequent use of (5.77) (in reverse) gives

$$a^\dagger_{-\ell}(\nu) = (-1)^\ell i^\ell \left(\frac{\nu}{|\nu|}\right)^{-\ell} \sum_{\substack{\nu = \lambda^{k+3}\zeta hwh'^3 \\ h,h',w\equiv 1 \pmod 3 \\ \mu^2(hw)=1 \\ h,h'|r^\infty \\ (w,r)=1}} a^\star(\lambda^k \zeta hwh'^3) \mu\left(\frac{r}{(hh'^3,r)}\right) \frac{\varphi(r)}{\varphi\left(\frac{r}{(hh'^3,r)}\right)}. \tag{5.79}$$

For a given ν, there is at most one summand on the right side on (5.79). This completes the proof. ∎

5.6. Voronoi formula. We are finally able to prove a variant of the Voronoi summation formula.

Proposition 5.3. *Let W be a smooth Schwartz function, compactly supported in $(0, \infty)$. Let $\ell \in \mathbb{Z}$. Then for $X > 0$ we have*

$$\sum_{\substack{c,d \equiv 1 \pmod 3 \\ (d,r)=1}} |d| \tilde{g}(c) \Big(\frac{cd^3}{|cd^3|}\Big)^\ell \cdot \overline{\Big(\frac{c}{r}\Big)_3} W\Big(\frac{N(cd^3)}{X}\Big)$$

$$= \delta_{\ell=0} \cdot X^{5/6} \widetilde{W}\Big(\frac{5}{6}\Big) \frac{(2\pi)^{5/3} \varphi(r) \overline{g(r)}}{3^{7/2} \Gamma(\frac{2}{3}) N(r)^{5/3}}$$

$$+ \frac{\overline{g(r)}}{3^{7/2}(2\pi)^2} \Big(\frac{\overline{r}}{r}\Big)^{-\ell} \sum_{\substack{\nu \in \lambda^{-1}\mathbb{Z}[\omega] \\ d \equiv 1 \pmod 3 \\ (d,r)=1}} \frac{a^\dagger(\nu) b_r^\dagger(\nu)}{N(\nu) N(d)^{5/2}} \Big(\frac{d^3 \nu}{|d^3\nu|}\Big)^{-\ell} \cdot \widetilde{W}_\ell\Big(\frac{(2\pi)^4 N(d^3\nu) X}{N(r)^2}\Big),$$

$$(5.80)$$

where the $a^\dagger(\cdot)$ and $b_r^\dagger(\cdot)$ are as in (5.67) and (5.68) respectively, and $\widetilde{W}_\ell : \mathbb{R}_{>0} \to \mathbb{C}$ is defined by

$$\widetilde{W}_\ell(u) := \frac{1}{2\pi i} \int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} u^s \frac{\Gamma(\frac{5}{6} + \frac{|\ell|}{2} - s)\Gamma(\frac{7}{6} + \frac{|\ell|}{2} - s)}{\Gamma(s + \frac{|\ell|}{2} - \frac{1}{6})\Gamma(s + \frac{|\ell|}{2} + \frac{1}{6})} \widetilde{W}(s) ds, \qquad (5.81)$$

for $\varepsilon \in (0, \frac{1}{10000})$. For any $A > 0$ we have

$$\widetilde{W}_\ell(u) \ll_{W,A} (1 + |\ell|)^2 \cdot \begin{cases} (u/(1+\ell^4))^{5/6} & \text{if } |u| \leqslant (1 + \ell^4) \\ (u/(1+\ell^4))^{-A} & \text{if } |u| > (1 + \ell^4). \end{cases} \qquad (5.82)$$

Proof. We have

$$\sum_{\substack{d,c \equiv 1 \pmod 3 \\ (d,r)=1}} |d| \tilde{g}(c) \overline{\Big(\frac{c}{r}\Big)_3} \Big(\frac{cd^3}{|cd^3|}\Big)^\ell W\Big(\frac{N(cd^3)}{X}\Big)$$

$$= \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta_{\mathbb{Q}(\omega)}(3s - \tfrac{1}{2}; \mathbf{1}_r, 3\ell) \mathcal{R}(s; \psi, \ell) X^s \widetilde{W}(s) ds, \qquad (5.83)$$

where $R(s; \psi, \ell)$ is given in (5.65). We shift the contour to $\operatorname{Re} s = -\varepsilon$. Proposition 5.2 tells us that we collect a pole at $s = \frac{5}{6}$ when $\ell = 0$. Thus (5.83) is equal to

$$\delta_{\ell=0} \cdot X^{5/6} \widetilde{W}\Big(\frac{5}{6}\Big) \frac{(2\pi)^{5/3} \overline{g(r)} \varphi(r)}{3^{7/2} \Gamma(\frac{2}{3}) N(r)^{5/3}} + \frac{1}{2\pi i} \int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} \zeta_{\mathbb{Q}(\omega)}(3s - \tfrac{1}{2}; \mathbf{1}_r, 3\ell) \mathcal{R}(s; \psi, \ell) X^s \widetilde{W}(s) ds.$$

$$(5.84)$$

We evaluate the integral in (5.84) by applying the functional equation in Proposition 5.2. We obtain

$$\frac{1}{2\pi i} \int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} \zeta_{\mathbb{Q}(\omega)}(3s - \tfrac{1}{2}; \mathbf{1}_r, 3\ell) \mathcal{R}(s; \psi, \ell) X^s \widetilde{W}(s) ds$$

$$= \frac{1}{2\pi i} \cdot \frac{i^\ell \overline{g(r)}}{3^{7/2}} \Big(\frac{\overline{r}}{r}\Big)^{-\ell} \int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} \frac{(2\pi)^{4s-2}}{N(r)^{2s}} \frac{\Gamma(\frac{5}{6} + \frac{|\ell|}{2} - s)\Gamma(\frac{7}{6} + \frac{|\ell|}{2} - s)}{\Gamma(s + \frac{|\ell|}{2} - \frac{1}{6})\Gamma(s + \frac{|\ell|}{2} + \frac{1}{6})}$$

$$\times \zeta_{\mathbb{Q}(\omega)}(\tfrac{5}{2} - 3s; \mathbf{1}_r, -3\ell) \mathcal{R}_r^\dagger(1 - s, -\ell) X^s \widetilde{W}(s) ds. \qquad (5.85)$$

Expanding the absolutely convergent series we see that (5.85) is equal to

$$\frac{\overline{g(r)}}{3^{7/2}(2\pi)^2}\Big(\frac{\overline{r}}{r}\Big)^{-\ell} \sum_{\substack{\nu\in\lambda^{-1}\mathbb{Z}[\omega] \\ d\equiv 1 \pmod 3 \\ (d,r)=1}} \frac{a^\dagger(\nu)b_r^\dagger(\nu)\big(\frac{d^3\nu}{|d^3\nu|}\big)^{-\ell}}{N(\nu)N(d)^{5/2}}$$

$$\times \frac{1}{2\pi i}\int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty}\Big(\frac{(2\pi)^4 N(d^3\nu)X}{N(r)^2}\Big)^s \frac{\Gamma(\frac{5}{6}+\frac{|\ell|}{2}-s)\Gamma(\frac{7}{6}+\frac{|\ell|}{2}-s)}{\Gamma(s+\frac{|\ell|}{2}-\frac{1}{6})\Gamma(s+\frac{|\ell|}{2}+\frac{1}{6})}\cdot\widetilde{W}(s)ds.$$

The above display can be expressed as

$$\frac{\overline{g(r)}}{3^{7/2}(2\pi)^2}\Big(\frac{\overline{r}}{r}\Big)^{-\ell} \sum_{\substack{\nu\in\lambda^{-1}\mathbb{Z}[\omega] \\ d\equiv 1 \pmod 3 \\ (d,r)=1}} \frac{a^\dagger(\nu)b_r^\dagger(\nu)\big(\frac{d^3\nu}{|d^3\nu|}\big)^{-\ell}}{N(\nu)N(d)^{5/2}}\cdot\widetilde{W}_\ell\Big(\frac{(2\pi)^4 N(d^3\nu)X}{N(r)^2}\Big).$$

This establishes (5.80). The decay bound (5.82) follows from a direct computation with Stirling's asymptotic [DLMF, (5.11.1)].

∎

6. Cancellations in sequences over primes

We begin with a remark about ordering integral ideals in number fields.

Remark 6.1. Let K be an algebraic number field of degree n, and let $\mathfrak{a}$ and $\mathfrak{b}$ denote integral ideals of K. The order on integral ideals of K, $<_K$, will essentially be given by the norm $N_{K/\mathbb{Q}}$. That is, $N_{K/\mathbb{Q}}(\mathfrak{a}) < N_{K/\mathbb{Q}}(\mathfrak{b})$ implies $\mathfrak{a} <_K \mathfrak{b}$. For ideals with equal norm the ordering can be arbitrary. For instance, if $N_{K/\mathbb{Q}}(\mathfrak{a}) = N_{K/\mathbb{Q}}(\mathfrak{b})$ and $\tilde{a}$ and $\tilde{b}$ are some fixed choice of ideal numbers of $\mathfrak{a}$ and $\mathfrak{b}$ respectively, then one could declare that $\mathfrak{a} <_K \mathfrak{b}$ if and only if $\arg\tilde{a} < \arg\tilde{b}$. We will abuse notation and denote $<_{\mathbb{Q}(\omega)}$ by $<$ when the meaning is clear from context

Lemma 6.1. *Assume the Generalized Riemann Hypothesis for the Dedekind zeta function attached to $\mathbb{Q}(\omega)$ twisted by Größencharaktern. Let W be a smooth test function with compact support in $(0,\infty)$ and $m_W := \max\{1,\max_{x\in(0,\infty)}|W(x)|\}$. Let $B\geqslant 10$, $10\leqslant w\leqslant M\leqslant N\ll B$, and $\pi\in\mathbb{Z}[\omega]$ satisfy $\pi\equiv 1$ (mod 3) be a prime or 1. If $R\leqslant\frac{\log B}{K\log\log B}$ with $K>1000$, then the sequence*

$$\beta_b = \frac{1}{m_W}W\Big(\frac{N(b)}{B}\Big)\sum_{\substack{b=\prod_{j=1}^{R}\varpi_j \\ \varpi_j\equiv 1 \pmod 3 \\ \varpi_1<\varpi_2<\ldots<\varpi_R \\ M\leqslant N(\varpi_j)\leqslant N \\ (b,\pi)=1}}1$$

belongs to $\mathcal{C}_\eta(B,w)$ (given in Definition 3.1) for all $\eta>\frac{100}{K}$.

Remark 6.2. Note that scaling m_W is a minor technicality introduced so that the given $\boldsymbol{\beta}$ satisfies the first axiom of Definition 3.1.

Proof. It is clear the first three properties in Definition 3.1 follow from definition of $\boldsymbol{\beta}$. After Mellin inversion of W,

$$\beta_b := \frac{1}{2\pi i}\frac{1}{m_W}\int_{2-i\infty}^{2+i\infty}\widetilde{W}(v)B^v\sum_{\substack{b=\prod_{j=1}^R\varpi_j\\ M\leqslant\varpi_1<\varpi_2<\ldots<\varpi_R\leqslant N\\ \varpi_j\equiv 1\pmod 3\\ \varpi_j\neq\pi}}\prod_{j=1}^R\frac{1}{N(\varpi_j)^v}dv,$$

it suffices to check the fourth property of Definition 3.1 for all $\eta\gg 1/K$. For $t\in\mathbb{R}$, $\ell\in\mathbb{Z}$ and $k,u\in\mathbb{Z}[\omega]$ with $k,u\equiv 1\pmod 3$, it suffices to estimate

$$\sum_{\substack{b\in\mathbb{Z}[\omega]\\ u|b}}\beta_b\left(\frac{b}{|b|}\right)^\ell\left(\frac{k}{b}\right)_3 N(b)^{it}, \tag{6.1}$$

provided that $\ell\neq 0$, or if $\ell=0$, then provided that $k\neq\boxdot$. Without loss of generality we can take $u=1$ since the case $u\neq 1$ reduces to this case after combinatorial manipulations. Thus (6.1) (with $u=1$) is equal to

$$\frac{1}{2\pi i}\frac{1}{m_W}\int_{2-i\infty}^{2+i\infty}\widetilde{W}(v)B^v\sum_{\substack{M\leqslant\varpi_1<\varpi_2<\ldots<\varpi_R\leqslant N\\ \varpi_j\equiv 1\pmod 3\\ \varpi_j\neq\pi}}\prod_{j=1}^R\frac{\left(\frac{k}{\varpi_j}\right)_3\left(\frac{\varpi_j}{|\varpi_j|}\right)^\ell}{N(\varpi_j)^{v-it}}dv. \tag{6.2}$$

The Newton-Girard identity [Mac95, (2.14′)] implies that

$$\sum_{\substack{M\leqslant\varpi_1<\varpi_2<\ldots<\varpi_R\leqslant N\\ \varpi_j\equiv 1\pmod 3\\ \varpi_j\neq\pi}}\prod_{j=1}^R\frac{\left(\frac{k}{\varpi_j}\right)_3\left(\frac{\varpi_j}{|\varpi_j|}\right)^\ell}{N(\varpi_j)^{v-it}}$$

$$=(-1)^R\sum_{\substack{m_1,\ldots,m_R\geqslant 0\\ m_1+2m_2+\cdots+Rm_R=R}}\prod_{j=1}^R\frac{(-1)^{m_j}}{m_j!j^{m_j}}\left(\sum_{\substack{M\leqslant N(\varpi)\leqslant N\\ \varpi\equiv 1\pmod 3\\ \varpi\neq\pi}}\frac{\left(\frac{k}{\varpi}\right)_3^j\left(\frac{\varpi}{|\varpi|}\right)^{j\ell}}{N(\varpi)^{j(v-it)}}\right)^{m_j}. \tag{6.3}$$

We can assume without loss of generality that M and N are half-integers. Using [Dav00, pg. 105] each sharp cut-off can be written as

$$\sum_{\substack{M\leqslant N(\varpi)\leqslant N\\ \varpi\equiv 1\pmod 3\\ \varpi\neq\pi}}\frac{\left(\frac{k}{\varpi}\right)_3^j\left(\frac{\varpi}{|\varpi|}\right)^{j\ell}}{N(\varpi)^{j(v-it)}}=\frac{1}{(2\pi i)^2}\int\!\!\!\int_{1/\log B-iB^{100}}^{1/\log B+iB^{100}}D\left(j(v-it)+s-w;\left(\frac{k}{\cdot}\right)_3^j\mathbf{1}_\pi,j\ell\right)N^s M^{-w}\frac{dsdw}{sw}$$

$$+O(B^{-50})\quad\text{say},\quad\operatorname{Re}(v)=2, \tag{6.4}$$

where (for $\operatorname{Re}(s)>1/2$) we have

$$D\left(s;\left(\frac{k}{\cdot}\right)_3^j\mathbf{1}_\pi,j\ell\right):=\log\zeta_{\mathbb{Q}(\omega)}\left(s;\left(\frac{k}{\cdot}\right)_3^j\mathbf{1}_\pi,j\ell\right)-\sum_{m\geqslant 2}\sum_{\substack{\varpi\equiv 1\pmod 3\\ \varpi\neq\pi}}\frac{\left(\frac{k}{\varpi}\right)_3^{mj}\left(\frac{\varpi}{|\varpi|}\right)^{mj\ell}}{mN(\varpi)^{ms}}.$$

We shift the v-contour in (6.2) to $\mathrm{Re}(v) = 1/2 + 1/\log B$. From (6.4), the Riemann hypothesis, and [IK04, Theorem 5.19], we deduce that (uniformly in $j \geqslant 1$)

$$\sum_{\substack{M \leqslant N(\varpi) \leqslant N \\ \varpi \equiv 1 \pmod 3 \\ \varpi \neq \pi}} \frac{\left(\frac{k}{\varpi}\right)_3^j \left(\frac{\varpi}{|\varpi|}\right)^{j\ell}}{N(\varpi)^{j(v-it)}} \ll (\log^2 B) \cdot \log^2\left(2 + (1+|v|+|t|)(1+|\ell|)N(k)B\right), \ \mathrm{Re}\, v \geqslant \frac{1}{2} + \frac{1}{\log B}.$$

Substitution of this bound into (6.3) shows that for $\mathrm{Re}(v) \geqslant 1/2 + 1/\log B$ we have

$$\sum_{\substack{M \leqslant \varpi_1 < \varpi_2 < ... < \varpi_R \leqslant N \\ \varpi_j \equiv 1 \pmod 3 \\ \varpi_j \neq \pi}} \prod_{j=1}^R \frac{\left(\frac{k}{\varpi_j}\right)_3 \left(\frac{\varpi_j}{|\varpi_j|}\right)^\ell}{N(\varpi_j)^{v-it}}$$

$$\ll C^R (\log^{2R} B) \cdot \log^{2R}\left(2 + (1 + |v| + |t|)(1 + |\ell|)N(k)B\right)$$

$$\ll (C/\varepsilon^4)^R (\log^{2R} B^\varepsilon) \cdot \log^{2R}\left(2 + \left((1 + |v| + |t|)(1 + |\ell|)N(k)B\right)^\varepsilon\right),$$

for some absolute constant $C > 1$ and any fixed $\varepsilon > 0$. Returning to (6.2) (and recalling that we shifted the contour to $\mathrm{Re}(v) = 1/2 + 1/\log B$) we see that (6.1) is

$$\ll (C/\varepsilon^4)^R B^{1/2 + 1/\log B} (\log^{2R} B^\varepsilon) \cdot \log^{2R}\left(2 + \left((1 + |t|)(1 + |\ell|)N(k)B\right)^\varepsilon\right). \tag{6.5}$$

We use the hypothesis $1 \leqslant R \leqslant \log B/(K \log \log B)$, the inequality

$$(\log x)^L \leqslant L! x, \quad x \geqslant 1, \quad L \geqslant 1,$$

and Stirling's asymptotic formula [DLMF, (5.11.1)] to conclude that (6.5) is

$$\ll_\varepsilon B^{1/2 + 100/K + 3\varepsilon} (1 + |t|)^\varepsilon (1 + |\ell|)^\varepsilon N(k)^\varepsilon,$$

say. This concludes the proof. ∎

A minor variation of the above proof gives a smoothed version of the Lemma.

Lemma 6.2. *Assume the Generalized Riemann Hypothesis for the Dedekind zeta function attached to $\mathbb{Q}(\omega)$ twisted by Größencharaktern. Let V and W be a smooth test functions with compact support in $(0, \infty)$, and let m_V and m_W be as in Lemma 6.1. Let $R \in \mathbb{N}$, $B \geqslant 10$, $10 \leqslant w \leqslant M \leqslant N \ll B$, and $P_1, \ldots, P_R > 1$ be such that $P_1 \cdots P_R \asymp B$. If $R \leqslant \frac{\log B}{K \log \log B}$ with $K > 1000$, then the sequence*

$$\beta_b = \frac{1}{m_W} W\left(\frac{N(b)}{B}\right) \sum_{\substack{b = \prod_{j=1}^R \varpi_j \\ \varpi_j \equiv 1 \pmod 3 \\ \varpi_1 < \varpi_2 < ... < \varpi_R \\ M \leqslant N(\varpi_j) \leqslant N}} \prod_{j=1}^R \left(\frac{1}{m_V} V\left(\frac{N(\varpi_j)}{P_j}\right)\right),$$

belongs to $\mathcal{C}_\eta(B, w)$ (given in Definition 3.1) for all $\eta > \frac{100}{K}$.

Remark 6.3. A sum over R running through any subset of $[1, \log B/(K \log \log B)]$ can be introduced in the definition of $\boldsymbol{\beta}$ occurring in both Lemma 6.1 and Lemma 6.2 with no change to the conclusions.

Lemma 6.3. *Suppose $A, B \geqslant 10$, $X := AB$, $0 < \eta_1, \eta_2 \leqslant 1/4$ and $w_1 > w_2 \geqslant 10$. Let $\boldsymbol{\alpha} = (\alpha_a) \in \mathcal{C}_{\eta_1}(A, w_1)$, and $\boldsymbol{\beta} = (\beta_b) \in \mathcal{C}_{\eta_2}(B, w_2)$ be such that*

$$\beta_b \neq 0 \implies (\varpi \mid b \implies w_2 < N(\varpi) \leqslant w_1). \tag{6.6}$$

Let $10 \leqslant M \leqslant N$ and $\boldsymbol{\gamma} := (\gamma_c)$ be given by

$$\gamma_c := \sum_{\substack{M \leqslant N(c) \leqslant N \\ a, b \in \mathbb{Z}[\omega] \\ a, b \equiv 1 \pmod 3 \\ ab = c}} \alpha_a \beta_b.$$

Then $\boldsymbol{\gamma} = (\gamma_c) \in \mathcal{C}_{\max\{\eta_1, \eta_2\}}(X, w_2)$.

Remark 6.4. It will be helpful to recall the definition of $\mathscr{P}(z)$ in (3.1).

Proof. Observe that the hypotheses (6.6) and $w_2 < w_1$ imply that $\boldsymbol{\gamma} = (\gamma_c)$ is supported on squarefree w_2-rough integers. We also have $\gamma_c \neq 0 \implies N(c) \asymp X$ from the supports of $\boldsymbol{\alpha} = (\alpha_a)$ and $\boldsymbol{\beta} = (\beta_b)$. Each $c \in \mathbb{Z}[\omega]$ with $c \equiv 1 \pmod 3$ has a unique factorisation $c = ab$ with $(a, \mathscr{P}(w_1)) = 1$ and $b \mid \mathscr{P}(w_1)$. Thus hypothesis (6.6) implies that $|\gamma_c| = |\alpha_a \beta_b| \leqslant 1$.

It only remains to prove inequality (3.2) in Definition 3.1 for $\boldsymbol{\gamma} = (\gamma_c)$. Without loss of generality we can assume that M and N are half-integers and that $M \asymp X$ (resp. $N \asymp X$), otherwise $\delta_{M \leqslant N(c)}$ (resp. $\delta_{N(c) \leqslant N}$) is a redundant condition. Using [Dav00, pg. 105], the sharp cut-off can be written as

$$\delta_{M \leqslant N(c) \leqslant N} = \frac{1}{(2\pi i)^2} \int\!\!\!\int_{1/\log X - iX^{100}}^{1/\log X + iX^{100}} \frac{N^s M^{-w}}{N(c)^{s-w}} \frac{ds\,dw}{sw} + O(X^{-50}),$$

say. The integrals incur an acceptable loss of $O((\log X)^2)$. Thus it suffices to show that

$$\tilde{\gamma}_c := \sum_{\substack{a, b \in \mathbb{Z}[\omega] \\ a, b \equiv 1 \pmod 3 \\ ab = c}} \alpha_a \beta_b$$

satisfies (3.2). In other words, for $t \in \mathbb{R}$, $\ell \in \mathbb{Z}$ and $k, u \in \mathbb{Z}[\omega]$ with $k, u \equiv 1 \pmod 3$, we need to estimate

$$\sum_{\substack{c \in \mathbb{Z}[\omega] \\ u \mid c}} \tilde{\gamma}_c \left(\frac{c}{|c|}\right)^\ell \left(\frac{k}{c}\right)_3 N(c)^{it}, \tag{6.7}$$

provided that $\ell \neq 0$, or if $\ell = 0$, then provided that $k \neq \boxdot$. Observe u has a unique factorisation $u = u_1 u_2$ such that $(u_1, \mathscr{P}(w_1)) = 1$ and $u_2 \mid \mathscr{P}(w_1)$. Hypothesis (6.6) implies that that (6.7) is equal to

$$\left(\sum_{\substack{a \\ u_1 \mid a}} \alpha_a \left(\frac{a}{|a|}\right)^\ell \left(\frac{k}{a}\right)_3 N(a)^{it}\right) \left(\sum_{\substack{b \\ u_2 \mid b}} \beta_b \left(\frac{b}{|b|}\right)^\ell \left(\frac{k}{b}\right)_3 N(b)^{it}\right). \tag{6.8}$$

Since $\boldsymbol{\alpha} = (\alpha_a) \in \mathcal{C}_{\eta_1}(A, w_1)$ and $\boldsymbol{\beta} = (\beta_b) \in \mathcal{C}_{\eta_2}(B, w_2)$, we see that (6.8) is

$$\ll_\varepsilon (1 + |\ell|)^\varepsilon N(k)^\varepsilon (1 + |t|)^\varepsilon \Big(\frac{A}{N(u_1)} \Big)^{1/2 + \eta_1 + \varepsilon} \Big(\frac{B}{N(u_2)} \Big)^{1/2 + \eta_2 + \varepsilon}$$

$$\ll_\varepsilon (1 + |\ell|)^\varepsilon N(k)^\varepsilon (1 + |t|)^\varepsilon \Big(\frac{X}{N(u)} \Big)^{1/2 + \max\{\eta_1, \eta_2\} + \varepsilon},$$

as required.

∎

7. Narrow Type II/III estimates

We establish estimates for type-II/III sums that are useful in narrow ranges corresponding to two or three variables of equal size respectively. In the three variable case, two variables are clumped together to reduce to a type-II analysis. These estimates will be in ranges where sharp bounds are required (but not asymptotics).

7.1. Sieve weights. We will need to use auxiliary sieve weights in the proof of our narrow range bounds.

Lemma 7.1. *Given $w \geqslant y^2 > 1$, there exists coefficients $(\lambda_d)_{d \in \mathbb{Z}[\omega]}$ such that*

(1) *$\lambda_1 = 1$ and $|\lambda_d| \ll_\varepsilon N(d)^\varepsilon$ for all $d \in \mathbb{Z}[\omega]$ and all $\varepsilon > 0$;*
(2) *$\lambda_d = 0$ if $N(d) > y^2$ or $d \not\equiv 1 \pmod 3$;*
(3) *For all $n \in \mathbb{Z}[\omega]$ we have*

$$\delta_{(n, \mathscr{P}(w)) = 1} \leqslant \sum_{\substack{d \mid n \\ d \mid \mathscr{P}(w) \\ N(d) \leqslant y^2}} \lambda_d; \tag{7.1}$$

(4) *They satisfy*

$$\sum_{d \in \mathbb{Z}[\omega]} \frac{\lambda_d}{N(d)} \ll \frac{1}{\log y}. \tag{7.2}$$

Proof. Given $d \equiv 1 \pmod 3$, define

$$\lambda_d := \sum_{\substack{N(e), N(f) \leqslant y \\ e, f \equiv 1 \pmod 3 \\ d = [e, f]}} \mu(e) \mu(f) \Big(1 - \frac{\log N(e)}{\log y} \Big) \Big(1 - \frac{\log N(f)}{\log y} \Big).$$

Properties (1) and (2) are immediate from the definition. Property (3) follows from

$$\sum_{\substack{d \mid n \\ d \mid \mathscr{P}(w) \\ N(d) \leqslant y^2}} \lambda_d = \Big(\sum_{\substack{N(e) \leqslant y \\ e \mid (n, \mathscr{P}(w)) \\ e \equiv 1 \pmod 3}} \mu(e) \Big(1 - \frac{\log N(e)}{\log y} \Big) \Big)^2. \tag{7.3}$$

It remains to check property (4). Observe that

$$\sum_{d \in \mathbb{Z}[\omega]} \frac{\lambda_d}{N(d)} = \sum_{\substack{N(e), N(f) \leqslant y \\ e, f \equiv 1 \pmod 3}} \frac{\mu(e)\mu(f)}{N([e,f])}\Big(1 - \frac{\log N(e)}{\log y}\Big)\Big(1 - \frac{\log N(f)}{\log y}\Big)$$

$$= \frac{1}{(2\pi i)^2} \int\limits_{1/\log y - i\infty}^{1/\log y + i\infty} \frac{H(s,w)\zeta_{\mathbb{Q}(\omega)}(1+s+w)}{\zeta_{\mathbb{Q}(\omega)}(1+s)\zeta_{\mathbb{Q}(\omega)}(1+w)} \cdot \frac{y^{s+w}}{s^2 w^2} \cdot \frac{ds\, dw}{(\log y)^2}, \tag{7.4}$$

where $H(s,w)$ is an analytic and absolutely convergent Euler product for $\operatorname{Re} s, \operatorname{Re} w > -1/4$. One can conclude that the display in (7.4) is $\ll 1/\log y$ by shifting contours or by carefully bounding the integral using a Taylor expansion around the pole. ∎

7.2. Narrow Type-II/III bound. We are now ready to state the main result of this section.

Proposition 7.1. *Let $\varepsilon \in (0, \frac{1}{10000})$, $A, B \geqslant 10$, $X := AB$, $10 \leqslant w < X^\varepsilon$, and $\eta > 0$. Suppose that $\boldsymbol{\alpha} = (\alpha_a)$ is a sequence supported on squarefree w-rough $a \in \mathbb{Z}[\omega]$ with $a \equiv 1 \pmod 3$ and $N(a) \in [A/10, 10A]$. Suppose that $\boldsymbol{\beta} = (\beta_b) \in \mathcal{C}_\eta(B, w)$. Then*

$$\Big| \sum_{a,b \in \mathbb{Z}[\omega]} \alpha_a \beta_b \tilde{g}(ab) \Big| \leqslant \frac{\mathcal{K}}{\sqrt{\log w}} \|\boldsymbol{\alpha}\|_2 \Big(A^{1/2} \|\boldsymbol{\beta}\|_2 + A^{1/3} \Big(\sum_{b \in \mathbb{Z}[\omega]} \frac{|\beta_b|}{N(b)^{1/6}} \Big) \Big) \tag{7.5}$$

$$+ O_\varepsilon\Big(B^\eta \cdot X^{1+\varepsilon} \Big(\frac{1}{A^{1/2}} + \frac{1}{B} \Big) \Big) + O_\varepsilon(X^{1+\varepsilon} \cdot (A/w)^{-1000}).$$

with $\mathcal{K} > 1$ an absolute constant.

Proof. Without loss of generality we can include the condition $(a,b) = 1$ on the left side (7.5) by (2.2). Application of (2.3) and Cauchy-Schwarz gives

$$\Big| \sum_{a,b \in \mathbb{Z}[\omega]} \alpha_a \beta_b \tilde{g}(ab) \Big| \leqslant \|\boldsymbol{\alpha}\|_2 \Big(\sum_{\substack{a \in \mathbb{Z}[\omega] \\ (a, \mathscr{P}(w))=1 \\ a \equiv 1 \pmod 3}} \Big| \sum_b \beta_b \tilde{g}(b) \overline{\Big(\frac{a}{b}\Big)_3} \Big|^2 \Big)^{1/2}. \tag{7.6}$$

Let $V : \mathbb{R} \to \mathbb{R}$ be a fixed smooth positive function with compact support in $[1/100, 100]$. We also stipulate that it satisfies $V \geqslant \delta_{[1/10,10]}$. By positivity of the right side of (7.6), we introduce both the smooth function V and the sieve weight (7.1) on the a-sum. Thus the right side of (7.6) is

$$\leqslant \|\boldsymbol{\alpha}\|_2 \Big(\sum_{\substack{a \in \mathbb{Z}[\omega] \\ a \equiv 1 \pmod 3}} V\Big(\frac{N(a)}{A}\Big) \Big| \sum_b \beta_b \tilde{g}(b) \overline{\Big(\frac{a}{b}\Big)_3} \Big|^2 \sum_{\substack{d|a \\ d|\mathscr{P}(w) \\ N(d) \leqslant y^2}} \lambda_d \Big)^{1/2}, \tag{7.7}$$

where $y^2 := w$. Expansion of the bracketed sum in (7.7) gives

$$\sum_{\substack{d|\mathscr{P}(w) \\ N(d) \leqslant y^2}} \lambda_d \sum_{b_1, b_2 \in \mathbb{Z}[\omega]} \beta_{b_1} \overline{\beta_{b_2}} \tilde{g}(b_1) \overline{\tilde{g}(b_2)} \sum_{\substack{a \in \mathbb{Z}[\omega] \ , \ d|a \\ a \equiv 1 \pmod 3}} V\Big(\frac{N(a)}{A}\Big) \overline{\Big(\frac{a}{b_1}\Big)_3} \Big(\frac{a}{b_2}\Big)_3. \tag{7.8}$$

Diagonal contribution to (7.8). The diagonal contribution $b_1 = b_2 =: b$ to (7.8) is

$$\sum_{\substack{a \in \mathbb{Z}[\omega] \\ a \equiv 1 \pmod 3}} V\left(\frac{N(a)}{A}\right) \sum_{\substack{b \in \mathbb{Z}[\omega] \\ (a,b)=1}} |\beta_b|^2 \left(\sum_{\substack{d | a \\ d | \mathscr{P}(w) \\ N(d) \leqslant y^2}} \lambda_d\right). \tag{7.9}$$

We can drop the condition $(a, b) = 1$ by non-negativity of (7.9) (the bracketed sieve weight divisor sum is non-negative by (7.1)). Thus

$$\begin{aligned}
(7.9) &\leqslant \sum_{\substack{d | \mathscr{P}(w) \\ N(d) \leqslant y^2}} \lambda_d \sum_{b \in \mathbb{Z}[\omega]} |\beta_b|^2 \sum_{\substack{a \in \mathbb{Z}[\omega] \\ a \equiv 1 \pmod 3}} V\left(\frac{N(a)N(d)}{A}\right) \\
&= \frac{4\pi A}{9\sqrt{3}} \|\boldsymbol{\beta}\|_2^2 \left(\sum_{\substack{d | \mathscr{P}(w) \\ N(d) \leqslant y^2}} \frac{\lambda_d}{N(d)}\right) \sum_{k \in \mathbb{Z}[\omega]} \check{e}\left(-\frac{k}{3\lambda}\right) \ddot{V}\left(k\sqrt{A/N(d)}\right) =: \mathscr{D}, \tag{7.10}
\end{aligned}$$

where (7.10) follows from Poisson summation (in the form of Lemma 4.3).

Non-diagonal contribution to (7.8). The non-diagonal $b_1 \neq b_2$ contribution to (7.8) is

$$\sum_{\substack{d | \mathscr{P}(w) \\ N(d) \leqslant y^2}} \lambda_d \sum_{\substack{b_1, b_2 \in \mathbb{Z}[\omega] \\ b_1 \neq b_2}} \beta_{b_1} \overline{\beta_{b_2}} \tilde{g}(b_1) \overline{\tilde{g}(b_2)} \overline{\left(\frac{d}{b_1}\right)_3} \left(\frac{d}{b_2}\right)_3 \sum_{\substack{a \in \mathbb{Z}[\omega] \\ a \equiv 1 \pmod 3}} V\left(\frac{N(a)N(d)}{A}\right) \overline{\left(\frac{a}{b_1}\right)_3} \left(\frac{a}{b_2}\right)_3. \tag{7.11}$$

For each fixed $b_1, b_2 \in \mathbb{Z}[\omega]$ occurring in (7.11), let $e := (b_1, b_2)$. Poisson summation (in the form of Corollary 4.1) tells us that

$$\begin{aligned}
&\sum_{\substack{a \in \mathbb{Z}[\omega] \\ a \equiv 1 \pmod 3}} V\left(\frac{N(a)N(d)}{A}\right) \overline{\left(\frac{a}{b_1}\right)_3} \left(\frac{a}{b_2}\right)_3 \\
&\qquad = \frac{4\pi \overline{\left(\frac{e}{b_1/e}\right)_3} \left(\frac{e}{b_2/e}\right)_3 A \overline{g(b_1/e)} g(b_2/e)}{9\sqrt{3} N(d) N(b_1 b_2/e)} \sum_{k \in \mathbb{Z}[\omega]} \tilde{c}_e(k) \left(\frac{k}{b_1/e}\right)_3 \overline{\left(\frac{k}{b_2/e}\right)_3} \ddot{V}\left(\frac{ke\sqrt{A}}{\sqrt{N(d)} b_1 b_2}\right). \tag{7.12}
\end{aligned}$$

Remark 7.1. Since $b_1 \neq b_2$, the character $\left(\frac{\cdot}{b_1/e}\right)_3 \overline{\left(\frac{\cdot}{b_2/e}\right)_3}$ in (7.12) has conductor with norm > 1. Hence the dual frequency $k = 0$ contributes zero to (7.12).

Observe that (2.3) and the squarefree property of b_1 and b_2 imply that

$$\begin{aligned}
\tilde{g}(b_1) \overline{\tilde{g}(b_2)} &= \tilde{g}(b_1/e) \tilde{g}(e) \overline{\left(\frac{e}{b_1/e}\right)_3} \overline{\tilde{g}(b_2/e)} \cdot \overline{\tilde{g}(e)} \left(\frac{e}{b_2/e}\right)_3 \\
&= \tilde{g}(b_1/e) \overline{\tilde{g}(b_2/e)} \overline{\left(\frac{e}{b_1/e}\right)_3} \left(\frac{e}{b_2/e}\right)_3. \tag{7.13}
\end{aligned}$$

Upon insertion of (7.12) and (7.13) into (7.11), we see that (7.11) becomes

$$\mathscr{N} := \frac{4\pi A}{9\sqrt{3}} \sum_{\substack{d|\mathscr{P}(w) \\ N(d) \leqslant y^2}} \frac{\lambda_d}{N(d)} \sum_{\substack{e\in\mathbb{Z}[\omega] \\ e\equiv 1 \pmod 3}} \sum_{\substack{b_1\neq b_2\in\mathbb{Z}[\omega] \\ (b_1,b_2)=e}} \frac{\beta_{b_1}\overline{\beta_{b_2}}}{\sqrt{N(b_1 b_2)}}$$

$$\times \sum_{k\in\mathbb{Z}[\omega]} \widetilde{c}_e(k) \Big(\frac{d^2 ek}{b_1/e}\Big)_3 \overline{\Big(\frac{d^2 ek}{b_2/e}\Big)_3} \ddot{V}\Big(\frac{ke\sqrt{A}}{\sqrt{N(d)}b_1 b_2}\Big). \tag{7.14}$$

Remark 7.2. Note that Poisson summation constitutes a key step in the proof - the dual side (7.14) has no Gauss sum weights.

Estimates for $\mathscr{D}$ and $\mathscr{N}$. We estimate $\mathscr{D}$ and $\mathscr{N}$ displayed in (7.10) and (7.14) respectively.

Consider $\mathscr{D}$. Lemma 4.4 tells us that

$$\mathscr{D} = \frac{4\pi A}{9\sqrt{3}} \|\boldsymbol{\beta}\|_2^2 \Big(\sum_{\substack{d|\mathscr{P}(w) \\ N(d)\leqslant y^2}} \frac{\lambda_d}{N(d)}\Big) \ddot{V}(0) + O_\varepsilon(X^{1+\varepsilon} \cdot (A/w)^{-2000}).$$

Application of (7.2) gives

$$\mathscr{D} \ll \frac{A}{\log w} \|\boldsymbol{\beta}\|_2^2 + O_\varepsilon(X^{1+\varepsilon} \cdot (A/w)^{-2000}). \tag{7.15}$$

Consider $\mathscr{N}$. For a given $d,e \in \mathbb{Z}[\omega]$ in (7.14), we split the $k\neq 0$ sum into two subsums:
- $0 \neq k \in \mathbb{Z}[\omega]$ such that $d^2 ek = \square$;
- $0 \neq k \in \mathbb{Z}[\omega]$ such that $d^2 ek \neq \square$.

Denote the contributions to $\mathscr{N}$ by each of these two cases by $\mathscr{N}_1$ and $\mathscr{N}_2$ respectively. Thus $\mathscr{N} = \mathscr{N}_1 + \mathscr{N}_2$.

Since $\mu^2(d) = \mu^2(e) = 1$ and $(d,e)=1$, we deduce that $d^2 ek = \square$ iff $k = de^2 h^3$ for some $h \in \mathbb{Z}[\omega]$. Notice that (4.8) and Lemma 5.5 imply that

$$\widetilde{c}_e(de^2 h^3) = \breve{e}\Big(-\frac{de^2 h^3}{3\lambda}\Big)\varphi(e) = \breve{e}\Big(-\frac{h^3}{3\lambda}\Big)\varphi(e),$$

where the last equality follows from the fact that $d \equiv e \equiv 1 \pmod 3$ and Remark 4.1. Thus

$$\mathscr{N}_1 = \frac{4\pi A}{3^3\sqrt{3}} \sum_{\substack{e\in\mathbb{Z}[\omega] \\ e\equiv 1 \pmod 3}} \varphi(e) \sum_{\substack{b_1\neq b_2\in\mathbb{Z}[\omega] \\ (b_1,b_2)=e}} \frac{\beta_{b_1}\overline{\beta_{b_2}}}{\sqrt{N(b_1 b_2)}}$$

$$\times \sum_{\substack{h\in\mathbb{Z}[\omega] \\ (h,b_1 b_2/e^2)=1}} \breve{e}\Big(-\frac{h^3}{3\lambda}\Big)\ddot{V}\Big(\frac{e^3 h^3\sqrt{A}}{b_1 b_2}\Big)\Big(\sum_{\substack{d|\mathscr{P}(w) \\ N(d)\leqslant y^2}} \frac{\lambda_d}{N(d)}\Big). \tag{7.16}$$

Note that the extra factor of $1/3$ in the above display for $\mathscr{N}_1$ accounts for the fact that $(\omega^i h)^3 = h^3$ for $i \in \{0,1,2\}$ and $0 \neq h \in \mathbb{Z}[\omega]$. Using Lemma 4.4 and recalling (7.2), we see that all this leads to

$$\mathscr{N}_1 \ll \frac{1}{\log w}\Big(A^{2/3}\Big(\sum_{b\in\mathbb{Z}[\omega]} \frac{|\beta_b|}{N(b)^{1/6}}\Big)^2 + A\|\boldsymbol{\beta}\|_2^2\Big). \tag{7.17}$$

We now focus on $\mathcal{N}_2$. We have

$$\mathcal{N}_2 := \frac{4\pi A}{9\sqrt{3}} \sum_{\substack{d | \mathscr{P}(w) \\ N(d) \leqslant y^2}} \frac{\lambda_d}{N(d)} \sum_{\substack{e \in \mathbb{Z}[\omega] \\ e \equiv 1 \pmod 3}} \sum_{\substack{b_1 \neq b_2 \in \mathbb{Z}[\omega] \\ (b_1, b_2) = e}} \frac{\beta_{b_1} \overline{\beta_{b_2}}}{\sqrt{N(b_1 b_2)}}$$

$$\times \sum_{\substack{k \in \mathbb{Z}[\omega] \\ d^2 e k \neq \square}} \widetilde{c}_e(k) \left(\frac{d^2 e k}{b_1 / e}\right)_3 \overline{\left(\frac{d^2 e k}{b_2 / e}\right)_3} \ddot{V}\left(\frac{k e \sqrt{A}}{\sqrt{N(d)} b_1 b_2}\right).$$

The term $\mathcal{N}_2$ is small because the characters $\left(\frac{d^2 e k}{\cdot}\right)_3$ and $\overline{\left(\frac{d^2 e k}{\cdot}\right)_3}$ are both non-principal. Using Lemma 4.4 and Lemma 5.5, we re-install the diagonal $b_1 = b_2$ in $\mathcal{N}_2$ with acceptable error $O(X^\varepsilon (A+B))$. After rescaling the variables $b_1 \to e b_1$ and $b_2 \to e b_2$ and using Lemma 4.4, we obtain

$$\mathcal{N}_2 = \frac{4\pi A}{9\sqrt{3}} \sum_{\substack{d | \mathscr{P}(w) \\ N(d) \leqslant y^2}} \frac{\lambda_d}{N(d)} \sum_{\substack{e \in \mathbb{Z}[\omega] \\ e \equiv 1 \pmod 3}} \frac{1}{N(e)} \sum_{\substack{b_1, b_2 \in \mathbb{Z}[\omega] \\ (b_1, b_2) = 1}} \frac{\beta_{e b_1} \overline{\beta_{e b_2}}}{\sqrt{N(b_1 b_2)}}$$

$$\times \sum_{\substack{k \in \mathbb{Z}[\omega] \\ d^2 e k \neq \square \\ N(k) \ll X^\varepsilon (1 + N(d) B^2 / (N(e) A))}} \widetilde{c}_e(k) \left(\frac{d^2 e k}{b_1}\right)_3 \overline{\left(\frac{d^2 e k}{b_2}\right)_3} \ddot{V}\left(\frac{k \sqrt{A}}{\sqrt{N(d)} e b_1 b_2}\right)$$

$$+ O(X^\varepsilon (A + B)).$$

After combining (4.11) and the Mellin–Barnes integral representation [DLMF, (10.9.22)] for the J-Bessel function, we obtain

$$\ddot{V}(u) = \frac{(-1)^L}{2\pi i} \int_0^\infty \int_{-\varepsilon - i\infty}^{-\varepsilon + \infty} V^{(L)}(r^2) r^{2L+1} \frac{\Gamma(-s)}{\Gamma(L + s + 1)} \left(\frac{2\pi r |u|}{3\sqrt{3}}\right)^{2s} ds\, dr, \qquad (7.18)$$

for $u \neq 0, L \in \mathbb{Z}_{\geqslant 1}$. For $L \in \mathbb{Z}_{\geqslant 1}$ sufficiently large and fixed depending on $\varepsilon > 0$, Stirling's asymptotic formula [DLMF, (5.11.9)] implies that

$$\ddot{V}(u) = \frac{(-1)^L}{2\pi i} \int_0^\infty \int_{-\varepsilon - iX^\varepsilon}^{-\varepsilon + iX^\varepsilon} V^{(L)}(r^2) r^{2L+1} \frac{\Gamma(-s)}{\Gamma(L + s + 1)} \left(\frac{2\pi r |u|}{3\sqrt{3}}\right)^{2s} ds\, dr + O_\varepsilon(X^{-2000}),$$
$$(7.19)$$

for $u \neq 0$.

We Möbius invert the condition $(b_1, b_2) = 1$ in the expression for $\mathcal{N}_2$ above, and then separate variables using (7.19). A subsequent interchange of the absolutely convergent finite (recall that $V^{(L)}$ is compactly supported) sums and integrations by Fubini's Theorem

gives

$$
\mathcal{N}_2 = (-1)^L \cdot \frac{2A}{9\sqrt{3}i} \int_0^\infty \int_{-\varepsilon-iX^\varepsilon}^{-\varepsilon+iX^\varepsilon} V^{(L)}(r^2)r^{2L+1}\frac{\Gamma(-s)}{\Gamma(L+s+1)}\Big(\frac{2\pi r\sqrt{A}}{3\sqrt{3}}\Big)^{2s}
$$

$$
\times \Big(\sum_{\substack{d\mid \mathscr{P}(w)\\ N(d)\leqslant y^2}} \frac{\lambda_d}{N(d)^{1+s}} \sum_{\substack{f\in\mathbb{Z}[\omega]\\ f\equiv 1 \pmod 3}} \frac{\mu(f)}{N(f)^{1+2s}}
$$

$$
\times \sum_{\substack{e\in\mathbb{Z}[\omega]\\ e\equiv 1 \pmod 3}} \sum_{\substack{k\in\mathbb{Z}[\omega]\\ d^2ek\neq\boxdot\\ N(k)\ll X^\varepsilon(1+N(d)B^2/(N(e)A))}} \frac{\widetilde{c}_e(k)N(k)^s}{N(e)^{1+s}}\Big(\frac{d^2ek}{f}\Big)_3\overline{\Big(\frac{d^2ek}{f}\Big)_3}
$$

$$
\times \Big(\sum_{b_1\in\mathbb{Z}[\omega]} \frac{\beta_{efb_1}}{\sqrt{N(b_1)}}\Big(\frac{d^2ek}{b_1}\Big)_3 N(b_1)^{-s}\Big)\Big(\sum_{b_2\in\mathbb{Z}[\omega]} \frac{\overline{\beta_{efb_2}}}{\sqrt{N(b_2)}}\overline{\Big(\frac{d^2ek}{b_2}\Big)_3} N(b_2)^{-s}\Big)\Big)ds\,dr
$$

$$
+ O(X^\varepsilon(A+B)).
$$

We use Axiom 4 of Definition 3.1 to estimate the sum over b_1 and b_2 (square root cancellation), and then estimate the remaining sums trivially using Lemma 5.5. We obtain

$$
\mathcal{N}_2 \ll AB^{2\eta}X^\varepsilon\Big(\sum_{\substack{d\mid\mathscr{P}(w)\\ N(d)\leqslant y^2}} \frac{1}{N(d)} \sum_{\substack{f\in\mathbb{Z}[\omega]\\ 1\leqslant N(f)\ll B}} \frac{\mu^2(f)}{N(f)} \sum_{\substack{e\in\mathbb{Z}[\omega]\\ 1\leqslant N(e)\ll B}} \frac{\mu^2(e)}{N(e)}
$$

$$
\times \sum_{g\mid e} \sum_{\substack{k\in\mathbb{Z}[\omega]\\ (k,e)=g\\ N(k)\ll X^\varepsilon(1+N(d)B^2/(N(e)A))}} \frac{\varphi(e)}{\varphi(e/g)}\Big) + X^\varepsilon(A+B)
$$

$$
\ll X^\varepsilon B^{2\eta}(A+B^2). \tag{7.20}
$$

7.2.1. *Conclusion.* Combining (7.15), (7.17) and (7.20) tells us that

$$
(7.8) \leqslant \frac{\mathcal{K}}{\log w}\Big(A^{2/3}\Big(\sum_{b\in\mathbb{Z}[\omega]} \frac{|\beta_b|}{N(b)^{1/6}}\Big)^2 + A\|\boldsymbol{\beta}\|_2^2\Big) + O_\varepsilon(B^{2\eta}(A+B^2)X^\varepsilon) + O_\varepsilon(X^{1+\varepsilon}\cdot(A/w)^{-2000}),
$$

for some absolute constant $\mathcal{K} > 1$. Chasing this bound through (7.7) and (7.6) gives the result. ∎

8. Type I estimates

We now establish Type-I estimates. In the Proposition below we use the Riemann Hypothesis for the Dedekind zeta function attached to $\mathbb{Q}(\omega)$ in order to restrict the sum to squarefree numbers.

Proposition 8.1. *Assume the Riemann Hypothesis for the family of Dedekind zeta functions attached to $\mathbb{Q}(\omega)$ twisted by Größencharaktern. Let $r\in\mathbb{Z}[\omega]$ be squarefree and satisfy $r\equiv 1 \pmod 3$. Let $\varepsilon\in(0,\frac{1}{10000})$, and $W:\mathbb{R}\to\mathbb{R}$ be a smooth function with compact*

support contained in $[1, 2]$. *Then there exists* $\rho(\varepsilon) \in (0, \frac{1}{10000})$ *such that*

$$\sum_{\substack{u \in \mathbb{Z}[\omega] \\ u \equiv 1 \pmod 3}} \tilde{g}(ur)\Big(\frac{u}{|u|}\Big)^{\ell} \cdot W\Big(\frac{N(u)}{U}\Big) = \delta_{\ell=0} \cdot \widetilde{W}\Big(\frac{5}{6}\Big) \cdot \frac{(2\pi)^{5/3}U^{5/6}}{3^{7/2}\Gamma(\frac{2}{3})\zeta_{\mathbb{Q}(\omega)}(2; \mathbf{1}_r)} \frac{\varphi(r)}{N(r)^{7/6}}$$

$$+ O_{\varepsilon}\Big((1 + |\ell|)^{\varepsilon} \cdot \Big(\frac{U^{5/6-\rho(\varepsilon)}}{N(r)^{1/6+\rho(\varepsilon)}} + U^{1/12+\varepsilon}N(r)^{7/12+\varepsilon} \cdot (1 + \ell^6)\Big)\Big),$$

for all $\ell \in \mathbb{Z}$ *and* $U \geqslant 1$.

Remark 8.1. The function $\rho(\varepsilon)$ is somewhat arbitrary. For instance, it follows from (8.6) that $\rho(\varepsilon) = \frac{11}{12}\varepsilon - \frac{1}{2}\varepsilon^2$ is an acceptable choice.

Remark 8.2. Mellin inversion of the smooth function, the Class number formula [Lan94, Chapter VIII, §2, Theorem 5], and a contour shift together imply that

$$\frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})} \sum_{\substack{u \equiv 1 \pmod 3 \\ (u,r)=1}} \frac{\mu^2(u)}{N(u)^{1/6}} W\Big(\frac{N(u)}{U}\Big) \cdot \frac{1}{N(r)^{1/6}} \tag{8.1}$$

$$= \widetilde{W}\Big(\frac{5}{6}\Big) \frac{(2\pi)^{5/3}U^{5/6}}{3^{7/2}\Gamma(\frac{2}{3})\zeta_{\mathbb{Q}(\omega)}(2; \mathbf{1}_r)} \frac{\varphi(r)}{N(r)^{7/6}} + O_{\varepsilon}\Big(\frac{U^{1/3+\varepsilon}}{N(r)^{1/6}}\Big).$$

Thus when $\ell = 0$, we can use the main term in (8.1) in Proposition 8.1 at negligible cost.

Proof. Möbius inversion implies that

$$\sum_{\substack{u \in \mathbb{Z}[\omega] \\ u \equiv 1 \pmod 3}} \tilde{g}(ur)\Big(\frac{u}{|u|}\Big)^{\ell} W\Big(\frac{N(u)}{U}\Big)$$

$$= \sum_{\substack{u,e \in \mathbb{Z}[\omega] \\ u,e \equiv 1 \pmod 3 \\ (e,r)=1}} \tilde{g}(ur)|e|\Big(\frac{e^3 u}{|e^3 u|}\Big)^{\ell} W\Big(\frac{N(e)^3 N(u)}{X}\Big)\Big(\sum_{\substack{e=cd \\ c,d \equiv 1 \pmod 3}} \mu(c)\Big). \tag{8.2}$$

Remark 8.3. Note that the factor of $|e|$ in (8.2) reflects the periodicity property possessed by the coefficients of the cubic theta function on cubes. See Remark 5.2.

On the right side of (8.2) we introduce a smooth partition of unity in the c variable. Let $V : \mathbb{R} \to \mathbb{R}$ be a fixed smooth function with compact support contained in $[1, 2]$ such that

$$\sum_{C \text{ dyadic}} V\Big(\frac{N(c)}{C}\Big) = 1 \quad \text{for all} \quad 0 \neq c \in \mathbb{Z}[\omega]. \tag{8.3}$$

Insertion of (8.3) into (8.2) yields

$$\sum_{\substack{u \in \mathbb{Z}[\omega] \\ u \equiv 1 \pmod 3}} \tilde{g}(ur)\Big(\frac{u}{|u|}\Big)^{\ell} W\Big(\frac{N(u)}{U}\Big) = \sum_{C \text{ dyadic}} \mathscr{M}(C, U), \tag{8.4}$$

where

$$
\mathscr{M}(C,U) := \sum_{\substack{u,d\in\mathbb{Z}[\omega] \\ u,d\equiv 1 \pmod 3 \\ (d,r)=1}} \tilde{g}(ur)|d|\left(\frac{u}{|u|}\right)^{\ell} \sum_{\substack{c\in\mathbb{Z}[\omega] \\ c\equiv 1 \pmod 3 \\ (c,r)=1}} \mu(c)|c|
$$

$$
\times \left(\frac{cd}{|cd|}\right)^{3\ell} V\left(\frac{N(c)}{C}\right) W\left(\frac{N(cd)^3 N(u)}{U}\right). \tag{8.5}
$$

We have suppressed the dependence of $\mathscr{M}(C,U)$ on the smooth functions V and W in the notation.

Large dyadic C. We estimate the contribution to the right side of (8.4) from all dyadic values of C satisfying

$$
C \geqslant (UN(r))^{1/12+\varepsilon/2}.
$$

We Mellin invert the smooth functions V and W in (8.5). We then use the rapid decay of their holomorphic Mellin transforms $\widetilde{W}$ and $\widetilde{V}$ in vertical strips to truncate the integrations appropriately. A subsequent interchange of the order of absolutely convergent finite sums and integrations by Fubini's Theorem gives

$$
\mathscr{M}(C,U) := \frac{1}{(2\pi i)^2} \int_{-i(C(1+|\ell|))^{\varepsilon/1000}}^{i(C(1+|\ell|))^{\varepsilon/1000}} \int_{-i(C(1+|\ell|))^{\varepsilon/1000}}^{i(C(1+|\ell|))^{\varepsilon/1000}} \widetilde{V}(s)\widetilde{W}(w) C^s U^w
$$

$$
\times \Bigg(\sum_{\substack{u,d\in\mathbb{Z}[\omega] \\ U/(100C^3)\leqslant N(ud^3)\leqslant 100U/C^3 \\ u,d\equiv 1 \pmod 3 \\ (d,r)=1}} \tilde{g}(ur)N(d)^{1/2-3w}\left(\frac{ud^3}{|ud^3|}\right)^{\ell} N(u)^{-w}
$$

$$
\times \sum_{\substack{c\in\mathbb{Z}[\omega] \\ C\leqslant N(c)\leqslant 2C \\ c\equiv 1 \pmod 3 \\ (c,r)=1}} \mu(c)\left(\frac{c}{|c|}\right)^{3\ell} N(c)^{1/2-s-3w} \Bigg) ds\, dw + O_\varepsilon\big((C(1+|\ell|))^{-2000}\big).
$$

To bound the sum over c we appeal to the Riemann Hypothesis for the Dedekind zeta function attached to $\mathbb{Q}(\omega)$ twisted by a Größencharakter. Estimating the other summations trivially, we obtain

$$
\sum_{\substack{C \text{ dyadic} \\ C\geqslant (UN(r))^{1/12+\varepsilon/2}}} \mathscr{M}(C,U) \ll_\varepsilon \left(\frac{U}{C^3}\right)C^{1+\varepsilon}(1+|\ell|)^\varepsilon \ll_\varepsilon (1+|\ell|)^\varepsilon \frac{U^{5/6-\rho(\varepsilon)}}{N(r)^{1/6+\rho(\varepsilon)}}, \tag{8.6}
$$

for some $\rho(\varepsilon) \in (0,\frac{1}{10000})$. See also Remark 8.1.

Small dyadic C. It remains to estimate

$$
\sum_{\substack{C \text{ dyadic} \\ 1/2\leqslant C\leqslant (UN(r))^{1/12+\varepsilon/2}}} \mathscr{M}(C,U).
$$

Rearranging the sums in (8.5), and then using (2.3), we obtain

$$\mathscr{M}(C,U) = \tilde{g}(r) \sum_{\substack{c\in\mathbb{Z}[\omega] \\ c\equiv 1 \pmod 3 \\ (c,r)=1}} |c|\Big(\frac{c}{|c|}\Big)^{3\ell}\mu(c)V\Big(\frac{N(c)}{C}\Big)$$

$$\times \sum_{\substack{d,u\in\mathbb{Z}[\omega] \\ d,u\equiv 1 \pmod 3 \\ (d,r)=1}} |d|\tilde{g}(u)\overline{\Big(\frac{u}{r}\Big)}_3\Big(\frac{ud^3}{|ud^3|}\Big)^\ell W\Big(\frac{N(ud^3)}{U/N(c)^3}\Big). \qquad (8.7)$$

Application of Voronoi summation (in the form of Proposition 5.3) gives

$$\sum_{\substack{d,u\in\mathbb{Z}[\omega] \\ d,u\equiv 1 \pmod 3 \\ (d,r)=1}} |d|\tilde{g}(u)\overline{\Big(\frac{u}{r}\Big)}_3\Big(\frac{ud^3}{|ud^3|}\Big)^\ell W\Big(\frac{N(ud^3)}{U/N(c)^3}\Big)$$

$$= \delta_{\ell=0}\frac{U^{5/6}}{N(c)^{5/2}}\widetilde{W}\Big(\frac{5}{6}\Big)\frac{(2\pi)^{5/3}\varphi(r)\overline{g(r)}}{3^{7/2}\Gamma(\frac{2}{3})N(r)^{5/3}}$$

$$+ \frac{\overline{g(r)}}{3^{7/2}(2\pi)^2}\Big(\frac{\overline{r}}{r}\Big)^{-\ell} \sum_{\substack{\nu\in\lambda^{-1}\mathbb{Z}[\omega] \\ d\equiv 1 \pmod 3 \\ (d,r)=1}} \frac{a^\dagger(\nu)b_r^\dagger(\nu)\Big(\frac{\nu d^3}{|\nu d^3|}\Big)^{-\ell}}{N(\nu)N(d)^{5/2}}\widetilde{W}_\ell\Big(\frac{(2\pi)^4 N(d^3\nu)U}{N(c^3 r^2)}\Big),$$

where the $a^\dagger(\cdot)$ and $b_r^\dagger(\cdot)$ are given by (5.67) and (5.68) respectively. Insertion of the above display into (8.7) gives

$$\mathscr{M}(C,U) = \mathscr{T}(C,U) + \mathscr{E}(C,U), \qquad (8.8)$$

where

$$\mathscr{T}(C,U) := \delta_{\ell=0}\frac{(2\pi)^{5/3}}{3^{7/2}\Gamma(\frac{2}{3})}\widetilde{W}\Big(\frac{5}{6}\Big)\frac{\varphi(r)}{N(r)^{7/6}}U^{5/6} \sum_{\substack{c\in\mathbb{Z}[\omega] \\ c\equiv 1 \pmod 3 \\ (c,r)=1}} \frac{\mu(c)}{N(c)^2}V\Big(\frac{N(c)}{C}\Big),$$

and

$$\mathscr{E}(C,U) := \frac{N(r)^{1/2}}{3^{7/2}(2\pi)^2}\Big(\frac{\overline{r}}{r}\Big)^{-\ell} \sum_{\substack{\nu\in\lambda^{-1}\mathbb{Z}[\omega] \\ d\equiv 1 \pmod 3 \\ (d,r)=1}} \frac{a^\dagger(\nu)b_r^\dagger(\nu)}{N(\nu)N(d)^{5/2}}\Big(\frac{d^3\nu}{|d^3\nu|}\Big)^{-\ell}$$

$$\times \sum_{\substack{c\in\mathbb{Z}[\omega] \\ c\equiv 1 \pmod 3 \\ (c,r)=1}} |c|\mu(c)\Big(\frac{c}{|c|}\Big)^{3\ell}V\Big(\frac{N(c)}{C}\Big)\widetilde{W}_\ell\Big(\frac{(2\pi)^4 N(d^3\nu)U}{N(c^3 r^2)}\Big).$$

We now collect the main term from the various $\mathscr{T}(C,U)$. We have

$$\sum_{\substack{C \text{ dyadic} \\ 1/2 \leqslant C \leqslant (UN(r))^{1/12+\varepsilon}}} \mathscr{T}(C,U) = \sum_{C \text{ dyadic}} \mathscr{T}(C,U) + O_\varepsilon\Big(\frac{U^{3/4}}{N(r)^{1/4}}\Big)$$

$$= \delta_{\ell=0}\widetilde{W}\Big(\frac{5}{6}\Big)\frac{(2\pi)^{5/3}U^{5/6}}{3^{7/2}\Gamma(\frac{2}{3})\zeta_{\mathbb{Q}(\omega)}(2;\mathbf{1}_r)}\frac{\varphi(r)}{N(r)^{7/6}} + O_\varepsilon\Big(\frac{U^{3/4}}{N(r)^{1/4}}\Big),$$
$$(8.9)$$

where the error term follows from a trivial estimation of the tail of $\zeta_{\mathbb{Q}(\omega)}$.

The various $\mathscr{E}(C,U)$ contribute the error term in the statement of the result. By the rapid decay of $\widetilde{W}$ in (5.82), we truncate the d and ν sums in $\mathscr{E}(C,U)$ to

$$N(d^3\nu) \ll (UN(r)(1+|\ell|))^{\varepsilon/1000}(1+\ell^4) \cdot \Big(1 + \frac{C^3 N(r)^2}{U}\Big) =: \mathcal{Z},$$

with negligible error. To separate variables, we subsequently use the definition (5.81) of $\widetilde{W}$ and Mellin inversion on V. We truncate the resulting integrations appropriately using the rapid decay of $\widetilde{V}$ and $\widetilde{W}$. A subsequent interchange of the order of absolutely convergent finite sums and integrations by Fubini's Theorem gives

$$\mathscr{E}(C,U) = \frac{N(r)^{1/2}}{3^{7/2}(2\pi)^4}\Big(\frac{\overline{r}}{r}\Big)^{-\ell}\int_{-i(UN(r)(1+|\ell|))^{\varepsilon/1000}}^{i(UN(r)(1+|\ell|))^{\varepsilon/1000}}\int_{-i(UN(r)(1+|\ell|))^{\varepsilon/1000}}^{i(UN(r)(1+|\ell|))^{\varepsilon/1000}}\frac{\Gamma(\frac{5}{6}+\frac{|\ell|}{2}-w)\Gamma(\frac{7}{6}+\frac{|\ell|}{2}-w)}{\Gamma(w+\frac{|\ell|}{2}-\frac{1}{6})\Gamma(w+\frac{|\ell|}{2}+\frac{1}{6})}$$

$$\times \Big(\frac{(2\pi)^4 U}{N(r)^2}\Big)^w C^s\widetilde{V}(s)\widetilde{W}(w)$$

$$\times \Big(\sum_{\substack{\nu\in\lambda^{-1}\mathbb{Z}[\omega] \\ d\equiv 1 \pmod 3 \\ (d,r)=1 \\ N(d^3\nu)\ll\mathcal{Z}}}\frac{a^\dagger(\nu)b_r^\dagger(\nu)}{N(\nu)^{1-w}N(d)^{5/2-3w}}\Big(\frac{d^3\nu}{|d^3\nu|}\Big)^{-\ell}\sum_{\substack{c\in\mathbb{Z}[\omega] \\ c\equiv 1 \pmod 3 \\ (c,r)=1 \\ C\leqslant N(c)\leqslant 2C}}N(c)^{1/2-s-3w}\mu(c)\Big(\frac{c}{|c|}\Big)^{3\ell}\Big)dsdw$$

$$+ O_\varepsilon\big((UN(r)(1+|\ell|))^{-1000}\big).$$

We estimate the sum over c using the Riemann hypothesis for the Dedekind zeta function attached to $\mathbb{Q}(\omega)$, and the quotient of Gamma factors using Stirling's asymptotic [DLMF, (5.11.1)] The other sums are estimated trivially using (5.67), (5.68), (5.7), (5.13) and (5.14). We obtain

$$\mathscr{E}(C,U) \ll_\varepsilon N(r)^{1/2+\varepsilon/10}\cdot(1+\ell^2)$$

$$\times \sum_{e|r}\sum_{\substack{\nu\in\lambda^{-1}\mathbb{Z}[\omega] \\ d\equiv 1 \pmod 3 \\ (d,r)=1 \\ (\lambda\nu,r)=e \\ N(d^3\nu)\ll\mathcal{Z}}}\frac{|a^\dagger(\nu)|}{N(\nu)N(d)^{5/2}}\frac{\varphi(r)}{\varphi(r/e)}C^{1+\varepsilon/10}(1+|\ell|)^{\varepsilon/10},$$

$$\ll N(r)^{1/2+\varepsilon/4}U^{\varepsilon/4}(1+\ell^6)C^{1+\varepsilon/4}(1+|\ell|)^{\varepsilon/4},$$

and so

$$\sum_{\substack{C \text{ dyadic} \\ 1/2 \leqslant C \leqslant (UN(r))^{1/12+\varepsilon/2}}} \mathscr{E}(C, U) \ll_\varepsilon U^{1/12+\varepsilon} N(r)^{7/12+\varepsilon} \cdot (1 + \ell^6)(1 + |\ell|)^\varepsilon. \tag{8.10}$$

After combining (8.6), (8.9) and (8.10), we obtain the result.

∎

We also record the following nearly immediate Corollary.

Corollary 8.1. *Assume the Riemann Hypothesis for the family of Dedekind zeta functions attached to $\mathbb{Q}(\omega)$ twisted by Größencharaktern. Let $r \in \mathbb{Z}[\omega]$ be squarefree and satisfy $r \equiv 1 \pmod 3$. Let $\varepsilon \in (0, \frac{1}{10000})$, and $V, W : \mathbb{R} \to \mathbb{R}$ be smooth functions with compact support contained in $[\frac{1}{4}, 4]$. Then there exists $\rho(\varepsilon) \in (0, \frac{1}{10000})$ such that*

$$\sum_{\substack{u \in \mathbb{Z}[\omega] \\ u \equiv 1 \pmod 3}} \tilde{g}(ur) \left(\frac{u}{|u|}\right)^\ell \cdot V\left(\frac{N(u)}{U}\right) W\left(\frac{N(ur)}{X}\right)$$

$$= \delta_{\ell=0} \cdot \frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})} \sum_{\substack{u \equiv 1 \pmod 3 \\ (u,r)=1}} \frac{\mu^2(u)}{N(u)^{1/6}} V\left(\frac{N(u)}{U}\right) W\left(\frac{N(ur)}{X}\right) \cdot \frac{1}{N(r)^{1/6}}$$

$$+ O_\varepsilon\left((1 + |\ell|)^\varepsilon \cdot \left(\frac{U^{5/6-\rho(\varepsilon)}}{N(r)^{1/6+\rho(\varepsilon)}} + U^{1/12+\varepsilon} N(r)^{7/12+\varepsilon}(1 + \ell^6)\right) + \frac{U^{1/3+\varepsilon}}{N(r)^{1/6}}\right),$$

for all $\ell \in \mathbb{Z}$ and $X \geqslant U \geqslant 1$.

Proof. If $UN(r) \asymp X$ then we simply apply the previous result with a different weight function and use the Remark 8.2. If $UN(r)$ is not of the order of magnitude of X then both main terms are zero. ∎

Remark 8.4. The main term in Corollary 8.1 is larger than the error term when $U \geqslant (N(r)(1 + \ell^6))^{1+o(1)}$.

9. Improved cubic large sieve

The cubic large sieve of Heath–Brown is as follows.

Theorem 9.1. [HB00, Theorem 2] *Let $A, B \geqslant 1$, $\varepsilon > 0$, and $(\beta_b)_{b \in \mathbb{Z}[\omega]}$ be an arbitrary sequence of complex numbers with support contained in the set of squarefree elements of $\mathbb{Z}[\omega]$. Then*

$$\sum_{\substack{N(a) \leqslant A \\ a \equiv 1 \pmod 3}} \mu^2(a) \left| \sum_{\substack{N(b) \leqslant B \\ b \equiv 1 \pmod 3}} \beta_b \left(\frac{b}{a}\right)_3 \right|^2 \ll_\varepsilon \left(A + B + (AB)^{2/3}\right)(AB)^\varepsilon \sum_{b \in \mathbb{Z}[\omega]} |\beta_b|^2. \tag{9.1}$$

Recall the operator norm $\mathcal{B}(A, B)$ defined in (1.12). The Duality Principle [IK04, (7.9)–(7.11)] and cubic reciprocity imply that

$$\mathcal{B}(A, B) = \mathcal{B}(B, A). \tag{9.2}$$

See also [HB00, Lemma 4]. We also have the following simple monotonicity property.

Lemma 9.1. [HB00, Lemma 5] *There exists an absolute constant $C \geqslant 1$ as follows. Let $A, B_1, B_2 \gg 1$ and $B_2 \geqslant C B_1 \log(2AB_1)$. Then,*

$$\mathcal{B}(A, B_1) \ll \mathcal{B}(A, B_2).$$

We now prove that Heath-Brown's cubic large sieve is optimal under the Generalized Riemann Hypothesis for Hecke L-functions over $\mathbb{Q}(\omega)$.

Proof of Theorem 1.4. The upper bounds in the given ranges follow from the cubic large sieve Theorem 9.1, and are unconditional.

We now focus on the conditional lower bounds. Let $\xi > 0$ be small and fixed, $A, B \geqslant 10$, $X := AB$ and $A \in [10, X^{1/2-\xi}]$. Consider the sequence

$$\beta_b = \overline{\tilde{g}(b)} W\left(\frac{N(b)}{B}\right), \tag{9.3}$$

where W is a smooth compactly supported function in $(1, 2)$. It is supported only on squarefree elements by (2.2). Then

$$\sum_{\substack{A < N(a) \leqslant 2A \\ a \equiv 1 \pmod 3}} \mu^2(a) \Big| \sum_{\substack{B < N(b) \leqslant 2B \\ b \equiv 1 \pmod 3}} \overline{\tilde{g}(b)} W\left(\frac{N(b)}{B}\right) \left(\frac{b}{a}\right)_3 \Big|^2$$

$$= \sum_{\substack{A < N(a) \leqslant 2A \\ a \equiv 1 \pmod 3}} \mu^2(a) \Big| \sum_{b \equiv 1 \pmod 3} \overline{W\left(\frac{N(b)}{B}\right)} \tilde{g}(ab) \Big|^2 \quad \text{(by (2.2) and (2.3))}$$

$$\gg A \left(\frac{B^{5/6}}{A^{1/6}}\right)^2 + O\left(X^{o(1)}(A^{17/12} B^{11/12} + A^{13/6} B^{1/6})\right) \tag{9.4}$$

$$\gg_{\xi, W} A^{2/3} B^{5/3}, \tag{9.5}$$

where display (9.4) follows from Voronoi summation (Proposition 8.1) and the GRH hypothesis, and (9.5) follows from the fact that we are in the range $A \ll (AB)^{1/2-\xi}$. Thus $\mathcal{B}(A, B) \gg_\xi (AB)^{2/3} \gg A + B$ for $A \ll (AB)^{1/2-\xi}$ and $A \in [X^{1/3}, X^{1/2}]$. Combining this result with (9.2) then gives the claim when $A \in [\sqrt{B}, B^2] \setminus [B^{1-\xi}, B^{1+2\xi}]$. The result in the range $A \in [B^{1-\xi}, B^{1+2\xi}]$ then follows from (9.2) and Lemma 9.1, so $\mathcal{B}(A, B) \gg \mathcal{B}(AX^{-3\xi}, B) \gg (AB)^{2/3-3\xi}$. ∎

In light of the proof of Theorem 1.4, we renormalise the sequences we consider in the cubic large sieve by setting $c_b := \tilde{g}(b) \beta_b$ where $\boldsymbol{\beta} := (\beta_b)_{b \in \mathbb{Z}[\omega]}$ is a sequence supported on squarefree $b \equiv 1 \pmod 3$. Note that $|c_b| = |\beta_b|$ on squarefree $b \equiv 1 \pmod 3$ by (2.2). We are able to refine Theorem 9.1 in a special case by:

(1) Introducing a non-trivial asymptotic main term;
(2) Assuming additional cancellations/density restrictions for the sequence $\boldsymbol{\beta} = (\beta_b)$.

Proposition 9.1. *Let $V : (0, \infty) \to \mathbb{R}_{\geqslant 0}$ be a smooth compactly supported function, $0 < \eta \leqslant 1/4$, $A, B, w \geqslant 10$ and $X := AB$. Suppose that $w > (\log X)^{10}$ and $\boldsymbol{\beta} = (\beta_b) \in \mathcal{C}_\eta(B, w)$. Let $\varepsilon \in (0, \frac{1}{10000})$ and $\pi \equiv 1 \pmod 3$ be a prime or 1. Then there exists*

$\rho(\varepsilon) \in (0, \frac{1}{10000})$ *such that uniformly in* $1 \leqslant N(\pi) < w$ *we have*

$$\sum_{\substack{a \in \mathbb{Z}[\omega] \\ a \equiv 1 \pmod 3 \\ \pi | a}} \mu^2(a) V\Big(\frac{N(a)}{A}\Big) \Big| \sum_{\substack{b \in \mathbb{Z}[\omega] \\ (b,a)=1}} \beta_b \tilde{g}(b) \overline{\Big(\frac{a}{b}\Big)_3} - \frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})} \frac{\overline{\tilde{g}(a)}}{N(a)^{1/6}} \sum_{\substack{b \in \mathbb{Z}[\omega] \\ (b,a)=1}} \frac{\beta_b}{N(b)^{1/6}} \Big|^2$$

$$\ll_\varepsilon \frac{A^{2/3}B^{5/3}}{N(\pi)}\Big(\frac{1}{w^{9/10}} + \frac{\delta_{\pi \neq 1}}{N(\pi)}\Big) + \frac{A^{2/3-\rho(\varepsilon)}B^{5/3-\rho(\varepsilon)}}{N(\pi)} + \frac{A^{1/6+\varepsilon}B^{5/3}}{N(\pi)^{1/2+\varepsilon}}$$

$$+ (N(\pi)X)^\varepsilon \Big(N(\pi)^{1/2}B^{29/12}A^{-1/12} + B^{2+2\eta} + \frac{X}{N(\pi)}\Big(1 + (B^2/A)^{-1000}\Big)\Big).$$

Proposition 9.1 will follow from a modification of the proof of Proposition 9.2 using sieve weights, and from Lemma 9.2 below. At the close of this section we sketch how Proposition 9.1 follows.

Proposition 9.2. *Let* $V : (0,\infty) \to \mathbb{R}_{\geqslant 0}$ *be a smooth compactly supported function,* $0 < \eta \leqslant 1/4$, $A, B, w \geqslant 10$ *and* $X := AB$. *Suppose that* $w > (\log X)^{10}$ *and* $\boldsymbol{\beta} = (\beta_b) \in \mathcal{C}_\eta(B, w)$. *Let* $\varepsilon \in (0, \frac{1}{10000})$, $\Delta \geqslant 1$, $\pi \equiv 1 \pmod 3$ *be a prime or* 1, *and* $\gamma \equiv 1 \pmod 3$ *be squarefree such that* $(\pi, \gamma) = 1$. *Then there exists* $\rho(\varepsilon) \in (0, \frac{1}{10000})$ *such that uniformly in* $1 \leqslant N(\gamma) \leqslant \Delta$ *and* $1 \leqslant N(\pi) < w$ *we have*

$$\sum_{\substack{a \in \mathbb{Z}[\omega] \\ a \equiv 1 \pmod 3 \\ \pi\gamma^2 | a}} V\Big(\frac{N(a)}{A}\Big) \Big| \sum_{\substack{b \in \mathbb{Z}[\omega] \\ (b,a)=1}} \beta_b \tilde{g}(b) \overline{\Big(\frac{b}{a}\Big)_3} - \frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})} \frac{\overline{\tilde{g}(a)}}{N(a)^{1/6}} \sum_{\substack{b \in \mathbb{Z}[\omega] \\ (b,a)=1}} \frac{\beta_b}{N(b)^{1/6}} \Big|^2 \tag{9.6}$$

$$\leqslant 2\Big(\frac{1}{N(\gamma)^2} - \frac{\delta_{\gamma=1}}{\zeta_{\mathbb{Q}(\omega)}(2; \mathbf{1}_\pi)}\Big) \tilde{V}\Big(\frac{2}{3}\Big) \frac{2\pi}{9\sqrt{3}} \Big(\frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})}\Big)^3 \frac{A^{2/3}}{N(\pi)} \Big| \sum_{b \in \mathbb{Z}[\omega]} \frac{\beta_b}{N(b)^{1/6}} \Big|^2$$

$$+ O_\varepsilon\Big(\frac{1}{N(\pi\gamma^2)}\Big(\frac{A^{2/3}B^{5/3}}{w^{9/10}} + (\Delta N(\pi)X)^\varepsilon\Big(B^{2+2\eta}N(\pi)\Delta^2 + X\Big(1 + (B^2/A)^{-1000}\Big)\Big)\Big)\Big)$$

$$+ \delta_{\gamma=1} \cdot \Big(O_\varepsilon\Big(\frac{A^{2/3-\rho(\varepsilon)}B^{5/3-\rho(\varepsilon)}}{N(\pi)} + \frac{A^{2/3}B^{5/3}}{N(\pi)}\Big(\frac{1}{w^{9/10}} + \frac{\delta_{\pi \neq 1}}{N(\pi)}\Big)$$

$$+ X^\varepsilon N(\pi)^{1/2}B^{29/12}A^{-1/12} + \frac{A^{1/6+\varepsilon}B^{5/3}}{N(\pi)^{1/2+\varepsilon}}\Big)\Big).$$

Proof. Using (2.2) and inclusion-exclusion with the condition $(a, b) = 1$, we see that the expression in (9.6) is equal to

$$\sum_{\substack{a \in \mathbb{Z}[\omega] \\ a \equiv 1 \pmod 3 \\ \pi\gamma^2 | a}} V\Big(\frac{N(a)}{A}\Big) \Big|\Big(\sum_{\substack{b \in \mathbb{Z}[\omega] \\ (b,\gamma a)=1}} \beta_b \tilde{g}(b) \overline{\Big(\frac{b}{a}\Big)_3} - \frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})} \frac{\delta_{\gamma=1}\overline{\tilde{g}(a)}}{N(a)^{1/6}} \sum_{b \in \mathbb{Z}[\omega]} \frac{\beta_b}{N(b)^{1/6}}\Big)$$

$$+ \Big(\frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})} \frac{\delta_{\gamma=1}\overline{\tilde{g}(a)}}{N(a)^{1/6}} \sum_{\substack{b \in \mathbb{Z}[\omega] \\ (b,a)\neq 1}} \frac{\beta_b}{N(b)^{1/6}}\Big)\Big|^2. \tag{9.7}$$

Since V is non-negative we can apply the parallelogram identity

$$|X + Y|^2 \leqslant 2(|X|^2 + |Y|^2) \quad \text{for all} \quad X, Y \in \mathbb{C}, \tag{9.8}$$

to the display above. This shows that (9.7) is

$$\leqslant 2\Big(\sum_{\substack{a \in \mathbb{Z}[\omega] \\ a \equiv 1 \pmod 3 \\ \pi\gamma^2 | a}} V\Big(\frac{N(a)}{A}\Big) \Big| \sum_{\substack{b \in \mathbb{Z}[\omega] \\ (b,\gamma a)=1}} \beta_b \tilde{g}(b) \overline{\Big(\frac{b}{a}\Big)_3} - \frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})} \frac{\delta_{\gamma=1} \overline{\tilde{g}(a)}}{N(a)^{1/6}} \sum_{b \in \mathbb{Z}[\omega]} \frac{\beta_b}{N(b)^{1/6}} \Big|^2 \tag{9.9}$$

$$+ \delta_{\gamma=1} \frac{(2\pi)^{4/3}}{9\Gamma(\frac{2}{3})^2} \sum_{\substack{a \in \mathbb{Z}[\omega] \\ a \equiv 1 \pmod 3 \\ \pi | a}} \frac{\mu^2(a)}{N(a)^{1/3}} V\Big(\frac{N(a)}{A}\Big) \Big| \sum_{\substack{b \in \mathbb{Z}[\omega] \\ (b,a) \neq 1}} \frac{\beta_b}{N(b)^{1/6}} \Big|^2 \Big), \tag{9.10}$$

where we used (2.2) to obtain the last display. The term in (9.10) is equal to

$$\delta_{\gamma=1} \frac{(2\pi)^{4/3}}{9\Gamma(\frac{2}{3})^2} \sum_{b_1,b_2 \in \mathbb{Z}[\omega]} \frac{\beta_{b_1} \overline{\beta_{b_2}}}{N(b_1 b_2)^{1/6}} \sum_{\substack{a \in \mathbb{Z}[\omega] \\ a \equiv 1 \pmod 3 \\ \pi | a \\ (a,b_1) \neq 1 \\ (a,b_2) \neq 1}} \frac{\mu^2(a)}{N(a)^{1/3}} V\Big(\frac{N(a)}{A}\Big) = \delta_{\gamma=1} O\Big(\frac{A^{2/3} B^{5/3}}{N(\pi) w^{9/10}}\Big). \tag{9.11}$$

The estimate in (9.11) follows from the triangle inequality, the fact $(b_1 b_2, \pi) = 1$ ($b_1 b_2$ is w-rough and π is w-smooth) and

$$\sum_{\substack{\varpi | b_1 b_2 \\ \varpi \text{ prime}}} \frac{1}{N(\varpi)} \ll \frac{\log B^2}{\log\log B^2} \frac{1}{w} \ll \frac{1}{w^{9/10}}, \quad \text{for} \quad w > (\log X)^{10} \quad \text{say.}$$

We repeatedly use this w-roughness argument in the course of the proof.

It suffices to compute the term in (9.9). We make the change of variable $a \mapsto \pi\gamma^2 a$. After using cubic reciprocity and (2.2), it suffices to compute

$$\sum_{\substack{a \in \mathbb{Z}[\omega] \\ a \equiv 1 \pmod 3}} V\Big(\frac{N(\pi\gamma^2)N(a)}{A}\Big) \Big| \sum_{\substack{b \in \mathbb{Z}[\omega] \\ (b,\pi\gamma a)=1}} \beta_b \tilde{g}(b) \overline{\Big(\frac{a}{b}\Big)_3} \Big(\frac{\pi^2\gamma}{b}\Big)_3 - \frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})} \frac{\delta_{\gamma=1} \overline{\tilde{g}(a\pi)}}{N(a\pi)^{1/6}} \sum_{b \in \mathbb{Z}[\omega]} \frac{\beta_b}{N(b)^{1/6}} \Big|^2. \tag{9.12}$$

Expansion of the square in (9.12) shows that we need to evaluate the diagonal term

$$\mathscr{D} := \sum_{\substack{a \in \mathbb{Z}[\omega] \\ a \equiv 1 \pmod 3}} V\Big(\frac{N(\pi\gamma^2)N(a)}{A}\Big) \Big| \sum_{\substack{b \in \mathbb{Z}[\omega] \\ (b,\pi\gamma a)=1}} \beta_b \tilde{g}(b) \overline{\Big(\frac{a}{b}\Big)_3} \Big(\frac{\pi^2\gamma}{b}\Big)_3 \Big|^2; \tag{9.13}$$

the cross term

$$\mathscr{C} := -2\frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})}\delta_{\gamma=1}\mathrm{Re}\Big(\sum_{b_1,b_2\in\mathbb{Z}[\omega]}\frac{\beta_{b_1}\tilde{g}(b_1)\overline{\beta_{b_2}}}{N(b_2)^{1/6}}$$

$$\times \sum_{\substack{a\in\mathbb{Z}[\omega]\\a\equiv 1\pmod 3\\(a\pi,b_1)=1}} V\Big(\frac{N(a)N(\pi)}{A}\Big)\overline{\Big(\frac{a}{b_1}\Big)_3}\Big(\frac{\pi^2}{b_1}\Big)_3\frac{\tilde{g}(a\pi)}{N(a\pi)^{1/6}}\Big); \quad (9.14)$$

and the trivial term,

$$\mathscr{T} := \Big(\frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})}\Big)^2\delta_{\gamma=1}\sum_{\substack{a\in\mathbb{Z}[\omega]\\a\equiv 1\pmod 3}} V\Big(\frac{N(a)N(\pi)}{A}\Big)\frac{\mu^2(a\pi)}{N(a\pi)^{1/3}}\Big|\sum_{b\in\mathbb{Z}[\omega]}\frac{\beta_b}{N(b)^{1/6}}\Big|^2. \quad (9.15)$$

The appearance of $\mu^2(a\pi)$ in $\mathscr{T}$ is due to $|\tilde{g}(a\pi)|^2 = \mu^2(a\pi)$ (a consequence of (2.2)).

In the course of this proof we will establish the three asymptotic estimates:

$$\mathscr{D} = \frac{2\pi}{9\sqrt{3}}\Big(\frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})}\Big)^2\widetilde{V}\Big(\frac{2}{3}\Big)\frac{A^{2/3}}{N(\pi\gamma^2)}\Big|\sum_{b\in\mathbb{Z}[\omega]}\frac{\beta_b}{N(b)^{1/6}}\Big|^2 + O\Big(\frac{A^{2/3}B^{5/3}}{N(\pi\gamma^2)w^{9/10}}\Big)$$

$$+ O_\varepsilon\Big((\Delta N(\pi)X)^\varepsilon\Big(\frac{B^{2+2\eta}N(\pi)\Delta^2}{N(\pi\gamma^2)} + \frac{X}{N(\pi\gamma^2)}\big(1 + (B^2/A)^{-1000}\big)\Big)\Big), \quad (9.16)$$

$$\mathscr{C} = -2\widetilde{V}\Big(\frac{2}{3}\Big)\delta_{\gamma=1}\frac{2\pi}{9\sqrt{3}\zeta_{\mathbb{Q}(\omega)}(2;\mathbf{1}_\pi)}\Big(\frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})}\Big)^2\frac{A^{2/3}}{N(\pi)}\Big|\sum_{b\in\mathbb{Z}[\omega]}\frac{\beta_b}{N(b)^{1/6}}\Big|^2$$

$$+ \delta_{\gamma=1}\Big(O_\varepsilon\Big(\frac{A^{2/3-\rho(\varepsilon)}B^{5/3-\rho(\varepsilon)}}{N(\pi)}\Big) + O_\varepsilon(X^\varepsilon N(\pi)^{1/2}B^{29/12}A^{-1/12})$$

$$+ O\Big(\frac{A^{2/3}B^{5/3}}{N(\pi)}\Big(\frac{1}{w^{9/10}} + \frac{\delta_{\pi\neq 1}}{N(\pi)}\Big)\Big)\Big), \quad (9.17)$$

and

$$\mathscr{T} = \delta_{\gamma=1}\widetilde{V}\Big(\frac{2}{3}\Big)\frac{2\pi}{9\sqrt{3}\zeta_{\mathbb{Q}(\omega)}(2;\mathbf{1}_\pi)}\Big(\frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})}\Big)^2\frac{A^{2/3}}{N(\pi)}\Big|\sum_{b\in\mathbb{Z}[\omega]}\frac{\beta_b}{N(b)^{1/6}}\Big|^2$$

$$+ \delta_{\gamma=1}\cdot\delta_{\pi\neq 1}\cdot O\Big(\frac{A^{2/3}B^{5/3}}{N(\pi)^2}\Big) + \delta_{\gamma=1}O_\varepsilon\Big(\frac{A^{1/6+\varepsilon}B^{5/3}}{N(\pi)^{1/2+\varepsilon}}\Big). \quad (9.18)$$

Thus $\mathscr{D} + \mathscr{C} + \mathscr{T}$ using the asymptotics (9.16), (9.17) and (9.18) respectively gives an asymptotic expression for (9.12). Substitution of this asymptotic expression into (9.9), and subsitition of (9.11) into (9.10), will give the result.

We now turn our attention to proving (9.16)–(9.18).

The diagonal term $\mathscr{D}$. After expansion of the square in (9.13), we obtain

$$\mathscr{D} = \sum_{\substack{b_1,b_2\in\mathbb{Z}[\omega]\\(b_1b_2,\pi\gamma)=1}} \beta_{b_1}\tilde{g}(b_1)\overline{\beta_{b_2}\tilde{g}(b_2)}\Big(\frac{\pi^2\gamma}{b_1}\Big)_3\overline{\Big(\frac{\pi^2\gamma}{b_2}\Big)_3}\sum_{\substack{a\in\mathbb{Z}[\omega]\\a\equiv 1\pmod 3}} V\Big(\frac{N(\pi\gamma^2)N(a)}{A}\Big)\overline{\Big(\frac{a}{b_1}\Big)_3}\Big(\frac{a}{b_2}\Big)_3.$$

If $(b_1, b_2) = d$, then recall that (7.13) tells us that

$$\tilde{g}(b_1)\overline{\tilde{g}(b_2)} = \tilde{g}(b_1/d)\overline{\tilde{g}(b_2/d)}\overline{\left(\frac{d}{b_1/d}\right)_3}\left(\frac{d}{b_2/d}\right)_3.$$

Thus an application of Poisson summation (in the form of Corollary 4.1) on the a sum shows that

$$\mathscr{D} = \frac{4\pi A}{9\sqrt{3}N(\pi\gamma^2)} \sum_{\substack{d\in\mathbb{Z}[\omega] \\ d\equiv 1 \pmod 3 \\ (d,\pi\gamma)=1}} \sum_{\substack{b_1,b_2\in\mathbb{Z}[\omega] \\ (b_1,b_2)=d}} \frac{\beta_{b_1}\overline{\beta_{b_2}}}{\sqrt{N(b_1b_2)}}$$

$$\times \sum_{k\in\mathbb{Z}[\omega]} \tilde{c}_d(k)\left(\frac{d\pi^2\gamma k}{b_1/d}\right)_3\overline{\left(\frac{d\pi^2\gamma k}{b_2/d}\right)_3}\ddot{V}\left(\frac{kd\sqrt{A}}{b_1b_2N(\pi\gamma^2)^{1/2}}\right). \tag{9.19}$$

For a given $d, \pi, \gamma \in \mathbb{Z}[\omega]$ in (9.19), we split the k sum into two subsums:

- $k \in \mathbb{Z}[\omega]$ such that $d\pi^2\gamma k = \square$;
- $k \in \mathbb{Z}[\omega]$ such that $d\pi^2\gamma k \neq \square$.

Denote the contributions to $\mathscr{D}$ from each of these two cases by $\mathscr{D}_1$ and $\mathscr{D}_2$ respectively. Thus $\mathscr{D} = \mathscr{D}_1 + \mathscr{D}_2$.

Consider $\mathscr{D}_1$. Since $\mu^2(d\pi\gamma) = 1$, we deduce that $d\pi^2\gamma k = \square$ if and only if $k = (d\gamma)^2\pi H$ for some $H \in \mathbb{Z}[\omega]$ with $H = \square$. Observe that (4.8) and Lemma 5.5 imply that $\tilde{c}_d((d\gamma)^2\pi H) = \check{e}\left(-\frac{(d\gamma)^2\pi H}{3\lambda}\right)\varphi(d)$. Thus

$$\mathscr{D}_1 = \frac{4\pi A}{9\sqrt{3}N(\pi\gamma^2)} \sum_{\substack{d\in\mathbb{Z}[\omega] \\ d\equiv 1 \pmod 3 \\ (d,\pi\gamma)=1}} \varphi(d) \sum_{\substack{b_1,b_2\in\mathbb{Z}[\omega] \\ (b_1,b_2)=d \\ (b_1b_2/d^2,\pi\gamma)=1}} \frac{\beta_{b_1}\overline{\beta_{b_2}}}{\sqrt{N(b_1b_2)}}$$

$$\times \sum_{\substack{H\in\mathbb{Z}[\omega] \\ H=\square \\ (H,b_1b_2/d^2)=1}} \check{e}\left(-\frac{(d\gamma)^2\pi H}{3\lambda}\right)\ddot{V}\left(\frac{d^3H\sqrt{A}}{b_1b_2}\right). \tag{9.20}$$

We further write $\mathscr{D}_1 = \mathscr{D}_1^\star + \mathscr{D}_1^\diamond$, where $\mathscr{D}_1^\star$ denotes the sum in (9.20) restricted to $d = 1$, and $\mathscr{D}_1^\diamond$ denotes the sum in (9.20) restricted to $d \neq 1$. The support of $\boldsymbol{\beta}$ guarantees that $d \neq 1$ implies that $N(d) > w$. Thus by Lemma 4.4 we have

$$\mathscr{D}_1^\diamond \ll \frac{A}{N(\pi\gamma^2)} \sum_{\substack{d\in\mathbb{Z}[\omega] \\ (d,\pi\gamma)=1 \\ d\equiv 1 \pmod 3 \\ N(d)>w}} \varphi(d) \sum_{\substack{b_1,b_2\in\mathbb{Z}[\omega] \\ (b_1,b_2)=d \\ (b_1b_2/d^2,\pi\gamma)=1}} \frac{|\beta_{b_1}\beta_{b_2}|}{\sqrt{N(b_1)N(b_2)}}\left(\frac{N(b_1b_2)^{1/3}}{N(d)A^{1/3}} + 1\right)$$

$$\ll_\varepsilon \frac{A^{2/3}B^{5/3}}{wN(\pi\gamma^2)} + \frac{ABX^\varepsilon}{N(\pi\gamma^2)} \ll_\varepsilon \frac{A^{2/3}B^{5/3}}{wN(\pi\gamma^2)} + \frac{X^{1+\varepsilon}}{N(\pi\gamma^2)}. \tag{9.21}$$

We now consider $\mathscr{D}_1^\star$. We write $H = h^3$ with $0 \neq h \in \mathbb{Z}[\omega]$ (h is necessarily non-zero in this case). We have

$$\check{e}\left(-\frac{\gamma^2\pi h^3}{3\lambda}\right) = \check{e}\left(-\frac{h^3}{3\lambda}\right) = 1. \tag{9.22}$$

This can be seen by writing $h = \zeta\lambda^i u$ with $u \equiv 1 \pmod 3$, $\zeta \in \{\pm 1, \pm\omega, \pm\omega^2\}$, and $i \in \mathbb{Z}_{\geqslant 0}$. Then the last equality in (9.22) follows from

$$\check{e}\left(-\frac{(\zeta\lambda^i u)^3}{3\lambda}\right) = \check{e}\left(\pm\frac{\lambda^{3i}}{3\lambda}\right) = 1.$$

Thus

$$\mathscr{D}_1^\star = \frac{A}{N(\pi\gamma^2)} \sum_{\substack{b_1,b_2\in\mathbb{Z}[\omega] \\ (b_1,b_2)=1 \\ (b_1b_2,\pi\gamma)=1}} \frac{\beta_{b_1}\overline{\beta_{b_2}}}{\sqrt{N(b_1)N(b_2)}} \cdot \frac{4\pi}{3^3\sqrt{3}} \sum_{\substack{h\in\mathbb{Z}[\omega] \\ (h,b_1b_2)=1}} \ddot{V}\left(\frac{h^3\sqrt{A}}{b_1b_2}\right). \tag{9.23}$$

Note that the extra factor of $1/3$ in the above display accounts for the fact that $(\omega^i h)^3 = h^3$ for $i \in \{0,1,2\}$ and $0 \neq h \in \mathbb{Z}[\omega]$. We remove the condition $(h, b_1b_2) = 1$ at negligible cost since $\boldsymbol{\beta}$ is supported on w-rough squarefree integers in $\mathbb{Z}[\omega]$ with $w > (\log X)^{10}$. Thus

$$\mathscr{D}_1^\star = \frac{A}{N(\pi\gamma^2)} \sum_{\substack{b_1,b_2\in\mathbb{Z}[\omega] \\ (b_1,b_2)=1 \\ (b_1b_2,\pi\gamma)=1}} \frac{\beta_{b_1}\overline{\beta_{b_2}}}{\sqrt{N(b_1)N(b_2)}} \cdot \frac{4\pi}{3^3\sqrt{3}} \sum_{h\in\mathbb{Z}[\omega]} \ddot{V}\left(\frac{h^3\sqrt{A}}{b_1b_2}\right)$$

$$+ O\left(\frac{A^{2/3}B^{5/3}}{N(\pi\gamma^2)w^{9/10}}\right) + O\left(\frac{AB}{N(\pi\gamma^2)w^{9/10}}\right). \tag{9.24}$$

Observe that $\ddot{V}(u) = \ddot{V}(|u|)$ is a Schwarz function by Lemma 4.4. Application of Poisson summation (in the form of Lemma 4.1) to the sum over $h \in \mathbb{Z}[\omega]$ yields

$$\frac{4\pi}{3^3\sqrt{3}} \sum_{h\in\mathbb{Z}[\omega]} \ddot{V}\left(\frac{h^3\sqrt{A}}{b_1b_2}\right) = \frac{8\pi}{3^4} \sum_{m\in\mathbb{Z}[\omega]} \int_{\mathbb{R}^2} \ddot{V}\left(\frac{(x+iy)^3\sqrt{A}}{b_1b_2}\right) \check{e}\left(\frac{m(x+iy)}{\lambda}\right) dxdy. \tag{9.25}$$

We simplify the right side of (9.25). Recall that $\ddot{V}(u) = \ddot{V}(|u|)$ is radial. After changing $x + iy$ into polar coordinates $re^{i\vartheta}$, the right side of (9.25) becomes

$$\frac{8\pi}{3^4} \sum_{m\in\mathbb{Z}[\omega]} \int_0^{2\pi} \int_0^\infty \ddot{V}\left(\frac{r^3\sqrt{A}}{\sqrt{N(b_1b_2)}}\right) \check{e}\left(\frac{mre^{i\vartheta}}{\lambda}\right) rdrd\vartheta$$

$$= \frac{8\pi}{3^4} \frac{N(b_1b_2)^{1/3}}{A^{1/3}} \sum_{m\in\mathbb{Z}[\omega]} \int_0^{2\pi} \int_0^\infty \ddot{V}(r^3) \check{e}\left(\frac{mre^{i\vartheta}}{\lambda} \frac{N(b_1b_2)^{1/6}}{A^{1/6}}\right) rdrd\vartheta. \tag{9.26}$$

For all $m \in \mathbb{Z}[\omega]$, Lemma 4.4 implies that

$$\int_0^{2\pi} \int_0^\infty \ddot{V}(r^3) \check{e}\left(\frac{mre^{i\vartheta}}{\lambda} \frac{N(b_1b_2)^{1/6}}{A^{1/6}}\right) rdrd\vartheta$$

$$= \int_0^{2\pi} \int_0^\infty \ddot{V}(r^3)\delta_{[0,X^\varepsilon]}(r) \check{e}\left(\frac{mre^{i\vartheta}}{\lambda} \frac{N(b_1b_2)^{1/6}}{A^{1/6}}\right) rdrd\vartheta + O_\varepsilon(X^{-2000}). \tag{9.27}$$

For $0 \neq m \in \mathbb{Z}[\omega]$, repeated integration by parts on the right side of (9.27) shows that

$$\int_0^{2\pi} \int_0^\infty \ddot{V}(r^3)\delta_{[0,X^\varepsilon]}(r) \check{e}\left(\frac{mre^{i\vartheta}}{\lambda} \frac{N(b_1b_2)^{1/6}}{A^{1/6}}\right) rdrd\vartheta \ll_\varepsilon X^\varepsilon (B^2/A)^{-1000} N(m)^{-1000}. \tag{9.28}$$

We substitute (9.27) and (9.28) into (9.26), and then sum trivially over $0 \neq m \in \mathbb{Z}[\omega]$. Chasing the result through (9.25) gives

$$\frac{4\pi}{3^3\sqrt{3}} \sum_{h \in \mathbb{Z}[\omega]} \ddot{V}\Big(\frac{h^3\sqrt{A}}{b_1 b_2}\Big) = \frac{16\pi^2}{3^4} \frac{N(b_1 b_2)^{1/3}}{A^{1/3}} \int_0^\infty \ddot{V}(r^3) r \, dr + O_\varepsilon\big(X^\varepsilon (B^2/A)^{-1000}\big). \quad (9.29)$$

We now evaluate the main term on the right side of (9.29). We open $\ddot{V}$ using the definition (4.4), and find that the main term is

$$\frac{16\pi^2}{3^4} \frac{N(b_1 b_2)^{1/3}}{A^{1/3}} \int_0^\infty u V(u^2) \int_0^\infty J_0\Big(\frac{4\pi r^3 u}{3\sqrt{3}}\Big) r \, dr \, du. \quad (9.30)$$

For each fixed $u \in (0, \infty)$, we make the change of variable $w = 4\pi r^3 u/(3\sqrt{3})$ in the r-integral. Thus (9.30) becomes

$$\frac{(4\pi)^{4/3}}{3^4} \frac{N(b_1 b_2)^{1/3}}{A^{1/3}} \int_0^\infty u^{1/3} V(u^2) \, du \int_0^\infty J_0(w) w^{-1/3} \, dw. \quad (9.31)$$

A change of variable shows that

$$\int_0^\infty u^{s-1} V(u^2) \, du = \frac{1}{2} \widetilde{V}\Big(\frac{s}{2}\Big), \quad \text{for} \quad s \in \mathbb{C}, \quad (9.32)$$

and [DLMF, (10.22.43)] implies that

$$\int_0^\infty w^{s-1} J_0(w) \, dw = \frac{2^{s-1}\Gamma(\frac{s}{2})}{\Gamma(1 - \frac{s}{2})}, \quad \text{for} -1 < \text{Re}(s-1) < 1/2. \quad (9.33)$$

Using (9.32), (9.33) and Euler's reflection formula [DLMF, (5.5.3)], we see that (9.31) becomes

$$\frac{2\pi}{9\sqrt{3}} \Big(\frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})}\Big)^2 \widetilde{V}\Big(\frac{2}{3}\Big) \frac{N(b_1 b_2)^{1/3}}{A^{1/3}}. \quad (9.34)$$

After retracing (9.29) (9.30), (9.31) and (9.34), we obtain

$$\frac{4\pi}{3^3\sqrt{3}} \sum_{h \in \mathbb{Z}[\omega]} \ddot{V}\Big(\frac{h^3\sqrt{A}}{b_1 b_2}\Big) = \frac{2\pi}{9\sqrt{3}} \Big(\frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})}\Big)^2 \widetilde{V}\Big(\frac{2}{3}\Big) \frac{N(b_1 b_2)^{1/3}}{A^{1/3}} + O_\varepsilon\big(X^\varepsilon (B^2/A)^{-1000}\big). \quad (9.35)$$

Substitution of (9.35) into (9.24) gives

$$\mathscr{D}_1^\star = \frac{2\pi}{9\sqrt{3}} \Big(\frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})}\Big)^2 \widetilde{V}\Big(\frac{2}{3}\Big) \frac{A^{2/3}}{N(\pi\gamma^2)} \sum_{\substack{b_1, b_2 \in \mathbb{Z}[\omega] \\ (b_1, b_2) = 1 \\ (b_1 b_2, \pi\gamma) = 1}} \frac{\beta_{b_1} \overline{\beta_{b_2}}}{N(b_1)^{1/6} N(b_2)^{1/6}}$$

$$+ O\Big(\frac{A^{2/3} B^{5/3}}{N(\pi\gamma^2) w^{9/10}}\Big) + O\Big(\frac{AB}{N(\pi\gamma^2) w^{9/10}}\Big) + O_\varepsilon\Big(\frac{X^{1+\varepsilon}}{N(\pi\gamma^2)} (B^2/A)^{-1000}\Big). \quad (9.36)$$

Using w-roughness of the support of $\boldsymbol{\beta}$, we drop the conditions $(b_1, b_2) = 1$ and $(b_1 b_2, \pi\gamma) = 1$ at the expense of the error term of the same order of magnitude of that occurring in

(9.36). After recalling that $\mathscr{D}_1 = \mathscr{D}_1^\star + \mathscr{D}_1^\diamond$, (9.21), and (9.36), we obtain

$$\mathscr{D}_1 = \frac{2\pi}{9\sqrt{3}}\Big(\frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})}\Big)^2 \widetilde{V}\Big(\frac{2}{3}\Big)\frac{A^{2/3}}{N(\pi\gamma^2)}\Big| \sum_{b\in\mathbb{Z}[\omega]} \frac{\beta_b}{N(b)^{1/6}}\Big|^2$$

$$+ O\Big(\frac{A^{2/3}B^{5/3}}{N(\pi\gamma^2)w^{9/10}}\Big) + O_\varepsilon\Big(\frac{X^{1+\varepsilon}}{N(\pi\gamma^2)}\big(1 + (B^2/A)^{-1000}\big)\Big). \tag{9.37}$$

We now consider $\mathscr{D}_2$,

$$\mathscr{D}_2 = \frac{4\pi A}{9\sqrt{3}N(\pi\gamma^2)} \sum_{\substack{d\in\mathbb{Z}[\omega]\\ d\equiv 1 \pmod 3\\ (d,\pi\gamma)=1}} \sum_{\substack{b_1,b_2\in\mathbb{Z}[\omega]\\ (b_1,b_2)=d}} \frac{\beta_{b_1}\overline{\beta_{b_2}}}{\sqrt{N(b_1 b_2)}}$$

$$\times \sum_{\substack{k\in\mathbb{Z}[\omega]\\ d\pi^2\gamma k\neq\boxdot}} \widetilde{c}_d(k)\Big(\frac{d\pi^2\gamma k}{b_1/d}\Big)_3 \overline{\Big(\frac{d\pi^2\gamma k}{b_2/d}\Big)_3} \ddot{V}\Big(\frac{kd\sqrt{A}}{b_1 b_2 N(\pi\gamma^2)^{1/2}}\Big). \tag{9.38}$$

We rescale $b_1 \to db_1$ and $b_2 \to db_2$ and use Lemma 4.4 in (9.38). We obtain

$$\mathscr{D}_2 = \frac{4\pi A}{9\sqrt{3}N(\pi\gamma^2)} \sum_{\substack{d\in\mathbb{Z}[\omega]\\ d\equiv 1 \pmod 3\\ (d,\pi\gamma)=1}} \frac{1}{N(d)} \sum_{\substack{b_1,b_2\in\mathbb{Z}[\omega]\\ (b_1 b_2,\pi\gamma)=1\\ (b_1,b_2)=1}} \frac{\beta_{db_1}\overline{\beta_{db_2}}}{\sqrt{N(b_1)N(b_2)}}$$

$$\times \sum_{\substack{k\in\mathbb{Z}[\omega]\\ d\pi^2\gamma k\neq\boxdot\\ N(k)\ll\mathcal{Z}}} \widetilde{c}_d(k)\Big(\frac{d\pi^2\gamma k}{b_1}\Big)_3 \overline{\Big(\frac{d\pi^2\gamma k}{b_2}\Big)_3} \ddot{V}\Big(\frac{k\sqrt{A}}{db_1 b_2 N(\pi\gamma^2)^{1/2}}\Big)$$

$$+ O_\varepsilon\big((N(\pi)\Delta X)^{-1000}\big), \tag{9.39}$$

where

$$\mathcal{Z} := (\Delta N(\pi)X)^\varepsilon\Big(1 + \frac{B^2 N(\pi)\Delta^2}{N(d)A}\Big).$$

We Möbius invert $(b_1, b_2) = 1$ and separate variables by opening $\ddot{V}$ using (7.18) and (7.19). Rearranging the absolutely convergent finite sums and integrals by Fubini's theorem gives

$$\mathscr{D}_2 = \frac{2A(-1)^L}{9i\sqrt{3}N(\pi\gamma^2)} \int_0^\infty \int_{-\varepsilon-iX^\varepsilon}^{-\varepsilon+iX^\varepsilon} V^{(L)}(r^2)r^{2L+1}\frac{\Gamma(-s)}{\Gamma(L+s+1)}\Big(\frac{2\pi\sqrt{A}}{3\sqrt{3}N(\pi)^{1/2}N(\gamma)}\Big)^{2s}$$

$$\times \sum_{\substack{f\in\mathbb{Z}[\omega]\\ f\equiv 1 \pmod 3\\ (f,\pi\gamma)=1}} \frac{\mu(f)}{N(f)^{1+2s}} \sum_{\substack{d\in\mathbb{Z}[\omega]\\ d\equiv 1 \pmod 3\\ (d,\pi\gamma)=1}} \frac{1}{N(d)^{1+s}} \sum_{\substack{k\in\mathbb{Z}[\omega]\\ d\pi^2\gamma k\neq\boxdot\\ N(k)\ll\mathcal{Z}}} \widetilde{c}_d(k)\Big(\frac{d\pi^2\gamma k}{f}\Big)_3 \overline{\Big(\frac{d\pi^2\gamma k}{f}\Big)_3} N(k)^s$$

$$\times \Big(\sum_{\substack{b_1\in\mathbb{Z}[\omega]\\ (b_1,\pi\gamma)=1}} \frac{\beta_{fdb_1}}{\sqrt{N(b_1)}}\Big(\frac{d\pi^2\gamma k}{b_1}\Big)_3 N(b_1)^{-s}\Big)\Big(\sum_{\substack{b_2\in\mathbb{Z}[\omega]\\ (b_2,\pi\gamma)=1}} \frac{\overline{\beta_{fdb_2}}}{\sqrt{N(b_2)}}\overline{\Big(\frac{d\pi^2\gamma k}{b_2}\Big)_3} N(b_2)^{-s}\Big) ds dr$$

$$+ O_\varepsilon\big((N(\pi)\Delta X)^{-1000}\big),$$

for any fixed $L \in \mathbb{Z}_{\geq 1}$. We use Axiom 4 of Definition 3.1 to estimate the sum over b_1 and b_2, and then estimate the remaining sums trivially using Lemma 5.5. We obtain

$$\mathscr{D}_2 \ll_\varepsilon (\Delta N(\pi)X)^\varepsilon \Big(\frac{AB^{2\eta}}{N(\pi\gamma^2)} + \frac{B^{2+2\eta}N(\pi)\Delta^2}{N(\pi\gamma^2)} \Big). \tag{9.40}$$

Since $\eta \leq \frac{1}{4}$ we have $AB^{2\eta} \leq X$. After recalling (9.37), (9.40) and the fact that $\mathscr{D} = \mathscr{D}_1 + \mathscr{D}_2$, we obtain (9.16).

The cross terms $\mathscr{C}$. Recall that (9.14) records the cross term. Observe that (2.3) tells us that

$$\tilde{g}(b_1)\tilde{g}(a\pi)\overline{\Big(\frac{a}{b_1}\Big)_3}\Big(\frac{\pi^2}{b_1}\Big)_3 = \tilde{g}(a\pi b_1). \tag{9.41}$$

Substituting (9.41) into (9.14) gives

$$\mathscr{C} := -2\frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})}\delta_{\gamma=1}\mathrm{Re}\Big(\sum_{b_1,b_2\in\mathbb{Z}[\omega]} \frac{\beta_{b_1}\overline{\beta_{b_2}}}{N(b_2)^{1/6}} \sum_{\substack{a\in\mathbb{Z}[\omega] \\ a\equiv 1 \pmod 3 \\ (a\pi,b_1)=1}} V\Big(\frac{N(a)N(\pi)}{A}\Big)\frac{\tilde{g}(a\pi b_1)}{N(a\pi)^{1/6}} \Big). \tag{9.42}$$

We now evaluate the sum over $a \in \mathbb{Z}[\omega]$ in (9.42) using our asymptotic formula for type-I sums in Proposition 8.1 (for level πb_1). Thus there exists $\rho(\varepsilon) \in (0, \frac{1}{10000})$ such that

$$\sum_{b_1\in\mathbb{Z}[\omega]} \beta_{b_1} \sum_{\substack{a\in\mathbb{Z}[\omega] \\ a\equiv 1 \pmod 3}} V\Big(\frac{N(a)N(\pi)}{A}\Big)\frac{\tilde{g}(a\pi b_1)}{N(a\pi)^{1/6}}$$

$$= \tilde{V}\Big(\frac{2}{3}\Big)\frac{(2\pi)^{5/3}A^{2/3}}{3^{7/2}\Gamma(\frac{2}{3})N(\pi)^{5/6}} \sum_{b_1\in\mathbb{Z}[\omega]} \frac{\beta_{b_1}\varphi(\pi b_1)}{\zeta_{\mathbb{Q}(\omega)}(2;\mathbf{1}_{\pi b_1})N(\pi b_1)^{7/6}}$$

$$+ O_\varepsilon\Big(\frac{A^{2/3-\rho(\varepsilon)}B^{5/6-\rho(\varepsilon)}}{N(\pi)}\Big) + O_\varepsilon\big(X^\varepsilon N(\pi)^{1/2}B^{19/12}A^{-1/12}\big). \tag{9.43}$$

We now use the fact that $\boldsymbol{\beta}$ is supported on w-rough squarefree elements of $\mathbb{Z}[\omega]$ that are congruent to 1 modulo 3. We have

$$\log\Big(\frac{\varphi(\pi b_1)}{N(\pi b_1)}\Big) = \sum_{\substack{\varpi|\pi b_1 \\ \varpi \text{ prime}}} \log\big(1 - N(\varpi)^{-1}\big) = -\sum_{\substack{\varpi|\pi b_1 \\ \varpi \text{ prime}}} \sum_{L=1}^\infty \frac{1}{LN(\varpi)^L} = O\Big(\frac{1}{w^{9/10}} + \frac{\delta_{\pi\neq 1}}{N(\pi)}\Big).$$

Thus

$$\frac{\varphi(\pi b_1)}{N(\pi b_1)} = 1 + O\Big(\frac{1}{w^{9/10}} + \frac{\delta_{\pi\neq 1}}{N(\pi)}\Big). \tag{9.44}$$

Similarly, we also have

$$\frac{1}{\zeta_{\mathbb{Q}(\omega)}(2;\mathbf{1}_{\pi b_1})} = \frac{1}{\zeta_{\mathbb{Q}(\omega)}(2;\mathbf{1}_\pi)} + O\Big(\frac{1}{w^{19/10}}\Big). \tag{9.45}$$

Insertion of (9.44) and (9.45) into (9.43) gives

$$\sum_{\substack{b_1 \in \mathbb{Z}[\omega]}} \beta_{b_1} \sum_{\substack{a \in \mathbb{Z}[\omega] \\ a \equiv 1 \pmod 3}} V\Big(\frac{N(a)N(\pi)}{A}\Big) \frac{\tilde{g}(a\pi b_1)}{N(a\pi)^{1/6}}$$

$$= \widetilde{V}\Big(\frac{2}{3}\Big) \frac{(2\pi)^{5/3} A^{2/3}}{3^{7/2}\Gamma(\frac{2}{3})\zeta_{\mathbb{Q}(\omega)}(2;\mathbf{1}_\pi)N(\pi)} \sum_{\substack{b_1 \in \mathbb{Z}[\omega]}} \frac{\beta_{b_1}}{N(b_1)^{1/6}} + O_\varepsilon\Big(\frac{A^{2/3-\rho(\varepsilon)}B^{5/6-\rho(\varepsilon)}}{N(\pi)}\Big)$$

$$+ O_\varepsilon(X^\varepsilon N(\pi)^{1/2}B^{19/12}A^{-1/12}) + O\Big(\frac{A^{2/3}B^{5/6}}{N(\pi)}\Big(\frac{1}{w^{9/10}} + \frac{\delta_{\pi \neq 1}}{N(\pi)}\Big)\Big). \qquad (9.46)$$

Insertion of (9.46) into (9.42) gives (9.17).

The trivial term $\mathscr{T}$. Recall that (9.15) records

$$\mathscr{T} = \Big(\frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})}\Big)^2 \delta_{\gamma=1} \sum_{\substack{a \in \mathbb{Z}[\omega] \\ a \equiv 1 \pmod 3}} V\Big(\frac{N(a)N(\pi)}{A}\Big) \frac{\mu^2(a\pi)}{N(a\pi)^{1/3}} \Big| \sum_{b \in \mathbb{Z}[\omega]} \frac{\beta_b}{N(b)^{1/6}} \Big|^2. \qquad (9.47)$$

Mellin inversion of the smooth function, the Class number formula [Lan94, Chapter VIII, §2, Theorem 5] and subsequent contour shift to the right of the 1/6-line (in the s-variable) gives

$$\sum_{\substack{a \in \mathbb{Z}[\omega] \\ a \equiv 1 \pmod 3}} V\Big(\frac{N(a)N(\pi)}{A}\Big) \frac{\mu^2(a\pi)}{N(a\pi)^{1/3}}$$

$$= \frac{1}{2\pi i}\int_{(2)} \widetilde{V}(s) \frac{\zeta_{\mathbb{Q}(\omega)}(s+1/3;\mathbf{1}_\pi)}{\zeta_{\mathbb{Q}(\omega)}(2s+2/3;\mathbf{1}_\pi)} \frac{A^s}{N(\pi)^{s+1/3}} ds$$

$$= \widetilde{V}\Big(\frac{2}{3}\Big)(1 - \delta_{\pi \neq 1} \cdot N(\pi)^{-1}) \frac{2\pi A^{2/3}}{9\sqrt{3}\zeta_{\mathbb{Q}(\omega)}(2;\mathbf{1}_\pi)N(\pi)} + O_\varepsilon\Big(\frac{A^{1/6+\varepsilon}}{N(\pi)^{1/2+\varepsilon}}\Big)$$

$$= \widetilde{V}\Big(\frac{2}{3}\Big) \frac{2\pi A^{2/3}}{9\sqrt{3}\zeta_{\mathbb{Q}(\omega)}(2;\mathbf{1}_\pi)N(\pi)} + \delta_{\pi \neq 1} O\Big(\frac{A^{2/3}}{N(\pi)^2}\Big) + O_\varepsilon\Big(\frac{A^{1/6+\varepsilon}}{N(\pi)^{1/2+\varepsilon}}\Big). \qquad (9.48)$$

Insertion of (9.48) into (9.47) gives (9.18). ∎

Lemma 9.2. *Given $y \geqslant 1$, there exists coefficients $(\lambda_d)_{d \in \mathbb{Z}[\omega]}$ such that*

(1) $\lambda_1 = 1$ and $|\lambda_d| \ll_\varepsilon N(d)^\varepsilon$ for all $d \in \mathbb{Z}[\omega]$ and all $\varepsilon > 0$;
(2) $\lambda_d = 0$ if $N(d) > y^2$ or $d \not\equiv 1 \pmod 3$;
(3) For all $n \in \mathbb{Z}[\omega]$ we have

$$\mu^2(n) \leqslant \sum_{d^2 | n} \lambda_d; \qquad (9.49)$$

(4) For any $\varepsilon > 0$ and $\pi \in \mathbb{Z}[\omega]$ a prime $\pi \equiv 1 \pmod 3$ (or 1) we have

$$\sum_{\substack{d \in \mathbb{Z}[\omega] \\ (d,\pi)=1}} \frac{\lambda_d}{N(d)^2} = \frac{1}{\zeta_{\mathbb{Q}(\omega)}(2;\mathbf{1}_\pi)} + O_\varepsilon(y^{-1/2+\varepsilon}). \qquad (9.50)$$

Proof. Given $d \equiv 1 \pmod{3}$, let

$$\lambda_d := \sum_{\substack{N(e),N(f) \leqslant y \\ e,f \equiv 1 \pmod{3} \\ d=[e,f]}} \mu(e)\mu(f)$$

Properties (1) and (2) are immediate from the definition. Property (3) follows from

$$\sum_{d^2 \mid n} \lambda_d = \Big(\sum_{\substack{N(e) \leqslant y \\ e^2 \mid n}} \mu(e) \Big)^2.$$

It remains to check property (4). We have

$$\sum_{\substack{d \in \mathbb{Z}[\omega] \\ (d,\pi)=1}} \frac{\lambda_d}{N(d)^2} = \sum_{\substack{N(e),N(f) \leqslant y \\ e,f \equiv 1 \pmod{3} \\ (ef,\pi)=1}} \frac{\mu(e)\mu(f)}{N([e,f])^2} = \sum_{\substack{e,f \equiv 1 \pmod{3} \\ (ef,\pi)=1}} \frac{\mu(e)\mu(f)}{N([e,f])^2} + O(y^{-1/2+\varepsilon}). \quad (9.51)$$

The main term in (9.51) is equal to

$$\prod_{\substack{\varpi \equiv 1 \pmod{3} \\ \varpi \neq \pi}} \Big(1 + \frac{1}{N(\varpi)^2}(-2+1) \Big) = \prod_{\substack{\varpi \equiv 1 \pmod{3} \\ \varpi \neq \pi}} \Big(1 - \frac{1}{N(\varpi)^2} \Big) = \frac{1}{\zeta_{\mathbb{Q}(\omega)}(2; \mathbf{1}_\pi)},$$

as required. ∎

Proof of Proposition 9.1. In the first display of the statement of Proposition 9.1 we use property (3) in Lemma 9.2 with $y = X^\varepsilon$ with $\varepsilon > 0$ small and fixed, and inclusion-exclusion on the condition $(a,b) = 1$. We see that the first display in Proposition 9.1 is

$$\leqslant \sum_{\substack{a \in \mathbb{Z}[\omega] \\ a \equiv 1 \pmod{3} \\ \pi \mid a}} V\Big(\frac{N(a)}{A} \Big) \Big| \Big(\sum_{\substack{b \in \mathbb{Z}[\omega] \\ (b,\gamma a)=1}} \beta_b \tilde{g}(b) \overline{\Big(\frac{b}{a} \Big)_3} - \frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})} \frac{\overline{\tilde{g}(a)}}{N(a)^{1/6}} \sum_{b \in \mathbb{Z}[\omega]} \frac{\beta_b}{N(b)^{1/6}} \Big)$$

$$+ \Big(\frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})} \frac{\overline{\tilde{g}(a)}}{N(a)^{1/6}} \sum_{\substack{b \in \mathbb{Z}[\omega] \\ (b,a)\neq 1}} \frac{\beta_b}{N(b)^{1/6}} \Big) \Big|^2 \Big(\sum_{\gamma^2 \mid a} \lambda_\gamma \Big). \quad (9.52)$$

Since V is non-negative and $\sum_{\gamma^2 \mid a} \lambda_\gamma \geqslant 0$, we can apply the parallelogram inequality (9.8) to (9.52). We see that the right side of (9.52) is

$$\leqslant 2 \sum_{\substack{a \in \mathbb{Z}[\omega] \\ a \equiv 1 \pmod{3} \\ \pi \mid a}} V\Big(\frac{N(a)}{A} \Big) \Big(\Big| \sum_{\substack{b \in \mathbb{Z}[\omega] \\ (b,\gamma a)=1}} \beta_b \tilde{g}(b) \overline{\Big(\frac{b}{a} \Big)_3} - \frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})} \frac{\overline{\tilde{g}(a)}}{N(a)^{1/6}} \sum_{b \in \mathbb{Z}[\omega]} \frac{\beta_b}{N(b)^{1/6}} \Big|^2$$

$$+ \frac{(2\pi)^{4/3}}{9\Gamma(\frac{2}{3})^2} \sum_{\substack{a \in \mathbb{Z}[\omega] \\ a \equiv 1 \pmod{3} \\ \pi \mid a}} \frac{\mu^2(a)}{N(a)^{1/3}} V\Big(\frac{N(a)}{A} \Big) \Big| \sum_{\substack{b \in \mathbb{Z}[\omega] \\ (b,a)\neq 1}} \frac{\beta_b}{N(b)^{1/6}} \Big|^2 \Big) \Big(\sum_{\gamma^2 \mid a} \lambda_\gamma \Big). \quad (9.53)$$

We interchange the order of summation and use (2.2) in (9.53). We see that the right side of (9.53) is equal to

$$2\Bigg(\sum_{\substack{\gamma \equiv 1 \pmod 3}} \lambda_\gamma$$

$$\times \Bigg(\sum_{\substack{a \in \mathbb{Z}[\omega] \\ a \equiv 1 \pmod 3 \\ \pi | a, \gamma^2 | a}} V\Big(\frac{N(a)}{A}\Big)\Big| \sum_{\substack{b \in \mathbb{Z}[\omega] \\ (b,\gamma a)=1}} \beta_b \tilde{g}(b) \overline{\Big(\frac{b}{a}\Big)_3} - \frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})} \frac{\delta_{\gamma=1} \overline{\tilde{g}(a)}}{N(a)^{1/6}} \sum_{b \in \mathbb{Z}[\omega]} \frac{\beta_b}{N(b)^{1/6}} \Big|^2 \quad (9.54)$$

$$+ \delta_{\gamma=1} \frac{(2\pi)^{4/3}}{9\Gamma(\frac{2}{3})^2} \sum_{\substack{a \in \mathbb{Z}[\omega] \\ a \equiv 1 \pmod 3 \\ \pi | a}} \frac{\mu^2(a)}{N(a)^{1/3}} V\Big(\frac{N(a)}{A}\Big)\Big| \sum_{\substack{b \in \mathbb{Z}[\omega] \\ (b,a) \neq 1}} \frac{\beta_b}{N(b)^{1/6}} \Big|^2 \Bigg)\Bigg). \quad (9.55)$$

Observe that (9.55) matches (9.10), and is estimated by (9.11). If $(\pi, \gamma) = 1$, then the divisibility condition in (9.54) becomes $\pi\gamma^2 \mid a$. Thus (9.54) in this case matches (9.9), and is estimated asymptotically by by $\mathscr{D} + \mathscr{C} + \mathscr{T}$, where $\mathscr{D}, \mathscr{C}$ and $\mathscr{T}$ have asymptotic expressions given by (9.16), (9.17), and (9.18) respectively. Summing the asymptotic expression for $\mathscr{D} + \mathscr{C} + \mathscr{T}$ over γ with $(\gamma, \pi) = 1$ (with sieve weights λ_γ) using properties (1), (2), and (4) of Lemma 9.2 yields the bound stated in Proposition 9.1. If $(\pi, \gamma) \neq 1$, then the divisibility condition in (9.54) becomes $\gamma^2 \mid a$ with $\pi \mid \gamma$ and $\pi \neq 1$. Performing a similar computation to the previous case gives the result. ∎

10. Broad Type II estimates

We prove the following type-II estimates for sequences in $\mathcal{C}_\eta(\cdot, w)$.

Proposition 10.1. *Let W be a smooth function compactly supported in $[1, 2]$, $0 < \eta \leqslant 1/4$, $A, B \geqslant 10$ and set $X := AB$. Let $\boldsymbol{\alpha}$ be a sequence supported in $N(a) \in [A/10, 10A]$ with $a \equiv 1 \pmod 3$ squarefree. Suppose that $w > (\log X)^{10}$, $\boldsymbol{\beta} = (\beta_b) \in \mathcal{C}_\eta(B, w)$, $\varepsilon \in (0, \frac{1}{10000})$ and $\pi \equiv 1 \pmod 3$ a prime or 1 satisfying $1 \leqslant N(\pi) < w$. Then there exists $\rho(\varepsilon) \in (0, \frac{1}{10000})$ such that*

$$\sum_{\substack{a,b \in \mathbb{Z}[\omega] \\ \pi | a}} \alpha_a \beta_b \tilde{g}(ab) W\Big(\frac{N(ab)}{X}\Big) = \frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})} \sum_{\substack{a,b \in \mathbb{Z}[\omega] \\ \pi | a}} \frac{\alpha_a \beta_b \mu^2(ab)}{N(ab)^{1/6}} W\Big(\frac{N(ab)}{X}\Big)$$

$$+ O_\varepsilon\Bigg(\Big(\sum_{\substack{a \in \mathbb{Z}[\omega] \\ \pi | a}} |\alpha_a|^2 \Big)^{1/2} \cdot \Big(\frac{A^{1/3} B^{5/6}}{N(\pi)^{1/2}} \Big(\frac{1}{w^{9/20}} + \frac{\delta_{\pi \neq 1}}{N(\pi)^{1/2}} \Big) + \frac{A^{1/3 - \rho(\varepsilon)/2} B^{5/6 - \rho(\varepsilon)/2}}{N(\pi)^{1/2}}$$

$$+ \frac{A^{1/12 + \varepsilon/2} B^{5/6}}{N(\pi)^{1/4 + \varepsilon/2}} + (X N(\pi))^{\varepsilon/2}\Big(N(\pi)^{1/4} B^{29/24} A^{-1/24} + B^{1+\eta} + \frac{X^{1/2}}{N(\pi)^{1/2}} \Big(1 + (B^2/A)^{-500} \Big) \Big) \Big) \Bigg).$$

Remark 10.1. Suppose $\pi = 1$ say. Then for dense sequences $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and given $\xi > 0$, the result is non-trivial in the range $X^{1/3 + \xi} \leqslant B \leqslant X^{1/2 - \xi}$, for some appropriate choice of ε, η and w.

Proof. Observe that (2.3) gives

$$\sum_{\substack{a,b\in\mathbb{Z}[\omega] \\ \pi|a}} \alpha_a\beta_b\Big(\tilde{g}(ab) - \frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})}\frac{\mu^2(ab)}{N(ab)^{1/6}}\Big)W\Big(\frac{N(ab)}{X}\Big)$$

$$= \sum_{\substack{a,b\in\mathbb{Z}[\omega] \\ \pi|a \\ (a,b)=1}} \alpha_a\beta_b\Big(\tilde{g}(ab) - \frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})}\frac{1}{N(ab)^{1/6}}\Big)W\Big(\frac{N(ab)}{X}\Big)$$

$$= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \widetilde{W}(s)X^s \sum_{\substack{a\in\mathbb{Z}[\omega] \\ \pi|a}} \frac{\alpha_a\tilde{g}(a)}{N(a)^s}\Big(\sum_{\substack{b\in\mathbb{Z}[\omega] \\ (b,a)=1}} \frac{\beta_b\tilde{g}(b)}{N(b)^s}\overline{\Big(\frac{a}{b}\Big)}_3 - \frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})}\frac{\overline{\tilde{g}(a)}}{N(a)^{1/6}}\sum_{\substack{b\in\mathbb{Z}[\omega] \\ (b,a)=1}} \frac{\beta_b}{N(b)^{1/6+s}}\Big)ds.$$

$$(10.1)$$

Application of triangle inequality and then Cauchy-Schwarz to the a-sum shows that

$$|(10.1)|^2 \ll \Big(\sum_{\substack{a\in\mathbb{Z}[\omega] \\ a\equiv 1 \pmod 3 \\ \pi|a}} |\alpha_a|^2\Big)\int_{-\infty}^{\infty} |\widetilde{W}(it)|\Big(\sum_{\substack{a\in\mathbb{Z}[\omega] \\ \pi|a}} \mu^2(a)V\Big(\frac{N(a)}{A}\Big)$$

$$\times \Big|\sum_{\substack{b\in\mathbb{Z}[\omega] \\ (b,a)=1}} \frac{\beta_b\tilde{g}(b)}{N(b)^{it}}\overline{\Big(\frac{a}{b}\Big)}_3 - \frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})}\frac{\overline{\tilde{g}(a)}}{N(a)^{1/6}}\sum_{\substack{b\in\mathbb{Z}[\omega] \\ (b,a)=1}} \frac{\beta_b}{N(b)^{1/6+it}}\Big|^2\Big)dt,$$

where $V : \mathbb{R} \to \mathbb{R}$ a smooth compactly supported function such that $V \geqslant \mathbf{1}_{[1/10,10]}$. Using Proposition 9.1 gives the result. ∎

11. Average Type-I estimates

In this section we prove an average Type-I estimate. This average Type-I estimate will be more directly useful to us than the pointwise Type-I estimate proved in Section 8. Recall that $\ell \in \mathbb{Z}$, $c \equiv 1 \pmod 3$, and

$$g_\ell(c) := \tilde{g}(c)\Big(\frac{c}{|c|}\Big)^\ell.$$

Proposition 11.1. *Assume the Generalized Riemann Hypothesis for the Dedekind zeta function attached to $\mathbb{Q}(\omega)$ twisted by Größencharaktern. Let $\ell \in \mathbb{Z}$, $\varepsilon \in (0, \frac{1}{1000000})$, $A > 1000$ be large and fixed, and $(\log X)^A \leqslant w \leqslant X^\varepsilon$. Let V, W be smooth functions, compactly supported in $[\frac{1}{2}, 4]$. Let $0 < \eta < 100/A$ and $\boldsymbol{\alpha} = (\alpha_r) \in \mathcal{C}_\eta(X, w)$. Then uniformly for $C > X^{2/3-\varepsilon}$ and $|\ell| \leqslant X^{1/100}$ we have*

$$\sum_{\substack{r\in\mathbb{Z}[\omega] \\ c\equiv 1 \pmod 3 \\ \varpi|c \Longrightarrow N(\varpi)>w}} \alpha_r\tilde{g}_\ell(cr)V\Big(\frac{N(c)}{C}\Big)W\Big(\frac{N(cr)}{X}\Big) = \frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})}\sum_{\substack{r\in\mathbb{Z}[\omega] \\ c\equiv 1 \pmod 3 \\ \varpi|c \Longrightarrow N(\varpi)>w}} \frac{\alpha_r\mu^2(cr)\big(\frac{cr}{|cr|}\big)^\ell}{N(cr)^{1/6}}$$

$$+ O_{\varepsilon,A}\Big(\frac{X^{5/6}}{(\log X)^{\frac{9}{20}A-10}}\Big). \quad (11.1)$$

Proof of Proposition 11.1. Möbius inversion asserts that

$$\delta_{\varpi|c \implies N(\varpi)>w} = \sum_{\substack{u|c \\ u\equiv 1 \pmod 3 \\ \varpi|u \implies N(\varpi)\leqslant w}} \mu(u). \tag{11.2}$$

Using (11.2), we express the left side of (11.1) as

$$\sum_{k\geqslant 0}(-1)^k \sum_{N \text{ dyadic}} \mathscr{B}_{k,\ell}(N), \tag{11.3}$$

where

$$\mathscr{B}_{k,\ell}(N) := \sum_{\substack{r,u,n\in\mathbb{Z}[\omega] \\ u,n\equiv 1 \pmod 3}} \alpha_r \tilde{g}_\ell(nru)\Big(\delta_{\substack{\omega(u)=k \\ \varpi|u \implies N(\varpi)\leqslant w}}\Big)V\Big(\frac{N(n)}{N}\Big)V\Big(\frac{N(nu)}{C}\Big)W\Big(\frac{N(nru)}{X}\Big). \tag{11.4}$$

Case 1: $N \geqslant X^{1/2+1/20}$. Consulting Remark 8.4 we see that we in the range where Corollary 8.1 is non-trivial. Thus Corollary 8.1 and Remark 8.1 guarantee a small fixed $\delta_0 > 0$ such that we have (uniformly in $|\ell| \leqslant X^{1/100}$),

$$\mathscr{B}_{k,\ell}(N) := \delta_{\ell=0} \cdot \frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})} \sum_{\substack{r,u,n\in\mathbb{Z}[\omega] \\ u,n\equiv 1 \pmod 3}} \frac{\alpha_r \mu^2(nru)\big(\frac{nru}{|nru|}\big)^\ell}{N(nru)^{1/6}}\Big(\delta_{\substack{\omega(u)=k \\ \varpi|u \implies N(\varpi)\leqslant w}}\Big)$$

$$\times V\Big(\frac{N(n)}{N}\Big)V\Big(\frac{N(nu)}{C}\Big)W\Big(\frac{N(nru)}{X}\Big) + O_\varepsilon(X^{5/6-\delta_0}). \tag{11.5}$$

We can drop the condition $\delta_{\ell=0}$ since if $\ell \neq 0$ then the sum over n majorised by the error term in the above display.

Case 2: $N < X^{1/2+1/20}$. Suppose we are given a squarefree $1 \neq u \in \mathbb{Z}[\omega]$ satisfying $u \equiv 1 \pmod 3$, and such that all prime factors of u have norm $\leqslant w$. Then, given a prime $\pi|u$, there is a unique factorisation $u = b\pi d$ such that all the prime factors of d (resp. b) satisfy $< \pi$ (resp. satisfy $> \pi$). See Remark 6.1 for the ordering $<$ on ideals. Since $N(r) \asymp X/C \ll X^{1/3+\varepsilon}$ and $N(ru) \asymp X/N \gg X^{1/2-1/20}$, we are guaranteed a unique prime $\pi \in \mathbb{Z}[\omega]$ such that $u = b\pi d$ with $N(rb) < X^{13/30}$ and $N(rb\pi) \geqslant X^{13/30}$. Hence

$$\mathscr{B}_{k,\ell}(N) := \sum_{\substack{r,n\in\mathbb{Z}[\omega] \\ n\equiv 1 \pmod 3}} \alpha_r V\Big(\frac{N(n)}{N}\Big) \sum_{\substack{u\in\mathbb{Z}[\omega] \\ u\equiv 1 \pmod 3}} \tilde{g}_\ell(nru)V\Big(\frac{N(nu)}{C}\Big)W\Big(\frac{N(nru)}{X}\Big)$$

$$\times \Big(\sum_{\substack{\pi\in\mathbb{Z}[\omega] \\ \pi \text{ prime} \\ N(\pi)\leqslant w \\ \pi\equiv 1 \pmod 3}} \sum_{\substack{u=b\pi d \\ \omega(u)=k \\ b,d\equiv 1 \pmod 3 \\ N(rb)<X^{13/30},N(rb\pi)\geqslant X^{13/30} \\ \varpi|b \implies w\geqslant N(\varpi)\geqslant N(\pi) \\ \varpi|d \implies N(\varpi)<N(\pi)}} 1\Big). \tag{11.6}$$

We introduce smooth partitions of unity in the $N(r), N(b)$ and $N(d)$ in (11.6). Then (2.3) and Mellin inversion imply that

$$\mathscr{B}_{k,\ell}(N) = \sum_{D,R,B \text{ dyadic}} \sum_{\substack{(k_1,k_2)\in(\mathbb{Z}_{\geqslant 0})^2 \\ k_1+k_2=k-1}} \mathscr{B}_{(k_1,k_2),\ell}(N,D,R,B), \qquad (11.7)$$

where

$$\mathscr{B}_{(k_1,k_2),\ell}(N,D,R,B) = \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} C^s X^v \widetilde{V}(s)\widetilde{W}(v)$$

$$\times \sum_{\substack{\pi\in\mathbb{Z}[\omega],\pi \text{ prime} \\ N(\pi)\leqslant w, \pi\equiv 1 \pmod 3}} \frac{\tilde{g}_\ell(\pi)}{N(\pi)^{s+v}} \Big(\sum_{\substack{j,h\in\mathbb{Z}[\omega] \\ j,h\equiv 1 \pmod 3}} \beta_j(\pi,k_1;s,v,\ell)\gamma_h(\pi,k_2;s,v,\ell)\tilde{g}(jh) \Big) ds\,dv,$$

$$(11.8)$$

and

$$\beta_j(\pi,k_1;s,v,\ell) := \frac{\mu^2(j)}{N(j)^v} \overline{\Big(\frac{\pi}{j}\Big)_3} \Big(\frac{j}{|j|}\Big)^\ell \sum_{\substack{j=rb \\ \omega(b)=k_1 \\ r,b\equiv 1 \pmod 3,(j,\pi)=1 \\ N(j)<X^{13/30},N(j\pi)\geqslant X^{13/30} \\ \varpi|b \Longrightarrow w\geqslant N(\varpi)\geqslant N(\pi)}} \alpha_r V\Big(\frac{N(r)}{R}\Big) V\Big(\frac{N(b)}{B}\Big) N(b)^{-s};$$

$$(11.9)$$

$$\gamma_h(\pi,k_2;s,v,\ell) := \frac{\mu^2(h)}{N(h)^{s+v}} \overline{\Big(\frac{\pi}{h}\Big)_3} \Big(\frac{h}{|h|}\Big)^\ell \sum_{\substack{h=nd \\ \omega(d)=k_2 \\ n,d\equiv 1 \pmod 3,(h,\pi)=1 \\ \varpi|d \Longrightarrow N(\varpi)<N(\pi)}} V\Big(\frac{N(n)}{N}\Big) V\Big(\frac{N(d)}{D}\Big).$$

We write

$$\beta_j(B,R,\pi,k_1;s,v,\ell) := \beta_j(\pi,k_1;s,v,\ell) \quad \text{and} \quad \gamma_h(N,D,k_2,\pi;s,v,\ell) := \gamma_h(\pi,k_2;s,v,\ell), \qquad (11.10)$$

when we care to emphasise the dyadic ranges B,R and N,D that are present in the definitions of $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ respectively.

For each given $\pi \in \mathbb{Z}[\omega]$ prime, the sum over j and h in (11.8) is empty unless

$$X^{13/30}/(1000N(\pi)) \leqslant RB \leqslant 1000X^{13/30}, \qquad (11.11)$$

and $NDRB \asymp X/N(\pi)$. Thus $ND \gg X^{17/30}$. Since $N < X^{1/2+1/20}$, we must have $D \gg X^{1/15-1/20}$ whenever the sum over j,h in (11.8) is non-zero. We now write

$$\mathscr{B}_{(k_1,k_2),\ell}(N,D,R,B) = \mathscr{B}^\dagger_{(k_1,k_2),\ell}(N,D,R,B) + \mathscr{B}^\star_{(k_1,k_2),\ell}(N,D,R,B), \qquad (11.12)$$

where $\mathscr{B}^\dagger_{(k_1,k_2),\ell}$ corresponds the part of (11.8) with $N(\pi) \leqslant (\log D)^A$, and $\mathscr{B}^\star_{(k_1,k_2),\ell}$ corresponds to $(\log D)^A < N(\pi) \leqslant w$.

Treatment of $\mathscr{B}^\dagger_{(k_1,k_2),\ell}(N,D,R,B)$. Since (α_r) is supported only on w-rough elements of $\mathbb{Z}[\omega]$, the factorisation $j = rb$ occurring in the definition of the sequence β_j is unique. Thus $|\beta_j(\pi,k_1;s,v,\ell)| \leqslant 1$ for $\operatorname{Re} s, \operatorname{Re} v = 0$.

On the other hand, the sequence $\gamma_h(N,D,\pi,k_2;s,v,\ell)$ is sparse when $N(\pi)$ is on log-power scales. We pause the proof to illustrate this in the following Lemma. We also

make the crude observations that γ_h is supported on $h \in \mathbb{Z}[\omega]$ with $N(h) \asymp ND$, and also satisfies $|\gamma_h(\pi, k_2; s, v, \ell)| \leqslant 2^{\omega(h)}$ for $\operatorname{Re} s, \operatorname{Re} v = 0$.

Lemma 11.1. *Let $N, D, A \geqslant 10$, $k_2 \in \mathbb{Z}_{\geqslant 0}$, and let $\pi \in \mathbb{Z}[\omega]$ be a prime that satisfies $\pi \equiv 1 \pmod 3$ and $N(\pi) \leqslant (\log D)^A$. Then*

$$\sum_{h \in \mathbb{Z}[\omega]} |\gamma_h(N, D, \pi, k_2; s, v, \ell)|^2 \ll_A (ND)^{o(1)} \cdot ND^{1-K/A},$$

with $K > 0$ a small absolute constant.

Proof of Lemma 11.1. We first refine our bound for $\|\boldsymbol{\gamma}\|_\infty$. We have

$$|\gamma_h(\pi, k_2; s, v, \ell)| \leqslant 2^{\omega(h)} \cdot \mathbf{1}_{h \in \mathscr{U}_\pi} \ll 2^{\frac{2 \log N(h)}{\log \log N(h)}} \cdot \mathbf{1}_{h \in \mathscr{U}_\pi},$$

where $\mathscr{U}_\pi$ is the set of squarefree integers of the form nd with $n, d \equiv 1 \pmod 3$, $(nd, \pi) = 1$, $N(n) \asymp N$, $N(d) \asymp D$, and such that all of the prime factors of d have norm $< N(\pi)$. Observe that d has necessarily $\geqslant \frac{\log D}{100 \log N(\pi)}$ (say) prime factors. Therefore

$$\sum_{h \in \mathbb{Z}[\omega]} |\gamma_h(N, D, \pi, k_2; s, v, \ell)|^2 \ll (ND)^{\frac{4}{\log \log (ND)}} \cdot |\mathscr{U}_\pi|. \tag{11.13}$$

Let $\rho > 0$ be chosen later. We have

$$|\mathscr{U}_\pi| \ll (ND) \sum_{\substack{d \equiv 1 \pmod 3 \\ \varpi | d \implies N(\varpi) < N(\pi) \\ \omega(d) \geqslant \frac{\log D}{100 \log N(\pi)}}} \frac{\mu^2(d)}{N(d)}$$

$$\leqslant (ND) \cdot \exp\left(-\frac{\rho \log D}{100 \log N(\pi)}\right) \sum_{\substack{d \equiv 1 \pmod 3 \\ \varpi | d \implies N(\varpi) < N(\pi)}} \frac{\mu^2(d) e^{\rho \omega(d)}}{N(d)}$$

$$\leqslant (ND) \cdot \exp\left(-\frac{\rho \log D}{100 \log N(\pi)}\right) \prod_{N(\varpi) < N(\pi)} \left(1 + \frac{e^\rho}{N(\varpi)}\right)$$

$$\ll (ND) \cdot \exp\left(2 e^\rho \log \log N(\pi) - \frac{\rho \log D}{100 \log N(\pi)}\right)$$

$$\ll (ND) \cdot \exp\left(2 e^\rho \log N(\pi) - \frac{\rho \log D}{100 \log N(\pi)}\right)$$

$$\ll (ND) \cdot \exp\left(2 A e^\rho \log \log D - \frac{\rho \log D}{100 A \log \log D}\right). \tag{11.14}$$

We choose

$$\rho := \log \log D - 1000 \log \log \log D.$$

Thus (11.14) implies that

$$|\mathscr{U}_\pi| \ll_A ND^{1-1/(1000A)} \tag{11.15}$$

for all D (hence X) sufficiently large. Thus (11.13) now implies the Lemma. ∎

We now resume the proof of Proposition 11.1. We now use (2.3), the Cauchy-Schwarz inequality, Heath-Brown's cubic large sieve (Theorem 9.1) Lemma 11.1, and the conditions (11.11), $NDRB \asymp X/N(\pi)$ and $D \gg X^{1/15-1/20}$ to obtain

$$
|\mathscr{B}^{\dagger}_{(k_1,k_2),\ell}(N,D,R,B)|
$$
$$
\ll_A X^{o(1)} \sum_{\substack{\pi \in \mathbb{Z}[\omega] \\ \pi \text{ prime} \\ N(\pi) \leqslant (\log D)^A \\ \pi \equiv 1 \pmod 3}} \left((RB)^{1/2} \Big(RB + ND + (RBND)^{2/3} \Big)^{1/2} N^{1/2} D^{1/2 - 1/(500A)} \right)
$$
$$
\ll_A X^{5/6 - 1/(1000000A)},
$$

say. We include a redundant main term of size that is absorbed by the error term i.e. we can write

$$
\mathscr{B}^{\dagger}_{(k_1,k_2),\ell}(N,D,R,B)
$$
$$
= \frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})} \sum_{\substack{r,u,n \in \mathbb{Z}[\omega] \\ n \equiv 1 \pmod 3}} \frac{\alpha_r \mu^2(nru)(\frac{nru}{|nru|})^{\ell}}{N(nru)^{1/6}} V\Big(\frac{N(r)}{R}\Big) V\Big(\frac{N(n)}{N}\Big) V\Big(\frac{N(nu)}{C}\Big) W\Big(\frac{N(nru)}{X}\Big)
$$
$$
\times \left(\sum_{\substack{\pi \in \mathbb{Z}[\omega] \\ \pi \text{ prime} \\ N(\pi) \leqslant (\log D)^A \\ \pi \equiv 1 \pmod 3}} \sum_{\substack{u = b\pi d \\ \omega(b)=k_1, \omega(d)=k_2 \\ b,d \equiv 1 \pmod 3 \\ N(rb) < X^{13/30} \\ N(rb\pi) \geqslant X^{13/30} \\ \varpi|b \implies w \geqslant N(\varpi) \geqslant N(\pi) \\ \varpi|d \implies N(\varpi) < N(\pi)}} V\Big(\frac{N(b)}{B}\Big) V\Big(\frac{N(d)}{D}\Big) \right) + O_A(X^{5/6 - 1/(1000000A)}).
$$

$$(11.16)$$

Treatment of $\mathscr{B}^{\star}_{k,\ell}(\cdots)$. Recall that $(\log X)^A \leqslant w \leqslant X^{\varepsilon}$, and that $(\log X)^A < N(\pi) \leqslant w$. We reassemble the integral in the v-variable in (11.8), and recover the smooth weight $W(N(jh\pi)/X)$. By Lemma 6.1 and Lemma 6.3 we have $\boldsymbol{\beta}(R,B;\pi;s,\ell) \in \mathcal{C}_{\eta}(RB, N(\pi))$ (from (11.9)) (after re-scaling by an appropriate absolute non-zero constant) for all $\eta > 100/A$. We then apply Proposition 10.1 and see that there is a $\rho(\varepsilon) \in (0, \frac{1}{10000})$ such that

$$
\mathscr{B}^{\star}_{(k_1,k_2),\ell}(N,D,R,B)
$$
$$
= \frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})} \sum_{\substack{r,u,n \in \mathbb{Z}[\omega] \\ n \equiv 1 \pmod 3}} \frac{\alpha_r \mu^2(nru)(\frac{nru}{|nru|})^{\ell}}{N(nru)^{1/6}} V\Big(\frac{N(r)}{R}\Big) V\Big(\frac{N(n)}{N}\Big) V\Big(\frac{N(nu)}{C}\Big) W\Big(\frac{N(nru)}{X}\Big)
$$
$$
\times \left(\sum_{\substack{\pi \in \mathbb{Z}[\omega] \\ \pi \text{ prime} \\ (\log D)^A < N(\pi) \leqslant w \\ \pi \equiv 1 \pmod 3}} \sum_{\substack{u = b\pi d \\ \omega(b)=k_1, \omega(d)=k_2 \\ b,d \equiv 1 \pmod 3 \\ N(rb) < X^{13/30} \\ N(rb\pi) \geqslant X^{13/30} \\ \varpi|b \implies w \geqslant N(\varpi) \geqslant N(\pi) \\ \varpi|d \implies N(\varpi) < N(\pi)}} V\Big(\frac{N(b)}{B}\Big) V\Big(\frac{N(d)}{D}\Big) \right) + E, \qquad (11.17)
$$

where

$$E = O_{A,\varepsilon}\Big(\log\log w \Big(\frac{X^{5/6}}{w^{9/20}} + X^{5/6 - \rho(\varepsilon)} \Big) + \frac{X^{5/6}}{(\log X)^{A/2}}$$
$$+ X^{83/120 + o(1)} w^{1/4} + X^{47/60 + o(1)} w^{3/4} + X^{17/60 + (13/30)(1 + 100/A) + o(1)} w^{1/2} \Big).$$

Note that both the error terms in (11.16) and (11.17) are uniform with respect to ℓ, k_1 and k_2.

Conclusion. After combining (11.16) and (11.17) in (11.12), we obtain an asymptotic expression for $\mathscr{B}_{(k_1,k_2),\ell}(N, D, R, B)$ for each dyadic value of N satisfying $N < X^{1/2 + 1/20}$. We reassemble the sum over $(k_1, k_2) \in (\mathbb{Z}_{\geqslant 0})^2$ (satisfying $k_1 + k_2 = k - 1$), as well as the partitions of unity in $N(b)$, $N(d)$ and $N(r)$ in (11.7). We then collapse the weights in the main term back to $\big(\delta_{\substack{\omega(u) = k \\ \varpi | u \implies N(\varpi) \leqslant w}} \big)$, and obtain an asymptotic expression for $\mathscr{B}_{k,\ell}(N)$ for each dyadic value N satisfying $N < X^{1/2 + 1/20}$. Recall that (11.5) gives an asymptotic expression for $\mathscr{B}_{k,\ell}(N)$ for each dyadic value N satisfying $N \geqslant X^{1/2 + 1/20}$. We combine these two results in (11.3), and reassemble the partition of unity over $N(n)$. Note that the reassembly of partitions of unity and the sums over k_i do not overwhelm the error terms (one only has losses of $O((\log X)^{10})$ say. Inserting this asymptotic expression into (11.3), and noting that

$$\sum_{k \geqslant 0} (-1)^k \big(\delta_{\substack{\omega(u) = k \\ \varpi | u \implies N(\varpi) \leqslant w}} \big) \mu^2(nru) = \mu^2(nru) \big(\delta_{\varpi | u \implies N(\varpi) \leqslant w} \big) \mu(u),$$

as well as (11.2), we obtain the result. ∎

12. Combinatorial decompositions

We will use the following combinatorial decomposition.

Lemma 12.1. *Let $W : \mathbb{R} \to \mathbb{R}$ be a smooth function compactly supported in $(0, C)$. Let $(s(n))_{n \in \mathbb{Z}[\omega]}$ be a sequence satisfying $|s(n)| \leqslant 1$ and have support on squarefree n satisfying $n \equiv 1 \pmod 3$. Then for $2 \leqslant w \leqslant CX^{1/3} \leqslant z$ we have*

$$\sum_{\substack{\varpi \equiv 1 \pmod 3 \\ N(\varpi) > z}} s(\varpi) W\Big(\frac{N(\varpi)}{X} \Big) = -\frac{1}{2} \sum_{\substack{\varpi_1, \varpi_2 \equiv 1 \pmod 3 \\ N(\varpi_1), N(\varpi_2) > z}} s(\varpi_1 \varpi_2) W\Big(\frac{N(\varpi_1 \varpi_2)}{X} \Big)$$

$$+ \sum_{k \geqslant 0} \frac{(-1)^k}{k!} \sum_{\substack{w < N(\varpi_1), \ldots, N(\varpi_k) \leqslant z \\ \forall i : \varpi_i \equiv 1 \pmod 3 \\ c \equiv 1 \pmod 3 \\ k = 0 \implies c \neq 1 \\ \varpi | c \implies N(\varpi) > w}} s(c\varpi_1 \ldots \varpi_k) W\Big(\frac{N(c\varpi_1 \ldots \varpi_k)}{X} \Big) + O(\sqrt{X}).$$

Proof. We assume that $\operatorname{Re} s > 1$ throughout this proof. We have

$$\zeta_{>z}(s) := \prod_{\substack{N(\varpi) > z \\ \varpi \equiv 1 \pmod 3}} \Big(1 - \frac{1}{N(\varpi)^s} \Big)^{-1},$$

and

$$\sum_{\substack{L \geqslant 1}} \frac{1}{L} \sum_{\substack{N(\varpi) > z \\ \varpi \equiv 1 \pmod 3}} \frac{1}{N(\varpi)^{Ls}} = \log \zeta_{>z}(s) = \log(1 + (\zeta_{>z}(s) - 1))$$

$$= (\zeta_{>z}(s) - 1) - \frac{1}{2} \cdot (\zeta_{>z}(s) - 1)^2 + \sum_{j \geqslant 3} \frac{(-1)^{j+1}}{j} \cdot (\zeta_{>z}(s) - 1)^j. \qquad (12.1)$$

Furthermore,

$$\zeta_{>z}(s) = \zeta_{>w}(s) \prod_{\substack{w < N(\varpi) \leqslant z \\ \varpi \equiv 1 \pmod 3}} \left(1 - \frac{1}{N(\varpi)^s} \right),$$

where

$$\zeta_{>w}(s) := \prod_{\substack{N(\varpi) > w \\ \varpi \equiv 1 \pmod 3}} \left(1 - \frac{1}{N(\varpi)^s} \right)^{-1} = \sum_{\substack{c \equiv 1 \pmod 3 \\ \varpi | c \implies N(\varpi) > w}} \frac{1}{N(c)^s}. \qquad (12.2)$$

The equation (12.2) is valid since every $c \equiv 1 \pmod 3$ has a unique factorisation $c = \varpi_1 \dots \varpi_k$ with $\varpi_i \equiv 1 \pmod 3$ for $i = 1, \dots, k$. Expand the product

$$\prod_{\substack{w < N(\varpi) \leqslant z \\ \varpi \equiv 1 \pmod 3}} \left(1 - \frac{1}{N(\varpi)^s} \right) = 1 + \sum_{k \geqslant 1} \frac{(-1)^k}{k!} \sum_{\substack{w < N(\varpi_1), \dots, N(\varpi_k) \leqslant z \\ \forall i : \varpi_i \equiv 1 \pmod 3 \\ \varpi_i \text{ all distinct}}} \frac{1}{N(\varpi_1 \dots \varpi_k)^s}.$$

Therefore

$$\zeta_{>z}(s) - 1 = \sum_{k \geqslant 0} \frac{(-1)^k}{k!} \sum_{\substack{w < N(\varpi_1), \dots, N(\varpi_k) \leqslant z \\ \forall \varpi_i \equiv 1 \pmod 3 \\ \varpi_i \text{ all distinct} \\ c \equiv 1 \pmod 3 \\ k = 0 \implies c \neq 1 \\ \varpi | c \implies N(\varpi) > w}} \frac{1}{N(c\varpi_1 \dots \varpi_k)^s}. \qquad (12.3)$$

Substitution of (12.3) into (12.1) gives

$$\sum_{\substack{L \geqslant 1}} \frac{1}{L} \sum_{\substack{N(\varpi) > z \\ \varpi \equiv 1 \pmod 3}} \frac{1}{N(\varpi)^{Ls}} = \sum_{k \geqslant 0} \frac{(-1)^k}{k!} \sum_{\substack{w < N(\varpi_1), \dots, N(\varpi_k) \leqslant z \\ \forall \varpi_i \equiv 1 \pmod 3 \\ \varpi_i \text{ all distinct} \\ c \equiv 1 \pmod 3 \\ k = 0 \implies c \neq 1 \\ \varpi | c \implies N(\varpi) > w}} \frac{1}{N(c\varpi_1 \dots \varpi_k)^s} \qquad (12.4)$$

$$- \frac{1}{2} \cdot (\zeta_{>z}(s) - 1)^2 + \sum_{j \geqslant 3} \frac{(-1)^{j+1}}{j} \cdot (\zeta_{>z}(s) - 1)^j.$$

The result follows from a comparison of coefficients. Observe that that the total contribution from terms $N(\varpi)^k \leqslant X$ with $k \geqslant 2$ on the left side of (12.4) is $O(\sqrt{X})$. Since $z > CX^{1/3}$ and W is compactly supported in $(0, C)$, we see that the contribution from all terms $(\zeta_{>z}(s) - 1)^j$ with $j \geqslant 3$ is zero. Notice that $s(c\varpi_1 \dots \varpi_k)$ is zero if $c\varpi_1 \dots \varpi_k$ is not squarefree by hypothesis, so we can drop the requirement that the ϖ_i are all distinct. ∎

13. Proof of Theorems 1.1 and 1.3

We first record a useful Lemma due to Polymath that classifies the Type-I, Type-II and Type-III information that occurs in the proof of our main theorems.

Lemma 13.1. [Pol14, Lemma 3.1] *Given an integer $n \geqslant 1$ and $\frac{1}{10} \leqslant \sigma < \frac{1}{2}$, let $t_1, \ldots, t_n$ be non-negative real numbers such that $t_1 + \ldots + t_n = 1$. Then at least one of the following three statement holds:*

(Type-I) There is an $i \in [1, n]$ such that $t_i \geqslant \frac{1}{2} + \sigma$;
(Type-II) There is a partition $\{1, \ldots, n\} = S \cup T$ such that

$$\frac{1}{2} - \sigma < \sum_{i \in S} t_i \leqslant \sum_{i \in T} t_i < \frac{1}{2} + \sigma;$$

(Type-III) There exists distinct $i, j, v \in [1, n]$ such that $2\sigma \leqslant t_i \leqslant t_j \leqslant t_v \leqslant \frac{1}{2} - \sigma$ and

$$t_i + t_j, t_j + t_v, t_v + t_i \geqslant \frac{1}{2} + \sigma.$$

Furthermore, if $\sigma > 1/6$, then the Type-III alternative can't occur.

Proof of Theorems 1.1 and 1.3. We first explain some initial manipulations.

Initial reduction. For any rational prime $p \equiv 1 \pmod 3$ we have

$$\frac{S_p}{2\sqrt{p}} = \operatorname{Re} \tilde{g}(\varpi),$$

where $\varpi \in \mathbb{Z}[\omega]$ is a prime such that $\varpi \equiv 1 \pmod 3$ and $p = \varpi\overline{\varpi}$. The number of primes $\varpi \equiv 1 \pmod 3$ for which $N(\varpi)$ is not prime is $O(\sqrt{X})$. Such primes are those that lie over rational primes $p \equiv 2 \pmod 3$. To prove Theorem 1.1 it suffices to estimate the quantity

$$\sum_{\varpi \equiv 1 \pmod 3} \tilde{g}(\varpi) W\left(\frac{N(\varpi)}{X}\right).$$

Observe that (2.2) implies that

$$\tilde{g}(\varpi)^3 = -\frac{\varpi}{|\varpi|}.$$

Thus

$$\tilde{g}(\varpi)^k = (-1)^\ell \left(\frac{\varpi}{|\varpi|}\right)^\ell \times \begin{cases} \tilde{g}(\varpi) & \text{if } k \equiv 1 \pmod 3 \text{ with } \ell = \frac{k-1}{3} \\ \overline{\tilde{g}(\varpi)} & \text{if } k \equiv 2 \pmod 3 \text{ with } \ell = \frac{k+1}{3} \\ 1 & \text{if } k \equiv 0 \pmod 3 \text{ with } \ell = \frac{k}{3} \end{cases}.$$

In particular, Theorem 1.3 with $k \equiv 0 \pmod 3$ follows directly from the assumption of the Generalized Riemann Hypothesis.

To establish Theorem 1.3, it suffices to show that

$$\sum_{\varpi \equiv 1 \pmod 3} \tilde{g}(\varpi) \left(\frac{\varpi}{|\varpi|}\right)^\ell W\left(\frac{N(\varpi)}{X}\right) = o\left(\frac{X^{5/6}}{\log X}\right),$$

as $X \to \infty$ and uniformly in $0 < |\ell| \leqslant X^{1/100}$. To prove both Theorem 1.1 and Theorem 1.3 simultaneously it is enough to estimate

$$\sum_{\varpi \equiv 1 \pmod{3}} \tilde{g}(\varpi)\Big(\frac{\varpi}{|\varpi|}\Big)^{\ell} W\Big(\frac{N(\varpi)}{X}\Big),$$

to a precision better than $o(X^{5/6}/\log X)$. For $c \equiv 1 \pmod{3}$ define

$$\tilde{g}_{\ell}(c) := \tilde{g}(c)\Big(\frac{c}{|c|}\Big)^{\ell}.$$

Let $\varepsilon \in (0, 10^{-6})$ be fixed. Let

$$w := X^{\varepsilon} \quad \text{and} \quad z := X^{1/3+\varepsilon}.$$

By Lemma 12.1 we have

$$\sum_{\varpi \equiv 1 \pmod{3}} \tilde{g}_{\ell}(\varpi)W\Big(\frac{N(\varpi)}{X}\Big) = -\frac{1}{2} \sum_{\substack{\varpi_1, \varpi_2 \equiv 1 \pmod{3} \\ N(\varpi_1), N(\varpi_2) > z}} \tilde{g}_{\ell}(\varpi_1\varpi_2)W\Big(\frac{N(\varpi_1\varpi_2)}{X}\Big) \quad (13.1)$$

$$+ \sum_{k \geqslant 0} \frac{(-1)^k}{k!} \sum_{(\varpi_1,\ldots,\varpi_k,c) \in \mathcal{S}(w,z)} \tilde{g}_{\ell}(\varpi_1 \ldots \varpi_k c)W\Big(\frac{N(\varpi_1 \ldots \varpi_k c)}{X}\Big) + O(\sqrt{X}), \quad (13.2)$$

where $\mathcal{S}(w,z)$ denotes the set of tuples $(\varpi_1, \ldots, \varpi_k, c)$ with $k \geqslant 0$ such that

- $\varpi_1, \ldots, \varpi_k$ are primes congruent to 1 (mod 3) (when $k \geqslant 1$);
- For all $1 \leqslant i \leqslant k$ we have $w < N(\varpi_i) \leqslant z$ (when $k \geqslant 1$);
- c is w-rough, $c \equiv 1 \pmod{3}$, and $k = 0 \implies c \neq 1$.

When $k = 0$, the sum is understood just to be over the variable c.

Let $\xi \in (0, 10^{-6})$ be a small fixed quantity to be decided at a later point in the proof (it will ultimately depend on ε).

Remark 13.1. Uniformity of error terms in ℓ is not an issue when deploying Type II/III estimates (i.e. Proposition 7.1 and Proposition 10.1). This is because (2.3) is applied to $\tilde{g}_{\ell}(ab)$, and the dependence on ℓ is absorbed into the coefficients $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ that satisfy $\|\boldsymbol{\alpha}\|_{\infty}, \|\boldsymbol{\beta}\|_{\infty} \leqslant 1$. The dependence on ℓ issue emanates from the application of the average Type-I estimate in Proposition 11.1.

Sum on the right side side of (13.1). We introduce a smooth partition of unity on each of the $N(\varpi_i)$ to evaluate the (Type-II) sum over $N(\varpi_1), N(\varpi_2) > z$. Thus it is sufficient to estimate

$$\mathscr{F}_{\ell}(X, P_1, P_2; z) := \sum_{\substack{\varpi_1, \varpi_2 \equiv 1 \pmod{3} \\ N(\varpi_1), N(\varpi_2) > z}} \tilde{g}_{\ell}(\varpi_1\varpi_2)W\Big(\frac{N(\varpi_1\varpi_2)}{X}\Big)V\Big(\frac{N(\varpi_1)}{P_1}\Big)V\Big(\frac{N(\varpi_2)}{P_2}\Big),$$

$$(13.3)$$

for all dyadic partitions (P_1, P_2) that satisfy $z/2 \leqslant P_1, P_2 \leqslant 2X$ and $P_1 P_2 \asymp X$. When $z/2 \leqslant \min\{P_1, P_2\} \leqslant X^{1/2-\xi}$ we can apply Proposition 10.1 with $\pi = 1$, and $\eta > 0$ arbitrarily small and fixed by Lemma 6.2 (the only requirement is that $\eta > 100 \log\log X/\log X$).

Thus there exists $\delta_0(\xi, \varepsilon) > 0$ such that

$$\mathcal{T}_\ell(X, P_1, P_2, z) = \frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})} \sum_{N(\varpi_1), N(\varpi_2) > z} \frac{\mu^2(\varpi_1 \varpi_2)\left(\frac{\varpi_1 \varpi_2}{|\varpi_1 \varpi_2|}\right)^\ell}{N(\varpi_1 \varpi_2)^{1/6}} V\left(\frac{N(\varpi_1)}{P_1}\right) V\left(\frac{N(\varpi_2)}{P_2}\right)$$

$$\times W\left(\frac{N(\varpi_1 \varpi_2)}{X}\right) + O_{\xi, \varepsilon}(X^{5/6 - \delta_0(\xi, \varepsilon)}), \quad \text{when} \quad z/2 \leqslant \min\{P_1, P_2\} \leqslant X^{1/2 - \xi}. \tag{13.4}$$

When $X^{1/2 - \xi} \leqslant P_1, P_2 \leqslant X^{1/2 + \xi}$, we appeal to Proposition 7.1. In particular, the smooth coefficients here are supported on $z = X^{1/3 + \varepsilon} > X^\varepsilon$ -rough integers. We obtain

$$\mathcal{F}_\ell(X, P_1, P_2; z) \ll \frac{X}{(\varepsilon \log X)^{3/2}} \frac{1}{\sqrt{\min(P_1, P_2)}} + \frac{X^{5/6}}{(\varepsilon \log X)^2},$$
$$\text{when} \quad X^{1/2 - \xi} \leqslant P_1, P_2 \leqslant X^{1/2 + \xi},$$

where the implied constant is absolute. We can include a redundant main term that is majorised by the error term i.e.

$$\mathcal{F}_\ell(X, P_1, P_2; z) = \frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})} \sum_{N(\varpi_1), N(\varpi_2) > z} \frac{\mu^2(\varpi_1 \varpi_2)\left(\frac{\varpi_1 \varpi_2}{|\varpi_1 \varpi_2|}\right)^\ell}{N(\varpi_1 \varpi_2)^{1/6}} V\left(\frac{N(\varpi_1)}{P_1}\right) V\left(\frac{N(\varpi_2)}{P_2}\right) W\left(\frac{N(\varpi_1 \varpi_2)}{X}\right)$$

$$+ O\left(\frac{X}{(\varepsilon \log X)^{3/2}} \cdot \frac{1}{\sqrt{\min(P_1, P_2)}}\right) + O\left(\frac{X^{5/6}}{(\varepsilon \log X)^2}\right),$$
$$\text{when} \quad X^{1/2 - \xi} \leqslant P_1, P_2 \leqslant X^{1/2 + \xi}. \tag{13.5}$$

Since $P_1 P_2 \asymp X$ there are $O(\xi \log X)$ choices of P_1, P_2 in the narrow range $X^{1/2 - \xi} \leqslant P_1, P_2 \leqslant X^{1/2 + \xi}$. Summing (13.4) and (13.5) over all possible dyadic tuples (P_1, P_2) gives

$$\sum_{\substack{\varpi_1, \varpi_2 \equiv 1 \pmod 3 \\ N(\varpi_1), N(\varpi_2) > z}} \tilde{g}_\ell(\varpi_1 \varpi_2) W\left(\frac{N(\varpi_1 \varpi_2)}{X}\right)$$

$$= \frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})} \sum_{\substack{\varpi_1, \varpi_2 \equiv 1 \pmod 3 \\ N(\varpi_1), N(\varpi_2) > z}} \frac{\mu^2(\varpi_1 \varpi_2)\left(\frac{\varpi_1 \varpi_2}{|\varpi_1 \varpi_2|}\right)^\ell}{N(\varpi_1 \varpi_2)^{1/6}} W\left(\frac{N(\varpi_1 \varpi_2)}{X}\right)$$

$$+ O\left(\frac{X^{3/4 + \xi/2}}{(\varepsilon \log X)^{3/2}}\right) + O\left(\frac{\xi X^{5/6}}{\varepsilon^2 \log X}\right) + O_{\xi, \varepsilon}(X^{5/6 - \delta_1(\xi, \varepsilon)}), \tag{13.6}$$

for any fixed $0 < \delta_1(\xi, \varepsilon) < \delta_0(\xi, \varepsilon)$.

Sum in (13.2). For each $0 \leqslant k \leqslant 1/\varepsilon$, we analyse the sum

$$\sum_{(\varpi_1, \ldots, \varpi_k, c) \in \mathcal{S}(w, z)} \tilde{g}_\ell(\varpi_1 \ldots \varpi_k c) W\left(\frac{N(\varpi_1 \ldots \varpi_k c)}{X}\right). \tag{13.7}$$

We insert a smooth partition of unity in $N(c)$ and each $N(\varpi_i)$ for $i = 1, \ldots, k$ in (13.7). Thus its suffices to estimate

$$
\mathscr{S}_\ell(P_1, \ldots, P_{k+1}) := \sum_{(\varpi_1, \ldots, \varpi_k, c) \in \mathcal{S}(w,z)} \tilde{g}_\ell(\varpi_1 \ldots \varpi_k c) W\left(\frac{N(\varpi_1 \ldots \varpi_k c)}{X}\right) V\left(\frac{N(c)}{P_{k+1}}\right) \prod_{i=1}^k V\left(\frac{N(\pi_i)}{P_i}\right),
$$
(13.8)

for all dyadic partitions $H = (P_1, \ldots, P_{k+1})$ satisfying $P_1 \ldots P_{k+1} \asymp X$, $w/2 \leqslant P_i \leqslant 2z$ for all $i = 1, \ldots, k$, and $P_{k+1} \geqslant 1/2$. Our goal will be to show that $\mathscr{S}(P_1, \ldots, P_{k+1})$ is asymptotically equal to (either for individual tuples $(P_1, \ldots, P_{k+1})$ or on average)

$$
\mathscr{M}_\ell(P_1, \ldots, P_{k+1})
$$
$$
= \frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})} \sum_{(\varpi_1, \ldots, \varpi_k, c) \in \mathcal{S}(w,z)} \frac{\mu^2(\varpi_1 \ldots \varpi_k c) \left(\frac{\varpi_1 \ldots \varpi_k c}{|\varpi_1 \ldots \varpi_k c|}\right)^\ell}{N(\varpi_1 \ldots \varpi_k c)^{1/6}} W\left(\frac{N(\varpi_1 \ldots \varpi_k c)}{X}\right) V\left(\frac{N(c)}{P_{k+1}}\right) \prod_{i=1}^k V\left(\frac{N(\varpi_i)}{P_i}\right).
$$

For a given $(P_1, \ldots, P_{k+1})$, let

$$
t_i := \frac{\log P_i}{\log(P_1 \ldots P_{k+1})} \geqslant 0 \quad \text{for} \quad i = 1, \ldots, k+1.
$$
(13.9)

We necessarily have

$$
t_1 + \ldots + t_{k+1} = 1;
$$
(13.10)

$$
\frac{\log w}{\log X} \leqslant t_i \leqslant \frac{\log z}{\log X} \quad \text{for} \quad i = 1, \ldots, k.
$$
(13.11)

We now apply Lemma 13.1 with choice $\sigma := 1/6 - \xi$ to decompose the proof into cases.

Narrow Type-III sums. In this case we necessarily have $k \geqslant 2$, and

$$
\exists \text{ three distinct indices } i, j, \ell \in \{1, \ldots, k+1\} \text{ such that } t_i, t_j, t_\ell \in (\tfrac{1}{3} - 2\xi, \tfrac{1}{3} + \xi). \quad (13.12)
$$

In particular, either

  (1) $\exists$ an index i such that $t_i \in [\tfrac{1}{3}, \tfrac{1}{3} + \xi)$, or
  (2) we have $t_i, t_j, t_\ell \in (\tfrac{1}{3} - 2\xi, \tfrac{1}{3})$.

The sum over all dyadic partitions $(P_1, \ldots, P_{k+1})$ for which there exists an index i such that $t_i \in [\tfrac{1}{3}, \tfrac{1}{3} + \xi)$ (and two additional indices j, ℓ such that $t_j, t_\ell \in (\tfrac{1}{3} - 2\xi, \tfrac{1}{3} + \xi)$) is

$$
\leqslant (k+1)! \cdot \sum_{\substack{X^{1/3} \leqslant P \leqslant X^{1/3+\xi} \\ P \text{ dyadic}}} \sup_{\substack{\|\boldsymbol{\alpha}\|_\infty, \|\boldsymbol{\beta}\|_\infty \leqslant 1 \\ \boldsymbol{\beta} \in \mathcal{C}_\eta(P,w)}} \left| \sum_{\substack{a,b \equiv 1 \pmod 3 \\ \pi | a \implies N(\pi) > w \\ \pi | b \implies N(\pi) > w \\ N(a) \asymp X/P, N(b) \asymp P}} \alpha_a \beta_b \tilde{g}_\ell(ab) W\left(\frac{N(ab)}{X}\right) \right|, \quad (13.13)
$$

where $\eta > 0$ is arbitrarily small and fixed by Lemma 6.2 (the only requirement is that $\eta > 100 \log \log X / (\varepsilon \log X)$). Notice that the factor $(k+1)! = k! \cdot (k+1)$ arises from the fact that there are $k+1$ ways of choosing the first index i for which $P_i \in [\tfrac{1}{3}, \tfrac{1}{3} + \xi)$ (and

this index becomes our P) and there are $k!$ ways of representing a as a product of the remaining k variables. Application of Proposition 7.1 shows that (13.13) is

$$\ll (k+1)! \sum_{\substack{X^{1/3} \leqslant P \leqslant X^{1/3+\xi} \\ P \text{ dyadic}}} \left(\frac{1}{(\varepsilon \log X)^{3/2}} \cdot \frac{X}{\sqrt{P}} + \frac{X^{5/6}}{(\varepsilon \log X)^2} \right)$$

$$\ll (k+1)! \left(\frac{X^{5/6}}{(\varepsilon \log X)^{3/2}} + \frac{\xi X^{5/6}}{\varepsilon^2 \log X} \right),$$

where the implied constants are absolute.

We now handle the remaining case in which $t_i, t_j, t_\ell \in (\frac{1}{3} - 2\xi, \frac{1}{3})$. We group together two variables coming from the indices i and j say. We sum over all dyadic partitions $(P_1, \ldots, P_{k+1})$ for which $t_i, t_j, t_\ell \in (\frac{1}{3} - 2\xi, \frac{1}{3})$. This sum is

$$\leqslant (k+1)! \cdot \sum_{\substack{X^{2/3-4\xi} \leqslant U \leqslant X^{2/3} \\ U \text{ dyadic}}} \sup_{\substack{\|\boldsymbol{\alpha}\|_\infty, \|\boldsymbol{\beta}\|_\infty \leqslant 1 \\ \boldsymbol{\beta} \in \mathcal{C}_\eta(X/U, w)}} \left| \sum_{\substack{a,b \equiv 1 \pmod 3 \\ \pi|a \implies N(\pi)>w \\ \pi|b \implies N(\pi)>w \\ N(a) \asymp U, N(b) \asymp X/U}} \alpha_a \beta_b \tilde{g}_\ell(ab) W\left(\frac{N(ab)}{X} \right) \right|, \quad (13.14)$$

where $\eta > 0$ is arbitrarily small and fixed by Lemma 6.2 (the only requirement is that $\eta > 100 \log\log X / (\varepsilon \log X)$). The factor $(k+1)! = (k-1)! \cdot 2\binom{k+1}{2}$ arises from the fact that there are $2\binom{k+1}{2}$ ordered choices of i and j such that $t_i, t_j \in (\frac{1}{3} - 2\xi, \frac{1}{3})$, and $(k-1)!$ ways of representing a as the product of the remaining $k-1$ variables. Applying Proposition 7.1 and arguing in a similar way to the above shows that (13.14) is

$$\ll (k+1)! \left(\frac{X^{5/6}}{(\varepsilon \log X)^{3/2}} + \frac{\xi X^{5/6}}{\varepsilon^2 \log X} \right),$$

where the implied constant is absolute.

Combining the two cases we conclude that

$$\sum_{\substack{(P_1,\ldots,P_{k+1}) \\ (13.12) \text{ holds}}} \mathscr{S}(P_1, \ldots, P_{k+1}) = \sum_{\substack{(P_1,\ldots,P_{k+1}) \\ (13.12) \text{ holds}}} \mathscr{M}(P_1, \ldots, P_{k+1}) + O\left(\frac{(k+1)! X^{5/6}}{(\varepsilon \log X)^{3/2}} \right)$$

$$+ O\left(\frac{(k+1)! \xi X^{5/6}}{\varepsilon^2 \log X} \right).$$

Notice that the main term is absorbed by the error term in this case.

Narrow Type-II sums. In this case we necessarily have $k \geqslant 1$, and

$$\exists \text{ a partition } S \cup T = \{1, \ldots, k+1\} \text{ such that } \tfrac{1}{2} - \xi < \sum_{i \in S} t_i \leqslant \sum_{j \in T} t_j < \tfrac{1}{2} + \xi. \quad (13.15)$$

The contribution of all such $(P_1, \ldots, P_{k+1})$ is

$$\leqslant (k+1)! \sum_{\substack{X^{1/2-\xi} \leqslant U \leqslant X^{1/2} \\ U \text{ dyadic}}} \sup_{\substack{\|\boldsymbol{\alpha}\|_\infty, \|\boldsymbol{\beta}\|_\infty \leqslant 1 \\ \boldsymbol{\beta} \in \mathcal{C}_\eta(U, w)}} \left| \sum_{\substack{a,b \equiv 1 \pmod 3 \\ \pi|a \implies N(\pi)>w \\ \pi|b \implies N(\pi)>w \\ N(a) \asymp X/U, N(b) \asymp U}} \alpha_a \beta_b \tilde{g}_\ell(ab) W\left(\frac{N(ab)}{X} \right) \right|, \quad (13.16)$$

where $\eta > 0$ is arbitrarily small and fixed by Lemma 6.2 (the only requirement is that $\eta > 100 \log\log X/(\varepsilon \log X)$). The term $(k+1)!$ arises from the fact that for each $1 \leqslant i \leqslant k$, there are $i!\binom{k+1}{i}$ ordered choices for the set S containing i elements, and there are $(k+1-i)!$ ways of representing b as a product of the remaining $k+1-i$ variables indicated by the set T. Applying Proposition 7.1, we see that (13.16) is

$$\ll (k+1)! \sum_{\substack{X^{1/2-\xi} \leqslant U \leqslant X^{1/2} \\ U \text{ dyadic}}} \left(\frac{1}{(\varepsilon \log X)^{3/2}} \cdot \frac{X}{\sqrt{U}} + \frac{X^{5/6}}{(\varepsilon \log X)^2} \right)$$

$$\ll (k+1)! \left(\frac{X^{3/4+\xi/2}}{(\varepsilon \log X)^{3/2}} + \frac{\xi X^{5/6}}{\varepsilon^2 \log X} \right),$$

where the implied constants are absolute.

In particular,

$$\sum_{\substack{(P_1,\ldots,P_{k+1}) \\ (13.15) \text{ holds}}} \mathscr{S}(P_1,\ldots,P_{k+1}) = \sum_{\substack{(P_1,\ldots,P_{k+1}) \\ (13.15) \text{ holds}}} \mathscr{M}(P_1,\ldots,P_{k+1}) + O\left(\frac{(k+1)! X^{3/4+\xi/2}}{(\varepsilon \log X)^{3/2}} \right)$$

$$+ O\left(\frac{(k+1)! \xi X^{5/6}}{\varepsilon^2 \log X} \right),$$

where the main term is absorbed by the error term.

Remaining ranges. We now consider all of the remaining dyadic partitions $(P_1,\ldots,P_{k+1})$ one by one. For each remaining tuple $(P_1,\ldots,P_{k+1})$ we will show that

$$\mathscr{S}_\ell(P_1,\ldots,P_{k+1}) = \mathscr{M}_\ell(P_1,\ldots,P_{k+1}) + O_{A,\xi,\varepsilon}\left(\frac{X^{5/6}}{\log^A X} \right), \tag{13.17}$$

for any given $A > 10$ (depending on $\varepsilon > 0$). Recall that $k \leqslant 1/\varepsilon$. Since there are at most $(\log X)^k$ dyadic partitions $(P_1,\ldots,P_{k+1})$ satisfying $P_1 \cdots P_{k+1} \asymp X$, we can sum over the error term in (13.17) without overwhelming the main term. Notice that each of the remaining configurations of $(P_1,\ldots,P_{k+1})$ now fall into either of two cases:

(1) $\exists i \in \{1,\ldots,k+1\}$ such that $t_i \geqslant \frac{2}{3} - \xi$;
(2) Or $\exists$ a partition $S \cup T = \{1,\ldots,k+1\}$ such that

$$\frac{1}{3} + \xi \leqslant \sum_{i \in S} t_i \leqslant \frac{1}{2} - \xi \leqslant \frac{1}{2} + \xi \leqslant \sum_{j \in T} t_j \leqslant \frac{2}{3} - \xi. \tag{13.18}$$

If $\exists i \in \{1,\ldots,k+1\}$ such that $t_i \geqslant \frac{2}{3} - \xi$, then $i = k+1$ by (13.11). This corresponds to the c variable appearing in $\mathscr{S}_\ell(P_1,\ldots,P_{k+1})$ in (13.8). After applying Proposition 11.1 (average Type-I estimate) we obtain (13.17) uniformly in $|\ell| \leqslant X^{1/100}$. If the second alternative holds, then (13.17) follows from Proposition 10.1 (broad Type-II estimate).

Assembly. Summing over all dyadic partitions $(P_1, \ldots, P_{k+1})$ we obtain

$$\sum_{0 \leqslant k \leqslant 1/\varepsilon} \frac{(-1)^k}{k!} \sum_{(\varpi_1, \ldots, \varpi_k, c) \in \mathcal{S}(w,z)} \tilde{g}_\ell(\varpi_1 \ldots \varpi_k c) W\Big(\frac{N(\varpi_1 \ldots \varpi_k c)}{X}\Big)$$

$$= \sum_{0 \leqslant k \leqslant 1/\varepsilon} \frac{(-1)^k}{k!} \sum_{(\varpi_1, \ldots, \varpi_k, c) \in \mathcal{S}(w,z)} \frac{\mu^2(\varpi_1 \ldots \varpi_k c)\big(\frac{\varpi_1 \ldots \varpi_k c}{|\varpi_1 \ldots \varpi_k c|}\big)^\ell}{N(\varpi_1 \ldots \varpi_k c)^{1/6}} W\Big(\frac{N(\varpi_1 \ldots \varpi_k c)}{X}\Big)$$

$$+ O\Big(\frac{\xi X^{5/6}}{\varepsilon^4 \log X}\Big) + O\Big(\frac{X^{5/6}}{\varepsilon^{7/2}(\log X)^{3/2}}\Big) + O\Big(\frac{X^{3/4+\xi/2}}{\varepsilon^{7/2}(\log X)^{3/2}}\Big) + O_{A,\xi,\varepsilon}\Big(\frac{X^{5/6}}{\log^{A-\varepsilon^{-1}} X}\Big), \tag{13.19}$$

uniformly in $|\ell| \leqslant X^{1/100}$. We now drop the third error term in (13.19) because it is majorised by the second one. Combining (13.19) and (13.6) in (13.1)–(13.2), and then applying Lemma 12.1 (in the reverse direction, and to the symbol $\mu^2(\cdot)(\cdot\cdot)^\ell$) gives

$$\sum_{\varpi \equiv 1 \pmod 3} \tilde{g}_\ell(\varpi) W\Big(\frac{N(\varpi)}{X}\Big) = \frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})} \sum_{\varpi \equiv 1 \pmod 3} \frac{\big(\frac{\varpi}{|\varpi|}\big)^\ell}{N(\varpi)^{1/6}} W\Big(\frac{N(\varpi)}{X}\Big)$$

$$+ O\Big(\frac{\xi X^{5/6}}{\varepsilon^4 \log X}\Big) + O\Big(\frac{X^{5/6}}{\varepsilon^{7/2}(\log X)^{3/2}}\Big) + O_{A,\xi,\varepsilon}\Big(\frac{X^{5/6}}{\log^{A-\varepsilon^{-1}} X}\Big). + O_{\xi,\varepsilon}\big(X^{5/6-\delta_1(\xi,\varepsilon)}\big), \tag{13.20}$$

uniformly in $|\ell| \leqslant X^{1/100}$. After choosing $\xi = \varepsilon^{1000}$ and $A = \varepsilon^{-1000}$ (say), the error terms in (13.20) are $O\big(\frac{\varepsilon X^{5/6}}{\log X}\big)$ as $X \to \infty$. We conclude by noticing that

$$\sum_{\varpi \equiv 1 \pmod 3} \frac{1}{N(\varpi)^{1/6}} W\Big(\frac{N(\varpi)}{X}\Big) \sim \int_0^\infty W(x) x^{-1/6} dx \cdot \frac{X^{5/6}}{\log X} \quad \text{as} \quad X \to \infty,$$

and for $\ell \neq 0$,

$$\sum_{\varpi \equiv 1 \pmod 3} \frac{\big(\frac{\varpi}{|\varpi|}\big)^\ell}{N(\varpi)^{1/6}} W\Big(\frac{N(\varpi)}{X}\Big) = o\Big(\frac{X^{5/6}}{\log X}\Big) \quad \text{as} \quad X \to \infty,$$

uniformly in $|\ell| \leqslant X^{1/100}$. This proves Theorem 1.1 and Theorem 1.3. $\blacksquare$

14. Proof of Theorem 1.2

Proof of Theorem 1.2. We expand f in a Fourier series

$$f(x) = \sum_{k \in \mathbb{Z}} \widehat{f}(k) e(kx).$$

For $p \equiv 1 \pmod 3$,

$$f(\theta_p) = \sum_{k \in \mathbb{Z}} \widehat{f}(k) e(k\theta_p) = \sum_{k \in \mathbb{Z}} \widehat{f}(k) \tilde{g}(\varpi)^k,$$

where ϖ is a prime in $\mathbb{Z}[\omega]$ such that $p = \varpi\overline{\varpi}$. Therefore

$$\sum_{p \equiv 1 \pmod 3} f(\theta_p) W\Big(\frac{p}{X}\Big)$$

is equal to

$$\widehat{f}(0) \sum_{p \equiv 1 \pmod 3} W\Big(\frac{p}{X}\Big) + \sum_{0 < |k| \leqslant X^{1/100}} \widehat{f}(k) \Big(\sum_{\varpi \equiv 1 \pmod 3} \tilde{g}(\varpi)^k W\Big(\frac{N(\varpi)}{X}\Big) + O(\sqrt{X}) \Big) + O_A(X^{-A})$$

for any given $A > 10$. We now appeal to Theorem 1.1 and Theorem 1.3 to see that the sum over $k \neq 0$ is equal to

$$(\widehat{f}(1) + \widehat{f}(-1)) \cdot \frac{(2\pi)^{2/3}}{3\Gamma(\frac{2}{3})} \cdot \int_0^\infty W(x) x^{-1/6} dx \cdot \frac{X^{5/6}}{\log X} + o\Big(\frac{X^{5/6}}{\log X}\Big),$$

as claimed.   ∎

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Appendix A. Appendix

This table completes the computation in [Pat77, Table III] where the values $k_j(E)$ were computed for all $1 \leqslant j \leqslant 27$. We supplement [Pat77, Table III] by also computing $k_j(T)$ and $k_j(P)$ for all $1 \leqslant j \leqslant 27$. We do not require these computations in any of our proofs.

j	$d_j(\mu)$	$k_j(E)$	$k_j(P)$	$k_j(T)$
1	$\tau(\mu)$	1	4	1
2	$\tau(\mu)\check{e}(\omega\mu)$	19	5	2
3	$\tau(\mu)\check{e}(-\omega\mu)$	10	6	3
4	$\tau(\omega\mu)$	7	7	6
5	$\tau(\omega\mu)\check{e}(-\mu)$	23	9	4
6	$\tau(\omega\mu)\check{e}(\mu)$	13	8	5
7	$\tau(\omega^2\mu)$	4	1	9
8	$\tau(\omega^2\mu)\check{e}(-\mu)$	14	3	7
9	$\tau(\omega^2\mu)\check{e}(\mu)$	22	2	8
10	$\omega\tau_2(\omega\mu)\check{e}(\mu)$	3	14	11
11	$\omega\tau_2(\omega^2\mu)$	12	17	12
12	$\omega\tau_2(\omega\mu)$	11	11	10
13	$\omega\tau_2(\mu)\check{e}(\omega^2\mu)$	6	10	14
14	$\omega\tau_2(\mu)\check{e}(-\omega\mu)$	8	13	15
15	$\tau_2(\omega^2\mu)\check{e}(-\omega^2\mu)$	24	16	13
16	$\tau_2(\mu)\check{e}(-\mu)$	25	18	17
17	$\tau_2(\mu)$	17	12	18
18	$\tau_2(\mu)\check{e}(\mu)$	27	15	16
19	$\omega^2\tau_1(\omega^2\mu)\check{e}(\mu)$	2	22	20
20	$\omega^2\tau_1(\omega\mu)$	21	21	21
21	$\omega^2\tau_1(\omega^2\mu)$	20	26	19
22	$\omega^2\tau_1(\mu)\check{e}(\omega\mu)$	9	23	23
23	$\omega^2\tau_1(\omega\mu)\check{e}(\omega^2\mu)$	5	19	24
24	$\omega^2\tau_1(\omega^2\mu)\check{e}(\omega^2\mu)$	15	27	22
25	$\tau_1(\mu)\check{e}(-\mu)$	16	24	26
26	$\tau_1(\mu)$	26	20	27
27	$\tau_1(\mu)\check{e}(\mu)$	18	25	25

We note that,

$$\langle\{(j, k_j(E)) : 1 \leqslant j \leqslant 27\}\rangle \simeq C_2^{12}$$
$$\langle\{(j, k_j(P)) : 1 \leqslant j \leqslant 27\}\rangle \simeq C_3^{9}$$
$$\langle\{(j, k_j(T)) : 1 \leqslant j \leqslant 27\}\rangle \simeq C_3^{8}.$$

These isomorphisms are easily seen from the table by following the cycle structure. The exponents 12 in C_2^{12} be explained by noticing that the forms $j = 1, 17, 26$ are invariant under E and all the other elements are of order two, giving us $\frac{27-3}{2} = 12$ generators. Likewise the exponent 8 in C_3^8 can be explained by noticing that the forms with $j = 1, 2, 3$ are invariant and there are $\frac{27-3}{3} = 8$ remaining generators all of order 3. Finally the

exponent 9 appears in the case of $k_j(P)$ because no forms is left invariant by P and P is of order three.

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