Computational Methods and Function Theory (2025) 25:629–644
https://doi.org/10.1007/s40315-024-00546-3

Diameter of Compact Riemann Surfaces

Huck Stepanyants[1,2,4] · Alan Beardon[3] · Jeremy Paton[1,2] · Dmitri Krioukov[1,2,4,5]

Received: 17 February 2023 / Accepted: 21 February 2024 / Published online: 27 June 2024

Abstract
Diameter is one of the most basic properties of a geometric object, while Riemann surfaces are one of the most basic geometric objects. Surprisingly, the diameter of compact Riemann surfaces is known exactly only for the sphere and the torus. For higher genuses, only very general but loose upper and lower bounds are available. The problem of calculating the diameter exactly has been intractable since there is no simple expression for the distance between a pair of points on a high-genus surface. Here we prove that the diameters of a class of simple Riemann surfaces known as *generalized Bolza surfaces* of any genus greater than 1 are equal to the radii of their fundamental polygons. This is the first exact result for the diameter of a compact hyperbolic manifold.

Keywords Diameter · Riemann surfaces · Hyperbolic manifolds

Mathematics Subject Classification Primary 53C22 (Geodesics in global differential geometry) · 30F10 (Compact Riemann surfaces and uniformization)

Communicated by Pekka Koskela.

✉ Huck Stepanyants
stepanyants.j@northeastern.edu

✉ Dmitri Krioukov
dima@northeastern.edu

Alan Beardon
afb@dpmms.cam.ac.uk

Jeremy Paton
paton.j@northeastern.edu

1 Department of Physics, Northeastern University, Boston, MA 02115, USA

2 Network Science Institute, Northeastern University, Boston, MA 02115, USA

3 Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Cambridge CB3 0WB, UK

4 Department of Mathematics, Northeastern University, Boston, MA 02115, USA

5 Department of Electrical and Computer Engineering, Northeastern University, Boston, MA 02115, USA

1 Introduction

A Riemann surface is any connected, one-dimensional complex manifold. According to the classification theorem of closed surfaces [1], any compact Riemann surface is homeomorphic to either the sphere or the connected sum of g tori, where g is the genus of the surface. Once equipped with a metric, a Riemann surface becomes a Riemannian manifold. The sphere ($g = 0$) admits the spherical metric, the torus ($g = 1$) admits the Euclidean metric, while the surfaces of genus $g > 1$ admit the hyperbolic metric.

The diameter $\mathcal{D}$ of a metric space is the maximum distance between a pair of points in it. Diameter is one of the most basic characteristics of any geometric object. Surprisingly, the diameter of compact Riemann surfaces is known exactly only for the sphere and the torus. The best results on the diameter of surfaces of genus $g > 1$ are only loose lower and upper bounds in terms of the total surface area, the systole, and the genus [2–5]. The *systole* is the length of a shortest non-contractible loop on the surface.

Our main result is a proof of the following theorem:

Theorem 1 *The diameter of a class of Riemann surfaces S_g, known as generalized Bolza surfaces, of genus $g > 1$ is* $\mathcal{D}_g = \operatorname{arccosh}\left(\cot^2(\pi/(4g))\right)$.

The definition of the surfaces S_g is in the next section. In particular, S_2 is known as the Bolza surface [6], one of the first compact hyperbolic manifolds ever considered. A free particle moving along a geodesic on the Bolza surface was the first dynamical system proven rigorously to be chaotic [7]. The Bolza surface is also known to maximize the systole across all genus-2 surfaces [8]. For $g > 2$, the S_g are the generalized Bolza surfaces [9]. They appear frequently in studies of hyperbolic surfaces due to their high degree of symmetry [9–12].

Knowing the diameter of a surface, we can efficiently compute the distance between any pair of points on it, a result to be published elsewhere. Theorem 1 states that the Bolza surface has diameter $\mathcal{D} = \operatorname{arccosh}(3 + 2\sqrt{2}) \approx 2.45$. To the best of our knowledge, this is the first ever exact result for the diameter of *a* compact hyperbolic manifold. The closest results to ours appear to be the ones in [13]. They apply to manifolds of dimension at least five.

We proceed by collecting all the necessary background information and definitions in Sect. 2. Section 3 contains the outline of the proof of Theorem 1 split into a sequence of theorems that we state in that section as well. We prove all those theorems in the concluding Sect. 4.

2 Background Information and Definitions

We use the Poincaré disk model of the hyperbolic plane $\mathbb{H}^2$. The isometries (distance-preserving maps) of $\mathbb{H}^2$ are given by the matrices

$$\begin{bmatrix} a & \overline{c} \\ c & \overline{a} \end{bmatrix}, \quad a, c \in \mathbb{C},\ |a|^2 - |c|^2 = 1, \tag{1}$$

which form a subgroup of $PSL(2, \mathbb{C})$, the invertible 2×2 complex matrices. The action of each matrix on a complex number z in the Poincaré disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is a fractional linear transformation:

$$\begin{bmatrix} a & \overline{c} \\ c & \overline{a} \end{bmatrix} (z) = \frac{az + \overline{c}}{cz + \overline{a}}. \tag{2}$$

A *Fuchsian group* $\mathcal{F}$ is a discrete subgroup of $PSL(2, \mathbb{C})$ that has an invariant disk in $\mathbb{C}^{\infty}$. Each $\mathcal{F}$ defines a hyperbolic Riemann surface S which is the quotient surface $S = \mathbb{H}^2/\mathcal{F}$. A *fundamental domain* is an open, connected set in $\mathbb{H}^2$ that contains at most one representative of each point on S, and whose closure contains at least one representative of each point on S [14]. A fundamental domain and its images under the actions of $\mathcal{F}$ tessellate $\mathbb{H}^2$.

Since points on a quotient surface S are cosets in $\mathbb{H}^2/\mathcal{F}$, the distance between two points (cosets) $[z] = \mathcal{F}z$, $[w] = \mathcal{F}w$ on S is given by

$$\delta^{\star}([z], [w]) = \inf\{\delta(z', w'), z' \in [z], w' \in [w]\}, \tag{3}$$

where $\delta(z, w)$ denotes the distance in $\mathbb{H}^2$. The diameter $\mathcal{D}$ of S is the largest distance between two points on S.

The best results on the diameters of Riemann surfaces of genus $g > 1$ are as follows. It was shown in [2] that for any such surface, the following inequalities hold, where ℓ is the systole, A is the total area of the surface, and $\mathcal{D}$ is the diameter:

$$2\ell \sinh(\mathcal{D}) \geq A, \text{ and} \tag{4}$$

$$2 \sinh\left(\frac{\ell}{4}\right) \mathcal{D} \leq A. \tag{5}$$

In [3] it was shown that

$$4 \cosh\left(\frac{\ell}{2}\right) \leq 3 \cosh(\mathcal{D}) - 1. \tag{6}$$

Another lower bound exists in terms of the area alone [4]:

$$\cosh \mathcal{D} \geq \frac{A}{2\pi} + 1, \tag{7}$$

and finally, there is the following lower bound in terms of the genus [5]:

$$\cosh \mathcal{D} \geq \frac{1}{\sqrt{3}} \cot\left(\frac{\pi}{6(2g-1)}\right). \tag{8}$$

All of the bounds above hold in general for any Riemann surface, as do other related spectral results [15–17]. However, none of them is tight for the S_g.

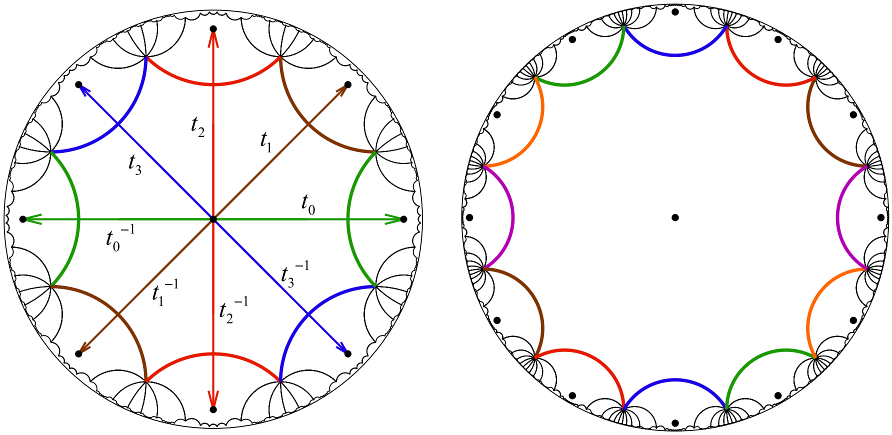

Fig. 1 Left: The Bolza surface (S_2) and its "gluing" scheme. Sides that are identified in the quotient space are labeled with the same color, and the arrows represent the actions of the generators $\{t_k\}$ and their inverses on the fundamental polygon. Right: S_3 and its gluing scheme

The *generalized Bolza surface* S_g of genus g is defined [9, 18] to be the surface obtained by identifying ("gluing") the opposite sides of the regular $4g$-gon with interior angles $\pi/(2g)$, Fig. 1, whose side length s and radius R satisfy [18]

$$\cosh^2\left(\frac{s}{2}\right) = \cosh R = \cot^2\left(\frac{\pi}{4g}\right), \tag{9}$$

and whose vertices $\{v_k\}$ are evenly spaced at distance R from the origin:

$$v_k = \tanh\left(\frac{R}{2}\right)\, e^{(k-1/2)\pi i/(2g)}, \quad k = 0, 1, \ldots, 4g-1. \tag{10}$$

This surface S_g is the quotient surface $\mathbb{H}^2/\mathcal{F}_g$, where $\mathcal{F}_g$ is the Fuchsian group generated by the following $2g$ generators and their inverses:

$$t_k = \begin{bmatrix} 1 & \tanh\left(\frac{s}{2}\right) e^{k\pi i/(2g)} \\ \tanh\left(\frac{s}{2}\right) e^{-k\pi i/(2g)} & 1 \end{bmatrix}, \quad k = 0, 1, \ldots, 2g-1. \tag{11}$$

These generators glue the opposite sides of the polygon by mapping it to its edge-adjacent polygons in the tessellation as shown in Fig. 1. By Poincaré's theorem [14], the interior of the polygon is a fundamental domain of S_g, while the polygon itself is called the *fundamental polygon*.

The surface S_2 of genus 2 is the well-known Bolza surface [6]. The surfaces S_g are also known as the *Wiman surfaces of type II* [11, 12]. They have $8g$ automorphisms, except the $g = 2$ Bolza surface, which has 48 automorphisms. The S_g also have automorphisms of order $4g$, which is the second largest possible order, as was proven in [10].

Our main result is the exact diameter of the surface S_g of genus $g > 1$, Theorem 1, which in view of Eq. (9) can be restated as:

Theorem 1 $\mathcal{D}_g = R$.

The problem of finding the diameter is not trivial thanks to the definition of the distance in Eq. (3), which is an infimum over the infinitely many elements in $\mathcal{F}_g$. Even though this infimum is actually a minimum since only a finite subset of group elements needs to be considered, there is no explicit expression for such a subset, which is the main difficulty in the problem. In fact, as mentioned in the introduction, our main motivation for this paper is that the knowledge of the diameter of a surface allows us to provide an explicit expression for this subset, leading to an efficient formula to compute distances between pairs of points on the surface, a result to appear in a follow-on paper.

Our proof of the main result, which we outline in the following section, is a combination of geometric and algebraic techniques. Specifically, we use some geometric symmetries of S_g to simplify the problem, and algebra to compute the optimal distances.

3 Proof Strategy

We first observe that every vertex of the fundamental polygon represents the same point in the quotient space; we will call this point the *quotient vertex*, $[v] \in S_g$. Similarly, we define $[0]$ as the point in the quotient space to which the origin maps. For an arbitrary point z in the Poincaré disk, let $d_0(z)$ and $d_v(z)$ denote the quotient distances from $[z]$ to $[0]$ and $[v]$, respectively.

The proof of Theorem 1 will be broken into several smaller, sequential theorems. First, we will show that $\delta^\star([0], [v]) = R$, thus proving the following theorem:

Theorem 2 *The diameter $\mathcal{D}_g$ of S_g is at least R, $\mathcal{D}_g \geq R$.*

The remaining theorems are aimed at proving $\mathcal{D}_g \leq R$, or equivalently, $\delta^\star([z], [w]) \leq R$ for every pair of points $[z], [w]$ in the quotient space. The symmetry of S_g allows us to make several simplifying assumptions about $[z]$ and $[w]$ without loss of generality.

The symmetries of the quotient surface S_g are isometric bijections that map S_g to itself. In the Poincaré disk $\mathbb{D}$, these symmetries can be thought of as the subgroup of isometries ϕ of $\mathbb{D}$ for which the projection $\phi^\star$ of ϕ onto the quotient surface, given by

$$\phi^\star([p]) = [\phi(p)], \tag{12}$$

is well-defined and isometric. These properties are satisfied by all ϕ for which the Fuchsian group $\mathcal{F}_g$, a subgroup of $PSL(2, \mathbb{C})$, is invariant under conjugation by ϕ: $\phi \mathcal{F}_g \phi^{-1} = \mathcal{F}_g$. Indeed, for such ϕ and any $f, f_1, f_2 \in \mathcal{F}_g$ and $p, p_1, p_2 \in \mathbb{D}$, there exist $f', f_1', f_2' \in \mathcal{F}_g$ such that

$$\phi^\star([fp]) = [\phi f(p)] = [f'\phi(p)] = [\phi(p)] = \phi^\star([p]), \text{ and} \tag{13}$$

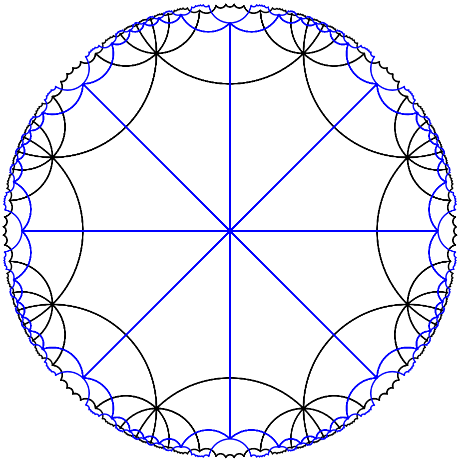

Fig. 2 The dual tessellation (blue) overlayed against the original tessellation (black) in the case $g = 2$

$$\begin{aligned}\delta^*(\phi^*([p_1]), \phi^*([p_2])) &= \delta^*([\phi(p_1)], [\phi(p_2)]) \\ &= \min\{\delta(f_1\phi(p_1), f_2\phi(p_2)), f_1, f_2 \in \mathfrak{F}_g\} \\ &= \min\{\delta(\phi f_1'(p_1), \phi f_2'(p_2)), f_1', f_2' \in \mathfrak{F}_g\} \\ &= \min\{\delta(f_1'(p_1)), \delta(f_2'(p_2)), f_1', f_2' \in \mathfrak{F}_g\} \\ &= \delta^\star([p_1], [p_2]),\end{aligned} \tag{14}$$

so ϕ is well-defined and an isometry.

The simplest such isometries are the elements of $\mathfrak{F}_g$. Indeed, for $\phi \in \mathfrak{F}_g$ we have $\phi\mathfrak{F}_g\phi^{-1} = \mathfrak{F}_g$. In this case, $\phi^\star$ is simply the identity function on S_g. Therefore, S_g is symmetric under actions by elements of $\mathfrak{F}_g$.

Another subgroup of isometries of $\mathbb{D}$ that are symmetries of S_g are rotations of $\mathbb{D}$ by $\pi/(4g)$. Although this fact is evident from the definition of S_g in Sect. 2, we give its formal proof in Sect. 4.

The final and most complicated subgroup of symmetries of S_g that we will need is its *dual symmetries*. We define them with the aid of new fundamental polygons that we call the *dual polygons*, and the corresponding *dual tessellation* of $\mathbb{H}^2$. Dual polygons are formed by joining by geodesics the centers of the $4g$ polygons around the vertices in the original tessellation, Fig. 2. We will prove the following proposition:

Proposition 1 *Let ϕ be any isometry of $\mathbb{D}$ that maps the fundamental polygon to any of its dual polygons. Then the projection $\phi^\star : S_g \to S_g$ given by*

$$\phi^\star([z]) = [\phi(z)] \tag{15}$$

is well-defined and is an isometric automorphism of S_g. In particular, dual polygons are fundamental polygons.

The rotational and dual symmetries of S_g allow us to reduce drastically the set of all the possibilities of where the pairs of points z and w can lie to the following subset:

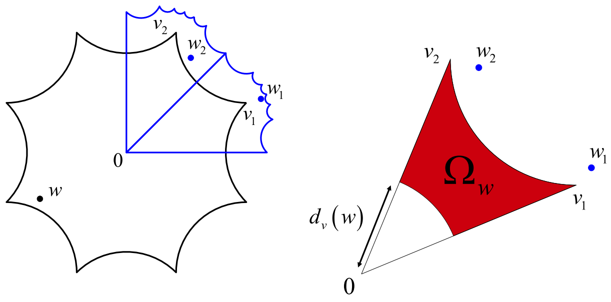

Fig. 3 Left: The representatives w_1 and w_2 of $[w]$ in the dual tessellation in the $g = 2$ case. They are images of w in the dual polygons centered at v_1 and v_2. Right: The domain Ω_w is the set of points z satisfying the conditions in Prop. 2 for a given point w

Proposition 2 *To prove Theorem 1, it suffices to consider only the pairs of points z and w satisfying the following conditions:*

1. *z lies in triangle $T = \Delta 0 v_1 v_2$, Fig. 3,*
2. *$d_0(z) \geq d_v(w)$,*
3. *$d_v(w) \leq s/2$.*

Assuming, thanks to Prop. 2, that the first point z lies in the triangle T with vertices O, v_1, v_2, let $w_1, w_2 \in [w]$ be the two dual-tessellation representatives of $[w]$ lying in the two dual polygons centered at v_1, v_2, respectively, Fig. 3. To prove our desired result $\delta^\star([z], [w]) \leq R$, it suffices to show that

$$\min\left[\delta(z, w_1), \delta(z, w_2)\right] \leq R, \tag{16}$$

simply because

$$\begin{aligned} \delta^\star([z], [w]) &= \min\{\delta(z', w'), z' \in [z], w' \in [w]\} \\ &\leq \min\left[\delta(z, w_1), \delta(z, w_2)\right]. \end{aligned} \tag{17}$$

Before we prove Eq. (16) using algebra, we take one final step to reduce the number of possibilities for the locations of z, w to consider. Given w such that $d_v(w) \leq s/2$, we define the domain Ω_w as the set of all $z \in T$ for which z and w meet the conditions listed in Prop. 2, Fig. 3. We will prove the following:

Theorem 3 *Given w for which $d_v(w) \leq s/2$, the function*

$$f(z) = \min\left[\delta(z, w_1), \delta(z, w_2)\right]$$

attains its global maximum over the domain Ω_w on its boundary $\partial\Omega_w$.

This theorem allows us to reduce the possibilities for the location of z even further, to include only the points on the boundary of Ω_w.

The final step in the proof of Theorem 1 is to show that Eq. (16) holds for all of the z, w pairs satisfying all the conditions above:

Theorem 4 *Given w for which $d_v(w) \leq s/2$, for all $z \in \partial\Omega_w$ we have*

$$\min\left[\delta(z, w_1), \delta(z, w_2)\right] \leq R.$$

Thus we will have shown that $\mathcal{D}_g \leq R$, and the combination of this result with that of Theorem 2 proves Theorem 1.

One important question is how generalizable our proof strategy outlined in this section is to other surfaces, such as those appearing in the Poincaré theorem [19] or the canonical surfaces [20]. For these surfaces, the gluing identifies not the opposite sides of a polygon, but its alternating sides: 1– 3, 2–4, 5–7, 6–8, and so on. One could hope that if the polygon is still a regular $4g$-gon, then this gluing would lead to a surface with the same diameter R. However, this is not true—the diameter of these surfaces appears to be greater than R in simulations. This observation is a reflection of the fact that the symmetries of a surface play a crucial role in defining its geometric properties including the diameter, so one should select very carefully a right set of symmetries of a surface in calculating its diameter. Our proof is an example of this strategy applied to a particular class of highly symmetric surfaces. Its generalization to other surfaces with other groups of symmetry presents an interesting open problem.

The next section contains the complete proofs of all the theorems and propositions above.

4 Proofs

Theorem 2 *The diameter $\mathcal{D}_g$ of S_g is at least R, $\mathcal{D}_g \geq R$.*

Proof Take $[z] = [0]$ and $[w] = [v]$. We will show that $\delta^\star([0], [v]) = R$, from which the theorem follows immediately. For this, we use that the fundamental polygon P is also the Dirichlet polygon of $[0]$,

$$D_0 = \{p \mid \delta(0, p) < \delta(f(0), p) \text{ for all } f \in \mathcal{F}_g,\ \ f \neq I\}. \tag{18}$$

where I is the identity. To see this, first note that points p on the geodesic segment $\overline{v_i v_{i+1}}$ satisfy $\delta(0, p) = \delta(t_i(0), p)$ owing to the symmetry of the fundamental polygon. Next, denote by H_i the half-plane bounded by this line and containing 0. Then

$$\begin{aligned} D_0 &= \{p \mid \delta(0, p) < \delta(f(0), p) \text{ for all } f \in \mathcal{F}_g,\ \ f \neq I\} \\ &\subseteq \{\delta(0, p) < \delta(t_i(0), p),\ 0 \leq i < 4g\} = \bigcap_{0 \leq i < 4g} H_i, \end{aligned} \tag{19}$$

and the last expression is just P, thus $D_0 \subseteq P$. Since Dirichlet polygons are fundamental polygons, which all have the same hyperbolic area [14, Thm. 9.1.3], we have $\text{area}(D_0) = \text{area}(P)$ and conclude $D_0 = P$.

Now, since the elements of $[v]$ contained in P, which are just the v_i, are the closest v-images to 0 and lie a distance R from 0, we have $\delta^\star([0], [v]) = R$. Therefore $\mathcal{D}_g \geq R$. □

Proposition 1 *Let ϕ be any isometry of $\mathbb{D}$ that maps the fundamental polygon to any of its dual polygons. Then the projection $\phi^\star : S_g \to S_g$ given by*

$$\phi^\star([z]) = [\phi(z)] \tag{20}$$

is well-defined and is an isometric automorphism of S_g. In particular, dual polygons are fundamental domains.

Proof First we will show the rotational symmetry of the quotient surface. It suffices to show that $\theta\mathfrak{F}_g\theta^{-1} = \mathfrak{F}_g$ by the argument given in Sect. 3, where θ is the counterclockwise rotation of $\mathbb{D}$ by $\pi/2g$. We will use the property of the generators of $\mathfrak{F}_g$ that $\theta t_i\theta^{-1} = t_{i+1}$ for all i, and the fact which follows from elementary group theory that $\theta\mathfrak{F}_g\theta^{-1} = \mathfrak{F}_g$ if and only if $\theta\mathfrak{F}_g\theta^{-1} \subseteq \mathfrak{F}_g$. We further note that it suffices to show $\theta t_i\theta^{-1} \subseteq \mathfrak{F}_g$ for all i, since this implies for all $f = t_{i_1}t_{i_2}\cdots t_{i_k} \in \mathfrak{F}_g$ that

$$\theta f\theta^{-1} = \theta t_{i_1}t_{i_2}\cdots t_{i_k}\theta^{-1} = (\theta_{t_{i_1}}\theta^{-1})(\theta_{t_{i_2}}\theta^{-1})\cdots(\theta_{t_{i_k}}\theta^{-1}) \in \mathfrak{F}_g. \tag{21}$$

For an arbitrary generator $t_i \in \mathfrak{F}_g$ we have

$$\theta t_i\theta^{-1} = t_{i+1} \in \mathfrak{F}_g, \tag{22}$$

and therefore $\theta\mathfrak{F}_g\theta^{-1} = \mathfrak{F}_g$. It follows that the isometry $\theta^\star$, given by $\theta^\star([p]) = [\theta(p)]$ for points $[p]$ on the quotient surface, is a well-defined isometry and therefore a symmetry of the quotient surface.

Next, we will show the same for $\phi^\star$. Suppose that ϕ maps the fundamental polygon to the dual polygon D centered at the vertex v. Then for any generator t_i, the map $t_i' = \phi t_i\phi^{-1}$ maps D to one of its neighboring dual polygons centered at vertex v', such that $t'(v) = v'$ and t' fixes the geodesic line between v and v'.

Now, choose $h \in \mathfrak{F}_g$ that maps v to v'. We will be done if we can show $t' = h$, since this implies $t' \in \mathfrak{F}_g$ as desired. Since h is orientation-preserving, we can write it as a product of t' and an arbitrary rotation by α around v'. Then h^{-1} is the product of $(t')^{-1}$ and a rotation by $-\alpha$ around v. Pick one of the representatives of the fundamental polygon that has $\overline{vv'}$ as an edge, such that the directed line segment $\overline{vv'}$ is oriented counterclockwise along the polygon boundary. Then by the rotational symmetry of $\mathfrak{F}_g$, every similar element of $\mathfrak{F}_g$ that maps a vertex to its neighbor in the counterclockwise direction has the same form, a translation followed by a rotation by α. Similarly, such maps in the clockwise direction are translations followed by rotations by $-\alpha$.

However, we could have chosen the other polygon that has edge $\overline{vv'}$ in the clockwise orientation, and drawn the same conclusions above with the orientations reversed. So we are forced to conclude $\alpha = -\alpha$, thus either $\alpha = 0$ or $\alpha = \pi$. But h cannot have fixed points, and if $\alpha = \pi$ then h fixes the midpoint between v and v', hence $\alpha = 0$, so $t' = h$. Therefore $\phi\mathfrak{F}\phi^{-1} = \mathfrak{F}$, so $\phi^\star$ is a symmetry of the quotient surface. □

Proposition 2 *To prove Theorem 1, it suffices to consider only the pairs of points z and w satisfying the following conditions:*

1. *z lies in triangle $T = \Delta 0 v_1 v_2$, Fig. 3,*

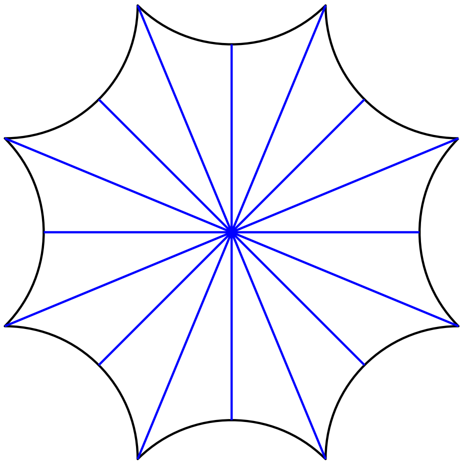

Fig. 4 Division of the fundamental polygon into $8g$ isosceles triangles for $g = 2$

2. $d_0(z) \geq d_v(w)$,
3. $d_v(w) \leq s/2$.

Proof For pairs z, w that do not meet at least one of the conditions above, our strategy will be to find another pair z', w' that do meet the conditions, such that $\delta(z, w) = \delta(z', w')$. Then the diameter of S_g, which is the maximum of δ over all z, w, is equal to the maximum of δ over just the z, w which satisfy the conditions above, so it will suffice to consider only those pairs.

Starting with an arbitrary pair z, w, let ϕ be any of the isometries from the fundamental polygon to a dual polygon, and $\phi^\star$ the corresponding isometry of the quotient surface. Note that $\phi^\star$ takes $[0] \to [v]$ and $[v] \to [0]$. Therefore $d_0(\phi(p)) = d_v(p)$ and $d_v(\phi(p)) = d_0(p)$ for points $p \in \mathbb{D}$. Now, for the given z, w consider the four quantities $d_0(z), d_v(z), d_0(w), d_v(w)$. We can divide the fundamental polygon into $8g$ congruent isosceles triangles (Fig. 4) each with one vertex at 0, one vertex at some v_i, and one vertex at the midpoint of an edge of the polygon, and side lengths $R, s/2, s/2$. Since z lies in one of these isosceles triangles, we have $d_0(z) + d_v(z) \leq s/2 + s/2 = s$, and similarly for w. It follows that

$$M = \min\{d_0(z), d_v(z), d_0(w), d_v(w)\} \leq \frac{s}{2}. \tag{23}$$

We now consider four cases, and in each case construct z', w' for which conditions (2) and (3) are satisfied and $\delta(z', w') = \delta(z, w)$.

1. $M = d_0(z)$. Let $z' = \phi(w)$ and $w' = \phi(z)$. Then

$$d_0(z') = d_v(w) \geq d_0(z) = d_v(w') \qquad \text{and} \tag{24}$$

$$d_v(w') = d_0(z) \leq \frac{s}{2}. \tag{25}$$

2. $M = d_v(z)$. Let $z' = w$ and $w' = z$. Then

$$d_0(z') = d_0(w) \geq d_v(z) = d_v(w') \quad \text{and} \tag{26}$$

$$d_v(w') = d_v(z) \leq \frac{s}{2}. \tag{27}$$

3. $M = d_0(w)$. Let $z' = \phi(z)$ and $w' = \phi(w)$. Then

$$d_0(z') = d_v(z) \geq d_0(w) = d_v(w') \quad \text{and} \tag{28}$$

$$d_v(w') = d_0(w) \leq \frac{s}{2}. \tag{29}$$

4. $M = d_v(w)$. Let $z' = z$ and $w' = w$. Then

$$d_0(z') = d_0(z) \geq d_v(w) = d_v(w') \quad \text{and} \tag{30}$$

$$d_v(w') = d_v(w) \leq \frac{s}{2}. \tag{31}$$

We have already demonstrated the rotational symmetry of S_g in the proof of Prop. 1. Starting with z', w' and rotating a sufficient number of times by $\pi/(2g)$, we can place z' in T. Furthermore, since rotations map vertices to vertices and fix 0, they preserve the quantities $d_0(z')$, $d_v(z')$, $d_0(w')$, $d_v(w')$, so the rotated z', w' now satisfy all three conditions. □

Theorem 3 *Given w for which $d_v(w) \leq s/2$, the function*

$$f(z) = \min\left[\delta(z, w_1), \delta(z, w_2)\right]$$

attains its global maximum over the domain Ω_w on its boundary $\partial\Omega_w$.

Proof Since $f(z)$ is continuous and bounded on the compact region $\Omega_w \cup \partial\Omega_w$, $f(z)$ attains a global maximum on this region.

Let M denote the midline of w_1 and w_2 (the locus of points equidistant from w_1 and w_2, also the perpendicular bisector of the geodesic segment $\overline{w_1 w_2}$). Then M divides Ω_w into some set of open regions, $\{S_i\}$. We will first show that for any i and $z_0 \in S_i$, z_0 cannot be a global maximum of $f(z)$.

Inside S_i we have $\delta(z, w_1) \neq \delta(z, w_2)$, and therefore either $\delta(z, w_1) < \delta(z, w_2)$ or $\delta(z, w_2) < \delta(z, w_1)$. Assume without loss of generality that $\delta(z, w_1) < \delta(z, w_2)$. Then $f(z_0) = \delta(z_0, w_1)$ and $f(z) = \delta(z, w_1)$ in some open set around z_0. Therefore, we can always increase the value of f by moving z_0 in some suitable direction (away from w_1).

It remains to consider $z_0 \in M \cap \Omega_w$. The set $M \cap \Omega_w$ is a union of segments $\{M_i\}$ which lie in Ω_w and have endpoints on $\partial\Omega_w$. Take $z_0 \in M_i$. In this case we also have $f(z_0) = \delta(z_0, w_1)$. It suffices to show that $\delta(z, w_1)$ is maximized at one of the endpoints of M_i. However, this is a well-known result on the distance from points to line segments (see [21, Thm. 2.2]). Since these endpoints are on $\partial\Omega_w$, we have shown that the global maximum of $f(z)$ is attained on $\partial\Omega_w$. □

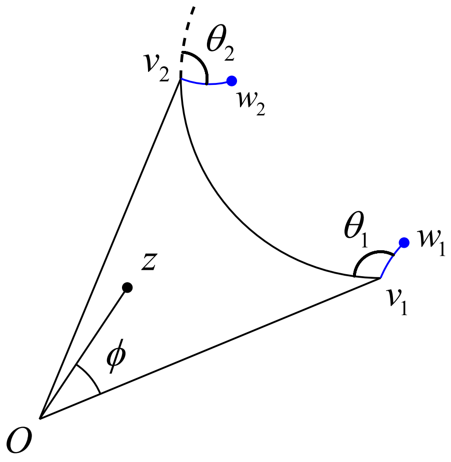

Fig. 5 Illustration of z, w_1, and w_2, and angles ϕ, θ_1, θ_2

Theorem 4 *Given w for which $d_v(w) \leq s/2$, for all $z \in \partial\Omega_w$ we have*

$$\min\left[\delta(z, w_1), \delta(z, w_2)\right] \leq R.$$

Proof We abbreviate $a = d_0(z)$, $b = d_v(w)$, $R' = \tanh(R/2)$, and $s' = \tanh(s/4)$.
Case I: We look at the radial segments of $\partial\Omega_w$ first, see Fig. 3, right. For z on $\overline{0v_0}$, consider the triangle with vertices z, v_0, and w_1 and side lengths b, $\delta(z, w_1)$, and $R - a$, respectively. By the triangle inequality,

$$b + (R - a) \geq \delta(z, w_1) \quad \Rightarrow \quad \delta(z, w_1) \leq R - (a - b) \leq R, \tag{32}$$

and a similar argument for z on $\overline{0v_2}$ gives $\delta(z, w_2) \leq R$, so that in either case $\min\left[\delta(z, w_1), \delta(z, w_2)\right] \leq R$.
Case II: Next we consider the circular arc of radius b centered at the origin (Fig. 3, right). For z on this arc we have $a = b$. Let ϕ denote the angle between $\overline{0z}$ and $\overline{0v_1}$, so that $\phi \in [0, \pi/(2\,g)]$. Let θ_1 denote the angle between $\overline{v_1w_1}$ and $\overline{v_1v_2}$, and θ_2 denote the angle between $\overline{v_2w_2}$ and the extension of $\overline{v_1v_2}$, Fig. 5.

We have the constraint $\theta_1 = \theta_2$, i.e. w_1 and w_2 make the same angle with the axis $\overline{v_1v_0}$, which is a consequence of Prop. 1. Now, we assume that $\delta(z, w_1) > R$ and $\delta(z, w_2) > R$ and seek a contradiction. Using the hyperbolic sine and cosine theorems to calculate the distances, we get after some algebra

$$\begin{aligned} &\delta(z, w_1) > R \Leftrightarrow \\ &\quad \left(1 + R'^2\right) - 2R' \coth a \left[\cos\left(\theta_1 + \frac{\pi}{4g}\right) + \cos\phi\right] \\ &\quad\quad + \cos\left(\theta_1 + \phi + \frac{\pi}{4g}\right) + R'^2 \cos\left(\theta_1 - \phi + \frac{\pi}{4g}\right) > 0, \end{aligned} \tag{33}$$

and a similar equation arises from $\delta(z, w_2) > R$. Solving Eq. (33) yields

$$\frac{\cos\left(\frac{\theta_1+\phi}{2}+\frac{\pi}{8g}\right)}{\sin\left(\frac{\theta_1-\phi}{2}+\frac{\pi}{8g}\right)} \in \left(-\infty, R'\tanh\left(\frac{a}{2}\right)\right) \cup \left(\frac{R'}{\tanh\left(\frac{a}{2}\right)}, \infty\right). \tag{34}$$

Notice that $R'\tanh(a/2)$ is an increasing function of a, while $R'/\tanh(a/2)$ is a decreasing function of a, so it suffices to consider the case where a is maximal, $a = s/2$. Plugging this in gives

$$\frac{\cos\left(\frac{\theta_1+\phi}{2}+\frac{\pi}{8g}\right)}{\sin\left(\frac{\theta_1-\phi}{2}+\frac{\pi}{8g}\right)} \in \left(-\infty, R's'\right) \cup \left(\frac{R'}{s'}, \infty\right). \tag{35}$$

Solving for $\alpha = \theta_1 - \phi + \pi/(4g)$ gives

$$\tan\left(\frac{\alpha}{2}\right) \in \left(\frac{\cos\phi - \frac{R'}{s'}}{\sin\phi}, \frac{\cos\phi - R's'}{\sin\phi}\right). \tag{36}$$

A similar analysis of the second inequality, $\delta(z, w_2) > R$ using $\theta_1 = \theta_2$ gives

$$\tan\left(\frac{\alpha}{2}\right) \in \left(-\infty, \frac{\sin\left(\frac{\pi}{2g}-\phi\right)}{\cos\left(\frac{\pi}{2g}-\phi\right)-\frac{R'}{s'}}\right) \cup \left(\frac{\sin\left(\frac{\pi}{2g}-\phi\right)}{\cos\left(\frac{\pi}{2g}-\phi\right)-R's'}, \infty\right). \tag{37}$$

However, since the inequalities

$$\frac{\cos\phi - \frac{R'}{s'}}{\sin\phi} \leq \frac{\sin\left(\frac{\pi}{2g}-\phi\right)}{\cos\left(\frac{\pi}{2g}-\phi\right)-\frac{R'}{s'}} \tag{38}$$

and

$$\frac{\cos\phi - R's'}{\sin\phi} \geq \frac{\sin\left(\frac{\pi}{2g}-\phi\right)}{\cos\left(\frac{\pi}{2g}-\phi\right)-R's'} \tag{39}$$

have no solutions in $\phi \in [0, \pi/(2g)]$, the sets in Eqs. (36, 37) are disjoint, and we conclude that the system

$$\begin{cases} \delta(z, w_1) > R \\ \delta(z, w_2) > R \end{cases} \tag{40}$$

has no solutions, so $\min\left[\delta(z, w_1), \delta(z, w_2)\right] \leq R$.

Case III: Finally, we consider z on the edge $\overline{v_1 v_2}$. Let $x = \delta(z, v_1)$ and $\alpha = \angle v_2 v_1 w_1$. Applying the first hyperbolic law of cosines to triangle $\Delta z v_1 w_1$ gives

$$\cosh\left(\delta(z, w_1)\right) = \cosh b \cosh x - \sinh b \sinh x \cos\alpha. \tag{41}$$

Assuming for the sake of contradiction that $\delta(z, w_1) > R$ and $\delta(z, w_2) > R$, then from the first inequality we find

$$\cosh b \cosh x - \sinh b \sinh x \cos\alpha > \cosh R, \tag{42}$$

which implies

$$\cos\alpha < \frac{\cosh b \cosh x - \cosh R}{\sinh b \sinh x}. \tag{43}$$

Similarly, the second inequality $\delta(z, w_2) > R$ yields

$$\cos\alpha > \frac{\cosh R - \cosh b \cosh(s - x)}{\sinh b \sinh(s - x)}. \tag{44}$$

However, for inequalities (43) and (44) to have solutions in α, we must have

$$\frac{\cosh b \cosh x - \cosh R}{\sinh b \sinh x} > \frac{\cosh R - \cosh b \cosh(s - x)}{\sinh b \sinh(s - x)}. \tag{45}$$

Simplifying yields

$$\cosh\left(\frac{s}{2}\right)\cosh\left(x - \frac{s}{2}\right) < \cosh b. \tag{46}$$

Notice that the right hand side is an increasing function of b, thus it suffices to disprove the inequality in the case where b is maximal, $b = s/2$. Plugging this in and simplifying the resulting inequality gives

$$\cosh\left(x - \frac{s}{2}\right) < 1, \tag{47}$$

which has no solutions in x. Therefore our assumption was false, so $\min\left[\delta(z, w_1), \delta(z, w_2)\right] \leq R$. □

Theorem 1 $\mathcal{D}_g = R$.

Proof We first show that for all $[z]$, $[w]$,

$$\delta^{\star}([z], [w]) \leq R. \tag{48}$$

By Prop. 2, it suffices to consider the case $z \in \Delta 0 v_1 v_2$, $d_0(z) \geq d_v(w)$, $d_v(w) \leq s/2$. Then by Theorem 3, for all such w the function $f(z) = \delta^{\star}([z], [w])$ defined on the domain Ω_w attains a global maximum M on the boundary $\partial\Omega_w$, and by Theorem 4, $M \leq R$. This proves Eq. (48), and therefore $\mathcal{D}_g \leq R$. But Theorem 2 says that $\mathcal{D}_g \geq R$. We conclude that $\mathcal{D}_g = R$. □

Acknowledgements We thank Florent Balacheff, Maxime Fortier Bourque, Antonio Costa, Benson Farb, Svetlana Katok, Gabor Lippner, Marissa Loving, Curtis McMullen, Hugo Parlier, John Ratcliffe, and Chaitanya Tappu for useful discussions and suggestions. This work was supported by NSF grant No. IIS-1741355 and CCF-2311160.

Funding Open access funding provided by Northeastern University Library

References

1. Seifert, H., Threlfall, W.: Lehrbuch der Topologie: Von H. Seifert und W. Threlfall. B.G. Teubner, Leipzig (1934). https://books.google.com/books?id=bRHRxQEACAAJ
2. Chang, F.-R.: On the diameters of compact Riemann surfaces. Proc. Am. Math. Soc. **65**(2), 274–276 (1977). https://doi.org/10.2307/2041905
3. Balacheff, F., Despré, V., Parlier, H.: Systoles and diameters of hyperbolic surfaces. Kyoto J. Math. **1**(1), 1–12 (2023). https://doi.org/10.1215/21562261-2022-0040
4. Budzinski, T., Curien, N., Petri, B.: On the minimal diameter of closed hyperbolic surfaces. Duke Math. J. **170**(2), 365–377 (2021). https://doi.org/10.48550/arXiv.1909.12283
5. Bavard, C.: Disques extrémaux et surfaces modulaires. Annales de la Faculté des Sciences de Toulouse. Mathématiques. Série VI **5**(2), 191–202 (1996). https://doi.org/10.5802/afst.827
6. Bolza, O.: On binary sextics with linear transformations into themselves. Am. J. Math. **10**(1), 47–70 (1887). https://doi.org/10.2307/2369402
7. Hadamard, J.: Les Surfaces à Courbures Opposées et Leurs Lignes Géodésiques. Journal de Mathématiques Pures et Appliquées **4**, 27–74 (1898). https://eudml.org/doc/235168
8. Jenni, F.: Über den ersten Eigenwert des Laplace-Operators auf ausgewählten Beispielen Kompakter Riemannscher Flächen. Commentarii Mathematici Helvetici **59**(2), 193–203 (1984). https://eudml.org/doc/139972
9. Ebbens, M., Iordanov, I., Teillaud, M., Vegter, G.: Delaunay triangulations of generalized Bolza surfaces. J. Comput. Geom. **13**(1), 125–177 (2022). https://doi.org/10.20382/jocg.v13i1a5
10. Wiman, A.: Über die hyperelliptischen Curven und diejenigen von Geschlechte $p = 3$, welche eindeutige Tiansformationen in sich zulassen. Bihang till Kongl. Svenska vetenskaps-akademiens handlingar **21**, 1–23, (1895). https://babel.hathitrust.org/cgi/pt?id=msu.31293024926804&view=1up&seq=11
11. Bujalance, E., Conder, M.D., Costa, A.F., Izquierdo, M.: On regular dessins d'enfants with 4g automorphisms and a curve of Wiman. Algebr. Curves Appl. Am. Math. Soc. [Providence], RI **724**, 225–233 (2019). https://doi.org/10.1090/conm/724/14592
12. Bujalance, E., Costa, A.F., Izquierdo, M.: On Riemann surfaces of genus g with 4g automorphisms. Topol. Appl. **218**, 1–18 (2017). https://doi.org/10.1016/j.topol.2016.12.013
13. Nabutovsky, A., Weinberger, S.: The fractal nature of Riem/Diff I. Geom. Dedicata. **101**(1), 1–54 (2003). https://doi.org/10.1023/A:1026358815492
14. Beardon, A.F.: The geometry of discrete groups. Springer, New York (2012). https://doi.org/10.1007/978-1-4612-1146-4
15. Cheng, X., Zhou, D.: First eigenvalue estimate on Riemannian manifolds. Hokkaido Math. J. **24**, 453–472 (1995). https://doi.org/10.14492/hokmj/1380892604
16. Cheng, S.Y.: Eigenvalue comparison theorems and its geometric applications. Math. Z. **143**(3), 289–297 (1975). https://doi.org/10.1007/BF01214381

17. Cheeger, J.: The relation between the Laplacian and the diameter for manifolds of non-negative curvature. Arch. Math. **19**, 558–560 (1968). https://doi.org/10.1007/BF01898781
18. Aurich, R., Steiner, F.: On the periodic orbits of a strongly chaotic system. Physica D **32**, 451–460 (1988). https://doi.org/10.1016/0167-2789(88)90068-1
19. Katok, S.: Fuchsian groups. University of Chicago Press, Chicago (1992). https://doi.org/10.2307/3619761
20. Abrams, A., Katok, S., Ugarcovici, I.: Rigidity and flexibility of entropies of boundary maps associated to Fuchsian groups. Math. Res. Reports **3**, 1–19 (2022). https://doi.org/10.5802/mrr.10
21. Ramsay, A., Richtmyer, R.D.: Introduction to hyperbolic geometry. Springer, New York (1995). https://doi.org/10.1007/978-1-4757-5585-5