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Realizing finite groups as automizers

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Abstract. It is shown that any finite group A is realizable as the automizer in a finite perfect group G of an abelian subgroup whose conjugates generate G. The construction uses techniques from fusion systems on arbitrary finite groups, most notably certain realization results for fusion systems of the type studied originally by Park.

1 Introduction

Not every finite group is realizable as $\mathrm{Aut}(U)$ for some finite group U. For example, no nontrivial cyclic group of odd order is the automorphism group of a group. We study here the realization of finite groups by automizers of subgroups of finite groups. That is, given a finite group A, we study when it is possible to find a finite group G and a subgroup $U \le G$ such that $A \cong \mathrm{Aut}_G(U) = N_G(U)/C_G(U)$. As it stands, the answer to this question is "always possible" for trivial reasons: choose a faithful action of A on an elementary abelian p-group U (for some prime p), and take for G the semidirect product of U by A. In this case, U is normal in G. Our main result shows that it is possible to realize A as $\mathrm{Aut}_G(U)$, where U is very far from being normal.

Theorem 1.1. *For each finite group A, there exist a finite perfect group G and a homocyclic abelian subgroup U of G such that $\langle U^G \rangle = G$ and $\mathrm{Aut}_G(U) \cong A$.*

Here, we write $\langle U^G \rangle$ for the normal closure of U in G, the subgroup of G generated by the G-conjugates of U. A group G is perfect if it coincides with its commutator subgroup. A homocyclic abelian group is a direct product of isomorphic cyclic groups.

We do not know whether more restrictions can be placed on G, up to and including whether G can be taken to be simple. Likewise, we do not know if whether more restrictions can be placed on U, such as requiring U to be an elementary abelian p-group for some prime p. Ultimately, the group G is constructed fairly explicitly as the commutator subgroup of a wreath product of the form $(U \rtimes A) \wr \Sigma_n$, but the embedding of U in G is not an obvious one.

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The method for constructing G and the embedding of U relies on certain constructions in fusion systems on arbitrary finite groups. A fusion system on a finite group S (not necessarily a p-group) is a category with objects the subgroups of S, and with morphism sets consisting of injective homomorphisms between subgroups, subject to two weak axioms which we recall in Section 2. The standard example is the fusion system $\mathcal{F}_S(G)$ of the group G on the finite subgroup S in which the morphisms are the G-conjugation homomorphisms between subgroups of S. The most important ingredient in our construction is a theorem of Warraich [8, Section 4] to the effect that every fusion system $\mathcal{F}$ on a finite group S is in fact of this form (see Theorem 2.14). Given such an $\mathcal{F}$, the group G is constructed as the group of automorphisms, as a right S-set, of a certain S-S biset associated with $\mathcal{F}$. Warraich's result and proof generalized the strategy of Park [6], who considered the case where S is finite p-group. We mention that Ünlü and Yalçin have also considered fusion systems on finite groups in the context of Park's result [7, Section 5].

In order to use the Park–Warraich theorem to prove Theorem 1.1, we need to be able to construct a suitable finite group S and fusion system $\mathcal{F}$ on S. One consequence of the way this fusion system is built is the following result.

Theorem 1.2. *For each finite group A, there exist a finite group S, a homocyclic abelian subgroup U of S, and a fusion system $\mathcal{F}$ on S such that $\mathfrak{foc}(\mathcal{F}) = S$, $Q(\mathcal{F}) = 1$, and $\mathrm{Aut}_{\mathcal{F}}(U) \cong A$.*

The definition of the focal subgroup $\mathfrak{foc}(\mathcal{F})$ of a fusion system is given in Section 2 and is the same as the definition for fusion systems on p-groups. The subgroup $Q(\mathcal{F})$ of S, which is a sort of replacement for $O_p(\mathcal{F})$ in a fusion system on an arbitrary finite group when compared with a fusion system on a p-group, is also introduced there.

When A is a p-group for some prime p, S is also a p-group in the construction of the fusion system $\mathcal{F}$ that we present. But we do not know whether it is possible to choose $\mathcal{F}$ to be a fusion system on a p-group independently of A in Theorem 1.2, much less whether $\mathcal{F}$ can be taken to be a *saturated* fusion system on a p-group.

A MathOverflow question of Peter Mueller asks [4]: is every finite group of the form $N_{\Sigma_n}(U)/U$ for a subgroup U of some finite symmetric group Σ_n? This work arose out of an attempt to say something about that question.

Here is a brief outline of the paper and some remarks on notation. In Section 2, we give some background on fusion systems and semicharacteristic bisets and give a definition of $Q(\mathcal{F})$. There we provide some discussion of the Park–Warraich theorem in order to set notation and prepare for the proof of the two theorems.

In Section 3, we prove a slightly more detailed version of Theorem 1.2 and combine it with the Park–Warraich theorem to prove Theorem 1.1. We use left-handed notation for conjugation $x \mapsto {}^g x = gxg^{-1}$. Our iterated commutators are right-associated: $[X, Y, Z] = [X, [Y, Z]]$, etc. The notation $\Phi(G)$ stands for the Frattini subgroup of a group G, and we sometimes write G' for the commutator subgroup.

2 Fusion systems on finite groups, semicharacteristic bisets, and the Park embedding

2.1 Fusion systems

Definition 2.1. Let S be a finite group. A *fusion system* on S is a category $\mathcal{F}$ with objects the set of subgroups of S, subject to the following two axioms: for all $P, Q \leq S$,

(1) $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ consists of a set of injective homomorphisms from P to Q, including all such morphisms induced by S-conjugation.

(2) Each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ is the composite of an $\mathcal{F}$-isomorphism from P to $\varphi(P)$ and the inclusion from $\varphi(P)$ to Q.

Axiom (1) implies that any inclusion ι_P^Q of subgroups $P \leq Q$ is a morphism in $\mathcal{F}$ from P to Q (being conjugation by $1 \in S$). Therefore, a morphism can be restricted to any subgroup of the source. Axiom (2) then implies for example that the target of any morphism can be restricted to a subgroup containing the image.

If G is a group and S is a finite subgroup of G, there is a fusion system $\mathcal{F}_S(G)$ of G on S with morphism sets $\mathrm{Hom}_G(P, Q) = \{c_g : t \mapsto {}^g t \mid {}^g P \leq Q\}$ consisting of the G-conjugation homomorphisms mapping P into Q. This is the standard example of a fusion system. The Park–Warraich Theorem 2.14 shows that indeed every fusion system on S is of this form, and G can be taken to be finite.

The notation $\mathrm{Aut}_{\mathcal{F}}(P)$ is short for $\mathrm{Hom}_{\mathcal{F}}(P, P)$ in a fusion system $\mathcal{F}$ on S. When $\mathcal{F} = \mathcal{F}_S(G)$ for some group G and $P \leq S$, then $\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Aut}_G(P)$ from the definitions.

We introduce now several properties of subgroups and morphisms in a fusion system that we will need, many of which are identical to their counterparts for fusion systems on p-groups [1,2].

Definition 2.2 (Generation of fusion systems). Let S be a finite group and let $\mathfrak{X}$ be a set of injective homomorphisms between subgroups of S. The *fusion system on S generated by* $\mathfrak{X}$, denoted $\langle \mathfrak{X} \rangle_S$, is the intersection of the fusion systems on S containing $\mathfrak{X}$.

If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two fusion systems on the finite group S, then the category $\mathcal{F}_1 \cap \mathcal{F}_2$ with objects the subgroups of S and with morphism sets

$$\mathrm{Hom}_{\mathcal{F}_1 \cap \mathcal{F}_2}(P, Q) := \mathrm{Hom}_{\mathcal{F}_1}(P, Q) \cap \mathrm{Hom}_{\mathcal{F}_2}(P, Q)$$

is again a fusion system on S. Thus, the definition makes sense. As in the case of fusion systems on finite p-groups, it is easy to see that an injective group homomorphism is in $\langle \mathfrak{X} \rangle_S$ if and only if it can be written as a composition of restrictions of homomorphisms in $\mathrm{Inn}(S) \cup \mathfrak{X}$. One consequence of this is the following lemma.

Lemma 2.3. *Let $\mathcal{F}$ be a fusion system on the finite group S, and $\mathcal{X}$ a collection of subgroups of S such that $\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}}(X) \mid X \in \mathcal{X} \rangle_S$. If P is a subgroup of S which is not $\mathcal{F}$-conjugate to a subgroup of any $X \in \mathcal{X}$, then each morphism in $\mathcal{F}$ defined on P is the restriction of an inner automorphism of S.*

We next define what is meant by the direct product of two fusion systems.

Definition 2.4 (Direct products). Let S_1 and S_2 be finite groups, and let $\mathcal{F}_i$ be a fusion system on S_i, $i = 1, 2$. The direct product $\mathcal{F}_1 \times \mathcal{F}_2$ is the fusion system over $S_1 \times S_2$ generated by the homomorphisms $(\varphi_1, \varphi_2)\colon P_1 \times P_2 \to S_1 \times S_2$, where $\varphi_i \in \mathrm{Hom}_{\mathcal{F}_i}(P_i, S_i)$.

We also need the definition of the focal subgroup of a fusion system.

Definition 2.5 (Focal subgroup). Let $\mathcal{F}$ be a fusion system on the finite group S. The *focal subgroup of $\mathcal{F}$* is the subgroup of S generated by elements of the form $[\varphi, s] := \varphi(s)s^{-1}$, where $s \in S$ and $\varphi\colon \langle s \rangle \to S$ is a morphism in $\mathcal{F}$.

Remark 2.6. By the Focal Subgroup Theorem [3, Theorem 7.3.4], if G is a finite group and S is a Sylow p-subgroup of G, then $\mathfrak{foc}(\mathcal{F}_S(G)) = S \cap [G, G]$. When S is an arbitrary subgroup of G, there is the obvious inclusion

$$\mathfrak{foc}(\mathcal{F}_S(G)) \leq S \cap [G, G]$$

since each generating element $\varphi(s)s^{-1} \in S$ is a commutator $gsg^{-1}s^{-1} = [g, s]$ for some $g \in G$, but in general, the reverse inclusion need not hold.

2.2 Nonextendable morphisms and the subgroup $Q(\mathcal{F})$

Definition 2.7 (Nonextendable morphisms). Let $\mathcal{F}$ be a fusion system on the finite group S, and let $P, Q \leq S$. A morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ is said to be *nonextendable* if it does not extend to a morphism defined on any subgroup of S properly containing P. That is, whenever $P \le R \le S$ and $\tilde{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(R, S)$ is such that $\iota_Q^S \circ \varphi = \tilde{\varphi} \circ \iota_P^R$, then $R = P$.

Definition 2.8. For a fusion system $\mathcal{F}$ on a finite group S, define $\mathcal{Q}(\mathcal{F})$ to be the set of all subgroups Q of S for which there is a nonextendable morphism $\varphi: Q \to S$ in $\mathcal{F}$, and let $Q(\mathcal{F})$ be the intersection of the family $\mathcal{Q}(\mathcal{F})$.

The relevance of the subgroup $Q(\mathcal{F})$ will be seen later in Lemma 2.15. By the same proof as for fusion systems on p-groups [2, Proposition 5.27 (c)], if $\mathcal{F}$ is a fusion system on a finite group S, there is a unique largest normal subgroup N of S having the property that each morphism $\varphi: P \to Q$ in $\mathcal{F}$ extends to a morphism $\tilde{\varphi}: PN \to QN$ with $\tilde{\varphi}|_N(N) = N$, which we might denote by $O_S(\mathcal{F})$. (If S is a p-group, then this is the largest normal p-subgroup $O_p(\mathcal{F})$ of $\mathcal{F}$.) It follows from the two definitions that $O_S(\mathcal{F})$ is a subgroup of each member of $\mathcal{Q}(\mathcal{F})$, and so $O_S(\mathcal{F}) \le Q(\mathcal{F})$. Thus, the property $Q(\mathcal{F}) = 1$ of $\mathcal{F}$ that appears in Theorem 1.2 is at least as restrictive as the property $O_S(\mathcal{F}) = 1$.

Remark 2.9. The direct product $\mathrm{Aut}_{\mathcal{F}}(S) \times \mathrm{Aut}_{\mathcal{F}}(S)$ acts on the left of the set of pairs (Q, φ) consisting of a subgroup $Q \in \mathcal{Q}(\mathcal{F})$ and a nonextendable morphism $\varphi: Q \to S$ via $(\alpha, \beta) \cdot (Q, \varphi) = (\alpha(Q), \beta\varphi\alpha^{-1})$. In particular, $Q(\mathcal{F})$ is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant.

2.3 Semicharacteristic bisets

For a finite group S, an S-S-biset X is a set with left and right S-actions such that $(sx)t = s(xt)$ for all $s, t \in S$, $x \in X$. An S-S-biset can be viewed as a left $(S \times S)$-set via $(s, t) \cdot x = sxt^{-1}$.

Let X be an S-S biset. Fix a subgroup Q of S and a group homomorphism $\varphi: Q \to S$. In this situation, the notation ${}_{\varphi}X$ refers to the Q-S biset obtained by having Q act on the left of X via the homomorphism φ, that is, $u \cdot x \cdot t = \varphi(u)xt$ for $u \in Q$, $t \in S$, and $x \in X$. The notation ${}_QX$ is short for the Q-S biset ${}_{\varphi}X$ with φ the inclusion map of Q into S.

Such a pair (Q, φ) consisting of a subgroup $Q \le S$ and a homomorphism $\varphi: Q \to S$ gives rise to a left Q-action on $S \times S$ via $u \cdot (x, y) = (xu^{-1}, \varphi(u)y)$ for $u \in Q$ and $x, y \in S$. We write

$$S \times_{(Q,\varphi)} S = (S \times S)/\sim$$

for the set of orbits under this action, and $\langle x, y\rangle \in S \times_{(Q,\varphi)} S$ for the Q-orbit of the point (x, y). Then $S \times_{(Q,\varphi)} S$ becomes an S-S biset where the left and right S-actions are given by $t\langle x, y\rangle = \langle tx, y\rangle$ and $\langle x, y\rangle t = \langle x, yt\rangle$, respectively, for each $t, x, y \in S$.

The S-S biset $S \times_{(Q,\varphi)} S$ is transitive, namely it has a single orbit when viewed as a left $S \times S$-set. The stabilizer in $S \times S$ of the point $\langle 1, 1\rangle$ is the subgroup $\Delta(Q,\varphi) := \{(u, \varphi(u)) \mid u \in Q\}$, and hence the map

$$(S \times S)/\Delta(Q,\varphi) \to S \times_{(Q,\varphi)} S,$$
$$(x, y)\Delta(Q,\varphi) \mapsto \langle x, y^{-1}\rangle$$

is an isomorphism of left $S \times S$-sets. We refer to a subgroup of $S \times S$ of the form $\Delta(Q,\varphi)$ as a *twisted diagonal subgroup*. If (R,ψ) is another pair consisting of a subgroup $R \leq S$ and a homomorphism $\psi\colon R \to S$, then of course $S \times_{(R,\psi)} S \cong S \times_{(Q,\varphi)} S$ if and only if the associated twisted diagonal subgroups are $S \times S$-conjugate, thus if and only if there are $s, t \in S$ such that

$$\Delta(R,\psi) = {}^{(s,t)}\Delta(Q,\varphi) = \Delta({}^{s}Q, c_t\varphi c_s^{-1}).$$

In particular, $S \times_{(Q,\varphi)} S \cong S \times_{(Q,c_t\varphi)} S$ for any $c_t \in \mathrm{Inn}(S)$.

The biset $S \times_{(Q,\varphi)} S$ is free as a right S-set, and it is also free as a left S-set if φ is injective. For example, assume φ is injective, and $s \in S$ fixes the point $\langle x, y\rangle$ from the left. Then there is some $u \in Q$ such that $(sx, y) = (xu^{-1}, \varphi(u)y)$. This forces $\varphi(u) = 1$, and so $u = 1$ by injectivity. Thus, $sx = x$, and hence $s = 1$.

Definition 2.10 (cf. [6, Definition 1.2]). Let $\mathcal{F}$ be a fusion system on a finite group S. A *left semicharacteristic biset* for $\mathcal{F}$ is a finite S-S-biset X satisfying the following properties.

- X is *$\mathcal{F}$-generated*, i.e., every transitive subbiset of X is of the form $S \times_{(Q,\varphi)} S$ for some $Q \leq S$ and some $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, S)$
- X is *left $\mathcal{F}$-stable*, i.e., ${}_QX \cong {}_\varphi X$ as Q-S-bisets for every $Q \leq S$ and every $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, S)$.

Here and later, when we use the language "of the form" $S \times_{(Q,\varphi)} S$, we mean an S-S biset which is isomorphic to $S \times_{(Q,\varphi)} S$ as a left $S \times S$-set.

In [6], Park showed that each fusion system on a finite p-group has a left semicharacteristic biset. Then Warraich [8] extended this to fusion systems on arbitrary finite groups.

Theorem 2.11. *Let $\mathcal{F}$ be a fusion system on the finite group S. Then there exists a left semicharacteristic biset X for $\mathcal{F}$, and X can be chosen to include at least one S-S orbit of the form $S \times_{(S,\alpha)} S$ for each $[\alpha] \in \mathrm{Out}_{\mathcal{F}}(S)$.*

The basic idea of the proof of Theorem 2.11 is to start with the $\mathcal{F}$-generated biset

$$\sum_{\alpha \in [\mathrm{Aut}_{\mathcal{F}}(S)/\,\mathrm{Inn}(S)]} S \times_{(S,\alpha)} S,$$

where $[\mathrm{Aut}_{\mathcal{F}}(S)/\,\mathrm{Inn}(S)]$ denotes a set of representatives for the cosets of $\mathrm{Inn}(S)$ in $\mathrm{Aut}_{\mathcal{F}}(S)$, and then inductively add orbits of the form $S \times_{(Q,\varphi)} S$ with $\varphi\colon Q \to S$ a morphism in $\mathcal{F}$ in order to build a biset which is left $\mathcal{F}$-stable. The inductive nature of the proof sometimes makes it difficult to understand precisely which orbits $S \times_{(Q,\varphi)} S$ occur in a semicharacteristic biset. The following lemma gives a sufficient condition on a pair (Q,φ) which forces the inclusion of the corresponding orbit.

Lemma 2.12. *Let X be a left semicharacteristic biset for $\mathcal{F}$ containing an orbit of the form $S \times_{(S,\mathrm{id})} S$. If $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ is nonextendable, then X contains an orbit isomorphic to $S \times_{(P,\varphi)} S$.*

Proof. Since X has an orbit of the form $S \times_{(S,\mathrm{id})} S$, the subgroup $\Delta(S,\mathrm{id})$ fixes a point in S, and hence so does $\Delta(P,\mathrm{id})$. It follows that $\Delta(P,\varphi)$ fixes a point, say $x \in X$, because X is left $\mathcal{F}$-stable. Since X is $\mathcal{F}$-generated, there is a morphism $\gamma\colon Q \to S$ in $\mathcal{F}$ such that $\Delta(Q,\gamma)$ is the full stabilizer in $S \times S$ of x, i.e., the $S \times S$ orbit of x is of the form $S \times_{(Q,\gamma)} S$. But then $\Delta(P,\varphi) \le \Delta(Q,\gamma)$, so $P \le Q$ and $\gamma|_P = \varphi$. Since φ is nonextendable, we have $Q = P$ and $\varphi = \gamma$. □

Lemma 2.13. *Let $X = \sum_{i=1}^{k} S \times_{(Q_i,\varphi_i)} S$ be a left semicharacteristic biset for a fusion system $\mathcal{F}$ on a finite group S. Then*

$$\bigcap_{i=1}^{k} \bigcap_{s\in S} {}^{s}Q_i \le Q(\mathcal{F}).$$

Proof. Let $\mathcal{Q}(X) = \{{}^{s}Q_i \mid 1 \le i \le k,\ s \in S\}$ and $Q(X) = \bigcap \mathcal{Q}(X)$ for short. Thus, we must show $Q(X) \le Q(\mathcal{F})$. By Lemma 2.12, for each nonextendable morphism $\varphi\colon Q \to S$ in $\mathcal{F}$, there is some point of X with stabilizer $\Delta(Q,\varphi)$ in $S \times S$. So, for each $s \in S$, there is some point in X with stabilizer $\Delta({}^{s}Q, c_s\varphi c_s^{-1})$ and $c_s\varphi c_s^{-1}$ is nonextendable by Remark 2.9. This shows $\mathcal{Q}(\mathcal{F}) \subseteq \mathcal{Q}(X)$, so $Q(X) \le Q(\mathcal{F})$. □

The reverse inclusion in Lemma 2.13 need not hold. If X is a left semicharacteristic biset for $\mathcal{F}$, then the disjoint union of X with a number of free $S \times S$-orbits (i.e., orbits of the form $S \times_{(1,\mathrm{id})} S$) is again left semicharacteristic. So there is always a semicharacteristic biset with some $Q_i = 1$.

2.4 The Park embedding

Let $\mathcal{F}$ be a fusion system on the finite group S, and let X be a left semicharacteristic biset for $\mathcal{F}$ which contains an orbit of the form $S \times_{(S,\mathrm{id})} S$. Consider the group $G = \mathrm{Aut}({}_1X)$ of automorphisms of X as a right S-set. We explain briefly Park's embedding of S into G with respect to which conjugation in G on the subgroups of S realizes the fusion system $\mathcal{F}$.

Fix a decomposition

$$X = \sum_{i=1}^{k} S \times_{(Q_i,\varphi_i)} S$$

such that $Q_i \le S$ and $\varphi_i \in \mathrm{Hom}_{\mathcal{F}}(Q_i, S)$ for all $1 \le i \le k$ and such that $Q_1 = S$ and $\varphi_1 = \mathrm{id}_S$. Following [5], define ι as

$$\begin{aligned} S &\xrightarrow{\iota} \mathrm{Aut}({}_1X) = G, \\ u &\mapsto (x \mapsto ux). \end{aligned}$$

This is indeed an injection because each orbit $S \times_{(Q_i,\varphi)} S$ is free as a left S-set.

Theorem 2.14 ([5], [8, Chapter 4]). *Let $\mathcal{F}$ be a fusion system on the finite group S, and let X be any left semicharacteristic biset for $\mathcal{F}$ which contains an orbit of the form $S \times_{(S,\mathrm{id})} S$. Let $G = \mathrm{Aut}({}_1X)$, the group of automorphisms of X as a right S-set. Then $G \cong S \wr \Sigma_n$ for some natural number n, and there is an injection $\iota\colon S \to G$ such that $\mathcal{F} \cong \mathcal{F}_{\iota(S)}(G)$.*

We next set up notation that will be needed later, looking more closely at the structure of G and the embedding ι. For each i, fix a collection $\{t_{ij}\}_{j\in J_i}$ of representatives of the left cosets of Q_i, and set $n_i = |S : Q_i| = |J_i|$. The action of $u \in S$ on the coset representatives is given by

$$ut_{ij}Q_i = t_{i\sigma_i(u)(j)}Q_i,$$

where $\sigma_i(u)\colon J_i \to J_i$ is the induced permutation on J_i. As a right S-set, the biset $S \times_{(Q_i,\varphi_i)} S$ decomposes as

$$S \times_{(Q_i,\varphi_i)} S = \sum_{j\in J_i} \langle t_{ij}, S\rangle,$$

where $\langle t_{ij}, S\rangle := \{\langle t_{ij}, y\rangle \mid y \in S\}$ is the set of ordered pairs with free and transitive right S-action given by $\langle t_{ij}, y\rangle \cdot s = \langle t_{ij}, ys\rangle$. Hence, also

$$X = \sum_{i=1}^{k} \sum_{j\in J_i} \langle t_{ij}, S\rangle$$

as a right S-set.

Since the right action of S on $\langle t_{ij}, S\rangle$ is regular, each automorphism of $\langle t_{ij}, S\rangle$ as a right S-set is left multiplication by an element of S, i.e., of the form

$$\langle t_{ij}, y\rangle \mapsto \langle t_{ij}, sy\rangle.$$

Thus, $\mathrm{Aut}({}_1\langle t_{ij}, S\rangle) \cong S$. It therefore follows from the above decompositions that

$$G_i := \mathrm{Aut}({}_1(S \times_{(Q_i,\varphi_i)} S)) \cong S \wr \Sigma_{n_i},$$

and

$$G = \mathrm{Aut}({}_1X) \cong S \wr \Sigma_n,$$

where $n = \sum_{i=1}^{k} n_i$.

We examine more closely the map ι. Now, the group S acts from the left on each $S \times_{(Q_i,\varphi_i)} S$, so $\iota(S) \le \prod G_i \le G$. Let $u \in S$. Since $ut_{ij} \in t_{i\sigma_i(u)(j)} Q_i$, we have $(t_{i\sigma_i(u)(j)})^{-1} ut_{ij} \in Q_i$, and

$$\begin{aligned} u\langle t_{ij}, y\rangle = \langle ut_{ij}, y\rangle &= \langle t_{i\sigma_i(u)(j)} \cdot (t_{i\sigma_i(u)(j)})^{-1} ut_{ij}, y\rangle \\ &= \langle t_{i\sigma_i(u)(j)}, \varphi_i((t_{i\sigma_i(u)(j)})^{-1} ut_{ij}) y\rangle. \end{aligned}$$

Thus, writing π_i for the projection $\Pi G_i \to G_i$, we have

$$\pi_i(\iota(u)) = \big((\varphi_i((t_{i\sigma_i(u)(j)})^{-1} ut_{ij}))_{j\in J_i}; \sigma_i(u)\big) \in S \wr \Sigma_{n_i}.$$

The following lemma gives some information on the intersection of $\iota(S)$ with the base subgroup of G.

Lemma 2.15. *Let $\mathcal{F}$ be a fusion system on the finite group S with left semicharacteristic biset X containing $S \times_{(S,\mathrm{id})} S$ and embedding*

$$\iota\colon S \to G = \mathrm{Aut}({}_1X) \cong S \wr \Sigma_n.$$

Let $B = S^n$ be the base subgroup of G. Then

$$B \cap \iota(S) \le \iota(Q(\mathcal{F})).$$

Proof. Write $X = \sum_{i=1}^{k} S \times_{(Q_i,\varphi_i)} S$. For each $u \in S$, the image $\iota(u) \in B$ if and only if $\sigma_i(u) = 1$ for all $1 \le i \le k$ in the notation above. That is, $\iota(u) \in B$ if and only if u fixes all cosets tQ_i, that is, if and only if $u \in \bigcap_i \bigcap_{t\in S} {}^t Q_i$. The result now follows from Lemma 2.13. □

3 Proof of Theorems 1.2 and 1.1

We now state and prove a slightly more detailed version of Theorem 1.2. Note that, in a homocyclic group $V = (C_e)^r$ of exponent e and rank r, the minimal generating sets for V are the bases for V when considered as a free $\mathbb{Z}/e\mathbb{Z}$-module (and all have size r). The automorphism group $\mathrm{Aut}(V)$ of V is transitive on such bases by the universal property for free modules. With respect to the primary decomposition of V, the automorphism group of V decomposes as a direct product of the factors, and since $\mathrm{Aut}(V)$ is transitive on bases, Nakayama's Lemma then shows that the restriction map $\mathrm{Aut}(V_p) \to \mathrm{Aut}(V_p/\Phi(V_p))$ is surjective for the Sylow p-subgroup V_p of V. This implies $[\mathrm{Aut}(V_p), V_p] = V_p$ and hence $[\mathrm{Aut}(V), V] = V$, which is the last thing we will need.

Theorem 3.1. *Let A be a finite group. Then there are a finite group S, a fusion system on S, and a homocyclic abelian subgroup U of S such that $Q(\mathcal{F}) = 1$, $\mathfrak{foc}(\mathcal{F}) = S$, and $\mathrm{Aut}_{\mathcal{F}}(U) = A$. Moreover, S, U, and $\mathcal{F}$ can be chosen so as to satisfy the following additional properties:*

(i) *S is the semidirect product of U by A with respect to a faithful action of A on U,*

(ii) *the exponent of U is the exponent of A, and*

(iii) *if $A > 1$, then there is $Q \in \mathcal{Q}(\mathcal{F})$ such that $|S : Q| > 2|A|$.*

Proof. In case $A = 1$, we take $G = S = U = 1$ and $\mathcal{F} = \mathcal{F}_S(G)$. So we may assume $A \neq 1$. Let e be the exponent of A. Consider the homocyclic group

$$U = U_1 \times U_2 = C_e^{|A|} \times C_e^{|A|}$$

of rank $2|A|$, where A acts freely on U_1 and U_2. Let $S := UA$ be the semidirect product with respect to this action. Thus, $\mathrm{Aut}_S(U) \cong A$ and (i) and (ii) are satisfied. Let $\mathcal{V}$ be the collection of all rank 2 homocyclic subgroups of S of order e^2, and define $\mathcal{F} = \langle \mathrm{Aut}(V) \mid V \in \mathcal{V}\rangle_S$.

We draw two consequences from Lemma 2.3 and the definition of $\mathcal{F}$. The first one is that $\mathrm{Aut}_{\mathcal{F}}(U) = \mathrm{Aut}_S(U) \cong A$. Indeed, $\mathrm{Aut}_S(U) \subseteq \mathrm{Aut}_{\mathcal{F}}(U)$ by definition of a fusion system, whereas U has strictly larger order than any member of $\mathcal{V}$ (since $|A| \geq 2$), so $\mathrm{Aut}_{\mathcal{F}}(U) \subseteq \mathrm{Aut}_S(U)$ by the lemma. A second consequence is that if $V \in \mathcal{V}$ and $\alpha \in \mathrm{Aut}_{\mathcal{F}}(V)$ extends to a proper overgroup of V in S, then $\alpha \in \mathrm{Aut}_S(V)$. This follows from the lemma because each $V \in \mathcal{V}$ is maximal under inclusion among the members of $\mathcal{V}$.

Let

$$\mathcal{V}_1 = \{V \in \mathcal{V} \mid V \text{ supports a nonextendable automorphism}\}.$$

We next want to show that $\langle \mathcal{V}_1 \rangle = S$ and $\bigcap \mathcal{V}_1 = 1$. Observe that we then have $Q(\mathcal{F}) = 1$ by definition of the collection $\mathcal{V}_1$. Since the focal subgroup of $\mathcal{F}$ contains $V = [\mathrm{Aut}(V), V]$ for each $V \in \mathcal{V}$, this will also show $\mathfrak{foc}(\mathcal{F}) = S$.

Since U is a free A-module of rank two, $C_U(A) = Z_1 \times Z_2$ with $Z_i = C_{U_i}(A)$ is cyclic of order e. Since $|A| \geq 2$, U has rank at least 4. There is a choice of a pair of cyclic subgroups $W_1 \leq U_1$, $W_2 \leq U_2$ of order e such that $W_i \cap C_U(A) = 1$ and $W_1 Z_1 \cap W_2 Z_2 = 1$. (For example, take W_i spanned by a coordinate in $U_i = C_e^{|A|}$.) For any such choice, and as $\mathrm{Aut}(W_i Z_i)$ is transitive on minimal generating sets, there is an automorphism of $W_i Z_i$ which interchanges W_i and Z_i and thus which does not extend to an S-automorphism of $W_i Z_i$ (because $Z_i \leq Z(S)$). So, by the above, we see that $W_i Z_i \in \mathcal{V}_1$ for $i = 1, 2$. In particular, this shows $\bigcap \mathcal{V}_1 = 1$, and also that $U \leq \langle \mathcal{V}_1 \rangle$ (coordinates generate). Since $Z_1 Z_2 \in \mathcal{Q}(\mathcal{F})$ for the same reasons and $|S : Z_1 Z_2| \geq |U : Z_1 Z_2| \geq e^2 |A| > 2|A|$, point (iii) is satisfied.

Let p be a prime dividing $|A|$, let p^a be the p-part of the exponent of A, and let C be any cyclic subgroup of A with generator c of order p^b. We claim that there is $V \in \mathcal{V}_1$ with $UC/U \leq UV/U$. Let $u \in U - [C, U]$ be any element of order p^{a-b}. Then uc has order p^a. Fix an element $w \in C_U(C)$ of order e/p^a and set $W = \langle wuc \rangle$. Since e/p^a is prime to p, W is cyclic of order e. Since the rank of $C_U(A)$ is 2, we can again find a cyclic subgroup $Z \leq C_U(A)$ of order e with $W \cap Z = 1$, and then $V = WZ$ is homocyclic of order e^2. As before, there is an automorphism of V interchanging W and Z, which therefore does not extend to an S-automorphism of V. This shows that $V \in \mathcal{V}_1$. By construction, $UC/U \leq UV/U$, and we saw above that $U \leq \langle \mathcal{V}_1 \rangle$. Since C was an arbitrary cyclic subgroup of p-power order, and the set of such subgroups generates A as p ranges over the primes dividing A, it follows that $\langle \mathcal{V}_1 \rangle = S$. □

Before giving the proof of Theorem 1.1, we prove a specialized lemma about the commutator subgroup of a wreath product.

Lemma 3.2. *Let S be a group, let K be a subgroup of Σ_n with $n > 1$, and let $\Gamma = S \wr K$ with base subgroup B and $G = \Gamma' = [\Gamma, \Gamma]$. Assume that K' is perfect and transitive. Then $[B, B] \leq [K, B] = [K', B] = [K', K', B]$ and $G = [K', B]K'$ is perfect.*

Proof. Although $n > 1$ was assumed initially, the further assumptions give implicitly that $n \geq 5$. Write $e_i(s)$ for the element

$$(1, \ldots, 1, s, 1, \ldots, 1) \in B = S_1 \times \cdots \times S_n$$

(with s in the i-th place), and $c^i_j(s) = e_j(s) e_i(s)^{-1}$. Let J be any transitive subgroup of Σ_n. Then $c^i_j(s) = [g, e_i(s)] \in [J, B]$ for each element $g \in J$ which sends i to j, so $c^i_j(S) \in [J, B]$ for each i and j. For i an index taken modulo n and

for $s, t \in S$,
$$[c_i^{i-1}(s), c_{i+1}^{i}(t^{-1})] = e_i([s, t]).$$
This shows that $[S_i, S_i] \leq [J, B]$ for each i, and hence $[B, B] = [S, S]^n \leq [J, B]$. Under the same assumptions on J, we just saw that $[J, B]$ contains all $c_j^i(s)$ with $s \in S$ and $1 \leq i, j \leq n$. These generate $\ker(\pi)$, where $\pi\colon B \to S/S'$ is the homomorphism sending an element of B to the product of its components. Since each generating element of $[J, B]$ is clearly in this kernel, we have $[J, B] = \ker(\pi)$. In particular, $[B, B] \leq [K', B] = [K, B]$.

Next, for any subgroup J, we have $[J, B, J] = [J, J, B]$. So if $J' = J$, then $[J, B] = [B, J] = [B, J'] = [B, J, J] \leq [J, J, B] \leq [J, B]$, the first inclusion by the Three Subgroups Lemma [3, Theorem 2.3 (ii)]. So $[J, J, B] = [J, B]$. In particular, $[K', K', B] = [K', B]$ since K' was assumed perfect.

Applying [3, Theorem 2.1] for example to $\Gamma = KB$, we see that
$$G = \Gamma' = K'[K, B][B, B],$$
and then
$$G = K'[K', B]$$
as $[B, B] \leq [K, B] = [K', B]$. Keeping in mind that $[[K', B], [K', B]] \leq [B, B]$ since $B/[B, B]$ is abelian, a similar argument gives
$$G' = [K', K'][K', K', B] = K'[K'B] = G,$$
so G is perfect. □

Proof of Theorem 1.1. Let A be any finite group. If $A = 1$, then we take $G = U = 1$, so we may assume $A > 1$. Fix a fusion system $\mathcal{F}$ on a finite group $S = U \rtimes A$ with U homocyclic and A faithful on U, satisfying the conclusion of Theorem 3.1. Let $X = \sum_{i=1}^{k} S \times_{(Q_i, \varphi_i)} S$ be any left semicharacteristic biset for $\mathcal{F}$ as in Theorem 2.11, and write $\iota\colon S \to \Gamma = \mathrm{Aut}({}_1X) \cong S \wr \Sigma_n$ for the Park embedding so that $\mathcal{F} \cong \mathcal{F}_{\iota(S)}(\Gamma)$ via ι. Set $G = \Gamma'$. To ease notation, we identify S with its image in Γ, and so we identify $\mathcal{F}$ and $\mathcal{F}_S(\Gamma)$. By choice of $\mathcal{F}$, we know
$$S \cap B \leq Q(\mathcal{F}) = 1$$
by Lemma 2.15, where B is the base subgroup of Γ as usual, while also
$$S = \mathfrak{foc}(\mathcal{F}) \leq S \cap G$$
by Remark 2.6. Thus, $U \leq S \leq G$. Since $\mathcal{F} = \mathcal{F}_S(\Gamma)$, we have
$$\mathrm{Aut}_\Gamma(U) = \mathrm{Aut}_{\mathcal{F}}(U) \cong A$$

again by choice of $\mathcal{F}$, and $\mathrm{Aut}_S(U) \cong A$ by construction. Since $S \leq G$, this shows $\mathrm{Aut}_G(U) \cong A$. We want to verify that G satisfies the conclusion of the theorem.

Let H be the alternating subgroup of Σ_n, and let $N = \langle U^G \rangle$ be the normal closure of U in G. We will see below that $n \geq 5$, so H is simple. By Lemma 3.2, G is perfect and $G = H[H, B]$. Thus, it remains to show that $N = G$.

Recall from the discussion of the Park embedding that $n = \sum_{i=1}^{k} |S : Q_i|$, so by Theorem 3.1 (iii) and Lemma 2.12, there is i such that $n \geq |S : Q_i| > 2|A|$. So, indeed, $n \geq 5$ and H is simple. Use Bertrand's postulate to get a prime p with $|A| < p < 2|A|$, and so a prime p that divides $|H|$ but not $|A|$. By Theorem 3.1 (ii), p divides $|H|$ but not $|S|$, so $|H|^2$ does not divide $|G|$. On the other hand, $U \cap B \leq S \cap B \leq Q(\mathcal{F}) = 1$, so as H is simple, N projects modulo B onto H. Thus, $|H|$ divides $|N|$. Since $N \cap H$ is normal in H, we have $H \leq N$ or $H \cap N = 1$. In the latter case, G contains the subgroup HN of order divisible by $|H|^2$, a contradiction, and hence $H \leq N$. As $H \leq N$, N contains the normal closure of H in G, which is $H[H, B] = G$, and this completes the proof of the theorem. $\square$

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