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Products and powers of principal symmetric ideals

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Principal symmetric ideals were recently introduced by Harada *et al.* in [The minimal free resolution of a general principal symmetric ideal, preprint (2023), arXiv:2308.03141], where their homological properties are elucidated. They are ideals generated by the orbit of a single polynomial under permutations of variables in a polynomial ring. In this paper, we determine when a product of two principal symmetric ideals is principal symmetric and when the powers of a principal symmetric ideal are again principal symmetric ideals. We characterize the ideals that have the latter property as being generated by polynomials invariant up to a scalar multiple under permutation of variables. Recognizing principal symmetric ideals is an open question for the purpose of which we produce certain obstructions. We also demonstrate that the Hilbert functions of symmetric monomial ideals are not all given by symmetric monomial ideals, in contrast to the non-symmetric case.

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1. Introduction

Principal symmetric ideals, abbreviated in this paper as psi's, were introduced in [2]. We recall this notion. Given an element f of the polynomial ring $R = \mathsf{k}[x_1, \ldots, x_d]$, the principal symmetric ideal $(f)_{S_d}$ is the ideal generated by the orbit of f under the natural action of the symmetric group S_d on R. In detail, the definition states

$$(f)_{S_d} = (f(x_{\sigma(1)}, \ldots, x_{\sigma(d)}) : \sigma \in S_d).$$

Throughout this paper, we make the assumption that the generator f of a psi is a *homogeneous* polynomial. In this case the psi $(f)_{S_d}$ is a homogeneous ideal and we may consider its invariants such as the minimal number of generators and the Hilbert function.

In this paper, we address a fundamental question.

Question 1.1. How can one determine whether a given ideal is a psi or not?

Whereas an ideal may be a psi despite not being written given in the form displayed above, this is a difficult question. An important ingredient in the remainder

of our results consists of developing obstructions for an ideal to be a psi based on its number of generators and other characteristics. We investigate this paradigm in more detail in Sec. 3. A highlight of our analysis is Theorem 3.6, whose statement is too technical to state here.

Next, we consider powers and products of psi's. Simple examples show that powers, and hence products of psi's need not be psi's; see Example 5.2. We investigate the questions.

Question 1.2. When is a product of psi's a psi? When are the powers of a psi psi's?

We settle the question regarding products completely in the case of monomial psi's and in the case of psi's in two-dimensional polynomial rings in Theorems 5.4 and 5.9, respectively. This characterization is in terms of k-symmetric polynomials, which are those only changed by a multiplication by a constant upon the action of any permutation of the variables. Regarding powers, we characterize psi's whose sufficiently large powers are also psi's as follows.

Theorem. (Abstracted from Corollary 5.5, Theorem 5.7, and Corollary 5.10) *Let I be a homogeneous principal symmetric ideal. Consider the statements*

(1) I^n *is a principal symmetric ideal for all $n \geq 1$.*
(2) I^n *is a principal symmetric ideal for all sufficiently large integers n.*
(3) I^2 *is a principal symmetric ideal.*
(4) I *is generated by a k-symmetric polynomial.*

Statements (1), (2), *and* (4) *are equivalent for all I. Statement* (3) *is equivalent to the rest when I is a monomial ideal and when I is an ideal of $k[x_1, x_2]$.*

Macaulay's classical work [3] yields that for every homogeneous ideal I there exists a monomial ideal having the same Hilbert function as I. We ask whether an analogous phenomenon takes place in the category of symmetric homogeneous ideals.

Question 1.3. Given a symmetric ideal I, does there exist a symmetric monomial ideal with the same Hilbert function as I?

In Theorem 6.2, we show that the answer to the question above is negative when I ranges over a large class of psi's. Once again the key to this observation is an analysis of the number of generators of general psi's performed in [2] versus that of monomial psi's in a large number of variables. We study monomial psi's in more detail in Sec. 4, where we show the following useful characterization:

Theorem. (Corollary 4.4) *A principal symmetric ideal I is a monomial ideal if and only if $I = (m)_{S_d}$ for some monomial m.*

Our work opens up avenues of investigation into a series of related questions which we list in Sec. 7.

2. Terminology

Throughout the paper, $\mathbb{N}$ denotes the set of natural numbers (including zero) and R denotes the polynomial ring $\mathsf{k}[x_1, \ldots, x_d]$. All facts stated in this paper are true for $d = 1$, but to avoid trivialities we assume henceforth that the number of variables is $d \geq 2$. The symmetric group S_d acts on R by means of setting for $\sigma \in S_d$ and $f \in R$

$$\sigma \cdot f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

Working with the symmetric group S_d leads naturally to partitions with d parts.

Definition 2.1. A *partition* is a tuple $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{N}^d$ such that $a_1 \geq \cdots \geq a_d$. Its number of parts is d and the sum of its parts is $|\mathbf{a}| = a_1 + \cdots + a_d$.

We denote by $P(n)$ the number of partitions of n into arbitrary number of parts, that is,

$$P(n) = \#\{\mathbf{a} : |\mathbf{a}| = n\}.$$

Partitions are in bijection with equivalence classes of monomials under the action of the symmetric group.

Definition 2.2. Let $\mathbf{a} = (a_1, \ldots, a_d)$ be a partition with d parts. A monomial $m \in \mathsf{k}[x_1, \ldots, x_d]$ is said to be of *order type* $\mathbf{a}$ if

$$m = x_1^{a_{\sigma(1)}} \cdots x_d^{a_{\sigma(d)}} \quad \text{for some } \sigma \in S_d.$$

We denote by $R_{\mathbf{a}}$ the k-linear span of all monomials of order type $\mathbf{a}$ in $R = \mathsf{k}[x_1, \ldots, x_d]$ and by $(R_{\mathbf{a}})$ the ideal generated by this set.

Let k denote the number of distinct values in the tuple $\mathbf{a}$ and let $\{n_i\}_{i=1}^k$ be the number of occurrences of each value in $\mathbf{a}$. Then we have

$$\dim_{\mathsf{k}} R_{\mathbf{a}} = \frac{d!}{n_1! n_2! \cdots n_k!}. \tag{2.1}$$

We single out a class of polynomials which generate psi's having well-controlled generators.

Definition 2.3. Let f be a homogeneous polynomial. We say that f is *strongly homogeneous* if all monomials in f have the same order type. If f is strongly homogeneous, then the order type of f is the order type of a monomial in f.

Note that the property of strong homogeneity and the order type of a polynomial are preserved under permutation of variables.

Definition 2.4. An ideal I is *strongly homogeneous* if I is generated by strongly homogeneous polynomials $f_1, \ldots, f_n$ such that all f_i's have the same order type. A *strongly homogeneous psi* is a principal symmetric ideal generated by a strongly homogeneous polynomial.

For example, principal symmetric ideals generated by a monomial are strongly homogeneous psi's. Note that a strongly homogeneous psi is also a strongly homogeneous ideal in the sense of Definition 2.4. Whether a homogeneous ideal is strongly homogeneous can be checked using any minimal generating set.

Definition 2.5. If I is a homogeneous ideal of a graded k-algebra, it admits a decomposition $I = \bigoplus_{n \geq 0} I_n$, where I_n is the k-vector space of elements of degree n in I. The *Hilbert function* of I is the function $H_I : \mathbb{N} \to \mathbb{N}$, $H_I(n) = \dim_{\mathsf{k}} I_n$.

3. Recognizing Principal Symmetric Ideals

In this section, we are concerned with the problem of identifying whether a given ideal is a psi. This is highly nontrivial as the ideal may have a different set of generators than the one expected in the definition of a psi. Thus, the difficulty of the problem consists in the non-uniqueness of minimal generating sets for homogeneous ideals.

In this section, we focus on obstructions preventing a homogeneous ideal from being a psi. For a homogeneous ideal I we denote by $\mu(I)$ its minimal number of generators. Since a psi I has all its generators in a single degree, say degree n, one can compute the minimal number of generators of I as $\mu(I) = \dim_{\mathsf{k}}(I_n)$. The number of generators, when too large, can be construed as an obstruction to being a psi by means of the following observation.

Lemma 3.1. *Let I is homogeneous psi of $\mathsf{k}[x_1, \ldots, x_d]$, then $\mu(I) \leq d!$.*

Proof. By definition, a set of (not necessarily minimal) generators of I is in bijection with the group S_d. $\qquad\square$

For a polynomial f we denote by $f(\mathbf{1})$ the element of k obtained by evaluating f at $x_i = 1$ for $1 \leq i \leq d$. Note that every element in the S_d orbit of f has the same value when evaluated at $\mathbf{1}$. More can be said regarding obstructions to being a psi in terms of this invariant.

Lemma 3.2. *Let f be a strongly homogeneous polynomial in $\mathsf{k}[x_1, \ldots, x_d]$ with order type $\mathbf{a}$. If $f(\mathbf{1}) = 0$ then $(f)_{S_d} \subsetneq (R_{\mathbf{a}})$. In particular $\mu((f)_{S_d}) \leq \dim_{\mathsf{k}} R_{\mathbf{a}} - 1$.*

Proof. The containment $(f)_{S_d} \subsetneq (R_{\mathbf{a}})$ follows because $f \in R_{\mathbf{a}}$ implies $\sigma \cdot f \in R_{\mathbf{a}}$ for any $\sigma \in S_d$. The condition $f(\mathbf{1}) = 0$ implies $(\sigma \cdot f)(\mathbf{1}) = 0$ for all $\sigma \in S_d$ and moreover $g(\mathbf{1}) = 0$ for any polynomial g in the degree $|\mathbf{a}|$ component of I. Therefore there are elements of $R_{\mathbf{a}}$ not in I, for example, no monomials of degree $\mathbf{a}$ are in I. This gives the strict containment and hence the claimed inequality. $\qquad\square$

The following technical lemmas are the main ingredients in the proof of Theorem 3.6 as well as Theorem 4.3.

Lemma 3.3. *Let f be a homogeneous polynomial and write $f = g_1 + \cdots + g_r$, where the monomials in each of the g_i's share no order types. Suppose there exists*

a homogeneous polynomial $g \in (f)_{S_d}$ such that $\deg(g) = \deg(f)$, g shares order types only with g_1 among all g_i's, and $g(\mathbf{1}) \neq 0$. Then for all $i \neq 1$, $g_i(\mathbf{1}) = 0$. Thus, $f(\mathbf{1}) = g_1(\mathbf{1})$.

Proof. Suppose that $g \in (f)_{S_d}$ with $g(\mathbf{1}) \neq 0$. Since $\deg(g) = \deg(f)$, there exist constants $c_\sigma \in \mathsf{k}$ such that

$$g = \sum_{\sigma \in S_d} c_\sigma \sigma \cdot (g_1 + \cdots + g_r) = \sum_{\sigma \in S_d} c_\sigma \sigma \cdot g_1 + \cdots + \sum_{\sigma \in S_d} c_\sigma \sigma \cdot g_r.$$

Since g shares order types only with g_1 and the g_i's share no order types, we conclude from the identity displayed above that $g = \sum_{\sigma \in S_d} c_\sigma \sigma \cdot g_1$ and $\sum_{\sigma \in S_d} c_\sigma \sigma \cdot g_i = 0$ for $i > 1$. Hence,

$$g(\mathbf{1}) = \sum_{\sigma \in S_d} c_\sigma (\sigma \cdot g_1)(\mathbf{1})$$

$$= \sum_{\sigma \in S_d} c_\sigma g_1(\mathbf{1})$$

$$= g_1(\mathbf{1}) \sum_{\sigma \in S_d} c_\sigma.$$

Since $g(\mathbf{1}) \neq 0$, it follows that $\sum_{\sigma \in S_d} c_\sigma \neq 0$. For $i > 1$, we deduce from $\sum_{\sigma \in S_d} c_\sigma \sigma \cdot g_i = 0$ that

$$0 = \sum_{\sigma \in S_d} c_\sigma (\sigma \cdot g_i)(\mathbf{1})$$

$$= \sum_{\sigma \in S_d} c_\sigma g_i(\mathbf{1})$$

$$= g_i(\mathbf{1}) \sum_{\sigma \in S_d} c_\sigma.$$

Since $\sum_{\sigma \in S_d} c_\sigma \neq 0$, this yields $g_i(\mathbf{1}) = 0$ for $i > 1$, as desired, and thus $f(\mathbf{1}) = g_1(\mathbf{1})$. $\qquad\square$

Lemma 3.4. *Let f be homogeneous and $g \in (f)_{S_d}$ be a homogeneous polynomial such that $\deg(g) = \deg(f)$. Then $f = g' + h$ for some g' which contains precisely the same order types as g and for some h which shares no order types with g. Also, if $g(\mathbf{1}) \neq 0$, then $g'(\mathbf{1}) \neq 0$.*

Proof. Since $g \in (f)_{S_d}$ and $\deg(g) = \deg(f)$, there exist $c_\sigma \in K$ such that

$$g = \sum_{\sigma \in S_d} c_\sigma \sigma \cdot f.$$

Since order types are invariant under permutation, f must contain a nonzero term for every order type present in g. Let g' contain precisely those terms of f whose

order type is present in g. Let $h = f - g'$, so that $f = g' + h$ and h shares no order types with g.

Now, suppose that $g(\mathbf{1}) \neq 0$. Note that

$$g = \sum_{\sigma \in S_d} c_\sigma \sigma \cdot (g' + h) = \sum_{\sigma \in S_d} c_\sigma \sigma \cdot g' + \sum_{\sigma \in S_d} c_\sigma \sigma \cdot h.$$

Since h shares no order types with g, $\sum_{\sigma \in S_d} c_\sigma \sigma \cdot g' = g$ and $\sum_{\sigma \in S_d} c_\sigma \sigma \cdot h = 0$. Thus,

$$g(\mathbf{1}) = \sum_{\sigma \in S_d} c_\sigma (\sigma \cdot g')(\mathbf{1}) = g'(\mathbf{1}) \sum_{\sigma \in S_d} c_\sigma,$$

so $g'(\mathbf{1}) \neq 0$. $\qquad\square$

We illustrate the previous lemma with an example.

Example 3.5. Let $f = x_1^3 - x_2^3 + x_1^2 x_3 + x_2 x_3 x_4 \in \mathbb{Q}[x_1, x_2, x_3, x_4]$. One can verify, for example by using Macaulay2 [1], that $g = x_1 x_2^2 + x_1 x_3 x_4 \in (f)_{S_4}$. Among the terms of f, those which share order types with g are $x_1^2 x_3$ and $x_2 x_3 x_4$, while x_1^3 and $-x_2^3$ do not. So, the proof of Lemma 3.4 tells us to set $g' = x_1^2 x_3 + x_2 x_3 x_4$ and $h' = x_1^3 - x_2^3$. Observe that $g(\mathbf{1}) \neq 0$ and also $g'(\mathbf{1}) \neq 0$.

Theorem 3.6. *Let I be a homogeneous ideal generated in degree n. Let g_1, g_2 be homogeneous polynomials of degree n that share no order types such that $g_1(\mathbf{1}) \neq 0$ and $g_2(\mathbf{1}) \neq 0$. If $g_1, g_2 \in I$ then I is not a principal symmetric ideal.*

Proof. Assume to the contrary that I is a principal symmetric ideal generated by some f. Then by Lemma 3.4, $f = g_1' + h_1$ and $f = g_2' + h_2$ where g_i' and h_i share no order types and moreover g_i' contains precisely the same order types as g_i. Thus, the identity $g_1' + h_1 = g_2' + h_2$ combined with the fact that g_1, g_2 share no order types shows that the terms of g_1' are a subset of the terms of h_2 and the terms of g_2' are a subset of the terms of h_1.

Let $h' = h_2 - g_1'$ and note that h' shares no order types with g_1', g_2'. Indeed, h' shares no order types with g_1' since all terms of g_1' cancel in $h_2 - g_1'$ and these are exactly the terms of h_2 of the same order type with terms in g_1'. Moreover, h' shares no order types with g_2' since the terms of h' are a subset of the terms of h_2 and h_2 shares no order types with g_2'.

Then, $f = g_1' + g_2' + h'$. Since $g_1(\mathbf{1}), g_2(\mathbf{1}) \neq 0$, we may deduce $g_1'(\mathbf{1}), g_2'(\mathbf{1}) \neq 0$ by Lemma 3.4. We now apply Lemma 3.3 for $g = g_1 \in (f)_{S_d}$ based on the decomposition $f = g_1' + g_2' + h'$. To verify the hypothesis of this lemma, note that g_1 shares order types only with g_1', but not with g_2' or h'. This is true since the order types of g_2' are precisely the order types of g_2 and g_1 shares no order types with g_2, while h' shares no order types with g_1', as established above, but g_1' has exactly the same order types as g_1. In view of this, Lemma 3.3 yields that $g_2'(\mathbf{1}) = 0$ and $h'(\mathbf{1}) = 0$. The former identity provides the desired contradiction. $\qquad\square$

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4. Monomial Principal Symmetric Ideals

As a matter of terminology, a *monomial principal symmetric ideal* may be interpreted to be a psi that is also a monomial ideal or a psi generated by a monomial, that is, an ideal of the form $(m)_{S_d}$, where m is a monomial. We prove in Corollary 4.4 that these two potential interpretations describe the same class of ideals.

Lemma 4.1. *Assume $(f)_{S_d}$ is a monomial ideal. If f contains a term of order type* **a**, *then $R_{\mathbf{a}} \subseteq (f)_{S_d}$. If, in addition, f is strongly homogeneous, then $(R_{\mathbf{a}}) = (f)_{S_d}$.*

Proof. Suppose that f contains a term which is a scalar multiple of a monomial m of order type **a** and let $I = (f)_{S_d}$. Since $f \in I$ and I is assumed to be a monomial ideal, we deduce $m \in I$. Since the ideal I is symmetric, $(m)_{S_d} \subseteq (f)_{S_d}$. Since the monomials of order type **a** are a basis for $R_{\mathbf{a}}$ and any monomial of order type **a** can be expressed as a permutation of m, we conclude that $(R_{\mathbf{a}}) = (m)_{S_d} \subseteq (f)_{S_d}$.

If f is strongly homogeneous order type **a**, then all terms of f are in $R_{\mathbf{a}}$, therefore $f \in R_{\mathbf{a}}$ and $(f)_{S_d} \subseteq (R_{\mathbf{a}})$. ∎

Lemma 4.2. *Let $(f)_{S_d}$ be a monomial ideal, where f is homogeneous. Decompose f into a sum of strongly homogeneous components of pairwise distinct order types, $f = g_1 + \cdots + g_t$. Then, for each i, $(g_i)_{S_d}$ is a monomial ideal and $(g_i)_{S_d} = (R_{\mathbf{a}})$ for some vector $\mathbf{a} \in \mathbb{N}^d$ that depends on i.*

Proof. Let **a** be the order type of g_i. By Lemma 4.1,

$$R_{\mathbf{a}} \subseteq (f)_{S_d} = (g_1 + \cdots + g_t)_{S_d} \subseteq (g_1)_{S_d} + \cdots + (g_t)_{S_d}.$$

Since g_i has a different order type than the other g_j's, $R_{\mathbf{a}} \subseteq (g_i)_{S_d}$. Also, the minimal generators of $(g_i)_{S_d}$ form a subset of $R_{\mathbf{a}}$, so $(g_i)_{S_d} = (R_{\mathbf{a}})$. Thus, $(g_i)_{S_d}$ is a monomial ideal. ∎

Theorem 4.3. *Assume $(f)_{S_d}$ is a monomial ideal. If f is homogeneous, then f is strongly homogeneous and $(f)_{S_d} = (m)_{S_d}$ for some monomial m.*

Proof. Assume to the contrary that f contains monomial terms of different order types, $\mathbf{a}_1, \ldots, \mathbf{a}_t$ say. Then we can write $f = g_1 + \cdots + g_t$, where each g_i is the sum of the terms of order type $\mathbf{a}_i$.

We show that

$$(g_1 + \cdots + g_t)_{S_d} = (f)_{S_d} = (g_1)_{S_d} + \cdots + (g_t)_{S_d}. \tag{4.1}$$

Clearly, one containment holds

$$(f)_{S_d} = (g_1 + \cdots + g_t)_{S_d} \subseteq (g_1)_{S_d} + \cdots + (g_t)_{S_d}.$$

By Lemma 4.2, we have $(g_i)_{S_d} = (R_{\mathbf{a}_i})$ and by Lemma 4.1 there are containments $R_{\mathbf{a}_i} \subseteq (f)_{S_d}$. Hence, there is a containment

$$(R_{\mathbf{a}_i}) + \cdots + (R_{\mathbf{a}_t}) = (g_1)_{S_d} + \cdots + (g_t)_{S_d} \subseteq (f)_{S_d} = (g_1 + \cdots + g_t)_{S_d}.$$

Since every term of f is in $(R_{\mathbf{a}_i}) + \cdots + (R_{\mathbf{a}_t})$, the claimed equality follows. In particular, we see that $g_i \in (f)_{S_d}$ for each i.

Since g_i is strongly homogeneous and $(g_i)_{S_d} = (R_{\mathbf{a}_i})$, Lemma 3.2 yields that $g_i(\mathbf{1}) \neq 0$ for all $1 \leq i \leq t$. If $t > 1$, then Theorem 3.6 contradicts the fact that I is a psi. Hence, $t = 1$ and therefore f is strongly homogeneous. $\qquad\square$

Corollary 4.4. *A principal symmetric ideal I is a monomial ideal if and only if $I = (m)_{S_d}$ for some monomial m.*

Proof. The forward direction follows from Theorem 4.3. For the backward direction, note that all ideals $(m)_{S_d}$, where m is any monomial, are both psi's and monomial ideals. $\qquad\square$

5. Products and Powers of Principal Symmetric Ideals

Simple examples show that powers, and hence products, of psi's need not be psi's.

Example 5.1. Consider the ideal $(x_1)_{S_2} = (x_1, x_2)$. Its square $I^2 = (x_1^2, x_1 x_2, x_2^2)$ cannot be a psi in $\mathsf{k}[x_1, x_2]$ since it has three minimal generators whereas a psi with respect to the action of the symmetric group S_2 has at most two generators by Lemma 3.1.

A contrasting behavior is offered by the following family of examples.

Example 5.2. Suppose that f is a k-symmetric polynomial and g is arbitrary. Then $(f)_{S_d}(g)_{S_d} = (fg)_{S_d}$ and $((f)_{S_d})^n = (f^n)_{S_d}$ are principal symmetric ideals.

In view of Example 5.2 one may wonder whether this class of examples characterizes the affirmative answer to Question 1.2. In Theorem 5.4, Corollary 5.5, Theorem 5.7, and Theorem 5.9 we confirm that this is indeed the case.

We start with a result on the number of generators of a product of psi's of importance to the future developments in this section.

Theorem 5.3. *Let $I = (f)_{S_d}$ be a principal symmetric ideal and let $J = (g_1, \ldots, g_m)$ be a homogeneous ideal generated in a single degree. Assume that f is not k-symmetric. Then*

$$\mu(IJ) \geq \mu(J) + 1.$$

In particular, if I, J are principal symmetric ideals neither of which is generated by a k-symmetric polynomial, then $\mu(IJ) \geq \max\{\mu(I), \mu(J)\} + 1$.

Proof. We may assume that f has no k-symmetric divisor and that the polynomials $g_1, \ldots, g_m$ have no non-constant common divisor since dividing by such divisors does not change any of the quantities involved. Then there is an irreducible factor of f, call it p, that does not divide all polynomials $\sigma \cdot f$. Indeed, if $p \mid \sigma \cdot f$ for all $\sigma \in S_d$

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then $\sigma^{-1} \cdot p \mid f$ and therefore $(\prod_{\sigma \in S_d} \sigma \cdot p) \mid f$. Since the product $\prod_{\sigma \in S_d} \sigma \cdot p$ is k-invariant we have obtained a contradiction.

Pick $\sigma \in S_d$ so that $p \nmid \sigma \cdot f$. Let U be the span of the minimal generators of IJ and let V be the span of the minimal generators of J. Let $h = \sigma \cdot f$. Then U contains $fV + hV$ and consequently we have

$$\mu(IJ) = \dim_{\mathsf{k}} U$$

$$\geq \dim_{\mathsf{k}}(fV + hV)$$

$$\geq \dim_{\mathsf{k}}(fV) + \dim_{\mathsf{k}}(hV) - \dim_{\mathsf{k}}(fV \cap hV)$$

$$\geq \dim_{\mathsf{k}}(fV) + 1 = \dim_{\mathsf{k}}(V) + 1$$

$$= \mu(J) + 1.$$

The only inequality that requires explanation is $\dim_{\mathsf{k}}(hV) - \dim_{\mathsf{k}}(fV \cap hV) \geq 1$, equivalently $\dim_{\mathsf{k}}(hV) > \dim_{\mathsf{k}}(fV \cap hV)$ or $hV \supsetneq fV \cap hV$. Assume towards a contradiction that $hV = fV \cap hV$. Then each element of hV is in fV. For example, $hg_i \in fV$ for each i, which yields that f, and therefore also p, divides hg_i for each i. However, p is coprime to h by assumption, so p divides each g_i. This contradicts the assumption that $g_1, \ldots, g_m$ have no non-constant common divisor, thus proving the claim. $\qquad\square$

5.1. *Products and powers of monomial principal symmetric ideals*

The following theorem answers Question 1.2 for products of monomial psi's.

Theorem 5.4. *Let I, J be monomial principal symmetric ideals. Then IJ is a principal symmetric ideal if and only if $I = ((x_1 \cdots x_d)^n)_{S_d}$ or $J = ((x_1 \cdots x_d)^n)_{S_d}$ for some $n \geq 0$.*

Proof. $\Leftarrow$ follows from Example 5.2.

$\Rightarrow$ Suppose that $I \neq ((x_1 \cdots x_d)^n)_{S_d}$ and $J \neq ((x_1 \cdots x_d)^n)_{S_d}$ for all $n \geq 0$. Let $\mathbf{a}, \mathbf{b}$ be partitions such that $I = (\mathbf{x}^{\mathbf{a}})_{S_d}$ and $J = (\mathbf{x}^{\mathbf{b}})_{S_d}$, where $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_d^{a_d}$ and $\mathbf{x}^{\mathbf{b}} = x_1^{b_1} \cdots x_d^{b_d}$. We now show that IJ is not strongly homogeneous and thus not a monomial psi in view of Theorem 4.3. By assumption, $\mathbf{a}, \mathbf{b} \neq (n, \ldots, n)$ for any n. Therefore, there exist i, j, which we pick to be smallest possible, such that $a_1 \neq a_i$ and $b_1 \neq b_j$. Without loss of generality assume that $j \geq i$. We can write

$$\mathbf{a} = (\underbrace{a_1, \ldots, a_1}_{i-1 \text{ times}}, a_i, \ldots),$$

$$\mathbf{b} = (\underbrace{b_1, \ldots, b_1}_{j-1 \text{ times}}, b_j, \ldots),$$

$$(1\, j) \cdot \mathbf{b} = (b_j, \underbrace{b_1, \ldots, b_1}_{j-1 \text{ times}}, \ldots).$$

Furthermore,

$$\mathbf{a} + \mathbf{b} = (\underbrace{a_1 + b_1, \ldots, a_1 + b_1}_{i - 1 \text{ times}}, a_i + b_1, \ldots)$$

and

$$\mathbf{a} + (1\,j) \cdot \mathbf{b} = (a_1 + b_j, \underbrace{a_1 + b_1, \ldots, a_1 + b_1}_{i - 2 \text{ times}}, a_i + b_1, \ldots).$$

Since $\mathbf{a}, \mathbf{b}$ are monotonic non-increasing vectors, $a_1 + b_1 \geq a_{i'} + b_{j'}$ for any i', j'. Furthermore, since $a_1 > a_i \geq a_{i'}$ and $b_1 > b_j \geq b_{j'}$ for $i' \geq i, j' \geq j$, every component of $\mathbf{a} + (1\,j) \cdot \mathbf{b}$ after the first i terms is strictly less than $a_1 + b_1$. Therefore, $\mathbf{a} + (1\,j) \cdot \mathbf{b}$ contains precisely $i - 2$ components with maximum value, while $\mathbf{a} + \mathbf{b}$ contains $i - 1$ components with maximum value. Hence, $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} + (1\,j) \cdot \mathbf{b}$ must have different order types. Therefore, since both $\mathbf{x}^{\mathbf{a}+\mathbf{b}}$ and $\mathbf{x}^{\mathbf{a}+(1\,j)\cdot\mathbf{b}}$ are minimal generators for IJ, IJ is not strongly homogeneous and thus not principal monomial symmetric. □

Corollary 5.5. *Let I be a monomial principal symmetric ideal of $\mathsf{k}[x_1, \ldots, x_d]$. The following statements are equivalent*:

(1) *I^n is a principal symmetric ideal for all integers $n \geq 1$.*
(2) *I^2 is a principal symmetric ideal.*
(3) *I is generated by a k-invariant monomial of the form $(x_1 \cdots x_d)^t$ for some $t \geq 1$.*

Proof. The implications $(1) \Rightarrow (2)$ and $(3) \Rightarrow (1)$ are clear. The implication $(2) \Rightarrow (3)$ follows from Theorem 5.4 by setting $J = I$. □

Another way to phrase Theorem 5.4 is that for monomial psi's I and J, IJ is a psi if and only if $\mu(I) = 1$ or $\mu(J) = 1$. Example 5.6 shows that in general, for (not necessarily monomial) psi's I, J, the invariants $\mu(I), \mu(J)$ are not sufficient to determine if IJ is principal symmetric.

Example 5.6. Let $I = (x_1 x_2)_{S_4}$ and $J = (x_1 x_2 + x_3 x_4)_{S_4}$ and $J' = (x_1 - x_2)_{S_4}$. Then $\mu(J) = \mu(J') = 3$ as computed using Macaulay2 [1]. Note that IJ is not principal symmetric by Theorem 3.6 since $x_1 x_2 x_3 x_4 + x_3^2 x_4^2$ and $x_2 x_3^2 x_4 + x_1 x_3 x_4^2$ are elements of minimal degree of IJ that share no order types and are nonzero when evaluated at $\mathbf{1}$. However, $IJ' = (x_1 x_2 (x_1 - x_3))_{S_4}$ is principal symmetric.

5.2. *Powers of principal symmetric ideals*

We now turn our attention to powers of arbitrary psi's.

Theorem 5.7. *Let I be a nonzero principal symmetric ideal. The following are equivalent*:

(1) *I^n is a principal symmetric ideal for all integers $n \geq 1$.*

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(2) I^n is a principal symmetric ideal for all sufficiently large integers n.

(3) I is generated by a k-*invariant polynomial.*

Proof. The implications $(1) \Rightarrow (2)$ and $(3) \Rightarrow (1)$ are clear. For $(2) \Rightarrow (3)$ assume towards a contradiction that I is not generated by a k-invariant polynomial and note that applying Theorem 5.3 inductively it follows that $\mu(I^n) \geq \mu(I) + n - 1$. For sufficiently large n we conclude that $\mu(I^n) > d!$, which contradicts Lemma 3.1. □

Comparing Corollary 5.5 and Theorem 5.7 we see that the result concerning monomial ideals is stronger. However, outside the case of monomial psi's it is not the case that I^2 being a psi forces I to be generated by a k-invariant polynomial, as the following example demonstrates.

Example 5.8. Let k be a field of characteristic not equal to 2 and consider $I = (x_1 - x_2)_{S_d}$. Note that $x_1 - x_2$ is *not* a k-invariant polynomial in $k[x_1, \ldots, x_d]$ for $d > 2$. We show that $I^2 = ((x_1 - x_2)^2)_{S_d}$ is a psi. Since the inclusion $((x_1 - x_2)^2)_{S_d} \subseteq I^2$ is evident, equality follows from expressing the generators of I^2 as elements of $((x_1 - x_2)^2)_{S_d}$ as shown below

$$2(x_i - x_j)(x_p - x_q)$$
$$= (2x_q - x_i - x_j)(x_j - x_i) + (-2x_p + x_i + x_j)(x_j - x_i)$$
$$= (x_q - x_i)^2 - (x_q - x_j)^2 + (x_j - x_p)^2 - (x_i - x_p)^2.$$

5.3. *The two-variable case*

The case of psi's in the ring $k[x_1, x_2]$ closely resembles the monomial case.

Theorem 5.9. *Let $I = (f)_{S_2}$ and $J = (g)_{S_2}$ be homogeneous principal symmetric ideals of $K[x_1, x_2]$. Then IJ is a principal symmetric ideal if and only if f or g is* k-*symmetric.*

Proof. Assume that neither f nor g are k-symmetric. Since f and g are not k-symmetric $\mu(I), \mu(J) \neq 1$, so $\mu(I) = 2$ and $\mu(J) = 2$ according to Lemma 3.1. Therefore, by Theorem 5.3, it follows that

$$\mu(IJ) \geq 2 + 1 = 3 > 2 = |S_2|.$$

Thus, by Lemma 3.1, IJ is not a principal symmetric ideal. □

We have obtained an analogue of Corollary 5.5.

Corollary 5.10. *Let $I = (f)_{S_2}$ be a homogeneous principal symmetric ideal of* $k[x_1, x_2]$. *Then I^n is a principal symmetric ideal for all integers $n \geq 1$ if and only if I^2 is a principal symmetric ideal if and only if f is* k-*symmetric.*

To see that the equivalence I^2 is a principal symmetric ideal if and only if f is k-symmetric does not extend to $d > 2$ variables, consider Example 5.8.

5.4. *Analysis of products in terms of stabilizers*

Definition 5.11. For $f \in \mathsf{k}[x_1, \ldots, x_d]$, let $\operatorname{Stab}_f = \{\sigma \in S_d : \sigma \cdot f = cf$ for some nonzero $c \in \mathsf{k}\}$ be the k-stabilizer of f.

Note that Stab_f is a subgroup of S_d. Also, let ι be the identity permutation of S_d.

Proposition 5.12. *Let $I = (f)_{S_d}$ and $J = (g)_{S_d}$. If $S_d = \operatorname{Stab}_f \operatorname{Stab}_g$, then $IJ = (fg)_{S_d}$, so IJ is a principal symmetric ideal.*

Proof. By definition one has $IJ = ((\sigma \cdot f)(\tau \cdot g) : \sigma, \tau \in S_d)$. For each $\sigma \cdot (fg)$ in the generating set of $(fg)_{S_d}$, $\sigma \cdot (fg) = (\sigma \cdot f)(\sigma \cdot g)$. Thus, $\sigma \cdot (fg) \in IJ$ yields $(fg)_{S_d} \subseteq IJ$.

Given $(\sigma \cdot f)(\tau \cdot g)$, by assumption, $\sigma^{-1}\tau \in \operatorname{Stab}_f \operatorname{Stab}_g$, so $\sigma^{-1}\tau = \gamma^{-1}\delta^{-1}$ for some $\gamma \in \operatorname{Stab}_f$ and $\delta \in \operatorname{Stab}_g$. Hence, $\iota = \gamma\sigma^{-1}\tau\delta$. By definition of Stab_f and Stab_g, $c_1(\sigma\gamma^{-1}) \cdot f = \sigma \cdot f$ and $c_2(\tau\delta) \cdot g = \tau \cdot g$ for some nonzero $c_1, c_2 \in \mathsf{k}$. Therefore,

$$
\begin{aligned}
(\sigma \cdot f)(\tau \cdot g) &= (c_1(\sigma\gamma^{-1}) \cdot f)(c_2(\tau\delta) \cdot g) \\
&= c_1 c_2 (\sigma\gamma^{-1} \cdot f)(\tau\delta \cdot g) \\
&= c_1 c_2 (\sigma\gamma^{-1}\gamma\sigma^{-1}) \cdot ((\sigma\gamma^{-1} \cdot f)(\tau\delta \cdot g)) \\
&= c_1 c_2 (\sigma\gamma^{-1}) \cdot ((\gamma\sigma^{-1}\sigma\gamma^{-1} \cdot f)(\gamma\sigma^{-1}\tau\delta \cdot g)) \\
&= c_1 c_2 (\sigma\gamma^{-1} \cdot (f(\gamma\sigma^{-1}\tau\delta \cdot g))) \\
&= c_1 c_2 (\sigma\gamma^{-1} \cdot (fg)) \quad (\text{since } \gamma\sigma^{-1}\tau\delta = \iota).
\end{aligned}
$$

The displayed computations show that $(\sigma \cdot f)(\tau \cdot g) \in (fg)_{S_d}$, thus, $IJ \subseteq (fg)_{S_d}$. $\qquad\square$

Corollary 5.13. *Let $I = (f)_{S_d}$ and $J = (g)_{S_d}$. If $\operatorname{Stab}_f = A_d$ (the alternating group) and $|\operatorname{Stab}_g| = 2$, then $IJ = (fg)_{S_d}$ is a principal symmetric ideal.*

Proof. Note that $\operatorname{Stab}_f \operatorname{Stab}_g = \{\sigma\tau : \sigma \in S_f, \tau \in S_g\}$. Furthermore, we show that all such products are distinct. Suppose that $\sigma\tau = \sigma'\tau'$ for some $\sigma, \sigma' \in \operatorname{Stab}_f$ and $\tau, \tau' \in \operatorname{Stab}_g$. If $\tau = \tau'$, then clearly, $\sigma = \sigma'$. If $\tau \neq \tau'$, then since $|\operatorname{Stab}_g| = 2$, one of τ, τ' is the identity ι. Without loss of generality assume that $\tau = \iota$, then $\sigma = \sigma'\tau'$, so $\tau' = \sigma'^{-1}\sigma \in \operatorname{Stab}_f$. Now, $\tau' \in \operatorname{Stab}_g$ has order 2, so it is a transposition, which is an odd permutation. Hence $\tau' \notin A_d = \operatorname{Stab}_f$, which is a contradiction.

Since all the products in $\operatorname{Stab}_f \operatorname{Stab}_g$ are distinct, $|\operatorname{Stab}_f \operatorname{Stab}_g| = |\operatorname{Stab}_f||\operatorname{Stab}_g| = |S_d|$. Hence, we have that $\operatorname{Stab}_f \operatorname{Stab}_g = S_d$, so by the above proposition, $IJ = (fg)_{S_d}$. $\qquad\square$

Example 5.14. Consider the polynomial ring $\mathsf{k}[x_1, x_2, x_3]$ and let $f = x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_1$ and $g = x_1 - x_2$. Then $\operatorname{Stab}_f = A_3, \operatorname{Stab}_g = \{(1\,2), \iota\}$ and thus $(f)_{S_3}(g)_{S_3} = (fg)_{S_3}$ by Corollary 5.13.

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6. Hilbert Functions of Principal Symmetric Ideals

A symmetric monomial ideal is a monomial ideal J that is closed under the action of the symmetric group, that is, $\sigma \cdot m \in J$ for any $m \in J$ and $\sigma \in S_d$. In this section, we consider Question 1.3 which aims to compare the collection of Hilbert functions of psi's to that of symmetric monomial ideals (not necessarily monomial psi's). We are not aware of a characterization of all Hilbert functions attained by psi's. However, in [2] the Hilbert function of "most" psi's is determined. We now review the relevant facts.

Fix $n \in \mathbb{N}$. A parameter space for the set of principal symmetric ideals generated in degree n is $\mathbb{P}^{N-1}$, where $N = \binom{n+d-1}{n}$. Indeed, let M_n denote the set of monomials of degree n in $R = \mathsf{k}[x_1, \ldots, x_d]$ listed as $M_n = \{m_1, \ldots, m_N\}$ in an arbitrary order. Points in $\mathbb{P}^{N-1}$ parametrize principal symmetric ideals generated in degree d via the assignment

$$\Phi : \mathbb{P}^{N-1} \to \text{ principal symmetric ideals}$$

$$\Phi(c_1 : \cdots : c_N) := (f_c)_{S_n}, \quad \text{where } f_c = \sum_{i=1}^{N} c_i m_i.$$

The map Φ above is onto, but not one-to-one. It allows us to formulate a notion of a general principal symmetric ideal, which we now formalize. We will say that *a general principal symmetric ideal generated in degree d satisfies property $\mathcal{P}$* if there exists a non-empty Zariski-open set U of $\mathbb{P}^N$ so that for each $c \in U$, the principal symmetric ideal $\Phi(c) = (f_c)_{S_n}$ satisfies the property $\mathcal{P}$. The following was shown in [2].

Theorem 6.1 (Partial statement of [2, Theorem 8.4]). *Suppose* $\mathrm{char}(\mathsf{k}) = 0$ *and fix an integer $n \geq 2$.*[a] *For sufficiently large d, a general principal symmetric ideal I of $\mathsf{k}[x_1, \ldots, x_d]$ generated in degree n yields a quotient ring with Hilbert function*

$$H_{R/I}(i) := \dim_{\mathsf{k}}(R/I)_i = \begin{cases} \dim_{\mathsf{k}} R_i & \text{if } i \leq d-1, \\ P(n) - 1 & \text{if } i = d, \\ 0 & \text{if } i > d. \end{cases}$$

In the following, we will refer to a psi I or to a polynomial f such that $I = (f)_{S_d}$ as general provided that it satisfies the conclusion of Theorem 6.1. Heuristically, since non-empty Zariski open sets are dense, "most" polynomials homogeneous f of a fixed degree are general.

Theorem 6.2. *Let k be a field of characteristic 0, and let $f \in R = \mathsf{k}[x_1, \ldots, x_d]$ be a polynomial of degree n. If d is sufficiently large satisfying in particular $d \geq n^n/n!$*

---

[a]Note that the meanings of n and d in this reference are interchanged compared to our notation here.

*and $d > n$, and if f is general then there exists no monomial symmetric ideal J
that satisfies $H_J(n) = H_{(f)_{S_d}}(n)$ for all $n \in \mathbb{N}$.*

Proof. Let J be a symmetric monomial ideal. Using the notation in Definition 2.2
we have a decomposition of the vector space R_n of homogeneous polynomials of
degree n as

$$R_n = \bigoplus_{|\mathbf{a}|=n} R_{\mathbf{a}},$$

where $\mathbf{a}$ ranges over the set of partitions of sum n. Due to the fact that J is
symmetric, $R_{\mathbf{a}} \cap J \neq 0$ implies $R_{\mathbf{a}} \subset J$. Thus, there is an induced decomposition
of J_n of the form

$$J_n = J \cap R_n = \bigoplus_{\substack{|\mathbf{a}|=n \\ R_{\mathbf{a}} \cap J \neq 0}} R_{\mathbf{a}}$$

and consequently we obtain

$$H_R(n) = \sum_{|\mathbf{a}|=n} \dim_k R_{\mathbf{a}} \quad \text{and} \quad H_J(n) = \sum_{\substack{|\mathbf{a}|=n \\ R_{\mathbf{a}} \cap J \neq 0}} \dim_k R_{\mathbf{a}}. \tag{6.1}$$

We write this concisely as

$$H_{R/J}(n) = H_R(n) - H_J(n) = \sum_{\substack{|\mathbf{a}|=n \\ R_{\mathbf{a}} \cap J = 0}} \dim_k R_{\mathbf{a}}. \tag{6.2}$$

Let $\mathbf{a}$ be a partition of n, let k denote the number of distinct values in $\mathbf{a}$ and $\{n_i\}_{i=1}^k$
the number of occurrences of each value. By Eq. (2.1), we have

$$\dim_k R_{\mathbf{a}} = \frac{d!}{n_1! n_2! \cdots n_k!}. \tag{6.3}$$

A partition $\mathbf{a}$ of n has at most n parts, where the possible value of the ith part
a_i satisfies $0 \leq a_i \leq \frac{n}{i}$. Indeed, the inequality $i a_i \leq |a| = n$ can be deduced from
the fact that the parts are ordered non-increasingly and the sum of the first i parts
does not exceed $|a|$. So, we must have the following upper bound for $P(n)$:

$$P(n) \leq \prod_{i=1}^{n} \frac{n}{i} = \frac{n^n}{n!} \leq d.$$

Since $\mathbf{a}$ is a partition of n having d parts and since $d > n$, no part can be
repeated d times as this would imply that $n = |a|$ is divisible by d, a contradiction.
Thus, $n_i < d$ for each $1 \leq i \leq k$ and in particular we have that $k \geq 2$. Since there
are d factors in the product $n_1! n_2! \cdots n_k!$ at least two of which are equal to 1, we
deduce that

$$n_1! n_2! \cdots n_k! \leq (d-1)!1!, \quad \text{i.e. } \dim_k R_{\mathbf{a}} = \frac{d!}{n_1! n_2! \cdots n_k!} \geq \frac{d!}{(d-1)!1!} = d$$

$$\geq P(n). \tag{6.4}$$

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Let $n = \deg f$ and assume that there is some symmetric monomial ideal J such that $H_J(n) = H_{(f)_{S_d}}(n)$ for all $n \in \mathbb{N}$. By Theorem 6.1, for sufficiently large d and general f we have

$$H_R(n) - H_J(n) = H_R(n) - H_{(f)_{S_d}}(n) = P(n) - 1. \qquad (6.5)$$

Equations (6.5) and (6.2) show there exists a partition $\mathbf{a}$ of n with d parts such that

$$\dim_{\mathsf{k}} R_{\mathbf{a}} \leq P(n) - 1.$$

This contradicts Eq. (6.4), which yields $\dim_{\mathsf{k}} R_{\mathbf{a}} \geq P(n)$ for every partition $\mathbf{a}$. $\qquad \square$

Next, we consider the case of two variables.

Definition 6.3. For $f \in \mathsf{k}[x_1, x_2]$ and $\sigma = (1\,2)$ let $g = \gcd(f, \sigma \cdot f)$. The *symmetric reduction* of f, or $\mathrm{sRed}(f)$ is defined as $\mathrm{sRed}(f) = f/g$.

Note that the symmetric reduction of a polynomial is only well defined up to multiplication by a unit. However its degree is well defined and this suffices for our purpose.

We observe that the Hilbert function of a psi in the two-dimensional ring depends only on the degree of the generator and its symmetric reduction. In the following, for a graded module M we employ the standard notation $M(-i)$ to mean the graded module which has the same underlying set as M and satisfies $M(-i)_n = M_{n-i}$.

Lemma 6.4. *Let $f \in R = \mathsf{k}[x_1, x_2]$ be homogeneous, let $k = \deg(f)$ and $\ell = \deg(\mathrm{sRed(f)})$. Then for every integer n the following holds[b]:*

$$H_I(n) = 2H_R(n-k) - H_R(n-k-\ell).$$

Proof. Set $I = (f)_{S_d}$ and $g = \gcd(f, \sigma \cdot f)$. Assume first that f is not k-symmetric, therefore $\ell > 0$ and $\deg(g) = k - \ell$. The Hilbert function of I can be deduced from the isomorphism

$$I = g(\mathrm{sRed}(f), \sigma \cdot \mathrm{sRed}(f)) \cong (\mathrm{sRed}(f), \sigma \cdot \mathrm{sRed}(f))(-k + \ell),$$

where $C = (\mathrm{sRed}(f), \sigma \cdot \mathrm{sRed}(f))$ is generated by a regular sequence. The isomorphism provides the equality $H_I(n) = H_C(n - k + \ell)$ and the Hilbert function of C is obtained from the Koszul complex

$$0 \to R(-2\ell) \to R^2(-\ell) \to C \to 0$$

which yields $H_C(n) = 2H_R(n-\ell) - H_R(n-2\ell)$. This establishes the claimed formula in the case $\ell > 0$.

Assume now that f is k-symmetric, therefore $\ell = 0$. Then, as $(f)_{S_2} = (f)$ is a principal ideal in the usual sense we have $H_{(f)_{S_2}}(n) = H_R(n - k)$. This agrees with the claim. $\qquad \square$

---

[b]Recall that $H_R(i) = 0$ for $i < 0$.

Proposition 6.5. *Let $f \in \mathsf{k}[x_1, x_2]$ be homogeneous, let $\ell = \deg(\mathrm{sRed}(f))$ and $k = \deg(f)$. There exists a monomial symmetric ideal J, such that $H_{(f)_{S_2}}(n) = H_J(n)$ if and only if $k + \ell$ is even.*

Proof. Set $I = (f)_{S_2}$. From Lemma 6.4 applied in degree $n > k + \ell$ with n odd we obtain

$$H_I(n) = 2H_R(n-k) - (n-k-\ell+1) \equiv n - k - \ell + 1 \pmod 2.$$

Suppose that J is a symmetric monomial ideal such that $H_{(f)_{S_2}}(n) = H_J(n)$ for all $n \in \mathbb{N}$. Fix n to be an odd integer. Then since $d = 2$ and there are no partitions $\mathbf{a}$ of an odd integer n with two equal parts we see by Eq. (2.1) that $\dim_{\mathsf{k}} R_{\mathbf{a}} = 2$ for all $\mathbf{a}$ with $|\mathbf{a}| = n$. Consequently by Eq. (6.1), $H_J(n)$ is even. Since $H_J(n) = H_I(n) \equiv k + \ell \pmod 2$ we conclude that $k + \ell$ must be even.

Conversely, assume $k + \ell$ is even, set $a = \frac{k-\ell}{2}$ and $g = x_1^{a+\ell} x_2^a$. By Lemma 6.4 the Hilbert function of $J = (g)_{S_2}$ is uniquely determined by $\deg(\mathrm{sRed}(g))$ and $\deg(g)$. Since $\mathrm{sRed}(g) = x_1^\ell$, which has degree ℓ and since $\deg(g) = k = \deg(f)$ we obtain $H_{(g)_{S_2}}(n) = H_I(n)$. □

7. Open Problems

Satisfactory partial answers to Questions 1.1 and 1.2 of Sec. 1 have been provided in this paper, however these questions are still open in full generality. We single out a few additional questions which result from our work.

In view of Example 5.8, where the polynomial $x_1 - x_2$ symmetrically generating the relevant ideal is k-invariant under the action of the subgroup S_2 of S_d we ask:

Question 7.1. Is it true that if I and I^2 are principal symmetric ideals, then $I = (f)_{S_d}$ for some polynomial f which is k-invariant under the action of a symmetric subgroup of S_d?

We remark that [2, Lemma 5.1] and Theorem 6.1 give an upper bound on the minimal number of generators of any principal symmetric ideal I generated by a polynomial of degree n in the ring $\mathsf{k}[x_1, \ldots, x_d]$ with d sufficiently large, namely

$$\mu(I) \leq \binom{n+d-1}{d-1} - P(n) + 1.$$

On the other hand several of our results proceed by way of estimating *lower bounds* on the number of minimal generators of various ideals as well as their products and powers; see for example, Theorem 5.3. It is therefore of interest to continue the search for such bounds.

Problem 7.2. *Find (nontrivial) upper and lower bounds for the number of minimal generators of the product of two homogeneous principal symmetric ideals.*

Find (nontrivial) upper and lower bounds for the number of minimal generators of the powers of a homogeneous principal symmetric ideal.

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Finally, Theorem 6.2 opens up the search for a non-monomial ideal analogue of Macaulay's theorem in the category of homogeneous symmetric ideals. Evidence from [2] suggests that binomial ideals could replace monomial ideals in the desired result.

Question 7.3. If k is a field of characteristic zero and I is a homogeneous symmetric ideal of $\mathsf{k}[x_1, \ldots, x_n]$ that does not have k-symmetric polynomials in any set of minimal generators, does there exists a symmetric ideal J, generated by monomials and binomials, so that $H_I(n) = H_J(n)$ for all $n \in \mathbb{N}$?

The answer to Question 7.3 is affirmative if I is a general psi by [2, Proposition 6.12] and Theorem 6.1. The answer is also affirmative in two-dimensional polynomial rings without any restriction on the generators of I, as we show next.

Proposition 7.4. *Let k be a field of characteristic different from two and let $f \in \mathsf{k}[x_1, x_2]$ be a homogeneous polynomial. Then there exists a symmetric ideal J generated by binomials or monomials such that $H_J(n) = H_{(f)_{S_2}}(n)$.*

Proof. Let $\ell = \deg(\mathrm{sRed}(f))$, $k = \deg f \geq \ell$. Consider the binomial
$$g = x_1^\ell x_2^{k-\ell} + x_1^k = x_1^\ell(x_2^{k-\ell} + x_1^{k-\ell}).$$
Thus $\deg(g) = k$ and since $\deg(\mathrm{sRed}(g)) = \ell$, Lemma 6.4 gives that $H_{(f)_{S_2}}(n) = H_{(g)_{S_2}}(n)$. □

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