

## BETTI NUMBERS FOR CONNECTED SUMS OF GRADED GORENSTEIN ARTINIAN ALGEBRAS

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**ABSTRACT.** The connected sum construction, which takes as input Gorenstein rings and produces new Gorenstein rings, can be considered as an algebraic analogue for the topological construction having the same name. We determine the graded Betti numbers for connected sums of graded Artinian Gorenstein algebras. Along the way, we find the graded Betti numbers for fiber products of graded rings; an analogous result was obtained in the local case by Geller [Proc. Amer. Math. Soc. 150 (2022), pp. 4159–4172]. We relate the connected sum construction to the doubling construction, which also produces Gorenstein rings. Specifically, we show that, for any number of summands, a connected sum of doublings is the doubling of a fiber product ring.

### 1. INTRODUCTION

The connected sum is a topological construction that takes two manifolds to produce a new manifold [Mas91, p. 7]. An algebraic analog of this surgery construction was introduced by H. Ananthnarayan, L. Avramov, and W.F. Moore in their paper [AAM12] in the local case. In this paper, we elucidate some properties of this construction in the graded case.

Let  $A$  and  $B$  be two graded Artinian Gorenstein (AG)  $K$ -algebras with the same socle degree  $d$ , let  $T$  be an AG  $K$ -algebra of socle degree  $k < d$ , and suppose there are surjective maps  $\pi_A: A \rightarrow T$ , and  $\pi_B: B \rightarrow T$ . From this data, one forms the fiber product algebra  $A \times_T B$  as the categorical pullback of  $\pi_A, \pi_B$ ; the connected sum algebra  $A \#_T B$  is the quotient of  $A \times_T B$  by a certain principal ideal  $\langle\langle \tau_A, \tau_B \rangle\rangle \subset A \times_T B$ . The connected sum is again an AG  $K$ -algebra (see Definition 2.10).

In [Gel22] and [CGS23], the authors determined the minimal free resolution of a two-factor fiber product  $A \times_T B$  of local rings. In this paper, we turn to the setting of fiber products of graded rings where we derive the minimal free resolution of a two-factor fiber product of graded rings using methods adapted to the graded setting. We extend this to fiber products involving multiple factors. In contrast to the local scenario, managing degrees leads not only to novel proofs, but also to a necessary increase in the complexity of the results especially in the multiple factor case. We also consider connected sums of graded rings, for which minimal graded

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free resolutions have not been constructed in the literature before this work. We pose Question 1.1:

**Question 1.1.** Fix  $A_1, \dots, A_r$  graded AG  $K$ -algebras with the same socle degree. What are the graded Betti numbers of their fiber product over  $K$ ? What are the graded Betti numbers of their connected sum over  $K$ ?

Our first series of main results answers the above question. For specific formulas we refer the reader to Theorem 3.2, Theorem 3.7, Theorem 3.13, Theorem 3.15 and their corollaries.

Celikbas, Laxmi and Weyman solved a particular case of Question 1.1 in [CLW19, Corollary 6.3]. Specifically, they determined a minimal free resolution of the connected sum of  $K$ -algebras  $A_i := K[x_i]/(x_i^{d_i})$  by using the doubling construction (see section 2.5). A second goal of this paper is to generalize their result and investigate conditions for a connected sum of AG  $K$ -algebras  $A_1, \dots, A_r$  with the same socle degree to be a doubling. More precisely we ask:

**Question 1.2.** Assume that  $A_1, \dots, A_r$  are graded AG  $K$ -algebras with the same socle degree. Is the connected sum  $A = A_1 \#_K \dots \#_K A_r$  a doubling? More precisely: if  $A_i$  is a doubling of  $\tilde{A}_i$ , is  $A$  a doubling of  $A_1 \times_K \dots \times_K \tilde{A}_r$ ?

We answer the above question in the affirmative in Theorem 4.3. While it might not come as a surprise that findings concerning the Betti numbers of connected sums can be generalized from two to multiple factors, the confirmation of a similar phenomenon for structural aspects, such as being a doubling, seems significantly less evident. In light of this, we find Theorem 4.3 pleasantly surprising.

Our paper is structured as follows: section 2 introduces the necessary background and develops the basic properties of multi-factor fiber products and connected sums, section 3 computes the graded Betti numbers for multi-factor fiber products and connected sums, and section 4 analyzes connected sums that arise as doublings of certain fiber products.

## 2. BACKGROUND

In this section, we fix some notation and recall some basic facts on Artinian Gorenstein (AG) algebras, fiber products, connected sums of graded Artinian algebras, as well as on Macaulay dual generators needed in the sequel.

**2.1. Oriented AG algebras.** Throughout this paper,  $K$  is an arbitrary field. Given a graded  $K$ -algebra  $A$ , its homogeneous maximal ideal is  $m_A = \bigoplus_{i \geq 1} A_i$ . A  $K$ -algebra  $A$  is called *Artinian* if it is a finite dimensional vector space over  $K$ . The *socle* of an Artinian  $K$ -algebra  $A$  is the ideal  $(0 : m_A)$ ; its *socle degree* is the largest integer  $d$  such that  $A_d \neq 0$ . The socle degree of an Artinian  $K$ -algebra agrees with its Castelnuovo-Mumford regularity, which is denoted by  $\text{reg}(A)$ . The *type* of  $A$  is the vector space dimension of its socle.

The *Hilbert series* of a graded  $K$ -algebra  $A$  is the generating function  $H_A(t) = \sum_{i \geq 0} \dim(A_i) t^i$ . The *Hilbert function*  $HF_A$  of a  $K$ -algebra  $A$  is the sequence of coefficients of its Hilbert series.

Suppose that  $A$  has a presentation  $A = R/I$  as a quotient of a graded  $K$ -algebra  $R$ . The graded Betti numbers of  $A$  over  $R$  are the integers  $\beta_{ij}^R(A) = \dim_K \text{Tor}_i^R(A, K)_j$ . These homological invariants are our main focus. The graded

Poincaré series of  $A$  over  $R$  is the generating function  $P_A^R(t, s) = \sum_{i,j} \beta_{ij}^R(A) t^i s^j$ . If  $R$  is regular, then the Poincaré series is in fact a polynomial.

A graded Artinian  $K$ -algebra  $A$  with socle degree  $d$  is said to be *Gorenstein* if its socle  $(0 : m_A)$  is a one dimensional  $K$ -vector space. For any Artinian Gorenstein graded  $K$ -algebra  $A$  with socle degree  $d$  and for any non-zero morphism of graded vector spaces  $f_A : A \rightarrow K(-d)$ , known as an orientation of  $A$ , there is a pairing

$$(2.1) \quad A_i \times A_{d-i} \rightarrow K \text{ defined by } (a_i, a_{d-i}) \mapsto f_A(a_i a_{d-i})$$

which is non-degenerate. We call the pair  $(A, f_A)$  an *oriented AG  $K$ -algebra*.

**Definition 2.1** ([IMS22, Lemma 2.1]). Let  $(A, f_A)$  and  $(T, f_T)$  be two oriented AG  $K$ -algebras with  $\text{reg}(A) = d$  and  $\text{reg}(T) = k$ , and let  $\pi : A \rightarrow T$  be a graded map. There exists a unique homogeneous element  $\tau_A \in A_{d-k}$  such that  $f_A(\tau_A a) = f_T(\pi(a))$  for all  $a \in A$ ; we call it the *Thom class* for  $\pi : A \rightarrow T$ .

*Remark 2.2.* Restating [IMS22, Remark 2.8], we have that  $\tau_A$  is the image of  $1 \in T$  under the composite map  $T(-k) \cong \text{Ext}^n(T, Q) \rightarrow \text{Ext}^n(A, Q) \cong A(-d)$ , where the middle map is  $\text{Ext}^n(\pi, Q)$ .

**Example 2.3.** Let  $(A, f_A)$  be an oriented AG  $K$ -algebra with socle degree  $\text{reg}(A) = d$ . Consider  $(K, f_K)$  where  $f_K : K \rightarrow K$  is the identity map. Then the Thom class for the canonical projection  $\pi : A \rightarrow K$  is the unique element  $s \in A_d$  such that  $f_A(s) = 1$ .

Note that the Thom class for  $\pi : A \rightarrow T$  depends not only on the map  $\pi$ , but also on the orientations chosen for  $A$  and  $T$ .

**2.2. Macaulay dual generators.** Let  $Q = K[x_1, \dots, x_n]$  be a polynomial ring and let  $Q' = K[X_1, \dots, X_n]$  be a divided power algebra, regarded as a  $Q$ -module with the contraction action

$$x_i \circ X_1^{j_1} \dots X_i^{j_i} \dots X_n^{j_n} = \begin{cases} X_1^{j_1} \dots X_i^{j_i-1} \dots X_n^{j_n} & \text{if } j_i > 0 \\ 0 & \text{otherwise.} \end{cases}$$

We regard  $Q$  as a graded  $K$ -algebra with  $\deg X_i = \deg x_i$ .

For each degree  $i \geq 0$ , the action of  $Q$  on  $Q'$  defines a non-degenerate  $K$ -bilinear pairing

$$(2.2) \quad Q_i \times Q'_i \longrightarrow K \text{ with } (f, F) \longmapsto f \circ F.$$

This implies that for each  $i \geq 0$  we have an isomorphism of  $K$ -vector spaces  $Q'_i \cong \text{Hom}_K(Q_i, K)$  given by  $F \mapsto \{f \mapsto f \circ F\}$ .

It is a classical result of Macaulay [Mac94] (cf. [IK99, Lemma 2.14]) that an Artinian  $K$ -algebra  $A = Q/I$  is Gorenstein with socle degree  $d$  if and only if  $I = \text{Ann}_Q(F) = \{f \in Q \mid f \circ F = 0\}$  for some homogeneous polynomial  $F \in Q'_d$ . Moreover, this polynomial, termed a *Macaulay dual generator* for  $A$ , is unique up to a scalar multiple.

A choice of orientation on  $A$  corresponds to a choice of Macaulay dual generator. Every orientation on  $A$  can be written as the function  $f_A : A \rightarrow K$  defined by  $f_A(g) \mapsto (g \circ F)(0)$  for some Macaulay dual generator  $F$  of  $A$  (the notation  $(g \circ F)(0)$  refers to evaluating the element  $g \circ F$  of  $Q'$  at  $X_i = 0$ ).

**2.3. Fiber product.** We start by recalling the definition of the fiber product.

**Definition 2.4.** Let  $A$ ,  $B$  and  $T$  be graded  $K$ -algebras and  $\pi_A : A \rightarrow T$  and  $\pi_B : B \rightarrow T$  morphisms of graded  $K$ -algebras. We define the *fiber product* of  $A$  and  $B$  over  $T$  as the graded  $K$ -subalgebra of  $A \oplus B$ :

$$A \times_T B = \{(a, b) \in A \oplus B \mid \pi_A(a) = \pi_B(b)\}.$$

If  $\pi_A$  and  $\pi_B$  are surjective, then there is a degree-preserving exact sequence

$$(2.3) \quad 0 \rightarrow A \times_T B \rightarrow A \oplus B \rightarrow T \rightarrow 0$$

which allows to compute the Hilbert series of the fiber product as

$$(2.4) \quad HF_{A \times_T B}(t) = HF_A(t) + HF_B(t) - HF_T(t).$$

While presentations of arbitrary fiber products can be unruly, the case  $T = K$  is best-behaved.

**Lemma 2.5.** Let  $R = K[x_1, \dots, x_m]$  and  $S = K[y_1, \dots, y_n]$  be polynomial rings over  $K$  with homogeneous maximal ideals  $\mathbf{x} = (x_1, \dots, x_m)$  and  $\mathbf{y} = (y_1, \dots, y_n)$ , respectively. Let  $Q = R \otimes_K S = K[x_1, \dots, x_m, y_1, \dots, y_n]$ . If  $A = R/\mathfrak{a}$  and  $B = S/\mathfrak{b}$  have canonical projections  $\pi_A : A \rightarrow K$  and  $\pi_B : B \rightarrow K$ , then the fiber product over  $K$  has presentation

$$(2.5) \quad A \times_K B = \frac{Q}{\mathbf{x} \cap \mathbf{y} + \mathfrak{a} + \mathfrak{b}},$$

where in (2.5)  $\mathfrak{a}, \mathfrak{b}, \mathbf{x}, \mathbf{y}$  denote extensions of the respective ideals to  $Q$ . In particular, if  $A$  and  $B$  are graded, then  $A \times_K B$  is a bigraded algebra with

$$[A \times_K B]_{(i,j)} = \begin{cases} K & \text{if } (i, j) = (0, 0), \\ A_i \oplus B_j & \text{if } (i, j) \neq (0, 0). \end{cases}$$

*Proof.* The presentation of the fiber product is given in [IMS22, Proposition 3.12]. The fact that the fiber product is bigraded follows from noticing that the relations in (2.5) are homogeneous with respect to the natural bigrading of  $Q$ . Finally, the formula for the graded components of  $A \times_K B$  follows from (2.3), which can be interpreted as an exact sequence of bigraded vector spaces.  $\square$

**Example 2.6.** Consider the standard graded complete intersection algebras

$$A = \frac{K[x, y, z]}{(x^3, y^4, z^4)} \text{ and } B = \frac{K[u, v]}{(u^5, v^5)}.$$

Their Hilbert functions are given by

$$HF_A = (1, 3, 6, 9, 10, 9, 6, 3, 1) \text{ and } HF_B = (1, 2, 3, 4, 5, 4, 3, 2, 1).$$

Set  $R = K[x, y, z]$  and  $S = K[u, v]$ . The minimal free resolutions of  $A$  and  $B$  are the Koszul complexes

$$0 \rightarrow R(-11) \rightarrow R(-7)^2 \oplus R(-8) \rightarrow R(-4)^2 \oplus R(-3) \rightarrow R \rightarrow A \rightarrow 0,$$

and

$$0 \rightarrow S(-10) \rightarrow S(-5)^2 \rightarrow S \rightarrow B \rightarrow 0.$$

The fiber product  $C = A \times_K B$  of  $A$  and  $B$  and its Hilbert function are

$$C = K[x, y, z, u, v]/(xu, xv, yu, yv, zu, zv, x^3, y^4, z^4, u^5, v^5),$$

TABLE 1. Betti table of  $C$  in Example 2.6

	0	1	2	3	4	5
total	1	11	25	24	11	2
0:	1	.	.	.	.	.
1:	.	6	9	5	1	.
2:	.	1	2	1	.	.
3:	.	2	4	2	.	.
4:	.	2	6	6	2	.
5:	.	.	2	4	2	.
6:	.	.	1	2	1	.
7:	.	.	.	.	.	.
8:	.	.	1	4	5	2

and

$$HF_C = (1, 5, 9, 13, 15, 13, 9, 5, 2).$$

The Betti table of  $C$  as a  $K[x, y, z, u, v]$ -module is shown in Table 1.

Note that  $C$  is an Artinian level  $K$ -algebra of type 2, i.e., all the elements of its socle have the same degree and the socle has dimension 2.

Recall that an Artinian  $K$ -algebra  $A$  has the *strong Lefschetz property* (SLP) if there exists a linear form  $\ell$  such that the multiplication map  $\times \ell^k : A_i \rightarrow A_{i+k}$  has maximal rank (i.e, it is injective or surjective) for all  $i$  and  $k$ . It is known that if  $A$  and  $B$  are two AG  $K$ -algebras with the same socle degree, and both have the SLP, then  $A \times_K B$  also has the SLP [IMS22, Proposition 5.6].

We also consider multi-factor fiber products, which we now define.

**Definition 2.7.** Let  $A_1, \dots, A_r$  and  $T$  be graded  $K$ -algebras and let  $\pi_i : A_i \rightarrow T$  morphisms of graded  $K$ -algebras. We define the *fiber product* of  $A_1, \dots, A_r$  over  $T$  as

$$(2.6) \quad A_1 \times_T \cdots \times_T A_r = \{(a_1, \dots, a_r) \in A_1 \oplus \cdots \oplus A_r \mid \pi_i(a_i) = \pi_j(a_j), 1 \leq i, j \leq r\}.$$

*Remark 2.8.* The multi-factor fiber product construction coincides with iteratively applying the two-factor fiber product construction to the list  $A_1, \dots, A_r$  a total of  $r - 1$  times, that is

$$A_1 \times_T \cdots \times_T A_r = ((A_1 \times_T A_2) \times_T \cdots) \times_T A_r.$$

We will need the following generalizations of equations (2.3) and (2.5), which describe a presentation for fiber products with arbitrarily many summands over the residue field.

**Lemma 2.9.** Let  $R_1, \dots, R_r$  be polynomial rings over  $K$  with maximal ideals  $\mathbf{x}_1, \dots, \mathbf{x}_r$ , and let  $Q = R_1 \otimes_K \cdots \otimes_K R_r$ . Suppose  $A_i = R_i/\mathfrak{a}_i$ , for some homogeneous ideal  $\mathfrak{a}_i$  of  $R_i$ . For  $r \geq 2$ ,  $A_1 \times_K \cdots \times_K A_r \cong Q/J$  with

$$J = \mathfrak{a}_1 + \cdots + \mathfrak{a}_r + \sum_{1 \leq i \neq j \leq r} (\mathbf{x}_i \cap \mathbf{x}_j)$$

and there is an exact sequence of graded  $Q$ -modules

$$(2.7) \quad 0 \rightarrow A_1 \times_K \cdots \times_K A_r \rightarrow A_1 \oplus \cdots \oplus A_r \rightarrow K^{r-1} \rightarrow 0.$$

*Proof.* This follows by induction on  $r$  with (2.5) settling the case  $r = 2$ .

Setting  $Q' = R_1 \otimes_K \cdots \otimes_K R_{r-1}$ , consider the ideal of  $Q'$

$$J' = \mathfrak{a}_1 + \cdots + \mathfrak{a}_{r-1} + \sum_{1 \leq i < j \leq r-1} (\mathbf{x}_i \cap \mathbf{x}_j).$$

Applying (2.5) to  $A_1 \times_K \cdots \times_K A_{r-1} \cong Q'/J'$ , we get

$$A_1 \times_K \cdots \times_K A_r \cong Q'/J' \times_K R_r/\mathfrak{a}_r \cong Q/J,$$

where the ideal  $J$  is given by

$$\begin{aligned} J &= J' + \mathfrak{a}_r + \mathbf{x}_r \cap (\mathbf{x}_1 + \cdots + \mathbf{x}_{r-1}) \\ &= \mathfrak{a}_1 + \cdots + \mathfrak{a}_{r-1} + \mathfrak{a}_r + \sum_{1 \leq i < j \leq r-1} (\mathbf{x}_i \cap \mathbf{x}_j) + \sum_{1 \leq i \leq r-1} (\mathbf{x}_i \cap \mathbf{x}_r) \\ &= \mathfrak{a}_1 + \cdots + \mathfrak{a}_r + \sum_{1 \leq i \neq j \leq r} (\mathbf{x}_i \cap \mathbf{x}_j). \end{aligned}$$

This yields the claimed presentation.

By Definition 2.7, we have that  $A_1 \times_K \cdots \times_K A_r$  is a  $K$ -subalgebra of  $A_1 \oplus \cdots \oplus A_r$ . The inclusion  $A_1 \times_K \cdots \times_K A_r \subseteq A_1 \oplus \cdots \oplus A_r$  gives the first nonzero map in (2.7), while the second map can be defined by

$$(a_1, \dots, a_r) \mapsto (\pi_2(a_2) - \pi_1(a_1), \pi_3(a_3) - \pi_1(a_1), \dots, \pi_r(a_r) - \pi_1(a_1)),$$

where the maps  $\pi_i : A_i \rightarrow K$  are the canonical projections. The claim regarding exactness of (2.7) follows from Definition 2.7.  $\square$

**2.4. Connected sum.** Let  $(A, f_A)$ ,  $(B, f_B)$  and  $(T, f_T)$  be oriented AG  $K$ -algebras with  $\text{reg}(A) = \text{reg}(B) = d$  and  $\text{reg}(T) = k$  and let  $\pi_A : A \rightarrow T$  and  $\pi_B : B \rightarrow T$  be surjective graded  $K$ -algebra morphisms with Thom classes  $\tau_A \in A_{d-k}$  and  $\tau_B \in B_{d-k}$ , respectively. We assume that  $\pi_A(\tau_A) = \pi_B(\tau_B)$ , so that  $(\tau_A, \tau_B) \in A \times_T B$ .

**Definition 2.10.** The *connected sum* of the oriented AG  $K$ -algebras  $A$  and  $B$  over  $T$  is the quotient ring of the fiber product  $A \times_T B$  by the principal ideal generated by the pair of Thom classes  $(\tau_A, \tau_B)$ , i.e.

$$A \#_T B = (A \times_T B) / \langle (\tau_A, \tau_B) \rangle.$$

Note that this definition depends on  $\pi_A$ ,  $\pi_B$  and the orientations on  $A$  and  $B$ .

By [IMS22, Lemma 3.7] the connected sum is characterized by the following exact sequence of vector spaces:

$$(2.8) \quad 0 \rightarrow T(k-d) \rightarrow A \times_T B \rightarrow A \#_T B \rightarrow 0.$$

Therefore, the Hilbert series of the connected sum satisfies

$$(2.9) \quad HF_{A \#_T B}(t) = HF_A(t) + HF_B(t) - (1 + t^{d-k})HF_T(t).$$

We recall the following characterization of connected sums.

**Theorem 2.11** ([IMS22, Theorem 4.6]). *Let  $Q = K[x_1, \dots, x_n]$  be a polynomial ring, and let  $Q' = K[X_1, \dots, X_n]$  be its dual ring (a divided power algebra). Let  $F, G \in Q'_d$  be two linearly independent homogeneous forms of degree  $d$ , and suppose that there exists  $\tau \in Q'_{d-k}$  (for some  $k < d$ ) satisfying*

- (a)  $\tau \circ F = \tau \circ G \neq 0$ , and
- (b)  $\text{Ann}(\tau \circ F = \tau \circ G) = \text{Ann}(F) + \text{Ann}(G)$ .

TABLE 2. Betti table of  $D$  in Example 2.12

	0	1	2	3	4	5
total	1	12	29	29	12	1
0:	1	.	.	.	.	.
1:	.	6	9	5	1	.
2:	.	1	2	1	.	.
3:	.	2	4	2	.	.
4:	.	2	6	6	2	.
5:	.	.	2	4	2	.
6:	.	.	1	2	1	.
7:	.	1	5	9	6	.
8:	.	.	.	.	.	1

Define the oriented AG  $K$ -algebras

$$A = \frac{Q}{\text{Ann}(F)}, \quad B = \frac{Q}{\text{Ann}(G)}, \quad T = \frac{Q}{\text{Ann}(\tau \circ F = \tau \circ G)},$$

and let  $\pi_A: A \rightarrow T$  and  $\pi_B: B \rightarrow T$  be the natural projection maps. Then there are  $K$ -algebra isomorphisms

$$(2.10) \quad A \times_T B \cong \frac{Q}{\text{Ann}(F) \cap \text{Ann}(G)}, \quad A \#_T B \cong \frac{Q}{\text{Ann}(F - G)}.$$

Conversely, every connected sum  $A \#_T B$  of graded AG  $K$ -algebras with the same socle degree over a graded AG  $K$ -algebra  $T$  arises in this way.

In particular, when  $T = K$  the polynomials  $F$  and  $G$  in the above theorem are polynomials expressed in disjoint sets of variables. The connected sum  $A \#_K B$  is a graded  $K$ -algebra, but it is not bigraded. Moreover, it is shown in [IMS22, Proposition 5.7] that if  $A$  and  $B$  satisfy the SLP and they have the same socle degree, then  $A \#_K B$  also satisfies the SLP.

**Example 2.12.** We will now build the connected sum of the standard graded complete intersection  $K$ -algebras  $A = K[x, y, z]/(x^3, y^4, z^4)$  and  $B = K[u, v]/(u^5, v^5)$  described in Example 2.6. The connected sum  $D = A \#_K B$  is isomorphic to

$$K[x, y, z, u, v]/(xu, xv, yu, yv, zu, zv, x^3, y^4, z^4, u^5, v^5, x^2y^3z^3 + u^4v^4).$$

Its Hilbert function is

$$HF_D = (1, 5, 9, 13, 15, 13, 9, 5, 1)$$

and its Betti table is given in Table 2.

So,  $D$  is an AG  $K$ -algebra with socle degree 8.

An important feature of the connected sum of AG  $K$ -algebras is that it is also an AG  $K$ -algebra with the same socle degree as  $A$  and  $B$  (see [IMS22, Lemma 3.8] or [AAM12, Theorem 1]), in contrast to the fiber product which is an algebra of type two, hence not Gorenstein.

As before, we consider multi-factor connected sums. The multi-factor connected sum construction defined below coincides with iteratively applying the two-factor construction to the list  $A_1, \dots, A_r$  a total of  $r - 1$  times. In order to define this, we need to define an appropriate orientation and find the Thom class of a connected sum.

**Lemma 2.13.** *Consider the setup of Section 2.4 and denote by  $\tau_A$  and  $\tau_B$  the Thom classes of  $\pi_A$  and  $\pi_B$  respectively. Then  $A \#_T B$  is an oriented AG  $K$ -algebra with orientation  $f : A \#_T B \rightarrow K$  defined by  $f(a, b) = f_A(a) - f_B(b)$ .*

*Moreover, provided that  $\pi_A(\tau_A) = 0$ , the surjective morphism*

$$\pi : A \#_T B \rightarrow T \text{ with } \pi(a, b) = \pi_A(a) = \pi_B(b)$$

*has Thom class  $\tau = (\tau_A, 0)$ .*

*Proof.* Recall from Theorem 2.11 that if the Macaulay dual generators of  $A$  and  $B$  are  $F$  and  $G$  respectively (chosen to correspond to the given orientations  $f_A$  and  $f_B$ ), then the Macaulay dual generator of  $A \#_T B$  is  $F - G$ . This defines an orientation by

$$g \mapsto (g \circ (F - G))(0) = (g \circ F)(0) - (g \circ G)(0).$$

If  $g = (a, b) \in A \#_T B$ , then  $(g \circ F)(0) = f_A(a)$  and  $(g \circ G)(0) = f_B(b)$ .

To establish the claim regarding the Thom class we verify that

$$f(\tau g) = f_T(\pi(g)) \text{ for all } g \in A \#_T B.$$

With  $g = (a, b)$ , we have  $\tau g = (\tau_A, 0)(a, b) = (\tau_A a, 0)$  since

$$f(\tau g) = f_A(\tau_A a) = f_A(\pi_A(a)) = f(\pi(g)).$$

□

We establish the convention that every connected sum in this paper will be oriented according to the orientation  $f$  in Lemma 2.13.

**Definition 2.14.** Let  $A_1, \dots, A_r$  and  $T$  be graded AG  $K$ -algebras with socle degrees  $\text{reg}(A_i) = d$  and  $\text{reg}(T) = k$  and let  $\pi_i : A_i \rightarrow T$  be morphisms of graded  $K$ -algebras with Thom classes  $\tau_1, \dots, \tau_r$ , respectively such that  $\pi_i(\tau_i) = 0$ . We define the multi-factor *connected sum*  $A_1 \#_T \dots \#_T A_r$  by

$$(2.11) \quad A_1 \#_T \dots \#_T A_r = A_1 \times_T \dots \times_T A_r / \langle (\tau_1, 0, \dots, 0, \tau_i, 0, \dots, 0) \mid 2 \leq i \leq r \rangle.$$

*Remark 2.15.* Note that Definition 2.14 coincides with iterating the two-factor connected sum construction. Indeed, repeatedly applying Lemma 2.13 shows that the Thom class of the iterated connected sum

$$((A_1 \#_T A_2) \#_T A_3) \cdots \#_T A_{i-1}$$

is  $(\tau_1, 0, \dots, 0)$ . Thus, to obtain

$$(((A_1 \#_T A_2) \#_T A_3) \cdots \#_T A_{i-1}) \#_T A_i,$$

one goes modulo  $\langle (\tau_1, 0, \dots, 0, \tau_i) \rangle$ .

**Lemma 2.16.** *In the setup of Definition 2.14, there is a short exact sequence*

$$(2.12) \quad 0 \rightarrow T(d - k)^{r-1} \xrightarrow{\alpha} A_1 \times_T \dots \times_T A_r \rightarrow A_1 \#_T \dots \#_T A_r \rightarrow 0,$$

*where the map  $\alpha$  sends the generator of the  $i$ -th summand  $T$  to the image of  $(\tau_1, 0, \dots, 0, \tau_{i+1}, 0, \dots, 0)$  in  $A_1 \times_T \dots \times_T A_r$ .*

*Proof.* It is clear from Definition 2.14 that the connected sum is the cokernel of  $\alpha$ . It remains to justify that this map is injective. This follows from [IMS22, Lemma 2.6], where it is shown that the Gysin map  $\iota : T(d - k) \rightarrow A_i$  that satisfies  $\iota(1) = \tau_i$  is injective. □



**2.5. Doubling.** Let us start by recalling the doubling construction and some basic facts needed later on.

**Definition 2.17.** Set  $R = K[x_1, \dots, x_n]$ . The *canonical module* of a graded  $R$ -module  $M$  is  $\omega_M = \text{Ext}_R^{n-\dim M}(M, R)$ .

For example, one has  $\omega_K \cong K$ .

**Definition 2.18** ([KKR<sup>+</sup>21, Section 2.5]). Let  $J \subset R$  be a homogeneous ideal of codimension  $c$ , such that  $R/J$  is Cohen-Macaulay and  $\omega_{R/J}$  is its canonical module. Furthermore, assume that  $R/J$  satisfies the condition  $G_0$  (i.e., it is Gorenstein at all minimal primes). Let  $I$  be an ideal of codimension  $c+1$ .  $I$  is called a *doubling* of  $J$  via  $\psi$  if there exists a short exact sequence of  $R/J$  modules

$$(2.13) \quad 0 \rightarrow \omega_{R/J}(-d) \xrightarrow{\psi} R/J \rightarrow R/I \rightarrow 0.$$

By [BH93, Proposition 3.3.18], if  $I$  is a doubling, then  $R/I$  is a Gorenstein ring.

Doubling plays an important role in the theory of Gorenstein liaison. Indeed, in [KMMR<sup>+</sup>01], doubling is used to produce suitable Gorenstein divisors on arithmetically Cohen-Macaulay subschemes in several foundational constructions. It is not true that every Artinian Gorenstein ideal of codimension  $c+1$  is a doubling of some codimension  $c$  ideal (see, for instance, [KKR<sup>+</sup>21, Example 2.19]).

Moreover, the mapping cone of  $\psi$  in (2.13) gives a resolution of  $R/I$ . If it is minimal, then one can read off the Betti table of  $R/I$  from the Betti table of  $R/J$ . This mapping cone is the direct sum of the minimal free resolution  $F_\bullet$  of  $R/J$  with its dual (reversed) complex  $\text{Hom}(F_\bullet, R)$  which justifies the terminology of “doubling”.

**Lemma 2.19.** *Let  $C = R/J$  be a Cohen-Macaulay  $K$ -algebra. Then:*

- (a)  $\text{reg } \omega_C = \dim C$ ; and
- (b) *if  $C$  is a doubling of a Cohen-Macaulay  $K$ -algebra  $\tilde{C}$ , i.e.,  $\dim \tilde{C} = \dim C + 1$  and there is an exact sequence of graded  $R$ -modules*

$$0 \rightarrow \omega_{\tilde{C}}(-t) \rightarrow \tilde{C} \rightarrow C \rightarrow 0,$$

*then  $t = \text{reg } C - \dim C$ .*

*Proof.* (a) Consider a graded minimal free resolution of  $C$  as an  $R$ -module

$$0 \rightarrow F_c \rightarrow \cdots \rightarrow F_1 \rightarrow R \rightarrow C \rightarrow 0,$$

where  $c = \dim R - \dim C$  denotes the codimension. Dualizing and then shifting it, we obtain a graded minimal free resolution of  $\omega_C$  of the form

$$0 \rightarrow R(-\dim R) \rightarrow F_1^*(-\dim R) \rightarrow \cdots \rightarrow F_c^*(-\dim R) \rightarrow \omega_C \rightarrow 0.$$

Claim (a) follows using the characterization of regularity by graded Betti numbers.

(b) The Tor sequence of the given short exact sequence begins

$$0 \rightarrow \text{Tor}_c^R(C, K) \rightarrow \text{Tor}_{c-1}^R(\omega_{\tilde{C}}, K)(-t) \rightarrow \cdots.$$

Since  $C$  is a Gorenstein algebra,  $\text{Tor}_c^R(C, K)$  is concentrated in degree  $c + \text{reg } C$ . The above resolution of  $\omega_C$  shows that  $\text{Tor}_{c-1}^R(\omega_{\tilde{C}}, K)(-t)$  is concentrated in degree  $t + \dim R$ . It follows that  $c + \text{reg } C = t + \dim R$ , which proves Claim (b).  $\square$

### 3. GRADED BETTI NUMBERS OF THE CONNECTED SUM

For notational ease in this section, given a  $K$ -algebra  $A$ , we write  $\text{Tor}_i(A)$  instead of  $\text{Tor}_i(A, K)$ .

**3.1. Two summands.** Let  $R = K[x_1, \dots, x_m]$  and  $S = K[y_1, \dots, y_n]$  be polynomial rings over  $K$  with the standard grading, and let  $\mathbf{x} = (x_1, \dots, x_m)$  and  $\mathbf{y} = (y_1, \dots, y_n)$  denote the homogeneous maximal ideals of  $R$  and  $S$ , respectively. Set

$$Q = R \otimes_K S \cong K[x_1, \dots, x_m, y_1, \dots, y_n].$$

Let  $A = R/\mathfrak{a}$  and  $B = S/\mathfrak{b}$  be standard graded  $K$ -algebras. We will assume  $\mathfrak{a}_1 = \mathfrak{b}_1 = 0$ , that is, the ideals  $\mathfrak{a}$  and  $\mathfrak{b}$  do not contain any non-zero linear forms. Note that  $R \times_K S = Q/(\mathbf{x} \cap \mathbf{y})$  by Lemma 2.5. We start by determining the Betti numbers of this ring. Note that the ideal  $\mathbf{x} \cap \mathbf{y}$  of  $Q$  is a so-called *Ferrers ideal* and admits a minimal graded free cellular resolution that is supported on the join of two simplices (see [CN09]). We show other useful fact about this resolution below.

Henceforth, we set  $\binom{a}{b} = 0$  if  $b > a$ .

**Lemma 3.1.** *The ideal  $\mathbf{x} \cap \mathbf{y}$  of  $Q$  has a 2-linear minimal free graded resolution over  $Q$  and, for  $i \geq 1$ , we have*

$$\begin{aligned} [\text{Tor}_i^Q(Q/\mathbf{x} \cap \mathbf{y})]_{i+1} &\cong \\ \text{coker}([\text{Tor}_{i+1}^Q(Q/\mathbf{x}) \oplus \text{Tor}_{i+1}^Q(Q/\mathbf{y})]_{i+1} &\rightarrow [\text{Tor}_{i+1}^Q(K)]_{i+1}). \end{aligned}$$

In particular, one has

$$\dim_K[\text{Tor}_i^Q(Q/\mathbf{x} \cap \mathbf{y})]_{i+1} = \binom{m+n}{i+1} - \binom{m}{i+1} - \binom{n}{i+1}$$

$$\text{and } P_{Q/\mathbf{x} \cap \mathbf{y}}^Q(t, s) = t^{-1}[(1+st)^m - 1][(1+st)^n - 1] + 1.$$

*Proof.* The first part is well-known (see, e.g., [CN09, Theorem 2.1]). It also follows from the Mayer-Vietoris sequence

$$0 \rightarrow Q/\mathbf{x} \cap \mathbf{y} \rightarrow Q/\mathbf{x} \oplus Q/\mathbf{y} \rightarrow Q/\mathbf{x} + \mathbf{y} \rightarrow 0.$$

Observe that  $Q/\mathbf{x} + \mathbf{y} \cong K$  as a  $Q$ -module. Thus, for any integer  $i \geq 0$ , the induced long exact sequence in Tor gives

$$\begin{aligned} [\text{Tor}_{i+1}^Q(Q/\mathbf{x} \cap \mathbf{y})]_{i+1} &\rightarrow [\text{Tor}_{i+1}^Q(Q/\mathbf{x}) \oplus \text{Tor}_{i+1}^Q(Q/\mathbf{y})]_{i+1} \rightarrow \\ [\text{Tor}_{i+1}^Q(K)]_{i+1} &\rightarrow [\text{Tor}_i^Q(Q/\mathbf{x} \cap \mathbf{y})]_{i+1} \rightarrow [\text{Tor}_i^Q(Q/\mathbf{x}) \oplus \text{Tor}_i^Q(Q/\mathbf{y})]_{i+1}. \end{aligned}$$

The left-most module in this sequence is zero because  $\mathbf{x} \cap \mathbf{y}$  has a 2-linear resolution. Similarly, the right-most module is zero because  $\mathbf{x}$  and  $\mathbf{y}$  have linear resolutions. Thus, the second claim follows. The last claim can be verified directly based on the second.  $\square$

The following result gives a formula to compute the graded Betti numbers of the fiber product  $A \times_K B$ .

*Notation.* Given a power series  $P(t, s) = \sum c_{ij} s^j t^i \in \mathbb{Z}[s][[t]]$ , we set  $\tilde{P}(t, s) = \sum_{j>i} c_{ij} s^j t^i$  to be the sum of the terms of  $P(t, s)$  having  $j > i$ .

**Theorem 3.2.** *For any integer  $i \geq 1$ , there is an isomorphism of graded  $K$ -vector spaces*

$$[\mathrm{Tor}_i^Q(A \times_K B)]_j \cong \begin{cases} 0 & \text{if } j \leq i, \\ [\mathrm{Tor}_i^Q(A)]_{i+1} \oplus [\mathrm{Tor}_i^Q(B)]_{i+1} \oplus [\mathrm{Tor}_i^Q(Q/\mathbf{x} \cap \mathbf{y})]_{i+1} & \text{if } j = i+1, \\ [\mathrm{Tor}_i^Q(A)]_j \oplus [\mathrm{Tor}_i^Q(B)]_j & \text{if } j \geq i+2. \end{cases}$$

In particular, one has

$$P_{A \times_K B}^Q(t, s) = \tilde{P}_A^Q(t, s) + \tilde{P}_B^Q(t, s) + P_{Q/\mathbf{x} \cap \mathbf{y}}^Q(t, s)$$

and

$$\mathrm{reg}(A \times_K B) = \max\{\mathrm{reg} A, \mathrm{reg} B\}.$$

*Proof.* Consider the exact sequence (2.3) of graded  $Q$ -modules

$$0 \rightarrow A \times_K B \rightarrow A \oplus B \rightarrow K \rightarrow 0.$$

Its long exact sequence in Tor gives exact sequences

$$(3.1) \quad \mathrm{Tor}_{i+1}^Q(K) \rightarrow \mathrm{Tor}_i^Q(A \times_K B) \rightarrow \mathrm{Tor}_i^Q(A) \oplus \mathrm{Tor}_i^Q(B) \rightarrow \mathrm{Tor}_i^Q(K).$$

As a  $Q$ -module,  $K$  is resolved by a Koszul complex, which shows in particular  $[\mathrm{Tor}_i^Q(K)]_j \neq 0$  if and only if  $0 \leq i = j \leq m + n$ . Considering the sequence (3.1) in degree  $j$ , we conclude that

$$[\mathrm{Tor}_i^Q(A \times_K B)]_j \cong [\mathrm{Tor}_i^Q(A)]_j \oplus [\mathrm{Tor}_i^Q(B)]_j$$

if  $j \geq i+2$ .

Using the fact that  $A \times_K B \cong Q/(\mathbf{x} \cap \mathbf{y}, \mathbf{a}, \mathbf{b})$ —as in (2.5)—and that the initial degree of  $(\mathbf{x} \cap \mathbf{y}, \mathbf{a}, \mathbf{b})$  is two since  $\mathbf{a}$  and  $\mathbf{b}$  do not contain any linear forms by assumption, we see that for  $j \leq i$ ,

$$[\mathrm{Tor}_i^Q(A \times_K B)]_j = 0.$$

It remains to consider  $[\mathrm{Tor}_i^Q(A \times_K B)]_{i+1}$ . To this end, we use a longer part of the exact Tor sequence and consider it in degree  $i+1$ :

$$\begin{aligned} [\mathrm{Tor}_{i+1}^Q(A) \oplus \mathrm{Tor}_{i+1}^Q(B)]_{i+1} &\xrightarrow{\gamma} [\mathrm{Tor}_{i+1}^Q(K)]_{i+1} \rightarrow \\ &[\mathrm{Tor}_i^Q(A \times_K B)]_{i+1} \rightarrow [\mathrm{Tor}_i^Q(A) \oplus \mathrm{Tor}_i^Q(B)]_{i+1} \rightarrow [\mathrm{Tor}_i^Q(K)]_{i+1}. \end{aligned}$$

The right-most module in this sequence is zero because  $K$  has a linear resolution as a  $Q$ -module. Note that there are isomorphisms of graded  $Q$ -modules  $A \cong Q/(\mathbf{a}, \mathbf{y})$  and  $B \cong Q/(\mathbf{x}, \mathbf{b})$ . Since  $\mathbf{a}_1 = \mathbf{b}_1 = 0$ , it follows that

$$[\mathrm{Tor}_{i+1}^Q(A) \oplus \mathrm{Tor}_{i+1}^Q(B)]_{i+1} \cong [\mathrm{Tor}_{i+1}^Q(Q/\mathbf{y}) \oplus \mathrm{Tor}_{i+1}^Q(Q/\mathbf{x})]_{i+1}.$$

Therefore, the cokernel of the map  $\gamma$  is equal to the cokernel of the map  $[\mathrm{Tor}_{i+1}^Q(Q/\mathbf{x}) \oplus \mathrm{Tor}_{i+1}^Q(Q/\mathbf{y})]_{i+1} \rightarrow [\mathrm{Tor}_{i+1}^Q(K)]_{i+1}$ . By Lemma 3.1, the latter is isomorphic to  $[\mathrm{Tor}_i^Q(Q/\mathbf{x} \cap \mathbf{y})]_{i+1}$ . Hence, the above sequence proves the claim for  $[\mathrm{Tor}_i^Q(A \times_K B)]_{i+1}$ .

The claim regarding the Poincaré series follows by taking the dimensions of the Tor modules and using the isomorphism in the first part of the theorem.  $\square$

*Remark 3.3.* The previous result allows us to write the Betti numbers of the fiber product in terms of those of the summands:

$$\beta_{i,j}^Q(A \times_K B) = \begin{cases} 0 & \text{if } j \leq i \\ \beta_{i,i+1}^Q(A) + \beta_{i,i+1}^Q(B) + \beta_{i,i+1}^Q(Q/\mathbf{x} \cap \mathbf{y}) & \text{if } j = i + 1 \\ \beta_{i,j}^Q(A) + \beta_{i,j}^Q(B) & \text{if } j \geq i + 2. \end{cases}$$

In Theorem 3.2, we have computed the graded Betti numbers of the fiber product  $A \times_K B$  in terms of the graded Betti numbers of  $A$  and  $B$  as  $Q$ -modules and we will now convert these formulas into formulas depending only on  $[\mathrm{Tor}_i^R(A)]_j$  and  $[\mathrm{Tor}_i^S(B)]_j$ .

*Notation.* We set  $[x]_+ = \max\{0, x\}$ .

**Proposition 3.4.** *The identities*

$$\begin{aligned} P_A^Q(t, s) &= P_A^R(t, s) \cdot (1 + st)^{\dim S}, \text{ and} \\ P_B^Q(t, s) &= P_B^S(t, s) \cdot (1 + st)^{\dim R} \end{aligned}$$

yield

$$\begin{aligned} \tilde{P}_A^Q(t, s) &= (P_A^R(t, s) - 1) \cdot (1 + st)^{\dim S}, \text{ and} \\ \tilde{P}_B^Q(t, s) &= (P_B^S(t, s) - 1) \cdot (1 + st)^{\dim R}. \end{aligned}$$

Thus, with  $m = \dim R$  and  $n = \dim S$ , we have

$$\begin{aligned} \beta_{i,j}^Q(A) &= \sum_{\ell=0}^{\min(i,m)-1} \binom{n}{[i-m]_+ + \ell} \beta_{\min(i,m)-\ell, j-\ell-[i-m]_+}^R(A), \text{ and} \\ \beta_{i,j}^Q(B) &= \sum_{\ell=0}^{\min(i,n)-1} \binom{m}{[i-n]_+ + \ell} \beta_{\min(i,n)-\ell, j-\ell-[i-n]_+}^S(B). \end{aligned}$$

*Proof.* Since  $Q$  is a free  $R$ -module, we have  $P_{Q/\mathfrak{a}}^Q(t, s) = P_{R/\mathfrak{a}}^R(t, s) = P_A^R(t, s)$ . We know  $A = Q/(\mathfrak{a} + \mathbf{y})$ , and  $\mathrm{Tor}_i^K(Q/\mathfrak{a}, Q/\mathbf{y}) = 0$  for  $i \geq 1$ , so the minimal free resolution of  $A$  as a  $Q$ -module is obtained by tensoring the minimal free resolutions of  $Q/\mathfrak{a}$  and  $Q/\mathbf{y}$ . This justifies the identity

$$P_A^Q(t, s) = P_{Q/\mathfrak{a}}^Q(t, s) \cdot P_{Q/\mathbf{y}}^Q(t, s) = P_A^R(t, s) \cdot (1 + st)^{\dim S}.$$

Since the terms with equal exponents for  $s$  and  $t$  in  $P_A^Q(t, s)$  arise from the constant term of the first factor multiplied with the second factor, removing this term yields  $\tilde{P}_A^Q(t, s)$ .

Therefore, using the convention  $\binom{n}{a} = 0$  if  $a > n$ , we conclude that

$$\beta_{i,j}^Q(A) = \begin{cases} \sum_{\ell=0}^{i-1} \binom{n}{\ell} \beta_{i-\ell, j-\ell}^R(A) & \text{if } 1 \leq i \leq m, \text{ and} \\ \sum_{\ell=0}^{m-1} \binom{n}{i-m+\ell} \beta_{m-\ell, j-i+m-\ell}^R(A) & \text{if } m < i \leq m+n. \end{cases}$$

The claims regarding  $B$  are justified similarly.  $\square$

Combining Lemma 3.1, Theorem 3.2 and Proposition 3.4, we get:

**Corollary 3.5.** *For integers  $a, b, c$  and a module  $M$  over a ring  $P$  denote*

$$s_{a,b,c,d}^P(M) = \sum_{\ell=0}^{\min(a,b)-1} \binom{c}{[i-b]_+ + \ell} \beta_{\min(a,b)-\ell, d-\ell-[a-b]_+}^P(M).$$

*With the above notation, we have*

$$(3.2) \quad P_{A \times_K B}^Q(t, s) = (P_A^R(t, s) - 1) \cdot (1 + st)^n + (P_B^S(t, s) - 1) \cdot (1 + st)^m + t^{-1}[(1 + st)^m - 1][(1 + st)^n - 1] + 1,$$

*that is,*

$$\beta_{i,j}^Q(A \times_K B) = \begin{cases} 0 & \text{if } j \leq i, (i, j) \neq (0, 0), \\ 1 & \text{if } i = j = 0, \\ s_{i,m,n,i+1}^R(A) + s_{i,n,m,i+1}^S(B) & \text{if } j = i + 1, \\ + \binom{m+n}{i+1} - \binom{m}{i+1} - \binom{n}{i+1} & \\ s_{i,m,n,j}^R(A) + s_{i,n,m,j}^S(B) & \text{if } j \geq i + 2. \end{cases}$$

*Remark 3.6.* The identity (3.2) extends [Gel22, Theorem 1.1] to the graded case. It can be checked that the two results agree upon substituting  $s = 1$  in (3.2).

We now turn to the graded Betti numbers of the connected sum. AG  $K$ -algebras with socle degree two have been classified by [Sal79]. Explicitly, if  $A = R/\mathfrak{a}$  has  $h$ -vector  $(1, n, 1)$  then, up to isomorphism, one has

$$\mathfrak{a} = (x_i x_j \mid 1 \leq i \neq j \leq n) + (x_1^2 - x_2^2, \dots, x_1^2 - x_n^2).$$

The graded minimal free resolution of  $A$  as an  $R$ -module has the form

$$0 \rightarrow R(-n-2) \rightarrow R(-n)^{\beta_{n-1}} \rightarrow \dots \rightarrow R(-2)^{\beta_1} \rightarrow R \rightarrow A \rightarrow 0,$$

and a straightforward computation gives us

$$\beta_i = \beta_{n-1-i} = i \binom{n}{i+1} + (n-i) \binom{n}{n-i+1}$$

for  $1 \leq i \leq n-1$ . Thus, it is harmless to consider AG  $K$ -algebras whose socle degree is at least three.

**Theorem 3.7.** *Assume that  $A$  and  $B$  are AG  $K$ -algebras such that  $\text{reg}(A) = \text{reg}(B) = e \geq 3$ . For any integer  $i \geq 1$ , there is an isomorphism of graded  $K$ -vector*

spaces

$$[\mathrm{Tor}_i^Q(A \#_K B)]_j \cong \begin{cases} 0 & \text{if } j \leq i \text{ and } (i, j) \neq (0, 0), \\ K & \text{if } (i, j) = (0, 0), \\ [\mathrm{Tor}_i^Q(A)]_{i+1} \oplus [\mathrm{Tor}_i^Q(B)]_{i+1} \\ \oplus [\mathrm{Tor}_i^Q(Q/\mathbf{x} \cap \mathbf{y})]_{i+1} & \text{if } j = i + 1, \\ [\mathrm{Tor}_i^Q(A)]_j \oplus [\mathrm{Tor}_i^Q(B)]_j & \text{if } i + 2 \leq j \leq i + e - 2, \\ [\mathrm{Tor}_i^Q(A)]_j \oplus [\mathrm{Tor}_i^Q(B)]_j \\ \oplus [\mathrm{Tor}_{m+n-i}^Q(Q/\mathbf{x} \cap \mathbf{y})]_{m+n-i+1} & \text{if } j = i + e - 1, \\ K & \text{if } (i, j) = (m + n, e + m + n), \\ 0 & \text{if } j \geq e + i \text{ and} \\ & (i, j) \neq (m + n, e + m + n); \end{cases}$$

equivalently,

$$P_{A \#_K B}^Q(t, s) = \tilde{P}_A^Q(t, s) + \tilde{P}_B^Q(t, s) + P_{Q/\mathbf{x} \cap \mathbf{y}}^Q(t, s) + s^{m+n+e} t^{m+n} P_{Q/\mathbf{x} \cap \mathbf{y}}^Q(t^{-1}, s^{-1}).$$

*Proof.* Consider the exact sequence (2.8) of graded  $Q$ -modules

$$0 \rightarrow K(-e) \rightarrow A \times_K B \rightarrow A \#_K B \rightarrow 0.$$

Its long exact Tor sequence gives exact sequences

$$[\mathrm{Tor}_i^Q(K)]_{j-e} \rightarrow [\mathrm{Tor}_i^Q(A \times_K B)]_j \rightarrow [\mathrm{Tor}_i^Q(A \#_K B)]_j \rightarrow [\mathrm{Tor}_{i-1}^Q(K)]_{j-e}.$$

Since  $\mathrm{Tor}_i^Q(K)$  is concentrated in degree  $i$  we conclude that

$$[\mathrm{Tor}_i^Q(A \#_K B)]_j \cong [\mathrm{Tor}_i^Q(A \times_K B)]_j$$

if  $j \notin \{e+i-1, e+i\}$ . Combined with Theorem 3.2, this determines  $[\mathrm{Tor}_i^Q(A \#_K B)]_j$  if  $j \leq e+i-2$ .

Using the fact that  $\mathrm{reg}(A \#_K B) = \mathrm{reg} A = \mathrm{reg} B = e$  which can be deduced from (2.9), we know that  $[\mathrm{Tor}_i^Q(A \#_K B)]_j = 0$  if  $j \geq e+i+1$ . It remains to determine  $[\mathrm{Tor}_i^Q(A \#_K B)]_j$  if  $j \in \{e+i-1, e+i\}$ . To this end we utilize the fact that  $A \#_K B$  is Gorenstein. Thus, its graded minimal free resolution is symmetric. In particular, since  $\dim Q = m+n$ , one has

$$[\mathrm{Tor}_i^Q(A \#_K B)]_j \cong [\mathrm{Tor}_{m+n-i}^Q(A \#_K B)]_{e+m+n-j}.$$

Similarly, for  $A$  and  $B$  we have  $\mathrm{Tor}_i^Q(A)_j \cong [\mathrm{Tor}_{m+n-i}^Q(A)]_{e+m+n-j}$  and  $\mathrm{Tor}_i^Q(B)_j \cong [\mathrm{Tor}_{m+n-i}^Q(B)]_{e+m+n-j}$ .

Combined with Theorem 3.2 and using  $e \geq 3$ , which implies that the degrees  $e+i-1, e+i$  are not self-dual under the isomorphisms given above, the claim regarding Tor modules follows.

The Poincaré series formula follows from the above considerations and the identities

$$\begin{aligned}
 & \sum_{i=0}^{m+n} \beta_{m+n-i, m+n-i+1}(Q/\mathbf{x} \cap \mathbf{y}) t^i s^{i+e-1} \\
 &= \sum_{j=0}^{m+n} \beta_{j, j+1}(Q/\mathbf{x} \cap \mathbf{y}) t^{m+n-j} s^{m+n-j+e-1} \\
 &= t^{m+n} s^{m+n+e} \sum_{j=0}^{m+n} \beta_{j, j+1}(Q/\mathbf{x} \cap \mathbf{y}) t^{-j} s^{-j-1} \\
 &= t^{m+n} s^{m+n+e} P_{Q/\mathbf{x} \cap \mathbf{y}}^Q(t^{-1}, s^{-1}).
 \end{aligned}$$

□

Using again Lemma 3.1 and Proposition 3.4, we can convert Theorem 3.7 into explicit formulas depending only on the Betti numbers of  $A$  as an  $R$ -module and of  $B$  as an  $S$ -module.

**Corollary 3.8.** *Consider the setup of Theorem 3.7. For integers  $a, b, c$  and a module  $M$  over a ring  $P$  denote*

$$s_{a,b,c,d}^P(M) = \sum_{\ell=0}^{\min(a,b)-1} \binom{c}{[i-b]_+ + \ell} \beta_{\min(a,b)-\ell, d-\ell-[a-b]_+}^P(M).$$

With the above notation we have:

$$\begin{aligned}
 & \beta_{i,j}^Q(A \#_K B) \\
 &= \begin{cases} 0 & \text{if } j \leq i, (i, j) \neq (0, 0), \\ 1 & \text{if } (i, j) = (0, 0), \\ s_{i,m,n,i+1}^R(A) + s_{i,n,m,i+1}^S(B) \\ \quad + \binom{m+n}{i+1} - \binom{m}{i+1} - \binom{n}{i+1} & \text{if } j = i + 1, \\ s_{i,m,n,j}^R(A) + s_{i,n,m,j}^S(B) & \text{if } i + 2 \leq j \leq i + e - 2, \\ s_{i,m,n,i+1}^R(A) + s_{i,n,m,i+1}^S(B) \\ \quad + \binom{m+n}{m+n-i-1} - \binom{m}{m-i-1} - \binom{n}{n-i-1} & \text{if } i \leq m + n, j = i + e - 1, \\ 1 & \text{if } (i, j) = (m + n, e + m + n), \\ 0 & \text{if } j \geq e + i \text{ and } \\ & (i, j) \neq (m + n, e + m + n). \end{cases}
 \end{aligned}$$

**3.2. Arbitrarily many summands.** For  $i = 1, \dots, r$ , consider standard graded polynomial rings  $R_i = K[x_{i,1}, \dots, x_{i,n_i}]$  with irrelevant maximal ideals  $\mathbf{x}_i = (x_{i,1}, \dots, x_{i,n_i})$ . Also, let  $Q = R_1 \otimes_K \dots \otimes_K R_r$ . In the following, we abuse notation to write  $\mathbf{x}_i$  to also denote the extensions of these ideals to ideals of  $Q$ .

**Lemma 3.9.** Consider ideals  $I_1, \dots, I_r$  of a commutative ring  $P$  and the map  $\varphi: P/\bigcap_{j=1}^r I_j \xrightarrow{\varphi} \bigoplus_{j=1}^r P/I_j$ , where  $\varphi$  maps the image of  $p \in P$  in  $P/\bigcap_{j=1}^r I_j$  onto the image of  $(p, \dots, p) \in P^r$  in  $\bigoplus_{j=1}^r P/I_j$ . The annihilator of  $\text{coker } \varphi$  as a  $P$ -module is  $\bigcap_{i=1}^r (I_i + \bigcap_{j \neq i} I_j)$ .

*Proof.* First, we prove the inclusion  $\text{Ann}(\text{coker } \varphi) \subseteq \bigcap_{i=1}^r (I_i + \bigcap_{j \neq i} I_j)$ . Consider  $a \in \text{Ann}(\text{coker } \varphi)$  and let  $m$  be the image of  $(1, 0, \dots, 0) \in P^r$  in  $\bigoplus_{j=1}^r P/I_j$ , so that  $am \in \text{im } \varphi$ . Then there exists  $p \in \bigcap_{j \neq 1} I_j$  such that  $a - p \in I_1$ , which shows that  $a \in I_1 + \bigcap_{j \neq 1} I_j$ . Repeating the argument with  $2 \leq i \leq r$  proves the desired inclusion.

For the reverse inclusion, observe that every  $a \in I_1 + \bigcap_{j \neq 1} I_j$  annihilates the image of  $(1, 0, \dots, 0) \in P^r$  in  $\text{coker } \varphi$ , and similarly for  $2 \leq i \leq r$ .  $\square$

Lemma 3.9 will be applied to the following family of ideals.

**Lemma 3.10.** Define the ideals  $I_j$  of  $Q$  by

$$I_j = \mathbf{x}_1 + \dots + \widehat{\mathbf{x}}_j + \dots + \mathbf{x}_r$$

so that  $Q/I_j \cong R_j$ . Then one has

$$\sum_{1 \leq i \neq j \leq r} \mathbf{x}_i \cap \mathbf{x}_j = \bigcap_{j=1}^r I_j.$$

*Proof.* We proceed by induction on  $r \geq 2$ , the base case follows immediately from the definitions.

Assume the statement holds for  $r-1$ , i.e.

$$\sum_{1 \leq i \neq j \leq r-1} \mathbf{x}_i \cap \mathbf{x}_j = \bigcap_{j=1}^{r-1} \tilde{I}_j,$$

where  $\tilde{I}_j = \mathbf{x}_1 + \dots + \widehat{\mathbf{x}}_j + \dots + \mathbf{x}_{r-1}$ . Then we have

$$\begin{aligned} \bigcap_{j=1}^r I_j &= \bigcap_{j=1}^{r-1} I_j \cap I_r = \bigcap_{j=1}^{r-1} (\tilde{I}_j + \mathbf{x}_r) \cap I_r = \left( \bigcap_{j=1}^{r-1} \tilde{I}_j + \mathbf{x}_r \right) \cap I_r \\ &= \left( \sum_{1 \leq i \neq j \leq r-1} \mathbf{x}_i \cap \mathbf{x}_j + \mathbf{x}_r \right) \cap I_r = \sum_{1 \leq i \neq j \leq r-1} \mathbf{x}_i \cap \mathbf{x}_j + \sum_{1 \leq i \leq r-1} \mathbf{x}_i \cap \mathbf{x}_r \end{aligned}$$

because  $\mathbf{x}_i \cap \mathbf{x}_j \subset I_r$  whenever  $1 \leq i \neq j \leq r-1$ .  $\square$

Combining Lemma 3.9 with Lemma 3.10, we obtain:

**Corollary 3.11.** There is an exact sequence of graded  $Q$ -modules

$$(3.3) \quad 0 \rightarrow Q / \left( \sum_{1 \leq i \neq j \leq r} \mathbf{x}_i \cap \mathbf{x}_j \right) \rightarrow \bigoplus_{j=1}^r R_j \rightarrow K^{r-1} \rightarrow 0.$$

*Proof.* Consider the ideal  $I_1, \dots, I_r$  of  $Q$  as defined in Lemma 3.10. Observe that  $Q/I_j \cong R_j$  and  $\bigcap_{j=1}^r I_j = \sum_{1 \leq i \neq j \leq r} \mathbf{x}_i \cap \mathbf{x}_j$ . Thus, the map in Lemma 3.9 becomes  $\varphi: Q / \left( \sum_{1 \leq i \neq j \leq r} \mathbf{x}_i \cap \mathbf{x}_j \right) \rightarrow \bigoplus_{j=1}^r R_j$ .



The definition of the ideals  $I_j$  implies that for each  $i$ ,  $I_i + \bigcap_{j \neq i} I_j$  is the maximal ideal  $\mathfrak{m}$  generated by all the variables of  $Q$ . Hence, Lemma 3.9 shows that  $\text{Ann}(\text{coker } \varphi) = \mathfrak{m}$ .

Notice that  $\text{im } \varphi$  is a cyclic  $Q$ -module whose minimal generator can also be taken as a minimal generator of  $\bigoplus_{j=1}^r R_j$ . It follows that  $\text{coker } \varphi$  is minimally generated by  $r - 1$  elements of degree zero. Since  $\text{Ann}(\text{coker } \varphi) = \mathfrak{m}$ , we conclude that  $\text{coker } \varphi \cong K^{r-1}$ , which completes the argument.  $\square$

We compute the Betti numbers for the leftmost term in the short exact sequence (3.3).

**Lemma 3.12.** *The ideal  $\sum_{i \neq j} \mathbf{x}_i \cap \mathbf{x}_j$  has a 2-linear minimal free resolution, and for  $t \geq 1$  we have that*

$$\left[ \text{Tor}_t^Q(Q / \sum_{i \neq j} \mathbf{x}_i \cap \mathbf{x}_j) \right]_{t+1} \cong \text{coker} \left( \left[ \bigoplus_{j=1}^r \text{Tor}_{t+1}^Q(Q/I_j) \right]_{t+1} \rightarrow \left[ \bigoplus_{j=1}^{r-1} \text{Tor}_{t+1}^Q(K) \right]_{t+1} \right).$$

Thus with  $N = n_1 + \cdots + n_r$ ,

$$\beta_{t,t+1}^Q \left( Q / \sum_{i \neq j} \mathbf{x}_i \cap \mathbf{x}_j \right) = (r-1) \binom{N}{t+1} - \sum_{k=1}^r \binom{N-n_k}{t+1}.$$

*Proof.* Since  $\text{reg}(Q/I_j) = 0$  for each  $1 \leq j \leq r$  and  $\text{reg}(K) = 0$ , the short exact sequence (3.3) implies, by means of the formula

$$\text{reg} \left( Q / \sum_{i \neq j} \mathbf{x}_i \cap \mathbf{x}_j \right) \leq \max \left\{ \text{reg} \left( \bigoplus_{j=1}^r Q/I_j \right), \text{reg}(K^{r-1}) + 1 \right\} = 1,$$

that  $\sum_{i \neq j} \mathbf{x}_i \cap \mathbf{x}_j$  has a 2-linear minimal free resolution. Moreover, for every  $t \geq 0$ , it induces the following long exact sequence.

$$\begin{aligned} \left[ \text{Tor}_{t+1}^Q \left( Q / \sum_{i \neq j} \mathbf{x}_i \cap \mathbf{x}_j \right) \right]_{t+1} &\rightarrow \left[ \bigoplus_{j=1}^r \text{Tor}_{t+1}^Q(Q/I_j) \right]_{t+1} \\ &\rightarrow \left[ \bigoplus_{j=1}^{r-1} \text{Tor}_{t+1}^Q(K) \right]_{t+1} \rightarrow \left[ \text{Tor}_t^Q \left( Q / \sum_{i \neq j} \mathbf{x}_i \cap \mathbf{x}_j \right) \right]_{t+1} \\ &\rightarrow \left[ \bigoplus_{j=1}^r \text{Tor}_t^Q(Q/I_j) \right]_{t+1}. \end{aligned}$$

Since  $\text{reg}(\sum_{i \neq j} Q/\mathbf{x}_i \cap \mathbf{x}_j) = 1$  and  $\text{reg}(Q/I_j) = 0$ , the left-most and right-most modules in the above long exact sequence are zero. Since the  $Q$ -modules  $Q/I_j$  and  $K$  are minimally resolved by Koszul complexes, the dimensions of the second and third terms of the sequence are given by sums of the appropriate binomial coefficients. Therefore, taking the difference of these dimensions yields the desired formula for the Betti numbers.  $\square$

For every  $i = 1, \dots, r$  we consider a standard graded ring  $A_i = R_i/\mathfrak{a}_i$  with  $\mathfrak{a}_i \subseteq (\mathbf{x}_i)^2$ . We abuse notation to write  $\mathfrak{a}_i$  to also denote the extensions of these ideals to ideals of  $Q$ .

**Theorem 3.13.** *For every  $t \geq 1$ , we have that*

$$[\mathrm{Tor}_t^Q(A_1 \times_K \cdots \times_K A_r)]_s = \begin{cases} 0 & \text{if } s \leq t, \\ \bigoplus_{i=1}^r [\mathrm{Tor}_t^Q(A_i)]_{t+1} \oplus \left[ \mathrm{Tor}_t^Q \left( Q / \sum_{i \neq j} \mathbf{x}_i \cap \mathbf{x}_j \right) \right]_{t+1} & \text{if } s = t+1, \\ \bigoplus_{i=1}^r [\mathrm{Tor}_t^Q(A_i)]_s & \text{if } s \geq t+2. \end{cases}$$

*Proof.* Consider the short exact sequence of graded  $Q$ -modules (2.7)

$$0 \rightarrow A_1 \times_K \cdots \times_K A_r \rightarrow A_1 \oplus \cdots \oplus A_r \rightarrow K^{r-1} \rightarrow 0.$$

For every  $t \geq 0$ , it induces the following long exact sequence.

$$\begin{aligned} \mathrm{Tor}_{t+1}^Q(K^{r-1}) \rightarrow \mathrm{Tor}_t^Q(A_1 \times_K \cdots \times_K A_r) \\ \rightarrow \mathrm{Tor}_t^Q(A_1 \oplus \cdots \oplus A_r) \rightarrow \mathrm{Tor}_t^Q(K^{r-1}). \end{aligned}$$

We have that  $[\mathrm{Tor}_t^Q(K^{r-1})]_s \neq 0$  if and only if  $0 \leq t = s \leq n_1 + \cdots + n_r$ . Thus, for every  $s \geq t+2$ , we get

$$[\mathrm{Tor}_t^Q(A_1 \times_K \cdots \times_K A_r)]_s \cong \bigoplus_{i=1}^r [\mathrm{Tor}_t^Q(A_i)]_s.$$

Restricting to degree  $s = t+1$ , we get the exact sequence:

$$\begin{aligned} \bigoplus_{i=1}^r [\mathrm{Tor}_{t+1}^Q(A_i)]_{t+1} \rightarrow [\mathrm{Tor}_{t+1}^Q(K^{r-1})]_{t+1} \\ \rightarrow [\mathrm{Tor}_t^Q(A_1 \times_K \cdots \times_K A_r)]_{t+1} \rightarrow \bigoplus_{i=1}^r [\mathrm{Tor}_t^Q(A_i)]_{t+1} \rightarrow 0. \end{aligned}$$

For every  $1 \leq i \leq r$ , we have  $A_i = Q/(I_i, \mathfrak{a}_i)$ , and since we assume that each  $\mathfrak{a}_i$  is generated in degrees at least two, we also have

$$\bigoplus_{i=1}^r [\mathrm{Tor}_{t+1}^Q(A_i)]_{t+1} \cong \bigoplus_{i=1}^r [\mathrm{Tor}_{t+1}^Q(Q/I_i)]_{t+1}.$$

This implies that

$$\begin{aligned} [\mathrm{Tor}_t^Q(A_1 \times_K \cdots \times_K A_r)]_{t+1} &\cong \bigoplus_{i=1}^r [\mathrm{Tor}_t^Q(A_i)]_{t+1} \\ &\oplus \mathrm{coker} \left( \bigoplus_{i=1}^r [\mathrm{Tor}_{t+1}^Q(Q/I_i)]_{t+1} \rightarrow \bigoplus_{i=1}^{r-1} [\mathrm{Tor}_{t+1}^Q(K)]_{t+1} \right). \end{aligned}$$

Using Lemma 3.12, we get the desired formula.  $\square$

From the previous result, we can compute the graded Betti numbers of the fiber product  $A_1 \times_K \cdots \times_K A_r$  in terms of the graded Betti numbers of the  $A_i$  as  $Q$ -modules. A straightforward computation allows us to translate into a formula depending only on the Betti numbers of the  $A_i$  as  $R_i$ -modules.

**Corollary 3.14.** *With  $N = n_1 + \cdots + n_r$ , we have*

$$\begin{aligned} P_{A_1 \times_K \cdots \times_K A_r}^Q(t, s) &= \sum_{i=1}^r (P_{A_i}^{R_i}(t, s) - 1)(1 + st)^{N-n_i} \\ &\quad + (r-1) \frac{(1 + st)^N - Nst - 1}{t} \\ &\quad - \sum_{i=1}^r \frac{(1 + st)^{N-n_i} - (N - n_i)st - 1}{t} + 1. \end{aligned}$$

*Proof.* The generating series is derived from Theorem 3.13, the numerical formula in Lemma 3.12, and an analogue of Proposition 3.4.  $\square$

Recall that it is harmless to assume that an AG  $K$ -algebra has socle degree at least three because AG algebras with smaller socle degrees are well understood (see the description above in Theorem 3.7).

**Theorem 3.15.** *Assume that  $A_1, \dots, A_r$  are AG  $K$ -algebras with  $\text{reg}(A_\ell) = e \geq 3$  for all  $1 \leq \ell \leq r$ . Then, for any integer  $t \geq 0$ , the graded Betti numbers of the connected sum  $A_1 \#_K \cdots \#_K A_r$  over the polynomial ring  $Q$  with  $\dim(Q) = N$  are given by*

$$[\text{Tor}_t^Q(A_1 \#_K \cdots \#_K A_r)]_s \cong \begin{cases} 0 & \text{if } s \leq t \text{ and } (t, s) \neq (0, 0), \\ K & \text{if } (t, s) = (0, 0), \\ \bigoplus_{i=1}^r [\text{Tor}_t^Q(A_i)]_{t+1} \\ \oplus [\text{Tor}_t^Q(Q / \sum_{i \neq j} \mathbf{x}_i \cap \mathbf{x}_j)]_{t+1} & \text{if } t = s + 1, \\ \bigoplus_{i=1}^r [\text{Tor}_t^Q(A_i)]_s & \text{if } t + 2 \leq s \leq t + e - 2, \\ \bigoplus_{i=1}^r [\text{Tor}_t^Q(A_i)]_s \\ \oplus [\text{Tor}_{N-t}^Q(Q / \sum_{i \neq j} \mathbf{x}_i \cap \mathbf{x}_j)]_{N-t+1} & \text{if } s = t + e - 1, \\ K & \text{if } (t, s) = (N, e + N), \\ 0 & \text{if } s \geq e + t \text{ and} \\ & (t, s) \neq (N, e + N). \end{cases}$$

*Proof.* We denote  $C = A_1 \#_K \cdots \#_K A_r$  and  $D = A_1 \times_K \cdots \times_K A_r$ , and consider the exact sequence of graded  $Q$ -modules (2.12)

$$0 \rightarrow K^{r-1}(-e) \rightarrow D \rightarrow C \rightarrow 0.$$

Its long exact Tor sequence gives exact sequences

$$[\text{Tor}_t^Q(K^{r-1})]_{s-e} \rightarrow [\text{Tor}_t^Q(D)]_s \rightarrow [\text{Tor}_t^Q(C)]_s \rightarrow [\text{Tor}_{t-1}^Q(K^{r-1})]_{s-e}.$$

Since  $\text{Tor}_t^Q(K)$  is concentrated in degree  $t$  we conclude that

$$[\text{Tor}_t^Q(C)]_s \cong [\text{Tor}_t^Q(D)]_s$$

if  $s \notin \{e+t-1, e+t\}$ . Combined with Theorem 3.13, this determines  $[\mathrm{Tor}_t^Q(C)]_s$  if  $s \leq e+t-2$ .

Using that  $\mathrm{reg}(C) = \mathrm{reg} A_i = e$ , we know  $[\mathrm{Tor}_t^Q(C)]_s = 0$  if  $s \geq e+t+1$ . It remains to determine  $[\mathrm{Tor}_t^Q(C)]_s$  if  $s \in \{e+t-1, e+i\}$ . To this end, we utilize the fact that  $C$  is Gorenstein. Thus, its graded minimal free resolution is symmetric. In particular, one has

$$[\mathrm{Tor}_t^Q(C)]_s \cong [\mathrm{Tor}_{N-t}^Q(C)]_{e+N-s}$$

and similarly, we have  $\mathrm{Tor}_t^Q(A_i)_s \cong [\mathrm{Tor}_{N-t}^Q(A)]_{e+N-s}$  for each  $A_i$ .

Combined with Theorem 3.13 and using  $e \geq 3$ , which implies that the degrees  $e+i-1, e+i$  are not self-dual under the isomorphisms given above, the claim regarding the Tor modules follows.  $\square$

**Corollary 3.16.** *With the notation of Theorem 3.15, we have*

$$\begin{aligned} P_{A_1 \#_K \dots \#_K A_r}^Q(t, s) &= \sum_{i=1}^r (P_{A_i}^{R_i}(t, s) - 1)(1+st)^{N-n_i} + 1 \\ &\quad + (r-1) \frac{(1+st)^N - Nst - 1}{t} \\ &\quad - \sum_{i=1}^r \frac{(1+st)^{N-n_i} - (N-n_i)st - 1}{t} \\ &\quad + (r-1)s^{N+e}t^{N+1}[(1+s^{-1}t^{-1})^N - \frac{N}{st} - 1] \\ &\quad - s^{N+e}t^{N+1} \sum_{i=1}^r \left[ (1+s^{-1}t^{-1})^{N-n_i} - \frac{N-n_i}{st} - 1 \right]. \end{aligned}$$

*Proof.* As a first step, we show

$$\begin{aligned} P_{A_1 \#_K \dots \#_K A_r}^Q(t, s) &= \sum_{i=1}^r P_{A_i}^Q(t, s) + P_{Q/\sum_{i \neq j} \mathbf{x}_i \cap \mathbf{x}_j}^Q(t, s) \\ &\quad + s^{N+e}t^N P_{Q/\sum_{i \neq j} \mathbf{x}_i \cap \mathbf{x}_j}^Q(t^{-1}, s^{-1}) - 2. \end{aligned}$$

This formula follows from Theorem 3.15 and the identities

$$\begin{aligned} &\sum_{u=0}^N \beta_{N-u, N-u+1} \left( Q / \sum_{i \neq j} \mathbf{x}_i \cap \mathbf{x}_j \right) t^u s^{u+e-1} \\ &= \sum_{v=0}^N \beta_{v, v+1} \left( Q / \sum_{i \neq j} \mathbf{x}_i \cap \mathbf{x}_j \right) t^{N-v} s^{N-v+e-1} \\ &= t^N s^{N+e} \sum_{v=0}^N \beta_{v, v+1} \left( Q / \sum_{i \neq j} \mathbf{x}_i \cap \mathbf{x}_j \right) t^{-v} s^{-v-1} \\ &= t^N s^{N+e} P_{Q/\sum_{i \neq j} \mathbf{x}_i \cap \mathbf{x}_j}^Q(t^{-1}, s^{-1}). \end{aligned}$$

Substituting the formulas of Corollary 3.14 and Lemma 3.12 into the formula above yields the claim.  $\square$

TABLE 3. Betti tables of  $R/I$  and  $R/J$  in Example 4.1

	0	1	2	3		0	1	2	3
total	1	3	3	1	total	1	3	3	1
0:	1	.	.	.	0:	1	.	.	.
1:	.	3	2	.	1:	.	3	2	.
2:	.	2	3	.					
3:	.	.	.	1					

## 4. CONNECTED SUM AS A DOUBLING

**4.1. Motivating examples.** We discuss examples of monomial complete intersections. Using the so-called doubling method, Celikbas, Laxmi and Weyman solved a particular case of Questions 1.1 and 1.2. Indeed, in [CLW19, Corollary 6.3], they determine a minimal free resolution of the connected sum of  $K$ -algebras  $A_i = K[x_i]/(x_i^{d_i})$  by using the doubling construction. The goal of this section is to generalize their result to AG  $K$ -algebras with the same socle degree. We start with a toy example.

**Example 4.1.** The Betti table of the connected sum

$$C = \frac{K[x]}{(x^4)} \#_K \frac{K[y]}{(y^4)} \#_K \frac{K[z]}{(z^4)}$$

is described on the left in Table 3.

It should be understood as follows. The connected sum  $C$  has the presentation

$$C = \frac{K[x, y, z]}{(xy, xz, yz, x^3 + y^3, x^3 + z^3)}.$$

Let  $Q = K[x, y, z]$ ,  $I = (xy, xz, yz, x^3 + y^3, x^3 + z^3)$  and  $J = (xy, xz, yz)$ . Then the Betti table of  $Q/J$  is given on the right in Table 3. It follows from this that  $\omega_{Q/J}$  has two generators and there is an exact sequence

$$0 \rightarrow \omega_{Q/J}(-3) \rightarrow Q/J \rightarrow C \rightarrow 0,$$

which maps the generators of  $\omega_{Q/J}$  to the elements  $x^3 + y^3$  and  $x^3 + z^3$  in  $Q/J$ . The resolution of  $C$  is obtained as a mapping cone from the previous exact sequence.

Each of the summands in  $C$  is obtained by doubling a polynomial ring. Indeed, the short exact sequence

$$0 \rightarrow \omega_{K[x]}(-3) \rightarrow K[x] \rightarrow \frac{K[x]}{(x^4)} \rightarrow 0$$

sending the generator of  $\omega_{K[x]} \cong K[x](-1)$  to  $x^4$ , shows that  $K[x]/(x^4)$  is a doubling of  $K[x]$ . Similarly, the remaining summands are doublings of  $K[y]$  and  $K[z]$ , respectively. Furthermore, the ring  $Q/J$  from above can be identified with the fiber product of the rings being doubled

$$Q/J = K[x] \times_K K[y] \times_K K[z].$$

Example 4.2 is the first generalization of the [CLW19, Corollary 6.3] to every monomial complete intersection.

**Example 4.2.** We focus on the connected sum  $A = A_1 \#_K \cdots \#_K A_r$  of complete intersection algebras

$$A_i := K[x_{i,1}, \dots, x_{i,n_i}] / (x_{i,1}^{d_{i,1}}, \dots, x_{i,n_i}^{d_{i,n_i}})$$

for  $i = 1, \dots, r$ , satisfying

$$(4.1) \quad \sum_{j=1}^{n_i} d_{i,j} - n_i = \sum_{j=1}^{n_{i'}} d_{i',j} - n_{i'} \text{ whenever } 1 \leq i, i' \leq r.$$

Let  $R_i = K[x_{i,1}, \dots, x_{i,n_i}]$ ,  $Q = R_1 \otimes_K \cdots \otimes_K R_r$  and let  $c$  be the quantity defined in (4.1). The connected sum of the  $K$ -algebras  $A_i$  admits the presentation  $A \cong Q/I$  where

$$\begin{aligned} I = & (x_{i,j_i} x_{h,j_h} \mid 1 \leq i < h \leq r, 1 \leq j_i \leq n_i, 1 \leq j_h \leq n_h) \\ & + \left( x_{i,l_i}^{d_{i,l_i}} \mid 1 \leq i \leq r, 1 \leq l_i \leq n_i - 1 \right) \\ & + \left( x_{i,1}^{d_{i,1}-1} \cdots x_{i,n_i}^{d_{i,n_i}-1} + x_{1,1}^{d_{1,1}-1} \cdots x_{1,n_1}^{d_{1,n_1}-1} \mid 2 \leq i \leq r \right). \end{aligned}$$

It can be verified that  $A$  is a doubling of  $\tilde{A} = Q/J$ , where  $J$  is an ideal defining  $r$  co-ordinate points in  $\mathbb{A}_K^{n_1} \times \cdots \times \mathbb{A}_K^{n_r}$  with multiplicity; more precisely,  $J = \bigcap_{i=1, \dots, r} J_i$ , where

$$J_i = \left( x_{j,h}, x_{i,l_i}^{d_{i,l_i}} \mid j \neq i, 1 \leq h \leq n_j, 1 \leq l_i \leq n_i - 1 \right).$$

More importantly, setting  $\tilde{A}_i = R_i / (x_{i,l_i}^{d_{i,l_i}} \mid 1 \leq l_i \leq n_i - 1)$ , we see that each ring  $A_i$  is a doubling of  $\tilde{A}_i$  via the sequence

$$0 \rightarrow \omega_{\tilde{A}_i}(-c) \rightarrow \tilde{A}_i \rightarrow A_i \rightarrow 0$$

sending the generator of  $\omega_{\tilde{A}_i}(-c) \cong \tilde{A}_i(-d_{i,n_i})$  to  $x_{i,n_i}^{d_{i,n_i}}$ , and that  $\tilde{A} = \tilde{A}_1 \times_K \cdots \times_K \tilde{A}_r$ . The Betti numbers of  $\tilde{A}$  can thus be obtained via Corollary 3.14.

We shall explain this observation as part of a general phenomenon in the following result.

**Theorem 4.3.** *Let  $A_1, \dots, A_r$  be graded AG  $K$ -algebras with  $\text{reg}(A_i) = d$  for all  $1 \leq i, j \leq r$ . Suppose that for each  $1 \leq i \leq r$ ,  $A_i$  is a doubling of some 1-dimensional Cohen-Macaulay algebra  $\tilde{A}_i$ , then the connected sum  $A_1 \#_K \cdots \#_K A_r$  is a doubling of  $\tilde{A}_1 \times_K \cdots \times_K \tilde{A}_r$ .*

*Proof.* We proceed by induction on  $r$ . We first prove the base case where  $r = 2$ . Set  $\tilde{A}_1 = R/\tilde{\mathfrak{a}}_1$  and  $\tilde{A}_2 = S/\tilde{\mathfrak{a}}_2$  and let  $Q = R_1 \otimes_K R_2$ . By [AAM12, Lemma 1.5] the ring  $\tilde{A}_1 \times_K \tilde{A}_2$  is Cohen Macaulay of dimension one. By Lemma 2.19, our assumptions imply that for each  $i$  we have exact sequences

$$(4.2) \quad 0 \rightarrow \omega_{\tilde{A}_i}(-d) \rightarrow \tilde{A}_i \rightarrow A_i \rightarrow 0.$$

Considering these in degree zero we conclude that

$$(4.3) \quad [\omega_{\tilde{A}_i}]_{-d} = 0.$$

Combining the exact sequences (4.2) for  $i \in \{1, 2\}$  with the sequence in (2.3), we obtain the following commutative diagram of  $Q$ -modules with exact rows and

middle column.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & (\omega_{\tilde{A}_1} \oplus \omega_{\tilde{A}_2})(-d) & \xrightarrow{=} & (\omega_{\tilde{A}_1} \oplus \omega_{\tilde{A}_2})(-d) & & \\
 & & \downarrow & & \downarrow & & \\
 (4.4) \quad 0 & \longrightarrow & \tilde{A}_1 \times_K \tilde{A}_2 & \xrightarrow{\sigma} & \tilde{A}_1 \oplus \tilde{A}_2 & \xrightarrow{\mu} & K \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow = \\
 0 & \longrightarrow & A_1 \times_K A_2 & \longrightarrow & A_1 \oplus A_2 & \longrightarrow & K \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

The vertical map  $\tilde{A}_1 \times_K \tilde{A}_2 \rightarrow A_1 \times_K A_2$  in (4.4) is uniquely determined by viewing  $A_1 \times_K A_2$  as a pullback in the category of  $K$ -algebras and utilizing the universal property of this categorical construction. Moreover, by the snake lemma, the kernel of this map is the module  $(\omega_{\tilde{A}_1} \oplus \omega_{\tilde{A}_2})(-d)$ .

Applying the functor  $\text{Hom}(-, Q)$  to the diagram (4.4) yields a new commutative diagram (4.5). The middle row in (4.5) comes from the top of (4.4), and the top row of (4.5) contains the non-vanishing Ext modules for the  $Q$ -modules in the middle row of (4.4). According to Remark 2.2, the map marked  $\nu$  satisfies  $\nu(1) = (\tau_{A_1}, \tau_{A_2})$  after identifying  $\omega_{A_1} \oplus \omega_{A_2} \cong A_1 \oplus A_2$ .

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longleftarrow & K & \longleftarrow & \omega_{\tilde{A}_1 \times_K \tilde{A}_2} & \longleftarrow & \omega_{\tilde{A}_1} \oplus \omega_{\tilde{A}_2} \longleftarrow 0 \\
 & & \downarrow \eta & & \downarrow & & \\
 (4.5) \quad & & (\tilde{A}_1 \oplus \tilde{A}_2)(d) & \xleftarrow{=} & (\tilde{A}_1 \oplus \tilde{A}_2)(d) & & \\
 & & \downarrow & & \downarrow \chi & & \\
 0 & \longleftarrow & \omega_{A_1 \times_K A_2} & \longleftarrow & \omega_{A_1} \oplus \omega_{A_2} & \longleftarrow & K \longleftarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

The snake lemma applied to (4.5) yields a connecting isomorphism  $\theta: K \rightarrow K$ . Let  $s \in \omega_{\tilde{A}_1 \times_K \tilde{A}_2}$  be such that  $\xi(s) = \theta(1)$ . Then  $\chi(\eta(s)) = \nu(1)$  can be identified with  $(\tau_{A_1}, \tau_{A_2}) \in A_1 \oplus A_2$ , that is,  $\eta(s)$  is equivalent to  $(\tau_{A_1}, \tau_{A_2})$  modulo the image of  $\omega_{\tilde{A}_1} \oplus \omega_{\tilde{A}_2}$ .

We want to compare the image of

$$\eta[-d]: \omega_{\tilde{A}_1 \times_K \tilde{A}_2}(-d) \rightarrow \tilde{A}_1 \oplus \tilde{A}_2$$

and the kernel of the map  $\mu$  from Diagram (4.4). The image of  $\eta[-d]$  is trivial in degree zero by Equation (4.3). Since  $K$  is concentrated in degree zero, the map  $\mu$

has zero image in every degree other than zero. It follows that the image of  $\eta[-d]$  is contained in  $\ker \mu = \text{im } \sigma \cong \tilde{A}_1 \times_K \tilde{A}_2$ . Hence  $\eta[-d]$  induces an injective graded  $Q$ -module homomorphism

$$\delta: \omega_{\tilde{A}_1 \times_K \tilde{A}_2}(-d) \rightarrow \tilde{A}_1 \times_K \tilde{A}_2.$$

Its existence proves that  $\omega_{\tilde{A}_1 \times_K \tilde{A}_2}(-d)$  can be identified with an ideal of  $\tilde{A}_1 \times_K \tilde{A}_2$ .

The following diagram combines the left column of Diagram (4.4) and the top row of (4.5). By previous considerations indicating that  $\delta(s) = \eta(s)$  is equivalent to  $(\tau_{A_1}, \tau_{A_2})$  modulo the image of  $\omega_{\tilde{A}_1} \oplus \omega_{\tilde{A}_2}$ , the diagram commutes provided that  $\xi(s)$  is mapped by  $\tau$  to  $(\tau_{A_1}, \tau_{A_2}) \in A_1 \times_K A_2$ . With this choice, the cokernel of  $\tau$  is  $A_1 \#_K A_2$  by Definition 2.10.

$$\begin{array}{ccccccc}
 & 0 & & 0 & & & \\
 & \downarrow & & \downarrow & & & \\
 & (\omega_{\tilde{A}_1} \oplus \omega_{\tilde{A}_2})(-d) & \xrightarrow{=} & (\omega_{\tilde{A}_1} \oplus \omega_{\tilde{A}_2})(-d) & & & \\
 & \downarrow & & \downarrow & & & \\
 0 \longrightarrow & \omega_{\tilde{A}_1 \times_K \tilde{A}_2}(-d) & \xrightarrow{\delta} & \tilde{A}_1 \times_K \tilde{A}_2 & \longrightarrow & C & \longrightarrow 0 \\
 & \downarrow \xi & & \downarrow & & & \\
 0 \longrightarrow & K(-d) & \xrightarrow{\tau} & A_1 \times_K A_2 & \longrightarrow & A_1 \#_K A_2 & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & & \\
 & 0 & & 0 & & & 
 \end{array}$$

Setting  $C$  be the cokernel of  $\delta$ , the snake lemma yields an isomorphism  $C \cong A_1 \#_K A_2$ . This shows that  $A_1 \#_K A_2$  is a doubling of  $\tilde{A}_1 \times_K \tilde{A}_2$ , as desired for the base case of induction.

Now, we assume that the AG  $K$ -algebra  $A_1 \#_K \cdots \#_K A_{r-1}$  is a doubling of  $\tilde{A}_1 \times_K \cdots \times_K \tilde{A}_{r-1}$ . The base case applied to AG  $K$ -algebras  $A_1 \#_K \cdots \#_K A_{r-1}$  and  $A_r$  implies by way of Remarks 2.8 and 2.15 that  $A_1 \#_K \cdots \#_K A_r$  is a doubling of  $\tilde{A}_1 \times_K \cdots \times_K \tilde{A}_r$  completing the proof.  $\square$

Theorem 4.3 generalizes [CLW19, Theorem 5.5], which considered the case of AG algebras  $A_1, \dots, A_r$  of embedding dimension one, establishing an analogous doubling result.

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