

Prolonged hysteresis in the Kuramoto model with inertia and higher-order interactions

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The inclusion of inertia in the Kuramoto model has long been reported to change the nature of a phase transition, providing a fertile ground to model the dynamical behaviors of interacting units. More recently, higher-order interactions have been realized as essential for the functioning of real-world complex systems ranging from the brain to disease spreading. Yet analytical insights to decipher the role of inertia with higher-order interactions remain challenging. Here, we study the Kuramoto model with inertia on simplicial complexes, merging two research domains. We develop an analytical framework in a mean-field setting using self-consistent equations to describe the steady-state behavior, which reveals a prolonged hysteresis in the synchronization profile. Inertia and triadic interaction strength exhibit isolated influence on system dynamics by predominantly governing, respectively, the forward and backward transition points. This paper sets a paradigm to deepen our understanding of real-world complex systems such as power grids modeled as the Kuramoto model with inertia.

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I. INTRODUCTION

The emergence of collective behavior in complex real-world systems has been of long-standing research interest [1]. It was initially in a landmark paper [2] that Kuramoto modeled the phenomenon of synchronization using a system of network-coupled oscillators in an analytically tractable setting, illustrating that the system underwent a second-order phase transition from an incoherent to a coherent state. Since then, numerous works on various extensions of the Kuramoto model have been done, revealing several phenomena [3–7]. Of particular interest to us is the Kuramoto model with inertia (also known as the second-order Kuramoto model). Inspired by the modeling of synchronized flashing in *Pterop- tix malaccae* by Ermentrout [8], a second-order extension of the Kuramoto model was proposed by Tanaka *et al.* [9,10]. They showed that the system experienced a first-order phase transition upon introducing inertia rather than the smooth second-order phase transition observed in the Kuramoto model. They put forth a self-consistent method akin to the one proposed by Kuramoto to study the steady-state behavior of the coupled oscillator system. Since then, the second-order Kuramoto model has been extensively explored in diluted networks [11] and various real-world complex systems like Josephson junctions [12] and power grids [13–17]. In Ref. [18], Filatrella *et al.* explained how the second-order Kuramoto model originates in power grids by simply accounting for power conservation at each node of the grid, rendering it more than just a mathematical convenience.

However, all these results were obtained by focusing on the interactions to be purely dyadic in nature. Recent research highlights that such a reductionist view might not reveal the

complete picture of the underlying mechanism of exotic phenomena observed in some real-world complex systems where the interactions between agents are inherently higher-order in nature [19–22]. Using the Ott-Antonsen (OA) dimensionality reduction method [23], Skardal and Arenas [24] showed that incorporating higher-order interactions into the Kuramoto model resulted in abrupt (de)synchronization transitions. It was remarkable to observe that adding an inertia term to the Kuramoto model or incorporating higher-order interactions independently gave rise to a first-order phase transition in the system. Here we are interested in understanding how the interplay of inertia and higher-order interactions manifests itself in the system and affects the synchronization profile.

This paper unifies these two disparate fields by providing a generalized analytical framework to predict the steady-state behavior of coupled oscillator systems with inertia interacting via higher-order interactions. The first challenge lies in the fact that using dimensionality reduction methods like the OA does not carry over easily towards analyzing second-order Kuramoto models [25] as the density function, which is usually Fourier expanded in terms of the phase, now also depends on the velocity of the oscillators. Hence, in this paper, we develop analytics using the self-consistency methodology originally proposed by Kuramoto and later extended to the second-order Kuramoto model by Tanaka *et al.* [9]. We show that the effect of inertia and higher-order interactions manifest independently, resulting in a prolonged hysteresis-driven synchronization transition within the system. The forward and backward transition points are found to be predominantly dependent on inertia and higher-order interaction, respectively, Fig. 1.

II. MODEL

We study the inertial effects in a globally coupled oscillator network considering the simultaneous presence of dyadic and

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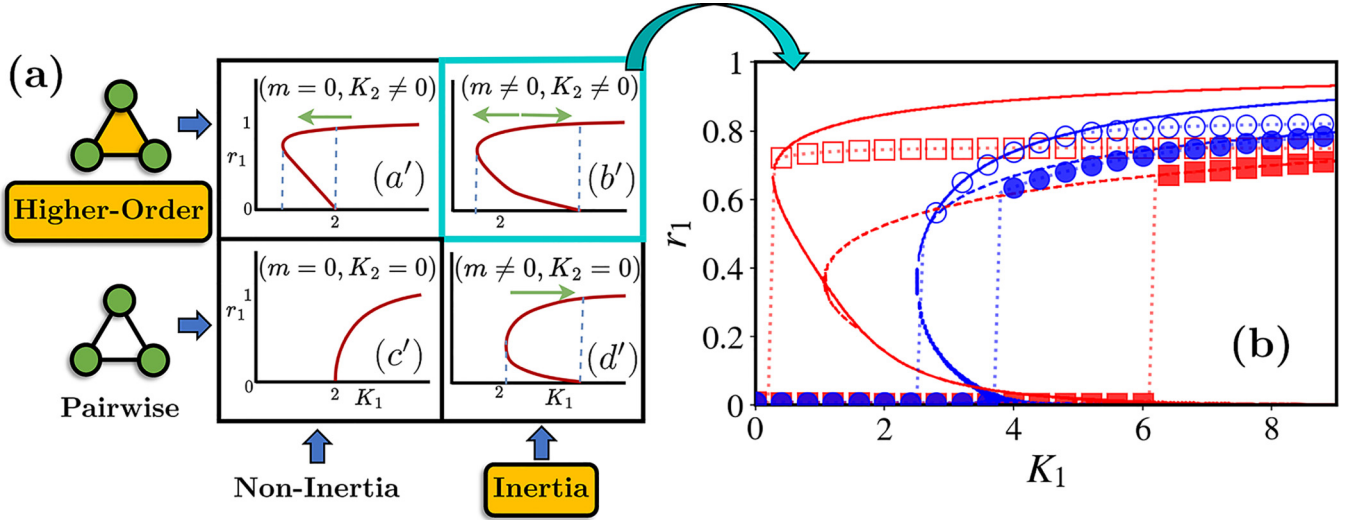


FIG. 1. Prolonged hysteresis. (a) Schematic depiction of emerging collective behavior in the Kuramoto model (KM). (a'), (c'), and (d') plot the usual behavior of KM [2] in the sole impression of higher-order [24] or inertia [9], whereas (b') illustrates a simultaneous forward and backward shift in the transition points upon introduction of m and K_2 in KM [Eq. (1)], revealing a prolonged hysteresis. The green arrow indicates the direction of the shift in the transition points. (b) r_1 versus K_1 plot for $K_2 = 1$ and $m = 1$ (blue circles) and $K_2 = 7$ and $m = 3$ (red squares). Filled circles and squares represent the simulation results for the forward, and hollow circles and squares represent the backward processes. The dashed and continuous curves represent the forward and backward analytical predictions, respectively.

triadic interactions. Phases of N -coupled oscillators, each with mass m , evolve based on the following coupled nonlinear equations:

$$m\ddot{\theta}_i = -\dot{\theta}_i + \omega_i + \frac{K_1}{N} \sum_{j=1}^N \sin(\theta_j - \theta_i) + \frac{K_2}{N^2} \sum_{j=1}^N \sum_{k=1}^N \sin(2\theta_j - \theta_k - \theta_i). \quad (1)$$

In Eq. (1), θ_i and $\dot{\theta}_i$ refer to the instantaneous phase and angular velocity of the i th oscillator, respectively. ω_i is the intrinsic frequency of the i th oscillator derived from a unimodal symmetric probability distribution $g(\omega)$ with mean Ω . The coupling constants $K_1 \geq 0$ and K_2 are the dyadic and triadic coupling strengths, respectively. The form of the higher order coupling term directly falls from the phase reduction of mean-field complex Ginzburg-Landau equation [26].

We decouple the differential equations in Eq. (1) and write them in terms of mean-field quantities by introducing the following general order parameter for $p \in \{1, 2\}$:

$$r_p e^{i\psi_p} = \frac{1}{N} \sum_{j=1}^N e^{ip\theta_j}. \quad (2)$$

From the above definition, r_1 measures the global phase coherence and can be interpreted as the centroid of phases of oscillators on a unit circle in the complex plane and ψ_1 measures the average phase of the oscillators. r_2 , referred to as the Daido order parameter [27], captures cluster synchronization. As we are interested in the steady-state behavior of the system, we omit the time dependence in the definition of the general order parameter. In the incoherent state, the phases of the oscillators are scattered uniformly on the unit circle and hence $r_1 \approx r_2 \approx 0$. Meanwhile, in the coherent state, a single group

of oscillators is formed locked to the mean phase ψ_1 rotating uniformly at angular velocity Ω , hence $r_1 \approx r_2 \approx 1$. Using Eq. (2), Eq. (1) can be written in terms of mean-field quantities as

$$m\ddot{\theta}_i = -\dot{\theta}_i + \omega_i + K_1 r_1 \sin(\psi_1 - \theta_i) + K_2 r_1 r_2 \sin(\psi_2 - \psi_1 - \theta_i). \quad (3)$$

Because of the rotational symmetry in the model, the mean of the $g(\omega)$ distribution can be set to zero by moving into the rotating frame at the frequency Ω . This can be facilitated by making the transformation $\theta_i \rightarrow \theta_i + \Omega t$ in Eq. (1). Once in the rotating frame, ψ_1 and ψ_2 can be set to zero by appropriately shifting the origin, i.e., $\theta_i(0) \rightarrow \theta_i(0) + \psi_1(0)$. Equation (3) now takes the following form:

$$m\ddot{\theta}_i = -\dot{\theta}_i + \omega_i - q \sin(\theta_i), \quad (4)$$

where, for the ease of notation, $q = r_1(K_1 + K_2 r_2)$.

III. ANALYTICAL RESULTS

Note that for fixed parameter values K_1 and K_2 , Eq. (4) has two variables r_1 and r_2 . Hence, to chalk out the steady-state behavior of Eq. (4), we develop a system of self-consistent equations and seek the values of (K_1, r_1, r_2) which simultaneously satisfy them. We start by taking the thermodynamic limit ($N \rightarrow \infty$); the coupled oscillator system in the steady state is then described by a probability density $\rho(\theta, \omega)$ where for a given intrinsic frequency ω , $\rho(\theta, \omega)d\theta$ represents the fraction of oscillators with their phase between θ and $\theta + d\theta$. The general order parameter in Eq. (2) takes the following form in the continuum limit:

$$r_p e^{i\psi_p} = \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} e^{ip\theta} \rho(\theta, \omega) g(\omega) d\omega d\theta.$$

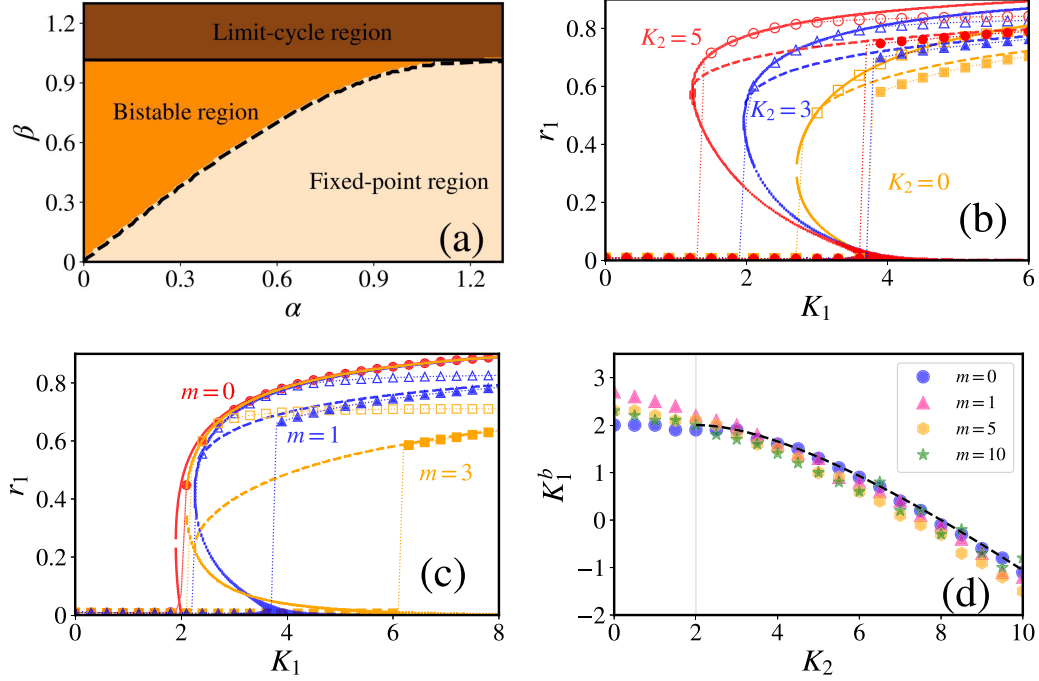


FIG. 2. (a) $\alpha(=1/\sqrt{qm}) - \beta(=\omega/q)$ parameter space. Different dynamical regimes are present in the $\dot{\theta}$ vs θ phase space of Eq. (5). (b) Synchronization profile r_1 versus K_1 for $m=1$ and different values of $K_2=0$ (orange squares), 3 (blue triangles), and 5 (red circles). (c) Synchronization profile for a fixed value of $K_2=2$ and different values of $m=3$ (orange squares), 1 (blue triangles), and 0 (red circles). In both (b) and (c), the filled and hollow symbols indicate the simulation results for the forward and backward cases, respectively. The dashed and continuous curves represent the analytically calculated values for the forward and backward processes, respectively. (d) Backward transition points. The dashed curve represents the analytical predictions of K_1^b for $m=0$ and different K_2 ($K_1^b = 2\sqrt{2K_2} - K_2$ as derived in Appendix A 1 for $K_2 \geq 2$). The scatter plots are K_1^b vs K_2 for different $m=0, 1, 5, 10$ obtained via numerical simulation.

In the steady state, the oscillator population splits into two groups, depending on their intrinsic frequency. One group of oscillators is locked to the mean phase; meanwhile, the other oscillators drift over the locked oscillators. Hence the overall phase coherence (r_p) can be split into contributions from the locked (r_p^l) and drifting (r_p^d) oscillators, i.e., $r_p = r_p^l + r_p^d$. Before calculating r_p^l and r_p^d , we point out that systems whose motion is governed by Eq. (4) are known to depict hysteresis and have been well studied in Refs. [9,10,28]. For completeness, we briefly summarize the reason for the hysteretic behavior here. Dropping subscript i and introducing a new timescale $\tau = \sqrt{\frac{q}{m}}t$, Eq. (4) is transformed to a second-order differential equation with just two parameters as

$$\ddot{\theta} = -\alpha\dot{\theta} + \beta - \sin(\theta), \quad (5)$$

where $\alpha = \frac{1}{\sqrt{qm}}$ and $\beta = \frac{\omega}{q}$. This equation has two fixed-point solutions, a saddle point and a stable node for $\beta < 1$, obtained by setting $\dot{\theta} = 0$ and $\ddot{\theta} = 0$. At $\beta = 1$, the system undergoes a saddle-node bifurcation, annihilating the two fixed-point solutions and admitting a unique stable limit cycle solution for all $\beta > 1$ [29]. However, it so happens that as we decrease the value of β to be less than one, the limit cycle persists for some small values of α . Hence, bistability exists in the system, where a stable limit cycle and a stable node coexist. A further decrease in β will disintegrate the limit cycle via a homoclinic bifurcation. Figure 2(a) displays these three dynamical regimes in the $\alpha - \beta$ parameter space. For small values of the damping term α , ensured by keeping finite

inertia, the homoclinic bifurcation curve is seen to be approximated by a straight line in Fig. 2(a). Upon implementing Melnikov's method [28,30], the equation of the straight line comes out to be $\beta = \frac{4}{\pi}\alpha$. In conclusion, we see the presence of three different dynamical regimes, namely, a limit-cycle regime ($\beta > 1$), a bistable regime which can be approximated by ($\frac{4}{\pi}\alpha < \beta \leq 1$), and a fixed point regime approximated by ($\beta \leq \frac{4}{\pi}\alpha$) [28].

The bistable region is responsible for hysteresis in systems governed by equations like Eq. (5). Hence, following Ref. [9], instead of studying the system in its full generality, we break down the self-consistency analysis for our model into forward (f) and backward (b) processes. In the forward process, we start from a small K_1 value and, therefore, the system is in an incoherent state ($r_1 \approx 0$). This leads to high α and β values, indicating that the oscillators are in the limit cycle regime. As we adiabatically increase K_1 , the oscillators stay in the basin of attraction of the stable limit cycle even after crossing $\beta = 1$ ($\omega = q$) and fall into the locked cluster only after $\beta \approx \frac{4}{\pi}\alpha$ ($\omega \approx \frac{4}{\pi}\sqrt{\frac{q}{m}}$), below which the limit cycle vanishes. For the backward process, we start from a high K_1 value, and hence the oscillators exist in the fixed point regime, i.e., the oscillators are locked in a cluster and therefore the system is in a coherent state ($0 \ll r_1 < 1$). As we adiabatically decrease K_1 , the oscillators remain in the basin of attraction of the stable node until $\beta = 1$, when the stable node vanishes via a saddle-node bifurcation. Thus, in the backward process, oscillators having $|\omega| \leq q = \omega_b$ contribute

to the locked oscillators, while in the forward process, only those with $|\omega| \leq \frac{4}{\pi} \sqrt{\frac{q}{m}} = \omega_f$ are in a locked state and all the oscillators with $\omega > \omega_{f,b}$ drift around the locked cluster. We point out that K_2 is concealed in q and directly affects the fraction of oscillators in a locked or drifting state.

The contribution of the locked oscillator (r_p^l) to overall coherence for the forward/backward process can now be calculated as $r_p^l = \int_{-\omega_{f,b}}^{\omega_{f,b}} e^{ip \sin^{-1}(\frac{\omega}{q})} g(\omega) d\omega$. The imaginary part of r_p^l is zero as $g(-\omega) = g(\omega)$. Hence, taking only the real part and noting that $\theta_{f,b} = \sin^{-1}(\omega_{f,b}/q)$, we arrive at the expression for r_p^l as follows:

$$r_p^l = q \int_{-\theta_{f,b}}^{\theta_{f,b}} \cos(\theta) \cos(p\theta) g(q \sin(\theta)) d\theta. \quad (6)$$

The contribution to overall coherence from the drifting oscillators can be accounted for by calculating $r_p^d = \int_{|\omega| > \omega_{f,b}} \int_{-\pi}^{\pi} e^{ip\theta} \rho_d(\theta, \omega) g(\omega) d\omega d\theta$, where $\rho_d(\theta, \omega)$ is the density of the drifting oscillator which satisfies $\rho_d(\theta, \omega) \propto 1/|\dot{\theta}|$ [9]. The normalization condition for $\rho_d(\theta, \omega)$ gives $\int_{-\pi}^{\pi} \rho_d(\theta, \omega) d\theta = \int_0^T \rho_d(\theta, \omega) \dot{\theta} dt = 1$ (for a given ω), where T is the time period of the limit cycle solution. Hence, we end up with the relation $\rho_d(\theta, \omega) = \frac{1}{|\dot{\theta}|T}$, which when plugged into the form of r_p^d gives us

$$r_p^d = \int_{|\omega| > \omega_{f,b}} \left[\frac{1}{T} \int_0^T e^{ip\theta} dt \right] g(\omega) d\omega. \quad (7)$$

To calculate r_p^d , we first need to obtain an approximate analytic expression for the limit cycle solution of Eq. (5). We follow the method specified in Ref. [31] for writing θ as a Fourier series in θ by only considering the first harmonics [$\dot{\theta}(\theta) = A_0 + A_1 \cos(\theta) + B_1 \sin(\theta)$]. On substituting this in Eq. (5), we find the expression of the coefficients in terms of $\alpha (= \frac{1}{\sqrt{qm}})$ and $\beta (= \frac{\omega}{q})$ such that the first harmonic vanishes, giving us

$$\dot{\theta}(\theta) = \frac{\beta}{\alpha} + \frac{\alpha^2}{\alpha^4 + \beta^2} \left[\frac{\beta}{\alpha} \cos(\theta) - \alpha \sin(\theta) \right], \quad (8)$$

and $\theta(t, \omega)$ by integrating Eq. (8) with time [31]. As it turns out that $\theta(t, -\omega) = -\theta(t, \omega)$, and $g(-\omega) = g(\omega)$, the imaginary part in Eq. (7) is zero. Thus,

$$r_p^d = \int_{|\omega| > \omega_{f,b}} \langle \cos(p\theta) \rangle g(\omega) d\omega. \quad (9)$$

The expression for $\langle \cos(p\theta) \rangle$ (for $p \in \{1, 2\}$) can now be readily calculated as $\langle \cos(p\theta) \rangle = \frac{1}{T} \int_0^T \cos(p\theta) dt = \int_0^{2\pi} \frac{\cos(p\theta)}{\dot{\theta}} d\theta / \int_0^{2\pi} \frac{1}{\dot{\theta}} d\theta$ to obtain

$$\begin{aligned} \langle \cos(\theta) \rangle &= \frac{\beta}{\alpha} \left[\sqrt{\frac{\beta^2}{\alpha^2} - \frac{\alpha^2}{\beta^2 + \alpha^4}} - \frac{\beta}{\alpha} \right], \\ \langle \cos(2\theta) \rangle &= \left[\frac{\beta^2 - \alpha^4}{\beta^2 + \alpha^4} \right] \\ &\times \left[\frac{2\beta(\beta^2 + \alpha^4)}{\alpha^3} \left(\frac{\beta}{\alpha} - \sqrt{\frac{\beta^2}{\alpha^2} - \frac{\alpha^2}{\beta^2 + \alpha^4}} \right) - 1 \right]. \end{aligned}$$

We are now finally ready to write the self-consistent equations that let us describe the steady state of the coupled oscillator system governed by Eq. (1). For the remainder of the paper, we consider the intrinsic frequency to be derived from Lorentz distribution, $g(\omega) = \frac{1}{\pi} \frac{1}{1+\omega^2}$ centered around zero. Noting that the integrands in Eqs. (6) and (9) for $p \in \{1, 2\}$ are even functions, we arrive at

$$\begin{aligned} r_p &= 2q \int_0^{\theta_{f,b}} \cos(\theta) \cos(p\theta) g(q \sin(\theta)) d\theta \\ &+ 2 \int_{\omega_{f,b}}^{\infty} \langle \cos(p\theta) \rangle g(\omega) d\omega. \end{aligned} \quad (10)$$

These two equations together describe the steady-state behavior of the system.

IV. NUMERICAL RESULTS

We numerically solve the set of self-consistent equations, Eq. (10), to find the nontrivial branch of solutions for the order parameter (both forward and backward processes). Figure 1(a) provides a schematic representation of the synchronization profiles of our result in comparison to previously explored models [2,9,24]. Figure 1(b) presents analytical and simulation results for the r_1 vs K_1 curves for $(m, K_2) = (1, 1)$ and $(m, K_2) = (3, 7)$. As for the simulation protocol, we simulate Eq. (3) on a network of $N = 10^4$ nodes by splitting it into a pair of first-order differential equations and integrating them using the Runge-Kutta 4 algorithm (time step 0.1). For a chosen value of m and K_2 , we start with random initial conditions for $\theta \in [0, 2\pi)$ and $\dot{\theta} \in [-1, 1]$ and $K_1 = 0$. We adiabatically increase K_1 in steps of $\Delta K_1 (= 0.1$ unless specified otherwise) till $K_1 = 12$ is reached (forward), followed by an adiabatic decrease till $K_1 = 0$ (backward). By adiabatic increase/decrease, we imply that for every K_1 except the first ($K_1 = 0$), the initial conditions are taken as the final state obtained for the previous K_1 value. At all coupling strengths K_1 , the order parameter values are calculated after discarding transients by averaging over the steady state.

Figure 1(b) displays good agreement between the simulation and analytical results. For the forward process, as K_1 is increased from zero, the system undergoes a first-order phase transition from the incoherent to coherent state at a finite critical coupling value (K_1^f). However, for the backward process, the system undergoes abrupt desynchronization at a value (K_1^b), which is less than K_1^f . Hence, hysteresis is observed where the system stays in two different states depending on the initial configuration. The derived self-consistency equations can also be used with other extended-tailed distributions like the Gaussian distribution. In the backward process, there exists a discrepancy between analytical and numerical values [Fig. 1(b)] as also reported in Ref. [9]. This happens because the maximum value of K_1 chosen for the simulations is 12, when the system starts from a partially coherent state, whereas the analytical solutions are for when the system starts from a fully coherent state. Hence, a better fit between analytical and numerical values for the backward process can be obtained by increasing the maximum value of K_1 in the simulation protocol. We point out that when m and K_2 values are both increased, K_1^f shifts to the right while K_1^b shifts to the

left, revealing a prolonged hysteresis region as illustrated in Fig. 1(b).

A natural question would then be to address the dependency of the forward and backward transition points on m and K_2 . To analytically obtain the expression for (K_1^f) , we evaluate Eq. (10) in the limit $r_1 \rightarrow 0^+$ ($q \rightarrow 0^+$). As we take this limit, we see that $\beta/\alpha (= \omega \sqrt{\frac{m}{q}})$ tends to very high value as compared to $\alpha^2/(\beta^2 + \alpha^4) (= \frac{qm}{1+\omega^2 m^2})$. This allows us to perform a Taylor series expansion of $\langle \cos(\theta) \rangle$ for $\epsilon = \alpha^2/(\beta^2 + \alpha^4) \ll 1$ which gives $\langle \cos(\theta) \rangle = \frac{-\alpha^2}{2(\beta^2 + \alpha^4)} + \mathcal{O}(\epsilon^4) \approx \frac{-mq}{2(1+m^2\omega^2)}$. However, in the limit $r_1 \rightarrow 0^+$, $r_2 \rightarrow 0^+$ and the parameter $\alpha \rightarrow \infty$, implying that the limit of the integrals for the forward and backward processes become the same as there exists no bistability region in the parameter space. Taking $\theta_{f,b} = \frac{\pi}{2}$, dividing both sides of Eq. (10) by q and evaluating the limit [at which the two equations in Eq. (10) decouple], we have

$$\frac{1}{K_1^f} = \frac{\pi}{2} g(0) - m \int_0^\infty \frac{1}{1 + m^2 \omega^2} g(\omega) d\omega.$$

After evaluating the integral and rearranging the terms, we end up with $K_1^f = 2(m+1)$. We see that the forward transition point is independent of K_2 and purely depends on m and, hence, matches the previously derived value of the forward transition point in Refs. [7,33]. Figure 2(b) illustrates the effect of varying K_2 (0.0, 3.0, 5.0) for the case of fixed m ($=1$). As expected, K_1^f remains the same for all three cases, validating our analytical result. At this K_1^f ($=4$), the magnitude of the first-order jump for fixed m increases with the value of K_2 . In Fig. 2(c), we study the effect of varying mass (0.0, 1.0, 3.0) for the case of fixed K_2 ($=2.0$). As inertia increases, K_1^f shifts to higher values, as predicted analytically. However, we note that the analytically calculated values of K_1^f do not match exactly with numerical simulations owing to the finite size effects. A detailed study has been done in Ref. [32].

A fairly good analytical approximation for K_1^b , as also pointed out in Ref. [32], would be to obtain the minimum value of K_1 along the nontrivial branch of the backward self-consistent curve. The simulation results in Figs. 1(b), 2(b), and 2(c) are seen to back up this observation for our model. However, obtaining a clean analytical expression for the same by calculating $\frac{dK_1}{dr_1} = 0$ is not possible because of the complexity of the integrand of the drift oscillator contribution in r_2 . Alternatively, we resort to simulation results to decipher the dependency of K_1^b on m and K_2 . From Fig. 2(b), it can be seen that for the backward process, the coherent branch persists until increasingly smaller values of K_1 , with an increase in the K_2 value, after which the system undergoes an abrupt transition to the incoherent state. Hence, it is clear that an increase in K_2 leads to a decrease in K_1^b . To study the effect of mass on K_1^b , we fix K_2 and vary m as in Fig. 2(c). It is observed that the backward branches for fixed K_2 ($=2$) and different m values merge for high K_1 values and get separated for low K_1 values. As there is an influence of m on the nature of the curve for low K_1 values, this indicates the possibility of dependency of K_1^b on m .

It was shown in Ref. [32] that for the pure dyadic case ($K_2 = 0$), K_1^b decreases with an increase in m and plateaus out for high m values. In Fig. 2(d), we address how this changes with the introduction of finite K_2 . The K_1^b obtained via simulation (performed for $N = 10^3$ number of nodes) for values of K_2 ranging from 0 to 10 and different values of m (0,1,5,10) are plotted. We see that for small values of K_2 and finite inertia, an increase in the values in m leads to a decrease in K_1^b . However, we point out that for higher values of K_2 , the effect of m on K_1^b becomes less pronounced and desynchronization happens at the same value irrespective of mass. An analytical prediction of K_1^b becomes possible following this observation by considering the $m = 0$ case. We derive self-consistent equations for this case in Appendix A 1 and obtain K_1^b as a function of K_2 ($K_1^b = 2\sqrt{2K_2} - K_2$ for $K_2 \geq 2$) by finding the minimum value of K_1 in the self-consistency curve. These analytically calculated K_1^b values for the $m = 0$ case are represented by the dashed line in Fig. 2(d). It can be clearly observed that for higher values of K_2 (≥ 2), the analytical predictions of K_1^b match closely with the ones obtained via simulation for different masses.

V. CONCLUSION AND DISCUSSION

In this paper, we have put forward a generalized analytical framework to study the steady-state behavior of coupled oscillator systems with inertia interacting via higher-order interactions. The analytical predictions, backed up by numerical simulation, show a prolonged hysteretic first-order phase transition to an (in)coherent state. We show that the forward transition point increases linearly with m and is independent of K_2 . Meanwhile, the backward transition point decreases with K_2 and is independent of m for high K_2 values.

We also highlight here the analytical challenges emanating from combining both the inertia and higher-order interaction terms in the Kuramoto model. For the coupled Kuramoto oscillator model, two widespread analytical techniques are the self-consistency method and the OA approach to studying the synchronization profile of the entire system. OA dimensionality reduction method provides a very easy way to obtain the time evolution form of r_1 , solving which yields a complete description of bifurcations leading to synchronization. While the OA approach was successfully extended for the Kuramoto model with higher-order interactions, for the second-order Kuramoto models, the density function containing phase term also depends on velocity, posing restrictions on practicing the OA method. Hence, the Kuramoto model with inertia has been analytically solved using the self-consistency method. The challenge in this method is first finding the limit of integration in the self-consistent equation by finding an approximate frequency bound for the oscillators, which are in the locked state as a function of external parameters and the order parameter. Further, the drifting oscillators also contribute to the overall coherence, which needs to be accounted for, unlike in the first-order Kuramoto model, where it is zero. Here, we have employed the self-consistency method to predict the steady-state behavior of the Kuramoto oscillators having both inertia and higher-order terms.

We have presented the results for triadic interactions; however, extending our analysis to other powers of higher-order

interactions is easy as long as the sinusoidal coupling function contains θ_i term only. For example, the detailed analysis of quartic interactions is presented in Appendix A 2. Further, developing the self-consistent method for other choices of higher-order coupling functions, such as $\sin(\theta_j + \theta_k - 2\theta_i)$ [34,35] along with pairwise coupling proves to be complicated because of the existence of higher order harmonics in the mean-field equation; however, the self-consistency approach can work for this form of the coupling in the absence of pairwise coupling (pure triadic case), which we have explored in Appendix A 3. An immediate future direction of our paper would be to extend our analysis to diluted simplicial complexes, which can provide fundamental insights into the dynamics of various real-world complex systems such as power grids.

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APPENDIX A: DERIVATION FOR $m = 0$

This Appendix elaborates on the self-consistency analysis and the derivation for the forward and backward transition points for $m = 0$. In this case, a change from a smooth (second-order) transition to synchronization to an abrupt (first-order) one, along with hysteresis, is observed with an increase in the K_2 value. The occurrence of hysteresis can be accounted for by the shift of the backward transition point to lower K_1 values with an increase in K_2 . Note that the results for the first-order Kuramoto model with higher-order interactions were reported in Ref. [24] using the OA dimensionality reduction method. Here, we derive the closed-form solution for the bifurcation curve using the self-consistency method.

The dynamical equation for our model takes the following form:

$$\begin{aligned} \dot{\theta}_i = \omega_i + \frac{K_1}{N} \sum_{j=1}^N \sin(\theta_j - \theta_i) \\ + \frac{K_2}{N^2} \sum_{j=1}^N \sum_{k=1}^N \sin(2\theta_j - \theta_k - \theta_i). \end{aligned} \quad (\text{A1})$$

Following the same procedure as in the main text, the mean-field equation in the rotating frame is obtained as $\dot{\theta}_i = \omega_i - q \sin(\theta_i)$, where $q = r_1(K_1 + K_2 r_2)$. All the oscillators with intrinsic frequency $|\omega_i| \leq q$ go to the fixed point state, while others are in a drift state. Unlike in the finite-inertia case, here the limit of the integrals for forward and backward cases become the same ($\theta_f = \theta_b = \frac{\pi}{2}$), enabling us to study the system in full generality. In the thermodynamic limit, the locked oscillator contribution has the same form as derived in the main text,

$$r_p^l = q \int_{-\pi/2}^{\pi/2} \cos(\theta) \cos(p\theta) g(q \sin(\theta)) d\theta, \quad (\text{A2})$$

which upon integrating for $p \in \{1, 2\}$ gives us

$$\begin{aligned} r_1^l &= \frac{\sqrt{q^2 + 1} - 1}{q}, \\ r_2^l &= \frac{2}{\pi q^2} [(q^2 + 2) \tan^{-1}(q) - 2q]. \end{aligned} \quad (\text{A3})$$

The contribution of drifting oscillators to the global order parameter, however, has the following expression:

$$\begin{aligned} r_p^d &= \int_{|\omega| > q} \int_{-\pi}^{\pi} e^{ip\theta} \rho_d(\theta, \omega) g(\omega) d\omega d\theta, \\ r_p^d &= \int_{|\omega| > q} \int_{-\pi}^{\pi} e^{ip\theta} \frac{\sqrt{\omega^2 - q^2}}{2\pi |\omega - q \sin(\theta)|} g(\omega) d\omega d\theta. \end{aligned}$$

The expression for the density of the drifting oscillators is derived by noting that $\rho_d(\theta, \omega) \propto 1/|\dot{\theta}| = c/|\omega - q \sin(\theta)|$, where c is the normalization constant such that $\int_{-\pi}^{\pi} \rho_d(\theta, \omega) d\theta = 1$. The value of r_1^d is zero since $\rho_d(\theta + \pi, -\omega) = \rho_d(\theta, \omega)$ and $g(-\omega) = g(\omega)$. Meanwhile, in r_2^d , only the imaginary term is zero, since $\rho_d(-\theta, -\omega) = \rho_d(\theta, \omega)$ and $g(-\omega) = g(\omega)$, while the real part is not zero. Since the integrand is even, we can evaluate the integral as follows:

$$\begin{aligned} r_2^d &= 2 \int_q^\infty \left[\int_{-\pi}^{\pi} \cos(2\theta) \frac{\sqrt{\omega^2 - q^2}}{2\pi (\omega - q \sin(\theta))} d\theta \right] g(\omega) d\omega, \\ &= 2 \int_q^\infty \left[\frac{2\omega}{q^2} (\sqrt{\omega^2 - q^2} - \omega) + 1 \right] g(\omega) d\omega, \\ &= -\frac{2}{\pi q^2} ((q^2 + 2) \tan^{-1}(q) - 2q) + \frac{2 + q^2 - 2\sqrt{q^2 + 1}}{q^2}. \end{aligned} \quad (\text{A4})$$

Using the self-consistency condition ($r_p = r_p^l + r_p^d$) and Eqs. (A3), in Eq. (A4) we arrive at a set of self-consistent equations which when solved give us the analytical predictions of the steady-state behavior,

$$r_1 = \frac{\sqrt{q^2 + 1} - 1}{q}, \quad (\text{A5a})$$

$$r_2 = \frac{2 + q^2 - 2\sqrt{q^2 + 1}}{q^2}, \quad (\text{A5b})$$

where $q = r_1(K_1 + r_2 K_2)$. Solving Eqs. (A5) gives us the relation $r_2 = r_1^2$, which when plugged back into Eq. (A5a) leads to the closed form solution for the bifurcation curve:

$$K_1 = \frac{2}{1 - r_1^2} - r_1^2 K_2. \quad (\text{A6})$$

We point out that this equation matches with the one obtained by Ref. [24] using the OA method.

Figure 3 presents analytical and simulation results for multiple values of triadic coupling strength K_2 . As predicted by our analysis, the forward synchronization transition happens at $K_1^f = 2$, while the backward transition point shifts to lower values of K_1 as K_2 increases. From Fig. 3, we note that the minima of the self-consistency curve fairly well approximates the backward desynchronization point. To this end, setting

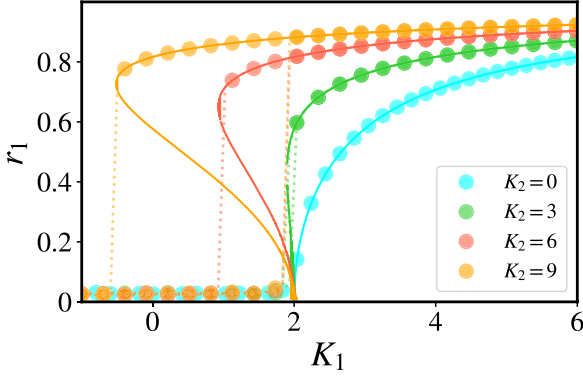


FIG. 3. Synchronization profile of Eq. (A1) for and $K_2 = 0$ (cyan), 3 (green), 6 (red), and 9 (yellow). The circles represent simulation results for the forward and backward cases, while the continuous line represents the analytically predicted values.

$\frac{dK_1}{dr_1} = 0$, we obtain $r_1^b = \sqrt{1 - \sqrt{\frac{2}{K_2}}}$ for $K_2 \geq 2$. Plugging the form of r_1^b back into Eq. (A6) gives $K_1^b = 2\sqrt{2K_2} - K_2$. This is used in Fig. 2(d).

APPENDIX B: EXTENSION TO QUARTIC INTERACTIONS

This section explains how the self-consistency analysis presented in the main text can be easily extended to include quartic interactions. We add a quartic interaction term to

our model as proposed in Ref. [24]. In our study, we find good agreement between the analytical predictions of our self-consistency analysis and the simulation results. The detailed analysis of the impact of quartic interactions on the dynamics is out of the scope of this paper. The dynamical equation for our model is given as follows:

$$\begin{aligned} m\ddot{\theta}_i = & -\dot{\theta}_i + \omega_i + \frac{K_1}{N} \sum_{j=1}^N \sin(\theta_j - \theta_i) \\ & + \frac{K_2}{N^2} \sum_{j=1}^N \sum_{k=1}^N \sin(2\theta_j - \theta_k - \theta_i) \\ & + \frac{K_3}{N^3} \sum_{j=1}^N \sum_{k=1}^N \sum_{l=1}^N \sin(\theta_j + \theta_k - \theta_l - \theta_i). \end{aligned} \quad (\text{B1})$$

Upon using the definition of the generalized order parameter as defined in the main text, we can convert the above equation into a mean-field form as

$$\begin{aligned} m\ddot{\theta}_i = & -\dot{\theta}_i + \omega_i + K_1 r_1 \sin(\psi_1 - \theta_i) \\ & + K_2 r_1 r_2 \sin(\psi_2 - \psi_1 - \theta_i) + K_3 r_1^3 \sin(\psi_1 - \theta_i). \end{aligned}$$

By moving to the rotating frame, we set ψ_1 and ψ_2 as zero, which gives $m\ddot{\theta}_i = -\dot{\theta}_i + \omega_i - q \sin(\theta_i)$, where $q = r_1(K_1 + K_2 r_2 + K_3 r_1^2)$ is the overall coupling constant. We proceed exactly as in the main text to arrive at the self-consistency equations defining the steady-state behavior of the model:

$$r_1 = 2q \int_0^{\theta_{f,b}} \cos^2(\theta) g(q \sin(\theta)) d\theta + 2 \int_{\omega_{f,b}}^{\infty} \omega \sqrt{\frac{m}{q}} \left[\sqrt{\omega^2 \frac{m}{q} - \frac{qm}{1 + \omega^2 m^2}} - \omega \sqrt{\frac{m}{q}} \right] g(\omega) d\omega, \quad (\text{B2a})$$

$$\begin{aligned} r_2 = & 2q \int_0^{\theta_{f,b}} \cos(\theta) \cos(2\theta) g(q \sin(\theta)) d\theta \\ & + 2 \int_{\omega_{f,b}}^{\infty} \left[\frac{\omega^2 m^2 - 1}{\omega^2 m^2 + 1} \right] \left[\frac{2\omega}{q} \left(\frac{\omega^2 m^2 + 1}{\sqrt{qm}} \right) \left(\sqrt{\omega^2 \frac{m}{q} - \frac{qm}{1 + \omega^2 m^2}} - \omega \sqrt{\frac{m}{q}} \right) - 1 \right] g(\omega) d\omega. \end{aligned} \quad (\text{B2b})$$

Figure 4 plots r_1 as a function of K_1 for different combinations of K_2 and K_3 , and we see that the analytical predictions match the numerical simulation results.

APPENDIX C: DERIVATION FOR THE $\sin(\theta_j + \theta_k - 2\theta_i)$ MODEL

In this section, we present the self-consistency analysis for the Kuramoto model with inertia involving purely triadic interactions via the $\sin(\theta_j + \theta_k - 2\theta_i)$ coupling function. Research on the dynamics of Kuramoto oscillators coupled via the said sinusoidal function [$\dot{\theta}_i = \omega_i + \frac{K_2}{N^2} \sum_{j=1}^N \sum_{k=1}^N \sin(\theta_j + \theta_k - 2\theta_i)$] [35] has shown the presence of cluster formation and a continuum of abrupt desynchronization transitions based on initial conditions. Crucially, no synchronization transition has been reported for this model. Surprisingly, our studies report that inertia has no effect on the synchronization profile of this system. We infer that considering purely triadic interactions removes the

distinction between inertialess and finite-inertia cases. The dynamical equations for this model are given as

$$m\ddot{\theta}_i = \omega_i - \dot{\theta}_i + \frac{K_2}{N^2} \sum_{j=1}^N \sum_{k=1}^N \sin(\theta_j + \theta_k - 2\theta_i). \quad (\text{C1})$$

Using the definition of the generalized order parameter, we write Eq. (C1) in the mean-field format as

$$m\ddot{\theta}_i = \omega_i - \dot{\theta}_i + K_2 r_1^2 \sin(2\psi_1 - 2\theta_i). \quad (\text{C2})$$

Upon simulating the dynamics of this equation, we observe that analogous to the studies reported in Ref. [35], there exists no forward synchronization in the system, rather a sequence of desynchronization transitions is observed based on the asymmetry in the initial conditions. Thus, we focus the analytical derivation only on the desynchronization profiles.

Dropping the subscript i , by going to a suitable rotating frame, we set $\psi = 0$, which gives us $m\ddot{\theta} = \omega - \dot{\theta} - K_2 r_1^2 \sin(2\theta)$, where the probability distribution $g(\omega)$ is

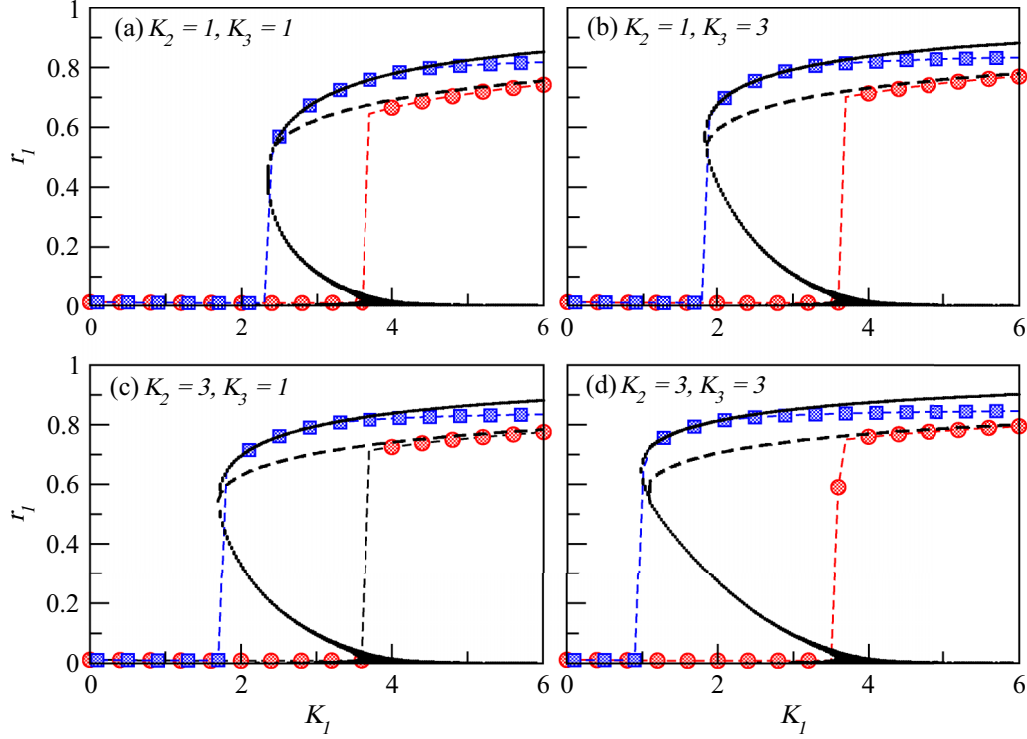


FIG. 4. Synchronization profiles for the Kuramoto model with inertia involving quartic interactions with $m = 1$ and different combinations of K_2 and K_3 values. The red circles and blue squares represent the simulation results for the forward and backward cases of Eq. (B1) for $N = 1000$ nodes, respectively, whereas the dashed and continuous curves are the analytical predictions from (B2) for the forward and backward cases.

unimodal and symmetric about the mean zero. To study the steady-state behavior, we invoke a time transformation as $\tau = \sqrt{\frac{K_2 r_1^2}{m}} t$, which gives

$$\ddot{\theta} = \beta - \alpha \dot{\theta} - \sin(2\theta), \quad (\text{C3})$$

where $\beta = \frac{\omega}{K_2 r_1^2}$ and $\alpha = \frac{1}{\sqrt{K_2 r_1^2 m}}$. The parameter space of Eq. (C3) is qualitatively similar to that of the model considered in the main text. The quantitative differences between the two are as follows: (i) For each β such that $|\beta| \leq 1$, we now have two stable fixed points given by $\theta_1^* = \frac{1}{2} \sin^{-1} \beta$, and $\theta_2^* = \frac{1}{2} \sin^{-1} \beta + \pi$ separated by two saddles. (ii) The separatrix equation is given by $\theta(t) = \sin^{-1} \tanh(\sqrt{2}t)$, $\dot{\theta}(t) = \sqrt{2} \frac{1}{\cosh(\sqrt{2}t)}$. Using this equation of separatrix and implementing Melnikov's method [30], the equation for homoclinic bifurcation can be obtained as $\beta = \frac{2\sqrt{2}}{\pi} \alpha$. Thus, as seen in Fig. 5:

- (1) If $|\beta| > 1$, the system goes to a limit cycle.
- (2) If $|\beta| \leq \frac{2\sqrt{2}}{\pi} \alpha$, the system goes to a stable fixed point state, where the choice of the fixed point depends upon the initial conditions.
- (3) Finally, if $\frac{2\sqrt{2}}{\pi} \alpha < |\beta| < 1$, we now have a tristable state with the simultaneous presence of one stable limit cycle and two stable fixed points.

In terms of ω , K_2 , and r_1 , it means that for a particular K_2 and corresponding steady-state value of r_1 , the oscillators with $|\omega| > K_2 r_1^2$ become drifting oscillators. The oscillators with $|\omega| \leq \frac{2\sqrt{2}}{\pi} \sqrt{\frac{K_2 r_1^2}{m}}$ contribute to the formation of two

diametrically opposite clusters of locked oscillators. Finally, the oscillators with $\frac{2\sqrt{2}}{\pi} \sqrt{\frac{K_2 r_1^2}{m}} < |\omega| \leq 1$ are in the tristable region. However, as we will deal only with the system desynchronization profile, the oscillators in the tristable region will also go to their respective fixed points and contribute to cluster formation.

The next step is to calculate the locked and drifting oscillator contribution to the order parameter. First, let us calculate the contribution from the locked oscillators. To account for the two clusters of locked oscillators, we introduce a variable $\eta(\omega) \in [0, 1]$. The values of $\eta(\omega)$ and $1 - \eta(\omega)$ are the probabilities that an oscillator with intrinsic frequency ω will go to the fixed points $\theta_1^* = \frac{1}{2} \sin^{-1} \beta$ and $\theta_2^* = \frac{1}{2} \sin^{-1} \beta + \pi$, respectively. For simplicity, we consider only symmetric cases of the function $\eta(\omega)$, such that $\eta(\omega) = \eta(-\omega)$. The contribution of locked oscillators is then given by $r_1^l = \int_{-K_2 r_1^2}^{K_2 r_1^2} [(1 - \eta(\omega))e^{i\theta(\omega)+\pi} + \eta(\omega)e^{i\theta(\omega)}] g(\omega) d\omega$. As $e^{i\theta(\omega)+\pi} = -e^{i\theta(\omega)}$ and $\sin(-x) = -\sin(x)$, we have

$$r_1^l = \int_{-K_2 r_1^2}^{K_2 r_1^2} (2\eta(\omega) - 1) \cos(\theta(\omega)) g(\omega) d\omega. \quad (\text{C4})$$

Next, let us consider the contribution of the drifting oscillators to the order parameter. Let $\rho_d(\theta, \omega)$ be the density of drifting oscillators which satisfies $\int_{-\pi}^{\pi} \rho_d(\theta, \omega) d\theta = 1$. The continuity equation for the conservation of the number of oscillators gives $\rho_d(\theta, \omega) = c/\dot{\theta}(\omega)$. An expression for $\dot{\theta}$ can be obtained by following an analogous method as in the main text by considering $\dot{\theta} = A_0 + A_1 \cos(2\theta) + B_1 \sin(2\theta)$.

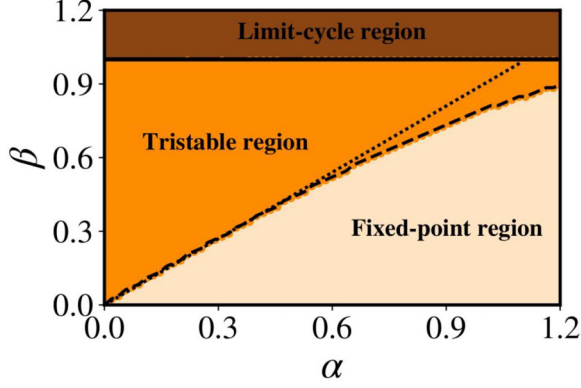


FIG. 5. The parameter space for Eq. (C3) representing the limit-cycle, tristable, and fixed-point regimes, respectively. The line at $\beta = 1$ represents the saddle-node bifurcation in the system. On the boundary of the tristable and fixed-point regimes, the scatter plot indicates the actual homoclinic bifurcation curve while the dotted line is the approximation obtained using Melnikov's method ($\beta = \frac{2\sqrt{2}}{\pi}\alpha$).

By substituting this in Eq. (C3), the expression for $\dot{\theta}$ can be obtained as

$$\dot{\theta}(\omega, \theta) = \frac{\beta}{\alpha} + \frac{2\beta\alpha}{\alpha^4 + 4\beta^2} \cos(2\theta) - \frac{\beta^3}{\alpha^4 + 4\beta^2} \sin(2\theta). \quad (\text{C5})$$

Equation (C5) can be simplified by considering $h_0 = \beta/\alpha$ and $\frac{e^{i\theta_2}}{h_1} = 2h_0 + i\alpha$, which gives $\dot{\theta} = h_0 + h_1 \cos(2\theta + h_2)$. By integrating over the normalization condition of $\rho_d(\theta, \omega)$, we get $c = \sqrt{h_0^2 - h_1^2/2\pi}$, which gives the following

equation for the density of drifting oscillators:

$$\rho_d(\theta, \omega) = \left| \frac{1}{2\pi} \sqrt{\frac{\omega^2 m}{K_2 r_1^2} - \frac{K_2 r_1^2 m}{1 + 4m^2 \omega^2}} \right| \times \frac{1}{\frac{\omega \sqrt{m}}{\sqrt{K_2 r_1^2}} + \sqrt{\frac{K_2 r_1^2 m}{1 + 4m^2 \omega^2}} \cos(2\theta(\omega) + \tan^{-1}(\frac{1}{2m\omega}))} \quad (\text{C6})$$

$$r_1^d = \int_{|\omega| > K_2 r_1^2} \int_{-\pi}^0 [e^{i\theta(\omega)} \rho_d(\theta, \omega) + e^{i(\theta(\omega) + \pi)} \rho_d(\theta + \pi, \omega)] \times g(\omega) d\theta d\omega. \quad (\text{C7})$$

From Eq. (C6), we note that $\rho_d(\theta + \pi, \omega) = \rho_d(\theta, \omega)$. Thus, the contribution of drifting oscillators to order parameter $r_1^d = \int_{|\omega| > K_2 r_1^2} \int_{-\pi}^{\pi} e^{i\theta(\omega)} \rho_d(\theta, \omega) g(\omega) d\theta d\omega$ can be simplified as Eq. (C7).

As $e^{i(\theta(\omega) + \pi)} = -e^{i\theta(\omega)}$, we get $r_1^d = 0$. Therefore, from Eq. (C4) and the self-consistency condition ($r_1 = r_1^l + r_1^d$), we get

$$r_1 = \int_{-K_2 r_1^2}^{K_2 r_1^2} (2\eta(\omega) - 1) \cos \theta(\omega) g(\omega) d\omega. \quad (\text{C8})$$

We now compare the results obtained by simulating Eq. (C2) with the analytical values predicted by Eq. (C8) in Fig. 6. For simplicity, we consider $\eta(\omega)$ to be a constant

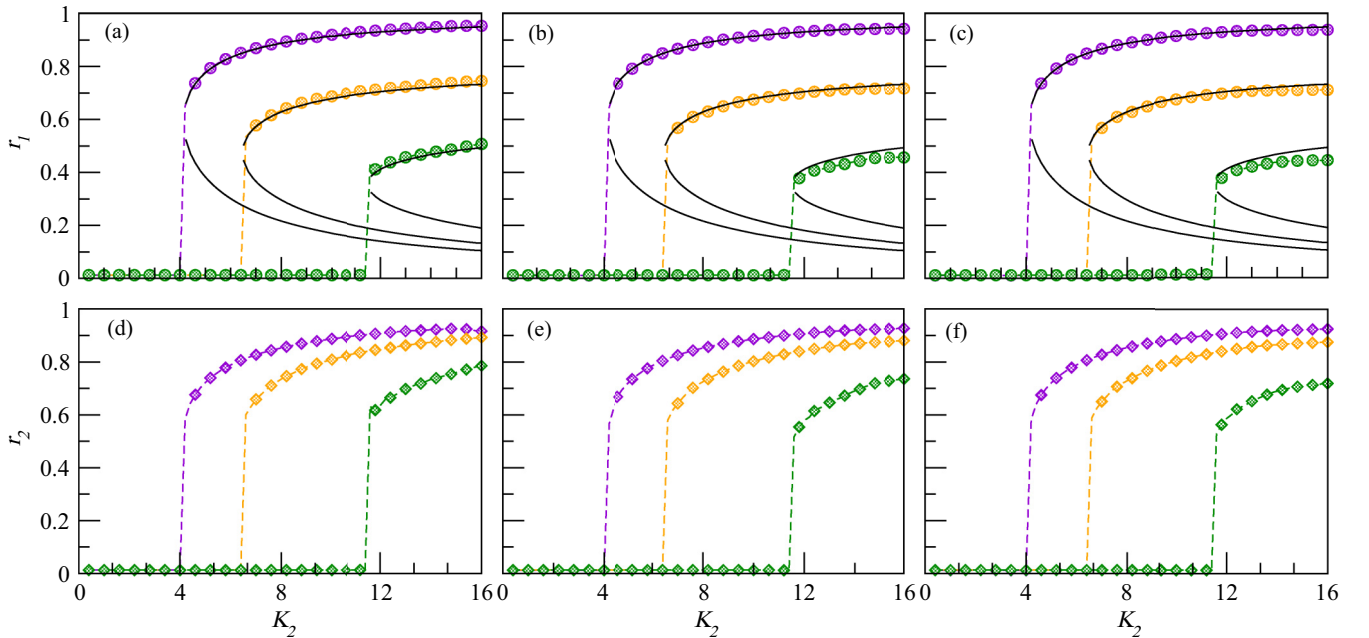


FIG. 6. The top panel displays the r_1 versus K_2 behavior for Eq. (C2) for (a) $m = 0$, (b) $m = 1$, and (c) $m = 3$, respectively. The violet, yellow, and green curves represent the de-synchronization profiles for η equal to 1, 0.9, and 0.8, respectively. For each case, the continuous curve represents the analytically obtained values. The bottom panel displays the simulation-obtained values for the Daido order parameter (r_2) for the corresponding cases (d) $m = 0$, (e) $m = 1$, and (f) $m = 3$.

function with respect to ω . While simulating for a given value of η , we consider the initial phase of the oscillators as 0 with probability η and π with probability $(1 - \eta)$. With these initial conditions, we start from $K_2 = K_{\max} = 16$ and then adiabatically decrease K_2 until $K_{\min} = 0$. At each K_2 , we integrate Eq. (C2) using the RK4 algorithm and calculate the values of r_1 and r_2 by averaging over all the iterations after removing the transients.

We note that due to the vanishing of the drift term, the form of the bifurcation curve Eq. (C8) is also the same for the $m = 0$ case. Figure 6 also shows that the r_1 and r_2 graphs are almost identical for the finite mass and the massless case, with identical backward transition points. Hence, considering purely triadic interactions eliminates the distinction between the finite inertia and inertialess case. For $\eta = 0.9$ (yellow) and $\eta = 0.8$ (green), we see that the r_2 values are higher than the r_1 values, indicating that a two-cluster state is present in the system, which is the expected outcome.

A more generalized approach to further this study would be to incorporate pairwise interactions along with the triadic ones to give a model like

$$m\ddot{\theta}_i = \omega_i - \dot{\theta}_i + \frac{K_1}{N} \sum_{j=1}^N \sin(\theta_j - \theta_i) + \frac{K_2}{N^2} \sum_{j=1}^N \sum_{k=1}^N \sin(\theta_j + \theta_k - 2\theta_i), \quad (\text{C9})$$

which in the mean-field format and rotating frame, with ψ_1 set to zero, reads as $m\ddot{\theta}_i = \omega_i - \dot{\theta}_i - K_1 r_1 \sin(\theta_i) - K_2 r_1^2 \sin(2\theta_i)$. However, the simultaneous presence of the first and second-order harmonics of the sinusoidal term makes it complicated to analytically study the parameter space of Eq. (C9).

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