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Families of jets of arc type and higher (co)dimensional Du Val singularities

Familles de jets de type arc et singularités de Du Val de (co)dimension supérieure

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In memory of Jean-Pierre Demailly

Abstract. Families of jets through singularities of algebraic varieties are here studied in relation to the families of arcs originally studied by Nash. After proving a general result relating them, we look at normal locally complete intersection varieties with rational singularities and focus on a class of singularities we call *higher Du Val singularities*, a higher dimensional (and codimensional) version of Du Val singularities that is closely related to Arnold singularities. More generally, we introduce the notion of *higher compound Du Val singularities*, whose definition parallels that of compound Du Val singularities. For such singularities, we prove that there exists a one-to-one correspondence between families of arcs and families of jets of sufficiently high order through the singularities. In dimension two, the result partially recovers a theorem of Mourtada on the jet schemes of Du Val singularities. As an application, we give a solution of the Nash problem for higher Du Val singularities.

Résumé. Les familles de jets à travers les singularités des variétés algébriques sont étudiées ici en relation avec les familles d'arcs initialement étudiées par Nash. Après avoir démontré un résultat général les concernant, nous examinons les variétés d'intersection localement complètes normales avec des singularités rationnelles et nous concentrons sur une classe de singularités que nous appelons « singularités de Du Val supérieures », une version de dimension (et codimension) supérieure des singularités de Du Val étroitement liée aux singularités d'Arnold. Plus généralement, nous introduisons la notion de « singularités de Du Val composées supérieures », dont la définition est parallèle à celle des singularités de Du Val composées. Pour de telles singularités, nous démontrons qu'il existe une correspondance bijective entre les familles d'arcs et les familles de jets d'ordre suffisamment élevé à travers les singularités. En dimension deux, le résultat récupère partiellement un théorème de Mourtada sur les schémas de jets des singularités de Du Val. En tant qu'application, nous proposons une solution au problème de Nash pour les singularités de Du Val supérieures.

Keywords. Jet scheme, arc space, Nash problem, rational singularity.

Mots-clés. Schémas de jet, espace d'arc, problème de Nash, singularité rationnelle.

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1. Introduction

The space of arcs through the singular locus of a complex variety decomposes into a finite union of irreducible components, each defining a distinct divisorial valuation, that is, a prime divisor on some resolution of singularities. These components were studied by Nash [38]; we will refer to them as *Nash families of arcs*, and to the valuations they define as *Nash valuations*. The problem of characterizing Nash families of arcs in terms of resolutions of singularities fits within the *Nash problem*, which was motivated by the desire of understanding what different resolutions would have in common.

It is natural to ask whether a similar picture holds for families of jets through the singular locus, at least when one looks at jets of sufficiently high order. (For clarity of exposition, in this introduction we restrict the discussion to the case where families of arcs and families of jets all stem from the singular locus of the variety; we refer to the main body of the paper for a more general formulation of the question.) As jets are parametrized by schemes of finite type, the fact that there are finitely many irreducible components of the set of jets of fixed order through the singular locus is clear. The question is how the families of jets defined by such components relate to the families of arcs identified by Nash.

Even though families of jets are introduced similarly to families of arcs, at the core there is an essential difference between the two: Nash families of arcs lift to resolutions of singularities and are naturally related to divisorial valuations; by contrast, families of jets through singularities do not lift to resolutions and cannot be related to valuations in any obvious way. In particular, the approach followed by Nash to study families of arcs using resolution of singularities does not apply to finite order jets.

Families of jets have been computed in several concrete examples, see, e.g., the works on plane curves and surface singularities [6, 28, 32–36]; in many of these works, the computation is carried out through a direct analysis of the defining equations. The problem of understanding families of jets is closely related to the *embedded Nash problem*, which aims to describe the irreducible components of contact loci of effective divisors on smooth ambient varieties in terms of embedded resolutions. A breakthrough in this direction was recently made in [3], where the problem was solved for unibranched plane curves; see also, e.g., [11, 21] for earlier work on this problem.

The purpose of this paper is to unveil a natural correspondence between families of arcs and certain families of jets of sufficiently high order. Our starting point is the following general property.

Theorem A (Theorem 4). *Among all families of jets of sufficiently high order stemming the singular locus of a variety, there is a selection of them that is in natural one-to-one correspondence with the Nash families of arcs.*

The correspondence is obtained by defining, in a geometric meaningful way, an injective map from the set of Nash families of arcs to the set of families of jets through the singular locus. We say that a family of jets is *of arc type* if it is in the image of this map.

We then address the question whether all families of jets of sufficiently high order through the singular locus are of arc type. Although in general there are more families of jets compared to families of arcs (see, e.g., the case of toric surface singularities [33, 35]), we will show that there is a one-to-one correspondence for certain rational singularities of arbitrary dimensions. One case we already understand, thanks to [34], is that of Du Val singularities, where there is a one-to-one correspondence. Here we extend the existence of such correspondence to a large class of locally complete intersection rational singularities of arbitrary dimensions which include isolated compound Du Val singularities.

For every normal locally complete intersection variety X there is a bound on embedding codimension in terms of minimal log discrepancy. The bound, which is proved in Proposition 16, is given by

$$\operatorname{ecodim}(\mathcal{O}_{X,x}) \le \dim(\mathcal{O}_{X,x}) - \operatorname{mld}_x(X)$$

for every $x \in X$. We say that X has *maximal embedding codimension* at x if the bound is achieved. Within this class of singularities, we have those for which

$$\operatorname{mld}_x(X) = \dim(\mathcal{O}_{X,x}) - \operatorname{ecodim}(\mathcal{O}_{X,x}) = 1.$$

It is easy to see that these are isolated singularities. We will see in a moment that these singularities have many properties that are natural higher dimensional analogues of properties characterizing Du Val singularities in dimension two (the analogy is also manifest in the examples provided in Proposition 25). For this reason, we call these singularities *higher Du Val singularities.* In dimension two, this class of singularities coincides with Du Val singularities.

We then look at rational singularities of maximal embedding dimension that reduce to higher Du Val singularities under generic hyperplane sections. One should think of this condition as an analogue of the definition of compound Du Val singularity. We call these singularities *higher compound Du Val singularities.* We have the following result.

Theorem B (Theorem 34). *On an isolated higher compound Du Val singularity $x \in X$, all families of jets of sufficiently high order stemming from x are of arc type.*

As a special case, we see that all families of jets of sufficiently high order stemming from an isolated compound Du Val singularities are of arc type. Theorem B addresses our motivating question on families of jets. Combined with Theorem A, the theorem relates to and partially recover a result of Mourtada on families of jets on Du Val singularities [34] (see Corollary 36). Mourtada asked whether for any locally complete intersection variety with rational singularities the number of families of jets of sufficiently high order stemming from the singular locus is the same as the number of Nash families of arcs [34, Question 4.5]. Our result provides evidence in this direction.

For higher Du Val singularities, we have a more precise result (see Theorem 28) which we use to solve the Nash problem for this class of singularities. In our solution, Nash valuations are characterized in terms of certain partial resolutions of the variety (the terminal models) that originate from the minimal model program. Valuations defined by the exceptional divisors on these models are called *terminal valuations.*

Theorem C (Corollary 29). *For a divisorial valuation* ord_E *on a variety X with higher Du Val singularities, the following are equivalent:*

(1) ord_E *is a Nash valuation.*
(2) ord_E *is a terminal valuation.*
(3) *E is a crepant exceptional divisor over X.*

This result is in line with the point of view proposed in [15]. It can be viewed as a higher dimensional generalization of one of the properties characterizing Du Val singularities among normal surface singularities.

In dimension two, there are four proofs of the Nash problem for Du Val singularities [10, 15, 39, 40]. While the proof given here follows a different path, relying on inversion of adjunction and the minimal model program, it also uses on the main theorem of [15] and therefore it should not be considered as providing a new proof in dimension two for Du Val singularities. In higher dimensions, however, Theorem C does not follow directly from [15].

Throughout the paper, we work over an algebraically closed field k of characteristic zero.

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2. Arc spaces and jet schemes

For a scheme X over k, we denote by X_∞ the *arc space* of X over k and by X_m the *m-th jet scheme* of X. We refer to [9, 13, 44] for general references on the subject. An arc $\alpha \in X_\infty$ is a morphism $\alpha\colon \operatorname{Spec} k_\alpha[\![t]\!] \to X$ and a jet $\beta \in X_m$ a morphism $\beta\colon \operatorname{Spec} k_\beta[t]/(t^{m+1}) \to X$. We denote by $\alpha(0)$ and $\beta(0)$ the images of the respective closed points, and by $\alpha(\eta)$ the image of the generic point of $\operatorname{Spec} k_\alpha[\![t]\!]$. There are truncation maps $\pi\colon X_\infty \to X$ and $\pi_m\colon X_m \to X$ sending an arc α (respectively, an m-jet β) to its special point $\alpha(0)$ (respectively, $\beta(0)$), as well as $\psi_m\colon X_\infty \to X_m$ and $\pi_{n,m}\colon X_n \to X_m$ for $n > m$. We denote these maps by π^X, π^X_m, ψ^X_m, and $\pi^X_{n,m}$ whenever there is a need to specify the underlying scheme X.

Let now X be a variety. Constructibility in X_∞ is defined as in [19] (see also [42, Tag 005G]): a subset $C \subset X_\infty$ is *constructible* if and only if it is a finite union of finite intersections of retrocompact open sets and their complements; equivalently, C is constructible if and only $C = \psi_m^{-1}(S)$ for some m and some constructible set $S \subset X_m$. An irreducible subset $C \subset X_\infty$ is *non-degenerate* if $C \not\subset (\operatorname{Sing} X)_\infty$.

When X is smooth, constructible sets are also called *cylinders*. Their *codimension* is defined by $\operatorname{codim}(C, X_\infty) := \operatorname{codim}(S, X_m)$ where, as before, $C = \psi_m^{-1}(S)$. Using the simple structure of the truncation maps $\pi_{n,m}$, it is easy to check that this is well defined. The codimension of C defined above agrees with topological codimension of the closure of C in X_∞; if C is irreducible and $\alpha \in C$ is the generic point, then this is the same as $\dim(\mathcal{O}_{X_\infty,\alpha})$.

When X is singular, one defines the *jet codimension* of a constructible set $C \subset X_m$ by setting $\operatorname{jet-codim}(C, X_\infty) := (m+1)\dim(X) - \dim(S)$ where, again, $C = \psi_m^{-1}(S)$ (cf. [16]). If C is irreducible and $\alpha \in C$ is the generic point, then this agrees with $\operatorname{edim}(\mathcal{O}_{X_\infty,\alpha})$.

Every arc $\alpha \in X_\infty$ defines a semi-valuation $\operatorname{ord}_\alpha\colon \mathcal{O}_{X,\alpha(0)} \to \mathbb{Z} \cup \{\infty\}$, given by $\operatorname{ord}_\alpha(h) = \operatorname{ord}_t(\alpha^\sharp(h))$, which extends to a valuation of the function field of X if and only if the generic point $\alpha(\eta)$ of the arc is the generic point of X. In a similar fashion, every jet $\beta \in X_m$ defines a function $\operatorname{ord}_\beta\colon \mathcal{O}_{X,\beta(0)} \to \{0,1,\dots,m\} \cup \{\infty\}$ given by $\operatorname{ord}_\beta(h) = \operatorname{ord}_t(\beta^\sharp(h))$, where we set $\operatorname{ord}_t(0) = \infty$.

A *prime divisor* over X is, by definition, a prime divisor E on a normal birational model $Y \to X$. Any such divisor E defines a valuation ord_E on X. A valuation on X of the form $v = q\operatorname{ord}_E$ where E is a prime divisor over X and q is a positive integer is called a *divisorial valuation*. The image in X of the generic point of E is called the *center* of the valuation (or of E), and is denoted by $c_X(v)$ or $c_X(E)$. For a divisorial valuation $v = q\operatorname{ord}_E$, the closure $C_X(v) \subset X_\infty$ of the set of arcs α such that $\operatorname{ord}_\alpha = v$ is called the *maximal divisorial set* associated to v. This is an irreducible closed constructible subset of X_∞. When $v = \operatorname{ord}_E$, we also denote this set by $C_X(E)$.

Let now X be a variety. As shown in [38] (see also, e.g., [13, 22]), the set $\pi^{-1}(\operatorname{Sing} X)$ decomposes as a finite union of irreducible components, and each component defines a divisorial valuation on X. These are called *Nash valuations* and the problem is to characterize them. Nash conjectured that, in dimension two, Nash valuations are precisely those defined by the exceptional divisors on the minimal resolution, and proposed the notion of *essential divisor* as a possible higher dimensional generalization which he speculated may characterize Nash valuations in all dimensions. These questions, which are generally referred to as the *Nash problem*, have generated a lot of activity.

Culminating the work of many people, the complete solution of the Nash problem in dimension two was eventually found by Fernandez de Bobadilla and Pe Pereira in [10], and before that, in the toric case by Ishii and Kollár [22]. A new, algebraic proof in the surface case was later found in [15], where it was proved that, in any dimension, all valuations defined by exceptional divisors on terminal models over X are Nash valuations; we call the valuations arising in this way *terminal valuations*. Nash's original guess of what the picture should be in dimension ≥ 3, however, turned out to be incorrect [12, 22, 23]. In view of this, one can reinterpret the Nash problem as asking for a characterization of Nash valuations in terms of resolution of singularities of a variety X and, more generally, its birational geometry.

3. Minimal log discrepancies

Let X be a normal variety, and assume that its canonical class K_X is $\mathbb{Q}$-Cartier. For every prime divisor E over X, if $f\colon Y \to X$ is the normal birational model where E lies, then we define the *log discrepancy* of E over X by $a_E(X) := \mathrm{ord}_E(K_{Y/X}) + 1$, and the *Mather log discrepancy* of E over X by $\widehat{a}_E(X) := \mathrm{ord}_E(\mathrm{Jac}_f) + 1$. These invariants of E only depends on the valuation ord_E, and they agree if X is smooth at the center of E.

An *effective* $\mathbb{R}$*-subscheme* Z of X is an expression $Z = \sum_{j=1}^s c_j Z_j$ where $Z_j \subset X$ is a proper closed subscheme and $c_j > 0$ for every j. Its *support* is the union of the support of the Z_j. For any effective $\mathbb{R}$-subscheme Z, we define the *log discrepancy* of E over the pair (X, Z) to be $a_E(X,Z) := a_E(X) - \sum c_j \,\mathrm{ord}_E(\mathscr{I}_{Z_j})$ where $\mathscr{I}_{Z_j} \subset \mathscr{O}_X$ is the ideal sheaf of Z_j. The *minimal log discrepancy* of (X,Z) at a point x is defined by

$$\mathrm{mld}_x(X,Z) := \inf_{c_X(E)=x} a_E(X,Z)$$

where the infimum is taken over all prime divisors E with center x. When there is no Z, we just write $\mathrm{mld}_x(X)$. We set $\mathrm{mld}_x(X,Z) = 0$ if x is the generic point of X. If $\dim X \geq 2$, then $\mathrm{mld}_x(X,Z) \in \{-\infty\} \cup [0,\infty)$. For sake of uniformity, it is convenient to declare that $\mathrm{mld}_x(X,Z) = -\infty$ whenever it is negative when $\dim X = 1$ as well.

The following is a slightly more general reformulation of the main theorem of [8]. The proof is essentially contained in [9]. We review the key part of the argument for completeness. A similar argument will also be used later in the paper, so it is useful to review it here anyway.

Theorem 1 (Inversion of adjunction [8]). *Let X be a smooth variety, $Y = H_1 \cap \cdots \cap H_e \subset X$ a normal positive dimensional subvariety defined by the complete intersection of e hypersurfaces $H_i \subset X$, and $Z = \sum c_j Z_j$ an effective $\mathbb{R}$-subscheme of X not containing Y in its support. Then for every $x \in Y$ we have*

$$\mathrm{mld}_x(Y, Z|_Y) = \mathrm{mld}_x(X, Z + eY) = \mathrm{mld}_x\Big(X, Z + \sum_{i=1}^e H_i\Big),$$

where $Z|_Y := \sum c_j (Z_j \cap Y)$.

Proof. We may assume that x is not the generic point of Y, the statement being elementary in that case. The proofs of the inequalities $\mathrm{mld}_x(Y, Z|_Y) \geq \mathrm{mld}_x(X, Z + eY) \geq \mathrm{mld}_x(X, Z + \sum H_i)$ are fairly standard and are omitted. We review the proof of the inequality

$$\mathrm{mld}_x(Y, Z|_Y) \leq \mathrm{mld}_x\Big(X, Z + \sum H_i\Big),$$

which is the hard part of the theorem. To this end, it suffices to show that for every divisorial valuation $v = \mathrm{ord}_F$ on X with center $c_X(v) = x$, there is a divisorial valuation $w = q\,\mathrm{ord}_E$ over Y with center $c_Y(w) = x$ such that

$$q\, a_E(Y, Z|_Y) \leq a_F\Big(X, Z + \sum H_i\Big).$$

We denote by Y_∞^x the reduced inverse image of x under the projection $\pi^Y\colon Y_\infty \to Y$. By definition, Y_∞^x is the set of arcs in Y stemming from x.

Let $C_X(v) \subset X_\infty$ be the maximal divisorial set associated to v. Note that $\pi^X(C_X(v))$ is an irreducible constructible set with generic point x. Consider the intersection

$$C_X(v) \cap Y_\infty.$$

As v is centered at x and $C_X(v)$ is closed under the action of the morphism $\Phi_\infty \colon \mathbb{A}^1 \times X_\infty \to X_\infty$ given by $(a, \alpha(t)) \mapsto \alpha(at)$ (cf. [8, Section 3]), we see that $C_X(v)$ contains the constant arc at x, hence $C_X(v) \cap Y^x_\infty \neq \emptyset$. It follows that x is the generic point of $\pi^X(C_X(v) \cap Y_\infty)$. Therefore we can pick an irreducible component W of $C_X(v) \cap Y_\infty$ such that $\pi^Y(W)$ has x as its generic point. Note that [9, Lemma 8.3] applies to $C_X(v) \cap Y^x_\infty$ since both $C_X(v)$ and Y^x_∞ are closed under the action of the morphism Φ_∞, hence $C_X(v) \cap Y^x_\infty \not\subset (\operatorname{Sing} Y)_\infty$. Therefore we can assume that W is not contained in $(\operatorname{Sing} Y)_\infty$. By construction W is the closure of an irreducible constructible set in Y_∞, hence, by [16], its generic point $\gamma \in W$ defines a divisorial valuation $w = q \operatorname{ord}_E$ on Y, and [9, Lemma 8.4] (its proof, to be precise) gives

$$\text{jet-codim}(W, Y_\infty) \leq \operatorname{codim}(C_X(v), X_\infty) + q \operatorname{ord}_E(\operatorname{Jac}_Y) - \sum \operatorname{ord}_F(\mathscr{I}_{H_i}).$$

Since $W \subset C_Y(w)$, [16, Theorem 3.8] implies that $\text{jet-codim}(W, Y_\infty) \geq q \cdot \widehat{a}_E(Y)$ where $\widehat{a}_E(Y)$ is the Mather log discrepancy. As Y is normal and locally complete intersection, we have $\widehat{a}_E(Y) = a_E(Y) + \operatorname{ord}_E(\operatorname{Jac}_Y)$ (see, e.g., [14, Corollary 3.5]), hence

$$\text{jet-codim}(W, Y_\infty) \geq q(a_E(Y) + \operatorname{ord}_E(\operatorname{Jac}_Y)).$$

On the other hand, as X is smooth, we have

$$\operatorname{codim}(C_X(v), X_\infty) = a_F(X).$$

Finally, by the semicontinuity of order of contact function induced by $\mathscr{I}_{Z_j}$ on X_∞, we have

$$q \operatorname{ord}_E(\mathscr{I}_{Z_j} \cdot \mathscr{O}_Y) \geq \operatorname{ord}_F(\mathscr{I}_{Z_j}).$$

By combining the above formulas, we conclude that $q\, a_E(Y, Z|_Y) \leq a_F(X, Z + \sum H_i)$. ☐

Remark 2. Going through the above proof (with $Z = 0$), suppose that $a_F(X, \sum H_i) = \operatorname{mld}_x(X, \sum H_i) \geq 0$. Then we necessarily have $q\, a_E(Y) = a_F(X, \sum H_i)$, since $q\, a_E(Y) \geq a_E(Y) \geq \operatorname{mld}_x(Y)$, hence $q\, a_E(Y) = a_E(Y) = \operatorname{mld}_x(Y)$. In particular, if $\operatorname{mld}_x(Y) > 0$ then $q = 1$. Furthermore, the inequalities in the formulas displayed in the proof must all be equalities, hence $W = C_Y(w)$.

Corollary 3. *Let X be a normal locally complete intersection variety, $Y = H_1 \cap \cdots \cap H_e \subset X$ a normal positive dimensional subvariety defined by the complete intersection of e hypersurfaces $H_i \subset X$, and $Z = \sum c_j Z_j$ an effective $\mathbb{R}$-subscheme of X not containing Y in its support. Then for every $x \in Y$ we have*

$$\operatorname{mld}_x(Y, Z|_Y) = \operatorname{mld}_x(X, Z + eY) = \operatorname{mld}_x\Big(X, Z + \sum_{i=1}^{e} H_i\Big).$$

Proof. Again, it suffices to prove that $\operatorname{mld}_x(Y, Z_Y) = \operatorname{mld}_x(X, Z + \sum H_i)$. Working locally near x, we can fix a closed embedding $X \subset A$ where A is a smooth variety, and hypersurfaces $D_1, \dots, D_r \subset A$ where $r = \operatorname{codim}(Y, A)$, such that $H_i = D_i \cap X$ for $i = 1, \dots, e$ and $X = D_{e+1} \cap \cdots \cap D_r$. Note that $Y = D_1 \cap \cdots \cap D_r$. By Theorem 1 (applied twice, to $Y \subset A$ and $X \subset A$), we have

$$\operatorname{mld}_x(Y, Z|_Y) = \operatorname{mld}_x\Big(A, Z + \sum_{i=1}^{r} D_i\Big) = \operatorname{mld}_x\Big(X, Z + \sum_{i=1}^{e} H_i\Big).$$

This completes the proof. ☐

4. Families of jets of arc type

Let X be a positive dimensional variety. For any subset $\Sigma \subset X$, we consider the sets

$$X_\infty^\Sigma := \pi^{-1}(\Sigma)_{\mathrm{red}} = \{\alpha \in X_\infty \mid \alpha(0) \in \Sigma\}$$

and

$$X_m^\Sigma := \pi_m^{-1}(\Sigma)_{\mathrm{red}} = \{\beta \in X_m \mid \beta(0) \in \Sigma\}.$$

By definition, X_∞^Σ is the set of arcs on X through Σ, and X_m^Σ is the set of m-jets through Σ.

Assume that $\Sigma \subset X$ is a closed subset. Since X_m is a scheme of finite type, each X_m^Σ decomposes into a finite union of irreducible components, and a generalization of Nash's theorem [38] tells us that the same happens for X_∞^Σ.

In the following, we denote by $\Gamma \subset X_\infty^\Sigma \setminus (\mathrm{Sing}\, X)_\infty$ the set of generic points; that is, $\alpha \in \Gamma$ if and only if α is the generic point of a non-degenerate irreducible component of X_∞^Σ. Let

$$\mu := \max_{\alpha \in \Gamma} \mathrm{ord}_\alpha(\mathrm{Jac}_X).$$

Note that $\mu < \infty$ since Γ is finite and each $\alpha \in \Gamma$ is non-degenerate.

We fix an integer $\nu \geq \mu$ such that the images $\psi_\nu(\alpha) \in X_\nu$, for $\alpha \in \Gamma$, are all distinct and there are no specializations within the set $\psi_\nu(\Gamma) \subset X_\nu$ (meaning that $\psi_\nu(\Gamma)$, with the induced topology, is discrete). The existence of such integer follows from the definition of X_∞ as inverse image of the jet schemes under the truncation maps.

Theorem 4. *Let X be a variety and $\Sigma \subset X$ a closed subset. Then for every $m \geq \mu + \nu$ there is a naturally defined injective map*

$$\Psi_m^\Sigma \colon \{\textit{non-degenerate irreducible components of } X_\infty^\Sigma\} \to \{\textit{irreducible components of } X_m^\Sigma\}$$

sending a non-degenerate irreducible component C of X_∞^Σ to the unique irreducible component D of X_m^Σ containing the image of C in X_m.

Definition 5. *We say that an irreducible component of X_m^Σ is* of arc type *if it is in the image of Ψ_m^Σ.*

Remark 6. There are two special cases about Theorem 4. The first is when we take $\Sigma = \mathrm{Sing}\, X$. In this case every irreducible component of $X_\infty^{\mathrm{Sing}\, X}$ is non-degenerate and the domain of $\Psi_m^{\mathrm{Sing}\, X}$ is the set of Nash families of arcs. The second special case is when $\Sigma = X$. In this case, the domain of Ψ_m^X is a singleton and the image of Ψ_m^X is the irreducible component of X_m dominating X, namely, the closure of $(X_{\mathrm{reg}})_m$.

We will break the proof of Theorem 4 into two steps: proving that Ψ_m^Σ is well-defined, and showing that it is injective. We may assume that Σ is nonempty, the statement being trivial otherwise.

We start with the basic observation that

$$\psi_m(X_\infty^\Sigma) \subset X_m^\Sigma.$$

This implies that for every non-degenerate irreducible component C of X_∞^Σ there exists an irreducible component D of X_m^Σ such that $\psi_m(C) \subset D$. Our goal is to prove that if $m \geq \mu + \nu$ then such component D is unique (proving well-definedness), and that a different component D of X_m^Σ occurs for each non-degenerate component C of X_∞^Σ (proving injectivity).

These properties follow by standard facts about the structure of the truncation maps, specifically from Greenberg's theorem on liftable jets [18] and from a result of Looijenga on the fibers of the truncation maps between jet schemes [29]. For convenience, we will cite these results from [9].

We start with the first assertion.

Lemma 7. *If $m \geq \mu+\nu$, then for every non-degenerate irreducible component C of X_∞^Σ there exists a unique irreducible component D of X_m^Σ such that $\psi_m(C) \subset D$.*

Proof. We proceed by contradiction and assume that there exists an integer $m \geq \mu+\nu$ and a non-degenerate irreducible component C of X_∞^Σ such that $\psi_m(C)$ is contained in the intersection of two distinct irreducible components D and D' of X_m^Σ. Whatever the value of m, we can find another integer n such that

(1) $n \geq \nu$ and
(2) $2n \geq m \geq \mu + n$.

A choice of n can be made by setting $n = \nu + k$ where k is defined by $m = \mu + \nu + k$.

Let $\alpha \in C$, $\beta \in D$ and $\beta' \in D'$ denote the respective generic points, and let $\alpha_n = \psi_n(\alpha)$, $\beta_n = \pi_{m,n}(\beta)$, and $\beta'_n = \pi_{m,n}(\beta')$ be their images in X_n. Note that both β_n and β'_n specialize to α_n. Since $\mathrm{ord}_\alpha(\mathrm{Jac}_X) \leq \mu \leq n$, we have

$$\mathrm{ord}_{\beta_n}(\mathrm{Jac}_X) \leq \mathrm{ord}_{\alpha_n}(\mathrm{Jac}_X) = \mathrm{ord}_\alpha(\mathrm{Jac}_X) \leq \mu \leq n,$$

hence [9, Proposition 4.1 (i)] implies that $\beta_n = \psi_n(\gamma)$ for some arc $\gamma \in X_\infty$. Similarly, we have $\beta'_n = \psi_n(\gamma')$ for some $\gamma' \in X_\infty$.

Note that $\gamma, \gamma' \in X_\infty^\Sigma$. In fact, as $n \geq \nu$, we see that $\gamma, \gamma' \in C$ since, by the definition of ν, no other irreducible component of X_∞^Σ contains a point whose image in X_m specializes to α_m. In particular, γ and γ' are specializations of α, hence β_n and β'_n are both generalizations and specializations of α_n, meaning that

$$\beta_n = \alpha_n = \beta'_n,$$

This means that β and β' belong to the same fiber of $X_m \to X_n$, namely, $\pi_{m,n}^{-1}(\alpha_n)$.

As $\alpha_n \in X_n^\Sigma$, the fiber $\pi_{m,n}^{-1}(\alpha_n)$ is contained in X_m^Σ, and since it contains the generic points β and β' of the irreducible components D and D' of X_m^Σ, it follows that D and D' are irreducible components of $\pi_{m,n}^{-1}(\alpha_n)$. This contradicts the fact that, by [9, Proposition 4.4 (ii)], this fiber is irreducible. □

We now turn to the second assertion.

Lemma 8. *If $m \geq \mu + \nu$, then for every irreducible component D of X_m^Σ there exists at most one non-degenerate irreducible component C of X_∞^Σ such that $\psi_m(C) \subset D$.*

Proof. We need to prove that if $m \geq \mu + \nu$ and $\alpha, \alpha' \in \Gamma$ are such that their images α_m and α'_m in X_m belongs to the same irreducible component D of X_m^Σ, then $\alpha = \alpha'$.

To prove this, let $\beta \in D$ be the generic point. Then β specializes to both α_m and α'_m, hence its image $\beta_{m-\mu} := \pi_{m,m-\mu}(\beta) \in X_{m-\mu}$ specializes to both images $\alpha_{m-\mu}$ and $\alpha'_{m-\mu}$ of α and α' in $X_{m-\mu}$. Note that $m - \mu \geq \nu \geq \mu$. By semicontinuity,

$$\mathrm{ord}_{\beta_{m-\mu}}(\mathrm{Jac}_X) \leq \mu$$

Then, by [9, Proposition 4.1 (i)], we see that $\beta_{m-\mu}$ lifts to an arc; that is, there exists $\gamma \in X_\infty$ such that $\psi_{m-\mu}(\gamma) = \beta_{m-\mu}$. By construction, $\gamma \in \pi^{-1}(\Sigma)$, hence there exists $\alpha'' \in \Gamma$ specializing to γ. It follows that the image of α'' in $X_{m-\mu}$ specializes to both $\alpha_{m-\mu}$ and $\alpha'_{m-\mu}$. As $m - \mu \geq \nu$, we conclude that $\alpha = \alpha'' = \alpha'$. □

Proof of Theorem 4. Lemma 7 implies that Ψ_m^Σ is well-defined for $m \geq \mu + \nu$, and Lemma 8 that this map is injective. □

Remark 9. The definition of the function Ψ_m^Σ constructed in Theorem 4 can be extended to all $m \geq 0$ as long as one is willing to regard them as multivalued function, sending each C to all components D containing the image of C.

5. The question of surjectivity

Given Theorem 4, it is natural to ask under which conditions on singularities one can guarantee that the maps Ψ_m^Σ are surjective. These functions are well-defined for $m \gg 1$, but if we are willing to regard them as a multivalued functions, then we can remove the constrain on m. The question of surjectivity still makes sense for multivalued functions.

Before we move to discuss the case we will be focusing on, it may be instructive to point out that there is already an interesting answer to the problem (a sufficient condition for surjectivity) in the special case where $\Sigma = X$. This comes from Mustaţă's theorem on locally complete intersection canonical singularities.

Theorem 10 ([37]). *Let X be a locally complete intersection variety with canonical singularities. Then Ψ_m^X is well defined and surjective for every m.*

Proof. As X_∞ has only one non-degenerate irreducible component (and in fact only one irreducible component since it is irreducible by Kolchin's theorem [24]), this is just a restatement of Mustaţă's theorem on the irreducibility of the jet schemes, since any additional irreducible component of X_m would lie over the singular locus of X and therefore would not contain the image of X_∞. □

Like in Mustaţă's theorem, we will be focusing on locally complete intersection canonical singularities. Our goal is to find a class of singularities for which $\Psi_m^{\operatorname{Sing} X}$ is surjective.

To get a sense of what one can expect, we start by reviewing some cases that are already understood.

Example 11 (Nodal curve). The case where X is a nodal curve already shows that one cannot expect $\Psi_m^{\operatorname{Sing} X}$ to be always surjective. Indeed, if $x \in X$ is a node, then for $m \ge 3$ the set X_m^x has $m-1$ irreducible components, and only two of them are in the image of Ψ_m^x.

Example 12 (Affine cones). Let $V \subset \mathbb{P}^{N-1}$ be a smooth complete intersection variety defined by equations of degree r, let $X \subset \mathbb{A}^N$ be the affine cone over V, and let $x \in X$ be the vertex. As the blow-up of x gives a resolution of X with a single exceptional divisor, one easily see that X_∞^x is irreducible. On the other hand, for every $m \ge r$ we have

$$\pi_m^{-1}(x) \cong X_{m-r} \times \mathbb{A}^{N(r-1)},$$

see, e.g., the proof of [17, Theorem 3.5]. By [37, Theorem 0.1], we know that if X is canonical then X_m is irreducible for all m, and conversely, using also [37, Proposition 1.6], if X is not canonical at x then X_m is reducible for all $m \gg 1$. It follows that X_m^x is irreducible (hence Ψ_m^x is surjective) for all $m \ge r$ if X is canonical, and is reducible (hence Ψ_m^x fails to be surjective) for all $m \gg 1$ if X is not canonical.

Mourtada, in part in collaboration with Plénat and Cobo, has studied the irreducible decomposition of $X_m^{\operatorname{Sing} X}$ in many explicit situations where X is a surface [6, 34–36]; see also [26] for related work. While in some cases these results indicate that the number of components continues to grow with m, there are also cases where the number of components stabilizes and matches the number of Nash families.

Example 13 (Toric surface singularities). The irreducible decomposition of $X_m^{\operatorname{Sing} X}$ was computed for toric surfaces by Mourtada [35], and the only case where we have the same number of components as Nash families is when X has A_n-singularities.

Example 14 (Du Val singularities). It is proved in [34] that, for $m \gg 1$, the number of families of m-jets through a Du Val singularity coincides with the number of exceptional divisors on the minimal resolution, hence with the number of Nash families of arcs. It follows in particular that in this case $\Psi_m^{\operatorname{Sing} X}$ is a bijection.

Example 15 (cA-type singularities). Another case where we can check directly that $\Psi_m^{\mathrm{Sing}\,X}$ is a bijection is that of cA-type singularities. Nash families of arcs on these singularities were described in [23], and the deformation argument used in their proof can be adapted to show that, for $m \gg 1$, there is the same number of families of m-jets through the singularity, proving that $\Psi_m^{\mathrm{Sing}\,X}$ is a bijection in this case as well. More specifically, suppose X is defined by an equation

$$xy = f(z_1,\dots,z_{d-1})$$

in $A = \mathbb{A}^{d+1}$, where $\mu := \mathrm{mult}_0(f) \ge 2$. The proof in [23] begins by identifying $\mu - 1$ irreducible open sets $U_i \subset X_\infty^0$, for $1 \le i \le \mu-1$, given by

$$U_i = \{\alpha \in X_\infty^0 \mid \mathrm{ord}_\alpha(x) = i, \mathrm{ord}_\alpha(y) = \mu - i, \mathrm{ord}_\alpha(f) = \mu\}.$$

The proof then goes by showing that every arc $\alpha \in X_\infty^0$ can be deformed (in X_∞^0) to an arc α^* with $\mathrm{ord}_{\alpha^*}(f) = \mu$. Clearly such arc must belong to one of the U_i, hence proving that the closures of these sets give all irreducible components of X_∞^0. The deformation is done in several steps: first, one deforms α to an arc α' with $\mathrm{ord}_{\alpha'}(f) < \infty$, and if $\mathrm{ord}_{\alpha'} > \mu$, then one deforms α' to an arc α'' with $\mathrm{ord}_{\alpha''}(f) < \mathrm{ord}_{\alpha'}(f)$. After a finite number of steps, this process produces the desired arc α^*.

This argument can be adapted to characterize the irreducible components of X_m^0, for any given $m \ge \mu$, as follows. For $1 \le i \le \mu - 1$, we consider the irreducible open sets

$$V_i = \{\beta \in X_m^0 \mid \mathrm{ord}_\beta(x) = i, \mathrm{ord}_\beta(y) = \mu - i, \mathrm{ord}_\beta(f) = \mu\}.$$

Given any $\beta \in X_m^0$, we take any lift $\alpha \in A_\infty^0$ (i.e., any arc α on A such that $\psi_m^A(\alpha) = \beta$) and apply the same deformation argument as in [23] to produce a new arc $\alpha^* \in A_\infty^0$ such that $\mathrm{ord}_{\alpha^*}(f) = \mu$. In fact, without loss of generality we can pick α so that $\mathrm{ord}_\alpha(f) < \infty$, hence skip the first deformation and just deform to reduce $\mathrm{ord}_\alpha(f)$ if the order of contact is larger than μ. The key observation here is that, just like in [23] the deformation keeps the arc on X, in this setting the deformation maintains the order of contact of the arc with X, hence the corresponding deformation at level m stays on X_m.

The above examples are mainly understood through their equations. Our goal is to identify a new class of examples of arbitrary dimensions where $\Psi_m^{\mathrm{Sing}\,X}$ is surjective, without having to rely on explicit equations. This will be done in the next two sections.

6. Singularities of maximal embedding codimension

For a local ring $(R,\mathfrak{m})$ we denote by $\dim(R)$ the Krull dimension, by $\mathrm{edim}(R)$ the embedding dimension (the dimension of the Zariski tangent space) and by $\mathrm{ecodim}(R)$ the embedding codimension (the codimension of the tangent cone in the Zaristi tangent space). When R is Noetherian, the latter is also known as the regularity defect [27] and is equal to $\mathrm{edim}(R) - \dim(R)$.

We start by establishing the following bound on embedding codimension for normal locally complete intersection singularities. The bound is likely known to experts.

Proposition 16. *Let X be a normal locally complete intersection variety. Then*

$$\mathrm{ecodim}(\mathscr{O}_{X,x}) \le \dim(\mathscr{O}_{X,x}) - \mathrm{mld}_x(X)$$

for every $x \in X$.

Proof. The assertion being trivial if $\mathrm{mld}_x(X) = -\infty$, we assume that $\mathrm{mld}_x(X) \ge 0$. Working locally in X, we may assume that X is embedded in an affine space $A := \mathbb{A}^N$. Let $d = \dim(X)$, $r = \dim(\mathscr{O}_{X,x})$, $e = \mathrm{ecodim}(\mathscr{O}_{X,x})$ and $c = \mathrm{codim}(X, A)$. By inversion of adjunction (see Theorem 1),

$$\mathrm{mld}_x(X) = \mathrm{mld}_x(A, cX).$$

Let $\mathfrak{m}_x \subset \mathscr{O}_{X,x}$ be the maximal ideal. By applying [31, Theorem 25.2] to the sequence $k \to \mathscr{O}_{X,x} \to k_x$, we get the exact sequence

$$0 \to \mathfrak{m}_x/\mathfrak{m}_x^2 \to \Omega_{X/k} \otimes k_x \to \Omega_{k_x/k} \to 0.$$

This gives

$$\dim_{k_x}(\Omega_{X/k} \otimes k_x) = \operatorname{edim}(\mathscr{O}_{X,x}) + d - r = d + e.$$

By the isomorphism $X_1 \cong \operatorname{Spec}(\operatorname{Sym}(\Omega_{X/k}))$ (see [9, Example 2.5] or [44, (1.4)]), we have $X_1^x \cong \operatorname{Spec}(\operatorname{Sym}(\Omega_{X/k} \otimes k_x))$, hence

$$\dim_k(\overline{X_1^x}) = \dim_{k_x}(X_1^x) + d - r = 2d + e - r.$$

The reduced inverse image $V \subset A_\infty$ of the closure $\overline{X_1^x} \subset A_1$ of X_1^x is a closed irreducible cylinder. Let v be the valuation defined by V (namely, $v = \operatorname{ord}_\alpha$ where $\alpha \in V$ is the generic point). By [7, Theorem C], v is a divisorial valuation, i.e., $v = p \operatorname{ord}_F$ where F is a prime divisor over A and p is a positive integer. Note that, by construction, we have $v(\mathscr{I}_X) \geq 2$. If $C_A(v) \subset A_\infty$ is the maximal divisorial set associated to the valuation, then we have $V \subset C(v)$, hence

$$\operatorname{codim}(V, A_\infty) \geq \operatorname{codim}(C_X(v), A_\infty) = p\, a_F(A)$$

(the last formula is implicit in [7]; for a direct reference, see [16, Theorem 3.8]). On the other hand,

$$\begin{aligned}
\operatorname{codim}(V, A_\infty) &= \operatorname{codim}(\overline{X_1^x}, A_1) \\
&= \dim(A_1) - \dim(\overline{X_1^x}) \\
&= 2(d+c) - (2d + e - r) \\
&= r - e + 2c.
\end{aligned}$$

It follows that

$$\operatorname{mld}_x(A, cX) \leq a_F(A, cX) \leq \frac{1}{p}\big(\operatorname{codim}(V, A_\infty) - 2c\big) \leq r - e,$$

where we use in the last inequality that $\operatorname{mld}_x(X) \geq 0$ to ensure that the inequality is preserved when we clear the denominator p. □

Definition 17. *In accordance with Proposition 16, we say that a normal locally complete intersection variety X has* maximal embedding codimension singularities *if*

$$\operatorname{ecodim}(\mathscr{O}_{X,x}) = \dim(\mathscr{O}_{X,x}) - \operatorname{mld}_x(X)$$

for every $x \in X$.

Remark 18. Smooth varieties have maximal embedding codimension singularities.

Remark 19. Every locally complete intersection variety with maximal embedding codimension singularities has log canonical singularities, since the condition implies that $\operatorname{mld}_x(X) \neq -\infty$ hence $\operatorname{mld}_x(X) \geq 0$ for all $x \in X$. Note that if X is a curve then normality already implies that X is smooth.

Example 20 (Hypersurface singularities). A normal hypersurface singularity $x \in X$ has maximal embedding codimension if and only if $\operatorname{mld}_x(X) = \dim(\mathscr{O}_{X,x}) - 1$. In particular Du Val singularities in dimension 2 and isolated cDV singularities of dimension 3 are all the examples in these dimensions of isolated hypersurface singularities of maximal embedding codimension (cf. [41]).

7. Higher Du Val singularities

We now identify a particular subclass of locally complete intersection varieties with maximal embedding codimension singularities which can be thought as a higher dimensional version of Du Val singularities.

Definition 21. *Let X be a normal locally complete intersection variety of dimension $d \geq 2$. We say that a point $x \in X$ is a* higher Du Val *(hDV)* singularity *if*

$$\mathrm{mld}_x(X) = \dim(\mathcal{O}_{X,x}) - \mathrm{ecodim}(\mathcal{O}_{X,x}) = 1.$$

By definition, hDV singularities are canonical but not terminal. They can be locally embedded as complete intersection singularities of codimension $d-1$ in $\mathbb{A}^{2d-1}$ (cf. [5, Theorem 3.15]) but not in any smaller affine space. In dimension two, these are the same as the Du Val singularities.

Remark 22. It is useful to compare the above definition with another classical way of generalizing Du Val singularities, namely, compound Du Val singularities. Compound Du Val singularities preserve two properties of Du Val singularities: being hypersurface singularities, and having minimal log discrepancy $\mathrm{mld}_x(X) = \dim(X) - 1$. By contrast, the definition of hDV singularities preserves the condition that $\mathrm{mld}_x(X) = 1$ and requires maximal embedding codimension. The attribute "higher" in hDV singularity reflects at the same time that these are higher dimensional and higher codimensional generalizations of Du Val singularities.

Remark 23. If we extended Definition 21 to the case $d = 1$, then in dimension one the definition would characterize smooth points on curves. This says something meaningful about the behavior of this notion as a function of dimension. We prefer to assume $d \geq 2$ as we want to regard this as defining a class of actual singular points.

Example 24 (Intersections of quadric cones). In higher codimensions, the simplest example of a hDV singularity is the cone $X \subset \mathbb{A}^{2e+1}$ over the transversal intersection of e smooth quadrics in $\mathbb{P}^{2e}$. The blow-up of the vertex x of the cone gives a log resolution of $(\mathbb{A}^{2e+1}, X)$, and

$$\mathrm{mld}_x(X) = \mathrm{mld}_x(\mathbb{A}^{2e+1}, eX) = 1$$

where the minimal log discrepancy is computed by the exceptional divisor of the blow-up.

More generally, we have the following set of examples, which shows the clear analogy with Du Val singularities.

Proposition 25. *Let $e \geq 1$, let $(u_1, \dots, u_{2e-2}, x, y, z)$ denote affine coordinates of $\mathbb{A}^{2e+1}$, and let $X \subset \mathbb{A}^{2e+1}$ be the subvariety defined by the vanishing of e general linear combinations of any finite set of generators of the ideal*

$$\mathfrak{a} = (u_1, \dots, u_{2e-2})^2 + \mathfrak{b}$$

of $k[u_1, \dots, u_{2e-2}, x, y, z]$, where $\mathfrak{b}$ is one of the following:

$$\mathfrak{b} = \begin{cases} (x^2, y^2, z^{n+1}) \quad (n \geq 1) & A_n\text{-type} \\ (z^2, x^2 y, y^{n-2}) \quad (n \geq 4) & D_n\text{-type} \\ (z^2, x^3, y^4) & E_6\text{-type} \\ (z^2, x^3, xy^3) & E_7\text{-type} \\ (z^2, x^3, y^5) & E_8\text{-type} \end{cases}$$

Then X has a hDV singularity at the origin $0 \in \mathbb{A}^{2e+1}$.

Proof. Clearly, X is a complete intersection variety with an isolated singularity at the origin, and $\operatorname{ecodim}(\mathscr{O}_{X,0}) = e$. What is left to show is that $\operatorname{mld}_0(X) = 1$. Note that $\operatorname{mld}_0(X) = \operatorname{mld}_0(\mathbb{A}^{2e+1}, eX)$. By looking at the exceptional divisor of the blow-up of $\mathbb{A}^{2e+1}$ at the origin, we see that $\operatorname{mld}_0(\mathbb{A}^{2e+1}, eX) \le 1$. On the other hand, a special case of the Thom–Sebastiani theorem (see [25, Proposition 8.21]) gives us the following formula for the log canonical thresholds of $\mathfrak{a}$:

$$\operatorname{lct}(\mathfrak{a}) = \operatorname{lct}((u_1, \dots, u_{2e-2})^2) + \operatorname{lct}(\mathfrak{b}) = e - 1 + \operatorname{lct}(\mathfrak{b}).$$

What we know about Du Val singularities already tells us that $\operatorname{lct}(\mathfrak{b}) > 1$; this can also be checked directly using Howald's formula for the log canonical threshold of monomial ideals [20]. Therefore $\operatorname{lct}(\mathfrak{a}) > e$, hence $\operatorname{mld}_0(\mathbb{A}^{2e+1}, eX) > 0$. We conclude that $\operatorname{mld}_0(\mathbb{A}^{2e+1}, eX) = 1$, as required. □

Remark 26. Assuming $k = \mathbb{C}$, hDV singularities are closely related certain hypersurface singularities studied by Arnol'd [1]. These are isolated hypersurface singularities characterized by the property that their versal deformations only contain finitely many analytically inequivalent singularities, and are known as *simple singularities.* They were classified in [1]; see also [4, Example (3.4)]. In the notation of Proposition 25, for any $\mathfrak{a}$ (which, according to the proposition, corresponds to an example of a hDV singularity) the vanishing of a general element $h \in \mathfrak{a}$ defines a simple singularity, and all simple singularities arise in this way. Conversely, the examples of hDV singularities provided by Proposition 25 are complete intersections of simple singularities of the same type.

Proposition 27. *Let X be a variety with hDV singularities. Then X has isolated singularities.*

Proof. Let $f\colon Y \to X$ be a log resolution that is an isomorphism over X_{reg}, and let E be the reduced exceptional locus. Note that $K_{Y/X} \ge 0$.

If $\dim(\operatorname{Sing} X) \ge 1$, then we can find a closed point $x \in \operatorname{Sing} X$ such that x is not the center of any component of E. On the other hand, $x \in f(E)$. Now, let F be an arbitrary prime divisor over X with center $c_X(F) = x$. We may assume that F lies on a nonsingular model $g\colon Z \to Y$. Since $f^{-1}(x)$ has codimension at least 2 in Y and contains the center of F in Y, we have $\operatorname{ord}_F(K_{Z/Y}) \ge 1$. It follows that $\operatorname{ord}_F(K_{Z/X}) \ge 1$, hence $a_F(X) \ge 2$. This contradicts the fact that, by hypothesis, $\operatorname{mld}_x(X) = 1$. □

Theorem 28. *Let $x \in X$ be a hDV singularity.*

(1) *The multivalued function Ψ^x_m is surjective for all m.*
(2) *An irreducible set $C \subset X^x_\infty$ is a non-degenerate irreducible component if and only if $C = C_X(E)$ for some prime divisor E over X with center $c_X(E) = x$ and log discrepancy $a_E(X) = 1$.*

Proof. By Proposition 27, $x \in X$ is an isolated singularity.

Let $d = \dim(X) = \dim(\mathscr{O}_{X,x})$ and $e = \operatorname{ecodim}(\mathscr{O}_{X,x})$. Note that, by our assumption, $e = d - 1$. Though not strictly necessary, to simplify the notation we apply [5, Theorem 3.15] to reduce to the case where X is embedded in $A := \mathbb{A}^{d+e}$.

Let $f_1, \dots, f_e \in k[x_1, \dots, x_{d+e}]$ be local generators of the ideal of X in A at the point x. For every $j \ge 1$, we denote by $f_i^{(j)}$ the j-th Hasse–Schmidt derivative of f_i. As $X^x_1 = A^x_1$ (by our choice of embedding), the polynomials f_i and f_i' vanish identically on A^x_1, hence on A^x_m. Therefore, the ideal of X^x_m in A^x_m is generated by the elements $f_i^{(j)}$ for $1 \le i \le e$ and $2 \le j \le m$. In particular, if D is any irreducible component of X^x_m, then

$$\operatorname{codim}(D, A^x_m) \le e(m-1).$$

Noticing that $\operatorname{codim}(A^x_m, A_m) = d + e = 2e + 1$, it follows that

$$\operatorname{codim}(D, A_m) \le e(m+1) + 1$$

Let $V \subset A_\infty$ be the cylinder over $D \subset A_m$. This is a closed irreducible cylinder of codimension

$$\operatorname{codim}(V, A_\infty) = \operatorname{codim}(D, A_m) \le e(m+1)+1.$$

If $v = p\operatorname{ord}_F$ is the divisorial valuation defined by the generic point of V, then $V \subset C(v)$, hence

$$\operatorname{codim}(V, A_\infty) \ge \operatorname{codim}(C_X(v), A_\infty) = p\, a_F(A).$$

Note that $v(\mathscr{I}_X) \ge m+1$. Then

$$\operatorname{mld}_x(A, eX) \le \frac{1}{p}\big(\operatorname{codim}(V, A_\infty) - e(m+1)\big) \le 1.$$

Since by our assumption on the singularity we have $\operatorname{mld}_x(X) = 1$, and $\operatorname{mld}_x(X) = \operatorname{mld}_x(A, eX)$ by inversion of adjunction, it follows that all inequalities in the above formula are equalities, and in particular $V = C_A(v)$.

We see from the proof of Theorem 1 (see also Remark 2) that there is a non-degenerate irreducible component W of $V \cap X_\infty$. Furthermore, any such component W is equal to $C_X(E)$ for some prime divisor E over X with center $c_X(E) = x$ and log discrepancy $a_E(X) = 1$. Note that $W \subset X_\infty^x$.

We may assume that E is an exceptional divisor on a log resolution $f\colon X' \to X$ of X. We apply [2, Corollary 1.4.3] to X and f, with $\Delta = \Delta_0 = 0$ and $\mathfrak{E}$ equal to the set of exceptional divisors with log discrepancy at most 1. The output of this operation is a terminal model Y over X where the center of val_E has codimension 1. This implies that val_E is a terminal valuation, hence, by [15, Theorem 1.1], a Nash valuation.

The fact that W is the maximal divisorial set of a Nash valuation implies that W is an irreducible component of X_∞^x. By construction, the image of W in X_m^x is contained in D, showing that D is in the image of Ψ_m^x. This proves (1).

To conclude, we use what we just proved and the injectivity of Ψ_m^x established in Theorem 4 for $m \gg 1$ to infer that every non-degenerate irreducible component of X_∞^x is of the form $C_X(E)$ for some prime divisor E over X with center $c_X(E) = x$ and log discrepancy $a_E(X) = 1$. Conversely, as explained above, [15, Theorem 1.1] implies that for every prime divisor E over X with center $c_X(E) = x$ and log discrepancy $a_E(X) = 1$, the set $C_X(E)$ is an irreducible component of X_∞^x. This gives (2). □

We apply this result to give a solution of the Nash problem for varieties with hDV singularities.

Corollary 29. *Let X be a variety with hDV singularities. For a divisorial valuation* ord_E *on X, the following are equivalent:*

(1) ord_E *is a Nash valuation.*
(2) ord_E *is a terminal valuation.*
(3) *E is exceptional over X and $a_E(X) = 1$.*

Proof. The implication (3) ⇒ (2) follows by [2, Corollary 1.4.3], the implication (2) ⇒ (1) follows by [15, Theorem 1.1], and the implication (1) ⇒ (3) follows by Theorem 28. □

This result illustrates how this class of singularities preserves some of the properties that characterize Du Val singularities. By [2, Corollary 1.4.3], there is a terminal model $Y \to X$ whose exceptional locus consists exactly of the divisors with log discrepancy 1 over X; from this perspective, this model should be regarded as the analogue of the minimal resolution of a Du Val singularity. Needless to say, it would be interesting to further study the structure of these higher dimensional singularities.

8. Higher compound Du Val singularities

In this section, we look again at rational singularities of maximal embedding codimension. We recall that these are normal, isolated, locally complete intersection singularities. A particular example of such singularities is given by isolated compound Du Val singularities. Compound Du Val singularities were originally introduced in dimension three in [41]. In general, they are defined as follows.

Definition 30. *We say that $x \in X$ is a* compound Du Val *(cDV)* singularity *if the surface $S \subset X$ cut out by* $\dim(X) - 2$ *general hyperplane sections through x has a Du Val singularity at x.*

The following property characterizes isolated cDV singularities (cf. [30] for an earlier result in this direction in dimension three).

Proposition 31. *Let $x \in X$ be an isolated hypersurface singularity of dimension $d \ge 3$. Then the following are equivalent:*

(1) *$x \in X$ is a cDV singularity.*
(2) $\mathrm{mld}_x(X) = d - 1$*, and for every divisor E over X computing* $\mathrm{mld}_x(X)$ *we have* $\mathrm{ord}_E(\mathfrak{m}_x) = 1$ *and E computes* $\mathrm{mld}_x(X, (d-2)\{x\})$.

In particular, isolated cDV singularities are normal locally complete intersection singularities of maximal embedding codimension, according to Definition 17.

Proof. First note that if $x \in X$ is a normal locally complete intersection singularity, then, by Proposition 16, we have $\mathrm{mld}_x(X) \le d-1$ and $\mathrm{ord}_E(\mathfrak{m}_x) \ge 1$ for any divisor E over X with center x. On the other hand, if S is cut out by $d-2$ general hyperplane sections through x, then $\mathrm{mld}_x(S) \le 1$, and $x \in S$ is a Du Val singularity if and only if $\mathrm{mld}_x(S) = 1$.

Assume (1) holds. If S is cut out by general hyperplane sections as in Definition 30, then $\mathrm{ord}_E(\mathscr{I}_S) = \mathrm{ord}_E(\mathfrak{m}_x)$ for any E computing $\mathrm{mld}_x(X)$ and

$$1 = \mathrm{mld}_x(S) = \mathrm{mld}_x(X, (d-2)S) \le a_E(X, (d-2)S) = \mathrm{mld}_x(X) - (d-2)\,\mathrm{ord}_E(\mathfrak{m}_x)$$

by inversion of adjunction (Corollary 3). The properties listed in (2) follows easily from this inequality.

Conversely, if (2) holds and E is any divisor computing $\mathrm{mld}_x(X)$, then we have

$$\mathrm{mld}_x(S) = \mathrm{mld}_x(X, (d-2)S) = a_E(X, (d-2)S) = a_E(X, (d-2)\{x\}) = 1,$$

hence S is a Du Val singularity. Here we used again that S is cut out by general hyperplane sections through x, hence $\mathrm{ord}_E(\mathscr{I}_S) = \mathrm{ord}_E(\mathfrak{m}_x)$. □

Proposition 31 implies in particular that cDV singularities are examples of rational singularities of maximal embedding codimension. However, they satisfy an additional property, namely, the condition that for every divisor E over X computing $\mathrm{mld}_x(X)$ we have $\mathrm{ord}_E(\mathfrak{m}_x) = 1$ and E computes $\mathrm{mld}_x(X, (d-2)\{x\})$. It is not clear to us whether this condition might follow from the definition of singularity of maximal embedding codimension.

By regarding hDV singularities as a higher dimensional version of Du Val singularities, we extend the notion of cDV singularity in the following way.

Definition 32. *We say that $x \in X$ is a* higher compound Du Val *(hcDV)* singularity *if, for some $r \ge 0$, the variety $Y \subset X$ cut out by r general hyperplane sections through x has a hDV singularity at x. (Alternatively, one could call these singularities* compound higher Du Val singularities.*)*

A straightforward adaptation of Proposition 31 gives the following property.

Proposition 33. *Let $x \in X$ be an isolated locally complete intersection singularity of dimension $d \geq 3$ and embedding codimension e. Then the following are equivalent:*

(1) *$x \in X$ is a hcDV singularity.*
(2) $\mathrm{mld}_x(X) = d - e$*, and for every divisor E over X computing* $\mathrm{mld}_x(X)$ *we have* $\mathrm{ord}_E(\mathfrak{m}_x) = 1$ *and E computes* $\mathrm{mld}_x(X, (d-e-1)\{x\})$.

In particular, isolated hcDV singularities are normal locally complete intersection singularities of maximal embedding codimension, according to Definition 17.

Theorem 34. *Let $x \in X$ be an isolated hcDV singularity. Then the function Ψ^x_m is surjective, hence a bijection, for all $m \gg 1$.*

Proof. With the case of hDV singularities already settled in Theorem 28, we may assume that $\mathrm{mld}_x(X) > 1$. Let $d = \dim(X)$ and $e = \mathrm{ecodim}(\mathscr{O}_{X,x})$. Note that $\mathrm{mld}_x(X) = d - e$. As in the proof of Theorem 28, for simplicity we reduce to the case where X is embedded in $A := \mathbb{A}^{d+e}$. Let $H := \mathbb{A}^{2e+1} \subset A$ a general linear subspace of codimension $d - e - 1$ through x, so that $Y := X \cap H$ is a variety with a hDV singularity at x.

Let m be any positive integer such that:

(1) Theorem 4 holds for Y (with $\Sigma = \{x\}$), and
(2) for every divisor E over X computing $\mathrm{mld}_x(X)$, we have

$$d(m+1) - \dim(\psi^X_m(C_X(E))) = \text{jet-codim}(C_X(E), X_\infty).$$

Note that these conditions hold for all $m \gg 1$. We can guarantee (1) because there are only finitely many divisorial valuations computing $\mathrm{mld}_x(X)$ since the minimal log discrepancy is positive.

Let D be an irreducible component of X^x_m, and pick an irreducible component D' of $D \cap Y^x_m$. If $h_1, \dots, h_{d-e-1}$ are linear forms on A cutting out H, then $D \cap Y^x_m$ is cut out off D by the equations $h^{(j)}_i = 0$ for $1 \leq i \leq d - e - 1$ and $1 \leq j \leq m$, hence

$$\mathrm{codim}(D', D) \leq (d - e - 1)m.$$

If $f_1 = \cdots = f_e = 0$ are local equations of X at x in A, then X^x_m is cut out in A^x_m by the equations $f^{(j)}_i = 0$ for $1 \leq i \leq e$ and $2 \leq j \leq m$. Here we are using that X is singular at x hence, for all i, both f_i and f'_i vanish identically on A^x_m. This implies that

$$\mathrm{codim}(D, A^x_m) \leq e(m-1).$$

Since $\mathrm{codim}(H^x_m, A^x_m) = (d - e - 1)m$, we obtain

$$\mathrm{codim}(D', H^x_m) \leq e(m-1),$$

hence

$$\mathrm{codim}(D', H_m) \leq e(m+1) + 1.$$

Let $V' \subset H_\infty$ the cylinder over D'. We have

$$\mathrm{codim}(V', H_\infty) \leq e(m+1) + 1.$$

Write $\mathrm{ord}_{V'} = p' \mathrm{ord}_{F'}$ for some divisor F' over H and some positive integer p'. The same argument as in the proof of Theorem 28 implies

$$1 = \mathrm{mld}_x(Y) = \mathrm{mld}_x(H, eY) \leq \frac{1}{p}\big(\mathrm{codim}(V', H_\infty) - e(m+1)\big) \leq 1.$$

This implies that $p' = 1$, $V' = C_H(F')$, and F' computes $\mathrm{mld}_x(H, eY)$. If $W' \subset Y_\infty$ is any non-degenerate irreducible component of $V' \cap Y_\infty$, then the argument also shows that W' is an irreducible component of Y^x_∞ and it is equal to $C_Y(E')$ for some divisor E' over Y with $a_{E'}(Y) = 1$. Furthermore, the argument implies that all inequalities above are equalities.

In particular, if $V \subset A_\infty$ is the cylinder over D then

$$\operatorname{codim}(V, A_\infty) = em + d.$$

Writing $\operatorname{ord}_V = p \operatorname{ord}_F$ for some divisor F over A and arguing again as in the proof of Theorem 28 (using now that, by Proposition 33, $\operatorname{mld}_x(A, eX) = d - e$), we conclude that $V = C_X(F)$ where F is a divisor over A computing $\operatorname{mld}_x(A, X)$. Moreover, there is an irreducible component W of $V \cap X_\infty$ that is not contained in $(\operatorname{Sing} X)_\infty$, and this component is of the form $W = C_X(E)$ for a divisor E over X computing $\operatorname{mld}_x(X)$.

By construction,

$$\psi^X_m(W) \subset D.$$

We do not know, however, that W is an irreducible component of X^x_∞. Note that we cannot apply [15] as we did in the proof of Theorem 28 (and, above, for W') since now E does not define a terminal valuation over X. The claim is that $Z \subset X^x_\infty$ is any irreducible component containing W, then

$$\psi^X_m(Z) \subset D.$$

This is all we need to conclude that D is in the image of Ψ^x_m.

To prove the claim, we proceed as follows. First, note that $W' \subset W \cap Y_\infty$. As discussed above, we have $W = C_X(E)$ and $W' = C_S(E')$ where E and E' are divisors over X and S, respectively, with center x and log discrepancies $a_E(X) = d - e$ and $a_{E'}(X) = 1$. In particular,

$$a_{E'}(X) = a_E(X) - (d - e - 1).$$

Since X and S are locally complete intersections at x, we have

$$a_E(X) = \widehat{a}_E(X) - \operatorname{ord}_E(\operatorname{Jac}_X),$$
$$a_{E'}(Y) = \widehat{a}_{E'}(Y) - \operatorname{ord}_{E'}(\operatorname{Jac}_Y)$$

by [14, Corollary 3.5]. By Teissier's Idealistic Bertini Theorem [43, 2.15 Corollary 3], we have $\overline{\mathrm{Jac_Y}} = \overline{\mathrm{Jac_X|_Y}}$ (the bar denoting integral closure), hence it follows by the inclusion $W' \subset W \cap Y_\infty$ that

$$\operatorname{ord}_{E'}(\operatorname{Jac}_Y) \geq \operatorname{ord}_E(\operatorname{Jac}_X).$$

Combining these formulas, we see that

$$\widehat{a}_{E'}(Y) \geq \widehat{a}_E(X) - (d - e - 1).$$

By [16] and the assumption (2) on our choice of m, we have

$$\widehat{a}_E(X) = d(m+1) - \dim\big(\psi^X_m(W)\big),$$
$$\widehat{a}_{E'}(Y) \leq (e+1)(m+1) - \dim\big(\psi^Y_m(W')\big).$$

Using the previous inequality, we get

$$\dim\big(\psi^Y_m(W')\big) \leq \dim\big(\psi^X_m(W)\big) - (d - e - 1)n.$$

Observe that $\psi^Y_m(W')$ is contained in $\psi^X_m(W) \cap Y^x_m$, which is cut out from $\psi^X_m(W)$ by the equations $h^{(j)}_i = 0$ for $1 \leq i \leq d - e - 1$ and $1 \leq j \leq m$. Here we are using that the polynomials h_i already vanish on X^x_m, hence on $\psi^X_m(W)$. It follows that

$$\dim\big(\psi^Y_m(W')\big) = \dim\big(\psi^X_m(W)\big) - (d - e - 1)m,$$

and the $h^{(j)}_i$ form a regular sequence at the generic point of $\psi^Y_m(W')$.

Now, let Z be an irreducible component of X^x_∞ containing W, and assume by contradiction that $\psi^X_m(Z) \not\subset D$. Then $\psi^X_m(Z)$ must be contained in another irreducible component of X^x_m. In particular, if $\widetilde{D}$ denote the union of all irreducible components of X^x_m containing $\psi^Y_m(W')$ and different from D, then

$$\psi^Y_m(W') \subset D \cap \widetilde{D}.$$

Note that $(D \cup \widetilde{D}) \cap Y_m^x$ is the union of the irreducible components of Y_m^x containing $\psi_m^Y(W')$. Since the elements $h_i^{(j)}$ form a regular sequence at each generic point of $D \cap \widetilde{D}$ and cut out Y_m^x on X_m^x, it follows that $(D \cup \widetilde{D}) \cap Y_m^x$ must be reducible. This means that $\psi_m^Y(W')$ is contained in more than one irreducible component of Y_m^x, contradicting Theorem 4, which is supposed to holds for Y by our assumption (1) on m.

We conclude that $\psi_m^X(Z) \subset D$, as claimed. This finishes the proof of the theorem. □

9. The graph generated by families of jets

Following [6, 34, 35], to any variety X we associate a directed graph Γ_X as follows.

Definition 35. *Given a variety X, let Γ_X be the directed graph whose vertices corresponds to the irreducible components of $X_m^{\operatorname{Sing} X}$ for $m \geq 0$; an edge is drawn from a vertex v to a vertex v' whenever v and v' correspond, respectively, to irreducible components $D \subset X_m^{\operatorname{Sing} X}$ and $D' \subset X_{m+1}^{\operatorname{Sing} X}$ with $\pi_{m+1,m}(D') \subset D$. We say that a vertex v has* order m, *and write* $\operatorname{ord}(v) = m$, *if v corresponds to an irreducible component of $X_m^{\operatorname{Sing} X}$. The orientation is defined by the order of the vertices. For every m, we denote by $\Gamma_X^{\geq m}$ and $\Gamma_X^{\leq m}$ the subgraphs of Γ_X obtained by removing all vertices of order $< m$, respectively, $> m$. We call the* root *of Γ_X the set of vertices of order zero. For any vertex v of Γ_X, the* branch *of Γ_X stemming from v is the subgraph $\Gamma_X^{\geq v}$ obtained by removing all vertices that are not reachable by v.*

By construction Γ_X is a directed acyclic graph, that is, a directed graph with no directed cycles. Due to the finiteness of the irreducible components of $X_m^{\operatorname{Sing} X}$, this graph has finitely many vertices of any given order. In particular, $\Gamma_X^{\leq m}$ is finite for every m.

Corollary 36. *Let X be a variety with isolated hcDV singularities, and let Γ_X be the associated graph.*

1. (Root). *The root of Γ_X is in natural bijection with the singular points of X. Each root in contained in a distinct connected component of Γ_X.*
2. (Finite branches). *There are no finite branches in Γ_X beyond a certain order. That is, there is an integer m_0 such that for every vertex v of Γ_X of order* $\operatorname{ord}(v) \geq m_0$ *and every* $m \geq \operatorname{ord}(v)$, *there exists a vertex u of order m that is reachable by v.*
3. (Infinite branches). *The infinite branches of Γ_X are in bijection with the Nash valuations on X. More precisely, for $m \gg 1$, the subgraph $\Gamma_X^{\geq m} \subset \Gamma_X$ is a disjoint union of infinite chains whose vertices have increasing orders $m, m+1, m+2, \dots$. The number of chains is the number of Nash valuations on X, and each chain is in natural correspondence with a distinct Nash valuation.*

In particular, for $m \geq 1$ the number of irreducible components of $X_m^{\operatorname{Sing} X}$ is equal to the number of irreducible components of $X_\infty^{\operatorname{Sing} X}$, and the function $\Psi_m^{\operatorname{Sing} X}$ is a bijection.

Proof. Property (1) is clear since the vertices in the root of Γ_X corresponds to the singular points of X, viewed as 0-jets on X. Properties (2) and (3) follow from Theorems 4 and 34, which establish that $\Psi_m^{\operatorname{Sing} X}$ is a bijection for $m \gg 1$. The correspondence is defined by associating to each chain of $\Gamma_X^{\geq m}$ the unique irreducible component C of $X_\infty^{\operatorname{Sing} X}$ such that for $n \geq m$ its image $\psi_n(C)$ is contained in the irreducible component of $X_n^{\operatorname{Sing} X}$ corresponding to the vertex of order n in the given chain.

Implicit in these arguments is the compatibility of the functions $\Psi_m^{\operatorname{Sing} X}$ as m varies. Specifically, in the range of application of Theorem 4, if $D = \Phi_m^{\operatorname{Sing} X}(C)$ and $D' = \Phi_{m+1}^{\operatorname{Sing} X}(C)$, then it follows by the geometric definition of these functions and their injectivity that $\pi_{m+1,m}(D') \subset D$, hence the corresponding vertices v and v' are joined by an edge. □

Remark 37. Regarding part (2) of Corollary 36, we should remark that bounded branching of arbitrary large order does occur for other singularities (e.g., see [6, 35]). As for (3), one can visualize the correspondence as attaching one vertex at the end of each chain, with such vertex corresponding to the Nash component. Thinking of the chain as consisting of the integers on $[m,\infty)$, with the intervals $[n,n+1]$ representing the edges, this is the same as adding ∞ to get $[m,\infty]$. Note that this extension of Γ_X is not a graph, since we want to see its geometric realization as a connected set but there is no edge ending at ∞.

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