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One can hear the shape of ellipses of small eccentricity

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Abstract

We show that if the eccentricity of an ellipse is sufficiently small, then up to isometries it is spectrally unique among all smooth domains. We do not assume any symmetry, convexity, or closeness to the ellipse, on the class of domains.

In the course of the proof we also show that for nearly circular domains, the lengths of periodic orbits that are shorter than the perimeter of the domain must belong to the singular support of the wave trace. As a result we also obtain a Laplace spectral rigidity result for the class of axially symmetric nearly circular domains using a similar result of De Simoi, Kaloshin, and Wei concerning the length spectrum of such domains.

1. Introduction

From the point of view of classical mechanics, elliptical billiards are very special because their billiard maps are completely integrable. In fact the Birkhoff conjecture asserts that ellipses are the only completely integrable strictly convex billiard tables. It is natural to expect this uniqueness property of ellipses to hold from the quantum mechanical point of view and ask, for example, whether the Laplace eigenvalues of ellipses with respect to Dirichlet or Neumann boundary conditions determine them uniquely. The only planar domains that are known to date to be determined by their spectrum among all smooth domains[1] are disks $D \subset \mathbb{R}^2$. In this article we show that nearly circular ellipses are spectrally determined among all smooth domains.

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[1]In fact disks are spectrally unique among all Lipschitz domains by the isoperimetric inequality, because area and perimeter are spectral invariants of a Lipschitz domain by the heat trace asymptotic of Brown [Bro93].

THEOREM 1.1. *There exists $\varepsilon_0 > 0$ such that any ellipse with eccentricity less than ε_0 is uniquely determined by its Dirichlet (or Neumman) Laplace spectrum, among all smooth domains.*

Henceforth, we use the term "nearly circular ellipse" as short for "eccentricity less than ε_0." This inverse spectral result should be compared with the recent dynamical inverse results of Avila-De Simoi-Kaloshin [ADK16] and Kaloshin-Sorrentino [KS18]. They prove a "local" version of the Birkhoff conjecture: if a strictly convex (finitely smooth) planar domain is sufficiently close to an ellipse and is *rationally integrable*, then it must be an ellipse. Rational integrabillity means that for every integer $q \geq 3$, there is a convex caustic of rotation number $\frac{1}{q}$ consisting of periodic orbits with q reflections. In fact, our proof is based in part on this result. To be able to use it, we need to prove that the hypothesis is valid. We first need an important definition.

Definition 1.2. Let $n \in \mathbb{N}$ and $\varepsilon > 0$. Let D be the unit disk and N_0 be its outward unit normal. A simply connected planar domain Ω with smooth boundary will be called "ε-nearly circular in C^n" if its boundary can be written as $\partial\Omega = \partial D + f(\theta)N_0$, with $\|f\|_{C^n(\partial D)} = \mathcal{O}_n(\varepsilon)$. Here $\mathcal{O}_n(\varepsilon)$ means that $\|f\|_{C^n(\partial D)}$ is bounded by $A_n\varepsilon$ for some A_n that depends only on n. If we only use "nearly circular," it means that ε is sufficiently small.

The main advance in this article is contained in the following:

THEOREM 1.3. *If Ω is a bounded smooth plane domain that is isospectral to a nearly circular ellipse of eccentricity ε, then Ω is ε-nearly circular in C^n for every $n \in \mathbb{N}$ (in particular, it must be strictly convex) and Ω is rationally integrable.*

The near circularity of Ω is proved in Proposition 2.1. The proof uses heat trace invariants to show that if a smooth domain Ω is isospectral to an ellipse E with small eccentricity ε, then Ω must be sufficiently close to E in the C^n norm for all n. In particular, Ω must itself be almost circular.

After this initial step, the proof of rational integrability is based on a study of the wave trace

$$w_\Omega(t) := \mathrm{Tr}\cos t\sqrt{\Delta_\Omega}. \tag{1}$$

It is well known that $w_\Omega(t)$ is a tempered distribution on $\mathbb{R}$ and that the positive singularities of w_Ω can only occur for $t \in \mathcal{L}(\Omega)$, the length spectrum (i.e., the closure of the set of lengths of closed billiard trajectories). Of particular importance here are the closed trajectories of type $\Gamma(1, q)$, i.e., with winding number 1 and with q bounces (reflections) off the boundary $\partial\Omega$. We denote the set of lengths of such closed trajectories by $\mathcal{L}_{1,q}(\Omega)$. For each q, the contribution to $w_\Omega(t)$ of closed trajectories $\Gamma(1, q)$ is denoted by $\hat{\sigma}_{1,q}$. In [MM82, Prop. 6.11], Marvizi-Melrose constructed microlocal parametrices, also denoted by $\hat{\sigma}_{1,q}$,

for the microlocal contribution of trajectories in $\Gamma(1,q)$ and proved that the parametrix was valid for $q \geq q_0(\Omega)$. By "valid" is meant that the wave trace is a sum of contributions from Lagrangian submanifolds Λ_q corresponding to q-bounce orbits and $\hat{\sigma}_{1,q}$ is the contribution from those with winding number $p = 1$ (see [MM82, §6]).

To apply the results of [ADK16], [KS18] it is essential to have analogous results for $q \geq 3$ bounces. One of the key results of this article is Theorem 5.3, which shows that the Marvizi-Melrose parametrices are in fact valid for closed billiard trajectories in $\Gamma(1,q)$ with $q \geq 2$ for nearly circular domains in C^8.

Theorem 5.3 is applied in two independent ways to prove Theorem 1.3. The first way is to combine it with a theorem of Soga [Sog81] for oscillatory integrals with degenerate phase functions to prove

THEOREM 1.4. *Let Ω be a nearly circular domain in C^8. Then, for all $q \geq 2$, one has*

$$\mathcal{L}_{1,q}(\Omega) \subset \operatorname{SingSupp} w_\Omega(t).$$

In other words, for such domains, the wave trace is singular at the length of every $(1,q)$ periodic orbit.

Let us present the application of Theorem 1.4 to Theorem 1.3. We let $\ell = |\partial\Omega|$ denote the circumference. It is well known to be a spectral invariant. It is proved that, for a nearly circular ellipse, the singular support of $w_\Omega(t)$ contained in $(0,\ell)$ is a discrete set whose *gap sequence* is monotonically decreasing. We refer to Lemma 4.1 for the definition and statement. On the other hand, if Ω is a nearly circular domain that is not rationally integrable, then the gap sequence of the singular support must fluctuate. By Theorem 1.4 the lengths in $(0,\ell)$ with $q \geq 2$ are spectral invariants. If Ω is isospectral to an ellipse of small eccentricity, then by Theorem 1.4 its gap sequence is monotonically decreasing and therefore it is rationally integrable.

We then apply the results of [ADK16] to show that Ω must be an ellipse. This step needs Ω to be nearly circular in C^n with $n = 39$, which is provided to us, in fact for any n, by Theorem 1.3. To conclude the proof, we use the easy result that if two ellipses are isospectral, then they must be isometric.

1.1. *Second approach.* The second application of Theorem 5.3 uses the following:

PROPOSITION 1.5. *If Ω is a nearly circular domain in C^8, then for $q \geq 2$, $\hat{\sigma}^{\Omega}_{1,q}(t)$ is a spectral invariant. Hence, if Ω is isospectral to an ellipse E_ε of small eccentricity ε, then for all $q \geq 2$, we have $\hat{\sigma}^{\Omega}_{1,q}(t) = \hat{\sigma}^{E_\varepsilon}_{1,q}(t)$.*

The statement is not obvious, because neither the winding number nor the bounce number are known to be spectral invariants. Moreover, if the length spectrum $\mathcal{L}(\Omega)$ of Ω is multiple, i.e., if there exists more than one connected

component in the set of closed billiard trajectories of some length L, then the contributions from the two components may cancel. Theorem 1.4 shows that complete cancellation cannot occur, but Proposition 1.5 asserts more.

Granted Proposition 1.5 the proof of Theorem 1.3 is rather simple: it is shown that the phase function of $\hat{\sigma}_{1,q}(t)$ has exactly one critical value. But that forces it to be constant, and from that one sees that Ω must be rationally integrable.

1.2. *Application to spectral rigidity of $\mathbb{Z}_2$-symmetric nearly circular domains.* In [dKW17] it is proved that the class $\mathcal{S}^8_\delta$ of axially symmetric planar domains that are δ-nearly circular in C^8, δ sufficiently small, are length spectrally rigid within this class. Length spectral rigidity means that if $\Omega_t \in \mathcal{S}^8_\delta$, $0 \leq t \leq 1$, is a C^1 family along which the length spectrum is preserved, then Ω_t must be trivial; that is, it consists of isometries of Ω_0. In fact the result of [dKW17] only uses the lengths of $(1, q)$-type periodic orbits. Now equipped with our Theorem 1.4, Lemma 4.4, and Corollary 4.8, we obtain the following result on the Laplace spectral rigidity of the class $\mathcal{S}^8_\delta$:

THEOREM 1.6. *There exists $\delta > 0$ universal such that the following statement holds. Let $\{\Omega_t\}_{t\in[0,1]}$ be a Δ isospectral deformation of domains in the class $\mathcal{S}^8_\delta$ of $\mathbb{Z}_2$-axially symmetric smooth domains that are δ-close in C^8 to the unit disk. Then Ω_t must be a trivial deformation.*

1.3. *Prospects for more general ellipses.* It is natural to try to extend the results to more general ellipses or even more general convex domains. However, there exist many obstructions to any generalization of the proofs in this article for higher eccentricities.

An important step is the proof of Theorem 5.3, which shows that the parametrix for the q-bounce wave trace (66) of [MM82, Prop. 6.11] is valid for $q \geq 3$ for a nearly circular domain. The proof of Theorem 5.3 is quite general but uses the "projectibility" of a certain portion of the q-bounce Lagrangian, namely the portion close to the diagonal consisting of orbits of winding number approximately one. It is possible that this portion of the Lagrangian is projectible for more general ellipses.

To explain the problem, we recall that the broken geodesic (billiard) flow induces a *billiard map* $\tilde{\beta} : B^*\partial\Omega \to B^*\partial\Omega$, where $B^*\partial\Omega$ is the unit "ball-bundle," which of course is an annulus in dimension 2. See Section 3.1 for background. In the case of a convex domain, $\tilde{\beta}$ is a *twist map* of the annulus. This means that a "vertical" $B^*_x\partial\Omega$ is mapped by $\tilde{\beta}$ to a horizontal curve $\tilde{\beta}(B^*_x\partial\Omega)$. Such a curve is of course a Lagrangian submanifold and may be parametrized by the differential of a function on the base $\partial\Omega$. However, q bounce periodic orbits are period q orbits of $\tilde{\beta}$ and $\tilde{\beta}^q$ fails to be a twist map. In fact the image $\tilde{\beta}^q(B^*_x\partial\Omega)$ folds over the base q times. The essence

of Theorem 5.3 is to show that the piece of $\tilde{\beta}^q(B_x^*\partial\Omega)$ corresponding to the image of small angles $\varphi \in B_x^*\partial\Omega$, i.e., to billiard geodesic loops of winding number 1, projects to $\partial\Omega$ without singularities near x. Hence this piece may be parametrized by the differential of a function on $\partial\Omega$, namely, the q-bounce loop-length function for billiard loops at x making q bounces.

Note that the result of [KS18] extends [ADK16] to arbitrary ellipses. However, the other steps of our arguments need closeness to a disk, so the proofs in this article do not extend without serious modifications to ellipses of arbitrary eccentricity. We also mention the works of Huang-Kaloshin-Sorrentino [HKS18] and a recent work of Koval [Kov21] that concern rational integrablity near the boundary of a nearly circular domain. In these results, it is only assumed that there exist caustics of rotations numbers $\frac{p}{q}$ where $\frac{p}{q} \leq \frac{1}{q_0}$ for a given (possibly large) q_0. There is hope that these methods can be extended to domains that are near ellipses of arbitrary eccentricity.

1.4. *Comparison to works of Marvizi-Melrose and Amiran.* In [MM82], Marvizi-Melrose used the parametrices to prove that there exists a two-parameter family of strictly convex domains that are spectrally determined among domains satisfying a certain non-coincidence condition. The domains are specified as solutions of extremal problems involving the so-called Marvizi-Melrose invariants. (See [Sib99], [Sib04] for the relation of these invariants to the marked length spectrum.) They even show that the curvature functions of the extremal domains are given by elliptic integrals. But they do not conclude that the domains are ellipses.

In [Ami93], [Ami96], Amiran does state the conclusions for the ellipse, but there appear to be serious gaps in the proof. The present article overlaps [Ami93], [Ami96] only in the proof of the *non-coincidence* condition. In [Ami96] (see Corollary 7), the author shows that the strong non-coincidence condition holds for an ellipse whose minor axis length exceeds $\frac{1}{4}\text{length}(\partial E)$. The proof does not appear to be complete and we give our own proof in the case of a nearly circular domain.

We briefly describe the approach of [Ami93], [Ami96]. In [Ami93], Amiran defines "caustics invariants" L, J_1, G and states (Theorem 9) that the extremals of G among domains with fixed L, J_1 are ellipses. The non-coincidence condition (Theorem 10 of [Ami93]) is used to show that sufficiently many caustics invariants are Δ-spectral invariants. The idea of the proof is to show that only curvature functions of ellipses solve the Euler-Lagrange equations for G, a nonlinear second order equation for the radius of curvature of the domain. We do not understand the proof given in [Ami93], [Ami96] that curvature functions of ellipses solve the equation, or that they are the only solutions. If indeed such ellipses are the only solutions of the extremal problem, then they would

be spectrally determined among domains whose curvature functions are near that of the ellipse (Corollary 7).

1.5. *Previous positive results.* To our knowledge, the results of this paper give the first "universal inverse spectral result" for any class of domains other than the circle. The result says that ellipses in a specific family ("almost circular") are determined by their spectra among all smooth domains without any further assumptions. In fact, there do not even exist prior "local spectral determination" results, which would say that an ellipse (or any other domain) is determined by its spectrum among domains that lie in a sufficiently small C^k neighborhood of the ellipse. The only prior positive result specific to the inverse Laplace spectral problem for ellipses is [HZ12] (see also [PT16] for ellipsoids), which says that ellipses are infinitesimally spectrally rigid among C^∞ domains with the same left-right and up-down symmetries. The progress in that article is to allow competing domains to be C^∞ and not real-analytic. To be precise, the rigidity result proved that any Dirichlet/Neumann isospectral deformation had to be "flat," i.e., all of its variational derivatives vanish. These results were generalized to all Robin boundary conditions in [Vig21].

The most general prior positive inverse results were that of [Zel09], where it is proved that a generic real analytic plane domain with one up-down symmetry is determined by its Dirichlet (or Neumann) spectrum among other such domains, and that of [dKW17], where a generic nearly circular domain with one reflection symmetry is shown to be Laplace spectrally rigid in the same class of domains. In [dKW17], the genericity was needed in drawing a conclusion on the Laplace spectral rigidity from the length spectral rigidity. The results of the present article, by comparison, do not make any symmetry assumptions and allow the competing domains to be general C^∞ domains. There also exists a sequence of results of Popov-Topalov [PT03], [PT12], [PT16] using the KAM structure of convex smooth plane domains to deduce spectral rigidity results for Liouville billiards (including ellipses) with two commuting reflection symmetries, and for analytic domains that are sufficiently close to an ellipse and possess the two reflection symmetries of the ellipse.

Prior inverse results for other classes of plane domains are surveyed in [Zel04], [DH13], [Zel14], with an emphasis on positive results. Negative results, such as the construction of isospectral polygonal domains of [GWW92], are surveyed in [Gor00].

1.6. *Organization of the paper.* In Section 2, we show that any smooth domain isospectral to a nearly circular domain must be nearly circular as well. The main tool in proving this result is heat trace invariants. Section 3 is dedicated to the existence of geodesics loops for nearly circular domains. The lengths of these loops (we call them loop functions) play a key role in our later

analysis of the wave trace via the Marvizi-Melrose parametrix. We also find a useful variational formula for the loop functions. Section 4 entirely involves the length spectrum of nearly circular domains. The important gap structure of the length spectrum is proved in this section. The Marvizi-Melrose parametrix is proved in Section 5 for nearly circular domains for all $q \geq 2$. We provide an independent proof of this theorem using Green's second identity. In Section 6, we show that the part of the length spectrum that is less than the perimeter is contained in the singular support of the wave trace hence, in fact, we obtain an equality in Poisson relation in the interval $(0, \ell)$. In Section 7, we prove our main result by showing that if a domain is isospectral to a nearly circular ellipse then it must be rationally integrable hence must be an ellipse by a result of Avila-De Simoi-Kaloshin. At the end of Section 7 we also provide an alternative proof of the rational integrability.

2. Isospectrality with a nearly circular ellipse implies closeness to the ellipse

Let E_ε be an ellipse of eccentricity ε. After a rescaling and a rigid motion, we can assume that E_ε is given by

$$E_\varepsilon = \left\{(x, y) \in \mathbb{R}^2;\ x^2 + \frac{y^2}{1-\varepsilon^2} \leq 1\right\}.$$

Then assume Ω is a smooth domain with

$$\mathrm{Spec}(\Omega) = \mathrm{Spec}(E_\varepsilon). \tag{2}$$

Here, Spec means the spectrum of the euclidean Laplacian with respect to Dirichlet (or Neumman) boundary condition. We know from the heat trace invariants that Ω must be simply-connected with the same perimeter as E_ε, which we shall call ℓ_ε, and use s for the arclength parameter. We will also use

$$\kappa(\Omega)(s) \quad \text{and} \quad \kappa(E_\varepsilon)(s)$$

for the curvature functions of $\partial\Omega$ and ∂E_ε respectively. Note that $\kappa(\Omega)$ and $\kappa(E_\varepsilon)$ belong to the same space $C^\infty[0, \ell_\varepsilon]$. We now have the following lemma:

Proposition 2.1. *Suppose Ω and E_ε are isospectral. Then for all integers $n \geq 0$, we have*

$$\|\kappa(\Omega) - 1\|_{C^n[0,\ell_\varepsilon]} = \mathcal{O}_n(\varepsilon).$$

In particular, for sufficiently small ε, Ω is strictly convex. Here, $\mathcal{O}_n(\varepsilon)$ means that the involved constant depends only on n.

As a corollary we will obtain

Corollary 2.2. *Suppose Ω and E_ε are isospectral. Then one can apply a rigid motion to Ω after which its boundary can be written as $\partial\Omega = \partial E_0 + f N_0$,*

with $\|f\|_{C^n(\partial D)} = \mathcal{O}_n(\varepsilon)$ *for all* $n \geq 0$. *Here* N_0 *is the outward unit normal of the unit disk* $D = E_0$. *In terms of Definition 1.2, it means that* Ω *is* ε*-nearly circular in* C^n.

Proof of Proposition 2.1. Let Δ_Ω be the positive Laplacian with Dirichlet (or Neumann) boundary condition on Ω. We recall the well-known heat trace asymptotic

$$\mathrm{Tr}(e^{-t\Delta_\Omega}) \sim t^{-1} \sum_{n\geq 0} a_n t^n + b_n t^{n+\frac{1}{2}}, \qquad t \to 0^+. \tag{3}$$

In [Mel84], the following structural property is proved for the heat invariants b_n:

$$b_{n+1} = c_n \int_0^\ell \kappa_n^2 \, ds + \sum_{\alpha\geq 0} d_\alpha \int_0^\ell \kappa^{\alpha_0} \kappa_1^{\alpha_1} \cdots \kappa_{n-1}^{\alpha_{n-1}} \, ds, \qquad n \geq 1, \tag{4}$$

where ℓ is the length of the boundary, κ_m denotes $\frac{d^m \kappa}{ds^m}$, $\alpha = (\alpha_0, \ldots, \alpha_{n-1})$ is a multi-index in $\mathbb{Z}^n$, $c_n \neq 0$ and d_α are universal constants. We also have $\sum_{j=0}^{k-1}(1+j)\alpha_j = 2n+2$, which in particular implies that the sum is finite and the number of terms is bounded by a constant dependent only on n. For us it is also important to know that

$$b_1 = c_1 \int_0^\ell \kappa^2 \, ds, \qquad c_1 \neq 0, \tag{5}$$

$$b_2 = c_2 \int_0^\ell \kappa_1^2 \, ds + c_2' \int_0^\ell \kappa^4 \, ds, \quad c_2 \neq 0. \tag{6}$$

Melrose used the trace invariants b_n to prove a pre-compactness for the class of isospectral domains to a given domain D. More precisely he showed that for each smooth domain Ω_0, and each $n \geq 0$, there is A_n such that for all Ω isospectral to Ω_0, we have

$$||\kappa(\Omega)||_{C^n} \leq A_n. \tag{7}$$

Suppose now that Ω is isospectral to E_ε. We would like to show that for all $n \geq 0$,

$$||\kappa(\Omega) - \kappa(E_\varepsilon)||_{C^n} = \mathcal{O}_n(\varepsilon). \tag{8}$$

Since $\kappa(E_0) = 1$, we have $\kappa(E_\varepsilon) = 1 + \mathcal{O}(\varepsilon)$ and $\kappa_n(E_\varepsilon) = \mathcal{O}_n(\varepsilon)$ for $n \geq 1$. Thus it is sufficient by the Sobolev embedding theorem to show that

$$\int_0^{\ell_\varepsilon} |\kappa(\Omega) - \kappa(E_\varepsilon)|^2 ds = \mathcal{O}(\varepsilon), \tag{9}$$

$$\int_0^{\ell_\varepsilon} \kappa_n^2(\Omega) \, ds = \mathcal{O}_n(\varepsilon), \quad n \geq 1. \tag{10}$$

To see (9), we first use the invariant b_1 in (5) and the fact $\ell_\varepsilon = 2\pi + \mathcal{O}(\varepsilon)$ to get

$$\int_0^{\ell_\varepsilon} \kappa^2(\Omega)\, ds = \int_0^{\ell_\varepsilon} \kappa^2(E_\varepsilon)\, ds = 2\pi + \mathcal{O}(\varepsilon).$$

This and the facts $\int_0^{\ell_\varepsilon} \kappa(\Omega) ds = 2\pi$ and $\kappa(E_\varepsilon) = 1 + \mathcal{O}(\varepsilon)$ imply that

$$\int_0^{\ell_\varepsilon} |\kappa(\Omega) - \kappa(E_\varepsilon)|^2\, ds = \int_0^{\ell_\varepsilon} \kappa^2(\Omega) + \kappa^2(E_\varepsilon) - 2\kappa(\Omega)\kappa(E_\varepsilon)\, ds = \mathcal{O}(\varepsilon).$$

To prove (10), we use (4) and argue by induction on $n \geq 1$. In the first step, we note that by the expression (6) for the invariant b_2, we have

$$\int_0^{\ell_\varepsilon} \kappa_1^2(\Omega)\, ds = \int_0^{\ell_\varepsilon} \kappa_1^2(E_\varepsilon)\, ds + \frac{c_2'}{c_2} \left(\int_0^{\ell_\varepsilon} \kappa^4(E_\varepsilon) - \kappa^4(\Omega)\, ds \right).$$

However, we can bound the last expression by $\mathcal{O}(\varepsilon)$ using (7), Cauchy-Schwartz inequality, and (9) as follows:

$$\begin{aligned}\int_0^{\ell_\varepsilon} \kappa_1^2(\Omega)\, ds \leq \mathcal{O}(\varepsilon) + \frac{c_2'}{c_2} \|\kappa(E_\varepsilon) - \kappa(\Omega)\|_{L^2} \\ \times \left\|\kappa^3(E_\varepsilon) + \kappa^2(E_\varepsilon)\kappa(\Omega) + \kappa(E_\varepsilon)\kappa^2(\Omega) + \kappa^3(\Omega)\right\|_{L^2} = \mathcal{O}(\varepsilon).\end{aligned}$$

Let us now assume $\int_0^{\ell_\varepsilon} \kappa_m^2(\Omega)\, ds = \mathcal{O}(\varepsilon)$ for $1 \leq m \leq n-1$. By (4), we have

$$\begin{aligned}\int_0^{\ell_\varepsilon} \kappa_n^2(\Omega) = \int_0^{\ell_\varepsilon} \kappa_n^2(E_\varepsilon) \\ + \frac{1}{c_n} \sum_{\alpha \geq 0} d_\alpha \int_0^{\ell_\varepsilon} \kappa^{\alpha_0}(E_\varepsilon) \cdots \kappa_{n-1}^{\alpha_{n-1}}(E_\varepsilon) - \kappa^{\alpha_0}(\Omega) \cdots \kappa_{n-1}^{\alpha_{n-1}}(\Omega).\end{aligned}$$

To conclude the induction, we need to show that the right-hand side of the above identity is $\mathcal{O}_n(\varepsilon)$. Obviously, by the induction hypothesis, and since for all $n \geq 1$, $\|\kappa_n(E_\varepsilon)\|_{C^0} = \mathcal{O}_n(\varepsilon)$, all terms involving at least one derivative of the curvature are of size $\mathcal{O}_n(\varepsilon)$. (Note that we still need the a priori bounds (7).) So it remains to estimate the part of the sum involving no derivatives, which is a sum of terms (up to multiplication by a constant) in the form:

$$\int_0^{\ell_\varepsilon} \kappa^{\alpha_0}(E_\varepsilon) - \kappa^{\alpha_0}(\Omega)\, ds.$$

Again, as in the first step, we factor $\kappa(E_\varepsilon) - \kappa(\Omega)$ in the integrand, apply Cauchy-Schwartz, and use the a priori bounds (7) to obtain the desired bound $\mathcal{O}_n(\varepsilon)$ for $\|\kappa(\Omega) - \kappa(E_\varepsilon)\|_{C^n}$. Note that this implies the proposition because $\|\kappa(E_\varepsilon) - 1\|_{C^n} = \mathcal{O}_n(\varepsilon)$. □

Proof of Corollary 2.2. We first apply a rigid motion to $\partial\Omega$ so that it becomes tangent to ∂D at $(1, 0)$ and stays on the left side of the line $x = 1$.

We identify the point of tangency $(1,0)$ with $s=0$. Then the parametrization γ_Ω is uniquely determined by its curvature by the expression

(11)
$$\gamma_\Omega(s)=\left(1-\int_0^s \sin\left(\int_0^{s'} \kappa(\Omega)(s'')\,ds''\right)\,ds'\,,\ \int_0^s \cos\left(\int_0^{s'} \kappa(\Omega)(s'')\,ds''\right)\,ds'\right).$$

Now let $\theta\in[0,2\pi]$ be the arc-length parametrization of the boundary of the unit disk D. We write

$$\gamma_\Omega(s(\theta)) = (r(\theta)\cos\theta, r(\theta)\sin\theta).$$

We want to show that $r(\theta)=1+f(\theta)$ with $f(\theta)=\mathcal{O}_n(\varepsilon)$ in C^n as this is exactly what the corollary requires. Here, $s(\theta)$ is given by

$$s(\theta)=\int_0^\theta \sqrt{r^2(\vartheta)+(r'(\vartheta))^2}\,d\vartheta=\int_0^\theta \sqrt{(1+f(\vartheta))^2+(f'(\vartheta))^2}\,d\vartheta. \tag{12}$$

We note that

$$f(\theta)=\|\gamma_\Omega(s(\theta)\|-1.$$

Since by Proposition 2.1 we have $\kappa(\Omega)(s)=1+h_0(s)$ with $h_0(s)=\mathcal{O}(\varepsilon)$ in $C^n[0,\ell_\varepsilon]$, by (11) we get

$$f(\theta)=\sqrt{1+h_1(s(\theta))}-1=h_2(s(\theta)), \tag{13}$$

with $h_1(s)=\mathcal{O}(\varepsilon)$ and $h_2(s)=\mathcal{O}(\varepsilon)$ in $C^n[0,\ell_\varepsilon]$. We emphasize that since $s(\theta)$ depends on $f(\theta)$, we cannot immediately conclude from this equation that $f(\theta)=\mathcal{O}(\varepsilon)$ in C^n. We proceed with induction. It is clear from (13) that $\|f\|_{C^0}=\mathcal{O}(\varepsilon)$. For f', we have

$$f'(\theta)=\sqrt{(1+f(\theta))^2+(f'(\theta))^2}\,h_2'(s(\theta)).$$

Solving this for $(f')^2$ we get

$$(f'(\theta))^2=(1+f(\theta))^2\frac{(h_2'(s(\theta)))^2}{1-(h_2'(s(\theta)))^2},$$

which is obviously $\mathcal{O}(\varepsilon)$ because $h_2=\mathcal{O}(\varepsilon)$ in C^n for all n. Now assume $\|f\|_{C^{n-1}}=\mathcal{O}(\varepsilon)$ for some $n\geq 2$. Differentiating (13) n times,

$$f^{(n)}(\theta)=\frac{f^{(n)}(\theta)f'(\theta)}{\sqrt{(1+f(\theta))^2+(f'(\theta))^2}}h_2'(s(\theta))+R_n(\theta),$$

where the remainder term R_n depends on $f^{(k)}$, $k\leq n-1$, and $h_2^{(k)}$, $k\leq n$. From the induction hypothesis and the form of R_n, one can easily see that $R_n=\mathcal{O}(\varepsilon)$. Solving the above equation for $f^{(n)}$ and using $\|f\|_{C^1}=\mathcal{O}(\varepsilon)$ and $h_2'=\mathcal{O}(\varepsilon)$, we conclude the induction and the corollary follows. □

3. The loop function and its first variation

This section focuses on the iterations of the billiard map of a nearly circular domain. Let us first introduce the billiard map and its periodic orbits.

3.1. *Billiard map and (p,q)-periodic orbits.* Consider a C^∞ strictly convex billiard table Ω with perimeter ℓ. We parametrize its boundary in the counter-clockwise direction by its arc-length s. We define the phase space by $S^*_{\text{inward}}\partial\Omega$, i.e., the inward vectors in the unit cotangent bundle of $\partial\Omega$. We identify the phase space with

$$\Pi = \mathbb{R}/\ell\mathbb{Z} \times [0,\pi],$$

and we use (s,φ) for a point in Π. Here, φ represents the angle that the inward unit vector at s makes with the positive unit tangent vector at s, i.e., the tangent vectors in the counter-clockwise direction. The billiard map is a smooth twist map on the closed annulus Π. We write it as

$$\begin{cases} \beta : \Pi \to \Pi, \\ \beta(s,\varphi) = (s_1(s,\varphi), \varphi_1(s,\varphi)) \,. \end{cases}$$

It is natural and convenient to lift β to $\hat{\Pi} = \mathbb{R} \times [0,\pi]$. We shall use (x,φ) for points in $\hat{\Pi}$. We fix the lift and call it $\hat{\beta}$ by requiring that $\hat{\beta}(x,0) = (x,0)$. Then by the continuity of the lift we have $\hat{\beta}(x,\pi) = (x+\ell,\pi)$. We shall write

$$\hat{\beta}(x,\varphi) = (x_1(x,\varphi), \varphi_1(x,\varphi)).$$

The billiard map satisfies the monotone twist property, meaning

$$\partial_\varphi x_1 > 0.$$

The map $\hat{\beta}$ also preserves the orientation and the boundaries of $\hat{\Pi}$. Moreover, $\hat{\beta}$ preserves the natural symplectic form $\sin\varphi\, dx \wedge d\varphi$ on the phase space $\hat{\Pi}$. A diffeomorphism β of the annulus Π whose lift $\hat{\beta}$ satisfies the above properties is called a twist map. We refer the readers to [MF94] for more on the properties of twist maps and Aubry-Mather theory.

Note that we can write

$$\hat{\beta}(x,\varphi) = (x + F(x,\varphi), G(x,\varphi)), \tag{14}$$

where F and G are smooth, ℓ-periodic in the x variable, and $F(x,0) = G(x,0) = 0$. We shall use $\hat{\beta}_0$ for the billiard map of the unit disk D. One can easily see that

$$\hat{\beta}_0(x,\varphi) = (x + 2\varphi, \varphi).$$

A point $(s,\varphi) \in \Pi$ is called a (p,q) periodic point of β if $\beta^q(s,\varphi) = (s,\varphi)$ and the orbit $\{\beta^j(s,\varphi)\}_{0\le j\le q-1}$ winds p times around $\partial\Omega$ in the positive direction.

This means that for any lift (x,φ) of (s,φ), we have

$$\hat{\beta}^q(x,\varphi) = (x + p\ell, \varphi).$$

The ratio $\frac{p}{q}$ is called the rotation number of (s,φ). Since the rotation number of the time reversal of a periodic orbit of rotation number $\frac{p}{q}$ is given by $\frac{q-p}{p}$, we always assume that $1 \leq p \leq \frac{q}{2}$. On the unit disk E_0, the (p,q) periodic points form an invariant circle given by $\{(s,\varphi)|\, \varphi = \pi p/q\}$.

3.2. *Expansions of the billiard map.* For small angles φ, the billiard map $\hat{\beta}(x,\varphi)$ of a smooth strictly convex domain has a useful expansion (via Taylor's theorem) in the form

$$\hat{\beta}(x,\varphi) = \left(x + \sum_{j=1}^{N-1} \alpha_j(x)\varphi^j + F_N(x,\varphi)\varphi^N, \sum_{j=1}^{N-1} \beta_j(x)\varphi^j + G_N(x,\varphi)\varphi^N \right).$$

By Proposition 14.2 of [Laz93], the remainder terms F_N and G_N are bounded by $\frac{\mathcal{O}_N(1)}{\kappa_{\min}} \|\frac{1}{\kappa}\|_{C^{N-1}}$, where $\kappa(x)$ is the curvature at x and $\kappa_{\min}$ is the minimum curvature. The coefficients $\alpha_j(x)$ and $\beta_j(x)$ can be calculated in terms of κ and its derivatives; see, for example, [Laz93] for the expressions of $\alpha_1,\ldots,\alpha_4$ and $\beta_1,\ldots,\beta_4$. In fact in certain coordinates, called Lazutkin coordinates, the billiard map can be written in a simpler form. More precisely, if we denote

$$\xi = C_1 \int_0^x \kappa^{2/3}(x')dx', \qquad \eta = C_2 \kappa^{-1/3}(x)\sin(\varphi/2),$$

where $C_1 = 1/\int_0^\ell \kappa^{2/3}(x')dx'$ and $C_2 = 4C_1$, then in these coordinates the billiard map takes the form

$$(\xi,\eta) \to \left(\xi + \eta + \sum_{j\geq 3} \tilde{\alpha}_j(\xi)\eta^j, \eta + \sum_{j\geq 4} \tilde{\beta}_j(\xi)\eta^j \right),$$

where the coefficient functions are 1-periodic and smooth. The infinite sums are understood as asymptotic expansions as $\eta \to 0$ and not as convergent power series. The remarkable fact is the vanishing of the η^2 term in the first component and of the η^2 and η^3 terms in the second component. In fact as Lemma 14.6 of [Laz93] shows, one can go further inductively and find for each $N \geq 3$, new coordinates (u,v) in which the billiard map is written as

$$(u,v) \to \left(u + v + \sum_{j\geq N} a_j(u)v^j, v + \sum_{j\geq N+1} b_j(u)v^j \right). \tag{15}$$

Moreover, the construction of this map reveals that the two coordinates (ξ,η) and (u,v) are related by

$$(\xi,\eta) = \left(u + v^2 A(u,v), v + v^3 B(u,v)\right). \tag{16}$$

Lazutkin coordinates can be very useful in proving existence of invariant curves as shown by Lazutkin but also in studying periodic orbits as done in [dKW17].

3.3. *Nearly circular deformations.* Suppose Ω is nearly circular in C^1. Recall that by Definition 1.2 this means that Ω is smooth and simply connected, and can be written as $\partial\Omega = \partial D + fN_0$, where f is a smooth function on ∂D sufficiently small in C^1 and N_0 is the outward unit normal field to ∂D. This also means that $\partial\Omega$ is a polar curve given by $r(\theta) = 1 + f(\theta)$, $\theta \in [0, 2\pi]$. In fact we will need to consider the linear deformation $\{\Omega_\tau\}_{0\le\tau\le1}$ defined by

$$\partial\Omega_\tau = \partial D + \tau f N_0, \tag{17}$$

or equivalently in polar coordinates, by

$$\partial\Omega_\tau : \ r(\tau,\theta) = 1 + \tau f(\theta), \quad \theta \in [0, 2\pi].$$

Hence by this notation, $\Omega_0 = E_0 = D$ and $\Omega_1 = \Omega$. We denote the arc-length parametrization of $\partial\Omega_\tau$ by $\gamma(\tau, s)$. The polar and arc-length parametrizations are related by

$$\gamma(\tau, s(\tau,\theta)) = \big(r(\tau,\theta)\cos\theta, r(\tau,\theta)\sin\theta\big), \tag{18}$$

where

$$s(\tau,\theta) = \int_0^\theta \sqrt{(\partial_\theta r)^2(\tau,\vartheta) + r^2(\tau,\vartheta)}\, d\vartheta = \int_0^\theta \sqrt{\tau^2 f'^2(\vartheta) + (1+\tau f(\vartheta))^2}\, d\vartheta. \tag{19}$$

Remark 3.1. Throughout the paper we identify $s = 0$ with $\theta = 0$.

3.4. *First variations of the deformation.* The first *normal variation* of $\partial\Omega_\tau$ at τ is defined by

$$n(\tau, s) = \partial_\tau\gamma(\tau, s) \cdot N(\tau, s), \tag{20}$$

where $N(\tau, s)$ is the outward unit normal at $\gamma(\tau, s)$. We also define the first *tangential variation* of $\partial\Omega_\tau$ by

$$t(\tau, s) = \partial_\tau\gamma(\tau, s) \cdot T(\tau, s), \tag{21}$$

with $T(\tau, s)$ being the unit tangent in the positive direction.

The next lemma bounds $n(\tau, s)$ and $t(\tau, s)$ in terms of the function $f(\theta)$.

LEMMA 3.2. *Suppose* $\|f\|_{C^1} \le 1$. *Then*

$$n(\tau, s(\tau,\theta)) = \mathcal{O}(\|f\|_{C^1}) \qquad \textit{and} \qquad t(\tau, s(\tau,\theta)) = \mathcal{O}(\|f\|_{C^1}).$$

Proof. We differentiate (18) and obtain

(22)
$$\partial_\tau(\gamma(\tau, s(\tau,\theta))) = (f(\theta)\cos\theta,\ f(\theta)\sin\theta),$$

(23)
$$\partial_\theta(\gamma(\tau, s(\tau,\theta))) = (\tau f'(\theta)\cos\theta - (1+\tau f(\theta))\sin\theta, \tau f'(\theta)\sin\theta + (1+\tau f(\theta))\cos\theta).$$

In particular, from (23) we find that the unit outward normal is given by

$$\begin{aligned} &N(\tau, s(\tau,\theta)) \\ &= \frac{(\tau f'(\theta)\sin\theta + (1+\tau f(\theta))\cos\theta,\ -\tau f'(\theta)\cos\theta + (1+\tau f(\theta))\sin\theta)}{\sqrt{\tau^2 f'^2(\theta) + (1+\tau f(\theta))^2}}. \end{aligned}$$

On the other hand,

$$(\partial_\tau\gamma)(\tau, s(\tau,\theta)) = \partial_\tau(\gamma(\tau, s(\tau,\theta))) - \partial_\tau s(\tau,\theta)(\partial_s\gamma)(\tau, s(\tau,\theta)),$$

which using $\partial_s\gamma \cdot N = 0$ and $\partial_s\gamma \cdot T = 1$, implies that

$$\begin{aligned} n(\tau, s(\tau,\theta))) &= \partial_\tau(\gamma(\tau, s(\tau,\theta))) \cdot N, \\ t(\tau, s(\tau,\theta))) &= \partial_\tau(\gamma(\tau, s(\tau,\theta))) \cdot T - \partial_\tau s(\tau,\theta). \end{aligned}$$

Thus by (22) and (23), we get

$$\begin{aligned} n(\tau, s(\tau,\theta)) &= \frac{f(\theta)(1+\tau f(\theta))}{\sqrt{\tau^2 f'^2(\theta) + (1+\tau f(\theta))^2}}, \\ t(\tau, s(\tau,\theta)) &= \frac{\tau f'(\theta) f(\theta)}{\sqrt{\tau^2 f'^2(\theta) + (1+\tau f(\theta))^2}} - \partial_\tau s(\tau,\theta). \end{aligned}$$

The lemma then follows easily from $\partial_\tau s(\tau,\theta) = \mathcal{O}(\|f\|_{C^1})$. □

3.5. *Loop function and its first variation.* Our primary purpose in this section is to study the $(1,q)$ periodic orbits of Ω_τ, hence in particular $\Omega = \Omega_1$, in terms of the ones of D. The main ingredients will be the loop functions and their linearizations in τ. We start by the following theorem that introduces what we will call the *loop angle*.

THEOREM 3.3. *Let $\partial\Omega = \partial D + fN_0$ be nearly circular in C^6. There exists $\varepsilon_0 > 0$ sufficiently small such that if $\|f\|_{C^6} \le \varepsilon_0$, then for each $\tau \in [0,1]$, s on $\partial\Omega_\tau$, and $q \ge 2$, there exists a unique angle $\varphi_q(\tau, s) \in (0,\pi)$ such that the orbit starting at $(s, \varphi_q(\tau,s))$ and making q reflections, winds around the boundary once in the counterclockwise direction, and returns to s (not necessarily in the same direction). Moreover, $\varphi_q(\tau,s)$ is smooth. We shall call $\varphi_q(\tau,s)$ the q-loop angle of Ω_τ.*

Remark 3.4. In [RR06] and [PRR13], a similar statement is proved, however since the proof is based on the implicit function theorem at $\tau = 0$, the above theorem is obtained only for τ small, and for φ near π/q, which is the

q-loop angle of the disk Ω_0. The size of the neighborhoods of $\tau = 0$ and $\varphi = \pi/q$ are not estimated in these references. To do this, one probably needs a *quantitative implicit function theorem*. (See, for example, the online notes of Liverani [Liv].) In this paper, we take a different route and estimate the q-iterations of the billiard map more directly.

In fact, for technical reasons, we will need a stronger result as follows. Below, ℓ_τ is the perimeter of Ω_τ.

THEOREM 3.5. *Let $\partial\Omega = \partial D + fN_0$ be nearly circular in C^6. There exists $\varepsilon_0 > 0$ sufficiently small such that if $\|f\|_{C^6} \le \varepsilon_0$, then for each $\tau \in [0,1]$, s and s' on $\partial\Omega_\tau$ with $|s - s'| < \frac{\ell_\tau}{100}$, and $q \ge 2$, there exists a unique angle $\alpha_q(\tau, s, s') \in (0, \pi)$, such that the orbit starting at $(s, \alpha_q(\tau, s, s'))$ and making q reflections winds around the boundary approximately once in the counterclockwise direction and ends at s. The function $\alpha_q(\tau, s, s')$ is smooth in (τ, s, s').*

Here, by "winding around the boundary approximately once in the counterclockwise direction," we precisely mean that if x and x' are lifts of s an s' with $|x - x'| < \ell_\tau/100$, then $\hat\beta_\tau^q(x, \alpha_q(\tau, x, x')) = x' + \ell_\tau$, where $\hat\beta_\tau$ is the natural lift of the billiard map of Ω_τ.

We will give the proof in Section 3.6. In the next few pages we draw some important consequences of these theorems.

Note that by our notations, on the diagonal $s = s'$ we have $\alpha_q(\tau, s, s) = \varphi_q(\tau, s)$. We shall call the angle $\varphi_q(\tau, s)$ the q-loop angle at s. We will use $\varphi_q(s)$ for the loop angle of $\Omega = \Omega_1$ instead of $\varphi_q(1, s)$. Obviously the loop angle satisfies

$$\hat\beta^q(s, \varphi_q(\tau, s)) = (s, \tilde\varphi_q(\tau, s))$$

for some angle $\tilde\varphi_q(\tau, s)$. The following lemma is then immediate.

LEMMA 3.6. *The point (s, φ) is a q-periodic point of $\partial\Omega$ if and only if*

$$\varphi_q(s) = \tilde\varphi_q(s) = \varphi.$$

We now define the main ingredients of this article, namely, the q-length function and its first variation. For the rest of the paper we assume that $\|f\|_{C^6}$ is small enough so that Theorems 3.3 and 3.5 hold.

Definition 3.7 (Length and loop functions). Let Ω be C^6 sufficiently close to the unit disk E_0, and let $q \ge 2$. The q-length function $\Psi_q(s, s')$, defined on $|s - s'| < \ell_\tau/100$, is the length of the unique q times reflected geodesic form s to s' defined in Theorem 3.5. The q-loop function $L_q(s)$ is the length of the unique q-loop at s defined by Theorem 3.3, i.e., $L_q(s) = \Psi_q(s, s)$. More

precisely, if $\gamma(s)$ is the arc-length parametrization of Ω, then

$$\Psi_q(s,s') = \sum_{j=0}^{q-1} \|\gamma(s_{j+1}) - \gamma(s_j)\|, \quad s_j = \mathrm{Proj}_1 \beta^j(s, \alpha_q(s,s')), \tag{24}$$

where Proj_1 is the projection map onto the base component. Similarly,

$$L_q(s) = \sum_{j=0}^{q-1} \|\gamma(s_{j+1}) - \gamma(s_j)\|, \quad s_j = \mathrm{Proj}_1 \beta^j(s, \varphi_q(s)). \tag{25}$$

Note that since $\alpha_q(s,s')$ is smooth, $\Psi_q(s,s')$ and $L_q(s)$ are also smooth. Correspondingly, we denote the q-loop functions of the deformation Ω_τ by $L_q(\tau, s)$.

The following lemma, although simple, gives a very useful characterization of q-periodic orbits in terms of the q-loop function.

LEMMA 3.8. *We have*

$$\{(s,q) \in \Pi;\ (s,\varphi)\ \textit{is } q\textit{-periodic}\} = \{(s, \varphi_q(s));\ L_q'(s) = 0\}.$$

In other words, q-periodic orbits correspond to the critical points of L_q.

Proof. Consider the loop $\{(s_j, \vartheta_j)\}_{0 \le j \le q}$ (not necessarily a periodic orbit) generated by $(s, \varphi_q(s))$, meaning

$$(s_j, \vartheta_j) = \beta^j(s, \varphi_q(s)).$$

By this notation, $(s_0, \vartheta_0) = (s, \varphi_q(s))$ and $(s_q, \vartheta_q) = (s, \tilde{\varphi}_q(s))$. Differentiating L_q we get

$$\begin{aligned} L_q'(s) &= \sum_{j=0}^{q-1} \left(\frac{ds_{j+1}}{ds} \frac{d\gamma}{ds}(s_{j+1}) - \frac{ds_j}{ds} \frac{d\gamma}{ds}(s_j) \right) \cdot \frac{\gamma(s_{j+1}) - \gamma(s_j)}{\|\gamma(s_{j+1}) - \gamma(s_j)\|} \\ &= \sum_{j=0}^{q-1} \cos\vartheta_{j+1} \frac{ds_{j+1}}{ds} - \cos\vartheta_j \frac{ds_j}{ds} \\ &= \cos\vartheta_q \frac{ds_q}{ds} - \cos\vartheta_0 \frac{ds_0}{ds} \\ &= \cos\vartheta_q - \cos\vartheta_0. \end{aligned}$$

This shows that s is a critical point of L_q if and only if $\vartheta_0 = \vartheta_q$, which by our notation means $\varphi_q(s) = \tilde{\varphi}_q(s)$. □

Next we define the first variation of L_q.

Definition 3.9. Let Ω_τ be a deformation of the unit disk as in Theorem 3.3 so that the loop angles and loop functions are defined. For each $q \ge 2$, the first variation of the q-loop function is denoted by $M_q(\tau, s) = \partial_\tau L_q(\tau, s)$.

More explicitly, we have

LEMMA 3.10. *Let $n(\tau,s)$ and $t(\tau,s)$ be the first normal and tangential variations of $\partial\Omega_\tau$ defined by (20) and (21), and let $\{(s_j(\tau,s),\vartheta_j(\tau,s))\}_{j=0}^q$ be the q-loop generated by $(s,\varphi_q(\tau,s))$, i.e., $(s_j(\tau,s),\vartheta_j(\tau,s)) = \beta_\tau^j(s,\varphi_q(\tau,s))$, where β_τ is the billiard map of Ω_τ. Then*

$$\begin{aligned} M_q(\tau,s) = {} & n(\tau,s)\big(\sin\tilde{\varphi}_q(\tau,s)+\sin\varphi_q(\tau,s)\big) \\ & + t(\tau,s)\big(\cos\tilde{\varphi}_q(\tau,s)-\cos\varphi_q(\tau,s)\big) \\ & + 2\sum_{j=1}^{q-1} n(\tau,s_j(\tau,s))\sin\vartheta_j(s,\tau). \end{aligned}$$

In particular, when $(s,\varphi_q(\tau,s))$ corresponds to a q-periodic orbit, i.e., $\varphi_q(\tau,s) = \tilde{\varphi}_q(\tau,s)$, we get

$$M_q(\tau,s) = 2\sum_{j=0}^{q-1} n(\tau,s_j(\tau,s))\sin\vartheta_j(s,\tau).$$

Proof. For simplicity, we use s_j and ϑ_j for $s_j(\tau,s)$ and $\vartheta_j(\tau,s)$. We then write

$$L_q(\tau,s) = \sum_{j=0}^{q-1} \|\gamma(\tau,s_{j+1})-\gamma(\tau,s_j)\|.$$

Taking the variation we get

$$\begin{aligned} \partial_\tau L_q(\tau,s) = {} & \sum_{j=0}^{q-1} (\partial_\tau s_{j+1}\partial_s\gamma(\tau,s_{j+1}) - \partial_\tau s_j\partial_s\gamma(\tau,s_j)) \cdot \frac{\gamma(\tau,s_{j+1})-\gamma(\tau,s_j)}{\|\gamma(\tau,s_{j+1})-\gamma(\tau,s_j)\|} \\ & + \sum_{j=0}^{q-1} (\partial_\tau\gamma(\tau,s_{j+1}) - \partial_\tau\gamma(\tau,s_j)) \cdot \frac{\gamma(\tau,s_{j+1})-\gamma(\tau,s_j)}{\|\gamma(\tau,s_{j+1})-\gamma(\tau,s_j)\|}. \end{aligned}$$

Let us denote the two sums by Σ_1 and Σ_2, respectively. For Σ_1, a similar computation as in the proof of Lemma 3.8 shows that

$$\Sigma_1 = \sum_{j=0}^{q-1} \cos\vartheta_{j+1}\partial_\tau s_{j+1} - \cos\vartheta_j\partial_\tau s_j = \cos\vartheta_q\partial_\tau s_q - \cos\vartheta_0\partial_\tau s_0.$$

However, because $s_q(\tau,s) = s_0(\tau,s) = s$, we get $\Sigma_1 = 0$. For Σ_2, we rearrange the sum into

$$\begin{aligned} \Sigma_2 = {} & \sum_{j=1}^{q-1} \left(\frac{\gamma(\tau,s_j)-\gamma(\tau,s_{j-1})}{\|\gamma(\tau,s_j)-\gamma(\tau,s_{j-1})\|} - \frac{\gamma(\tau,s_{j+1})-\gamma(\tau,s_j)}{\|\gamma(\tau,s_{j+1})-\gamma(\tau,s_j)\|} \right) \cdot \partial_\tau\gamma(\tau,s_j) \\ & + \left(\frac{\gamma(\tau,s_q)-\gamma(\tau,s_{q-1})}{\|\gamma(\tau,s_q)-\gamma(\tau,s_{q-1})\|} - \frac{\gamma(\tau,s_1)-\gamma(\tau,s_0)}{\|\gamma(\tau,s_1)-\gamma(\tau,s_0)\|} \right) \cdot \partial_\tau\gamma(\tau,s_0). \end{aligned}$$

Now let $N(\tau,s)$ and $T(\tau,s)$ be the unit outward normal and unit positive tangent of $\partial\Omega_\tau$ at s, respectively. Then

$$\Sigma_2 = \sum_{j=1}^{q-1} \partial_\tau\gamma(\tau,s_j)\cdot\big(N(\tau,s_j)\sin\vartheta_j + T(\tau,s)\cos\vartheta_j + N(\tau,s_j)\sin\vartheta_j - T(\tau,s)\cos\vartheta_j\big)$$
$$+\partial_\tau\gamma(\tau,s_0)\cdot\big(N(\tau,s_0)\sin\vartheta_q + T(\tau,s_0)\cos\vartheta_q + N(\tau,s_0)\sin\vartheta_0 - T(\tau,s_0)\cos\vartheta_0\big),$$

and the lemma follows by noting that $\vartheta_0 = \varphi_q(s)$ and $\vartheta_q = \tilde{\varphi}_q(s)$. □

The following estimate on M_q will be useful.

LEMMA 3.11. *For sufficiently small $\|f\|_{C_2}$, we have $M_q(\tau,s) = \mathcal{O}(\|f\|_{C^1})$.*

Proof. By Lemmas 3.10 and 3.2, we get

$$M_q(\tau,s) = \left(1 + \sum_{j=1}^{q-1}\vartheta_j(s,\tau)\right)\mathcal{O}(\|f\|_{C^1}).$$

By (28), for sufficiently small $\|f\|_{C_2}$, we get $\sum_{j=1}^{q-1}\vartheta_j(s,\tau) \le \frac{1}{2}l_\tau \max\kappa_\tau \le 3\pi$. □

3.6. *Proof of Theorem 3.3.* Let Ω_τ, $0 \le \tau \le 1$, be as in the previous section, meaning

$$\partial\Omega_\tau = \partial E_0 + \tau f(\theta)N_0,$$

where E_0 is the unit disk and $f(\theta)$ is a smooth function. We will always use the notation $\ell_\tau = \text{length}(\partial\Omega_\tau)$. Instead of the billiard maps β_τ, it is more convenient to use the natural lifts $\hat{\beta}_\tau$ because they are all defined on the same space $\hat{\Pi} = \mathbb{R}\times[0,\pi]$. To prove Theorem 3.3, we need to study the q-iterates $\hat{\beta}_\tau^q$, but before doing this we need the following important perturbative lemma for $\hat{\beta}_\tau$. We recall that $\hat{\beta}_0(x,\varphi) = (x+2\varphi,\varphi)$.

LEMMA 3.12. *Let $||f||_{C^6} \le 1$, and let $||f||_{C^2}$ be sufficiently small. Then $\hat{\beta}_\tau(x,\varphi)$ can be written as*

$$\hat{\beta}_\tau(x,\varphi) = (x + 2\varphi + P_\tau(x,\varphi), \varphi + Q_\tau(x,\varphi)),$$

where P_τ and Q_τ are analytic families of ℓ_τ periodic functions in x and

$$i,j,k = 0,1: \quad \partial_\tau^i\partial_x^j\partial_\varphi^k P_\tau(x,\varphi) = \varphi^{1-k}\mathcal{O}(||f||_{C^6}),$$
$$i,j,k = 0,1: \quad \partial_\tau^i\partial_x^j\partial_\varphi^k Q_\tau(x,\varphi) = \varphi^{2-k}\mathcal{O}(||f||_{C^6}),$$

uniformly for $x\in\mathbb{R}$, $\varphi\in[0,\frac{\pi}{2}]$ and $\tau\in[0,1]$. When $\varphi\in[\frac{\pi}{2},\pi]$, we need to replace φ with $\pi-\varphi$ in the above estimates.

Remark 3.13. We do not claim that C^6 is the optimal choice here. In fact, probably C^4 is sufficient, however since C^6 is more than good enough for our main theorem, we do not attempt to optimize this lemma.

Proof. By symmetry, it is enough to prove the lemma for $\varphi \in [0, \frac{\pi}{2}]$, hence we assume this throughout the proof. Let $\hat{\beta}$ be the lift of the billiard map of a strictly convex domain Ω, with $\hat{\beta}(x, 0) = (x, 0)$. Also, let κ be the curvature function of $\partial\Omega$. By Proposition 14.1 of [Laz93], we know that if we define

$$(x_1, \varphi_1) = \hat{\beta}(x, \varphi),$$

then

$$\int_x^{x_1} \sin\left(\varphi - \int_x^{x'} \kappa(x'')dx''\right) dx' = 0, \tag{26}$$

$$\varphi_1 = \int_x^{x_1} \kappa(x')dx' - \varphi. \tag{27}$$

Moreover,

$$\frac{2}{\kappa_{\max}}\varphi \le x_1 - x \le \frac{2}{\kappa_{\min}}\varphi, \tag{28}$$

$$\frac{1}{2\kappa_{\max}/\kappa_{\min} - 1}\varphi \le \varphi_1 \le (2\kappa_{\max}/\kappa_{\min} - 1)\,\varphi. \tag{29}$$

Now consider the the deformation $\partial\Omega_\tau$, $\tau \in [0, 1]$. By our notation,

$$(x_1, \varphi_1) = \hat{\beta}_\tau(x, \varphi) = (x + 2\varphi + P_\tau(x, \varphi), \varphi + Q_\tau(x, \varphi)).$$

Since

$$\kappa_\tau = 1 + \mathcal{O}(||f||_{C^2}), \tag{30}$$

by (28) we obtain

$$P_\tau(x, \theta) = x_1 - x - 2\varphi = \varphi\mathcal{O}(||f||_{C^2}). \tag{31}$$

Next, as in [Laz93], we study P_τ (sometimes we write P) as an implicit function defined, using (26), by

$$I(\tau, x, \varphi, P) = \int_0^1 \sin\left(\varphi - \int_x^{x+t(2\varphi+P)} \kappa_\tau(x'')dx''\right) dt = 0. \tag{32}$$

Let us compute $|\partial_P I|$ and estimate it from below. We have

$$-\partial_P I = \int_0^1 \cos\left(\varphi - \int_x^{x+t(2\varphi+P)} \kappa_\tau(x'')dx''\right) \kappa_\tau\big(x + t(2\varphi + P)\big)\, t\, dt.$$

By (30) and (31), we get

$$\begin{aligned} -\partial_P I &= \int_0^1 \cos\left(\varphi - \int_x^{x+2t\varphi} dx''\right) t dt + \mathcal{O}(||f||_{C^2}) \\ &= \int_0^1 \cos(\varphi(1 - 2t))\, t\, dt + \mathcal{O}(||f||_{C^2}) \\ &= \frac{\sin\varphi}{\varphi} + \mathcal{O}(||f||_{C^2}). \end{aligned}$$

Since for $\varphi \in [0, \frac{\pi}{2}]$ we have $\frac{\sin\varphi}{\varphi} \geq \frac{2}{\pi}$, we obtain that for $||f||_{C^2}$ sufficiently small, uniformly for $0 \leq \tau \leq 1$, $0 \leq \varphi \leq \frac{\pi}{2}$, and $x \in \mathbb{R}$, we have

$$|\partial_P I| \geq \frac{1}{\pi}. \tag{33}$$

Consequently, by the implicit function theorem there is a unique $P_\tau(x,\varphi)$ satisfying (32) and it is differentiable in τ, x and φ. Moreover,

$$\begin{cases} \partial_\tau P_\tau = -\frac{\partial_\tau I}{\partial_P I}, \\ \partial_x P_\tau = -\frac{\partial_x I}{\partial_P I}, \\ \partial_\varphi P_\tau = -\frac{\partial_\varphi I}{\partial_P I}. \end{cases} \tag{34}$$

On the other hand, since $P_0(x,\varphi) = 0$, we have $P_\tau(x,\varphi) = \tau \tilde{P}_\tau(x,\varphi)$. But since $P_\tau(x,0) = 0$, we have $\tilde{P}_\tau(x,0) = 0$, and so

$$P_\tau(x,\varphi) = \tau\varphi R_\tau(x,\varphi).$$

In fact

$$R_\tau(x,\varphi) = \int_0^1 \int_0^1 (\partial_\tau \partial_\varphi P)(u\tau, x, v\varphi) du dv.$$

Here, to ease the notation for the integrand, we have set $P(\tau,x,\varphi) := P_\tau(x,\varphi)$.

Similarly, for Q_τ we know that $Q_0(x,\varphi) = 0$, so $Q_\tau(x,\varphi) = \tau\tilde{Q}_\tau(x,\varphi)$. It is known by the asymptotic expansion of the billiard map near $\varphi = 0$ (see, for example, page 145 of [Laz93]), that

$$Q_\tau(x,0) = \partial_\varphi Q_\tau(x,0) = 0,$$

so we must have

$$Q_\tau(x,\varphi) = \tau\varphi^2 S_\tau(x,\varphi).$$

By the integral remainder formula of the Taylor's theorem, we have

$$S_\tau(x,\varphi) = \frac{1}{2}\int_0^1 \int_0^1 (1-v)(\partial_\tau \partial_\varphi^2 Q)(u\tau, x, v\varphi) du dv.$$

Again, here for convenience we have denoted $Q(\tau,x,\varphi) := Q_\tau(x,\varphi)$. Thus, to prove the lemma it suffices to prove the estimates

$$\partial_\varphi^k \partial_x^j \partial_\tau^i P = \mathcal{O}(||f||_{C^6}), \quad i = 1,2;\ j = 0,1;\ k = 1,2,3 \tag{35}$$

and

$$\partial_\varphi^k \partial_x^j \partial_\tau^i Q = \mathcal{O}(||f||_{C^6}), \quad i = 1,2;\ j = 0,1;\ k = 2,3. \tag{36}$$

We first show the estimates for P. An important observation is that for $n \leq 4$,

$$\partial_x^n \partial_\tau \kappa_\tau(x) = \mathcal{O}(||f||_{C^{n+2}}).$$

One then immediately sees by (32) that

$$\partial_\tau^i I = \mathcal{O}(||f||_{C^2}), \quad i = 1,2.$$

Furthermore, by taking derivatives with respect to x, φ, and P, we get

$$\partial_P^m \partial_\varphi^k \partial_x^j \partial_\tau^i I = \mathcal{O}(||f||_{C^6}), \quad i = 1, 2; j + k + m \leq 5.$$

The estimates (35) can be concluded by differentiating the first equation of (34), then using the lower bound (33), and the above estimates for the derivatives of I.

The estimates for Q follow from the ones for P, and the relation

$$Q(\tau, x, \varphi) = \int_x^{x+2\varphi+P(\tau,x,\varphi)} \kappa_\tau(x')dx' - 2\varphi,$$

which is obtained from (27). □

Equipped with Lemma 3.12, we are in position to start the proof of Theorems 3.3 and 3.5. Let $\hat{\beta}_\tau$ be the lift of the billiard map of $\partial\Omega_\tau = \partial E_0 + \tau f N_0$, and let Proj_1 be the projection onto the x component of $\Pi = \mathbb{R} \times [0, \pi]$. Our strategy is to show that

$$x_q(\tau, x, \varphi) := \mathrm{Proj}_1 \hat{\beta}_\tau^q(x, \varphi)$$

is strictly increasing as a function of φ on the interval $[0, C/q]$. More precisely,

THEOREM 3.14. *For any $C > 0$, there exists ε_0 such that for all perturbations $\partial\Omega_\tau = \partial D + \tau f N_0$ of the unit disk D with $\|f\|_{C^6} \leq \varepsilon_0$, for all $q \geq 2$, $0 \leq \tau \leq 1$, $x \in \mathbb{R}$, and $\varphi \leq C/q$, we have*

$$\partial_\tau x_q(\tau, x, \varphi) = q\varphi C^2 e^{2C} \mathcal{O}(\|f\|_{C^6}),$$
$$\partial_\varphi x_q(\tau, x, \varphi) = 2q + qC^2 e^{2C} \mathcal{O}(\|f\|_{C^6}),$$

and

$$x_q(\tau, x, \varphi) = x + 2q\varphi + q\varphi C^2 e^{2C} \mathcal{O}(\|f\|_{C^6}).$$

Note that the last statement follows by integrating the the second statement and the fact $x_q(\tau, x, 0) = x$. Before proving this theorem, we need to state and prove a lemma and its corollary.

LEMMA 3.15. *Let $C > 0$ and $\delta > 0$. Suppose $\varphi \leq C/q$. Then for $\|f\|_{C^6}$ sufficiently small in terms of C and δ, we have that for all $x \in \mathbb{R}$ and $\tau \in [0, 1]$, all angles of reflections of the orbit $\{\hat{\beta}_\tau^j(x, \varphi)\}_{j=0}^q$ are bounded above by $C(1 + \delta)/q$. Moreover, if $\varphi \geq C/q$, then all angles of reflections are bounded from below by $\frac{C}{(1+\delta)q}$.*

Proof. Let φ_j be the j-th angle of reflection. We shall prove by induction that for all $0 \leq j \leq q$, we have

$$\varphi_j \leq \frac{C}{q}\left(1 + \frac{A}{q}\right)^j, \tag{37}$$

where $A > 0$ is chosen such that $e^A = 1 + \delta$. Once proved, this estimate would imply the lemma immediately because

$$\frac{C}{q}\left(1 + \frac{A}{q}\right)^q \leq \frac{C}{q}e^A \leq \frac{C(1+\delta)}{q}.$$

The estimate (37) is obviously true for $j = 0$ by assumption of the lemma. Assume it is true for some $j \geq 0$. Then using Lemma 3.12, we have

$$\varphi_{j+1} = \varphi_j \left(1 + \varphi_j \mathcal{O}(\|f\|_{C^6})\right).$$

Therefore, the proof is concluded if we choose $\|f\|_{C^6}$ small enough (in terms of C and δ) so that

$$\varphi_j \mathcal{O}(\|f\|_{C^6}) \leq \frac{C(1+\delta)}{q}\mathcal{O}(\|f\|_{C^6}) \leq \frac{A}{q}.$$

The second statement follows from the first. □

As a corollary we have

COROLLARY 3.16. *For $\|f\|_{C^6}$ sufficiently small, all angles of reflections of a q-reflected path in Ω_τ from x to x' with $|x - x'| \leq \frac{\ell_\tau}{100}$ are less than $\frac{3\pi}{2q}$. In particular, all angles of reflections of q- loops of winding number one are bounded by $\frac{3\pi}{2q}$.*

Proof. By definition of a q-reflected path from x to x', we have $x_q(\tau, x, \alpha) = x' + \ell_\tau$. Let $\delta \geq \frac{1}{100}$, to be determined later. Since $\ell_\tau = 2\pi + \mathcal{O}(\|f\|_{C^2})$, if we define

$$x_j(\tau, x, \varphi) = \mathrm{Proj}_1 \hat{\beta}_\tau^j(x, \varphi)$$

for small enough $\|f\|_{C^2}$ in terms of δ, there must exist $0 \leq j^* \leq q - 1$ such that

$$x_{j^*+1} - x_{j^*} \leq \frac{x' - x + \ell_\tau}{q} \leq \frac{2(1+\delta)^2\pi}{q}.$$

By (28), for sufficiently small $\|f\|_{C^6}$ (in terms of δ), we get $\varphi_{j^*} \leq \frac{(1+\delta)^3\pi}{q}$. We then start with the point (x_{j^*}, φ_{j^*}) in the phase space and apply the billiard map β_τ, $q - j^*$ times, and we apply its inverse β_τ^{-1}, j^* times. Using Lemma 3.15 we obtain $\varphi_j \leq \frac{(1+\delta)^4\pi}{q}$, which is less than $\frac{3\pi}{2q}$, for example, for $\delta = \frac{1}{10}$. □

Proof of Theorem 3.14. To obtain estimates on the φ and τ derivatives of $x_q(\tau, x, \varphi)$ we take its difference with the corresponding function for the unit disk and differentiate with respect to φ and τ and denote it by $A(x, \varphi)$, i.e.,

$$A_q(\tau, x, \varphi) = \partial_\tau \partial_\varphi \left(\mathrm{Proj}_1 \hat{\beta}_\tau^q(x, \varphi) - \mathrm{Proj}_1 \hat{\beta}_0^q(x, \varphi)\right).$$

Recall that $\hat{\beta}_\tau$ is the billiard map of Ω_τ and $\hat{\beta}_0$ is the billiard map of the unit disk $D = E_0$. Since

$$\hat{\beta}_0^q(x, \varphi) = (x + 2q\varphi, \varphi),$$

we have

$$\partial_\tau \partial_\varphi \mathrm{Proj}_1 \hat{\beta}_0^q(x,\varphi) = \partial_\tau(2q) = 0,$$

so in fact

$$A_q(\tau,x,\varphi) = \partial_\tau \partial_\varphi \mathrm{Proj}_1 \hat{\beta}_\tau^q(x,\varphi).$$

We claim that that for all $\tau \in [0,1]$, $x \in \mathbb{R}$, and $\varphi \in [0, \frac{C}{q}]$,

$$A_q(\tau,x,\varphi) = C^2 e^{2C} q \mathcal{O}(\|f\|_{C^6}).$$

To prove this, we write

$$\begin{aligned} \partial_\tau \partial_\varphi \mathrm{Proj}_1 \hat{\beta}_\tau^q(x,\varphi) &= \partial_\varphi \mathrm{Proj}_1 \sum_{j=0}^{q-1} \hat{\beta}_\tau^j \circ (\partial_\tau \hat{\beta}_\tau) \circ \hat{\beta}_\tau^{q-j-1}(x,\varphi) \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \sum_{j=0}^{q-1} D\hat{\beta}_\tau^j \circ (D\partial_\tau \hat{\beta}_\tau) \circ D\hat{\beta}_\tau^{q-j-1}(x,\varphi) \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned} \tag{38}$$

By Lemma 3.15, all angles of iterations are bounded by $2C/q$. By Lemma 3.12, we have uniformly for all $\tau \in [0,1]$, $\varphi \in [0, \frac{2C}{q}]$, and $x \in \mathbb{R}$,

$$D\hat{\beta}_\tau(x,\varphi) = \begin{bmatrix} 1 + \frac{C}{q}\mathcal{O}(\|f\|_{C^6}) & 2 + \mathcal{O}(\|f\|_{C^6}) \\ \frac{C^2}{q^2}\mathcal{O}(\|f\|_{C^6}) & 1 + \frac{C}{q}\mathcal{O}(\|f\|_{C^6}) \end{bmatrix}$$

and

$$D\partial_\tau \hat{\beta}_\tau(x,\varphi) = \begin{bmatrix} \frac{C}{q}\mathcal{O}(\|f\|_{C^6}) & \mathcal{O}(\|f\|_{C^6}) \\ \frac{C^2}{q^2}\mathcal{O}(\|f\|_{C^6}) & \frac{C}{q}\mathcal{O}(\|f\|_{C^6}) \end{bmatrix}. \tag{39}$$

We shall need to estimate the powers of the matrix $D\hat{\beta}_\tau$. Breaking it into the diagonal and the off-diagonal part, and factoring the diagonal part, we see that

$$|D\hat{\beta}_\tau^j| \le \left(1 + \frac{C}{q}\mathcal{O}(\|f\|_{C^6})\right)^j \left(I + \begin{bmatrix} 0 & 4 + \mathcal{O}(\|f\|_{C^6}) \\ \frac{C^2}{q^2}\mathcal{O}(\|f\|_{C^6}) & 0 \end{bmatrix}\right)^j. \tag{40}$$

Let us denote

$$B = \begin{bmatrix} 0 & 4 + \mathcal{O}(\|f\|_{C^6}) \\ \frac{C^2}{q^2}\mathcal{O}(\|f\|_{C^6}) & 0 \end{bmatrix}.$$

We note that

$$B^2 = \begin{bmatrix} \frac{C^2}{q^2}\mathcal{O}(\|f\|_{C^6}) & 0 \\ 0 & \frac{C^2}{q^2}\mathcal{O}(\|f\|_{C^6}) \end{bmatrix}.$$

By the binomial expansion,

$$(I+B)^j = \sum_{m=0}^{j} \binom{j}{m} B^m = \sum_{k=0}^{[j/2]} \binom{j}{2k} B^{2k} + \sum_{k=0}^{[(j-1)/2]} \binom{j}{2k+1} B^{2k+1}.$$

Thus,

$$(I+B)^j \leq \sum_{k=0}^{[j/2]} \binom{j}{2k} \frac{C^{2k}}{q^{2k}} \mathcal{O}(\|f\|_{C^6}^k) I + \sum_{k=0}^{[(j-1)/2]} \binom{j}{2k+1} \frac{C^{2k}}{q^{2k}} \mathcal{O}(\|f\|_{C^6}^k) B.$$

We choose $\|f\|_{C^6}$ small enough so that all $\mathcal{O}(\|f\|_{C^6}^k)$ terms are bounded by one. To estimate the second sum we note that for any $a > 0$,

$$\sum_{k=0}^{[(j-1)/2]} \binom{j}{2k+1} a^{2k} = \frac{(1+a)^j - (1-a)^j}{2a} \leq j(1+a)^{j-1}.$$

Hence,

$$(I+B)^j \leq \begin{bmatrix} \left(1+\frac{C}{q}\right)^j & 10j\left(1+\frac{C}{q}\right)^{j-1} \\ \frac{C}{q}\left(1+\frac{C}{q}\right)^j & \left(1+\frac{C}{q}\right)^j \end{bmatrix}.$$

Plugging this into (40), we get

$$|D\hat{\beta}_\tau^j| \leq \left(1+\frac{C}{q}\right)^{2j} \begin{bmatrix} 1 & 10j \\ \frac{C}{q} & 1 \end{bmatrix}.$$

Inserting this estimate into (38) and using (39), we arrive at

$$\begin{aligned} |\partial_\tau \partial_\varphi \mathrm{Proj}_1 \hat{\beta}_\tau^q(x,\varphi)| &\leq \left(1+\frac{C}{q}\right)^{2q} \sum_{j=0}^{q-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \begin{bmatrix} 1 & 10j \\ \frac{C}{q} & 1 \end{bmatrix} \\ &\quad \times \begin{bmatrix} \frac{C}{q} & 1 \\ \frac{C^2}{q^2} & \frac{C}{q} \end{bmatrix} \begin{bmatrix} 1 & 10(q-j-1) \\ \frac{C}{q} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathcal{O}(\|f\|_{C^6}) \\ &\leq C^2 e^{2C} q \mathcal{O}(\|f\|_{C^6}). \end{aligned}$$

The theorem follows by integrating this with respect to φ and τ separately, and the facts $x_q(\tau,x,0) = x$ and $x_q(0,x,\varphi) = x + 2q$. □

For future reference we record that our estimates also show the following bounds for the φ derivative of $\vartheta_q(\tau,x,\varphi) = \mathrm{Proj}_2 \hat{\beta}_\tau^q(x,\varphi)$, where Proj_2 is the projection onto the second component.

LEMMA 3.17. *We have* $\partial_\varphi \vartheta_q(\tau,x,\varphi) = 1 + C^2 e^{2C} \mathcal{O}(\|f\|_{C^6})$.

3.6.1. *Concluding the proof of Theorems 3.3 and 3.5.* Theorem 3.14 shows that $x_q(\tau,x,\varphi)$ is monotonically increasing in φ on $[0, 3\pi/2q]$ and

$$x_q(\tau,x,\varphi) = x + 2q\varphi + q\varphi \mathcal{O}(\|f\|_{C^6}).$$

Since $x_q(\tau,x,0) = 0$, and because for sufficiently small $\|f\|_{C^6}$,

$$x_q(\tau,x,3\pi/2q) = x + 3\pi + \mathcal{O}(\|f\|_{C^6}) > x' + \ell_\tau,$$

by the intermediate value theorem there must be a unique $\varphi = \alpha_q(\tau,x,x') \leq 3\pi/2q$ such that

$$x_q(\tau,x,\alpha_q(\tau,x,x')) = x' + \ell_\tau.$$

By the implicit function theorem, $\alpha_q(\tau, x, x')$ is smooth in (τ, x, x'). This together with Corollary 3.16 conclude the proof of Theorem 3.5, thus also Theorem 3.3.

4. **Length spectrum**

Let Ω be a smooth strictly convex domain. For $1 \le p \le \frac{q}{2}$, we denote $\mathcal{L}_{p,q}(\Omega)$ to be the set of lengths of periodic orbits of type (p, q), i.e., periodic orbits that make q reflections and wind around the boundary of Ω, p times, in the counterclockwise direction. The length spectrum of Ω is

$$\mathcal{L}(\Omega) = \text{closure} \bigcup_{1 \le p \le q/2} \mathcal{L}_{p,q}(\Omega).$$

We also denote $T_{p,q}$ and $t_{p,q}$ to be the sup and inf of $\mathcal{L}_{p,q}(\Omega)$, respectively. Marvizi-Melrose [MM82] proved that for a fixed p, as $q \to \infty$, one has

$$T_{p,q} - t_{p,q} = \mathcal{O}(q^{-k}), \tag{41}$$

and moreover, there are constants $c_{k,p}(\Omega)$ such that

$$T_{p,q} \sim p\,\ell + \sum_{k=1}^{\infty} c_{k,p}(\Omega) q^{-2k}, \tag{42}$$

where ℓ is the perimeter of Ω. Note that this in particular shows that natural integer multiples of ℓ belong to the length spectrum as they are limit points of closed geodesics.

The Mather function $ß(\omega)$ is a strictly convex function on $[0, \frac{1}{2}]$ (see, for example, [MF94], [Sib04]) whose values at the rational numbers are given by

$$ß(p/q) = -\frac{1}{q} T_{p,q}. \tag{43}$$

The following lemma will be useful for us. From now on we shall use $T_q = T_{1,q}$ and $t_q = t_{1,q}$.

LEMMA 4.1. *The sequence $\{T_q\}_{q \ge 2}$ is strictly increasing to ℓ, and its gap sequence*

$$\{T_{q+1} - T_q\}_{q \ge 2}$$

is strictly decreasing.

Proof. Since ß is strictly convex, the slopes must strictly increase on its graph, hence

$$\frac{ß\left(\frac{1}{q+1}\right) - ß\left(\frac{1}{q+2}\right)}{\frac{1}{q+1} - \frac{1}{q+2}} < \frac{ß\left(\frac{1}{q}\right) - ß\left(\frac{1}{q+1}\right)}{\frac{1}{q} - \frac{1}{q+1}}.$$

The lemma follows quickly by (43). □

Remark 4.2. One can try to prove this using the asymptotic (42), however this method would only prove the lemma for large q.

For our purposes we will need the following rough, but quantitative, version of estimates (41) and (42):

LEMMA 4.3. *Let $\partial\Omega_\tau = \partial E_0 + \tau f N_0$ be a nearly circular deformation in C^6. Assume that $\|f\|_{C^8} \leq 1$ and that $\|f\|_{C^2}$ is sufficiently small so that $\kappa_\tau = 1 + \mathcal{O}(\|f\|_{C^2}) \geq \frac{1}{2}$. Then uniformly for $\tau \in [0,1]$, we have*

$$T_q - t_q = q^{-3}\mathcal{O}(\|f\|_{C^6}) + \mathcal{O}(q^{-4}), \tag{44}$$

$$T_q = \ell_\tau - \frac{1}{4}\left(\int_0^{\ell_\tau} \kappa_\tau^{2/3}(s)\, ds\right)^3 q^{-2} + q^{-3}\mathcal{O}(\|f\|_{C^6}) + \mathcal{O}(q^{-4}). \tag{45}$$

Here, the constants in all $\mathcal{O}$ remainders are universal.

Proof. We provide a proof using "Lazutkin coordinate" and the Euler-Maclaurin formula. Note that this is a quantitative version of a result in [MM82] that was not obtained in this reference.

It is sufficient to prove this lemma for $\tau = 1$. We shall use Ω for Ω_1, κ for κ_1, and ℓ for ℓ_1. We first recall the Lazutkin coordinate that is a diffeomorphism from $\mathbb{R}/\ell\mathbb{Z}$ to $\mathbb{R}/\mathbb{Z}$ defined by

$$\xi = \frac{\int_0^s \kappa^{2/3}(s')ds'}{\int_0^\ell \kappa^{2/3}(s')ds'}. \tag{46}$$

Here we have used ξ instead of x, the later being the standard notation for the Lazutkin coordinate, to avoid confusion with our x used for the lift of s to $\mathbb{R}$. The periodic orbits of type $(1,q)$ in Ω have a rather nice description in the Lazutkin coordinate. To present this feature, let $\{(s_j,\varphi_j)\}_{j=0}^{q-1}$ be any such periodic orbit, and let $\{\xi_j\}_{j=0}^{q-1}$ correspond to $\{s_j\}_{j=0}^{q-1}$ in Lazutkin coordinate. From Appendix A of [dKW17], one can show that there exists a 1-periodic smooth function α defined only in terms of κ, with $\|\alpha\|_{C^m} = \mathcal{O}(\|(1/\kappa)'\|_{C^{m+1}})$, such that

$$\xi_j = \xi_0 + \frac{j}{q} + \frac{\alpha\,(\xi_0 + j/q)}{q^2} + \frac{\mathcal{O}(\|(1/\kappa)'\|_{C^3})}{q^4}, \qquad 1 \leq j \leq q-1. \tag{47}$$

We note that by our lower bound assumption $\kappa \geq \frac{1}{2}$, we can replace the term $\mathcal{O}(\|(1/\kappa)'\|_{C^3})$ in the remainder by $\mathcal{O}(\|\kappa'\|_{C^3}) = \mathcal{O}(\|f\|_{C^6})$. We shall use (47) to find an asymptotic for the length T of the orbit $\{s_j\}_{j=0}^{q-1}$. Let $s(\xi)$ be the inverse function of $\xi = \xi(s)$ defined by (46), and let $\tilde{\gamma}(\xi) = \gamma(s(\xi))$. We write

$$T = \sum_{j=0}^{q-1} \|\gamma(s_{j+1}) - \gamma(s_j)\| = \sum_{j=0}^{q-1} \|\tilde{\gamma}(\xi_{j+1}) - \tilde{\gamma}(\xi_j)\|.$$

Inserting (47) and using the mean value theorem,

$$T = \sum_{j=0}^{q-1} \left\| \tilde{\gamma}\left(\xi_0 + \frac{j+1}{q} + \frac{\alpha\left(\xi_0 + (j+1)/q\right)}{q^2}\right) - \tilde{\gamma}\left(\xi_0 + \frac{j}{q} + \frac{\alpha\left(\xi_0 + j/q\right)}{q^2}\right)\right\| + \frac{\mathcal{O}(\|f\|_{C^6})}{q^3}.$$

For the sum, we shall use the Euler-Maclaurin formula, which asserts that if $g(\xi) \in C^\infty[0,1]$ and $g^{(k)}(0) = g^{(k)}(1)$ for all $k \geq 0$, then for all $m \geq 1$,

$$\frac{1}{q}\sum_{j=0}^{q-1} g(j/q) = \int_0^1 g(\xi)d\xi + R_m(g),$$

with

$$|R_m(g)| \leq \frac{2\zeta(m)}{(2\pi q)^m}\int_0^1 |g^{(m)}(\xi)|d\xi.$$

In our situation, we have

$$g(\xi) = \left\| \tilde{\gamma}\left(\xi_0 + \xi + \frac{1}{q} + \frac{\alpha(\xi_0 + \xi + 1/q)}{q^2}\right) - \tilde{\gamma}\left(\xi_0 + \xi + \frac{\alpha(\xi_0 + \xi)}{q^2}\right)\right\|,$$

which is a smooth 1-periodic function on $[0,1]$. Thus if we choose $m = 4$, we obtain

$$T = q\int_0^1 \left\| \tilde{\gamma}\left(\xi + \frac{1}{q} + \frac{\alpha(\xi + 1/q)}{q^2}\right) - \tilde{\gamma}\left(\xi + \frac{\alpha(\xi)}{q^2}\right)\right\| d\xi + \frac{\mathcal{O}(1 + \|f\|_{C^8})}{q^4} + \frac{\mathcal{O}(\|f\|_{C^6})}{q^3}.$$

Taylor expanding the integrand we arrive at

$$T = a_0 + \frac{a_1}{q} + \frac{a_2}{q^2} + \frac{a_3}{q^3} + \frac{\mathcal{O}(\|f\|_{C^6})}{q^3} + \frac{\mathcal{O}(1 + \|f\|_{C^8})}{q^4}.$$

It is clear that $a_0 = \int_0^1 \|\tilde{\gamma}'(\xi)\|d\xi = \ell$. That $a_1 = 0$ and $a_2 = -\frac{1}{4}\left(\int \kappa^{2/3}\right)^3$ follows from [MM82]. Since by (42) only even powers of q^{-1} appear in the expansion, we must have $a_3 = \mathcal{O}(\|f\|_{C^6})$, so

$$T = \ell + \frac{a_2}{q^2} + \frac{\mathcal{O}(\|f\|_{C^6})}{q^3} + \frac{\mathcal{O}(1 + \|f\|_{C^8})}{q^4}.$$

4.1. *A proof independent of* [dKW17]. The proof of (47) in [dKW17] is given under the assumption of axial symmetry. However, this assumption is not essential, even for the proof in [dKW17]. For the sake of independence, we present another proof of Lemma 4.3, at the mild cost of replacing $\|f\|_{C^6}$ with $\|f\|_{C^7}$, and $\|f\|_{C^8}$ with $\|f\|_{C^9}$ in the lemma.

For the proof of Lemma 4.3, we only need a weaker statement than (47), which states that there exist 1-periodic smooth functions α_1 and α_2 defined

only in terms of κ, with $\|\alpha_1\|_{C^m} = \mathcal{O}(\|(1/\kappa)'\|_{C^{m+1}})$, $\|\alpha_2\|_{C^m} = \mathcal{O}(\|(1/\kappa)'\|_{C^{m+2}})$ such that

(48)
$$\xi_j = \xi_0 + \frac{j}{q} + \frac{\alpha_1(\xi_0 + j/q)}{q^2} + \frac{\alpha_2(\xi_0 + j/q)}{q^3} + \frac{\mathcal{O}(\|(1/\kappa)'\|_{C^3})}{q^4}, \qquad 1 \le j \le q-1.$$

We emphasize that the results of [dKW17] imply that $\alpha_2 \equiv 0$, but we do not need this stronger statement. Assuming (48), the proof of Lemma 4.3 follows line-by-line as before, but at the end we would obtain

$$T = \ell + \frac{a_2}{q^2} + \frac{\mathcal{O}(\|f\|_{C^7})}{q^3} + \frac{\mathcal{O}(1 + \|f\|_{C^9})}{q^4}.$$

So if throughout we assume near circularity in C^9, the rest of the proof goes through without modification.

To prove (48), we shall use Lazutkin coordinates of order 5 as defined in (15). Let (u, v) be such coordinates. Then the billiard map is of the form

$$\beta(u, v) = \big(u + v + v^5 a(u, v), v + v^6 b(u, v)\big) \tag{49}$$

for smooth functions a and b that are bounded by $\mathcal{O}(\|(1/\kappa)'\|_{C^3})$. These bounds follow from similar bounds for the derivatives of the billiard map in (s, φ) coordinates, and also from the construction of the coordinates (u, v) from (s, φ), which are provided in [Laz93, Prop. 14.2, Lemma 14.6].

Let (u_0, v_0) correspond to a $(1, q)$ periodic orbit, meaning $\hat{\beta}^q(u_0, v_0) = (u_0 + 1, v_0)$. By (49) and finite induction, we obtain

$$u_0 + qv_0 + q\mathcal{O}(|v_0|^5 \|(1/\kappa)'\|_{C^3}) = u_0 + 1.$$

This implies that

$$v_0 + \mathcal{O}(|v_0|^5 \|(1/\kappa)'\|_{C^3}) = \frac{1}{q},$$

which in turn gives $v_0 = \frac{1}{q} + \frac{\mathcal{O}(\|(1/\kappa)'\|_{C^3})}{q^5}$.

Now for each $1 \le j \le q$, let $(u_j, v_j) = \hat{\beta}^j(u_0, v_0)$. Then by finite induction on j, we can show that

$$u_j = u_0 + \frac{j}{q} + \frac{\mathcal{O}(\|(1/\kappa)'\|_{C^3})}{q^4}, \qquad v_j = \frac{1}{q} + \frac{\mathcal{O}(\|(1/\kappa)'\|_{C^3})}{q^4}.$$

In the final step, we recall from (16) that (ξ, η) and (u, v) are related by

$$(\xi, \eta) = \big(u + v^2 A(u, v), v + v^3 B(u, v)\big),$$

where A and B are smooth and

$$\|A\|_{C^m} = \mathcal{O}(\|(1/\kappa)'\|_{C^{m+1}}), \quad \|B\|_{C^m} = \mathcal{O}(\|(1/\kappa)'\|_{C^{m+2}}).$$

If we write

$$A(u, v) = A_0(u) + A_1(u)v + \mathcal{O}(\|(1/\kappa)'\|_{C^3})|v|^2,$$

then

$$\xi_j = u_j + v_j^2 A(u_j, v_j) = u_0 + \frac{j}{q} + \frac{A_0\,(u_0 + j/q)}{q^2} + \frac{A_1\,(u_0 + j/q)}{q^3} + \frac{\mathcal{O}(\|(1/\kappa)'\|_{C^3})}{q^4}.$$

Writing u_0 in terms of ξ_0 and η_0, we obtain (48). □

We now focus on the part of the length spectrum that is less than the length of the boundary. While this does not inclusively correspond to $(1, q)$ periodic orbits (consider a very thin ellipse for example), as we show it does for nearly circular domains.

LEMMA 4.4. *Let Υ be a (p, q) periodic orbit with $p \geq 2$ of a nearly circular deformation Ω_τ of D in C^6. Then for $\|f\|_{C^6}$ sufficiently small, the length of Υ is bounded below by ℓ_τ, uniformly for $\tau \in [0, 1]$. Hence, in particular,*

$$\mathcal{L}(\Omega_\tau) \cap (0, \ell_\tau) = \bigcup_{q \geq 2} \mathcal{L}_{1,q}(\Omega_\tau). \tag{50}$$

Proof of Lemma 4.4. Proposition 5 of [Ami96] would imply this lemma easily, however the proof of [Ami96] is not correct. Hence we give an independent proof by means of variations.

To clarify the idea, we first verify Lemma 4.4 in the case of the unit disc D. It is well known that every link of a billiard trajectories of D (not necessarily a periodic trajectory) intersects the boundary with the same angle of incidence say φ. One can then easily verify that length of each link is $2\sin\varphi$, thus if the trajectory makes q bounces, its length must be $2q\sin\varphi$. The angle for a (p, q) periodic orbit on a circle is given by $\varphi = \frac{p}{q}\pi$. Therefore, the length of a (p, q) orbit on the circle is $2q\sin\frac{p}{q}\pi$. Since $\sin x \geq \frac{2}{\pi}x$ on the interval $[0, \pi/2]$, we get $2q\sin\frac{p}{q}\pi \geq 4p > 2\pi$ for $p \geq 2$.

Now let $\partial\Omega_\tau = \partial D + \tau f N_0$ be a nearly circular deformation of the unit circle ∂D with f sufficiently small in C^6. We wish to approximate the lengths of (p, q) periodic orbits of $\partial\Omega_\tau$ by the ones of the disk ∂D using a variational method. Let (x_0, φ_0) be an initial point in the phase space of $\partial\Omega_\tau$ of a (p, q) periodic orbit of β_τ with $p \geq 2$. We keep in mind that $q \geq 4$ because $\frac{p}{q} \leq \frac{1}{2}$. Since we can choose $\varphi_0 \in [0, \pi/2]$ (otherwise consider $\pi - \varphi_0$), there is a unique $k_0 \in \{0, 1, 2, \ldots, q\}$ such that

$$\frac{(k_0 - \frac{1}{2})\pi}{2q} \leq \varphi_0 \leq \frac{(k_0 + \frac{1}{2})\pi}{2q}.$$

We first claim that $k_0 \geq 4$. To prove this let $(x_j(\tau), \varphi_j(\tau)) = \hat{\beta}_\tau^j(x_0, \varphi_0)$ with $0 \leq j \leq q$. Since (x_0, φ_0) is a (p, q) periodic orbit, we have $x_q(\tau) - x_0 = p\ell_\tau$. Suppose $k_0 \leq 3$. Then by Lemma 3.15, for a given $\delta > 0$, we get $\varphi_j(\tau) \leq$

$\frac{7\pi(1+\delta)}{4q}$ for sufficiently small f in C^6. In particular,

$$\sum_{j=0}^{q-1} \varphi_j(\tau) \leq \frac{7\pi(1+\delta)}{4}.$$

On the other hand, by (28) this sum must be larger than or equal to

$$\frac{1-\delta}{2}(x_q(\tau) - x_0) = \frac{p\ell_\tau(1-\delta)}{2} \geq 2\pi(1-\delta)^2,$$

which leads to a contradiction for δ small.

From now on we assume $k_0 \geq 4$. We consider the (partial) orbit

$$\{(x_j(\tau), \varphi_j(\tau))\}_{j=0}^{q_0}$$

in Ω_τ, where we define

$$q_0 := \begin{cases} \left[\frac{4q}{k_0}\right] + 1 & \text{if} \quad k_0 \geq 5, \\ q & \text{if} \quad k_0 = 4. \end{cases}$$

One can easily verify, using $q \geq k_0 \geq 4$, that q_0 never exceeds q. As a result, a lower bound for the length of this partial orbit provides a lower bound on the length of the desired full orbit $\{(x_j(\tau), \varphi_j(\tau))\}_{j=0}^{q}$. So let us set

$$b(\tau) = \sum_{j=0}^{q_0-1} \|\gamma(\tau, x_{j+1}(\tau)) - \gamma(\tau, x_j(\tau))\|.$$

Obviously for the case of the unit disk D, i.e., $\tau = 0$, we have $b(0) = 2q_0 \sin\varphi_0$. By the mean value theorem, for all $\tau \in [0,1]$,

$$|b(\tau) - 2q_0 \sin\varphi_0| \leq \sup_{\tau\in[0,1]} |\partial_\tau b(\tau)|. \tag{51}$$

We compute the variation of $b(\tau)$ in the same manner as in the proof of Lemma 3.10. We first write

$$\begin{aligned} \partial_\tau b_\tau = &\sum_{j=0}^{q_0-1} (\partial_\tau x_{j+1} \partial_s \gamma(\tau, x_{j+1}) - \partial_\tau x_j \partial_s \gamma(\tau, x_j)) \cdot \frac{\gamma(\tau, x_{j+1}) - \gamma(\tau, x_j)}{\|\gamma(\tau, x_{j+1}) - \gamma(\tau, x_j)\|} \\ &+ \sum_{j=0}^{q_0-1} (\partial_\tau \gamma(\tau, x_{j+1}) - \partial_\tau \gamma(\tau, x_j)) \cdot \frac{\gamma(\tau, x_{j+1}) - \gamma(\tau, x_j)}{\|\gamma(\tau, x_{j+1}) - \gamma(\tau, x_j)\|}. \end{aligned}$$

Denote the two sums by Σ_1 and Σ_2, respectively. For Σ_1, a similar computation as in the proof of Lemma 3.8 shows that

$$\Sigma_1 = \sum_{j=0}^{q_0-1} \cos\varphi_{j+1}(\tau)\partial_\tau x_{j+1}(\tau) - \cos\varphi_j(\tau)\partial_\tau x_j(\tau) = \cos\varphi_{q_0}(\tau)\partial_\tau x_{q_0}(\tau).$$

For Σ_2, following the proof of Lemma 3.10, we obtain

$$\begin{aligned}\Sigma_2 &= \partial_\tau\gamma(\tau,x_0)\cdot\big(N(\tau,x_0)\sin\varphi_0 - T(\tau,x_0)\cos\varphi_0\big)\\ &\quad+\partial_\tau\gamma(\tau,x_q)\cdot\big(N(\tau,x_q)\sin\varphi_q + T(\tau,x_q)\cos\varphi_q\big)\\ &\quad+2\sum_{j=1}^{q_0-1}\partial_\tau\gamma(\tau,x_j)\cdot N(\tau,x_j)\sin\varphi_j\\ &= n(\tau,x_0)\sin\varphi_0 - t(\tau,x_0)\cos\varphi_0 + n(\tau,x_{q_0}(\tau))\sin\varphi_{q_0}(\tau)\\ &\quad+t(\tau,x_{q_0}(\tau))\cos\varphi_{q_0}(\tau)\\ &\quad+2\sum_{j=1}^{q_0-1}n(\tau,x_j(\tau))\sin\varphi_j(\tau).\end{aligned}$$

Let us estimate $|\Sigma_1|$ and $|\Sigma_2|$. Since

$$\varphi_0 \le \frac{(k_0+\frac12)\pi}{2q} \le \frac{4\pi}{q_0-1},$$

by Theorem 3.14 we have

$$|\Sigma_1| \le |\partial_\tau x_{q_0}(\tau)| \le C_1 q_0\varphi_0\|f\|_{C^6} \le 8\pi C_1\|f\|_{C^6},$$

where C_1 is universal constant.

To estimate $|\Sigma_2|$ we first observe that by Lemma 3.2, each $n(\tau,x)$ and $t(\tau,x)$ is of size $\mathcal{O}(\|f\|_{C^1})$. On the other hand, by (28) and Theorem 3.14, for $\|f\|_{C^6}$ small enough, we get

$$\sum_{j=1}^{q_0-1}|\sin\varphi_j(\tau)| \le \sum_{j=1}^{q_0-1}\varphi_j(\tau) \le \frac{1+\delta}{2}(x_{q_0}(\tau)-x_0) \le 2(1+\delta)\,q_0\varphi_0 \le 32\pi.$$

Revisiting inequality (51), we have just proved that

$$b(\tau) = 2q_0\sin\varphi_0 + \mathcal{O}(\|f\|_{C^6}).$$

Since $\varphi_0 \ge \frac{(k_0-\frac12)\pi}{2q}$, we get

$$b(\tau) \ge \frac{4q_0}{\pi}\varphi_0 + \mathcal{O}(\|f\|_{C^6}) \ge 8 - \frac{4}{k_0} + \mathcal{O}(\|f\|_{C^6}) \ge 7 + \mathcal{O}(\|f\|_{C^6}),$$

which is strictly larger than $\ell_\tau = 2\pi+\mathcal{O}(\|f\|_{C^1})$ for $\|f\|_{C^6}$ sufficiently small. □

In the next section we study the $(1,q)$ length spectrum in terms of the q-loop function and its first variation.

4.2. *Length spectrum and loop functions.* Let Ω_τ be as in the previous sections. It is then obvious from the definition of the q-loop function $L_q(\tau, s)$ that

$$\mathcal{L}_{1,q}(\Omega_\tau) = \text{critical values of } L_q(\tau, s). \tag{52}$$

Thus, in particular,

$$T_q(\tau) = \max L_q(\tau, s), \quad t_q(\tau) = \min L_q(\tau, s).$$

The following lemma will be very useful for small values of q:

LEMMA 4.5. *For all $q \geq 2$, uniformly in $\tau \in [0, 1]$ we have*

$$L_q(s) = 2q \sin(\pi/q) + \mathcal{O}(\|f\|_{C^6}).$$

In particular,

$$T_q(\tau) - t_q(\tau) = \mathcal{O}(\|f\|_{C^6}).$$

Proof. By the mean value theorem and the definition (3.9) of M_q, we have

$$|L_q(\tau, s) - L_q(0, s)| \leq \sup_{\tau \in [0,1]} |\partial_\tau L_q(\tau, s)| = \sup_{\tau \in [0,1]} |M_q(\tau, s)|.$$

However, for the unit disk, the loop function is constant. In fact $L_q(0, s) = 2q \sin(\pi/q)$. Also,

$$T_q(\tau) - t_q(\tau) = \max L_q(\tau, s) - \min L_q(\tau, s) \leq 2 \sup_{\tau \in [0,1]} |M_q(\tau, s)|.$$

The lemma follows quickly from Lemma 3.11. □

Let us now state a key structural result for the q-length spectrum.

LEMMA 4.6. *Let $\partial\Omega_\tau = \partial D + \tau f N_0$ be an ε-nearly circular deformation in C^6, i.e., $\|f\|_{C^6} = \mathcal{O}(\varepsilon)$. In addition, assume that $\|f\|_{C^8} \leq 1$. Then for ε sufficiently small, the q-length spectra $\mathcal{L}_{1,q}(\Omega_\tau)$ are disjoint for distinct values of $q \geq 2$. Moreover, there exists q_0 uniform in τ and f such that*

(a) *For $q \geq q_0$,*

$$t_{q+1}(\tau) - T_q(\tau) = \min \mathcal{L}_{1,q+1}(\Omega_\tau) - \max \mathcal{L}_{1,q}(\Omega_\tau) \geq \frac{1}{10(q+1)^3}, \tag{53}$$

$$T_q(\tau) - t_q(\tau) = \max \mathcal{L}_{1,q}(\Omega_\tau) - \min \mathcal{L}_{1,q}(\Omega_\tau) \leq \frac{1}{100(q+1)^3}. \tag{54}$$

(b) *For $2 \leq q \leq q_0$, for sufficiently small $\|f\|_{C^2}$ that is uniform in τ, we have*

$$t_{q+1}(\tau) - T_q(\tau) = \min \mathcal{L}_{1,q+1}(\Omega_\tau) - \max \mathcal{L}_{1,q}(\Omega_\tau) \geq \frac{\eta_0}{10}, \tag{55}$$

$$T_q(\tau) - t_q(\tau) = \max \mathcal{L}_{1,q}(\Omega_\tau) - \min \mathcal{L}_{1,q}(\Omega_\tau) \leq \frac{\eta_0}{100}, \tag{56}$$

where

$$\eta_0 = 2(q_0+1)\sin(\pi/(q_0+1)) - 2q_0\sin(\pi/q_0). \tag{57}$$

Remark 4.7. We note that the estimate (54) is rather rough. In fact one has estimates of the form $\mathcal{O}(q^{-N})$, but this would force us to use more derivatives of f, which will be unnecessary for our purposes.

Proof. It is obvious from Lemma 4.3 that we can find a universal q_0 (hence, in particular, uniform in τ) so that estimates (53) and (54) hold true. In fact we choose $\|f\|_{C^6}$ small enough and q_0 large enough so that the remainder terms in Lemma 4.3 satisfy

$$q^{-3}\mathcal{O}(\|f\|_{C^6}) + \mathcal{O}(q^{-4}) < \frac{1}{100}(q+1)^{-3}. \tag{58}$$

Estimate (56) follows from Lemma 4.5 by choosing $\|f\|_{C^2}$ small enough in terms of the universal constant η_0. It only remains to prove (55). For this, we note that using Lemma 4.5, we have

$$t_{q+1}(\tau) - T_q(\tau) = 2(q+1)\sin(\pi/(q+1)) - 2q\sin(\pi/q) + \mathcal{O}(\|f\|_{C^2}).$$

On the other hand, by Lemma 4.1 and the definition (57) of η_0, for all $2 \le q \le q_0$, we have

$$2(q+1)\sin(\pi/(q+1)) - 2q\sin(\pi/q) \ge \eta_0. \qquad \square$$

Let us state a very interesting corollary of this lemma. It shows that for nearly circular domains, the number of bounces q can be heard from the length spectrum.

COROLLARY 4.8. *Let Ω_1 and Ω_2 be two nearly circular domains in C^6 satisfying the conditions of Theorem 4.6. Suppose*

$$\mathcal{L}(\Omega_1) \cap (0, 2\pi + 1/10) = \mathcal{L}(\Omega_2) \cap (0, 2\pi + 1/10). \tag{59}$$

Then $\ell(\partial\Omega_1) = \ell(\partial\Omega_2)$ and for all $q \ge 2$,

$$\mathcal{L}_{1,q}(\Omega_1) = \mathcal{L}_{1,q}(\Omega_2).$$

We comment that instead of $2\pi + 1/10$, one can use $2\pi + \delta$ for any $\delta > 0$, but the smallness of f in C^6 would depend on δ.

Proof. As we saw in the proof of Lemma 4.4, the length of every periodic orbit of type $(p \ge 2, q)$ must be larger than $7 + \mathcal{O}(\|f\|_{C^6})$. Let $\partial\Omega_1 = \partial D + f_1 N_0$ and $\partial\Omega_2 = \partial D + f_2 N_0$. We choose f_1 and f_2 small enough so that

$$2\pi + 1/10 < 7 + \mathcal{O}(\|f_1\|_{C^6}) \le \inf \cup_{p\ge 2} \mathcal{L}_{p,q}(\Omega_1),$$
$$2\pi + 1/10 < 7 + \mathcal{O}(\|f_2\|_{C^6}) \le \inf \cup_{p\ge 2} \mathcal{L}_{p,q}(\Omega_2).$$

In addition, we also choose them small enough such that

$$\ell(\partial\Omega_1) = 2\pi + \mathcal{O}(\|f_1\|_{C^1}) < 2\pi + 1/10,$$
$$\ell(\partial\Omega_2) = 2\pi + \mathcal{O}(\|f_2\|_{C^1}) < 2\pi + 1/10.$$

Then under these conditions, if we take supremum of (59), we obtain $\ell(\partial\Omega_1) = \ell(\partial\Omega_2)$. Now by Lemma 4.4, we get

$$\bigcup_{q\geq 2} \mathcal{L}_{1,q}(\Omega_1) = \bigcup_{q\geq 2} \mathcal{L}_{1,q}(\Omega_2).$$

Let us denote this common union by U. Let q_0 and η_0 be as in Lemma 4.6. Recall that they are identical for Ω_1 and Ω_2. We first show by finite induction that for each $q \leq q_0$, one has $\mathcal{L}_{1,q}(\Omega_1) = \mathcal{L}_{1,q}(\Omega_2)$. Clearly $t_2(\Omega_1) = t_2(\Omega_2)$ because this is the infimum of the union U. We then move in U starting at t_2 and record all gaps. By Lemma 4.6, the first gap that is larger than or equal $\eta_0/10$ must take place at $T_2(\Omega_1)$ and $T_2(\Omega_2)$, hence these two quantities must agree. We then take the infimum of $U \cap (T_2, \ell)$ to obtain $t_3(\Omega_1) = t_3(\Omega_2)$. By continuing this procedure we get that for all $2 \leq q \leq q_0$,

$$t_q(\Omega_1) = t_q(\Omega_2) \quad \text{and} \quad T_q(\Omega_1) = T_q(\Omega_2). \tag{60}$$

The argument for $q \geq q_0$ is very similar. We shall use induction. Obviously (60) holds for $q = q_0$. Suppose now that (60) holds for some $\tilde{q} > q_0$. Then by taking inf of $U \cap (T_{\tilde{q}}, \ell)$ we obtain $t_{\tilde{q}+1}(\Omega_1) = t_{\tilde{q}+1}(\Omega_2)$. By Lemma 4.6 again, starting at $t_{\tilde{q}+1}$ the first gap of size at least $(1/10)(\tilde{q}+2)^{-3}$ happens at $T_{\tilde{q}+1}(\Omega_1) = T_{\tilde{q}+1}(\Omega_2)$. Thus (60) holds for all $q \geq 2$ and the corollary follows. □

4.3. *Length spectrum of a nearly circular ellipse.* Let E_ε be an ellipse of eccentricity ε. We choose ε small enough so that no periodic orbits of type (p, q) with $p \geq 2$ contribute to the part of the length spectrum that is less than ℓ_ε, the perimeter of E_ε. This is possible by Lemma 4.4. Thus we will only focus on the q-spectrum, i.e., $\mathcal{L}_{1,q}$. Since the ellipse is completely integrable, for each $q \geq 3$, all $(1, q)$ periodic orbits have the same length. Therefore for all $q \geq 3$, we have $T_q = t_q$, or in other words the q-loop functions of the ellipse collapse to a constant. In the case $q = 2$, it is known that the only $(1, 2)$ periodic orbits are the bouncing ball orbits on the major and minor axes, whose lengths correspond to T_2 and t_2, respectively. Note that $T_2 \neq t_2$ if the ellipse is not a disk. In summary,

$$\mathcal{L}(E_\varepsilon) \cap (0, \ell_\varepsilon) = \{t_2(\varepsilon) \leq T_2(\varepsilon) < T_3(\varepsilon) < \cdots < T_q(\varepsilon) < T_{q+1}(\varepsilon) < \cdots\}, \tag{61}$$

and the gaps sequence $\{T_{q+1}(\varepsilon) - T_q(\varepsilon)\}_{q=2}^{\infty}$ is strictly decreasing by Lemma 4.1.

5. **Wave trace and Marvizi-Melrose parametrices**

5.1. *Background on wave trace.* Suppose Ω is a smooth planar domain. Let

$$w_\Omega(t) = \operatorname{Tr} \cos(t\sqrt{\Delta_\Omega})$$

be the wave trace of Δ_Ω, the positive Laplacian associated to Ω with Dirichlet (or Neumman) boundary condition. Let also $\operatorname{SingSupp} w_\Omega(t)$ denote the singular support of $w_\Omega(t)$. By a result of Andersson-Merlose [AM77] (in the convex case) and Petkov-Stoyanov [PS92] (for general smooth domains), we have the so-called Poisson relation

$$\operatorname{SingSupp} w_\Omega(t) \subset -\mathcal{L}(\Omega) \cup \{0\} \cup \mathcal{L}(\Omega). \tag{62}$$

The wave trace is the distribution integral,

$$w_\Omega(t) = \int_\Omega E(t, x, x)dx,$$

where $E(t, x, y)$ is the Schwartz kernel of $\cos t\sqrt{\Delta_\Omega}$ with the prescribed Dirichlet or Neumann boundary conditions. By distribution integral it is meant that the integral $w_\Omega(\rho) := \int_{\mathbb{R}} w_\Omega(t)\rho(t)dt$ is a temperate distribution. By Poisson relation, $w_\Omega(t)$ is a C^∞ function on the complement of the length spectrum. The nature of its singularities at the lengths depends on the structure of the periodic billiard trajectories of Ω. Only convex smooth plane domains, and only lengths of periodic trajectories in $(0, |\partial\Omega|)$ are relevant to this article, so we restrict our attention to them.

In general, the singularities of $w_\Omega(t)$ at $t \in \mathcal{L}(\Omega)$ can be extremely complicated since the set of lengths and the set of periodic orbits themselves can be extremely complicated. The simplest periodic orbits are the non-degenerate ones, i.e., isolated non-degenerate critical points of the length function on the configuration space $(\partial\Omega)^q$ of q points minus the "diagonals" where two points are equal. The major and minor axes of the ellipse are of this type. The next simplest are "clean fixed point sets," i.e., a smooth curve of periodic orbits of length L satisfying the cleanliness condition (see [GM79b]), such as the periodic orbits of an ellipse (except the major and minor axes). Equivalently, *clean* is in the Bott-Morse sense that for each q, the fixed point set of β^q is a submanifold of $B^*\partial\Omega$, and the tangent space to the fixed point set is the fixed point set of $(d\beta)^q$. In general, the set of periodic orbits of length L may be as complicated as the critical point set of a smooth function.

As discussed in [MM82, (6.8)], there exists a Lagrangian (i.e., oscillatory integral) parametrix $E'_\epsilon(t, x, y)$ for the wave kernel $E(t, x, y)$ away from the boundary and modulo a C^∞ error; $w_\Omega(t)$ may be expressed as the trace of this parametrix away from the boundary plus an additional boundary region term. We will not use this expression directly. We refer to [MM82, (6.8)]

for background. An alternative oscillatory integral formula based on layer potentials and boundary integral operators was introduced by Balian-Bloch and exploited in [Zel09] (to which we refer for background and references).

In the case of non-degenerate transversally reflecting periodic billiard trajectories, or clean curves of periodic billiard trajectories, of length L, there exists a microlocal parametrix due to Chazarain that one can use to calculate the singularities of $w_\Omega(t)$ near $t = L$. In [GM79b], Guillemin-Melrose review this parametrix construction and use it to derive a singularity expansion for $w_\Omega(t)$ in these cases; see [GM79b, Ths. 1 and 2]. The non-degeneracy (or cleanliness) assumptions make it possible to apply stationary phase (on a manifold with boundary) to the integral $\int_{\mathbb{R}} \hat\rho(t) e^{it\lambda} w_\Omega(t) dt$; see [GM79b, Lemma 5.2].

In the generic case where all periodic orbits of length $< L(\partial\Omega)$ are non-degenerate transversal periodic reflecting rays, the length spectrum is discrete in $[0, L(\partial\Omega))$ and accumulates only at $L(\partial\Omega)$. The wave trace admits a decomposition, on $\mathbb{R}^{\geq 0}$, into terms with singular support at a single length $L \in \mathcal{L}(\Omega)$:

$$w_\Omega(t) = \hat\sigma(t) = \ e_0(t) + \textstyle\sum_{L\in\mathcal{L}(\Omega)} e_L(t), \quad \text{SingSupp } e_L = \{L\}. \tag{63}$$

The term $e_0(t)$ is singular only at $t = 0$ and admits the asymptotic expansion,

$$e_0(t) = C_n |\Omega| \,\mathrm{Re}\{(t+i0)^{-n+1}\} + C_{n-1} |\partial\Omega| \,\mathrm{Re}\{(t+i0)^{-n-\frac{1}{2}}\} + \text{lower order terms},$$

in terms of homogeneous Lagrangian singularities decreasing in singularity by unit steps. When the billiard flow of Ω has *clean fixed point sets*,

$$e_L(t) = \sum_{\gamma : L_\gamma = L} e_\gamma(t), \tag{64}$$

where the last sum is over components of the closed billiard trajectories of length L or equivalently over components of the fixed points of iterates of the billiard map. In this case, the terms $e_\gamma(t)$ admit singularity expansions depending on the dimension d_γ of the component of the fixed point set. They have the form

$$e_\gamma(t) = \mathrm{Re}\{a_{\gamma,1}(t - L + i0^+)^{-n_\gamma}\} + \text{lower order terms}, \tag{65}$$

where the exponent n_γ (the "excess") equals $1 + d_\gamma/2$. Since $\dim B^*\partial\Omega = 2$, the cleanliness means that either the fixed point set consists of isolated non-degenerate fixed points or else of smooth curves of transversally non-degenerate fixed points.

Much of the difficulty of inverse spectral theory is caused by multiplicities in the length spectrum. In the non-degenerate or clean case, the Poisson formula (63) then expresses a singularity at $t = L$ as the sum of contributions from all closed orbits of length L. Since the coefficients are signed, the terms in this sum may cancel.

We do not use the Poisson formula (63), because cleanliness is not even a generic condition, and does not hold for the fixed point set of β^q for general almost circular domains. Instead, we follow the idea of Marvizi-Melrose [MM82] to break up $w_\Omega(t)$ into a sum of q-bounce contributions, and then to express the q-bounce contribution as an oscillatory integral whose phase is, roughly speaking, the q-bounce loop length function for loops of winding number one. In the nearly circular case, we can prove that the Melrose-Marvizi type parametrices are valid for all bounce numbers $q \geq 3$. Since the closed trajectories do not necessarily form clean sets, it is not generally possible to apply stationary phase to these oscillatory integral. But we can use them effectively in the inverse problem.

5.2. *Marvizi-Melrose Parametrices for nearly circular domains.* Let Ω be a nearly circular domain in C^6; that is, $\partial\Omega = \partial E_0 + f(\theta)N_0$ for some smooth function $f(\theta)$ on the unit circle ∂E_0 with $\|f\|_{C^6}$ sufficiently small.

By Lemmas 4.6 and 4.4, for $\tau = 1$, the intervals $[t_q, T_q]$ are disjoint from each other for $q \geq 2$, and they are also disjoint from $[t_{p,q}, T_{p,q}]$ for all $p > 1$ and $q \geq 2$. Hence we can choose a cutoff function $\hat{\chi}_q(t) \in C_0^\infty(\mathbb{R})$ that equals one on the interval $[t_q, T_q]$, whose support does not contain any lengths in $\mathcal{L}_{p,m}$ with $(p, m) \neq (1, q)$. Then as in [MM82], we denote

$$\hat{\sigma}_{1,q}(t) = \hat{\chi}_q(t) w_\Omega(t). \tag{66}$$

The distribution $\hat{\sigma}_{1,q}(t)$ is the localization of the wave trace to the interval $[t_q, T_q]$; it satisfies

$$\mathrm{SingSupp}\hat{\sigma}_{1,q}(t) \subset [t_q, T_q]$$

and

$$\hat{\sigma}_{1,q}(t) = w_\Omega(t) \quad \text{near } [t_q, T_q].$$

Thus instead of studying the wave trace, we study the localized wave trace $\hat{\sigma}_{1,q}(t)$.

Marvizi-Melrose [MM82] proved that (in fact for any smooth strictly convex domain)

THEOREM 5.1 (Proposition 6.11 of [MM82]). *For $q \geq q_0(\Omega)$ sufficiently large, one has a parametrix of the form*

$$\hat{\sigma}_{1,q}(t) = \int_0^\infty \int_{\partial\Omega} \mathrm{Re}\left(e^{i\pi r_q/4} e^{i\xi(t - L_q(s))} \xi^{\frac{1}{2}} a(q, t, s, \xi) \right) ds d\xi + R_q(t), \tag{67}$$

where $R_q(t)$ is a smooth function, $L_q(s)$ is the q-loop function, r_q is a Maslov index that depends on q and the boundary condition,[2] *and $a(q, t, s, \xi)$ is a*

[2]The indices r_q for Dirichlet and Neumman boundary conditions differ from each other by $4q$.

smooth classical symbol in ξ, and periodic in s, of the form

$$a(q,t,s,\xi) \sim \sum_{j=0}^{\infty} a_j(q,t,s)\xi^{-j}, \qquad (\xi \to +\infty),$$

whose principal symbol $a_0(q,t,s) = a_0(q,s)$ is independent of t and is a positive function on $\partial\Omega$.

Remark 5.2. In [MM82], the factor $\xi^{\frac{1}{2}}$ is missing from the integrand of (67). In [Pop94], there is instead a factor ξ. None of these are correct. The correct factor in the principal term must be $\xi^{\frac{1}{2}}$, as one can easily inspect by the wave trace asymptotic of Guillemin-Melrose [GM79a] of simple non-degenerate periodic orbits. An independent proof of this parametrix, given by Vig [Vig19], also confirms the factor $\xi^{\frac{1}{2}}$.

We prove that the Marvizi-Melrose parametrix (67) is valid for all $q \geq 2$ for nearly circular domains.

THEOREM 5.3. *Suppose Ω is nearly circular in C^8, meaning that $\partial\Omega = \partial E_0 + fN_0$ with $\|f\|_{C^8}$ sufficiently small. Then the parametrix (67) for the wave trace $w_\Omega(t)$ is valid for all $q \geq 2$.*

Proof. As in [MM82], the first step in the proof is the reduction of the wave trace to a boundary integral for which we provide a simple proof independent of [MM82]. We shall use a Rellich-type identity (essentially Green's second identity). Let X be a suitable vector field defined in a neighborhood of Ω. Let φ_j be an L^2-normalized eigenfunction of $\Delta = -\partial_x^2 - \partial_y^2$, with Dirichlet boundary condition on Ω, associated to the eigenvalue λ_j^2. First, by Green's second identity, we have

$$\begin{aligned} \langle \varphi_j, [\Delta, X]\varphi_j\rangle_{L^2(\Omega)} = &\int_\Omega (\Delta - \lambda_j^2)\varphi_j\, X\varphi_j - \varphi_j X(\Delta - \lambda_j^2)\varphi_j \\ &+ \int_{\partial\Omega} \partial_n\varphi_j\, X\varphi_j - \varphi_j\, \partial_n(X\varphi_j). \end{aligned} \tag{68}$$

But since $(\Delta - \lambda_j^2)\varphi_j = 0$ and $\varphi_j|_{\partial\Omega} = 0$, we get

$$\langle \varphi_j, [\Delta, X]\varphi_j\rangle_{L^2(\Omega)} = \int_{\partial\Omega} \partial_n\varphi_j\, X\varphi_j.$$

Now if we choose the vector field $X = x\partial_x + y\partial_y$, we get $[\Delta, X] = 2\Delta$, which implies that

$$2\lambda_j^2 = \int_{\partial\Omega} (X\cdot n)|\partial_n\varphi_j|^2. \tag{69}$$

Here, the function $(X.n)(s)$ is the dot product of the position vector $X(s)$ on the boundary with the outer unit normal $n(s)$ to $\partial\Omega$ at s. Using this equation,

the second time derivative of the trace of $E(t) = \cos(t\sqrt{\Delta_\Omega})$ can be written as (cf. (1))

$$\partial_t^2 w_\Omega(t) = -\frac{1}{2}\int_{\partial\Omega} (X\cdot n)\partial_{n_x}\partial_{n_y} E(t,x,y)|_{x=y}, \tag{70}$$

where $E(t,x,y)$ is the distributional kernel of $E(t)$ and $\partial_{n_x}, \partial_{n_y}$ are outer normal derivatives in the variables x, y, respectively. Equation (70) is our reduction to the boundary. We now need to find a parametrix for the distribution $\partial_{n_x}\partial_{n_y} E(t,x,y)|_{x=y}$ and plug it into (70). This distribution was studied thoroughly in [HZ12]. Away from the tangential directions to $\partial\Omega$, it is a Lagrangian distribution whose wavefront relation satisfies

$$\mathrm{WF}'(\partial_{n_x}\partial_{n_y} E(t,x,y)) \subset \bigcup_{q\geq 0} \Gamma^{\partial}_{q,\pm},$$

where

$$\Gamma^{\partial}_{q,\pm} = \left\{ (t, \pm\tau, x, \xi, y, \eta) \in T^*(\mathbb{R}\times\partial\Omega\times\partial\Omega) \;\middle|\; \begin{array}{c} \tau > 0,\ |\xi| < \tau,\ |\eta| < \tau, \\ \tilde{\beta}^q\left(x, \frac{\xi}{\tau}\right) = \left(y, \frac{y}{\eta}\right), \\ t = t\left(q, x, \frac{\xi}{\tau}\right) \end{array} \right\}. \tag{71}$$

Here, $\tilde{\beta}$ is the billiard map as a map on the ball bundle $B^*\Omega = \{(x,\xi) \in T^*\partial\Omega : |\xi| \leq 1\}$ instead of $\Pi = \partial\Omega \times [0,\pi]$. Note that $\tilde{\beta} : B^*\Omega \to B^*\Omega$ and $\beta : \Pi \to \Pi$ are related by

$$\tilde{\beta}(x,\xi) = \beta(x, \arccos(\xi)).$$

Furthermore, in (71), $t(q,x,\xi/\tau)$ is defined as follows. Let $\xi' \in T^*_x\bar{\Omega}$ be the unique inward unit vector whose orthogonal projection onto the cotangent space of the boundary $T^*_x\partial\Omega$ is ξ/τ. Then $t(q,x,\xi/\tau)$ is the time that it takes to travel from (x,ξ') and make precisely q reflections along the billiard trajectory and of course end at (y,η').

Since for nearly circular domains $\mathrm{SingSupp}\hat{\sigma}_{1,q}(t) \subset [t_q, T_q]$, we restrict our attention to a (time) neighborhood of $[t_q, T_q]$. We recall that we are only interested in a parametrix near the diagonal. These two conditions force the billiard trajectories from x to y to wind around the boundary almost once. By our Theorem 3.5, for each $q \geq 2$, there is a unique such trajectory whose length we denoted by $\Psi_q(x,y)$ (see Definition 3.7). In particular, for $(t, \pm\tau, x, \xi, y, \eta) \in \Gamma^{\partial}_{j,\pm}$ and $t \in [t_q, T_q]$, we have $t(q,x,\xi/\tau) = \Psi_q(x,y)$. In fact an easy computation (see, for example, [Vig21], [Vig19]) shows that $\Psi_q(x,y)$ generates $\tilde{\beta}^q$, in the sense that,

$$\tilde{\beta}^q(x, -\partial_x\Psi_q) = (y, \partial_y\Psi_q).$$

Now let us define the phase functions

$$\Phi_{\pm,q} := \pm\tau(t - \Psi_q(x,y)).$$

The critical point set of $\Phi_\pm$ with respect to τ is given by

$$C(\Phi_{\pm,q}) = \{(t, \pm\tau, x, y) |\ t = \Psi_q(x,y)\}.$$

Hence the image of $C(\Phi_{\pm,q})$ under the canonical embedding

$$\begin{aligned}\iota(t,\tau,x,y) &= (t, \partial_t\Phi_{\pm,q}, x, \partial_x\Phi_{\pm,q}, y, -\partial_y\Phi_{\pm,q}) \\ &= (t, \pm\tau, x, \mp\tau\partial_x\Psi_q, y, \pm\tau\partial_y\Psi_q)\end{aligned}$$

is precisely $\Gamma^\partial_{q,\pm}$ defined by (71). In other words, $\Phi_{\pm,q}$ parametrizes $\Gamma^\partial_{q,\pm}$. By [HZ12, Prop 4], the principal symbol of $\partial_{n_x}\partial_{n_y}E(t,x,y)$ on $\Gamma^\partial_{q,\pm}$, as a half-density, is of the form

$$e^{i\pi m_q/4} g(q,x,\xi,\tau)|dx\wedge d\xi\wedge d\tau|^{1/2}, \tag{72}$$

where m_q is a Maslov index and g is a certain positive symbol. Now by [Hör85, IV, Prop 25.1.5], and because $\partial_{n_x}\partial_{n_y}E(t,x,y)$ is a real-valued Lagrangian distribution, there exists a classical symbol $b(q,t,x,y,\tau)$ such that for $t\in[t_q,T_q]$ and (x,y) near the diagonal of the boundary,

$$\partial_{n_x}\partial_{n_y}E(t,x,y) = \mathrm{Re}\int_0^\infty e^{i\pi m_q/4} e^{i\tau(t-\Psi_q(x,y))} b(q,t,x,y,\tau)d\tau + (\text{smooth}).$$

By expanding b into a power series of $t-\psi_q$, and performing integration by parts, we can in fact eliminate the t variable from the amplitude b. The principal term b_0 of b can be calculated in terms of g in (72). Since $g>0$, we have $b_0>0$. Plugging this parametrix into (70) and noticing that $\Psi_q(x,x) = L_q(x)$, we obtain a parametrix for $\partial_t^2 w_\Omega(t)$ in the form

$$\partial_t^2 w_\Omega(t)|_{[t_q,T_q]} = \int_0^\infty\int_{\partial\Omega} \mathrm{Re}\left(e^{i\pi r_q/4} e^{i\tau(t-L_q(s))}\tau^{\frac12}\tilde b(q,s,\tau)\right) dsd\tau + (\text{smooth}),$$

where $r_q = m_q+4$ and

$$\tau^{\frac12}\tilde b(q,s,\tau) = \frac12 b(q,s,s,\tau)(X\cdot n)(s).$$

Because $b_0>0$ and because $X.n>0$ for a strictly convex domain (if the origin is chosen to be inside Ω), the principal term $\tilde b_0$ must be positive as well. Now let

$$\tilde w_\Omega(t) := \int_0^\infty\int_{\partial\Omega} \mathrm{Re}\left(e^{i\pi r_q/4} e^{i\tau(t-L_q(s))}\tau^{\frac12} a(q,s,\tau)\right) dsd\tau,$$

with amplitude $a(q,s,\tau) := -\tau^{-2}\tilde b(q,s,\tau)$. It is then clear that $\partial_t^2(w_\Omega(t) - \tilde w_\Omega(t))$ is smooth on $[t_q,T_q]$, thus $w_\Omega(t) = \tilde w_\Omega(t) + (\text{smooth})$, and the result follows. □

Remark 5.4. The article [Vig19] calculates the principal term a_0 explicitly. Also, it is possible to use the layer potential formulae of [Zel09] for the semi-classical resolvent kernel to prove Theorem 5.3. Since this proof is much longer and more complicated than the proof above, we do not present it here.

5.3. $\hat{\sigma}_{1,q}$ *as a spectral invariant*: *Proof of Proposition 1.5.* In this section we state the following more formally:

PROPOSITION 5.5. *If Ω is a nearly circular domain in C^8, then the distribution $\hat{\sigma}_{1,q}(t)$ is a spectral invariant of Ω.*

The key ingredient in the proof is Corollary 4.8. The wave trace on the open interval $(0, |\partial\Omega|) = (0, \ell)$ has the following *q-bounce* decomposition, introduced in [MM82]:

$$\hat{\sigma}(t)|_{(0,\ell)} = \sum_{q\geq 2} \hat{\sigma}_{1,q}(t). \tag{73}$$

By Corollary 4.8, $\mathcal{L}_{1,q} \cap \mathcal{L}_{1,q'} = \emptyset$ if $q \neq q'$ and $\mathcal{L}_{1,q} \cap \mathcal{L}_{p,q'} = \emptyset$ if $p \geq 2$. Hence, given a singularity L of the wave trace in the interval $(0, \ell)$, we know exactly to which interval $[t_q, T_q]$ it belongs to, in other words the number of bounces q of L can be heard. Thus $\hat{\sigma}_{1,q}$ is a spectral invariant, proving Proposition 1.5.

Remark 5.6. Proposition 5.5 does not rule out that there might exist two distinct $(1, q)$ orbits of Ω of the same length. It also does not imply that the set of lengths in $\mathcal{L}_{1,q}$ is finite, nor that the corresponding fixed point sets are clean, nor in the clean case that the individual terms $e_\gamma(t)$ in (64) are spectral invariants. If Ω is isospectral to E, it says that $\hat{\sigma}^{\Omega}_{1,q}(t) = \hat{\sigma}^{E}_{1,q}(t)$. It is not a priori clear that the fixed point sets of the billiard map of Ω must be clean, nor that $\mathcal{L}(\Omega) = \mathcal{L}(E)$, since cancellations may occur in the sums.

6. **Length spectrum and wave trace: Proof of Theorem 1.4**

For inverse spectral problems, it is important to know which lengths are in the singular support of the wave trace. The following proposition was proved in [MM82] for any smooth strictly convex domain (satisfying a non-coincidence condition) but only for q sufficiently large. For nearly circular domains, we improve their result by showing that it holds for all $q \geq 2$.

PROPOSITION 6.1. *If Ω is nearly circular in C^8, then for all $q \geq 2$, we have*

$$\{t_q(\Omega), T_q(\Omega)\} \subset \operatorname{SingSupp} w_\Omega(t).$$

In fact, we prove the stronger Theorem 1.4.

Remark 6.2. For a smooth strictly convex domain Ω that is not necessarily nearly circular, if we add the assumption that

$$|\partial\Omega| \text{ is not a limit point from below of the set } \bigcup_{p\geq 2, q\geq 2} \mathcal{L}_{p,q}(\Omega), \tag{74}$$

then there exists $q_0(\Omega)$ sufficiently large such that

$$\forall q \geq q_0(\Omega), \qquad \mathcal{L}_{1,q} \subset \operatorname{SingSupp} w_\Omega(t). \tag{75}$$

Although formally speaking [MM82] only proves the inclusion $\{t_q, T_q\} \subset \mathrm{SingSupp}\, w_\Omega(t)$, essentially the same argument follows to show (75). As shown in [MM82], condition (74) holds for a generic class of smooth strictly convex domains, hence the inclusion (75) holds generically.

Proof. We argue by contradiction. Fix $q \geq 2$. To prove Theorem 1.4, assume t_0 belongs to $\mathcal{L}_{1,q}(\Omega)$ but not to the singular support of $w_\Omega(t)$. Then, there is an open interval J_1 near t_0 such that $\hat{\sigma_{1,q}}(t)$ is smooth in J_1. We then choose any non-negative cutoff function $\rho_q(t)$ supported in J_1, which is positive exactly on a proper open subinterval J_2 of J_1 containing t_0. In particular, $\rho_q(t_0) > 0$. In addition we assume that the boundary points of $\mathrm{supp}\, \rho_q(t) = \bar{J}_2$ are not critical values of $L_q(s)$. This can be done because by Sard's theorem the set of critical values of $L_q(s)$ has measure zero. By our assumption on t_0, the inverse Fourier transform of $\rho_q(t)\hat{\sigma_{1,q}}(t)$ must be rapidly decaying. We will see that this leads us to a contradiction via the following theorem of Soga:

THEOREM 6.3 (Soga [Sog81]). *Consider an oscillatory integral*

$$I(\lambda) = \int_{\mathbb{R}} e^{i\lambda\varphi(x)} a(x)dx, \quad \lambda \geq 1,$$

where $\varphi(x)$ and $a(x)$ are smooth and $a(x)$ is compactly supported. Furthermore, assume that $a(x) \geq 0$ for all x, and $a(x) > 0$ for at least one degenerate critical point of $\varphi(x)$. Then $I(\lambda) \neq \mathcal{O}(\lambda^{-\infty})$. In fact, $\lambda^m I(\lambda)$ is not in $L^2(1, +\infty)$ for some $m < \frac{1}{2}$.

To exploit this result, let us compute the inverse Fourier transform of $\rho_q(t)\hat{\sigma}_{1,q}(t)$ using the parametrix (67) as follows:

$$\begin{aligned} I_q(\lambda) = &\int_{\mathbb{R}} \rho_q(t)\hat{\sigma}_{1,q}(t)e^{i\lambda t}\, dt + \mathcal{O}(\lambda^{-\infty}) \\ =&\frac{\lambda^{3/2}}{2}\int_{\mathbb{R}}\int_0^\infty\int_{\partial\Omega} e^{i\lambda(t+\xi(t-L_q(s))+\frac{i\pi r_q}{4}} a(q,t,s,\lambda\xi)\,\xi^{\frac{1}{2}}\rho_q(t)dsd\xi dt \\ &+ \frac{\lambda^{3/2}}{2}\int_{\mathbb{R}}\int_0^\infty\int_{\partial\Omega} e^{i\lambda(t-\xi(t-L_q(s))-\frac{i\pi r_q}{4}} \bar{a}(q,t,s,\lambda\xi)\,\xi^{\frac{1}{2}}\rho_q(t)dsd\xi dt \\ &+ \mathcal{O}(\lambda^{-\infty}). \end{aligned}$$

We perform the stationary phase lemma in the $d\xi dt$ integral. We have two phase functions, namely,

$$\Phi_1(t,s,\xi) = t + \xi(t - L_q(s)), \quad \Phi_2(t,s,\xi) = t - \xi(t - L_q(s)).$$

Since the critical point of Φ_1 is given by $\xi = -1$ and $t = L_q(s)$, the first integral must be rapidly decaying as $\xi = -1$ is not in the domain of the integral. The critical points of Φ_2 are given by $\xi = 1$ and $t = L_q(s)$, therefore

by the stationary phase lemma we get

$$I_q(\lambda) = \pi\lambda^{1/2} e^{\frac{-i\pi r_q}{4}} \int_{\partial\Omega} e^{i\lambda L_q(s)} a_0(q,s)\rho_q(L_q(s))ds + \mathcal{O}(\lambda^{-1/2}).$$

We know by Lemma 3.8 that the critical points of $L_q(s)$ correspond to the $(1,q)$ periodic orbits of β. Let s_0 be a critical point of $L_q(s)$ with $L_q(s_0)=t_0$. If s_0 is degenerate, then by Theorem 6.3 we get a contradiction because $\lambda^{m-1} \in L^2(1,\infty)$ for $m < \frac{1}{2}$. Now suppose s_0 is non-degenerate. The only remaining case is when all the periodic orbits whose lengths are in the support of $\rho(t)$ are also non-degenerate and are finite. This is because if any such critical point is degenerate, we get a contradiction by the same argument as above. Also, if there are infinitely many such non-degenerate critical points, they must accumulate at a degenerate critical point. This accumulation point cannot be in the interior of the support of $\rho_q(t)$ or we would get a contradiction again, so it must be on its boundary. But we chose $\rho_q(t)$ so that the boundary points of its support are not critical values of $L_q(s)$. Finally suppose the set of critical points of $L_q(s)$, whose corresponding critical values are in $\operatorname{supp}\rho_q(t)$, is finite and consists of non-degenerate orbits. (Hence each must be a local max or a local min.) We shrink the support of $\rho_q(t)$ so it contains only the critical value t_0. We shall use $\{s_0,\dots,s_r\}$ for the set of critical points of $L_q(s)$ with critical value t_0. By the stationary phase lemma, we get

$$I_q(\lambda) = \sqrt{2}\pi^{3/2} e^{-\frac{i\pi r_q}{4}+i\lambda t_0}\rho_q(t_0)\sum_{j=0}^{r} e^{i\frac{\pi}{4}\operatorname{sign}(L_q''(s_j))}\frac{a_0(q,s_j)}{\sqrt{|L_q''(s_j)|}} + \mathcal{O}(\lambda^{-1/2}).$$

Since $\operatorname{sign}(L_q''(s_j))=1$ or -1, and $a_0(q,s)>0$, the sum cannot cancel to zero. □

Before we present the proof of the main theorem, we state a key corollary of Theorem 1.4.

COROLLARY 6.4. *For nearly circular domains in C^8, one has*

$$\operatorname{SingSupp} w_\Omega(t) \cap (0, |\partial\Omega|) = \bigcup_{q\geq 2} \mathcal{L}_{1,q}(\Omega), \tag{76}$$

$$\operatorname{SingSupp} w_\Omega(t) \cap (5, |\partial\Omega|) = \bigcup_{q\geq 3} \mathcal{L}_{1,q}(\Omega). \tag{77}$$

Proof. The first statement follows from (62), Theorem 1.4, and Lemma 4.4. To show the second statement, we note that by Lemma 4.5, we have

$$L_q(s) = 2q\sin(\pi/q) + \mathcal{O}(\|f\|_{C^2}).$$

Therefore,

$$T_2(\Omega) = 4 + \mathcal{O}(\|f\|_{C^2}), \quad t_3(\Omega) = 3\sqrt{3} + \mathcal{O}(\|f\|_{C^2}).$$

Clearly if we choose $\mathcal{O}(\|f\|_{C^2})$ sufficiently small, we have $T_2(\Omega) < 5 < t_3(\Omega)$. Then, (77) follows from this and (76). □

7. **Proof of the main theorem**

Suppose Ω is a smooth domain, whose Δ spectrum with respect to Dirichlet (or Neumann) boundary condition is identical with the one of an ellipse E_ε of eccentricity $\varepsilon < \varepsilon_0$. By Lemma 2.1 and Corollary 2.2, there is a rigid motion after which Ω is C^n for every $n \in \mathbb{N}$. Let us denote the wave traces of Ω and E_ε by $w_\Omega(t)$ and $w_{E_\varepsilon}(t)$, respectively. Since Ω and E_ε are isospectral, we must in particular have

$$\text{Sing Supp } w_\Omega(t) \cap (0, \ell_\varepsilon) = \text{Sing Supp } w_{E_\varepsilon}(t) \cap (0, \ell_\varepsilon).$$

Here ℓ_ε is the length of E_ε that equals the length of Ω by the known fact that the perimeter of a domain is a spectral invariant. By Corollary 6.4, for ε sufficiently small, we must have

$$\bigcup_{q\geq 3} \mathcal{L}_{1,q}(\Omega) = \bigcup_{q\geq 3} \mathcal{L}_{1,q}(E_\varepsilon).$$

As we saw in (61), for an ellipse E_ε, we have

$$\bigcup_{q\geq 3} \mathcal{L}_{1,q}(E_\varepsilon) = \{T_3(\varepsilon) < T_4(\varepsilon) < \cdots < T_q(\varepsilon) < T_{q+1}(\varepsilon) < \cdots\},$$

which is a monotonically increasing sequence converging to ℓ_ε whose gaps sequence $T_{q+1}(\varepsilon) - T_q(\varepsilon)$ is monotonically decreasing. Thus $\bigcup_{q\geq 3} \mathcal{L}_{1,q}(\Omega)$ must be a sequence with the same properties. We claim that this implies that for all $q \geq 3$, we have $T_q(\Omega) = t_q(\Omega)$. Assume not. So for some $q \geq 3$, $t_q(\Omega) \neq T_q(\Omega)$; i.e., the q-loop function $L_q(s)$ is not a constant. Then by Lemma 4.6, if $q \geq q_0$, we have

$$T_q(\Omega) - t_q(\Omega) \leq \frac{1}{100(q+1)^3} < \frac{1}{10(q+1)^3} \leq t_{q+1}(\Omega) - T_q(\Omega),$$

and if $q < q_0$, we have

$$T_q(\Omega) - t_q(\Omega) \leq \frac{\eta_0}{100} < \frac{\eta_0}{10} \leq t_{q+1}(\Omega) - T_q(\Omega).$$

Either way, we get

$$0 < T_q(\Omega) - t_q(\Omega) < t_{q+1}(\Omega) - T_q(\Omega).$$

This shows that the sequence of gaps of $\bigcup_{q\geq 3} \mathcal{L}_{1,q}(\Omega)$ fluctuates and is not decreasing, thus a contradiction. Therefore, for all $q \geq 3$, we must have that $L_q(s)$ is a constant function of s, or equivalently that there is a smooth convex caustic Γ_q of rotation number $1/q$ consisting of $(1, q)$ periodic orbits. In fact the caustic Γ_q in the phase space Π of Ω is given by

$$\Gamma_q = \{(s, \varphi_q(s))\},$$

where $\varphi_q(s)$ is the q-loop angle defined by Theorem 3.3. In the language of [ADK16], this precisely means that Ω is rationally integrable. The following dynamical theorem of [ADK16] is the final major step in our argument.

THEOREM 7.1 (Avila, De Simoi, and Kaloshin). *Let Ω be a C^{39} smooth domain that is rationally integrable and is C^{39} sufficiently close to the unit disk. Then Ω is an ellipse.*

Hence if we choose $n = 39$, we obtain that Ω must be an ellipse. By Proposition 6.1, we know that for sufficiently small ε, the lengths t_2 and T_2 are spectral invariants. For an ellipse, these correspond to the bouncing ball orbits on the minor and major axes, respectively, thus Ω and E_ε must be isometric. This concludes the proof of our main theorem.

7.1. *Second proof of the Theorem 1.3.* Assume that Ω is nearly circular in C^8 and that it is isospectral to an ellipse E_ε of small eccentricity. We recall from Proposition 5.5 that

$$\hat{\sigma}^{\Omega}_{1,q}(t) = \hat{\sigma}^{E_\varepsilon}_{1,q}(t).$$

We remember from (66) that $\hat{\sigma}_{1,q}(t) = \hat{\chi}_q(t)w(t)$, where $\hat{\chi}_q(t)$ is a cutoff supported near $[t_q, T_q]$ and equals one there. Taking Fourier transform of this equation and inserting the Marvizi-Melrose parametrix, we get

$$\int_0^{\ell_\varepsilon} e^{i\lambda L_q(s)} a(q,s,\lambda)ds = \int_0^{\ell_\varepsilon} e^{i\lambda L_q^{E_\varepsilon}(s)} a^{E_\varepsilon}(q,s,\lambda)ds + \mathcal{O}(\lambda^{-\infty})$$

for $\lambda > 0$, where $L_q(s)$ and $L_q^{E_\varepsilon}(s)$ are the q-length functions of Ω and E_ε, respectively, and $a(q,s,\lambda)$ and $a^{E_\varepsilon}(q,s,\lambda)$ are the corresponding complete amplitudes of the trace parametrix (67). They are classical symbols of order zero, i.e., polyhomogeneous functions of λ with orders descending by unit steps. We denote their symbol expansions as $\lambda \to \infty$ as follows:

$$\begin{cases} a(q,s,\lambda) \sim \sum_{j=0}^{\infty} a_j(q,s)\lambda^{-j}, \\ a^{E_\varepsilon}(q,s,\lambda) \sim \sum_{j=0}^{\infty} a_j^{E_\varepsilon}(q,s)\lambda^{-j}. \end{cases} \tag{78}$$

The asymptotics are the standard ones for symbols, i.e., $a - \sum_{j\le N} a_j \in S^{-(N-1)}$. We comment that when $\lambda < 0$ we need to replace $a(q,s,\lambda)$ and $a^{E_\varepsilon}(q,s,\lambda)$ with their complex conjugates. Note that $L_q^{E_\varepsilon}(s)$ is the constant $t_q(\varepsilon) = T_q(\varepsilon)$. Moving the constant phase factor to the left side gives

LEMMA 7.2. *The integral*

$$b_{1,q}(\lambda) := \int_0^{\ell_\varepsilon} e^{i\lambda(L_q(s)-T_q(\varepsilon))} a(q,s,\lambda)ds = \sum_{j=0}^{\infty} \int_0^{\ell_\varepsilon} a_j^{E_\varepsilon}(q,s)ds\ \lambda^{-j} + \mathcal{O}(\lambda^{-\infty})$$

is a poly-homogeneous symbol of order zero; i.e., it belongs to $S^0 := S^0_{1,0}(\mathbb{R})$.

COROLLARY 7.3. *The Fourier transform*

$$\hat{b}_{1,q}(t) = \int_{\mathbb{R}} \int_{S^1} e^{-i\lambda t} e^{i\lambda(L_q(s)-T_q(\varepsilon))} a(q,s,\lambda) ds d\lambda \tag{79}$$

of $b_{1,q}$ is a co-normal distribution in the class $I^{1/4}(\mathbb{R},\{0\})$, with principal symbol $\int a_0^{E_\varepsilon}(q,s)ds$ times $|d\xi|^{\frac{1}{2}}$ on $T_0^*\mathbb{R}$. In particular, its principal symbol is strictly positive and $\hat{b}_{1,q}(t)$ is singular at, and only at, $t=\{0\}$.

Let us recall the definitions: In the notation of [Hör07, §18.2] (see pp. 100–101), $u \in I^{1/4}(\mathbb{R},\{0\})$ is a co-normal distribution co-normal to $\{0\}$ if $(xD_x)^k u$ belongs to the same Sobolev space as u (see [Hör07, Def. 18.2.6]). By [Hör07, Th. 18.2.8], $u \in I^{1/4}(\mathbb{R},\{0\})$ if and only if

$$u(x) = \int_{\mathbb{R}} e^{i\tau x} a(\tau) d\tau,$$

where $a \in S^0$. Such co-normal distributions are sums of homogeneous distributions of the form $x_+^s, (x \pm i0)^s$ that are singular only at $x=0$. Moreover, a co-normal distribution to $\{0\}$ has a "symbol," namely, a half density $a_0(\xi)|d\xi|^{\frac{1}{2}}$ on $T_0^*\mathbb{R}$, where $a_0(\xi)$ is the leading order term of the symbol expansion of $a(\xi)$. With these definitions in hand, we give the simple proof of Corollary 7.3.

Proof. Corollary 7.3 follows from Lemma 7.2 and the definition of the class $I^{1/4}(\mathbb{R},\{0\})$. The fact that $a_0 > 0$ follows from [MM82, Prop. 6.11]. □

From Lemma 7.2 and Corollary 7.3, we deduce the fundamental fact allowing us to give a second proof of the main theorem.

LEMMA 7.4. *The loop function $L_q(s)$ has exactly one critical value. Hence, $L_q(s) \equiv L_q^{E_\varepsilon}$.*

Proof. Suppose that L_q is non-constant. Then it has distinct maxima and minima, and at least one of these must differ from T_q. We denote the corresponding critical value by $t_1 \neq 0$. With no loss of generality, we assume that t_1 is the minimum value. Let $\psi(t) \in C_0^\infty(\mathbb{R})$ be a bump function equal to 1 in a neighborhood of t_1 and equal to zero in a neighborhood of 0. Then $\hat{b}_{1,q}(t) = (1-\psi(t))\hat{b}_{1,q}(t) + \psi(t)\hat{b}_{1,q}(t)$. By Corollary 7.3 we have $\psi(t)\hat{b}_{1,q}(t) \in C_0^\infty(\mathbb{R})$. We will see that this would lead to a contradiction. We first take inverse Fourier transform and obtain

$$\begin{aligned}
&\mathcal{F}^*_{t\to\tau}\left(\psi(t)\hat{b}_{1,q}(t)\right) = (\check{\psi} * b_{1,q})(\tau) \\
&= \int_{\mathbb{R}} \check{\psi}(\lambda) b_{1,q}(\tau-\lambda) d\lambda \\
&= \int_0^{\ell_\varepsilon} e^{i\tau(L_q(s)-T_q(\varepsilon))} \left(\int_{\mathbb{R}} \check{\psi}(\lambda) e^{-i\lambda(L_q(s)-T_q(\varepsilon))} a(q,s,\tau-\lambda) d\lambda \right) ds
\end{aligned}$$

is rapidly decaying in τ.

We now claim that $A_q(s,\tau) := \int_{\mathbb{R}} \check{\psi}(\lambda) e^{-i\lambda(L_q(s)-T_q(\varepsilon))} a(q,s,\tau-\lambda)d\lambda$ is an element of $S^0(T^*S^1)$. Indeed, using the symbol expansion (78) one has, as $\tau \to \infty$,

$$A_q(s,\tau) \sim \sum_{j=0}^{\infty} \tilde{a}_j(q,s) \int_{\mathbb{R}} \check{\psi}(\lambda) e^{-i\lambda(L_q(s)-T_q(\varepsilon))} \langle \tau-\lambda\rangle^{-j} d\lambda,$$

with $\langle x\rangle := (1+x^2)^{\frac{1}{2}}$, and for new amplitudes $\tilde{a}_j$ that depend linearly on $\{a_k\}_{k\le j}$. Then we write this as

$$A_q(s,\tau) \sim \sum_{j=0}^{\infty} \tilde{a}_j(q,s)\tau^{-j} \int_{\mathbb{R}} \check{\psi}(\lambda) e^{-i\lambda(L_q(s)-T_q(\varepsilon))} \sigma^{-j}(\tau,\lambda)d\lambda,$$

where $\sigma(\lambda,\tau) := (|\tau|^{-2} + |\frac{\lambda}{\tau} - 1|^2)^{1/2}$. Note that $\langle \tau-\lambda\rangle^{-1} = |\tau|^{-1}\sigma(\lambda,\tau)^{-1}$ and that $\sigma(\lambda,\tau)^{-1} \to 1$ uniformly on compact sets in λ as $\tau \to \infty$, and indeed there is an asymptotic expansion in τ. Since $\check{\psi} \in \mathcal{S}(\mathbb{R})$, the asymptotic expansion is integrable in $d\lambda$ and some arrangement gives a symbol expansion for $A_q(s,\tau)$. In conclusion,

$$\int_0^{\ell_\varepsilon} e^{i\tau(L_q(s)-T_q(\varepsilon))} A_q(s,\tau)ds = O(\tau^{-N}) \ \ \forall N > 0.$$

However, the phase has a critical point, so that the conclusion contradicts [Sog86, Th. 2]. This concludes the second proof of Theorem 1.3. For the sake of completeness, we state [Sog86, Th. 2] in the relevant dimension one.

THEOREM 7.5. *Let $I(\tau) := \int_{\mathbb{R}} e^{i\tau\varphi(x)}\rho(x,\tau)dx$, where $\rho(x,\tau) \sim \rho_0(x) + \rho_1(x)(i\tau)^{-1} + \cdots$ as $\tau \to \infty$. Assume that $\rho_0 \ge 0$ with $\rho_0(x) > 0$ on the minimum set of $\varphi(x)$. Then for some $m \in \mathbb{R}$ depending only on the dimension, $\tau^m I(\tau) \notin L^2(\mathbb{R}_+)$.* □

In the final section we provide a general result that makes our article independent of [Sog81].

7.2. *A refined lemma.* In this part we present the following more general theorem. which is of independent interest. It shows that many assumptions that we previously used can be relaxed.

LEMMA 7.6. *Let $\varphi(s)$ and $a(s,\lambda)$ be two smooth functions on $S^1 = \mathbb{R}/\mathbb{Z}$ with $a(s,\lambda)$ satisfying*

$$a(s,\lambda) = a_0(s) + \mathcal{O}(\lambda^{-1/2-\epsilon}), \qquad a_0(s) \text{ smooth and positive}$$

for some $\epsilon > 0$. Assume that for $|\lambda| \ge \lambda_0 > 0$ we have

$$I(\lambda) = \int_{S^1} e^{-i\lambda\varphi(s)} a(s,\lambda)ds = c_0 + \mathcal{O}(\lambda^{-\frac{1}{2}-\epsilon})$$

for a constant c_0. Then $\varphi(s) \equiv 0$ on S^1.

Proof. We follow the proof of [Sog81] closely. First let us discard the remainder term in $a(s,\lambda)$ and call the resulting integral $J(\lambda)$, i.e.,

$$J(\lambda) = \int_{S^1} e^{-i\lambda\varphi(s)} a_0(s) ds.$$

Obviously we have $J(\lambda) = c_0 + \mathcal{O}(\lambda^{-1/2-\epsilon})$. Now, let $\psi(t)$ be a cutoff function in $\mathbb{R}$ that equals one on an open set containing the range of φ. Let $H(t)$ be the Heaviside function at zero, and define

$$g_0(t) = \psi(t) \int_{\mathbb{R}} H(t-\varphi(s)) a_0(s) ds = \psi(t) \int_{\varphi(s)\le t} a_0(s) ds,$$
$$g_1(t) = \psi'(t) \int_{\mathbb{R}} H(t-\varphi(s)) a_0(s) ds = \psi'(t) \int_{\varphi(s)\le t} a_0(s) ds.$$

The function g_0 is in H^m for every $m < 1/2$, and because it is a compactly supported distribution, its Fourier transform $\widehat{g_0}(\lambda)$ is an analytic function. Also since $\int_{\mathbb{R}} H(t-\varphi(s)) a(s) ds$ is smooth at the regular values t of $\varphi(s)$, and since $\psi'(t)$ vanishes on the range of $\varphi(s)$, $g_1(t)$ is smooth, and hence $\widehat{g_1}(\lambda)$ is rapidly decaying. We note that by a simple integration by parts, $I(\lambda)$ can be written as

$$\begin{aligned} J(\lambda) &= i\lambda \int_{\mathbb{R}} e^{-i\lambda t} g_0(t) dt - \int_{\mathbb{R}} e^{-i\lambda t} g_1(t) dt \\ &= i\lambda \widehat{g_0}(\lambda) - \widehat{g_1}(\lambda). \end{aligned}$$

Thus, by our assumption on $I(\lambda)$ (so the same for $J(\lambda)$), we obtain

$$\widehat{g_0}(\lambda) = -\frac{iJ(\lambda)}{\lambda} + \mathcal{O}(\lambda^{-\infty}) = -\frac{ic_0}{\lambda} + \mathcal{O}(\lambda^{-3/2-\epsilon}), \qquad |\lambda| \ge \lambda_0.$$

By taking inverse Fourier transform and using the Sobolev Embedding Theorem, we get

$$g_0(t) = c_0 H(t) + f(t), \tag{80}$$

where $H(t)$ is the Heaviside function at $t = 0$ (co-normal distribution) with a jump discontinuity at $t = 0$, and $f(t)$ is continuous at every t and in fact belongs to the Hölder class $C^{0,\alpha}$ for every $0 < \alpha < 1/2 + \epsilon$. However, we will show that every critical value t_0 of $\varphi(s)$ is a "big singularity" of $g_0(t)$. By this we mean that $g_0(t)$ is not Hölder continuous $C^{0,\alpha}$ at t_0 for any $\alpha > 1/2$. This together with the decomposition (80) would imply that the only critical value of $\varphi(s)$ is 0. Since φ is a function on S^1, it must be zero everywhere.

So assume $t_0 = \varphi(s_0)$ is a critical value of φ and s_0 is a critical point in its inverse image. For each $h > 0$, we denote

$$A_h = \{s \in S^1;\ -h < \varphi(s) - t_0 \le h\}.$$

We recall that $\psi(t) = 1$ in an open set containing the image of φ, so we can choose h small enough so that $\psi(t_0 - h) = \psi(t_0 + h) = 1$. By the definition of g_0, we have

$$|g_0(t_0 + h) - g_0(t_0 - h)| = \int_{A_h} a_0(s)ds.$$

However, since s_0 is a critical point of φ, we have $|\varphi(s) - t_0| \leq c|s - s_0|^2$ for some $c > 0$, and thus we have the inclusion

$$\{s \in S^1;\ c|s - s_0|^2 < h\} \subset A_h.$$

We then write

$$|g_0(t_0 + h) - g_0(t_0 - h)| \geq \int_{c|s-s_0|^2<h} a_0(s)ds \geq \sqrt{h/c}\, \min a_0(s),$$

which by letting $h \to 0$, implies that g_0 is not Hölder continuous $C^{0,\alpha}$ at t_0 for $\alpha > 1/2$. □

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References

[Ami93] E. Y. Amiran, A dynamical approach to symplectic and spectral invariants for billiards, *Comm. Math. Phys.* **154** no. 1 (1993), 99–110. MR 1220949. Zbl 0786.58034. https://doi.org/10.1007/BF02096834.

[Ami96] E. Y. Amiran, Noncoincidence of geodesic lengths and hearing elliptic quantum billiards, *J. Statist. Phys.* **85** no. 3-4 (1996), 455–470. MR 1413669. Zbl 0929.37007. https://doi.org/10.1007/BF02174214.

[AM77] K. G. Andersson and R. B. Melrose, The propagation of singularities along gliding rays, *Invent. Math.* **41** no. 3 (1977), 197–232. MR 0494322. Zbl 0373.35053. https://doi.org/10.1007/BF01403048.

[ADK16] A. Avila, J. De Simoi, and V. Kaloshin, An integrable deformation of an ellipse of small eccentricity is an ellipse, *Ann. of Math.* (2) **184** no. 2 (2016), 527–558. MR 3548532. Zbl 1379.37104. https://doi.org/10.4007/annals.2016.184.2.5.

[Bro93] R. M. Brown, The trace of the heat kernel in Lipschitz domains, *Trans. Amer. Math. Soc.* **339** no. 2 (1993), 889–900. MR 1134755. Zbl 0853.35083. https://doi.org/10.2307/2154304.

[DH13] K. Datchev and H. Hezari, Inverse problems in spectral geometry, in *Inverse Problems and Applications: Inside Out. II*, *Math. Sci. Res. Inst. Publ.* **60**, Cambridge Univ. Press, Cambridge, 2013, pp. 455–485. MR 3114967. Zbl 1316.35001.

[dKW17] J. de Simoi, V. Kaloshin, and Q. Wei, Dynamical spectral rigidity among $\mathbb{Z}_2$-symmetric strictly convex domains close to a circle, *Ann. of Math.* (2) **186** no. 1 (2017), 277–314, Appendix B coauthored with H. Hezari. MR 3665005. Zbl 1377.37062. https://doi.org/10.4007/annals.2017.186.1.7.

[Gor00] C. Gordon, Survey of isospectral manifolds, in *Handbook of Differential Geometry, Vol. I*, North-Holland, Amsterdam, 2000, pp. 747–778. MR 1736857. Zbl 0959.58039. https://doi.org/10.1016/S1874-5741(00)80009-6.

[GWW92] C. S. Gordon, D. Webb, and S. Wolpert, Isospectral plane domains and surfaces via Riemannian orbifolds, *Invent. Math.* **110** no. 1 (1992), 1–22. MR 1181812. Zbl 0778.58068. https://doi.org/10.1007/BF01231320.

[GM79a] V. Guillemin and R. Melrose, An inverse spectral result for elliptical regions in $\mathbf{R}^2$, *Adv. in Math.* **32** no. 2 (1979), 128–148. MR 0535619. Zbl 0415.35062. https://doi.org/10.1016/0001-8708(79)90039-2.

[GM79b] V. Guillemin and R. Melrose, The Poisson summation formula for manifolds with boundary, *Adv. in Math.* **32** no. 3 (1979), 204–232. MR 0539531. Zbl 0421.35082. https://doi.org/10.1016/0001-8708(79)90042-2.

[HZ12] H. Hezari and S. Zelditch, C^∞ spectral rigidity of the ellipse, *Anal. PDE* **5** no. 5 (2012), 1105–1132. MR 3022850. Zbl 1264.35150. https://doi.org/10.2140/apde.2012.5.1105.

[Hör85] L. Hörmander, *The Analysis of Linear Partial Differential Operators. IV, Fourier Integral Operators*, *Grundlehren math. Wissen.* **275**, Springer-Verlag, Berlin, 1985. MR 781537. Zbl 0612.35001. https://doi.org/10.1007/978-3-642-00136-9.

[Hör07] L. Hörmander, *The Analysis of Linear Partial Differential Operators. III, Pseudo-Differential Operators*; reprint of the 1994 edition, *Classics in Math.*, Springer, Berlin, 2007. MR 2304165. Zbl 1115.35005. https://doi.org/10.1007/978-3-540-49938-1.

[HKS18] G. Huang, V. Kaloshin, and A. Sorrentino, Nearly circular domains which are integrable close to the boundary are ellipses, *Geom. Funct. Anal.* **28** no. 2 (2018), 334–392. MR 3788206. Zbl 1395.37041. https://doi.org/10.1007/s00039-018-0440-4.

[KS18] V. Kaloshin and A. Sorrentino, On the local Birkhoff conjecture for convex billiards, *Ann. of Math.* (2) **188** no. 1 (2018), 315–380. MR 3815464. Zbl 1394.37093. https://doi.org/10.4007/annals.2018.188.1.6.

[Kov21] I. Koval, Domains which are integrable close to the boundary and close to the circular ones are ellipses, 2021. arXiv 2111.12171.

[Laz93] V. F. Lazutkin, *KAM Theory and Semiclassical Approximations to Eigenfunctions*, *Ergeb. Math. Grenzgeb.* **24**, Springer-Verlag, Berlin, 1993, With an addendum by A. I. Shnirel′man. MR 1239173. Zbl 0814.58001. https://doi.org/10.1007/978-3-642-76247-5.

[Liv] C. Liverani, Implicit function theorem, a quantitative version, online note, Available on author's webpage.

[MM82] S. Marvizi and R. Melrose, Spectral invariants of convex planar regions, *J. Differential Geom.* **17** no. 3 (1982), 475–502. MR 0679068. Zbl 0492.53033. https://doi.org/10.4310/jdg/1214437138.

[MF94] J. N. Mather and G. Forni, Action minimizing orbits in Hamiltonian systems, in *Transition to Chaos in Classical and Quantum Mechanics* (Montecatini Terme, 1991), *Lecture Notes in Math.* **1589**, Springer, Berlin, 1994, pp. 92–186. MR 1323222. Zbl 0822.70011. https://doi.org/10.1007/BFb0074076.

[Mel84] R. B. Melrose, Isospectral sets of drumheads are compact in C^∞, 1984, unpublished, MSRI preprint.

[PS92] V. M. Petkov and L. N. Stoyanov, *Geometry of Reflecting Rays and Inverse Spectral Problems*, *Pure Appl. Math.* (*New York*), John Wiley & Sons, Ltd., Chichester, 1992. MR 1172998. Zbl 0761.35077.

[PRR13] S. Pinto-de-Carvalho and R. Ramírez-Ros, Non-persistence of resonant caustics in perturbed elliptic billiards, *Ergodic Theory Dynam. Systems* **33** no. 6 (2013), 1876–1890. MR 3122155. Zbl 1408.37065. https://doi.org/10.1017/S0143385712000417.

[Pop94] G. Popov, Invariants of the length spectrum and spectral invariants of planar convex domains, *Comm. Math. Phys.* **161** no. 2 (1994), 335–364. MR 1266488. Zbl 0797.58070. https://doi.org/10.1007/BF02099782.

[PT03] G. Popov and P. Topalov, Liouville billiard tables and an inverse spectral result, *Ergodic Theory Dynam. Systems* **23** no. 1 (2003), 225–248. MR 1971204. Zbl 1042.37043. https://doi.org/10.1017/S0143385702001190.

[PT12] G. Popov and P. Topalov, Invariants of isospectral deformations and spectral rigidity, *Comm. Partial Differential Equations* **37** no. 3 (2012), 369–446. MR 2889558. Zbl 1245.58015. https://doi.org/10.1080/03605302.2011.641051.

[PT16] G. Popov and P. Topalov, From KAM tori to isospectral invariants and spectral rigidity of billiard tables, 2016. arXiv 1602.03155.

[RR06] R. Ramírez-Ros, Break-up of resonant invariant curves in billiards and dual billiards associated to perturbed circular tables, *Phys. D* **214** no. 1 (2006), 78–87. MR 2200796. Zbl 1099.37027. https://doi.org/10.1016/j.physd.2005.12.007.

[Sib99] K. F. Siburg, Aubry-Mather theory and the inverse spectral problem for planar convex domains, *Israel J. Math.* **113** (1999), 285–304. MR 1729451. Zbl 0996.37051. https://doi.org/10.1007/BF02780181.

[Sib04] K. F. Siburg, *The Principle of Least Action in Geometry and Dynamics*, *Lecture Notes in Math.* **1844**, Springer-Verlag, Berlin, 2004. MR 2076302. Zbl 1060.37048. https://doi.org/10.1007/b97327.

[Sog81] H. Soga, Oscillatory integrals with degenerate stationary points and their application to the scattering theory, *Comm. Partial Differential Equations*

6 no. 3 (1981), 273–287. MR 0607549. Zbl 0463.35049. https://doi.org/10.1080/03605308108820177.

[Sog86] H. Soga, Conditions against rapid decrease of oscillatory integrals and their applications to inverse scattering problems, *Osaka J. Math.* **23** no. 2 (1986), 441–456. MR 0856897. Zbl 0617.35104. Available at http://projecteuclid.org/euclid.ojm/1200779334.

[Vig19] A. Vig, The wave trace and Birkhoff billiards, 2019. arXiv 1910.06441.

[Vig21] A. Vig, Robin spectral rigidity of the ellipse, *J. Geom. Anal.* **31** no. 3 (2021), 2238–2295. MR 4225807. Zbl 1460.35243. https://doi.org/10.1007/s12220-019-00339-4.

[Zel04] S. Zelditch, The inverse spectral problem, in *Surveys in Differential Geometry. Vol. IX*, *Surv. Differ. Geom.* **9**, Int. Press, Somerville, MA, 2004, with an appendix by Johannes Sjöstrand and Maciej Zworski, pp. 401–467. MR 2195415. Zbl 1061.58029. https://doi.org/10.4310/SDG.2004.v9.n1.a12.

[Zel09] S. Zelditch, Inverse spectral problem for analytic domains. II. $\mathbb{Z}_2$-symmetric domains, *Ann. of Math.* (2) **170** no. 1 (2009), 205–269. MR 2521115. Zbl 1196.58016. https://doi.org/10.4007/annals.2009.170.205.

[Zel14] S. Zelditch, Survey on the inverse spectral problem, *ICCM Not.* **2** no. 2 (2014), 1–20. MR 3314780. https://doi.org/10.4310/ICCM.2014.v2.n2.a1.

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