J. Spectr. Theory 10 (2020), 1303–1332
DOI 10.4171/JST/328

Journal of Spectral Theory

Schrödinger trace invariants for homogeneous perturbations of the harmonic oscillator

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Abstract. Let $H = H_0 + P$ denote the harmonic oscillator on $\mathbb{R}^d$ perturbed by an isotropic pseudodifferential operator P of order 1 and let $U(t) = \exp(-itH)$. We prove a Gutzwiller-Duistermaat-Guillemin type trace formula for $\operatorname{Tr} U(t)$. The singularities occur at times $t \in 2\pi\mathbb{Z}$ and the coefficients involve the dynamics of the Hamilton flow of the symbol $\sigma(P)$ on the space $\mathbb{CP}^{d-1}$ of harmonic oscillator orbits of energy 1. This is a novel kind of sub-principal symbol effect on the trace. We generalize the averaging technique of Weinstein and Guillemin to this order of perturbation, and then present two completely different calculations of $\operatorname{Tr} U(t)$. The first proof directly constructs a parametrix of $U(t)$ in the isotropic calculus, following earlier work of Doll–Gannot–Wunsch. The second proof conjugates the trace to the Bargmann–Fock setting, the order 1 of the perturbation coincides with the 'central limit scaling' studied by Zelditch–Zhou for Toeplitz operators.

Mathematics Subject Classification (2010). 35Q40, 35S05.

Keywords. Harmonic oscillator, trace formula, isotropic calculus, threshold sub-principal perturbation.

Contents

[1] Steve Zelditch gratefully acknowledge the support of NSF grant DMS-1810747.

1. Introduction

In a recent article [3], a two-term Weyl law with remainder was proved for special types of perturbations $H = H_0 + P$ of the homogeneous isotropic harmonic oscillator

$$H_0 = (1/2)(-\Delta + |x|^2)$$

with symbol $p_2 = (1/2)(|x|^2 + |\xi|^2)$. The perturbation P is a classical isotropic pseudodifferential operator of order 1, $P = \mathrm{Op}^w(p) \in G^1_{\mathrm{cl}}$ with (classical) isotropic symbol

$$p \sim p_1 + p_0 + p_{-1} + \cdots \in \Gamma^1_{\mathrm{cl}}.$$

The salient feature of the perturbation is that its order is at a threshold or critical level, so that the perturbation has a strong contribution to the Weyl law. The purpose of this note is to go further in the spectral analysis, by considering the Duistermaat–Guillemin–Gutzwiller trace formula for the propagator $U(t) = e^{-itH}$. It was shown in [3] that the distribution trace $\mathrm{Tr}\, U(t)$ has singularities at the same points $t \in 2\pi\mathbb{Z}$ as for the unperturbed propagator. We give a new proof of this result and, more importantly, determine the singularity coefficients at non-zero singular times $t = 2\pi k$. It turns out that the coefficients involve the Hamiltonian mechanics of p on the space $\mathbb{CP}^{d-1}$ of Hamilton orbits of H_0. After generalizing the averaging technique of Weinstein [10] and Guillemin [5], we prove the result in two ways: (i) using special parametrices for the isotropic propagators $U(t)$ on $L^2(\mathbb{R}^d)$ developed in [3], and (ii) by using an eigespace decomposition for H_0 and conjugating the traces of $U(t)$ on eigenspaces to the Bargmann–Fock representation and then to holomorphic sections of line bundles over the space $\mathbb{CP}^{d-1}$ of Hamilton orbits of H_0. It turns out that the eigenspace traces of $U(t)$ involve the central limit scaling of Toeplitz propagators studied in [11, 12]. We then use the trace asymptotics of [12] to obtain a second proof. The two approaches are in some sense dual, the first leading to Fourier integral representations, and the second to Fourier series representations, of $\mathrm{Tr}\, U(t)$. The second approach shows (see Corollary 1.4) that $\mathrm{Tr}\, U(t)$ trace is a sum of distributions on $\mathbb{R}$ of the form $e^{ita|D|^{\frac{1}{2}}} e_r(x)$ where $D = \frac{1}{i}\frac{d}{dt}$, $a \in \mathbb{R}$ and $e_r(t)$ is a periodic classical homogeneous distribution of order r. The operator $e^{iat|D|^{\frac{1}{2}}}$ is a pseudo-differential operator on $\mathbb{R}$ with symbol $e^{ita\sqrt{|\xi|}}$ in the class $S^0_{\frac{1}{2},0}$, i.e. each derivative decreases the order by $\frac{1}{2}$. The proofs actually show that the unusual contribution of P to the wave trace occurs whenever one perturbs an operator H_0

with periodic bicharacteristic flow and maximally high multiplicities by an operator of 'one-half lower order'. See Section 1.3.[1]

To state the results, we introduce some notation. Let H_0 denote the Hamiltonian vector field of $p_2 = (1/2)(|x|^2+|\xi|^2)$, whose flow $(x(t), \xi(t)) = \exp(t\mathsf{H}_0)(x_0, \xi_0)$ satisfies

$$x(t) = \cos(t)x_0 + \sin(t)\xi_0,$$

$$\xi(t) = \cos(t)\xi_0 - \sin(t)x_0.$$

We define

$$\mathsf{X}f(x,\xi) = \frac{1}{2\pi}\int_0^{2\pi} f(\exp(t\mathsf{H}_0)(x,\xi))\,dt. \tag{1}$$

For $f \in \mathcal{C}^\infty(\mathbb{S}^{2d-1})$, where $\mathbb{S}^{2d-1} = \{p_2 = 1\}$, we use the same notation[2] $\mathsf{X}f$. The map X induces a map

$$\widetilde{\mathsf{X}}\colon \mathcal{C}^\infty(\mathbb{R}^{2d}) \longrightarrow \mathcal{C}^\infty(\mathbb{CP}^{d-1}),$$

where $\mathbb{CP}^{d-1} = \{p_2 = 1\}/\sim$ and $z_1 \sim z_2$ if and only if there exists a $\theta \in \mathbb{R}$ such that $z_1 = e^{i\theta}z_2$. Note, that we can write $\widetilde{\mathsf{X}}$ as

$$\widetilde{\mathsf{X}}f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta}z)d\theta$$

using the natural identification $\mathbb{C}^d = \mathbb{R}^{2d}$.

In both approaches we critically use an averaging method originally due to Weinstein [10] to simplify the perturbation (cf. Lemma 3.1). If $p_1 = \sigma^1(P)$, then the principal symbol of the averaged perturbation can be identified with $\widetilde{\mathsf{X}}p_1\colon \mathbb{CP}^{d-1} \to \mathbb{R}$. Assume that $\widetilde{\mathsf{X}}p_1$ is a Morse function on $\mathbb{CP}^{d-1}$. The set of stationary points is given by $\Pi_{2\pi} = \{z_j\}_{j=1}^n = \{z \in \mathbb{CP}^{d-1}\colon d\widetilde{\mathsf{X}}p_1(z) = 0\}$. Furthermore, we set $d_j = |\det Dd\widetilde{\mathsf{X}}p_1(z_j)| \neq 0$ and σ_j the signature of the Hessian, $\operatorname{sgn} Dd\widetilde{\mathsf{X}}p_1(z_j)$.

1.1. Statement of results in the homogeneous isotropic calculus. It was shown in [3, Theorem 1.2] that under the assumption that $\widetilde{\mathsf{X}}p_1$ is a Morse function on $\mathbb{CP}^{d-1}$, the singularity at $2\pi k$ is of order $O(\lambda^{(d-1)/2})$.

[1] The order convention in this article is that H_0 has order 2 and P has order 1. But it is also natural to define the order of H_0 to be 1 and that of P to be $\frac{1}{2}$ and this order relation generalizes to many other settings.

[2] Note that the definition differs from the one in [3] by a factor of 2π.

The content of the main theorem is to calculate the principal term of the distribution at $t = 2\pi k$.

Theorem 1.1. *Let H be as above and $w(t) = \operatorname{Tr} e^{-itH}$ its Schrödinger trace. For $k \in \mathbb{Z} \setminus \{0\}$ and $\epsilon \in (0, \pi)$ choose $\chi_k \in \mathcal{C}_c^\infty((2\pi k - \epsilon, 2\pi k + \epsilon))$ with $\chi_k(t) = 1$ on $(2\pi k - \epsilon/2, 2\pi k + \epsilon/2)$. Then,*

$$\mathcal{F}^{-1}\{\chi_k \cdot w\}(\lambda) \sim \lambda^{(d-1)/2} e^{2\pi k i \lambda} \sum_{j=1}^{n} e^{-2\pi i k \lambda^{1/2} \widetilde{\mathfrak{X}}_{p_1}(z_j)} \sum_{l=0}^{\infty} \lambda^{-l/2} \gamma_{k,j,l}$$

and

$$\gamma_{k,j,0} = (\pi k)^{-(d-1)} d_j^{-1/2} e^{\pi i(-\sigma_j/4 + dk)} e^{-2\pi i k \widetilde{\mathfrak{X}}_{p_0}(z_j)}.$$

1.2. Statement of results in the Bargmann–Fock setting. We give a different proof of Theorem 1.1 in which we first expand $\operatorname{Tr} U(t)$ into traces on eigenspaces of H_0 and then conjugate by the Bargmann transform to obtain traces of semi-classical Toeplitz operators in the Bargmann–Fock setting. The advantage of the conjugation is that the spectral projections for H_0 conjugate to the simpler Bergman kernel projections for holomorphic sections of the standard ample line bundles over projective space. In that setting, Theorem 1.1 can be reduced to the trace asymptotics calculated in [12] for semi-classial Toeplitz operators acting on holomorphic sections of line bundles over Kähler manifolds.

Let $L^2(\mathbb{R}^d) = \bigoplus_{N=0}^{\infty} \mathcal{H}_N$ be the decomposition of $L^2(\mathbb{R}^d)$ into eigenspaces of H_0, and let

$$\Pi_N \colon L^2(\mathbb{R}^d) \longrightarrow \mathcal{H}_N \tag{2}$$

be the orthogonal projection, i.e. the spectral projections for H_0. By the averaging Lemma 3.2, $H_0 + P$ is unitarily equivalent to $H_0 + B + R$ where $[B, H_0] = 0$ and $R \colon \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$. We can assume that that $R = 0$ (cf. Corollary 3.3). Using the known spectrum of H_0, we have the following:

Lemma 1.2. *The trace $\operatorname{Tr} e^{itH}$ is given by*

$$\operatorname{Tr} e^{it(H_0+P)} = \operatorname{Tr} e^{it(H_0+B)} = \sum_{N=0}^{\infty} \operatorname{Tr} e^{itB}|_{\mathcal{H}_N}\, e^{it(N+\frac{d}{2})}.$$

It therefore suffices to determine the asymptotics as $N \to \infty$ of

$$\operatorname{Tr} e^{itB}|_{\mathcal{H}_N} = \operatorname{Tr} \Pi_N e^{itB} \Pi_N \tag{3}$$

for each t and in particular for $t = 2\pi k$ for some $k \in \mathbb{Z}$. Using the trace asymptotics of [12], we prove (in the notation of Theorem 1.1):

Theorem 1.3. *Let $N > 0$, $m = d - 1$ and $t = 2\pi k$. Then,* $\mathrm{Tr}\,\Pi_N e^{itB}$ *admits a complete asymptotic expansion as $N \to \infty$ in powers of $(t\sqrt{N})$ with leading term,*

$$\mathrm{Tr}\,\Pi_N e^{itB} = \Big(\frac{N}{2\pi}\Big)^m \Big(\frac{t\sqrt{N}}{4\pi}\Big)^{-m} \sum_{j=1}^{n} d_j^{-1/2} \cdot e^{it\sqrt{N}\widetilde{X}p_1(z_j)} e^{i\pi\sigma_j/4} (e^{it\widetilde{X}p_0(z_j)} + O(|t|^3 N^{-1/2})).$$

To prove Theorem 1.3 we relate the unitary group $\Pi_N e^{itB}\Pi_N$ to semi-classical Toeplitz Fourier integral operators. Since B has order 1, it does not generate a semi-classical Toeplitz Fourier integral operator. Rather,

$$V_N(t) = \Pi_N e^{it\sqrt{N}B}\Pi_N$$

is a semi-classical Toeplitz Fourier integral operator. We then employ the trick that

$$\Pi_N e^{itB}\Pi_N = V_N\Big(\frac{t}{\sqrt{N}}\Big)$$

to express $\Pi_N e^{itB}\Pi_N$ as a time-scaled Toeplitz Fourier integral unitary group. Under conjugation by the Bargmann transform, $V_N(t)$ is carried to a Toeplitz Fourier integral operator $U_N(t)$ in the complex domain. Therefore, combining with Lemma 1.2 gives,

$$\mathrm{Tr}\, e^{it(H_0+P)} = \sum_{N=0}^{\infty} \mathrm{Tr}\, U_N\Big(\frac{t}{\sqrt{N}}\Big)\, e^{it(N+\frac{d}{2})}.$$

The time-scaled Toeplitz Fourier integral operators $U_N(\frac{t}{\sqrt{N}})$ are studied in [11, 12] for unrelated reasons, and the pointwise and integrated asymptotics of those articles gives Theorem 1.3.

We then substitute the asympotics of Theorem 1.3 into Lemma 1.2. When $B = 0$, $\mathrm{Tr}\, e^{-itH}$ is a sum of Hardy distributions of the form,[3]

$$\mu_{N,d,r}(t) := \sum_{N=0}^{\infty} N^r e^{it(N+\frac{d}{2})} \quad \Big(r \in \frac{1}{2}\mathbb{Z}\Big), \tag{4}$$

with $r = d - 1$. When $B \neq 0$ and σ_B descends to a Morse function on $\mathbb{CP}^{d-1}$, the order is $\frac{d-1}{2}$ and the fact $e^{it\sqrt{k}H(z_c)}$ changes the singularity type to a sum of distributions of the type

$$e_{N,d,r}(t,a) = \sum_{N=1}^{\infty} N^r e^{it(N+\sqrt{N}a+\frac{d}{2})} \quad (a \in \mathbb{R}).$$

[3] By a Hardy distribution is meant one with only positive frequencies

These are non-standard and apparently novel types of homogeneous Lagrangian distributions (the authors have not found them in prior articles). We obtain an interesting description of $\mathrm{Tr}\, e^{-itH}$ and its singularities in terms of these distributions,

Corollary 1.4. *The leading order singularity in t of $\mathrm{Tr}\, e^{it(H_0+B)}$ at $t = 2\pi k$ is the same as the leading order singularity of the finite sum,*

$$\sum_{j=1}^{n} \frac{e^{i\pi\sigma_j/4}}{d_j^{1/2}} e_{N,d,(d-1)/2}(t, \widetilde{\mathrm{X}} p_1).$$

This explict expansion of the trace in terms of non-standard Lagrangian distributions does not seem to follow easily from the calculation in the Schroedinger representation.

We further observe that $e^{ita\sqrt{|D|}} \in \Psi^0_{\frac{1}{2},0}(\mathbb{R})$, i.e. is a pseudo-differential operator on $\mathbb{R}$ of order zero, with symbol $e^{ita\sqrt{|\xi|}}$. Each derivative decreases the order of the symbol of by $\frac{1}{2}$. It follows that although $e_{N,d}(t, a, (d-1)/2)$ is not a homogeneous distribution, it has the same properties as a homogeneous distribution. For instance, since $e^{ita\sqrt{|D|}}$ is pseudo-local, $e_{N,d}(t, a)$ has singularities at the same points $t \in 2\pi\mathbb{Z}$ as $e_{N,d}(t, 0)$, as first proved in [3].

To relate the expansions of Theorem 1.1 and Theorem 1.3, it is proved in Section 6 that the asymptotics $\lambda \to \infty$ in Theorem 1.1 and the asymptotics $N \to \infty$ in Theorem 1.3 agree to leading order.

1.3. Related problems. The trace asymptotics for the propagator e^{-itH} of the perturbation $H = H_0 + P$ has many analogues in settings where one has a first order elliptic pseudo-differential operator H_0 with periodic bicharacteristic flow and maximally high multiplicities and P is a $\frac{1}{2}$-order below that of H_0. Without trying to state the most general result, we note that it is valid for $H_0 = \sqrt{-\Delta}$ on $\mathbb{S}^n$, the standard sphere, and P is a pseudo-differential operator of order $\frac{1}{2}$. Indeed, as in the proof of Theorem 3, it suffices to work out the semi-classical trace asymptotics in the eigenspaces, which only uses that $U_N(t)$ is a semi-classical unitary Fourier integral operator. The main results pertain to the special effect of perturbation of the special order $\frac{1}{2}$ and hold for such perturbations on all rank one symmetric spaces. The results can also be adapted to general Zoll manifolds if $\sqrt{-\Delta}$ is replaced by an operator $\mathcal{N}$ which indexes the 'bands' in the sense of [6].

2. Isotropic calculus and quantization

The class of isotropic symbols of order $m \in \mathbb{R}$, $\Gamma^m(\mathbb{R}^d)$, consists of the smooth functions $a \in \mathcal{C}^\infty(\mathbb{R}^{2d})$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \lesssim_{\alpha,\beta} \langle (x,\xi)\rangle^{m-(|\alpha|+|\beta|)}, \quad \text{for all } \alpha,\beta \in \mathbb{N}^d.$$

We will mainly be interested in the subclass of *classical* symbols $a \in \Gamma^m_{\mathrm{cl}}(\mathbb{R}^d)$, which admit an asymptotic expansion in homogeneous terms of order $m-j$. To any isotropic symbol $a \in \Gamma^m(\mathbb{R}^d)$ we associate an isotropic pseudodifferential operator $A \in G^m(\mathbb{R}^d)$ by the *Weyl-quantization*,

$$A = \operatorname{Op}^w(a) = (2\pi)^{-d} \int e^{i(x-y)\xi} a((x+y)/2,\xi)\,d\xi.$$

The integral is defined as an oscillatory integral and we further note that any bounded linear operator $A\colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ has a Weyl-symbol $a \in \mathcal{S}'(\mathbb{R}^{2d})$, which satisfies $\operatorname{Op}^w(a) = A$. This follows directly from the Fourier inversion formula and the Schwartz kernel theorem. We also use that the class of residual isotropic operators, $G^{-\infty}(\mathbb{R}^d) = \bigcap_{m\in\mathbb{R}} G^m(\mathbb{R}^d)$ is the space of (*globally*) *smoothing* operators $A\colon \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$. For the main properties of the isotropic calculus, we refer to [3] (see also Helffer [7] and Shubin [9]).

For later reference, we recall the formula for composing isotropic pseudodifferential operators.

Proposition 2.1. *Let $a \in \Gamma^{m_1}_{\mathrm{cl}}$, $b \in \Gamma^{m_2}_{\mathrm{cl}}$ two isotropic symbols. Then*

$$\operatorname{Op}^w(a)\operatorname{Op}^w(b) = \operatorname{Op}^w(c),$$

where

$$c(x,\xi) = (a\#b)(x,\xi) = e^{iA(D)} a(x,\xi) b(y,\eta)\big|_{y=x,\eta=\xi}.$$

The exponential is defined as an Fourier multiplier and the operator $A(D)$ is given by $A(D) = (\langle D_\xi, D_y\rangle - \langle D_x, D_\eta\rangle)/2$. Moreover, there exists an asymptotic expansion

$$(a\#b)(x,\xi) \sim \sum_k \frac{i^k}{k!} A(D)^k a(x,\xi) b(y,\eta)\big|_{y=x,\eta=\xi}.$$

3. Averaging

In this section we prove that there exists a $B \in G^1_{\mathrm{cl}}(\mathbb{R}^d)$ and $R \in G^{-\infty}(\mathbb{R}^d)$ such that

$$H \cong H_0 + B + R, \quad [H_0, B] = 0$$

and calculate its symbol. We will follow the arguments of [10, 5].

For the proof of Theorem 1.1, the following lemma suffices.

Lemma 3.1. *Let $H = H_0 + P$ with $P \in G^1_{\mathrm{cl}}(\mathbb{R}^d)$. For any $N \in \mathbb{N}$ there exists a unique $B_{-N} \in G^1_{\mathrm{cl}}(\mathbb{R}^d)$ modulo $G^{-N}_{\mathrm{cl}}(\mathbb{R}^d)$ and an unitary operator U_{-N} such that*

$$[H_0, B_{-N}] \in G^{-N}_{\mathrm{cl}}(\mathbb{R}^d),$$

$$U^{-1}_{-N} H U_{-N} = H_0 + B_{-N}.$$

Furthermore, for $N > 1$ the Weyl quantized symbol of B_{-N} has an asymptotic expansion

$$b = \mathsf{X}p + \Gamma^{-1}_{\mathrm{cl}}(\mathbb{R}^d)$$

where p is the Weyl-quantized symbol of P.

Proof. First, we define for $F \in G^k_{\mathrm{cl}}$ the operator

$$\mathrm{ad}(F)\colon G^m_{\mathrm{cl}} \longrightarrow G^{m+k-2}_{\mathrm{cl}}$$

by

$$\mathrm{ad}(F)A = [F, A].$$

If $k < 2$ we obtain, using the Taylor expansion of the exponential, that

$$e^{-i\,\mathrm{ad}(F)}A = A - i[F, A] + G^{m-2(2-k)}_{\mathrm{cl}}.$$

We define the pseudodifferential operator F_1 by

$$F_1 = \frac{1}{2\pi} \int_0^{2\pi} \int_0^t e^{isH_0} P e^{-isH_0}\, ds\, dt.$$

It has the property that

$$[F_1, H_0] = P - B_1,$$

where

$$B_1 = \frac{1}{2\pi} \int_0^{2\pi} e^{itH_0} P e^{-itH_0} \, dt.$$

Therefore, we calculate

$$e^{-i\,\mathrm{ad}(F_1)} H = H_0 + B_1 + [F_1, B_1] + G_{\mathrm{cl}}^{-1}.$$

Now, we inductively lower the order of the remainder term. Set $R_0 = [F_1, B_1] \in G_{\mathrm{cl}}^0$ then define F_0 and B_0 as above, but with P replaced by R_0. Then we find that

$$e^{-i\,\mathrm{ad}(F_0)} e^{-i\,\mathrm{ad}(F_1)} H = H_0 + B_1 + B_0 + G_{\mathrm{cl}}^{-1},$$

thus concluding

$$e^{-i\,\mathrm{ad}(F_{-N})} \cdots e^{-i\,\mathrm{ad}(F_1)} H = H_0 + B_1 + \cdots + B_N + G_{\mathrm{cl}}^{-N-1}$$

with $[H_0, B_1 + \cdots + B_N] = 0$. The operator U_{-N} is given by $e^{iF_{-N}} \cdots e^{iF_1}$. This follows from the fact that

$$\mathrm{Ad}_{e^{-iF_j}} = e^{-i\,\mathrm{ad}(F_j)},$$

where $\mathrm{Ad}_U\, A = UAU^{-1}$.

Finally, we show that $B_0 \in G_{\mathrm{cl}}^{-1}(\mathbb{R}^d)$. The Weyl-quantized symbol of B_1 is given by

$$b_1 = \mathsf{X}p = \mathsf{X}p_1 + \mathsf{X}p_0 + \Gamma_{\mathrm{cl}}^{-1}.$$

Observe that

$$\begin{aligned} B_0 &= \frac{1}{2\pi} \int_0^{2\pi} U_0(-t)[F_1, B_1] U_0(t) \, dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} [U_0(-t) F_1 U_0(t), B_1] \, dt \\ &= \left[\frac{1}{2\pi} \int_0^{2\pi} U_0(-t) F_1 U_0(t) \, dt, B_1 \right]. \end{aligned}$$

Here, we have used that B_1 commutates with $U(t)$. By Fubini, we calculate the principal symbol of the left entry in the commutator,

$$\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\int_0^{t} p_1 \circ \exp((s+t')\mathsf{H}_0)\, ds\, dt\, dt' = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{t} \mathsf{X}p_1\, ds\, dt$$
$$= \pi \cdot \mathsf{X}p_1.$$

The second equality follows from the fact that $\mathsf{X}p_1$ is constant along the Hamiltonian flow and that $\int_0^{2\pi}\int_0^t ds\; dt = 2\pi^2$. Now, the principal symbol of $U_0(-t)F_1U_0(t)$ and B_1 are the same up to a constant, therefore its commutator has zero principal symbol. □

Now, we have to make sure that we can use asymptotic summation.

Lemma 3.2. *Let $H \in G^2_{\mathrm{cl}}(\mathbb{R}^d)$ as above. There exists a unitary pseudodifferential operator $U \in G^0_{\mathrm{cl}}(\mathbb{R}^d)$ and a self-adjoint $B \in G^1_{\mathrm{cl}}(\mathbb{R}^d)$ such that*

$$[H_0, B] = 0,$$
$$U^*HU = H_0 + B + G^{-\infty}.$$

Proof. Let $\widetilde{U} \in G^0_{\mathrm{cl}}(\mathbb{R}^d)$ and $B \in G^1_{\mathrm{cl}}(\mathbb{R}^d)$ such that

$$\widetilde{U} - U_{-k} \in G^{-(k+1)}_{\mathrm{cl}}(\mathbb{R}^d),$$
$$B - B_{-k} \in G^{-k}_{\mathrm{cl}}(\mathbb{R}^d),$$

for all $k \in \mathbb{N}$. Here, U_{-k} and B_{-k} are as constructed in the proof of the previous lemma. Consider the bounded self-adjoint operator $F = \widetilde{U}\widetilde{U}^* - \mathrm{I}$. Since U_k is unitary for all k, it follows that $F \in G^{-\infty}$ and hence compact. Thus, we may assume that its L^2-norm is bounded by $C \in (0,1)$, by modifying it on a finite-dimensional subspace. We let $K = \sum_{j=1}^{\infty} c_j F^j$, where c_j are the Taylor coefficients of the expansion of $(1+t)^{-1/2}$ at the origin. The operator K is well defined since the series converges for $|t| < 1$ and smoothing. Putting $U = (\mathrm{I}+K)\widetilde{U}$, we see that

$$UU^* = (\mathrm{I}+K)\widetilde{U}\widetilde{U}^*(\mathrm{I}+K)$$
$$= (\mathrm{I}+K)^2(\mathrm{I}+F)$$
$$= \mathrm{I}.$$

We can apply the averaging to B again so that B commutes with H_0 and $U^*HU = H_0 + B + G^{-\infty}$ as claimed. □

It follows from Duhamel's formula (see Proposition A.1 of the Appendix) that one has,

Corollary 3.3. *With the same notation as in Lemma 3.2,*

$$\operatorname{Tr} e^{-itH} = \operatorname{Tr} e^{-it(H_0+B)} \mod \mathcal{C}^\infty(\mathbb{R}).$$

4. Fourier integral operators

In what follows, we will encounter operators of the form $A = \operatorname{Op}^w(e^{i\phi}a)$, where ϕ is homogeneous of degree one outside a compact set in $\mathbb{R}^{2d}$. These operators are clearly not isotropic pseudodifferential operators, because they are not pseudolocal. Since spatial derivatives of isotropic symbols gain decay in both x and ξ, we can still compose A with and any isotropic pseudodifferential operator and explicitly calculate the asymptotic expansion.

Proposition 4.1. *Let $p \in \Gamma^m_{\mathrm{cl}}$, $a \in \Gamma^0_{\mathrm{cl}}$, and $\phi \in \Gamma^1_{\mathrm{cl}}$. Then*

$$p\# e^{i\phi}a = e^{i\phi}c,$$

where

$$\begin{aligned}
c_m &= p_m a_0,\\
c_{m-1} &= p_m a_{-1} + p_{m-1}a_0 + (1/2)\{p_m,\phi_1\}a_0,\\
c_{m-2} &= p_m a_{-2} + p_{m-1}a_{-1} + p_{m-2}a_0\\
&\quad + (1/2)\left(\{p_{m-1},\phi_1\}a_0 + \{p_m,\phi_1\}a_{-1} + \{p_m,\phi_0\}a_0 - i\{p_m,a_0\}\right)\\
&\quad + (1/4)\Big(\sum_{j,k} \partial_{x_j x_k} p_m \partial_{\xi_j}\phi_1 \partial_{\xi_k}\phi_1 - \partial_{\xi_j\xi_k} p_m \partial_{x_j}\phi_1 \partial_{x_k}\phi_1\Big)a_0.
\end{aligned}$$

Proof. The asymptotic expansion follows from the composition theorem, Proposition 2.1, and ordering the terms by homogeneity. ☐

Moreover, we need a composition result for quadratic phase functions:

Proposition 4.2. *Let $A \in \mathbb{R}^{d\times d}$ be symmetric and $a, b \in \mathcal{S}(\mathbb{R}^d)$. There is an integral representation*

$$\begin{aligned}
&(e^{i\langle A\cdot,\cdot\rangle}a\# b)(z)\\
&\quad = \pi^{-2d}\int_{\mathbb{R}^{4d}} e^{-2i\langle Qw,w\rangle} a(z+w_1)b(z+w_2+(1/2)JA(w_1+z))\,dw_1\,dw_2,
\end{aligned}$$

where

$$Q = \begin{pmatrix} A/2 & -J \\ J & 0 \end{pmatrix}.$$

Proof. By Zworski [14, Theorem 4.11] we have the integral representation

$$(e^{i\langle A\cdot,\cdot\rangle}a\#b)(z) \\ = \pi^{-2d}\int_{\mathbb{R}^{4d}} e^{-2i\sigma(w_1,w_2)}e^{i\langle A(z+w_1),z+w_1\rangle}a(z+w_1)b(z+w_2)\,dw_1\,dw_2.$$

Define the phase function

$$\begin{aligned}\Phi(w_1,w_2,z) &= -2\sigma(w_1,w_2) + \langle A(z+w_1), z+w_1\rangle \\ &= -2\left\langle \begin{pmatrix} 0 & -J \\ J & 0\end{pmatrix}\begin{pmatrix} w_1 \\ w_2\end{pmatrix}, \begin{pmatrix} w_1 \\ w_2\end{pmatrix}\right\rangle + \langle A(z+w_1), z+w_1\rangle.\end{aligned}$$

Changing coordinates

$$\begin{aligned}\tilde{w}_1 &= w_1, \\ \tilde{w}_2 &= w_2 - (1/2)JA(w_1+z)\end{aligned}$$

yields

$$\Phi(w_1,w_2,z) = \widetilde{\Phi}(\tilde{w}_1,\tilde{w}_2,z) := -2\left\langle \begin{pmatrix} A/2 & -J \\ J & 0\end{pmatrix}\begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2\end{pmatrix}, \begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2\end{pmatrix}\right\rangle + \langle Az, z\rangle.$$

Hence, we have

$$\begin{aligned}&(e^{i\langle A\cdot,\cdot\rangle/2}a\#b)(z) \\ &= \pi^{-2d}\int_{\mathbb{R}^{4d}} e^{i\widetilde{\Phi}(w_1,w_2)}a(z+w_1)b(z+w_2+(1/2)JA(w_1+z))dw_1\,dw_2 \\ &= \pi^{-2d}e^{i\langle Az,z\rangle}\int_{\mathbb{R}^{4d}} e^{-2i\langle Qw,w\rangle}a(z+w_1) \\ &\qquad\qquad b(z+w_2+(1/2)JA(w_1+z))dw_1\,dw_2.\end{aligned}$$ □

We also have to calculate how quadratic exponentals act on oscillating functions:

Proposition 4.3. *Let* $\phi \in \Gamma^1_{\mathrm{cl}}(\mathbb{R}^d)$ *homogeneous of degree* 1 *outside a compact set,* $a \in \Gamma^{m_1}_{\mathrm{cl}}(\mathbb{R}^d)$*, and* $b \in \Gamma^{m_2}_{\mathrm{cl}}(\mathbb{R}^d)$*. For any symmetric matrix* $A \in \mathbb{R}^{d\times d}$ *we have*

$$(e^{i\langle A\cdot,\cdot\rangle}a\#e^{i\phi}b)(z) = e^{i\langle Az,z\rangle}e^{i\phi(z)}\tilde{a},$$

where $\tilde{a} \in \Gamma^{m_1+m_2}_{\mathrm{cl}}(\mathbb{R}^d)$.

Proof. Using an approximation argument, we can show that the previous calculation also holds for isotropic symbols and the evaluation of the oscillatory integral is essentially the same as in the proof of Lemma 4.2 in [3]. ☐

4.1. Parametrix for $U(t)$. We use Lemma 3.1 to simplify the computations. The trace is invariant under conjugation by unitary operators, therefore we may assume that for $N \gg 0$,

$$H = H_0 + B,$$
$$[H_0, B] \in G_{\mathrm{cl}}^{-N}(\mathbb{R}^d).$$

For our purposes it will suffice to have $N = 1$. By Lemma 3.1, the Weyl-quantized symbol b of B is given by

$$b = \mathsf{X}p + \Gamma_{\mathrm{cl}}^{-1}(\mathbb{R}^d).$$

Since the symbol p is assumed to be classical, we have an asymptotic expansion

$$b \sim \sum_{j=0}^{\infty} b_{1-j}$$

and $b_1 = \mathsf{X}\sigma^1(P)$.

In this section, we construct a parametrix for $U(t) = e^{-itH}$, where $H = H_0 + B$. As before, we first consider the reduced propagator $F(t) = U_0(-t)U(t)$, where $U_0(t) = e^{-itH_0}$ is the propagator of the harmonic oscillator.

The reduced propagator $F(t)$ satisfies

$$\begin{cases} (i\partial_t - B(t))F(t) = 0, \\ F(0) = \mathrm{I}. \end{cases} \tag{5}$$

Here, $B(t) = U_0(-t)BU_0(t) = \mathrm{Op}^w(b(t))$, where

$$b(t) = b \circ \exp(t\mathsf{H}_0).$$

Note that the first terms in the asymptotic expansion are b_1 and b_0, repectively, because due to the averaging b is invariant under the flow $\exp(t\mathsf{H}_0)$ modulo $\Gamma_{\mathrm{cl}}^{-N}$.

Our ansatz is

$$\widetilde{F}(t) = \mathrm{Op}^w(e^{i\phi_1(t)}a(t)),$$

where ϕ_1 is homogeneous of degree 1 and $a \in \mathcal{C}^\infty(\mathbb{R}_t, \Gamma_{\mathrm{cl}}^0)$. Applying $i\partial - B(t)$ to $\widetilde{F}(t)$ yields a Weyl-quantized operator with full "symbol"

$$-e^{i\phi}a(t)\partial_t\phi_1 + ie^{i\phi}\partial_t a - b(t)\# e^{i\phi_1}a.$$

Thus, $\widetilde{F}(t)$ solves (5) if

$$-e^{i\phi}a(t)\partial_t\phi_1 + ie^{i\phi}\partial_t a - b(t)\# e^{i\phi_1}a = 0.$$

Ordering the the terms by homogeneity, we obtain by Proposition 4.1 for the leading order:

$$\partial_t\phi_1 + b_1 = 0,$$

which is our usual eikonal equation. Next order gives the first transport equation:

$$i\partial_t a_0 = (b_0 - (1/2)\{b_1, \phi_1(t)\})a_0.$$

The higher transport equations for a_{-k} contain as usual inhomogeneous terms depending on the derivatives of a_{-j}, $j < k$.

Hence, the parametrix is given on an interval $(2\pi k - \epsilon, 2\pi k + \epsilon)$ by

$$\widetilde{F}(t) = \mathrm{Op}^w(e^{i\phi_1(t)}a(t)),$$

where

$$\begin{aligned}\phi_1(t,x,\eta) &= -tb_1(x,\eta),\\ a_0(t,x,\eta) &= e^{-itb_0(x,\eta)}.\end{aligned}$$

Proposition 4.4. *There is an oscillatory integral operator* $\widetilde{U} \in \mathcal{C}^\infty((2\pi k - \epsilon, 2\pi k + \epsilon), \mathcal{L}(\mathcal{S}', \mathcal{S}'))$ *such that* $\widetilde{U}(t)$ *is a parametrix for* $U(t)$*, that is*

$$\widetilde{U}(t) - U(t) \in \mathcal{C}^\infty((2\pi k - \epsilon, 2\pi k + \epsilon), \mathcal{L}(\mathcal{S}', \mathcal{S}))$$

and

$$\widetilde{U}(t) = \mathrm{Op}^w(e^{i\phi_2(t)}e^{i\phi_1(t)}a(t)),$$

where

$$\begin{aligned}\phi_2(t,x,\eta) &= -2\tan(t/2)p_2(x,\eta),\\ \phi_1(t,x,\eta) &= -tb_1(x,\eta),\end{aligned}$$

and $a \in \mathcal{C}^\infty((2\pi k - \epsilon, 2\pi k + \epsilon), \Gamma^0_{\mathrm{cl}}(\mathbb{R}^{2d}))$ *with*

$$\sigma^0(a(2\pi k)) = (-1)^{dk}\exp(-2\pi i k b_0).$$

Proof. It is well known (cf. Hörmander [8, p. 427]) that the propagator of the harmonic oscillator can be written as a Weyl-quantized operator. Namely, we have for $t \notin \pi + 2\pi\mathbb{Z}$ that

$$U_0(t) = \cos(t/2)^{-d} \operatorname{Op}^w(e^{i\phi_2(t)}),$$

where $\phi_2(t,x,\eta) = -2\tan(t/2)p_2(x,\eta)$. A suitable parametrix is therefore given by

$$\widetilde{U}(t) = U_0(t)\widetilde{F}(t).$$

By Proposition 4.3, $\widetilde{U}(t)$ can be represented as a Weyl-quantized operator with symbol

$$e^{i(\phi_2(t)+\phi_1(t))}\tilde{a}(t),$$

where $\tilde{a} \in \mathcal{C}^\infty((2\pi k - \epsilon, 2\pi k + \epsilon), \Gamma^0_{\mathrm{cl}})$. Thus, it remains to show that $\tilde{a}(2\pi k) = (-1)^{dk} a_0(2\pi k) + \Gamma^{-1}_{\mathrm{cl}}$, but this obviously true, since $U_0(2\pi k) = (-1)^{dk}\,\mathrm{I}$. □

4.2. Proof of Theorem 1.1. We consider an oscillatory integral of the form

$$I(\lambda) = \int e^{i(t\lambda + \psi_2(t,x,\eta)+\psi_1(t,x,\eta))}\chi(t)a(t,x,\eta)\,dt\,dx\,d\eta.$$

We follow the proof of [3, Proposition 5.1], but keeping track of the leading order of the amplitude. We assume that

- $\chi \in \mathcal{C}^\infty_c(\mathbb{R})$,
- ψ_j is homogeneous of degree j outside a compact neighborhood of 0,
- $a \in \mathcal{C}^\infty(\mathbb{R}, \Gamma^0_{\mathrm{cl}}(\mathbb{R}^d))$, and ψ_j are smooth on the support of a,
- there exists unique $t_0 \in \operatorname{supp}\chi$ and $r_0 > 0$ such that

  $$\psi_2(t_0, r_0, \theta) = 0,$$
  $$\partial_t\psi_2(t_0, r_0, \theta) = -1$$

  for all $\theta \in \mathbb{S}^{2d-1}$,
- ψ_2 is normalized in the sense that $|\partial_r\partial_t\psi_2(t_0, r_0, \theta)| = r_0$.

The extension of [3, Proposition 5.1] is the following:

Proposition 4.5. *Under the assumptions above and assuming that $\psi_1(t_0, r_0, \bullet)$ is Morse–Bott with $2d-2$ non-degenerate directions, the integral $I(\lambda)$ has an asymptotic expansion*

$$I(\lambda) = \lambda^{(d-1)/2} e^{it_0\lambda} \sum_{j=1}^{n} e^{i\lambda^{1/2}\psi_1(t_0,r_0,\theta_j)} \sum_{l=0}^{\infty} \lambda^{-l/2}\gamma_{j,l},$$

where

$$\gamma_{j,0} = \frac{(2\pi)^d}{|\det D_\theta d_\theta \psi_1(t_0, r_0, \theta_j)|^{1/2}} r_0^{2(d-1)} e^{\pi i\sigma_j/4} \int \sigma^0(a(t_0))(r_0, \theta_j) d\theta.$$

Here, σ_j is the signature of $D_\theta d_\theta \psi_1(t_0, r_0, \theta_j)$ and the integral is over the 1-dimensional manifold on which ψ_1 is constant.

Proof. It already follows from the proof of [3, Proposition 5.1] that we have the claimed asymptotic expansion and we only have to calculate the leading coefficient. As in [3], we have that

$$I(\lambda) = \int_{\mathbb{S}^{2d-1}} J(\lambda, \lambda^{-1/2}, \theta) d\theta,$$

where

$$J(\lambda, \mu, \theta) = \lambda^d \int e^{i\lambda\Psi_\mu(t,r,\theta)} \chi(t) a(t, \lambda^{1/2} r, \theta) r^{2d-1} dt\, dr,$$
$$\Psi_\mu(t, r, \theta) = \psi_2(t, r, \theta) + \mu\psi_1(t, r, \theta) + t.$$

By assumption, the determinant of $D_{r,t} d_{r,t} \psi_2(t_0, r_0, \theta)$ has absolute value r_0 and the signature is zero, hence, by the stationary phase formula, we obtain for any $M \geq 1$,

$$J(\lambda, \mu, \theta) = \lambda^{d-1} e^{i\lambda(t_0+\mu\psi_1(t_0,r_0,\theta))} a_M(\lambda^{1/2}, \mu, \theta) + O(\lambda^{d-1-M})$$

with

$$a_M(\lambda^{1/2}, \lambda^{-1/2}, \theta) = 2\pi r_0^{2(d-1)} \sigma^0(a(t_0))(r_0, \theta) + O(\lambda^{-1/2}).$$

This leads to the asymptotic formula

$$I(\lambda) = \lambda^{d-1} e^{i\lambda t_0} \cdot 2\pi r_0^{2(d-1)} \int e^{i\lambda^{1/2}\psi_1(t_0,r_0,\theta)} (\sigma^0(a(t_0))(r_0, \theta) + O(\lambda^{-1/2})) d\theta.$$

By assumption, $\psi_1(t_0, r_0, \theta)$ is Morse–Bott, so we may apply the stationary phase formula again and obtain

$$I(\lambda) = \lambda^{(d-1)/2} e^{i\lambda t_0} \sum_{j=1}^{n} e^{i\lambda^{1/2}\psi_1(t_0,r_0,\theta_j)}(\gamma_{j,0} + O(\lambda^{-1/2})). \qquad \square$$

Now, we are able to prove the main theorem.

Proof of Theorem 1.1. By [2, Proposition 13], we can calculate the trace of a Weyl-quantized operator by integration along the diagonal. Thus, the inverse Fourier transform of the Schrödinger trace is given by

$$\begin{aligned}
&\mathcal{F}^{-1}_{t\to\lambda}\, \chi(t) \operatorname{Tr} U(t) \\
&\quad = \mathcal{F}^{-1}_{t\to\lambda}\, \chi(t) \operatorname{Tr} \widetilde{U}(t) + O(\lambda^{-\infty}) \\
&\quad = (2\pi)^{-(d+1)} \int e^{i(\phi_2(t,x,\xi)+\phi_1(t,x,\xi)+t\lambda)} \chi(t) a(t,x,\xi)\, dt\, dx\, d\xi + O(\lambda^{-\infty}),
\end{aligned}$$

where ϕ_2, ϕ_1, and a are given by Proposition 4.4. The phase function ϕ_2 has an expansion $\phi(t - 2\pi k, r, \theta) = -(1/2) r^2 t + O(t^3)$ and therefore we may apply Proposition 4.5 with

$$I_k(\lambda) = (2\pi)^{-(d+1)} \int e^{i(\phi_2(t,x,\xi)+\phi_1(t,x,\xi)+t\lambda)} \chi(t) a(t,x,\xi)\, dt\, dx\, d\xi.$$

The stationary point of ϕ_2 is at $t_0 = 2\pi k$ and $r_0 = \sqrt{2}$. On the stationary point, we have that $\phi_1(t_0, r_0, \cdot) = -2\pi k b_1(\sqrt{2}, \cdot)$. By Proposition 4.5, we see that there is an asymptotic expansion

$$I_k(\lambda) = \lambda^{(d-1)/2} e^{2\pi i k\lambda} \sum_{j=1}^{n} e^{-2\pi i k \lambda^{1/2} b_1(\sqrt{2}, z_j)} \sum_{l=0}^{\infty} \lambda^{-l/2} \gamma_{k,j,l}.$$

The determinant and signature of the Hessian of the phase function ϕ_1 at a critical point z_j are given by $(2\pi k)^{2d-2} d_j$ and $-\sigma_j$, respectively, where

$$\begin{aligned}
d_j &= |\det(D d b_1(z_j))| \neq 0, \\
\sigma_j &= \operatorname{sgn}(D d b_1(z_j)).
\end{aligned}$$

Since $\sigma^0(a(2\pi k)) = (-1)^{dk} e^{-2\pi i k b_0}$ is invariant under the flow, we have for the leading order term at one critical point z_j that

$$\begin{aligned}
\gamma_{k,j,0} &= (2\pi)^{-(d+1)} \frac{(2\pi)^d}{(2\pi k)^{d-1} |d_j|^{1/2}} 2^{d-1} e^{-\pi i \sigma_j/4} \int \sigma^0(a(2\pi k))(\sqrt{2}, z_j)\, d\theta \\
&= (\pi k)^{-(d-1)} d_j^{-1/2} e^{-\pi i \sigma_j/4} \cdot \frac{1}{2\pi} \int \sigma^0(a(2\pi k))(\sqrt{2}, z_j)\, d\theta \\
&= (\pi k)^{-(d-1)} d_j^{-1/2} e^{\pi i(-\sigma_j/4 + dk)} e^{-2\pi i k b_0(z_j)}. \qquad \square
\end{aligned}$$

5. Conjugation to the Bargmann–Fock model

In this section, we prove Theorem 1.3 as outlined in Section 1.2. The proof is based on results of [11, 12] and involves Toeplitz Fourier integral operators acting on holomorphic sections of the standard line bundles $\mathcal{O}(N) \to \mathbb{CP}^{d-1}$ over the projective space. Since it is our second proof of the main result, we do not provide detailed background on Toeplitz Fourier integral operators and refer to [11, 12] for further background and references. Our goal to describe the conjugation to the holomorphic setting and to connect the asymptotics of Theorem 1.3 to those of [11, 12].

In this section, we assume that the perturbation B commutes with H_0. As in Lemma 3.2, by iterated averaging we may assume that $H = H_0 + B + R$ where R is smoothing and $[H_0, B] = 0$. By Corollary 3.3 and Proposition A.1, the smoothing operator does not change singularities or asymptotics of the trace; it is omitted for simplicity of exposition.

As in (2), we denote by $\mathcal{H}_N$ the eigenspace of H_0 with the eigenvalue $N + d/2$ and by Π_N the orthogonal projection onto $\mathcal{H}_N$. The perturbation P and the unitarily equivalent B are isotropic pseudo-differential operators of order 1. When we conjugate to the Bargmann–Fock setting, orders in Toeplitz calculus are traditionally defined by powers of N. Thus, H_0 is considered to have order 1 and P, B are considered to have order $\frac{1}{2}$. To keep track of the orders, it is convenient to scale B to have order 0, i.e. we define zeroth order isotropic pseudo-differential operator

$$\widetilde{B} := H_0^{-1/2} B$$

with Weyl-quantized symbol $\tilde{b} = p_2^{-1/2} \# b$. Then $[\widetilde{B}, H_0] = 0$, and

$$\operatorname{Tr} e^{itB}|_{\mathcal{H}_N} = \operatorname{Tr} e^{itH_0^{1/2}\widetilde{B}}|_{\mathcal{H}_N} = \operatorname{Tr} e^{it\sqrt{N}\widetilde{B}}|_{\mathcal{H}_N}.$$

As mentioned in the introduction, $e^{it\sqrt{N}\widetilde{B}}|_{\mathcal{H}_N}$ is not a standard type of Fourier integral operator, which would be the exponential of a first order operator, and that is why we view it as $U(\frac{t}{\sqrt{N}})$ (in the notation of Section 1.2).

5.1. Conjugation to Bargmann–Fock space. Consider the weight function $\Phi(z) = |z|^2/2$. The L^2-space of weighted entire functions,

$$H_\Phi(\mathbb{C}^d) := L^2(\mathbb{C}^d, e^{-2\Phi(z)/h} d^{2d} z) \cap \operatorname{Hol}(\mathbb{C}^d)$$

is called the *Bargmann–Fock space*. Here, $d^{2d} z$ denotes the Lebesgue measure on $\mathbb{C}^d$.

There exists a standard unitary intertwining operator, the *Bargmann transform*[4], from $L^2(\mathbb{R}^d)$ to the Bargmann–Fock space,

$$\mathcal{B}\colon L^2(\mathbb{R}^d) \longrightarrow H_\Phi(\mathbb{C}^d),$$

defined by

$$\mathcal{B}u(z;h) = 2^{-d/4}(\pi h)^{-3d/4} \int_{\mathbb{R}^d} e^{i\varphi(z,y)/h} u(y)dy$$

with phase function

$$\varphi(z,x) = i\Big(\frac{1}{2}(z^2 + x^2) - \sqrt{2}x \cdot z\Big).$$

It is a Fourier integral operator with positive complex phase. The phase function and the weight are related by

$$\Phi(z) = \sup_{y \in \mathbb{R}^d} (-\operatorname{Im} \varphi(z,x)).$$

We denote by Π^{BF} the orthogonal projection $L^2(\mathbb{C}^d, e^{-2\Phi} d^{2d} z) \to H_\Phi(\mathbb{C}^d)$. Its Schwartz kernel is known as the Bargmann–Fock Bergman kernel.

Remark 5.1. The operator $\mathcal{B}$ can be written as $\mathcal{B} f(z;h) = (\mathscr{B}_{1/(2h)} f)(\sqrt{2}z)$, where $\mathscr{B}_\alpha$ is defined as in Zhu [13, Section 6.2].

Remark 5.2. In what follows, we will always take $h = 1$.

Define the linear complex canonical transformation

$$\begin{aligned} \kappa_\varphi\colon \mathbb{C}^{2d} &\longrightarrow \mathbb{C}^{2d}, \\ (x,\xi) &\longmapsto \frac{1}{\sqrt{2}}(x - i\xi, \xi - ix), \end{aligned}$$

which satisfies $\kappa_\varphi(x, -\partial_x \varphi(z,x)) = (z, \partial_z \varphi(z,x))$ for all $x, z \in \mathbb{C}^d$. The canonical transformation maps $\mathbb{R}^{2d}$ bijectively onto the totally real IR Lagrangian subspace

$$\Lambda_\Phi := \{(z, -2i\partial_z \Phi(z))\colon z \in \mathbb{C}^d\} = \{(z, -i\bar{z})\colon z \in \mathbb{C}^d\} \subset \mathbb{C}^{2d}.$$

[4] It is a special case of the FBI transform.

For a proof, we refer to [14, Theorem 13.5]. The inverse of $\kappa_\varphi \colon \mathbb{R}^{2d} \to \Lambda_\Phi$ is given by

$$\begin{aligned} \kappa_\varphi^{-1} \colon \Lambda_\Phi &\longrightarrow \mathbb{R}^{2d}, \\ (z,\zeta) &\longmapsto \frac{1}{\sqrt{2}}(z+i\zeta, \zeta+iz). \end{aligned}$$

The Bargmann transform is a quantization of κ_φ in the sense that

$$\mathcal{B}^* a^w(z, D_z)\mathcal{B} = (\kappa_\varphi^* a)^w(x, D_x), \quad a \in \Gamma(\Lambda_\Phi)$$

(cf. Zworski [14, Theorem 13.9] in the semiclassical setting).

It is a classical fact (cf. Zhu [13] or Zworski [14, Theorem 4.5]) that $H_0 - d/2$ is conjugated to the degree operator $\mathcal{N} := \langle z, \partial_z \rangle$ on Bargmann–Fock space. Note that $(\kappa_\varphi^* p_2)(z,\zeta) = \langle z, i\zeta \rangle$, where $p_2(x,\xi) = (1/2)(|x|^2 + |\xi|^2)$. This gives a short proof that $\mathcal{B}^*(H_0 - d/2)\mathcal{B} = \mathcal{N}$, since $\mathrm{Op}^w(\langle z, i\zeta \rangle) = \mathcal{N} + d/2$.

The eigenspace $\mathcal{H}_N^{\mathrm{BF}}$ of eigenvalue N is spanned by the monomials z^α with $|\alpha| = N$. Comparing the spectral decompositions of H_0 and the degree operator gives

Lemma 5.3. *The operator $\Pi_N^{\mathrm{BF}} := \mathcal{B}\Pi_N\mathcal{B}^*$ is the orthogonal projection onto $\mathcal{H}_N^{\mathrm{BF}}$.*

Under conjugation by $\mathcal{B}$, $H_0\widetilde{B}$ transforms as $\mathcal{B}H_0\widetilde{B}\Pi_N\mathcal{B}^* = \mathcal{N}\mathcal{B}\widetilde{B}\mathcal{B}^*$. It follows that

$$\mathcal{B}\Pi_N H_0 \widetilde{B}\Pi_N \mathcal{B}^* = \Big(N + \frac{d}{2}\Big)\Pi_N^{\mathrm{BF}}\mathcal{B}\widetilde{B}\mathcal{B}^*\Pi_N^{\mathrm{BF}}. \tag{6}$$

5.2. Toeplitz operators. The next step is to determine $\Pi_N^{\mathrm{BF}}\mathcal{B}\widetilde{B}\mathcal{B}^*\Pi_N^{\mathrm{BF}}$. We now review the Bargmann conjugation of Weyl pseudo-differential operators to Toeplitz operators, following [6, 1, 14]. Our presentation differs from these references in two ways: (i) we are dealing with isotropic pseudo-differential operators, which are considered in [6] but not in the other two references; (ii) we are interested in semi-classical asymptotics in N rather than in homogeneous operators. Hence, we need to reformulate results of the references in terms of semi-classical symbol expansions. The basic relation is that if $a_j(z,\bar{z})$ is homogeneous in $(z,\bar{z})$ of order j, then the semi-classical Toeplitz operator $\Pi_N a_j \Pi_N$ is of order $N^{j/2}$ (cf. Lemma 5.5 below).

We also recall that there exist two notions of complete symbol for a Toeplitz operator: (i) its contravariant symbol q, and (ii) its covariant symbol $\Pi^{\mathrm{BF}} q \Pi^{\mathrm{BF}}(z,z)$, i.e. the value of the Schwartz kernel on the (anti-)diagonal. The transform from the contravariant symbol to the covariant symbol is known as the Berezin transform.

The Bargmann transform conjugates isotropic Weyl pseudo-differential operators $\mathrm{Op}^w(a)$ to Toeplitz operators $\Pi^{\mathrm{BF}} q \Pi^{\mathrm{BF}}$ with isotropic symbols. The complete contravariant symbol of an isotropic Toeplitz operator $\Pi^{\mathrm{BF}} q \Pi^{\mathrm{BF}}$ of order 0 is an isotropic symbol on $\mathbb{C}^d \simeq T^*\mathbb{R}^d$ of the form,

$$q \sim q_0 + q_{-1} + q_{-2} + \cdots,$$

where q_{-j} is homogeneous of degree $-j$.

By a well-known result of Berezin (see [1] for a recent proof), the relation between the complete Weyl symbol a of $\mathrm{Op}^w(a)$ to the contravariant symbol q is given by

$$a(x, -2i\partial_x \Phi(x)) = \left(\exp\left(\frac{1}{4}(\partial_{x\bar{x}}\Phi)^{-1}\partial_x \cdot \partial_{\bar{x}}\right)q\right)(x), \quad x \in \mathbb{C}^d. \tag{7}$$

The operator $(\partial_{x\bar{x}}\Phi)^{-1}\partial_x \cdot \partial_{\bar{x}}$ is a constant coefficient second order differential operator on $\mathbb{C}^n$ whose symbol is a negative definite quadratic form; so this is a forward heat flow acting on q. The Berezin transform is the inverse heat flow. In our case (cf. [1, 6, 14]), we have that $(1/4)(\partial_{x\bar{x}}\Phi)^{-1}\partial_x \cdot \partial_{\bar{x}} = (1/8)\Delta$ (the standard Euclidean Laplacian on $\mathbb{R}^{2d}$) and therefore

$$q = e^{-\frac{1}{8}\Delta} a \circ \kappa_\varphi. \tag{8}$$

A general $\mathcal{C}^\infty$ isotropic symbol of order 0 does not lie in the domain of $e^{-\frac{1}{8}\Delta}$. However, the expression (8) makes sense as an isotropic symbol since Δ lowers the order of an isotropic symbol by two orders. That is, we invert the transform (7) in the topology of symbols and view $e^{-\frac{1}{8}\Delta}$ as an operator taking complete (formal) isotropic Weyl symbols to complete (formal) isotropic contravariant symbols. If we Taylor expand $e^{-\frac{1}{8}\Delta}$ to order M, and a is isotropic of order m, then

$$e^{-\frac{1}{8}\Delta} a = \sum_{k=0}^{M} \frac{(-1)^k}{8^k k!}(\Delta^k a) \mod \Gamma^{m-2M}.$$

Summing up, we have the following:

Lemma 5.4. *Let $A = \mathrm{Op}^w(a)$ be a zeroth order isotropic pseudo-differential operator on $\mathbb{R}^d$. Then $\mathcal{B}A\mathcal{B}^*$ is a Toeplitz operator $\Pi^{\mathrm{BF}} q \Pi^{\mathrm{BF}}$ on $H_\Phi(\mathbb{C}^d)$, where the symbol q has an asymptotic expansion*

$$q \sim \sum_{k=0}^{\infty} \frac{(-1)^k}{8^k k!}(\Delta^k a) \circ \kappa_\varphi$$

in the sense of isotropic symbols.

If we assume that $a \in \Gamma^0_{\mathrm{cl}}$ *and* $a \sim \sum_k a_{-k}$, *then* $q \in \Gamma^0_{\mathrm{cl}}$ *with expansion* $q \sim \sum_k q_{-k}$, *where*

$$\begin{aligned} q_0 &= a_0 \circ \kappa_\varphi, \\ q_{-1} &= a_{-1} \circ \kappa_\varphi, \\ q_{-2} &= (a_{-2} - (1/8)\Delta a_0) \circ \kappa_\varphi. \end{aligned}$$

In particular, we have that

$$\mathcal{N} = \Pi^{\mathrm{BF}}(|z|^2 - d)\Pi^{\mathrm{BF}},$$

since the symbol of $\mathcal{N}$ restricted to Λ_Φ is $a(z) = |z|^2 - d/2$ and the Berezin transform of a is $q(z) = |z|^2 - d$.

Returning to the symbol $\tilde{b} \in \Gamma^0_{\mathrm{cl}}$ with $\{\tilde{b}, p_2\} = 0$ and asymptotic expansion

$$\tilde{b} \sim \sum_{j=0}^{\infty} \tilde{b}_{-j},$$

where $\tilde{b}_j = p_2^{-1/2} \chi p_{j+1}$ for $j = 0, -1$. We now compress $\Pi^{\mathrm{BF}}\tilde{b}\Pi^{\mathrm{BF}}$ with Π^{BF}_N and exponentiate. The order of a homogeneous isotropic symbol coincides with its eigenvalue under the degree operator $\mathcal{N}$. This is an immediate consequence of Euler's homogeneity theorem. If we write $z = (x + i\xi)$, then

$$\begin{aligned} \mathcal{N}\tilde{b} &= \frac{1}{2}(x\partial_x + \xi\partial_\xi)\tilde{b} + \frac{i}{2}(\xi\partial_x - x\partial_\xi)\tilde{b} \\ &= \frac{1}{2}(x\partial_x + \xi\partial_\xi)\tilde{b} + \frac{i}{2}\{p_2, \tilde{b}\}. \end{aligned}$$

Since $\tilde{b}$ was assumed to Poisson-commute with p_2, the second term vanishes and we obtain that $\mathcal{N}\tilde{b}_j = (j/2)\tilde{b}_j$. We may write

$$\begin{aligned} \Pi^{\mathrm{BF}}_N \tilde{b}_j \Pi^{\mathrm{BF}}_N &= \Pi^{\mathrm{BF}}_N (\mathcal{N} + d)^{j/2} |z|^{-j} \tilde{b}_j \Pi^{\mathrm{BF}}_N \\ &= (N + d)^{j/2} \Pi^{\mathrm{BF}}_N |z|^{-j} \tilde{b}_j \Pi^{\mathrm{BF}}_N. \end{aligned}$$

Note that $|z|^{-j}\tilde{b}_j$ is bounded with norm independent of N. This agrees with the statement at the beginning of this section that isotropic orders get multiplied by $\frac{1}{2}$ when we conjugate to the Bargmann–Fock model and use homogeneity in $H_0 \simeq \mathcal{N}$ to define orders. It follows that the polyhomogeneous expansion of an isotropic symbol coincides with its expansion in powers of $(N + d)^{-1/2}$ when compressed by Π^{BF}_N. We thus have:

Lemma 5.5. *The operator* $\Pi_N^{\mathrm{BF}} \mathcal{B} \widetilde{B} \mathcal{B}^{-1} \Pi_N^{\mathrm{BF}}$ *is a semi-classical Toeplitz operator whose complete contravariant expansion has the form*

$$q = \tilde{b}_0 \circ \kappa_\varphi + N^{-1/2} \tilde{b}_{-1} \circ \kappa_\varphi + N^{-1} (\tilde{b}_{-2} - (1/8) \Delta \tilde{b}_0) \circ \kappa_\varphi + O(N^{-3/2}).$$

In particular, we have that $\Pi_N \widetilde{B} \Pi_N$ *is conjugated to* $\Pi_N^{\mathrm{BF}} q \Pi_N^{\mathrm{BF}}$ *under the Bargmann transform.*

5.3. Trace asymptotics. Our aim is to determine the large N asymptotics of

$$\operatorname{Tr} \Pi_N e^{it\sqrt{N}\widetilde{B}} \Pi_N. \tag{9}$$

We observe that (9) is the trace of the rescaled propagator $U_N\big(\frac{t}{\sqrt{N}}\big)$, where

$$U_N(t) := \Pi_N e^{itN\widetilde{B}} \Pi_N. \tag{10}$$

It follows from Lemma 5.5 that $U_N(t) = \mathcal{B}^* \exp(itN \Pi_N^{\mathrm{BF}} q \Pi_N^{\mathrm{BF}}) \mathcal{B}$ is a semi-classical isotropic Fourier integral operator. Clearly,

$$\operatorname{Tr} \Pi_N e^{itB} = \operatorname{Tr} U_N\Big(\frac{t}{\sqrt{N}}\Big). \tag{11}$$

The main value of the Bargmann–Fock conjugation is that the propagator of the harmonic oscillator and the projection Π_N become much simpler on the Bargmann–Fock side. The trace $\operatorname{Tr} U_N\big(\frac{t}{\sqrt{N}}\big)$ may be further simplified by noting that $\mathcal{H}_N$ is the same as the space $H^0(\mathbb{CP}^{d-1}, \mathcal{O}(N))$ of holomorphic sections of the Nth power of the natural line bundle $\mathcal{O}(1) \to \mathbb{CP}^{d-1}$. The identification is to lift holomorphic sections, $s \to \hat{s}$, of $\mathcal{O}(N)$ to homogeneous functions on $\mathbb{C}^d \setminus \{0\}$. It follows that

$$H^2(\mathbb{C}^d, e^{-|z|^2} d^{2d} z) \simeq \bigoplus_{N=0}^{\infty} \mathcal{H}_N^{\mathrm{BF}} \simeq \bigoplus_{N=0}^{\infty} H^0(\mathbb{CP}^{d-1}, \mathcal{O}(N)). \tag{12}$$

We refer to [4] for background. Tracing through the identifications, we see that $\mathcal{H}_N \simeq H^0(\mathbb{CP}^{d-1}, \mathcal{O}(N))$. This identification explains why the trace formula is an integral over the space of Hamilton orbits of H_0.

We use the last identification to determine the asymtotics of the trace of $U_N\big(\frac{t}{\sqrt{N}}\big)$ in the model $H^0(\mathbb{CP}^{d-1}, \mathcal{O}(N))$. The advantage of conjugating to this model is that the calculations have mostly been done in this setting in [11, 12] (in fact, on any Kähler manifold M). It would be equivalent to work directly with the Fourier components $\Pi_N^{\mathrm{BF}} \widetilde{B} \Pi_N^{\mathrm{BF}}$.

We let

$$\Pi_N^{\mathbb{CP}^{d-1}} \colon L^2(\mathbb{CP}^{d-1}, \mathcal{O}(N)) \longrightarrow H^0(\mathbb{CP}^{d-1}, \mathcal{O}(N))$$

denote the orthogonal projection. Since $\tilde{b}$ is invariant under the natural $\mathbb{S}^1$ action on $\mathbb{C}^d$ defining $\mathbb{C}^d \setminus \{0\} \to \mathbb{CP}^{d-1}$, $\sigma^0(\tilde{b})$ descends to a multiplication operator on $L^2(\mathbb{CP}^{d-1}, \mathcal{O}(N))$.

We briefly recall the setting of [11, 12]. Consider a polarized Kähler manifold M with positive Hermitian line bundle L (cf. [12] for definitions) and let $H^0(M, L^N)$ denote the space of holomorphic sections of the N-th power of the line bundle L and Π_N^M is the orthogonal projection $L^2(M, L^N) \to H^0(M, L^N)$. For any function $H \colon M \to \mathbb{R}$ we define

$$\hat{H}_N = \Pi_N^M \, H \, \Pi_N^M \colon H^0(M, L^N) \longrightarrow H^0(M, L^N)$$

the corresponding semiclassical Toeplitz operator [5] and

$$U_N(t) = \exp itN\hat{H}_N \tag{13}$$

its propagator, which is a semiclassical Fourier integral operator. In [11, 12] it is shown that for any $z \in M$, the following pointwise asymptotics hold:

Proposition 5.6 ([11, Proposition 5.3]). *Let (M, ω) be a Kähler manifold of complex dimension m and $H \colon M \to \mathbb{R}$ a Morse function. If $z \in M$, then for any $\tau \in \mathbb{R}$,*

$$U_N(t/\sqrt{N}, z, z) = \left(\frac{N}{2\pi}\right)^m e^{it\sqrt{N}H(z)} e^{-t^2\frac{\|dH(z)\|^2}{4}} (1 + O(|t|^3 N^{-1/2})),$$

where the constant in the error term is uniform as t varies over compact subset of $\mathbb{R}$.

Remark 5.7. It is emphasized that the asymptotics are valid at critical points of H.

The asymptotics of the trace follow from Proposition 5.6 and the method of stationary phase.

[5] In [11] the authors in fact define the quantization of H by $\hat{H}_N := \Pi_N(\frac{i}{N}\nabla_{\mathsf{H}} + H)\Pi_N$, where H is the Hamilton vector field of H.

Theorem 5.8 (cf. [12, Theorem 1.7]). *Let (M,ω) be a Kähler manifold of complex dimension m. If $t \neq 0$, the trace of the scaled propagator $U_N(t/\sqrt{N}) = e^{i\sqrt{N}t\hat{H}_N}$ admits the following aymptotic expansion*

$$\int_{z\in M} U_N(t/\sqrt{N}, z, z)d\,\mathrm{vol}_M(z)$$
$$= N^m\Big(\frac{t\sqrt{N}}{4\pi}\Big)^{-m}\sum_{z_c\in\mathrm{crit}(H)}\frac{e^{it\sqrt{N}H(z_c)}e^{(i\pi/4)\,\mathrm{sgn}(DdH(z_c))}}{\sqrt{|\det(DdH(z_c)))|}}(1+O(|t|^3N^{-1/2})).$$

Remark 5.9. We note that the Gaussian factor equals 1 at the critical points.

We see that the operator $U_N(t/\sqrt{N})$ is given by

$$U_N(t/\sqrt{N}) = \exp(it\Pi_N^{\mathrm{BF}}(\sqrt{N}\tilde{b}_0 + \tilde{b}_{-1})\Pi_N^{\mathrm{BF}}) + O(1/\sqrt{N}).$$

If $\widetilde{X}p_0 = 0$, then we may directly apply Theorem 5.8 with $M = \mathbb{CP}^{d-1}$, $L = \mathcal{O}(1)$, and $H = \tilde{b}_0|_{\mathbb{CP}^{d-1}} = \widetilde{X}p_1$ to obtain Theorem 1.3. A complete asymptotic expansion with remainder could be obtained by the same method if one wished to have lower order terms.

Corollary 5.10. *If $\widetilde{X}p_1 = \tilde{b}_0|_{\mathbb{CP}^{d-1}}$ is a Morse function on $\mathbb{CP}^{d-1}$, the asymptotics of (11) are given by Theorem 5.8 with $m = d-1$.*

If $\widetilde{X}p_0 \neq 0$, then the contribution of $\tilde{b}_{-1}|_{\mathbb{CP}^{d-1}} = \widetilde{X}p_0$ may be absorbed into the amplitude and repeating the stationary phase calculation as in [11], we obtain Theorem 1.3.

6. Equivalence of the expansions

In this section we show that the large λ expansion of Theorem 1.1 agrees with the large N expansion of Theorem 1.3 to leading order.

Consider the Fourier series

$$w_r(t,a) := \sum_{N=1}^{\infty} N^{-r}e^{-i(N+d/2)t}e^{-iat\sqrt{N}}.$$

We introduce a cutoff $\hat{\rho}(t)$ supported near $t = 2\pi k_0$ with $\hat{\rho}(2\pi k_0) = 1$ and calculate asymptotics as $\lambda \to \infty$ of

$$\begin{aligned}\mathcal{F}^{-1}_{t\to\lambda}\{\hat{\rho}(t)w_r(t,a)\}(\lambda) &= (2\pi)^{-1}\int_{\mathbb{R}} \hat{\rho}(t)e^{it\lambda}w_r(t,a)dt \\ &= (2\pi)^{-1}\sum_{N=1}^{\infty} N^{-r}\int_{\mathbb{R}} \hat{\rho}(t)e^{it\lambda}e^{-it(N+d/2)}e^{-iat\sqrt{N}}dt. \end{aligned} \tag{14}$$

We claim that the large λ asymptotics of the integral (14) coincides with the large N asymptotics of the Fourier coefficients.

Proposition 6.1. *Let w_r and ρ be as above. Then*

$$\mathcal{F}^{-1}_{t\to\lambda}\{\hat{\rho}(t)w_r(t,a)\}(\lambda) = \lambda^{-r}e^{i\pi k_0 d}e^{2\pi i k_0 a\lambda^{1/2}} + O(\lambda^{-r-1/2}).$$

Proof. Let $f_r \in \mathcal{C}^\infty(\mathbb{R})$ such that $f_r(\xi) = \xi^{-r}$ for $\xi > 1/2$ and $f_r(\xi) = 0$ for $\xi \leq 0$. By applying the Poisson summation formula to the function

$$\xi \longmapsto f_r(\xi)\,\mathcal{F}^{-1}_{t\to\lambda}\{e^{it(\xi+d/2)}e^{iat|\xi|^{1/2}}\}(\lambda),$$

we have that

$$\begin{aligned} I(\lambda) &:= \mathcal{F}^{-1}_{t\to\lambda}\{\hat{\rho}(t)w_r(t,a)\}(\lambda) \\ &= (2\pi)^{-1}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\int_0^\infty \hat{\rho}(t)e^{it\lambda}e^{2\pi i k\xi}f_r(\xi)e^{-iat|\xi|^{1/2}}e^{-it(\xi+d/2)}d\xi\,dt. \end{aligned}$$

Changing variables $\xi \mapsto \lambda\xi$ for $\lambda > 0$ yields

$$\begin{aligned} I(\lambda) = {} & (2\pi)^{-1}\lambda^{1-r}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\int_0^\infty \hat{\rho}(t)e^{i\lambda(-t\xi+t+2\pi k\xi)}f_r(\xi)e^{-iat\lambda^{1/2}|\xi|^{1/2}}e^{-itd/2}d\xi\,dt \\ & + O(\lambda^{-\infty}). \end{aligned}$$

We set

$$a_\lambda(t,\xi) = (2\pi)^{-1}\hat{\rho}(t)f_r(\xi)e^{-iat\lambda^{1/2}|\xi|^{1/2}}e^{-itd/2}$$

and

$$\phi_k(t,\xi) = (1-\xi)t + 2\pi k\xi$$

and note that derivatives in ξ decrease the order in ξ by $1/2$ while increasing it in λ by $1/2$. This suffices, since integration by parts decreases the orders by 1. We have

$$I(\lambda) = \lambda^{1-r} \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \int_0^\infty e^{i\lambda\phi_k(t,\xi)} a_\lambda(t,\xi) d\xi\, dt + O(\lambda^{-\infty}).$$

We may decompose the integral as follows:

$$\mathcal{F}_{t\to\lambda}\{\hat{\rho}(t) w_r^{2\pi k_0}(t,a)\}(\lambda) = \lambda^{1-r}(I_1(\lambda) + I_2(\lambda) + I_3(\lambda) + O(\lambda^{-\infty})),$$

where

$$I_1(\lambda) = \int_{\mathbb{R}} \int_0^\infty \psi(\xi) e^{i\lambda\phi_{k_0}(t,\xi)} a_\lambda(t,\xi) d\xi dt,$$

$$I_2(\lambda) = \int_{\mathbb{R}} \int_0^\infty (1-\psi(\xi)) e^{i\lambda\phi_{k_0}(t,\xi)} a_\lambda(t,\xi) d\xi dt,$$

$$I_3(\lambda) = \sum_{k\neq k_0} \int_{\mathbb{R}} \int_0^\infty \psi(\xi) e^{i\lambda\phi_k(t,\xi)} a_\lambda(t,\xi) d\xi dt,$$

$$I_4(\lambda) = \sum_{k\neq k_0} \int_{\mathbb{R}} \int_0^\infty (1-\psi(\xi)) e^{i\lambda\phi_k(t,\xi)} a_\lambda(t,\xi) d\xi dt.$$

First, we consider the integral $I_2(\lambda)$. We have that $\partial_t\phi_k(t,\xi) = 1-\xi \neq 0$ on $\operatorname{supp}\psi$. Thus, by integration by parts we obtain arbitrary decay in both λ and ξ, which yields that the integral converges and is $O(\lambda^{-\infty})$.

The $I_3(\lambda)$ integral posses no problem, since by integration by parts For $I_3(\lambda)$, we use that $\partial_\xi\phi_k = 2\pi k - t \neq 0$ for $k \neq k_0$ on the support of $\hat{\rho}$. Hence, we obtain arbtrary decay in k to make the series converge, this gives also rapid decay in λ, showing that we have indeed $I_3(\lambda) = O(\lambda^{-\infty})$.

The last integral, $I_4(\lambda)$, can be treated by a combination of the previous arguments to make the integral and series converge with arbitrary decay in λ.

Hence, we have that

$$\begin{aligned}\mathcal{F}_{t\to\lambda}\{\hat{\rho}(t) w_r^{2\pi k_0}(t,a)\}(\lambda) &= \lambda^{1-r} I_1(\lambda) + O(\lambda^{-\infty}) \\ &= \lambda^{1-r} \int e^{i\lambda\phi_{k_0}(t,\xi)} \psi(\xi) a_\lambda(t,\xi) d\xi\, dt + O(\lambda^{-\infty}).\end{aligned}$$

The phase function ϕ_{k_0} is stationary on $t = 2\pi k_0$ and $\xi = 1$ and its Hessian is given by

$$Dd\phi_{k_0}|_{d\phi_{k_0}=0} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The method of stationary phase yields

$$I_1(\lambda) = \lambda^{-1} e^{2\pi i k_0} e^{-2\pi i k_0 a \lambda^{1/2}} e^{i\pi k_0 d} + O(\lambda^{-3/2}). \qquad \square$$

A. Duhamel's formula

We recall the Duhamel principle for general isotropic evolution equations. Let $H \in G^2(\mathbb{R}^d)$ elliptic and $R \in G^{-\infty}(\mathbb{R}^d)$ such that H and $H + R$ are self-adjoint. Consider the propagators $U(t) = e^{-itH}$ and $V(t) = e^{-it(H+R)}$.

Proposition A.1. *The difference $U(t) - V(t)$ is a smoothing operator.*

Proof. The difference of the propagators, $F(t) = U(t) - V(t)$ solves the equation

$$\begin{cases} i\partial_t F(t) = (H + R)F(t) + R(t), \\ F(0) = 0, \end{cases} \tag{15}$$

where $R(t) = RU(t)$. By the Duhamel principle,

$$F(t) = \int_0^t V(t-s)R(s)ds.$$

Since $V(t)$ and $U(t)$ are unitary, we obtain that for any $N \in \mathbb{N}$,

$$\begin{aligned} \|H_0^N F(t)\|_{L^2} &= \left\| \int_0^t H_0^N V(t-s)R(s)ds \right\|_{L^2} \\ &\leq \int_0^t \|RH_0^N\|_{L^2} ds \\ &\leq \sup_{s\in[0,t]} \|RH_0^N\|_{L^2} \\ &< \infty. \end{aligned}$$

This shows that $F(t)$ is bounded in every isotropic Sobolev space (cf. [3] for the definition) and this implies that $F(t) \in G^{-\infty}(\mathbb{R}^d)$. $\square$

Acknowledgments. We would like to thank Jared Wunsch for many useful discussions and inviting the first author to Northwestern University, where the project was started. We would also like to thank Peng Zhou for many discussions related to the Bargmann–Fock setting.

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Received November 19, 2018

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