

STANDARD MONOMIALS AND INVARIANT THEORY FOR ARC SPACES I: GENERAL LINEAR GROUP

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ABSTRACT. This is the first in a series of papers on standard monomial theory and invariant theory of arc spaces. For any algebraically closed field K , we construct a standard monomial basis for the arc space of the determinantal variety over K . As an application, we prove the arc space analogue of the first and second fundamental theorems of invariant theory for the general linear group.

1. INTRODUCTION

Classical invariant theory. Classical invariant theory has a long history that began in the 19th century in work of Cayley, Gordan, Klein, and Hilbert. Given an algebraically closed field K , a reductive algebraic group G over K , and a finite-dimensional G -module W , the ring of invariant polynomial functions $K[W]^G$ is the main object of study. It is often useful to consider invariant rings $K[V]^G$, where $V = W^{\oplus p} \oplus W^{*\oplus q}$ is the direct sum of p copies of W and q copies of the dual G -module W^* . In the terminology of Weyl, a *first fundamental theorem of invariant theory* (FFT) for the pair (G, W) is a generating set for $K[V]^G$, and a *second fundamental theorem* (SFT) for (G, W) is a generating set for the ideal of relations among the generators of $K[V]^G$. When $\text{char } K = 0$, if G is one of the classical groups and W is the standard representation, the FFTs and SFTs are due to Weyl [36]. The analogous results in arbitrary characteristic were proven by de Concini and Procesi in [7]. Explicit FFTs and SFTs are in general difficult to obtain and are known only in a few other cases, such as the adjoint representations of the classical groups which is due to Procesi [29], the 7-dimensional representation of G_2 and the 8-dimensional representation of Spin_7 , which are due to Schwarz [30].

The main example in this paper is the case where G is the general linear group $GL_h(K)$ over K , and $W = K^{\oplus h}$ is its standard representation. For $V = W^{\oplus p} \oplus W^{*\oplus q}$ as above, the affine coordinate ring is

$$K[V] = K[a_{il}^{(0)}, b_{jl}^{(0)} \mid 1 \leq i \leq p, 1 \leq j \leq q, 1 \leq l \leq h].$$

Theorem 1.1. (FFT and SFT for $G = GL_h(K)$ and $W = K^{\oplus h}$)

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(1) The ring of invariants $K[V]^{GL_h(K)}$ is generated by

$$\{X_{ij}^{(0)} = \sum_l a_{il}^{(0)} b_{jl}^{(0)} \mid 1 \leq i \leq p, 1 \leq j \leq q\}.$$

(2) The ideal of relations among the generators in (1) is generated by

$$\begin{vmatrix} X_{u_1 v_1}^{(0)} & X_{u_1 v_2}^{(0)} & \cdots & X_{u_1 v_{h+1}}^{(0)} \\ X_{u_2 v_1}^{(0)} & X_{u_2 v_2}^{(0)} & \cdots & X_{u_2 v_{h+1}}^{(0)} \\ \vdots & \vdots & \vdots & \vdots \\ X_{u_{h+1} v_1}^{(0)} & X_{u_{h+1} v_2}^{(0)} & \cdots & X_{u_{h+1} v_{h+1}}^{(0)} \end{vmatrix},$$

for all $u_1, u_2, \dots, u_h$ and $v_1, v_2, \dots, v_h$ with $1 \leq u_i < u_{i+1} \leq p$ and $1 \leq v_i < v_{i+1} \leq q$.

Standard monomial theory. Standard monomial theory was initiated in the 1970s by Seshadri, Musili and Lakshmibai [31, 17, 18, 19], generalizing earlier work of Hodge [12]. It involves nice combinatorial bases for the affine coordinate rings of Schubert varieties inside quotients of classical groups by parabolic subgroups. In this paper, we only need the case of determinantal varieties.

For positive integers p and q , let

$$(1.1) \quad R = R_{p,q} = \mathbb{Z}[x_{ij}^{(0)} \mid 1 \leq i \leq p, 1 \leq j \leq q],$$

be the ring of polynomial functions with integer coefficients on the space of $p \times q$ matrices. Consider the h -minor

$$(1.2) \quad B = \begin{vmatrix} x_{u_1 v_1}^{(0)} & x_{u_1 v_2}^{(0)} & \cdots & x_{u_1 v_h}^{(0)} \\ x_{u_2 v_1}^{(0)} & x_{u_2 v_2}^{(0)} & \cdots & x_{u_2 v_h}^{(0)} \\ \vdots & \vdots & \vdots & \vdots \\ x_{u_h v_1}^{(0)} & x_{u_h v_2}^{(0)} & \cdots & x_{u_h v_h}^{(0)} \end{vmatrix},$$

with $u_i < u_{i+1}$, $v_i < v_{i+1}$. Throughout this paper, we will represent B by the pair of ordered h -tuples

$$(u_h, \dots, u_2, u_1 \mid v_1, v_2, \dots, v_h).$$

There is a partial ordering on the set of these minors given by

$$(u_h, \dots, u_2, u_1 \mid v_1, v_2, \dots, v_h) \leq (u'_h, \dots, u'_2, u'_1 \mid v'_1, v'_2, \dots, v'_h),$$

if $h' \leq h$, $u_i \leq u'_i$, $v_i \leq v'_i$.

R has a standard monomial basis (cf. [16]) with respect to this partially ordered set of minors: the ordered products $A_1 A_2 \cdots A_k$ of minors A_i with $A_i \leq A_{i+1}$, form a basis of R . Similarly, let $R[h]$ be the ideal of R generated by all h -minors in the form of (1.2), and let

$$(1.3) \quad R_h = R/R[h+1].$$

Then R_h has a basis consisting of ordered products $A_1 A_2 \cdots A_k$ of h_i -minors A_i with $h_i < h$ and $A_i \leq A_{i+1}$.

For an arbitrary algebraically closed field K , let $M_{p,q} = M_{p,q}(K)$ be the space of $p \times q$ matrices with entries in K . The affine coordinate ring $K[M_{p,q}]$ is obtained from R by base change, that is, $K[M_{p,q}] \cong R \otimes_{\mathbb{Z}} K$. Let $K[M_{p,q}][h]$ be the ideal generated by all h -minors. The determinantal variety $D_h = D_h(K)$ is a closed subvariety of $M_{p,q}$ with $K[M_{p,q}][h]$ as the defining ideal. Then the affine coordinate ring $K[D_h] \cong K[M_{p,q}]/K[M_{p,q}][h]$, has a standard monomial basis: the ordered products $A_1 A_2 \cdots A_k$ of h_i -minors A_i with $h_i < h$ and $A_i \leq A_{i+1}$ form a basis of $K[D_h]$. With $G = GL_h(K)$ and V as in Theorem 1.1, we have $V//G \cong D_{h+1}$, and the proof of Theorem 1.1 in [7] makes use of this standard monomial basis. A uniform treatment of the FFT and SFT for all the classical groups using standard monomial theory can also be found in the book [16].

Arc spaces. For a scheme X of finite type over K , the arc space $J_{\infty}(X)$ is defined as the inverse limit of the finite jet schemes $J_n(X)$ [11]. By Corollary 1.2 of [5], it is determined by its functor of points : for every K -algebra A , we have a bijection

$$\mathrm{Hom}(\mathrm{Spec} A, J_{\infty}(X)) \cong \mathrm{Hom}(\mathrm{Spec} A[[t]], X).$$

If $i : X \rightarrow Y$ is a morphism of schemes, we get a morphism of schemes $i_{\infty} : J_{\infty}(X) \rightarrow J_{\infty}(Y)$. Arc spaces were first studied by Nash in [28], and carry important information about the singularities of X . The *Nash problem* asks whether there is a bijection between the irreducible components of $J_{\infty}(X)$ lying over the singular locus of X , and the essential divisors over X . This has been answered affirmatively for many classes of varieties, although counterexamples are known [13]. Arc spaces are also important in Kontsevich's theory of *motivic integration*, which was used to prove that birationally equivalent Calabi-Yau manifolds have the same Hodge numbers [15]. This theory has been developed by many authors including Batyrev, Craw, Denef, Ein, Loeser, Looijenga, Mustata, and Veys; see for example [4, 6, 8, 9, 11, 26, 27, 35]. More recently, arc spaces have turned out to have applications to the theory of vertex algebras, which in many cases can be viewed as quantizations of arc spaces [1, 3, 2, 21, 32, 33, 34].

Standard monomials for arc spaces. Let

$$(1.4) \quad \mathfrak{R} = \mathfrak{R}_{p,q} = \mathbb{Z}[x_{ij}^{(k)} \mid 1 \leq i \leq p, 1 \leq j \leq q, k \geq 0],$$

which has a derivation ∂ characterized by $\partial x_{ij}^{(k)} = (k+1)x_{ij}^{(k+1)}$. It can be regarded as the ring of polynomial functions with integer coefficients on the arc space of $p \times q$ matrices; in particular, $K[J_{\infty}(M_{p,q})] \cong \mathfrak{R} \otimes_{\mathbb{Z}} K$.

Let $\mathfrak{R}[h]$ be the ideal of $\mathfrak{R}$ generated by all h -minors B of the form (1.2) and their normalized derivatives $\frac{1}{n!} \partial^n B$. Let

$$(1.5) \quad \mathfrak{R}_h = \mathfrak{R}/\mathfrak{R}[h+1].$$

Let $\mathcal{J}_r$ be the set of h -minors of the form (1.2) with $h \leq r$ and their normalized derivatives. Note that R and R_h are naturally subrings of $\mathfrak{R}$ and $\mathfrak{R}_h$,

respectively. In Section 2, we will define a notion of standard monomial on $\mathfrak{R}_h$ that extends the above notion on R_h , and we will prove the following result.

Theorem 1.2. $\mathfrak{R}_h$ has a $\mathbb{Z}$ -basis given by the standard monomials of $\mathcal{J}_h$.

Let $J_\infty(D_h)$ be the arc space of the determinantal variety D_h . Then the affine coordinate ring $K[J_\infty(D_h)]$ is $\mathfrak{R}_{h-1} \otimes_{\mathbb{Z}} K$, so we immediately have

Corollary 1.3. $K[J_\infty(D_h)]$ has a K -basis given by the standard monomials of $\mathcal{J}_{h-1}$.

When $K = \mathbb{C}$, the arc space $J_\infty(D_h)$, as well as the finite jet schemes $J_n(D_h)$, were also studied by Docampo in [10]. He gave an explicit description of the decomposition of $J_\infty(D_h)$ and $J_n(D_h)$ as a union of orbits for the action of $J_\infty(GL_p(\mathbb{C}) \times GL_q(\mathbb{C}))$ and $J_n(GL_p(\mathbb{C}) \times GL_q(\mathbb{C}))$, respectively.

Application in invariant theory. Given an algebraic group G over K , $J_\infty(G)$ is again an algebraic group. If V is a finite-dimensional G -module, there is an induced action of $J_\infty(G)$ on $J_\infty(V)$, and the invariant ring $K[J_\infty(V)]^{J_\infty(G)}$ was studied in our earlier paper [20] with Schwarz in the case $K = \mathbb{C}$. The quotient morphism $V \rightarrow V//G$ induces a morphism $J_\infty(V) \rightarrow J_\infty(V//G)$, so we have a morphism

$$(1.6) \quad J_\infty(V)//J_\infty(G) \rightarrow J_\infty(V//G).$$

In particular, we have a ring homomorphism

$$(1.7) \quad K[J_\infty(V//G)] \rightarrow K[J_\infty(V)]^{J_\infty(G)}.$$

If $V//G$ is smooth or a complete intersection and $K[V]$ has no nontrivial one-dimensional G -invariant subspaces, it was shown in [20] that (1.7) is an isomorphism, although in general it is neither injective nor surjective.

We specialize to the case $G = GL_h(K)$, $W = K^{\oplus h}$, and $V = W^{\oplus p} \oplus W^{*\oplus q}$, as above. Then

$$K[J_\infty(V)] = K[a_{il}^{(k)}, b_{jl}^{(k)} \mid 1 \leq i \leq p, 1 \leq j \leq q, 1 \leq l \leq h, k \in \mathbb{Z}_{\geq 0}],$$

which has an induced action of $J_\infty(GL_h(K))$ as above. We have the following theorem, which is the arc space analogue of Theorem 1.1.

Theorem 1.4. Fix integers $h \geq 1$ and $p, q \geq 0$, and let $W = K^{\oplus h}$ and $V = W^{\oplus p} \oplus W^{*\oplus q}$ be as above. Let $\bar{\partial}^k = \frac{1}{k!} \partial^k$ be the k^{th} normalized derivative.

(1) The ring of invariants $K[J_\infty(V)]^{J_\infty(GL_h(K))}$ is generated by

$$(1.8) \quad \{X_{ij}^{(k)} = \bar{\partial}^k \sum_l a_{il}^{(0)} b_{jl}^{(0)} \mid 1 \leq i \leq p, 1 \leq j \leq q, k \geq 0\}.$$

(2) The ideal of relations among the generators (1.8) is generated by

$$(1.9) \quad \bar{\partial}^k \begin{vmatrix} X_{u_1 v_1}^{(0)} & X_{u_1 v_2}^{(0)} & \cdots & X_{u_1 v_{h+1}}^{(0)} \\ X_{u_2 v_1}^{(0)} & X_{u_2 v_2}^{(0)} & \cdots & X_{u_2 v_{h+1}}^{(0)} \\ \vdots & \vdots & \ddots & \vdots \\ X_{u_{h+1} v_1}^{(0)} & X_{u_{h+1} v_2}^{(0)} & \cdots & X_{u_{h+1} v_{h+1}}^{(0)} \end{vmatrix},$$

for all $u_1, u_2, \dots, u_h$ and $v_1, v_2, \dots, v_h$ with $1 \leq u_i < u_{i+1} \leq p$ and $1 \leq v_i < v_{i+1} \leq q$, and all integers $k \geq 0$.

(3) $K[J_\infty(V)]^{J_\infty(GL_h(K))}$ has a K -basis given by standard monomials of $\mathcal{J}_h$.

Corollary 1.5. For all $h \geq 1$ and $p, q \geq 0$, the map $K[J_\infty(V//GL_h(K))] \rightarrow K[J_\infty(V)]^{J_\infty(GL_h(K))}$ given by (1.7) is an isomorphism. In particular, we have

$$J_\infty(V)//J_\infty(GL_h(K)) \cong J_\infty(V//GL_h(K)).$$

Corollary 1.5 is a generalization of Theorem 4.6 of [20], which deals with the following special cases for $K = \mathbb{C}$.

- (1) $p \leq h$ or $q \leq h$, so that $V//GL_h(\mathbb{C})$ is an affine space,
- (2) $p = h + 1 = q$, so that $V//GL_h(\mathbb{C})$ is a hypersurface.

In the second paper in this series [22], we will prove a similar theorem for the symplectic group $Sp_h(K)$ for h an even integer: for $W = K^{\oplus h}$ and $V = W^{\oplus p}$, (1.7) is an isomorphism for all h and p . In the third paper [23], we will study the case $G = SL_h(K)$, $W = K^{\oplus h}$ and $V = W^{\oplus p} \oplus W^{*\oplus q}$. This case is more subtle since (1.7) is always surjective, but fails to be injective if $\max(p, q) - 2 > h$. We will completely determine its kernel, which coincides with the nilradical of $K[J_\infty(V//G)]$ when $\text{char } K = 0$. Unfortunately we are unable to prove similar results for the orthogonal and special orthogonal groups using these methods.

Our results on the invariant theory of arc spaces have significant applications to vertex algebras which we will develop in separate papers [24, 25]. These include the structure of cosets of affine vertex algebras inside free field algebras, classical freeness of the affine vertex algebras $L_k(\mathfrak{sp}_{2n})$ for all positive integers n and k , new level-rank dualities involving affine vertex superalgebras, and the complete description of the vertex algebra of global sections of the chiral de Rham complex of an arbitrary compact Ricci-flat Kähler manifold.

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2. STANDARD MONOMIALS

Fix integers $p, q \geq 1$, and recall the ring

$$\mathfrak{R} = \mathfrak{R}^{p,q} = \mathbb{Z}[x_{ij}^{(k)} \mid 1 \leq i \leq p, 1 \leq j \leq q, k \geq 0],$$

with derivation ∂ defined on generators by $\partial x_{ij}^{(k)} = (k+1)x_{ij}^{(k+1)}$. As above, this is an integral version of the coordinate ring of the arc space of the space of $p \times q$ matrices, that is, $K[J_\infty(M_{p,q})] = \mathfrak{R} \otimes_{\mathbb{Z}} K$, for any field K .

Normalized derivatives. For $l \geq 0$, we define the l^{th} normalized derivative $\bar{\partial}^l = \frac{1}{l!} \partial^l$ on $\mathcal{R}$, which satisfies

$$\bar{\partial}^l x_{ij}^{(k)} = C_{k+l}^l x_{ij}^{(k+l)} \in \mathfrak{R}.$$

Here for $k, n \in \mathbb{Z}_{\geq 0}$,

$$C_n^k = \begin{cases} \frac{n!}{k!(n-k)!}, & 0 \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

The following propositions are easy to verify.

Proposition 2.1. *For any $a, b \in \mathfrak{R}$,*

$$\bar{\partial}^l(ab) = \sum_{i=0}^l \bar{\partial}^i a \bar{\partial}^{l-i} b,$$

and $\bar{\partial}^l a \in \mathfrak{R}$.

Proposition 2.2. *For a minor B of the form (1.2),*

$$(2.1) \quad \bar{\partial}^n B = \sum_{\substack{n_1 + \dots + n_h = n \\ n_i \in \mathbb{Z}_{\geq 0}}} \sum_{\sigma} \text{sign}(\sigma) x_{u_1 v_{\sigma(1)}}^{(n_1)} x_{u_2 v_{\sigma(2)}}^{(n_2)} \cdots x_{u_h v_{\sigma(h)}}^{(n_h)}.$$

Generators. Recall that the minor B in (1.2) can be represented by the pair of ordered h -tuples $(u_h, \dots, u_2, u_1 | v_1, v_2, \dots, v_h)$, where $1 \leq u_i < u_{i+1} \leq p$ and $1 \leq v_i < v_{i+1} \leq q$. Similarly, let

$$(2.2) \quad J = \bar{\partial}^n(u_h, \dots, u_2, u_1 | v_1, v_2, \dots, v_h)$$

represent $\bar{\partial}^n B \in \mathfrak{R}$, the n^{th} normalized derivative of the minor B . For convenience, we shall call such expressions $\bar{\partial}$ -lists throughout this paper. We call $wt(J) = n$ the weight of J and call $sz(J) = h$ the size of J . Let $\mathcal{J}$ be the set of these $\bar{\partial}$ -lists, and

$$\mathcal{J}_h = \{J \in \mathcal{J} | sz(J) \leq h\}$$

be the set of elements of $\mathcal{J}$ with size less than or equal to h . Let $\mathcal{E}$ be the set of pairs of ordered h -tuples of ordered pairs of the form

$$(2.3) \quad E = ((u_h, k_h), \dots, (u_2, k_2), (u_1, k_1) | (v_1, l_1), (v_2, l_2), \dots, (v_h, l_h))$$

with $1 \leq u_i \leq p$, $1 \leq v_i \leq q$, $u_i \neq u_j$ if $i \neq j$, $v_i \neq v_j$ if $i \neq j$, and $k_i, l_j \in \mathbb{Z}_{\geq 0}$. For each E , there are unique permutations σ, σ' of $\{1, 2, \dots, h\}$ such that $u_{\sigma(i)} < u_{\sigma(i+1)}$ and $v_{\sigma'(i)} < v_{\sigma'(i+1)}$. Let

$$||E|| = \bar{\partial}^n(u_{\sigma(h)}, \dots, u_{\sigma(2)}, u_{\sigma(1)} | v_{\sigma'(1)}, v_{\sigma'(2)}, \dots, v_{\sigma'(h)}) \in \mathcal{J}.$$

Here $n = \sum k_i + \sum l_i$ and σ, σ' are the above permutations. Let

$$wt(E) = wt(||E||), \quad sz(E) = sz(||E||).$$

Let

$$\mathcal{E}_h = \{E \in \mathcal{E} \mid sz(E) \leq h\}.$$

For $J \in \mathcal{J}$, let

$$(2.4) \quad \mathcal{E}(J) = \{E \in \mathcal{E} \mid \|E\| = J\}.$$

$\mathcal{J}$ is a set of generators of $\mathfrak{R}$ and we can use the elements in $\mathcal{E}(J)$ to represent J .

Ordering. For any set S , let $\mathcal{M}(S)$ be the set of ordered products of elements of S . If S is an ordered set, we order $\mathcal{M}(S)$ lexicographically, that is

$$S_1 S_2 \cdots S_m \prec S'_1 S'_2 \cdots S'_n \quad \text{if } S_i = S'_i, i < i_0, \text{ with } S_{i_0} \prec S'_{i_0} \text{ or } i_0 = m+1, n > m.$$

We order $\mathcal{M}(\mathbb{Z})$, the set of ordered product of integers, lexicographically.

There is an ordering on the set $\mathcal{J}$:

$$\bar{\partial}^k(u_h, \dots, u_2, u_1 | v_1, v_2, \dots, v_h) \prec \bar{\partial}^{k'}(u'_{h'}, \dots, u'_2, u'_1 | v'_1, v'_2, \dots, v'_{h'})$$

if

- $h' < h$;
- or $h' = h$ and $k < k'$;
- or $h' = h, k = k'$ and $u_h \cdots u_1 v_h \cdots v_1 \prec u'_{h'} \cdots u'_1 v'_{h'} \cdots v'_1$. Here we order the words of natural numbers lexicographically.

We order the pairs $(u, h) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ by

$$(u, h) \leq (u', h'), \text{ if } h < h' \text{ or } h = h' \text{ and } u \leq u'.$$

There is a partial ordering on the set $\mathcal{E}$:

$$((u_h, k_h), \dots, (u_1, k_1) | (v_1, l_1), \dots, (v_h, l_h)) \leq ((u'_{h'}, k'_{h'}), \dots, (u'_1, k'_1) | (v'_1, l'_1), \dots, (v'_{h'}, l'_{h'}))$$

if $h' \leq h$ and $(u_i, k_i) \leq (u'_i, k'_i), (v_i, l_i) \leq (v'_i, l'_i)$, for $1 \leq i \leq h'$.

Finally, there is an ordering on $\mathcal{E}$:

$$((u_h, k_h), \dots, (u_1, k_1) | (v_1, l_1), \dots, (v_h, l_h)) \prec ((u'_{h'}, k'_{h'}), \dots, (u'_1, k'_1) | (v'_1, l'_1), \dots, (v'_{h'}, l'_{h'}))$$

if

- $h > h'$;
- or $h = h'$ and $\sum(k_i + l_i) < \sum(k'_i + l'_i)$;
- or $h = h', \sum(k_i + l_i) = \sum(k'_i + l'_i)$ and

$$(u_h, k_h) \cdots (u_1, k_1)(v_h, l_h) \cdots (v_1, l_1) \prec (u'_{h'}, k'_{h'}) \cdots (u'_1, k'_1)(v'_{h'}, l'_{h'}) \cdots (v'_1, l'_1).$$

Here we order the words of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ lexicographically.

Lemma 2.3. If $E \leq E'$, then $\|E\| \prec \|E'\|$.

Proof. If $sz(E') < sz(E)$ or $sz(E) = sz(E')$ and $wt(E) < wt(E')$, then $\|E\| \prec \|E'\|$. If $sz(E) = sz(E')$ and $wt(E) = wt(E')$, we must have $k_i = k'_i$ and $l_j = l'_j$. So $u_i \leq u'_i$ and $v_j \leq v'_j$, we have $\|E\| \prec \|E'\|$. $\square$

Relations. Let

$$\bar{\partial}^k(u_h, \dots, u_1 | v_1, \dots, v_h) = 0$$

if there is $1 \leq i < j \leq h$ such that $u_i = u_j$ or $v_i = v_j$. For a $\bar{\partial}$ -list $J \in \mathcal{J}$ of the form (2.2), let

$$\bar{\partial}^k(u_{\sigma(h)}, \dots, u_{\sigma(2)}, u_{\sigma(1)} | v_{\sigma'(1)}, v_{\sigma'(2)}, \dots, v_{\sigma'(h)}) = \text{sign}(\sigma) \text{sign}(\sigma') J.$$

Here σ, σ' are permutations of $\{1, 2, \dots, h\}$, and $\text{sign}(\sigma), \text{sign}(\sigma')$ are the signs of these permutations. We have the following relations, which we will prove later in Section 6.

Lemma 2.4. *For $i_1, i_2, j_1, j_2, h, h', k_0, m \in \mathbb{Z}_{\geq 0}$ with $h \geq h', i_1, j_1 \leq h, i_2, j_2 \leq h'$ and $k_0 \leq m$, let $l_0 = i_1 + i_2 + j_1 + j_2 - 2h - 1$. Given any integers $a_k, k_0 \leq k \leq k_0 + l_0$, there are integers $a_k, 0 \leq k < k_0$ or $k_0 + l_0 < k \leq m$, such that*

$$(2.5) \quad \sum_{k=0}^m a_k \sum_{\sigma, \sigma'} \frac{1}{i_1! i_2! j_1! j_2!} \text{sign}(\sigma) \text{sign}(\sigma')$$

$$\left(\begin{array}{c|c} \bar{\partial}^{m-k}(u_h, \dots, u_{i_1+1}, \sigma(u_{i_1}), \dots, \sigma(u_1) & \sigma'(v_1), \dots, \sigma'(v_{j_1}), v_{j_1+1}, \dots, v_h) \\ \bar{\partial}^k(u'_{h'}, \dots, u'_{i_2+1}, \sigma(u'_{i_2}), \dots, \sigma(u'_1) & \sigma'(v'_1), \dots, \sigma'(v'_{j_2}), v'_{j_2+1}, \dots, v'_{h'}) \end{array} \right)$$

is in $\mathfrak{R}[h+1]$. Here the second summation is over all pairs of permutations σ of $u_{i_1}, \dots, u_1, u'_{i_2}, \dots, u'_1$ and permutations σ' of $v_{i_1}, \dots, v_1, v'_{i_2}, \dots, v'_1$, and $\text{sign}(\sigma)$ and $\text{sign}(\sigma')$ are the signs of the permutations.

For simplicity, we write Equation (2.5) in the following way,

$$(2.6) \quad \sum \epsilon a_k \left(\begin{array}{c|c} \bar{\partial}^{m-k}(u_h, \dots, u_{i_1+1}, \underline{u_{i_1}, \dots, u_1} & \underline{v_1, \dots, v_{j_1}}, v_{j_1+1}, \dots, v_h) \\ \bar{\partial}^k(u'_{h'}, \dots, u'_{i_2+1}, \underline{u'_{i_2}, \dots, u'_1} & \underline{v'_1, \dots, v'_{j_2}}, v'_{j_2+1}, \dots, v'_{h'}) \end{array} \right) \in \mathfrak{R}[h+1].$$

Remark 2.5. *Since the second summation in Equation (2.5) is over all permutations, each monomial in the equation will appear $i_1! i_2! j_1! j_2!$ times, so the coefficient of each monomial will be $\pm a_k$.*

Standard monomials. Now we give a definition of the standard monomials of $\mathcal{J}$.

Definition 2.6. *An ordered product $E_1 E_2 \cdots E_m$ of elements of $\mathcal{E}$ is said to be standard if*

- (1) $E_a \leq E_{a+1}, 1 \leq a < m$,
- (2) E_1 is the largest in $\mathcal{E}(\|E_1\|)$ under the order $\prec$, where $\mathcal{E}(\|E_1\|)$ is defined by (2.4),
- (3) E_{a+1} is the largest in $\mathcal{E}(\|E_{a+1}\|)$ such that $E_a \leq E_{a+1}$.

An ordered product $J_1 J_2 \cdots J_m$ of elements of $\mathcal{J}$ is said to be standard if there is a standard ordered product $E_1 E_2 \cdots E_m$ such that $E_i \in \mathcal{E}(J_i)$.

Let $\mathcal{SM}(\mathcal{J}) \subset \mathcal{M}(\mathcal{J})$ be the set of standard monomials of $\mathcal{J}$;
let $\mathcal{SM}(\mathcal{E}) \subset \mathcal{M}(\mathcal{E})$ be the set of standard monomials of $\mathcal{E}$;
let $\mathcal{SM}(\mathcal{J}_h) = \mathcal{M}(\mathcal{J}_h) \cap \mathcal{SM}(\mathcal{J})$ be the set of standard monomials of $\mathcal{J}_h$;
let $\mathcal{SM}(\mathcal{E}_h) = \mathcal{M}(\mathcal{E}_h) \cap \mathcal{SM}(\mathcal{E})$ be the set of standard monomials of $\mathcal{E}_h$.

By Definition 2.6, if $J_1 J_2 \cdots J_m$ is a standard monomial, the standard monomial $E_1 \cdots E_m \in \mathcal{SM}(\mathcal{E})$ corresponding to $J_1 \cdots J_m$ is unique and E_1 has the form

$$((u_h, wt(E_1)), (u_{h-1}, 0), \dots, (u_1, 0) | (v_1, 0), \dots, (v_h, 0)) \in \mathcal{E}$$

with $u_i < u_{i+1}$ and $v_i < v_{i+1}$. So the map

$$\pi_h : \mathcal{SM}(\mathcal{E}_h) \rightarrow \mathcal{SM}(\mathcal{J}_h), \quad E_1 E_2 \cdots E_m \mapsto ||E_1|| ||E_2|| \cdots ||E_m||$$

is a bijection.

We order $\mathcal{M}(\mathcal{J})$, the set of ordered products of elements of $\mathcal{J}$, lexicographically. The following lemma will be proved later in Section 7.

Lemma 2.7. *If $J_1 \cdots J_b \in \mathcal{M}(\mathcal{J})$ is not standard, J can be written as a linear combination of elements of $\mathcal{M}(\mathcal{J})$ preceding $J_1 \cdots J_{b-1}$ with integer coefficients.*

Recall that $\mathfrak{R}[h]$ denotes the ideal generated by $J \in \mathcal{J}$ with $sz(J) = h$, and $\mathfrak{R}_h = \mathfrak{R}/\mathfrak{R}[h+1]$, as in (1.5). If $h \geq \min\{p, q\}$, then $\mathcal{J}_h = \mathcal{J}$ and $\mathfrak{R}_h = \mathfrak{R}$. By the above lemma, we immediately have

Lemma 2.8. *Any element of $\mathfrak{R}_h$ can be written as a linear combination of standard monomials of $\mathcal{J}_h$ with integer coefficients.*

Proof. We only need to show that any element of $\mathfrak{R}$ can be written as a linear combination of standard monomials of $\mathcal{J}$ with integer coefficients. Recall that $\mathcal{J}$ generates $\mathfrak{R}$. If the lemma is not true, there must be a smallest element $J \in \mathcal{M}(\mathcal{J})$, which cannot be written as a linear combination of elements of $\mathcal{SM}(\mathcal{J})$ with integer coefficients. So J is not standard. By Lemma 2.7, $J = \sum_{\alpha} \pm J_{\alpha}$ with $J_{\alpha} \in \mathcal{M}(\mathcal{J})$ and $J_{\alpha} \prec J$. Since J_{α} can be written as a linear combination of elements of $\mathcal{SM}(\mathcal{J})$ with integer coefficients, J can also be written as such a linear combination, which is a contradiction. $\square$

3. A CANONICAL BASIS

A ring homomorphism. Let

$$\mathcal{S}_h = \{a_{il}^{(k)}, b_{jl}^{(k)} \mid 1 \leq i \leq p, 1 \leq j \leq q, 1 \leq l \leq h, k \in \mathbb{Z}_{\geq 0}\},$$

and let

$$(3.1) \quad \mathfrak{B} = \mathbb{Z}[\mathcal{S}_h],$$

the polynomial ring generated by $\mathcal{S}_h$. For later use, we mention that for a field K , if $W = K^{\oplus h}$, and $V = W^{\oplus p} \oplus W^{*\oplus q}$, the affine coordinate ring $K[J_{\infty}(V)]$ is obtained from $\mathfrak{B}$ by base change, i.e., $K[J_{\infty}(V)] = \mathfrak{B} \otimes_{\mathbb{Z}} K$.

Let ∂ be the derivation on $\mathfrak{B}$ given by $\partial a_{ij}^{(k)} = (k+1)a_{ij}^{(k+1)}$, $\partial b_{ij}^{(k)} = (k+1)b_{ij}^{(k+1)}$. We have a homomorphism of rings

$$\tilde{Q}_h : \mathfrak{R} \rightarrow \mathfrak{B}, \quad x_{ij}^{(k)} \mapsto \bar{\partial}^k \sum_{l=1}^h a_{il}^{(0)} b_{jl}^{(0)}.$$

For any $J \in \mathcal{J}$ with $sz(J) > h$, we have $\tilde{Q}_h(J) = 0$, so $\tilde{Q}_h$ induces a ring homomorphism

$$(3.2) \quad Q_h : \mathfrak{R}_h \rightarrow \mathfrak{B}.$$

Double tableaux. Let $\tilde{\mathcal{S}}_h = \mathcal{S}_h \cup \{*\}$. We define an ordering on the set $\tilde{\mathcal{S}}_h$: for $X_{ij}^{(k)}, Y_{i'j'}^{(k')} \in \mathcal{S}_h$, $X_{ij}^{(k)} < * \text{ and } X_{ij}^{(k)} \geq Y_{i'j'}^{(k')}$ if

- $X = a, Y = b$;
- or $X = Y, k > k'$;
- or $X = Y, k = k', i > i'$;
- or $X = Y, k = k', i = i', j \geq j'$.

We use double tableaux to represent the monomials of $\mathfrak{B}$. Let $\mathcal{T}$ be the set of the following double tableaux:

$$(3.3) \quad \left| \begin{array}{ccc|ccc} y_{1,h}, \dots, y_{1,2}, y_{1,1} & & & z_{1,1}, z_{1,2}, \dots, z_{1,h} \\ & \vdots & & \vdots \\ y_{m,h}, \dots, y_{m,2}, y_{m,1} & & & z_{m,1}, z_{m,2}, \dots, z_{m,h} \end{array} \right|.$$

Here $y_{s,l}$ are some $a_{il}^{(k)}$ or $*$ and $z_{s,l}$ are some $b_{jl}^{(k)}$ or $*$; every row of the tableau has elements in $\mathcal{S}_h$; and

$$y_{s,j} \leq y_{s+1,j}, \quad z_{s,j} \leq z_{s+1,j}.$$

We use the tableau (3.3) to represent a monomial in $\mathfrak{B}$, which is the product of $a_{ij}^{(k)}$'s and $b_{ij}^{(k)}$'s in the tableau. It is easy to see that the representation is a one-to-one correspondence between $\mathcal{T}$ and the set of monomials of $\mathfrak{B}$. We associate to the tableau (3.3) the word:

$$y_{1,h} \cdots y_{1,1} z_{1,h} \cdots z_{1,1} y_{2,h} \cdots y_{2,1} z_{2,h} \cdots z_{2,1} \cdots z_{m,h} \cdots z_{m,1}$$

and order these words lexicographically. For a polynomial $f \in \mathfrak{B}$, let $Ld(f)$ be its leading monomial in f under the order we defined on $\mathcal{T}$.

For $E_i = ((u_{h_1}^i, k_{h_1}^i), \dots, (u_2^i, k_2^i), (u_1^i, k_1^i) | (v_1^i, l_1^i), (v_2^i, l_2^i), \dots, (v_{h_1}^i, l_{h_1}^i)) \in \mathcal{E}$, $1 \leq i \leq m$, we use a double tableau to represent $E_1 \cdots E_m \in \mathcal{SM}(\mathcal{E})$,

$$(3.4) \quad \left(\begin{array}{ccc|ccc} (u_{h_1}^1, k_{h_1}^1), \dots, (u_2^1, k_2^1), (u_1^1, k_1^1) & & & (v_1^1, l_1^1), (v_2^1, l_2^1), \dots, (v_{h_1}^1, l_{h_1}^1) \\ (u_{h_2}^2, k_{h_2}^2), \dots, (u_2^2, k_2^2), (u_1^2, k_1^2) & & & (v_1^2, l_1^2), (v_2^2, l_2^2), \dots, (v_{h_2}^2, l_{h_2}^2) \\ & \vdots & & \vdots \\ (u_{h_m}^m, k_{h_m}^m), \dots, (u_2^m, k_2^m), (u_1^m, k_1^m) & & & (v_1^m, l_1^m), (v_2^m, l_2^m), \dots, (v_{h_m}^m, l_{h_m}^m) \end{array} \right).$$

Let $T : \mathcal{SM}(\mathcal{J}_h) \rightarrow \mathcal{T}$ with

$$T(E_1 \cdots E_m) = \left(\begin{array}{c|c} *, \dots, *, a_{u_{h_1}^1 h_1}^{(k_{h_1}^1)}, \dots, a_{u_1^1 1}^{(k_1^1)} & b_{v_1^1 1}^{(l_1^1)}, \dots, b_{v_{h_1}^1 h_1}^{(l_{h_1}^1)}, *, \dots, * \\ *, \dots, *, a_{u_{h_2}^2 h_2}^{(k_{h_2}^2)}, \dots, a_{u_1^2 1}^{(k_1^2)} & b_{v_1^2 1}^{(l_1^2)}, \dots, b_{v_{h_2}^2 h_2}^{(l_{h_2}^2)}, *, \dots, * \\ \vdots & \vdots \\ *, \dots, *, a_{u_{h_m}^m h_m}^{(k_{h_m}^m)}, \dots, a_{u_1^m 1}^{(k_1^m)} & b_{v_1^m 1}^{(l_1^m)}, \dots, b_{v_{h_m}^m h_m}^{(l_{h_m}^m)}, *, \dots, * \end{array} \right).$$

Obviously, T is an injective map and $T(E_1) \prec T(E_2)$ if $E_1 \prec E_2$.

Lemma 3.1. *Let $J_1 \cdots J_m \in \mathcal{SM}(\mathcal{J}_h)$ and $E_1 \cdots E_m \in \mathcal{SM}(\mathcal{E}_h)$ be its associated standard monomial. Assume the double tableau representing $E_1 \cdots E_m$ is (3.4). Then the leading monomial of $Q_h(J_1 \cdots J_m)$ is represented by the double tableau $T(E_1 E_2 \cdots E_m)$. Thus*

$$Ld \circ Q_h = T \circ \pi_h^{-1} : \mathcal{SM}(\mathcal{J}_h) \rightarrow \mathcal{T}$$

is injective. Moreover, the coefficient of the leading monomial of $Q_h(J_1 \cdots J_m)$ is ± 1 .

Proof. Let W_m be the monomial corresponding to the tableau $T(E_1 \cdots E_m)$. Let

$$M_m = a_{u_{h_m}^m h_m}^{(k_{h_m}^m)} \cdots a_{u_1^m 1}^{(k_1^m)} b_{v_1^m 1}^{(l_1^m)} \cdots b_{v_{h_m}^m h_m}^{(l_{h_m}^m)}$$

be the monomial corresponding to the double tableau $T(E_m)$. Then $W_m = W_{m-1} M_m$. We prove the lemma by induction on m . If $m = 1$, the lemma is obviously true. Assume the lemma is true for $J_1 \cdots J_{m-1}$, then its leading monomial $Ld(Q_h(J_1 \cdots J_{m-1})) = W_{m-1}$, the monomial corresponding to $T(E_1 \cdots E_{m-1})$, and the coefficient of W_{m-1} in $Q_h(J_1 \cdots J_{m-1})$ is ± 1 .

$$Q_h(J_m) = \sum \pm a_{u_1^m s_1}^{(k_1)} a_{u_2^m s_2}^{(k_2)} \cdots a_{u_{h_m}^m s_{h_m}}^{(k_{h_m})} b_{v_1^m t_1}^{(l_1)} b_{v_2^m t_2}^{(l_2)} \cdots b_{v_{h_m}^m t_{h_m}}^{(l_{h_m})}.$$

The summation is over all $l_i, k_i \geq 0$ with $\sum (l_i + k_i) = wt(E_m)$, all s_i with $1 \leq s_1, s_2, \dots, s_{h_m} \leq h$ and they are different from each other, and all $t_1, \dots, t_{h_m}$, which are permutations of $s_1, s_2, \dots, s_{h_m}$. M_m is one of the monomials in $Q_h(J_m)$ with coefficient ± 1 . All of the monomials in the polynomial $Q_h(J_1 \cdots J_{m-1})$ except W_{m-1} are less than W_{m-1} , so any monomial in $Q_h(J_1 \cdots J_{m-1})$ except W_{m-1} times any monomial in $Q_h(J_m)$, is less than W_{m-1} . Since $W_{m-1} \prec W_m$, the coefficient of W_m in $Q_h(J_1 \cdots J_m)$ is not zero. Now

$$W_{m-1} \prec W_m \prec Ld(Q_h(J_1 \cdots J_m)).$$

The leading monomial $Ld(Q_h(J_1 \cdots J_m))$ must have the form

$$W = W_{m-1} a_{u_1^m s_1}^{(k_1)} a_{u_2^m s_2}^{(k_2)} \cdots a_{u_{h_m}^m s_{h_m}}^{(k_{h_m})} b_{v_1^m t_1}^{(l_1)} b_{v_2^m t_2}^{(l_2)} \cdots b_{v_{h_m}^m t_{h_m}}^{(l_{h_m})}.$$

If some s_i or t_i greater than h_{m-1} , then $W \prec W_{m-1}$. If there is some $h_{m-1} \geq s_i > h_m$, there is $1 \leq j \leq h_m$, with $j \notin \{s_1, \dots, s_{h_m}\}$, if we replace s_i by j in W , we get a larger monomial in $Q_h(J_1 \cdots J_m)$. So we can assume

$s_1, \dots, s_{h_m}$ is a permutation of $1, 2, \dots, h_m$. We must have $a_{u_i^m s_i}^{(k_i)} \geq a_{u_{s_i}^{m-1} s_i}^{(k_{s_i}^{m-1})}$ and $b_{v_i^m t_i}^{(k_i)} \geq b_{v_{t_i}^{m-1} t_i}^{(k_{t_i}^{m-1})}$, otherwise $W \prec W_{m-1}$. These kind of monomials in $Q_h(J_m)$ are in one-to-one correspondence with $E'_m \in \mathcal{E}(J_m)$ such that $E_{m-1} \leq E'_m$. Finally, E_m is the largest in $\mathcal{E}(J_m)$ with $E_{m-1} \leq E_m$ since E is standard, so W_m is the leading term of $Q_h(J_1 \cdots J_m)$. The coefficient of W_m in $Q_h(J_1 \cdots J_m)$ is ± 1 since the coefficients of W_{m-1} in $Q_h(J_1 \cdots J_{m-1})$ and M_m in $Q_h(J_m)$ are ± 1 . $\square$

Proof of Theorem 1.2. By Lemma 3.1, $Ld(Q_h(\mathcal{SM}(\mathcal{J}_h)))$ are linearly independent, so $\mathcal{SM}(\mathcal{J}_h)$ is a linearly independent set. By Lemma 2.8, $\mathcal{SM}(\mathcal{J}_h)$ generates $\mathfrak{R}_h$. So $\mathcal{SM}(\mathcal{J}_h)$ is a $\mathbb{Z}$ -basis of $\mathfrak{R}_h$. $\square$

Theorem 3.2. $Q_h : \mathfrak{R}_h \rightarrow \mathfrak{B}$ is injective. So we may identify $\mathfrak{R}_h$ with the image $Im(Q_h)$, which is the subring of $\mathcal{B}$ generated by $\bar{\partial}^k \sum_{i=1}^r a_{il}^{(0)} b_{jl}^{(0)}$. In particular, $Q_h(\mathcal{SM}(\mathcal{J}_h))$ is a $\mathbb{Z}$ -basis of $Im(Q_h)$.

Proof. By Lemma 3.1, $Ld(Q_h(\mathcal{SM}(\mathcal{J}_h)))$ are linearly independent. Since $\mathcal{SM}(\mathcal{J}_h)$ is a $\mathbb{Z}$ -basis of $\mathfrak{R}_h$, $Q_h : \mathfrak{R}_h \rightarrow \mathfrak{B}$ is injective. $\square$

Since Q_h is injective and $\mathfrak{B}$ is an integral domain, we obtain

Corollary 3.3. $\mathfrak{R}_h$ is an integral domain.

4. APPLICATION

In this section, we give the main application of the standard monomial basis we have constructed, which is the arc space analogue of Theorem 1.1.

Arc spaces. Suppose that X is a scheme of finite type over K . Its arc space (cf.[11]) $J_\infty(X)$ is determined by its functor of points. For every K -algebra A , we have a bijection

$$\mathrm{Hom}(\mathrm{Spec} A, J_\infty(X)) \cong \mathrm{Hom}(\mathrm{Spec} A[[t]], X).$$

If $i : X \rightarrow Y$ is a morphism of schemes, we get a morphism of schemes $i_\infty : J_\infty(X) \rightarrow J_\infty(Y)$. If i is a closed immersion, then i_∞ is also a closed immersion.

If $X = \mathrm{Spec} K[x_1, \dots, x_n]$, then $J_\infty(X) = \mathrm{Spec} K[x_i^{(k)} | 1 \leq i \leq n, k \in \mathbb{Z}_{\geq 0}]$. The identification is made as follows: for a K -algebra A , a morphism $\phi : K[x_1, \dots, x_n] \rightarrow A[[t]]$ determined by $\phi(x_i) = \sum_{k=0}^{\infty} a_i^{(k)} t^k$ corresponds to a morphism $K[x_i^{(k)}] \rightarrow A$ determined by $x_i^{(k)} \rightarrow a_i^{(k)}$. Note that $K[x_1, \dots, x_n]$ can naturally be identified with the subalgebra $K[x_1^{(0)}, \dots, x_n^{(0)}]$ of $K[x_i^{(k)}]$, and from now on we use $x_i^{(0)}$ instead of x_i .

The polynomial ring $K[x_i^{(k)}]$ has a derivation ∂ defined on generators by

$$(4.1) \quad \partial x_i^{(k)} = (k+1)x_i^{(k+1)}.$$

It is more convenient to work with the normalized k -derivation $\frac{1}{k!}\partial^k$, but this is a priori not well-defined on $K[x_i^{(k)}]$ if $\text{char } K$ is positive. However, ∂ is well-defined on $\mathbb{Z}[x_i^{(k)}]$ and $\bar{\partial}^k = \frac{1}{k!}\partial^k$ maps $\mathbb{Z}[x_i^{(k)}]$ to itself, so for any K , there is an induced K -linear map

$$(4.2) \quad \bar{\partial}^k : K[x_i^{(k)}] \rightarrow K[x_i^{(k)}],$$

obtained by tensoring with K .

Proposition 4.1. *If X is the affine space $\text{Spec } K[x_1^{(0)}, \dots, x_n^{(0)}]/(f_1, \dots, f_r)$, then $J_\infty(X)$ is an affine space*

$$\text{Spec } K[x_1^{(0)}, \dots, x_n^{(0)}, \dots, x_i^{(k)}, \dots]/(f_1, \dots, f_r, \bar{\partial}^k f_1, \dots, \bar{\partial}^k f_r, \dots).$$

Proof. Let $\bar{\partial}^k : A[[t]] \rightarrow A[[t]]$ be a morphism of A -modules with $\bar{\partial}^k t^n = C_n^k t^{n-k}$. Then for any $a(t), b(t) \in A[[t]]$, we have

$$\bar{\partial}^n(a(t)b(t)) = \sum_{k=0}^n \bar{\partial}^k a(t) \bar{\partial}^{n-k} b(t),$$

and the coefficient of t^k in $a(t)$ is $\bar{\partial}^k a(t)|_{t=0}$. Any morphism

$$\phi : K[x_1^{(0)}, \dots, x_n^{(0)}] \rightarrow A[[t]]$$

determined by $\phi(x_i^{(0)}) = \sum_{k=0}^\infty a_i^{(k)} t^k$ induces a morphism

$$\tilde{\phi} : K[x_i^{(k)}] \rightarrow A[[t]], \quad \text{given by } x_i^{(k)} \mapsto \bar{\partial}^k \phi(x_i^{(0)}).$$

Then $\tilde{\phi} \bar{\partial}^k = \bar{\partial}^k \tilde{\phi}$ and $\tilde{\phi}(x_i^{(k)})|_{t=0} = a_i^{(k)}$.

For every $f \in K[x_i^{(0)}]$,

$$\bar{\partial}^k \phi(f)|_{t=0} = \tilde{\phi}(\bar{\partial}^k f)|_{t=0} = (\bar{\partial}^k f)(\tilde{\phi}(x_1^{(0)}), \dots, \tilde{\phi}(x_n^{(k)}))|_{t=0} = (\bar{\partial}^k f)(a_1^{(0)}, \dots, a_n^{(k)}),$$

we have

$$\phi(f) = \sum_{k=0}^\infty \bar{\partial}^k f(a_1^{(0)}, \dots, a_n^{(k)}) t^k.$$

It follows that ϕ induces a morphism $K[x_i^{(0)}]/(f_1, \dots, f_r) \rightarrow A[[t]]$ if and only if

$$\bar{\partial}^k f_i(a_1^{(0)}, \dots, a_n^{(k)}) = 0, \text{ for all } i = 1, \dots, r, k \geq 0.$$

□

If Y is the affine scheme $\text{Spec } K[y_1^{(0)}, \dots, y_m^{(0)}]/(g_1, \dots, g_s)$, a morphism $P : X \rightarrow Y$ induces a ring homomorphism $P^* : K[Y] \rightarrow K[X]$. Then the induced morphism of arc spaces $P_\infty : J_\infty(X) \rightarrow J_\infty(Y)$ is given by $P_\infty^*(y_i^{(k)}) = \bar{\partial}^k P^*(y_i^{(0)})$; in particular, P_∞^* commutes with $\bar{\partial}^k$ for all $k \geq 0$.

Arc space of the determinantal variety. Recall that the space $M_{p,q}$ of $p \times q$ matrices over K has affine coordinate ring

$$K[M_{p,q}] = K[x_{ij}^{(0)} \mid 1 \leq i \leq p, 1 \leq j \leq q],$$

which is just $R \otimes_{\mathbb{Z}} K$, where R is given by (1.1). The determinantal variety D_h is the subvariety of $M_{p,q}$ determined by the ideal $K[M_{p,q}][h]$ generated by all h -minors, so $K[D_h] = K[M_{p,q}]/K[M_{p,q}][h] = R_{h-1} \otimes_{\mathbb{Z}} K$, where R_{h-1} is given by (1.3). Similarly, recall that

$$K[J_{\infty}(M_{p,q})] = K[x_{ij}^{(k)} \mid 1 \leq i \leq p, 1 \leq j \leq q, k \in \mathbb{Z}_{\geq 0}] = \mathfrak{R} \otimes_{\mathbb{Z}} K,$$

where $\mathfrak{R}$ is given by (1.4). Then

$$K[J_{\infty}(D_h)] = K[J_{\infty}(M_{p,q})]/K[J_{\infty}(M_{p,q})][h],$$

where $K[J_{\infty}(M_{p,q})][h]$ is the ideal generated by the elements $\bar{\partial}^n J$, where J is an h -minor. Note that $K[J_{\infty}(D_h)] = \mathfrak{R}_{h-1} \otimes_{\mathbb{Z}} K$, where $\mathfrak{R}_{h-1}$ is given by (1.5).

Proof of Corollary 1.3. By Theorem 1.2, $\mathcal{SM}(\mathcal{J}_{h-1})$ is a $\mathbb{Z}$ -basis of $\mathfrak{R}_{h-1}$. So it is a K -basis of $K[J_{\infty}(D_h)]$. $\square$

Invariant theory for $J_{\infty}(GL_h(K))$. Let $G = GL_h(K)$ be the general linear group of degree h over K . The group structure $G \times G \rightarrow G$ induces the group structure on its arc space

$$J_{\infty}(G) \times J_{\infty}(G) \rightarrow J_{\infty}(G),$$

so $J_{\infty}(G)$ is an algebraic group. Recall the G -modules $W = K^{\oplus h}$ and $V = W^{\oplus p} \oplus W^{*\oplus q}$. Recall that V has affine coordinate ring

$$K[V] = K[a_{il}^{(0)}, b_{jl}^{(0)} \mid 0 \leq i \leq p, 1 \leq j \leq p, 1 \leq l \leq h].$$

The action $G \times V \rightarrow V$ induces the action of $J_{\infty}(G)$ on $J_{\infty}(V)$,

$$J_{\infty}(G) \times J_{\infty}(V) \rightarrow J_{\infty}(V).$$

This induces an action of $J_{\infty}(G)$ on the affine coordinate ring

$$K[J_{\infty}(V)] = K[a_{il}^{(k)}, b_{jl}^{(k)} \mid 0 \leq i \leq p, 1 \leq j \leq p, 1 \leq l \leq h, k \in \mathbb{Z}_{\geq 0}],$$

which is identified with $\mathfrak{B} \otimes_{\mathbb{Z}} K$ where $\mathfrak{B}$ is given by (3.1).

Recall the map $Q_h : \mathfrak{R}_h \rightarrow \mathfrak{B}$ given by (3.2). It extends naturally to a map

$$(4.3) \quad Q_h^K : K[J_{\infty}(D_{h+1})] \rightarrow K[J_{\infty}(V)],$$

where $K[J_{\infty}(D_{h+1})]$ and $K[J_{\infty}(V)]$ are identified with $\mathfrak{R}_h \otimes_{\mathbb{Z}} K$ and $\mathfrak{B} \otimes_{\mathbb{Z}} K$, respectively, and $Q_h^K = Q_h \otimes Id$.

Theorem 4.2. Q_h^K is injective, so we may identify $K[J_{\infty}(D_{h+1})]$ with the subring $\text{Im}(Q_h^K)$ of $K[J_{\infty}(V)]$. In particular, $K[J_{\infty}(D_{h+1})]$ is integral.

Proof. By Lemma 3.1, $Ld(Q_h(\mathcal{SM}(\mathcal{J}_h)))$ are linearly independent. By Corollary 1.3, $\mathcal{SM}(\mathcal{J}_h)$ is a K -basis of $\mathfrak{R}_h$, so Q_h^K is injective. Since $K[J_\infty(V)]$ is integral, so is $K[J_\infty(D_{h+1})]$. $\square$

Remark 4.3. In general, if $\text{char } K = 0$, the arc space of an integral scheme is irreducible [14], but it may not be reduced. The determinantal varieties are examples whose arc spaces are integral.

If $p, q \geq h$, let $\Delta = Q_h^K((h, \dots, 1|1, \dots, h))$. Let $K[J_\infty(V)]_\Delta$ and $\text{Im}(Q_h^K)_\Delta$ be the localizations of $K[J_\infty(V)]$ and $\text{Im}(Q_h^K)$ at Δ , respectively.

Lemma 4.4. If $p, q \geq h$,

$$K[J_\infty(V)]_\Delta^{J_\infty(GL_h(K))} = \text{Im}(Q_h^K)_\Delta.$$

Proof. Let $K[V]_\Delta$ be the localization of $K[V]$ at Δ and $V_\Delta = \text{Spec } K[V]_\Delta$. Let H be the subvariety of V_Δ given by the ideal generated by $a_{il} - \delta_i^l$ with $1 \leq i, l \leq h$. The composition of the imbedding $\iota : H \hookrightarrow V_\Delta$ and the affine quotient $q : V_\Delta \rightarrow V_\Delta // G = (V // G)_\Delta$ gives the isomorphism $q \circ \iota : H \rightarrow (V // G)_\Delta$. So as morphisms of their arc spaces,

$$q_\infty \circ \iota_\infty : J_\infty(H) \rightarrow J_\infty((V // G)_\Delta) = J_\infty(V // G)_\Delta.$$

The map q_∞ induces a morphism $\bar{q}_\infty : J_\infty(V_\Delta) // J_\infty(G) \rightarrow J_\infty(V // G)_\Delta$. The action of G on V gives a G -equivariant isomorphism

$$G \times H \rightarrow V_\Delta.$$

So we have a $J_\infty(G)$ -equivariant isomorphism

$$J_\infty(G) \times J_\infty(H) \rightarrow J_\infty(V_\Delta) = J_\infty(V)_\Delta.$$

and an isomorphism of their affine quotients

$$i : J_\infty(H) = J_\infty(G) \times J_\infty(H) // G \cong J_\infty(V) // J_\infty(G).$$

$\bar{q}_\infty \circ i = q_\infty \circ \iota_\infty$ is an isomorphism, so $\bar{q}_\infty$ is an isomorphism since i is an isomorphism, which is equivalent to the lemma. $\square$

Theorem 4.5. $K[J_\infty(V)]^{J_\infty(GL_h(K))} = \text{Im}(Q_h^K)$.

Proof. If $p, q \geq h$, we regard $K[J_\infty(V)]$ and $\text{Im}(Q_h^K)_\Delta$ as subrings of $K[J_\infty(V)]_\Delta$. By Lemma 4.4, we have

$$K[J_\infty(V)]_\Delta^{J_\infty(G)} = K[J_\infty(V)]_\Delta \cap \text{Im}(Q_h^K)_\Delta.$$

Now for any $f \in K[J_\infty(V)]_\Delta \cap \text{Im}(Q_h^K)_\Delta$, $f = \frac{g}{\Delta^n}$ with $\Delta^n f = g \in \text{Im}(Q_h^K)$. The leading monomial of g is

$$Ld(g) = (a_{11}^{(0)} \cdots a_{hh}^{(0)} b_{11}^{(0)} \cdots b_{hh}^{(0)})^n Ld(f)$$

with coefficient $C_0 \neq 0$. Since $g \in \text{Im}(Q_h^K)$, there is a standard monomial $J \in \mathcal{SM}(\mathcal{J}_h)$, with $Ld(Q_h(J)) = Ld(g)$. Since J has the factor $(h, \dots, 1|1, \dots, h)^n$, $Q_h^K(J)$ has the factor Δ^n . Thus $f - C_0 \frac{Q_h^K(J)}{\Delta^n} \in K[J_\infty(V)]_\Delta \cap \text{Im}(Q_h^K)_\Delta$ with a

lower leading monomial and $\frac{Q_h^K(J)}{\Delta^n} \in \text{Im}(Q_h^K)$. By induction on the leading monomial of f , $f \in \text{Im}(Q_h^K)$. So

$$K[J_\infty(V)] \cap \text{Im}(Q_h^K)_\Delta = \text{Im}(Q_h^K),$$

and $K[J_\infty(V)]^{J_\infty(G)} = \text{Im}(Q_h^K)$.

More generally, let $V' = W^{\oplus p+h} \oplus (W^*)^{\oplus q+h}$, where $W = K^{\oplus h}$ as before. Its arc space has affine coordinate ring

$$K[J_\infty(V')] = K[a_{il}^{(k)}, b_{jl}^{(k)} \mid 1 \leq i \leq p+h, 1 \leq j \leq q+h, k \in \mathbb{Z}_{\geq 0}],$$

which contains $K[J_\infty(V)]$ as a subalgebra, and has an action of $J_\infty(G)$. By the above argument, $K[J_\infty(V')]^{J_\infty(G)}$ is generated by $X_{ij}^{(k)} = \bar{\partial}^k \sum_l a_{il} b_{jl}$. Let $\mathcal{I}$ be the ideal of $K[J_\infty(V')]$ generated by $a_{il}^{(k)}, b_{jl}^{(k)}$ with $i > p, j > q$. Then

$$K[J_\infty(V')] = K[J_\infty(V)] \oplus \mathcal{I}.$$

Note that $K[J_\infty(V)]$ and $\mathcal{I}$ are $J_\infty(G)$ -invariant subspaces of $K[J_\infty(V')]$, and

$$K[J_\infty(V')]^{J_\infty(G)} = K[J_\infty(V)]^{J_\infty(G)} \oplus \mathcal{I}^{J_\infty(G)}.$$

If $i > p$ or $j > q$, $X_{ij}^{(k)} \in \mathcal{I}^{J_\infty(G)}$. So

$$K[J_\infty(V)]^{J_\infty(G)} \cong K[J_\infty(V')]^{J_\infty(G)} / \mathcal{I}^{J_\infty(G)}$$

is generated by $X_{ij}^{(k)}, 1 \leq i \leq p, 1 \leq j \leq q$. It follows that $K[J_\infty(V)]^{J_\infty(G)} = \text{Im}(Q_h^K)$, as claimed. $\square$

Proof of Theorem 1.4. By Theorem 4.5 and Theorem 4.2, $K[J_\infty(V)]^{J_\infty(GL_h(K))} = \text{Im}(Q_h^K) \cong K[J_\infty(D_{h+1})]$. $\square$

Proof of Corollary 1.5. This is immediate from Theorem 1.4 because $V // GL_h(K)$ is isomorphic to the determinantal variety D_{h+1} . $\square$

5. SOME PROPERTIES OF STANDARD MONOMIALS

By the definition of standard monomials, if $E_1 E_2 \cdots E_n \in \mathcal{SM}(\mathcal{E})$, then E_{i+1} is the largest element in $\|\mathcal{E}(E_{i+1})\|$ such that $E_i \leq E_{i+1}$. In this section, we study the properties of $\|\mathcal{E}(E_{i+1})\|$ and E_{i+1} that need to be satisfied to make $E_1 E_2 \cdots E_n$ a standard monomial.

Let

$$E = ((u_h, k_h), \dots, (u_1, k_1) | (v_1, l_1), \dots, (v_h, l_h)) \in \mathcal{E},$$

$$J' = \bar{\partial}^{n'}(u'_{h'}, \dots, u'_1 | v'_1, \dots, v'_{h'}) \in \mathcal{J}.$$

$L(E, J')$ and $R(E, J')$. For $h' \leq h$, let σ_L and σ_R be the permutations of $\{1, 2, \dots, h'\}$ such that $u_{\sigma_L(i)} < u_{\sigma_L(i+1)}$ and $v_{\sigma_R(i)} < v_{\sigma_R(i+1)}$. Let $L(E, J')$ and $R(E, J')$ be the smallest non-negative integers i_0 and j_0 such that $u'_i \geq u_{\sigma_L(i-i_0)}$, $i_0 < i \leq h'$ and $v'_j \geq v_{\sigma_R(j-j_0)}$, $j_0 < j \leq h'$, respectively. Let

$$(5.1) \quad E(h') = ((u_{h'}, k_{h'}), \dots, (u_1, k_1) | (v_1, l_1), \dots, (v_{h'}, l_{h'})).$$

Then $L(E, J') = L(E(h'), J')$ and $R(E, J') = R(E(h'), J')$.

The following lemma is obvious.

Lemma 5.1. For $J'' = \bar{\partial}^k(u''_1, \dots, u''_1 | v''_1, \dots, v''_1) \in \mathcal{J}$, if there are at least s elements in $\{u''_1, \dots, u''_1\}$ from the set $\{u'_{h'}, \dots, u'_1\}$, then $L(E, J'') \geq L(E, J') - h' + s$; if there are at least s elements in $\{v''_1, \dots, v''_1\}$ from the set $\{v'_{h'}, \dots, v'_1\}$, then $R(E, J'') \geq R(E, J') - h' + s$.

A criterion for J' to be greater than E . We say J' is greater than E if there is an element $E' \in \mathcal{E}(J')$ with $E \leq E'$. Then J' is greater than E if and only if J' is greater than $E(h')$. The following lemma is a criterion for J' to be greater than E .

Lemma 5.2. J' is greater than E if and only if $wt(E(h')) - wt(J') \geq L(E, J') + R(E, J')$.

Proof. Let $i_0 = L(E, J')$ and $j_0 = R(E, J')$. Let σ and σ' be permutations of $\{1, 2, \dots, h'\}$ such that $u_{\sigma_L(i)} < u_{\sigma_L(i+1)}$ and $v_{\sigma_R(i)} < v_{\sigma_R(i+1)}$.

If $wt(E(h')) - wt(J') \geq L(E, J') + R(E, J')$, let

$$\tilde{u}'_{\sigma(i)} = \begin{cases} u'_{i+i_0}, & \sigma(i) + i_0 \leq h' \\ u'_{i+i_0-h_b}, & \sigma(i) + i_0 > h' \end{cases}, \quad k'_{\sigma(i)} = \begin{cases} k_{\sigma(i)}, & i + i_0 \leq h', i \neq h' \\ k_{\sigma(i)} + 1, & i + i_0 > h', i \neq h' \end{cases},$$

$$\tilde{v}'_{\sigma'(j)} = \begin{cases} v'_{j+j_0}, & j + j_0 \leq h' \\ v'_{j+j_0-h_b}, & j + j_0 > h' \end{cases}, \quad l'_{\sigma'(j)} = \begin{cases} l_{\sigma'(j)}, & j + j_0 \leq h' \\ l_{\sigma'(j)} + 1, & j + j_0 > h' \end{cases},$$

$$k'_{\sigma(h')} = wt(J') - \sum_{i=1}^{h'-1} k'_i - \sum_{j=1}^{h'} l'_j.$$

Then

$$k'_{\sigma(h')} = wt(J') - wt(E(h')) - i_0 - j_0 + k_{\sigma(h')} + 1 - \delta_{i_0}^0 \geq k_{\sigma(h')} + 1 - \delta_{i_0}^0.$$

$$(\tilde{u}'_{\sigma(i)}, k'_{\sigma(i)}) \geq (u_{\sigma(i)}, k_{\sigma(i)}), \quad (\tilde{v}'_{\sigma'(j)}, l'_{\sigma'(j)}) \geq (v_{\sigma'(j)}, l_{\sigma'(j)}).$$

So

$$\tilde{E}' = ((\tilde{u}'_{h'}, k'_{h'}), \dots, (\tilde{u}'_2, k'_2), (\tilde{u}'_1, k'_1) | (\tilde{v}'_1, l'_1), (\tilde{v}'_2, l'_2), \dots, (\tilde{v}'_{h'}, l'_{h'}))$$

is an element in $\mathcal{E}(J')$ with $\tilde{E}' \geq E$.

On the other hand, suppose that $\tilde{E}' \in \mathcal{E}(J')$ with $\tilde{E}' \geq E$. Assume

$$\tilde{E}' = ((\tilde{u}'_{h'}, k'_{h'}), \dots, (\tilde{u}'_2, k'_2), (\tilde{u}'_1, k'_1) | (\tilde{v}'_1, l'_1), (\tilde{v}'_2, l'_2), \dots, (\tilde{v}'_{h'}, l'_{h'})).$$

We have $(\tilde{u}'_i, k'_i) \geq (u_i, k_i)$ i.e. $k'_i > k_i$ or $k'_i = k_i, \tilde{u}'_i \geq u_i$ and $(\tilde{v}'_i, l'_i) \geq (v_i, v_i)$ for $1 \leq i \leq h'$ i.e. $l'_i > l_i$ or $l'_i = l_i, \tilde{v}'_i \geq v_i$. So

$$\sum_{i=1}^{h'} (k'_i - k_i) + \#\{\tilde{u}'_i \geq u_i, i | 1 \leq i \leq h'\} \geq h',$$

$$\sum_{i=1}^{h'} (l'_i - l_i) + \#\{\tilde{v}'_i \geq v_i, i | 1 \leq i \leq h'\} \geq h'.$$

Let $i'_0 = h' - \#\{\tilde{u}'_i \geq u_i, i | 1 \leq i \leq h'\}$ and $j'_0 = h' - \#\{\tilde{v}'_i \geq v_i, i | 1 \leq i \leq h'\}$. Then

$$i'_0 + j'_0 \leq \sum_{i=1}^{h'} (k'_i - k_i) + \sum_{i=1}^{h'} (l'_i - l_i) = wt(J') - wt(E(h')).$$

Here $\tilde{u}'_1, \dots, \tilde{u}'_{h'}$ is a permutation of $u'_1, \dots, u'_{h'}$ and $\tilde{v}'_1, \dots, \tilde{v}'_{h'}$ is a permutation of $v'_1, \dots, v'_{h'}$. By the definition of i'_0 and j'_0 , it is easy to see that $u'_i \geq u_{\sigma(i-i'_0)}$, $i'_0 < i \leq h'$ and $v'_j \geq v_{\sigma'(j-j'_0)}$, $j'_0 < j \leq h'$. So $i'_0 \geq L(E, J')$ and $j'_0 \geq R(E, J')$. Thus

$$wt(E(h')) - wt(J') \geq i'_0 + j'_0 \geq L(E, J') + R(E, J').$$

□

Corollary 5.3. J' is greater than E if and only if $\|E(h')\|J'$ is standard.

Proof. By Lemma 5.2, J' is greater than E if and only if $wt(E(h')) - wt(J') \geq L(E, J') + R(E, J')$ and $\|E(h')\|J'$ is standard if and only if $wt(E(h')) - wt(J') \geq L(E(h'), J') + R(E(h'), J') = L(E, J') + R(E, J')$. □

The property “largest”. Let

$$\mathcal{W}_s(E, J') = \{J = \bar{\partial}^k(u'_{i_s}, \dots, u'_{i_1} | v'_{j_1}, \dots, v'_{j_s}) | 1 \leq i_l, j_l \leq h', J \text{ is greater than } E\}.$$

Lemma 5.4. If E' is the largest element in $\mathcal{E}(J')$ such that $E \leq E'$, then for $s < h'$, $\|E'(s)\|$ is the smallest element in $\mathcal{W}_s(E, J')$.

Proof. Assume

$$E' = ((u'_{h'}, k_{h'}), \dots, (u'_2, k_2), (u'_1, k_1) | (v'_1, l'_1), (v'_2, l'_2), \dots, (v'_h, l'_h)).$$

For $s < h'$, let J_s be the smallest element in $\mathcal{W}_s(E, J')$. Let

$$E_s = ((u'_{i_s}, \tilde{k}_s), \dots, (u'_{i_2}, \tilde{k}_2), (u'_{i_1}, \tilde{k}_1) | (v'_{j_1}, \tilde{l}_1), (v'_{j_2}, \tilde{l}_2), \dots, (v'_{j_s}, \tilde{l}_s))$$

be the largest element in $\mathcal{E}(J_s)$ such that $E(s) \leq E_s$.

Assume l is the largest number such that $(u'_j, k'_j) = (u'_{i_j}, \tilde{k}_j)$ for $j < l \leq s + 1$.

- (1) If $l \leq s$, then $i_l \geq l$ and $(u'_{i_l}, \tilde{k}_l) \neq (u'_l, k'_l)$. If $i_l = l$, by the maximality of E' and the minimality of J_s , we must have $(u'_{i_l}, \tilde{k}_l) = (u'_l, k'_l)$. This is a contradiction, so $i_l > l$.

If $(u'_{i_l}, \tilde{k}_l) < (u'_l, k'_l)$, then $(u'_l, k'_l + k'_{i_l} - \tilde{k}_l) > (u'_{i_l}, k'_{i_l})$. Let E'' be the element in $\mathcal{E}(J')$ obtained by replacing (u'_l, k'_l) and (u'_{i_l}, k'_{i_l}) in E' by $(u'_{i_l}, \tilde{k}_l)$ and $(u'_l, k'_l + k'_{i_l} - \tilde{k}_l)$, respectively. We have $E' \prec E''$ and $E(h') \leq E''$. But $E' \neq E''$ is the largest element in $\mathcal{E}(\|E'\|)$ such that $E \leq E'$, which is a contradiction.

Assume $(u'_{i_l}, \tilde{k}_l) > (u'_l, k'_l)$. If $l \notin \{i_1, \dots, i_s\}$, replacing $(u'_{i_l}, \tilde{k}_l)$ in E_s by (u'_l, k'_l) , we get E'_s with $E(s) \leq E'_s$ and $\|E'_s\| \prec J_s$. This is impossible since $J_s \neq \|E'_s\|$ is the smallest element in $\mathcal{W}_s(E, E')$. If $l = i_j \in \{i_1, \dots, i_s\}$, $(u'_{i_l}, \tilde{k}_l + \tilde{k}_j - k'_l) > (u'_{i_j}, \tilde{k}_j)$. Let E'_s be the element in $\mathcal{E}(J_s)$ obtained by replacing $(u'_{i_l}, \tilde{k}_l)$ and $(u'_{i_j}, \tilde{k}_j)$ in E'_s by (u'_l, k'_l) and $(u'_{i_l}, \tilde{k}_l + \tilde{k}_j - k'_l)$, respectively. We have $E_s \prec E'_s$ and $E \leq E'_s$. But $E'_s \neq E_s$ is the largest element in $\mathcal{E}(J_s)$ such that $E \leq E'_s$, a contradiction.

(2) If $l = s + 1$, then the left part of E_s is equal to the left part of $E'(s)$.

Assume m is the largest number such that $(v'_i, l'_i) = (v'_{j_i}, \tilde{k}_i)$ for $i < m \leq s + 1$. By the same argument of (1), we can show that $m = s + 1$.

So $E_s = E'(s)$ and $\|E'(s)\|$ is the smallest element in $\mathcal{W}_s(E, J')$. $\square$

Corollary 5.5. *If E' is the largest element in $\mathcal{E}(\|E'\|)$ such that $E \leq E'$, then for $s < h'$,*

$$L(E, \|E'(s)\|) + R(E, \|E'(s)\|) = wt(E'(s)) - wt(E(s)).$$

Proof. Since $E \leq E'$, $E \leq E'(s)$. By Lemma 5.4, $\|E'(s)\|$ is the smallest element in $\mathcal{W}_s(E, J')$. By Lemma 5.2, $L(E, \|E'(s)\|) + R(E, \|E'(s)\|) = wt(E'(s)) - wt(E(s))$. $\square$

Corollary 5.6. *If E' is the largest element in $\mathcal{E}(\|E'\|)$ such that $E \leq E'$, then for $s < h'$ and any $J \in \mathcal{W}_s(E, E')$,*

$$L(E, \|E'(s)\|) \leq L(E, J), \quad R(E, \|E'(s)\|) \leq R(E, J).$$

Proof. Assume

$$J = \bar{\partial}^k(u_s, \dots, u_1 | v_1, \dots, v_s), \quad \|E'(s)\| = \bar{\partial}^l(u'_s, \dots, u'_1 | v'_1, \dots, v'_s).$$

If $m = L(E, J) - L(E, \|E'(s)\|) < 0$, let $J'' = \bar{\partial}^{l+m}(u_s, \dots, u_1 | v'_1, \dots, v'_s)$. By Lemma 5.2, $J'' \in \mathcal{W}_s(E, J')$. By Lemma 5.4, $\|E'(s)\|$ is the smallest element in $\mathcal{W}_s(E, \|E'\|)$. But $wt(\|E'(s)\|) > wt(J'')$, a contradiction. Similarly, we can show $R(E, \|E'(s)\|) \leq R(E, J)$. $\square$

Lemma 5.7. *Let $E_i = ((u_{h_i}^i, k_{h_i}^i), \dots, (u_1^i, k_1^i) | (v_1^i, l_1^i), \dots, (v_{h_i}^i, l_{h_i}^i))$, $i = a, b$. Suppose that $E_b \leq E_a$ and that E_a is the largest element in $\mathcal{E}(\|E_a\|)$ such that $E_b \leq E_a$. Let $1 \leq h < h_a$ and σ_i, σ'_i be permutations of $\{1, \dots, h\}$, such that $u_{\sigma_i(1)}^i < u_{\sigma_i(2)}^i < \dots < u_{\sigma_i(h)}^i$ and $v_{\sigma'_i(1)}^i < v_{\sigma'_i(2)}^i < \dots < v_{\sigma'_i(h)}^i$. Let $v'_1, \dots, v'_{h_a}$ be a permutation of $v_1^a, \dots, v_{h_a}^a$ such that $v'_1 < v'_2 < \dots < v'_{h_a}$. Let $u'_1, \dots, u'_{h_a}$ be a permutation of $u_1^a, \dots, u_{h_a}^a$ such that $u'_1 < u'_2 < \dots < u'_{h_a}$.*

(1) Assume $u'_{i_2} = u_{\sigma(i_1)}^a$ with $i_2 > i_1$, then for any

$$K = \bar{\partial}^k(u_h, \dots, u_{s+1}, u'_{t_s}, \dots, u'_{t_1} | v_1, \dots, v_h),$$

with $t_1 < t_2 < \dots < t_s < i_2$, $L(E_b, K) > L(E_b, ||E_a(h)||) + s - i_1$.

(2) Assume $v'_{j_2} = v_{\sigma(j_1)}^a$ with $j_2 > j_1$, then for any

$$K = \bar{\partial}^k(u_h, \dots, u_1 | v'_{t_1}, \dots, v'_{t_s}, v_{s+1}, \dots, v_h),$$

with $t_1 < t_2 < \dots < t_s < j_2$, $R(E_b, K) > R(E_b, ||E_a(h)||) + s - j_1$.

Proof. We prove (2). The proof for (1) is similar.

Let $n = wt(E_a(h))$.

Let $j_0 = R(E_b, K)$, then $v'_{t_j} \geq v_{\sigma_b(j-j_0)}^b$, for $s \geq j > j_0$.

Let $j'_0 = R(E_b, ||E_a(h)||)$, then $v_{\sigma_a(j)}^a \geq v_{\sigma_b(j-j'_0)}^b$, for $h \geq j > j'_0$.

We have $j_1 > j'_0$. Otherwise, $j_1 \leq j'_0$. Replacing $v_{\sigma_a(j_1)}^a$ in $||E^a(h)||$ by some $v_j^a < v_{\sigma_a(j_1)}^a$ with $j > h$, (such v_j^a exists since $j_2 > j_1$), we get

$$J = \bar{\partial}^n(u_{\sigma_a(h)}^a, \dots, u_{\sigma_a(1)}^a | v_{\sigma'_a(1)}^a, \dots, v_j^a, \dots, v_{\sigma'_a(h)}^a)$$

with $R(E_b, J) \leq j'_0$. By Lemma 5.2, J is greater than E_b . This is impossible by Lemma 5.4 since $J \prec ||E_a(h)||$ and E_a is the largest element in $\mathcal{E}(||E_a||)$ such that $E_b \leq E_a$.

If $s \geq j_1$, let

$$J' = \bar{\partial}^n(u_{\sigma_a(h)}^a, \dots, u_{\sigma_a(1)}^a | v'_{t_{s-j_1+1}}, \dots, v'_{t_s} v_{\sigma_a(j_1+1)}^a, \dots, v_{\sigma_a(h)}^a).$$

If $R(E_b, K) \leq R(E_b, ||E_a||) + s - j_1$,

$$v'_{t_j} \geq v_{\sigma_b(j-j_0)}^b \geq v_{\sigma_b(j-j'_0-s+j_1)}^b, \text{ for } j \leq s.$$

We have $R(E_b, J') \leq j'_0$. By Lemma 5.2, J' is greater than E_b . By Lemma 5.4, this is impossible since $J' \prec ||E_a(h)||$ and E_a is the largest element in $\mathcal{E}(||E_a||)$ such that $E_b \leq E_a$. So when $s \geq j_1$,

$$R(E_b, K) \geq R(E_b, ||E_a||) + s - j_1.$$

If $s < j_1$, let $t'_1 < t'_2 < \dots < t'_{j_1} < j_2$ with $\{t_1, \dots, t_s\} \subset \{t'_1, \dots, t'_{j_1}\}$. Let

$$K' = \bar{\partial}^k(u_h, \dots, u_1 | v'_{t'_1}, \dots, v'_{t'_{j_1}}, v_{j_1+1}, \dots, v_h).$$

By Lemma 5.1, we have

$$R(E_b, K) \geq E(E_b, K') + s - j_1 \geq R(E_b, ||E_a||) + s - j_1.$$

So for any $s > 0$, we have $R(E_b, K) > R(E_b, ||E_a||) + s - j_1$. $\square$

The following lemmas are obvious.

Lemma 5.8. If $J_1 \prec J_2 \prec \dots \prec J_n$, and σ is a permutation of $\{1, \dots, n\}$, then

$$J_1 J_2 \dots J_n \prec J_{\sigma(1)} \dots J_{\sigma(n)}.$$

Lemma 5.9. If $K_1 K_2 \dots K_k \prec J_1 \dots J_l$, then

$$K_1 K_2 \dots K_{s-1} J K_s \dots K_k \prec J_1 \dots J_{s-1} J J_s \dots J_l.$$

6. PROOF OF LEMMA (2.4)

In this section we will prove Lemma (2.4). Since the relation in Lemma (2.4) does not depend on p and q if $p, q \geq h + h'$, we can assume $p = q = s = h + h'$. Let $S = \{1, 2, \dots, s\}$. For a subset $N \subset S$, let $|N|$ be the number of elements of N . Let $\bar{N} = S \setminus N$. For $l \in S$, let ∂_l and ∂_{*l} be the differentials on $\mathfrak{R}$ given by

$$\partial_l x_{ij}^{(k)} = \delta_i^l (k+1) x_{ij}^{(k+1)}, \quad \partial_{*l} x_{ij}^{(k)} = \delta_j^l (k+1) x_{ij}^{(k+1)}.$$

Let

$$\partial_N = \sum_{l \in N} \partial_l, \quad \partial_{*N} = \sum_{l \in N} \partial_{*l}.$$

So $\partial = \partial_S = \partial_{*S}$. Let

$$\bar{\partial}_i^l = \frac{1}{l!} \partial_i^l, \quad \bar{\partial}_N^l = \frac{1}{l!} \partial_N^l, \quad \bar{\partial}_{*N}^l = \frac{1}{l!} \partial_{*N}^l.$$

For $a, b \in \mathfrak{R}$, we have $\bar{\partial}_N^l(ab) = \sum_{i=0}^l \bar{\partial}_N^i a \bar{\partial}_N^{l-i} b$ and $\bar{\partial}_N^l a$ and $\bar{\partial}_{*N}^l a \in \mathfrak{R}$.

If $I = \{i_1, \dots, i_k\} \subset S$ and $J = \{j_1, \dots, j_k\} \subset S$ with $i_a < i_{a+1}$ and $j_a < j_{a+1}$, let

$$\mathcal{A}(I, J) = \begin{vmatrix} x_{i_1 j_1}^{(0)} & x_{i_1 j_2}^{(0)} & \cdots & x_{i_1 j_k}^{(0)} \\ x_{i_2 j_1}^{(0)} & x_{i_2 j_2}^{(0)} & \cdots & x_{i_2 j_k}^{(0)} \\ \vdots & \vdots & \ddots & \vdots \\ x_{i_k j_1}^{(0)} & x_{i_k j_2}^{(0)} & \cdots & x_{i_k j_k}^{(0)} \end{vmatrix}$$

be the determinant of the $k \times k$ matrix with entries $x_{ij}^{(0)}$ for $i \in I$ and $j \in K$. For example,

$$\mathcal{A}(\{1, 2\}, \{1, 3\}) = x_{11}^{(0)} x_{23}^{(0)} - x_{13}^{(0)} x_{21}^{(0)}.$$

Let $\epsilon(I, J) = (-1)^{\sum_{i \in I} i + \sum_{j \in J} j}$.

Lemma 6.1. For $I, K, L \subset S$ with $L \subset I$, $|K| = |I| = k$ and $|L| \leq n \leq |I|$, if $l < 2n - |L|$,

$$\sum_{L \subset N \subset I, |N|=n} \bar{\partial}_N^l \mathcal{A}(I, K) \in \mathfrak{R}[k - n + 1].$$

Proof. We say $a \sim b$ if $a - b \in \mathfrak{R}[k - n + 1]$. It is an equivalence relation on $\mathfrak{R}$.

$$\bar{\partial}_N^l \mathcal{A}(I, K) = \sum_{\substack{\sum_{i \in N} l_i = l \\ l_i \geq 0}} \left(\prod_{i \in N} \bar{\partial}_i^{l_i} \right) \mathcal{A}(I, K).$$

We have the following properties.

Property 1: if there is some $i_0 \in N$ with $l_{i_0} = 0$, then $(\prod_{i \in N} \bar{\partial}_i^{l_i}) \mathcal{A}(I, K) \in \mathfrak{R}[k - n + 1]$.

Since

$$\left(\prod_{i \in N} \bar{\partial}_i^{l_i} \right) \mathcal{A}(I, K) = \sum_{J \subset K, |J|=k-n+1} \pm \mathcal{A}((I \setminus N) \cup \{i_0\}, J) \left(\prod_{i \in N} \bar{\partial}_i^{l_i} \right) \mathcal{A}(N \setminus \{i_0\}, K \setminus J).$$

Property 2: if $L \subset M \subset I$ with $|M| = m < n$ and $\sum_{i \in M} l_i = l - n + m$, then

$$(6.1) \quad \sum_{M \subset N \subset I, |N|=n} \left(\prod_{j \in N \setminus M} \partial_j \right) \left(\prod_{i \in M} \bar{\partial}_i^{l_i} \right) \mathcal{A}(I, K) \in \mathfrak{R}[k - n + 1].$$

Since on one hand by property 1,

$$\bar{\partial}_M^{n-m} \left(\prod_{i \in M} \bar{\partial}_i^{l_i} \right) \mathcal{A}(I, K) \sim \sum_{M \subset N \subset I, |N|=n} \left(\prod_{j \in N \setminus M} \partial_j \right) \left(\prod_{i \in M} \bar{\partial}_i^{l_i} \right) \mathcal{A}(I, K);$$

and on the other hand,

$$\begin{aligned} \bar{\partial}_M^{n-m} \left(\prod_{i \in M} \bar{\partial}_i^{l_i} \right) \mathcal{A}(I, K) &= \sum_{J \subset K, |J|=|M|} \pm \left(\prod_{i \in M} \bar{\partial}_i^{l_i} \right) \mathcal{A}(M, J) \bar{\partial}^{n-m} \mathcal{A}(I \setminus M, K \setminus J) \\ &\in \mathfrak{R}[k - m]. \end{aligned}$$

Now

$$\sum_{L \subset N \subset I, |N|=n} \bar{\partial}_N^l \mathcal{A}(I, K) = \sum_{L \subset N \subset I, |N|=n} \sum_{\sum_{i \in N} l_i = l} \left(\prod_{i \in N} \bar{\partial}_i^{l_i} \right) \mathcal{A}(I, K)$$

(take out $\bar{\partial}_j^{l_j}$ with $l_j = 1$ and $j \notin L$.)

$$= \sum_{L \subset N \subset I, |N|=n} \sum_{L \subset M \subset N} \sum_{\substack{l_i \neq 1, i \in M \setminus L; \\ \sum_{i \in M} l_i = l - n + |M|}} \left(\prod_{j \in N \setminus M} \partial_j \right) \left(\prod_{i \in M} \bar{\partial}_i^{l_i} \right) \mathcal{A}(I, K)$$

(switch the order of the summation)

$$= \sum_{L \subset M \subset I, |M| \leq n} \sum_{\substack{l_i \neq 1, i \in M \setminus L; \\ \sum_{i \in M} l_i = l - n + |M|}} \sum_{M \subset N \subset I, |N|=n} \left(\prod_{j \in N \setminus M} \partial_j \right) \left(\prod_{i \in M} \bar{\partial}_i^{l_i} \right) \mathcal{A}(I, K)$$

(by property 2)

$$\sim \sum_{L \subset M \subset I, |M|=n} \sum_{\substack{l_i \neq 1, i \in M \setminus L; \\ \sum_{i \in M} l_i = l}} \left(\prod_{i \in M} \bar{\partial}_i^{l_i} \right) \mathcal{A}(I, K)$$

(by property 1, since $l < 2n - |L|$, there must some $l_i = 0$)

$$\sim 0$$

□

Lemma 6.2. For $T, J, K \subset S$ with $J \cap T = \emptyset$, if $0 \leq a \leq l$, then

$$\sum_{\substack{|I|=|K| \\ J \subset I \subset \bar{T}}} \epsilon(I, K) \bar{\partial}^a \mathcal{A}(I, K) \bar{\partial}_T^{l-a} \mathcal{A}(\bar{I}, \bar{K}) \in \mathfrak{R}[s - |J| - |T| - a].$$

Proof. Let $\mathcal{B}(T, J, K)$ be the determinant of the $s \times s$ matrix with entries y_{ij} , where

$$y_{ij} = x_{ij}^{(0)}, \text{ if } (i, j) \notin (T \times K) \cup (J \times \bar{K}); \quad y_{ij} = 0, \text{ if } (i, j) \in (T \times K) \cup (J \times \bar{K}).$$

On one hand,

$$\bar{\partial}_T^{l-a} \bar{\partial}_{*K}^a \mathcal{B}(T, J, K) = \sum_{\substack{|I|=|K| \\ J \subset I \subset T}} \epsilon(I, K) \bar{\partial}^a \mathcal{A}(I, K) \bar{\partial}_T^{l-a} \mathcal{A}(\bar{I}, \bar{K}).$$

On the other hand $\bar{\partial}_T^{l-a} \bar{\partial}_{*K}^a \mathcal{B}(T, J, K)$

$$= \sum_{b=0}^a \sum_{I \subset \bar{K}, |I|=|T|} \sum_{M \subset K, |M|=|J|} \pm \bar{\partial}^{l-a} \mathcal{A}(T, I) \bar{\partial}^b \mathcal{A}(J, M) \bar{\partial}_{*K}^{a-b} \mathcal{A}(\bar{T} \cap \bar{J}, \bar{I} \cap \bar{M}).$$

It is easy to see that

$$\bar{\partial}_{*K}^{a-b} \mathcal{A}(\bar{T} \cap \bar{J}, \bar{I} \cap \bar{M}) \in \mathfrak{R}[s - |J| - |T| - a].$$

So

$$\sum_{\substack{|I|=|K| \\ J \subset I \subset T}} \epsilon(I, K) \bar{\partial}^a \mathcal{A}(I, K) \bar{\partial}_T^{l-a} \mathcal{A}(\bar{I}, \bar{K}) \in \mathfrak{R}[s - |J| - |T| - a].$$

□

Lemma 6.3. For $L, K \subset S$ with $|L|, |K| \leq s - n$, if $0 \leq l \leq 2(s - n) - |L| - |K| - 1$, then

$$\sum_{N \subset \bar{L}, |N|=n} \sum_{J \subset \bar{K}, |J|=n} \epsilon(N, J) \mathcal{A}(N, J) \bar{\partial}^l \mathcal{A}(\bar{N}, \bar{J}) \in \mathfrak{R}[n + 1].$$

Proof. Let $|K| = k$. For $N \subset S$ with $|N| = n$, let $\mathcal{D}(N, K)$ be the determinant of the $s \times s$ matrix with entries

$$y_{ij} = x_{ij}^{(0)}, \text{ if } i \notin N \text{ or } j \notin K; \quad y_{ij} = 0, \text{ if } i \in N \text{ and } j \in K.$$

We have

$$\mathcal{D}(N, K) = \sum_{J \subset \bar{K}, |J|=n} \epsilon(N, J) \mathcal{A}(N, J) \mathcal{A}(\bar{N}, \bar{J})$$

and

$$\mathcal{D}(N, K) = \sum_{I \subset \bar{N}, |I|=k} \epsilon(I, K) \mathcal{A}(I, K) \mathcal{A}(\bar{I}, \bar{K}).$$

By taking the summation of $\bar{\partial}^l \mathcal{D}(N, K)$ over $N \subset \bar{L}$, we have

$$\begin{aligned}
& \sum_{N \subset \bar{L}, |N|=n} \sum_{J \subset \bar{K}, |J|=n} \epsilon(N, J) \mathcal{A}(N, J) \bar{\partial}^l \mathcal{A}(\bar{N}, \bar{J}) \\
&= \sum_{N \subset \bar{L}, |N|=n} \bar{\partial}_N^l \mathcal{D}(N, K) \\
&= \sum_{N \subset \bar{L}, |N|=n} \sum_{I \subset \bar{N}, |I|=k} \sum_{a=0}^l \epsilon(I, K) \bar{\partial}^a \mathcal{A}(I, K) \bar{\partial}_{\bar{N} \cap \bar{I}}^{l-a} \mathcal{A}(\bar{I}, \bar{K}) \\
&\quad \text{(switching the order of the summation)} \\
&= \sum_{a=0}^l \sum_{I \subset S, |I|=k} \epsilon(I, K) \bar{\partial}^a \mathcal{A}(I, K) \sum_{N \subset \bar{I} \cap \bar{L}, |N|=n} \bar{\partial}_{\bar{N} \cap \bar{I}}^{l-a} \mathcal{A}(\bar{I}, \bar{K}).
\end{aligned}$$

If $l - a < 2(s - k - n) - |\bar{I} \cap L|$, let $J = \bar{N} \cap \bar{I}$. By Lemma 6.1,

$$\sum_{N \subset \bar{I} \cap \bar{L}, |N|=n} \bar{\partial}_{\bar{N} \cap \bar{I}}^{l-a} \mathcal{A}(\bar{I}, \bar{K}) = \sum_{L \cap \bar{I} \subset J \subset \bar{I}, |J|=s-k-n} \bar{\partial}_J^{l-a} \mathcal{A}(\bar{I}, \bar{K}) \in \mathfrak{R}[n+1].$$

If $l - a \geq 2(s - k - n) - |\bar{I} \cap L|$, let $J = I \cap L$, $T = \bar{N} \cap \bar{I}$. By Lemma 6.2,

$$\begin{aligned}
& \sum_{I \subset S, |I|=k} \epsilon(I, K) \bar{\partial}^a \mathcal{A}(I, K) \sum_{N \subset \bar{I} \cap \bar{L}, |N|=n} \bar{\partial}_{\bar{N} \cap \bar{I}}^{l-a} \mathcal{A}(\bar{I}, \bar{K}) \\
&= \sum_{\substack{J \subset L, T \subset \bar{J} \\ |T|=s-n-k}} \sum_{\substack{T \subset \bar{I} \subset \bar{J} \\ |I|=|K|}} \bar{\partial}^a \mathcal{A}(I, K) \bar{\partial}_T^{l-a} \mathcal{A}(\bar{I}, \bar{K}) \in \mathfrak{R}[n+1].
\end{aligned}$$

This completes the proof. $\square$

For $L_0, L_1, K_0, K_1 \subset S$ with $L_0 \cap L_1 = \emptyset = K_0 \cap K_1$, $|L_0| + |L_1| \leq s - n$ and $|K_0| + |K_1| \leq s - n$, let

$$\mathcal{F}_l^k(L_0, L_1, K_0, K_1, n) = \sum_{\substack{|N|=n \\ L_0 \subset N \subset \bar{L}_1}} \sum_{\substack{|J|=n \\ K_0 \subset J \subset \bar{K}_1}} \epsilon(N, J) \bar{\partial}^l \mathcal{A}(N, J) \bar{\partial}^{k-l} \mathcal{A}(\bar{N}, \bar{J}).$$

We have

(6.2)

$$\bar{\partial}^m \mathcal{F}_l^k(L_0, L_1, K_0, K_1, n) = \sum_{a=0}^m C_{l+a}^l C_{k+m-l-a}^{k-l} \mathcal{F}_{l+a}^{k+m}(L_0, L_1, K_0, K_1, n).$$

Lemma 6.4. *If $l \leq 2(s - n) - |L_0| - |L_1| - |K_0| - |K_1| - 1$, then*

$$\mathcal{F}_0^l(L_0, L_1, K_0, K_1, n) \in \mathfrak{R}[n+1].$$

Proof. We can show this by induction on $|L_0| + |K_0|$. If $|L_0| + |K_0| = 0$, $L_0 = K_0 = \emptyset$. By Lemma 6.3, the lemma is true.

Suppose the lemma is true for $|L_0| + |K_0| = m$. For $|L_0| + |K_0| = m + 1$, assume $L_0 \neq \emptyset$ (similarly for $K_0 \neq \emptyset$). Let $i_0 \in L_0$ and $L = L_0 \setminus \{i_0\}$, $\mathcal{F}_0^l(L_0, L_1, K_0, K_1, n) = \mathcal{F}_0^l(L, L_1, K_0, K_1, n) - \mathcal{F}_0^l(L, L_1 \cup \{i_0\}, K_0, K_1, n) \in \mathfrak{R}[n+1]$ by induction. $\square$

Now we have the relations for the determinant of the matrix.

Lemma 6.5. *For $L_0, L_1, K_0, K_1 \subset S$ with $L_0 \cap L_1 = \emptyset = K_0 \cap K_1$, $|L_0| + |L_1| \leq s - n$ and $|K_0| + |K_1| \leq s - n$, let $l_0 = 2(s - n) - |L_0| - |L_1| - |K_0| - |K_1| - 1$. Given a fixed integer $0 \leq k_0 \leq m - l_0$ and integers $a_{m, k_0+l}, 0 \leq l \leq l_0$, there are integers $a_{m, k}, 0 \leq k < k_0$ or $k_0 + l_0 < k \leq m$ such that*

$$\sum_{k=0}^m a_{m, k} \mathcal{F}_k^m(L_0, L_1, K_0, K_1, n) \in \mathfrak{R}[n+1].$$

Proof. For $0 \leq l \leq l_0$, by acting $\bar{\partial}^{m-l}$ on the relations of Lemma 6.4, we get

$$\bar{\partial}^{m-l} \mathcal{F}_0^l(L_0, L_1, K_0, K_1, n) = \sum_{k=0}^{m-l} C_{m-k}^l \mathcal{F}_k^m(L_0, L_1, K_0, K_1, n) \in \mathfrak{R}[n+1].$$

Now the $(l_0 + 1) \times (l_0 + 1)$ integer matrix with entries $c_{ij} = C_{m-k_0-i}^j, 0 \leq i, j \leq l_0$ is invertible since the determinant of this matrix is ± 1 . Let b_{ij} be the entries of the inverse matrix; clearly b_{ij} are integers. Let $a_{m, k} = \sum_{l=0}^{l_0} \sum_{j=0}^{l_0} C_{m-k}^l b_{l, j} a_{m, k_0+j}$. These integers satisfy the lemma. $\square$

Proof of Lemma 2.4. We only need to show the lemma when $v_i = u_i = i, v'_i = u'_i = h + i$. Let

$$\begin{aligned} L_0 &= \{i_1 + 1, \dots, h\}, & L_2 &= \{h + i_2 + 1, \dots, s\}, \\ K_0 &= \{j_1 + 1, \dots, h\}, & K_2 &= \{h + j_2 + 1, \dots, s\}. \end{aligned}$$

By the definition of $\mathcal{F}_l^k$,

$$\begin{aligned} \mathcal{F}_{m-k}^m(L_0, L_1, K_0, K_1, h) &= \sum_{\sigma, \sigma'} \frac{1}{i_1! i_2! j_1! j_2!} \text{sign}(\sigma) \text{sign}(\sigma') \\ &\left(\begin{array}{c|c} \bar{\partial}^{m-k}(u_h, \dots, u_{i_1+1}, \sigma(u_{i_1}), \dots, \sigma(u_1)) & \sigma'(v_1), \dots, \sigma'(v_{j_1}), v_{j_1+1}, \dots, v_h \\ \bar{\partial}^k(u'_{h'}, \dots, u'_{i_2+1}, \sigma(u'_{i_2}), \dots, \sigma(u'_1)) & \sigma'(v'_1), \dots, \sigma'(v'_{j_2}), v'_{j_2+1}, \dots, v'_{h'} \end{array} \right). \end{aligned}$$

Let $a_k = a_{m, m-k}$, by Lemma 6.5, we have Equation (2.5) $\square$

7. PROOF OF LEMMA 2.7

In this section we prove Lemma 2.7. By Lemma 5.8, we can assume the monomials are expressed as an ordered product $J_1 J_2 \cdots J_b$ with $J_a \prec J_{a+1}$. For $\alpha \in \mathcal{M}(\mathcal{J})$, let

$$\mathfrak{R}(\alpha) = \left\{ \sum c_i \beta_i \in \mathfrak{R} \mid c_i \in \mathbb{Z}, \beta_i \in \mathcal{M}(\mathcal{J}), \beta_i \prec \alpha, \beta_i \neq \alpha \right\},$$

be the space of linear combinations of elements preceding α in $\mathcal{M}(\mathcal{J})$ with integer coefficients.

Lemma 7.1. *If $J_1 J_2$ is not standard, $J_1 J_2 \in \mathfrak{R}(J_1)$.*

Proof. Assume $J_i = \bar{\partial}^{n_i}(u_{h_i}^i, \dots, u_2^i, u_1^i | v_1^i, v_2^i, \dots, v_{h_i}^i)$, for $i = 1, 2$. Let $E_1 = ((u_{h_1}^1, n_1), \dots, (u_1^1, 0) | (v_1^1, 0), \dots, (v_{h_1}^1, 0))$. Let $i_0 = L(E_1, J_2)$, $j_0 = R(E_1, J_2)$ and $l_0 = i_0 + j_0$. If $i_0 \neq 0$, there is $i_0 \leq i_1 \leq h_2$, such that $u_{i_1}^2 < u_{i_1-i_0+1}^1$. If $i_0 = 0$, let $i_1 = 0$. If $j_0 \neq 0$, there is $j_0 \leq j_1 \leq h$, such that $v_{j_1}^2 < v_{j_1-j_0+1}^1$. If $j_0 = 0$, let $j_1 = 0$. Let $i_* = i_1 - i_0$, $j_* = j_1 - j_0$ and $m = n_1 + n_2$. By Lemma 2.4, there are integers a_k with $a_{n_2-l} = \delta_l^0$ for $0 \leq l \leq l_0 - 1$.

(7.1)

$$\sum \epsilon a_k \left(\begin{array}{c|c} \bar{\partial}^{m-k}(u_{h_1}^1, \dots, u_{i_*+1}^1, u_{i_*}^1, \dots, u_1^1 & v_1^1, \dots, v_{j_*}^1, v_{j_*+1}^1, \dots, v_{h_1}^1) \\ \bar{\partial}^k(u_{h_2}^2, \dots, u_{i_1+1}^2, u_{i_1}^2, \dots, u_1^2 & v_1^2, \dots, v_{j_1}^2, v_{j_1+1}^2, \dots, v_{h_2}^2) \end{array} \right) \in \mathfrak{R}[h_1 + 1].$$

(1) If $h_1 = h_2$, then $n_1 \leq n_2$. Since $J_1 J_2$ is not standard, J_2 is not greater than E_1 . By Lemma 5.2, $l_0 > n_2 - n_1 \geq 0$. $J_1 J_2 \in \mathfrak{R}(J_1)$ since in Equation (7.1):

- All the terms with $k = n_2$ precede J_1 except $J_1 J_2$ itself;
- All the terms with $k = n_2 - 1, \dots, n_1$ vanish since $a_k = 0$;
- All the terms with $k = n_2 + 1, \dots, m$ precede J_1 since the weight of the upper $\bar{\partial}$ -list is $m - k < n_1$;
- All the terms with $k = 0, \dots, n_1 - 1$ precede J_1 after exchanging the upper $\bar{\partial}$ -list and the lower $\bar{\partial}$ -list since the weight of the lower $\bar{\partial}$ -list is $k < n_1$;
- The terms in $\mathfrak{R}[h_1 + 1]$ precede J_1 since they have bigger sizes.

(2) If $h_1 > h_2$. Since $J_1 J_2$ is not standard, by Lemma 5.2, $l_0 > n_2$. $J_1 J_2 \in \mathfrak{R}(J_1)$ since in Equation (7.1):

- All the terms with $k = n_2$ precede J_1 in the lexicographic order except $J_1 J_2$ itself;
- All the terms with $k = n_2 - 1, \dots, 0$ vanish since $a_k = 0$;
- All the terms with $k = n_2, \dots, m$ precede J_1 since the weight of the upper $\bar{\partial}$ -list is $m - k < n_1$;
- The terms in $\mathfrak{R}[h_1 + 1]$ precede J_1 since they have bigger sizes.

□

Proof of Lemma 2.7. We prove the lemma by induction on b .

If $b = 1$, J_1 is standard.

If $b = 2$, by Lemma 7.1, the lemma is true.

For $b \geq 3$, assume the lemma is true for $b - 1$. We can assume $J_1 \cdots J_{b-1}$ is standard by induction and Lemma 5.9. Let $E_1 \cdots E_{b-1} \in SM(\mathcal{E})$ be the standard order product of element of $\mathcal{E}$ corresponding to $J_1 \cdots J_{b-1}$. If $J_1 \cdots J_b$ is not standard, then J_b is not greater than E_{b-1} . By Lemma 7.2 (below), $J_{b-1} J_b = \sum K_i f_i$ with $K_i \in \mathcal{J}$, $f_i \in \mathfrak{R}$ such that K_i is either smaller than J_{b-1} or K_i is not greater than E_{b-2} . If K_i is smaller than J_{b-1} , then $J_1 \cdots J_{b-1} K_i f \in \mathfrak{R}(J_1 \cdots J_{b-1})$. If K_i is not greater than E_{b-2} , $J_1 \cdots J_{b-2} K_i$

is not standard, so it is in $\mathfrak{R}(J_1 \cdots J_{b-2})$ by induction. Then $J_1 \cdots J_{b-2} K_i f \in \mathfrak{R}(J_1 \cdots J_{b-1})$. So $J_1 \cdots J_b = \sum J_1 \cdots J_{b-2} K_i f_i \in \mathfrak{R}(J_1 \cdots J_{b-1})$. $\square$

Lemma 7.2. *Let $E \in \mathcal{E}$, J_a and J_b in $\mathcal{J}$ with $J_a \prec J_b$, and suppose that E_a is the largest element in $\mathcal{E}(J_a)$ such that $E \leq E_a$. If J_b is not greater than E_a , then $J_a J_b = \sum K_i f_i$ with $K_i \in \mathcal{J}$, $f_i \in \mathfrak{R}$ such that K_i is either smaller than J_a , or K_i is not greater than E .*

Proof. Assume

$$J_a = \bar{\partial}^{n_a}(u'_{h_a}, \dots, u'_1 | v'_1, \dots, v'_{h_a}), \quad J_b = \bar{\partial}^{n_b}(u^b_{h_b}, \dots, u^b_1 | v^b_1, \dots, v^b_{h_b}).$$

$J_a \prec J_b$, so $h_a \geq h_b$. Assume $\|E_a(h_b)\| = \bar{\partial}^{m_a}(u^a_{h_b}, \dots, u^a_1 | v^a_1, \dots, v^a_{h_b})$. Let $m = n_b + n_a$. Let $i_0 = L(E_a, J_b)$, $j_0 = (E_a, J_b)$ and $l_0 = i_0 + j_0$. If $i_0 \neq 0$, there is $i_0 \leq i_1 \leq h_b$, such that $u^b_{i_1} < u^a_{i_1-i_0+1}$. If $i_0 = 0$, let $i_1 = 0$. If $j_0 \neq 0$, there is $j_0 \leq j_1 \leq h_b$, such that $v^b_{j_1} < v^a_{j_1-j_0+1}$. If $j_0 = 0$, let $j_1 = 0$. Since J_b is not greater than E_a , by Lemma 5.2,

$$(7.2) \quad l_0 = i_0 + j_0 > n_b - m_a.$$

By definition, $\{u^a_{h_b}, \dots, u^a_1\}$ is a subset of $\{u'_{h_a}, \dots, u'_1\}$ with $u'_i < u'_{i+1}$ and $u^a_i < u^a_{i+1}$. If we assume $u'_{i_2} = u^a_{i_1-i_0+1}$, we have $i_2 \geq i_1 - i_0 + 1$. Similarly, $\{v^a_{h_b}, \dots, v^a_1\}$ is a subset of $\{v'_{h_a}, \dots, v'_1\}$ with $v'_j < v'_{j+1}$ and $v^a_i < v^a_{i+1}$; if we assume $v'_{j_2} = v^a_{j_1-j_0+1}$, then $j_2 \geq j_1 - j_0 + 1$.

Now we prove the lemma. The proof is quite long and it is divided into three cases.

Case 1: $h_a = h_b$. Let $a_{n_b-l} = \delta_l^0$ for $0 \leq l \leq l_0 - 1$. By Lemma 2.4, there are integers a_k , such that

$$(7.3) \quad \sum \epsilon a_k \left(\begin{array}{c|c} \bar{\partial}^{m-k}(u'_{h_a}, \dots, u'_{i_1-i_0+1}, u'_{i_1-i_0}, \dots, u'_1 & v'_1, \dots, v'_{j_1-j_0}, v'_{j_1-j_0+1}, \dots, v'_{h_a}) \\ \hline \bar{\partial}^k(u^b_{h_b}, \dots, u^b_{i_1+1}, u^b_{i_1}, \dots, u^b_1 & v^b_1, \dots, v^b_{j_1}, v^b_{j_1+1}, \dots, v^b_{h_b}) \end{array} \right) \in \mathfrak{R}[h_1 + 1].$$

$J_a J_b \in \mathfrak{R}(J_a)$ since in the above equation,

- All the terms with $k = n_b$ precede J_a in the lexicographic order except $J_a J_b$ itself;
- All the terms with $k = n_b - 1, \dots, n_a$ vanish since $n_a = m_a$, $a_k = 0$;
- All the terms with $k = n_b + 1, \dots, m$ precede J_a since the weight of the upper $\bar{\partial}$ -list is $m - k < n_a$;
- All the terms with $k = 0, \dots, n_a - 1$ precede J_a after exchanging the upper $\bar{\partial}$ -list and the lower $\bar{\partial}$ -list since the weight of the lower $\bar{\partial}$ -list is $k < n_a$;
- The terms in $\mathfrak{R}[h + 1]$ precede J_a since they have bigger sizes.

Case 2: $h_b < h_a$ and $n_b < m_a$.

By Lemma 2.4,

$$(7.4) \quad \sum_{\substack{0 \leq i, j \leq h_b \\ i+j < 2h_b}} \sum_{\sigma, \sigma'} \frac{(-1)^{i+j} \text{sign}(\sigma) \text{sign}(\sigma')}{i!(h_b - i)!j!(h_b - j)!} \sum \epsilon a_k^{i,j} \\ \left(\begin{array}{c|c} \bar{\partial}^{m-k}(u'_{h_a}, \dots, u'_{i+1}, u_{\sigma(i)}^b, \dots, u_{\sigma(2)}^b, u_{\sigma(1)}^b) & v_{\sigma'(1)}^b, v_{\sigma'(2)}^b, \dots, v_{\sigma'(j)}^b, \underline{v'_{j+1}, \dots, v'_{h_a}} \\ \hline \bar{\partial}^k(u_{\sigma(h_b)}^b, \dots, u_{\sigma(i+1)}^b, u'_i, \dots, u'_1) & \underline{v'_1, \dots, v'_j, v_{\sigma'(j+1)}^b, \dots, v_{\sigma'(h_b)}^b} \end{array} \right) \\ \in \mathfrak{R}[h_a + 1].$$

Here $a_k^{i,j}$ are integers and $a_{n_b-l}^{i,j} = \delta_{0,l}$ for $0 \leq l < 2h_b - (i + j)$. The second summation is over all pairs of permutations σ and σ' of $\{1, \dots, h_b\}$. In the above equation:

- The terms in $\mathfrak{R}[h_a + 1]$ precede J_a since they have bigger sizes.
- All the terms with $k = n_b, \dots, m$ precede J_a since the weight of the upper $\bar{\partial}$ -list is $m - k < n_a$.
- The terms with $k = n_b$ are $J_a J_b$ and the terms with the lower $\bar{\partial}$ -lists

$$K_0 = \bar{\partial}^{n_b}(u'_{i_{h_b}}, \dots, u'_{i_2}, u'_{i_1} | v'_{j_1}, v'_{j_2}, \dots, v'_{j_{h_b}}) \in \mathcal{J}.$$

All the other terms cancel. By Corollary 5.5 and 5.6,

$$L(E_{b-2}, K_0) + R(E_{b-2}, K_0) \geq L(E_{b-2}, E_a(h_b)) + R(E_{b-2}, E_a(h_b)) \\ = m_a - m_{b-2} > n_b - m_{b-2}.$$

By Lemma 5.2, K_0 is not greater than E_{b-2} .

- The terms with $k < n_b$ vanish unless $2h_b - (i + j) \leq n_b - k$. In this case, the lower $\bar{\partial}$ -lists of the terms are

$$K_1 = \bar{\partial}^k(u_{\sigma(h_b)}^b, \dots, u_{\sigma(i+1)}^b, u'_{s_i}, \dots, u'_{s_1} | v'_{t_1}, \dots, v'_{t_j}, v_{\sigma(j+1)}^b, \dots, v_{\sigma'(h_b)}^b).$$

By Lemma 5.1,

$$L(E_{b-2}, K_1) \geq L(E_{b-2}, K_0) - (h_b - i), \quad R(E_{b-2}, K_1) \geq R(E_{b-2}, K_0) - (h_b - j).$$

So

$$L(E_{b-2}, K_1) + R(E_{b-2}, K_1) > n_b - m_{b-2} - (2h_b - i - j) \geq k - m_{b-2}.$$

By Lemma 5.2, K_1 is not greater than E_{b-2} .

Case 3: $h_a > h_b$ and $n_b \geq m_a$.

- (1) If $i_0 = 0$, then $j_0 > 0$. Since if $j_0 = 0$, then J_b is greater than E_a and $J_1 \cdots J_b$ is standard.

By Equation (2.4),

$$(7.5) \quad \sum_{0 \leq i \leq h_b} \sum_{\substack{j_1 \geq t \\ j_2 > t \\ j_1 + h_b \neq i + t}} \sum_{\sigma, \sigma'} \frac{(-1)^{i+t} \text{sign}(\sigma) \text{sign}(\sigma')}{i!(h_b - i)!t!(j_2 - 1 - t)!} \sum_{k=0}^m \epsilon a_k^{i,t} \\ \bar{\partial}^{m-k}(\underline{u'_{h_a}, \dots, u'_{i+1}}, u_{\sigma(i)}^b, \dots, u_{\sigma(2)}^b, u_{\sigma(1)}^b | \underline{v_1^b, \dots, v_t^b, v_{\sigma'(t+1)}^b, \dots, v_{\sigma'(j_2-1)}^b, v'_{j_2}, \dots, v'_h})$$

$$\begin{aligned} & \bar{\partial}^k(\underline{u_{\sigma(h_b)}^b, \dots, u_{\sigma(i+1)}^b, u'_i, \dots, u'_1} | \underline{v'_{\sigma'(1)}, \dots, v'_{\sigma'(t)}, v_{t+1}^b, \dots, v_{j_1}^b, v_{j_1+1}^b, \dots, v_{h_b}^b}) \\ & \in \mathfrak{R}[h_a + 1]. \end{aligned}$$

Here $a_k^{i,t}$ are integers with $a_{n_b-l}^{i,t} = \delta_{0,l}$ for $0 \leq l < h_b - i + j_1 - t$, σ are permutations of $\{1, \dots, h_b\}$ and σ' are permutations of $\{1, \dots, j_2 - 1\}$. Next,

(7.6)

$$\sum_{\substack{0 \leq i \leq h_b \\ j_1 \leq j \leq h_b}} \sum_{\substack{j_1 \geq t \\ j_2 > t \\ h_b + j_1 \neq i + t}} \sum_{\sigma, \sigma', \sigma_2} \frac{(-1)^{i+j+t} \text{sign}(\sigma) \text{sign}(\sigma') \text{sign}(\sigma_2)}{i!(h_b - i)!(j - j_1)!(h_b - j)!t!(j_2 - 1 - t)!} \sum_{k=0}^m \epsilon a_k^{i,j,t}$$

$$\begin{aligned} & \bar{\partial}^{m-k}(\underline{u'_{h_a}, \dots, u'_{i+1}, u_{\sigma(i)}^b, \dots, u_{\sigma(2)}^b, u_{\sigma(1)}^b} \\ & \quad | \underline{v'_{h_b+1}, \dots, v'_{h_a}, v_1^b, \dots, v_{j_1}^b, v_{\sigma'(j_1+1)}^b, \dots, v_{\sigma'(j)}^b, \dots, v_{\sigma'(h_b)}^b}) \\ & \bar{\partial}^k(\underline{u_{\sigma(h_b)}^b, \dots, u_{\sigma(i+1)}^b, u'_i, \dots, u'_1} \\ & \quad | \underline{v'_{\sigma(1)}, \dots, v'_{\sigma(t)}, v'_{\sigma(t+1)}, \dots, v'_{\sigma(j_2-1)}, v'_{j_2}, \dots, v'_{h_b+1}}) \\ & \in \mathfrak{R}[h_a + 1]. \end{aligned}$$

Here $a_k^{i,j,t}$ are integers and $a_{n_b-l}^{i,j,t} = \delta_{0,l}$ for $0 \leq l < h_b - i + j - t$, σ are permutations of $\{1, \dots, h_b\}$, σ' are permutations of $\{h_b, \dots, j_1 + 1\}$, and σ_2 are permutations of $\{1, \dots, j_2 - 1\}$.

We use Equation (7.5) if $j_2 > j_1 - j_0 + 1$ and use Equation (7.6) if $j_2 = j_1 - j_0 + 1$. In the above equations:

- (a) The terms in $\mathfrak{R}[h_a + 1]$ precede J_a since they have bigger sizes;
- (b) All the terms with $k = n_b + 1, \dots, m$ precede J_a since the weight of upper $\bar{\partial}$ -list is $m - k < n_a$.
- (c) The terms with $k = n_b$ are $J_a J_b$, the terms with upper $\bar{\partial}$ -list preceding J_a (the upper $\bar{\partial}$ -lists are the $\bar{\partial}$ -lists given by replacing some v'_j , $j \geq j_2$ in J_a by some u_k^b , $k \leq j_1$), and the terms with the lower $\bar{\partial}$ -lists

$$K_0 = \bar{\partial}^k(\underline{u'_{i_{h_b}}, \dots, u'_{i_1}} | \underline{v'_{\sigma(1)}, \dots, v'_{\sigma(j_1)}, v_{j_1+1}^b, \dots, v_{h_b}^b}).$$

All of the other terms cancel. If $j_2 - 1 < j_1$, the terms of the form K_0 do not appear in Equations (7.5), (7.6). Otherwise, $j_2 - 1 \geq j_1$. In this case $j_2 \geq j_1 + 1 > j_1 - j_0 + 1$. By Corollary 5.6,

$$(7.7) \quad L(E_{b-2}, ||E_a(h_b)||) \leq L(E_{b-2}, K_0).$$

By Lemma 5.7,

$$(7.8) \quad R(E_{b-2}, K_0) > R(E_{b-2}, ||E_a(h_b)||) + j_1 - (j_1 - j_0 + 1).$$

By Corollary 5.5,

$$(7.9) \quad L(E_{b-2}, ||E_a(h_b)||) + R(E_{b-2}, ||E_a(h_b)||) = wt(E_a(h_b)) - wt(E_{b-2}(h_b)).$$

So by Equations (7.7), (7.8), (7.9), and (7.2),

$$L(E_{b-2}, K_0) + R(E_{b-2}, K_0) \geq m_a - m_{b-2} + j_0 > n_b - m_{b-2}.$$

By Lemma 5.2, K_0 is not greater than E_{b-2} .

- (d) If $j_2 > j_1 - j_0 + 1$, the terms with $k < n_b$ in Equation (7.5) vanish unless $h_b - i + j_1 - t \leq n_b - k$. The lower $\bar{\partial}$ -lists of these terms are

$$K_1 = \bar{\partial}^k(\underline{u_{\sigma(h_b)}^b, \dots, u_{\sigma(i+1)}^b, u_i', \dots, u_1'} | \underline{v_{\sigma'(1)}', \dots, v_{\sigma'(t)}'}, \underline{v_{t+1}^b, \dots, v_{j_1}^b, v_{j_1+1}^b, \dots, v_{h_b}^b})$$

Underlined u and v can be any underlined u and v in Equation (7.5), respectively.

By Lemma 5.1 and Equation (7.7),

$$(7.10) \quad L(E_{b-2}, K_1) \geq L(E_{b-2}, K_0) - (h_b - i) \geq L(E_{b-2}, ||E_a(h_b)||) - (h_b - i).$$

Since $j_2 > j_1 - j_0 + 1$, by Lemma 5.7,

$$(7.11) \quad R(E_{b-2}, ||E_a(h_b)||) + t - (j_1 - j_0 + 1).$$

So by Equation (7.9), (7.10), (7.11), and (7.2),

$$L(E_{b-2}, K_1) + R(E_{b-2}, K_1) \geq m_a - m_{b-2} - (h_b - i) + t - (j_1 - j_0) > k - m_{b-2}.$$

By Lemma 5.2, K_1 is not greater than E_{b-2} .

- (e) If $j_2 = j_1 - j_0 + 1$, the terms with $k < n_b$ in Equation (7.6) vanish unless $h_b - i + j - t \leq n_b - k$. In this case, the lower $\bar{\partial}$ -lists of the terms are

$$K_1 = \bar{\partial}^k(\underline{u_{\sigma(h_b)}^b, \dots, u_{\sigma(i+1)}^b, u_i', \dots, u_1'} | \underline{v_{\sigma_2(1)}', \dots, v_{\sigma_2(t)}'}, \underline{v_{\sigma_2(t+1)}', \dots, v_{\sigma_2(j_2-1)}'}, v_{j_2}', \dots, v_{h_b+1}')$$

Underlined u and v can be any underlined u and v in Equation (7.6).

Let

$$K_1' = \bar{\partial}^k(\underline{u_{\sigma(h_b)}^b, \dots, u_{\sigma(i+1)}^b, u_i', \dots, u_1'} | \underline{v_{\sigma_2(1)}', \dots, v_{\sigma_2(j_2-1)}'}, v_{l_{j_2}}', \dots, v_{l_{h_b}}')$$

Here $j_2 \leq l_{j_2} < l_{j_2+1} < \dots < l_{h_b} \leq h_a$. By Lemma 5.1 and Equation (7.7),

$$(7.12) \quad L(E_{b-2}, K_1) \geq L(E_{b-2}, K_0) - (h_b - i) \geq L(E_{b-2}, ||E_a(h_b)||) - (h_b - i).$$

By Lemma 5.1, there is some K_1' such that

$$(7.13) \quad R(E_{b-2}, K_1) \geq R(E_{b-2}, K_1') - (j_2 - 1 - t) - (j - j_1)$$

since in K_1 the number of v_j^b with $v_j^b > v_{j_2}'$ is at most $j - j_1$. By Corollary 5.6,

$$(7.14) \quad R(E_{b-2}, K_1') \geq R(E_{b-2}, ||E_a(h_b)||).$$

So by Equations (7.12), (7.13), (7.14), (7.9), and (7.2),

$$\begin{aligned} L(E_{b-2}, K_1) + R(E_{b-2}, K_1) &\geq m_a - m_{b-2} - (h_b - i) - (j_2 - 1 - t) - (j - j_1) \\ &> k - wt(E_{b-2}(h_b)). \end{aligned}$$

By Lemma 5.2, K_1 is not greater than E_{b-2} .

(2) $j_0 = 0$, then $i_0 > 0$. The proof is similar to the case of $i_0 = 0$.

(3) $i_0 > 0$ and $j_0 > 0$. By Lemma 2.4, we have

$$(7.15) \quad \sum_{\substack{i_1 \geq s \\ i_2 > s}} \sum_{\substack{j_1 \geq t \\ j_2 > t \\ j_1 + i_1 \neq s+t}} \sum_{\sigma, \sigma'} \frac{(-1)^{s+t} \text{sign}(\sigma) \text{sign}(\sigma')}{s!(i_2 - 1 - s)!t!(j_2 - 1 - t)!} \sum_{k=0}^m \epsilon a_k^{s,t}$$

$$\bar{\partial}^{m-k}(u'_h, \dots, u'_{i_2}, u'_{\sigma(i_2-1)}, \dots, u'_{\sigma(s+1)}, u_s^b, \dots, u_1^b | v_1^b, \dots, v_t^b, v'_{\sigma'(t+1)}, \dots, v'_{\sigma'(j_2-1)}, v'_{j_2}, \dots, v'_h)$$

$$\bar{\partial}^k(u_{h_b}^b, \dots, u_{i_1+1}^b, \underline{u_{i_1}^b, \dots, u_{s+1}^b}, u'_{\sigma(s)}, \dots, u'_{\sigma(1)} | v'_{\sigma'(1)}, \dots, v'_{\sigma'(t)}, \underline{v_{t+1}^b, \dots, v_{j_1}^b}, v_{j_1+1}^b, \dots, v_{h_b}^b)$$

$$\in \mathfrak{R}[h_a + 1].$$

Here $a_k^{s,t}$ are integers and $a_{n_b-l}^{s,t} = \delta_{0,l}$ for $0 \leq l < i_1 - s + j_1 - t$, σ are permutations of $\{1, \dots, i_2 - 1\}$, and σ' are permutations of $\{1, \dots, j_2 - 1\}$.

$$(7.16) \quad \sum_{\substack{i_1 \geq s \\ i_2 > s}} \sum_{\substack{j \geq j_1 \geq t \\ j_2 > t \\ i_1 + j_1 > s+t}} \sum_{\sigma, \sigma', \sigma_2} \frac{(-1)^{j+s+t} \text{sign}(\sigma) \text{sign}(\sigma') \text{sign}(\sigma_2)}{(j - j_1)!(h_b - j)!s!(i_2 - 1 - s)!t!(j_2 - 1 - t)!} \sum_{k=0}^m \epsilon a_k^{j,s,t}$$

$$\bar{\partial}^{n-k}(u'_h, \dots, u'_{i_2}, u'_{\sigma(i_2-1)}, \dots, u'_{\sigma(s+1)}, u_s^b, \dots, u_1^b$$

$$| \underline{v'_{h_b+1}, \dots, v'_{h_a}}, v_1^b, \dots, v_{j_1}^b, v_{\sigma'(j_1+1)}^b, \dots, v_{\sigma'(j)}^b, \dots, v_{\sigma'(h_b)}^b)$$

$$\bar{\partial}^k(u_{h_b}^b, \dots, u_{i_1+1}^b, \underline{u_{i_1}^b, \dots, u_{s+1}^b}, u'_{\sigma(s)}, \dots, u'_{\sigma(1)}$$

$$| \underline{v'_{\sigma_2(1)}, \dots, v'_{\sigma_2(t)}, v'_{\sigma_2(t+1)}, \dots, v'_{\sigma_2(j_2-1)}, v'_{j_2}, \dots, v'_{h_b+1}})$$

$$\in \mathfrak{R}[h_a + 1].$$

Here $a_k^{j,s,t}$ are integers and $a_{n_b-l}^{j,s,t} = \delta_{0,l}$ for $0 \leq l < (i_1 + j - s - t)$, σ' are permutations of $\{h_b, \dots, j_1 + 1\}$, σ are permutations of $\{1, \dots, i_2 - 1\}$, and σ_2 are permutations of $\{1, \dots, j_2 - 1\}$.

$$(7.17) \quad \sum_{\substack{i \geq i_1 \geq s \\ i_2 > s}} \sum_{\substack{j_1 \geq t \\ j_2 > t \\ i_1 + j_1 > s+t}} \sum_{\sigma, \sigma', \sigma_1} \frac{(-1)^{i+s+t} \text{sign}(\sigma) \text{sign}(\sigma') \text{sign}(\sigma_1)}{(i - i_1)!(h_b - i)!s!(i_2 - 1 - s)!t!(j_2 - 1 - t)!} \sum_{k=0}^m \epsilon a_k^{i,s,t}$$

$$\bar{\partial}^{n-k}(u_{\sigma(h_b)}^b, \dots, \underline{u_{\sigma(i)}^b, \dots, u_{\sigma(i_1+1)}^b}, u_{i_1}^b, \dots, u_1^b, u'_{h_a}, \dots, u'_{h_b+1}$$

$$| \underline{v_1^b, \dots, v_t^b, v'_{\sigma'(t+1)}, \dots, v'_{\sigma'(j_2-1)}, v'_{j_2}, \dots, v'_h})$$

$$\bar{\partial}^k(u'_{h_b+1}, \dots, u'_{i_2}, \underline{u'_{\sigma_1(i_2-1)}, \dots, u'_{\sigma_1(s+1)}}, u'_{\sigma_1(s)}, \dots, u'_{\sigma_1(1)}$$

$$| \underline{v'_{\sigma'(1)}, \dots, v'_{\sigma'(t)}, \underline{v_{t+1}^b, \dots, v_{j_1}^b}, v_{j_1+1}^b, \dots, v_{h_b}^b})$$

$$\in \mathfrak{R}[h_a + 1].$$

Here $a_k^{i,s,t}$ are integers and $a_{n_b-l}^{i,s,t} = \delta_{0,l}$ for $0 \leq l < (i+j_1-s-t)$, σ are permutations of $\{h_b, \dots, i_1 + 1\}$, σ_1 are permutations of $\{1, \dots, i_2 - 1\}$, and σ' are permutations of $\{1, \dots, j_2 - 1\}$.

(7.18)

$$\sum_{\substack{i \geq i_1 \geq s \\ i_2 > s}} \sum_{\substack{j \geq j_1 \geq t \\ j_2 > t \\ i_1 + j_1 > s + t}} \sum_{\sigma, \sigma', \sigma_1, \sigma_2} \frac{(-1)^{i+j+s+t} \text{sign}(\sigma) \text{sign}(\sigma') \text{sign}(\sigma_1) \text{sign}(\sigma_2)}{(i-i_1)!(h_b-i)!(j-j_1)!(h_b-j)!s!(i_2-1-s)!t!(j_2-1-t)!} \sum_{k=0}^m \epsilon a_k^{i,j,s,t}$$

$$\bar{\partial}^{n-k}(u_{\sigma(h_b)}^b, \dots, u_{\sigma(i)}^b, \dots, u_{\sigma(i_1+1)}^b, u_{i_1}^b, \dots, u_1^b, u_{h_a}^b, \dots, u_{h_b+1}^b$$

$$\frac{|v_{h_b+1}'^b, \dots, v_{h_a}'^b, v_1^b, \dots, v_{j_1}'^b, v_{\sigma'(j_1+1)}^b, \dots, v_{\sigma'(j)}^b, \dots, v_{\sigma'(h_b)}^b|}{\bar{\partial}^k(u_{h_b+1}'^b, \dots, u_{i_2}'^b, u_{\sigma_1(i_2-1)}^b, \dots, u_{\sigma_1(s+1)}^b, u_{\sigma_1(s)}^b, \dots, u_{\sigma_1(1)}^b$$

$$\frac{|v_{\sigma_2(1)}'^b, \dots, v_{\sigma_2(t)}'^b, v_{\sigma_2(t+1)}'^b, \dots, v_{\sigma_2(j_2-1)}'^b, v_{j_2}'^b, \dots, v_{h_b+1}'^b|}{\in \mathfrak{R}[h_a + 1].}$$

Here $a_k^{i,j,s,t}$ are integers and $a_{n_b-l}^{i,j,s,t} = \delta_{0,l}$ for $0 \leq l < (i+j-s-t)$, σ are permutations of $\{h_b, \dots, i_1 + 1\}$, σ' are permutations of $\{h_b, \dots, j_1 + 1\}$, σ_1 are permutations of $\{1, \dots, i_2 - 1\}$, and σ_2 are permutations of $\{1, \dots, j_2 - 1\}$.

- We use Equation (7.15) when $i_2 > i_1 - i_0 + 1$ and $j_2 > j_1 - j_0 + 1$;
- We use Equation (7.16) when $i_2 > i_1 - i_0 + 1$ and $j_2 = j_1 - j_0 + 1$;
- We use Equation (7.17) when $i_2 = i_1 - i_0 + 1$ and $j_2 > j_1 - j_0 + 1$;
- We use Equation (7.18) when $i_2 = i_1 - i_0 + 1$ and $j_2 = j_1 - j_0 + 1$.

In the above relations:

- The terms in $\mathfrak{R}[h_a + 1]$ precede J_a since they have bigger sizes.
- All the terms with $k = n_b + 1, \dots, n$ precede J_a since the weight of the upper $\bar{\partial}$ -list is $n - k < n_a$.
- The terms with $k = n_b$ are $J_a J_b$, the terms with the upper $\bar{\partial}$ -list preceding J_a (the upper $\bar{\partial}$ -lists are the $\bar{\partial}$ -lists given by replacing some u_i' , $i \geq i_2$ in J_a by some u_k^b , $k \leq i_1$ or v_j' , $j \geq j_2$ by v_k^b , $k \leq j_1$), and the terms with the lower $\bar{\partial}$ -lists

$$K_0 = \bar{\partial}^{n_b}(u_{h_b}^b, \dots, u_{i_1+1}^b, u_{\sigma_1(i_1)}^b, \dots, u_{\sigma_1(1)}^b | v_{\sigma_2(1)}'^b, \dots, v_{\sigma_2(j_1)}'^b, v_{j_1+1}^b, \dots, u_{h_b}^b).$$

All of the other terms cancel. By Lemma 5.7,

$$L(E_{b-2}, K_0) > L(E_{b-2}, ||E_a(h)||) + i_1 - (i_1 - i_0 + 1);$$

$$R(E_{b-2}, K_0) > R(E_{b-2}, ||E_a(h)||) + j_1 - (j_1 - j_0 + 1).$$

By the above two inequalities,

$$\begin{aligned} L(E_{b-2}, K_0) + R(E_{b-2}, K_0) &\geq L(E_{b-2}, ||E_a(h_b)||) + R(E_{b-2}, ||E_a(h_b)||) + i_0 + j_0 \\ (\text{by Corollary 5.5}) &= wt(E_a(h_b)) - wt(E_{b-2}) + i_0 + j_0 \\ (\text{by Equation (7.2)}) &> wt(E_b) - wt(E_{b-2}(h_b)). \end{aligned}$$

By Lemma 5.2, K_0 is not greater than E_{b-2} .

- (d) When $i_2 > i_1 - i_0 + 1$ and $j_2 > j_1 - j_0 + 1$; The terms with $k < n_b$ in Equation (7.15) vanishes unless $i_1 + j_1 - s - t \leq n_b - k$. In this case, the lower $\bar{\partial}$ -lists of the terms are

$$K_1 = \bar{\partial}^k(u_{h_b}^b, \dots, u_{i_1+1}^b, \underline{u_{i_1}^b, \dots, u_{s+1}^b}, u'_{\sigma(s)}, \dots, u'_{\sigma(1)} \\ |v'_{\sigma'(1)}, \dots, v'_{\sigma'(t)}, \underline{v_{t+1}^b, \dots, v_{j_1}^b}, v_{j_1+1}^b, \dots, v_{h_b}^b)$$

The underlined u and v in K_1 can be any underlined u and v in Equation (7.15). By Lemma 5.7,

$$L(E_{b-2}, K_1) > L(E_{b-2}, E_a(h_b)) + s - (i_1 - i_0 + 1);$$

$$R(E_{b-2}, K_1) > R(E_{b-2}, E_a(h_b)) + t - (j_1 - j_0 + 1);$$

$$\begin{aligned} L(E_{b-2}, K_1) + R(E_{b-2}, K_1) &\geq L(E_{b-2}, E_a(h_b)) + R(E_{b-2}, E_a(h_b)) \\ &\quad + k - n_b + i_0 + j_1 \\ \text{(by Corollary 5.5)} &= wt(E_a(h_b)) - wt(E_{b-2}(h_b)) + k - n_b + i_0 + j_0 \\ \text{(by Equation (7.2))} &> k - wt(E_{b-2}(h_b)) \end{aligned}$$

By Lemma 5.2, K_1 is not greater than E_{b-2} .

- (e) When $i_2 > i_1 - i_0 + 1$ and $j_2 = j_1 - j_0 + 1$, the terms with $k < n_b$ are the terms in Equation (7.16), such that $i_1 + j - s - t \leq n_b - k$.

$$K_1 = \bar{\partial}^k(u_{h_b}^b, \dots, u_{i_1+1}^b, \underline{u_{i_1}^b, \dots, u_{s+1}^b}, u'_{\sigma(s)}, \dots, u'_{\sigma(1)} \\ |v'_{\sigma_2(1)}, \dots, v'_{\sigma_2(t)}, \underline{v'_{\sigma_2(t+1)}, \dots, v'_{\sigma_2(j_2-1)}}, v'_{j_2}, \dots, v'_{h_b+1})$$

The underlined u and v in K_1 can be any underlined u and v in Equation (7.15). By Lemma 5.7

$$(7.19) \quad L(E_{b-2}, K_1) > L(E_{b-2}, ||E_a(h_b)||) + s - (i_1 - i_0 + 1).$$

Let

$$K'_1 = \bar{\partial}^k(u_{h_b}^b, \dots, u_{i_1+1}^b, \underline{u_{i_1}^b, \dots, u_{s+1}^b}, u'_{\sigma(s)}, \dots, u'_{\sigma(1)} \\ |v'_{\sigma_2(1)}, \dots, v'_{\sigma_2(j_2-1)}, v'_{l_{j_2}}, \dots, v'_{l_{h_b}})$$

Here $j_2 \leq l_{j_2} < l_{j_2+1} < \dots < l_{h_b} \leq h_a$. By Lemma 5.1, there is some K'_1 such that

$$(7.20) \quad R(E_{b-2}, K_1) \geq R(E_{b-2}, K'_1) - (j_2 - 1 - t) - (j - j_1)$$

since in K_1 the number of v_j^b with $v_j^b > v'_{j_2}$ is at most $j - j_1$. By Corollary 5.6,

$$(7.21) \quad R(E_{b-2}, K'_1) \geq R(E_{b-2}, ||E_a(h_b)||).$$

So by Equation (7.19), (7.20), and (7.21),

$$\begin{aligned}
L(E_{b-2}, K_1) + R(E_{b-2}, K_1) &\geq L(E_{b-2}, ||E_a(h_b)||) + R(E_{b-2}, ||E_a(h_b)||) \\
&\quad + s + t - (i_1 - i_0) - (j - j_0) \\
(\text{by Corollary 5.5}) &\geq wt(E_a(h_b)) - wt(E_{b-2}(h_b)) + k - n_b + i_0 + j_0 \\
(\text{by Equation (7.2)}) &> k - wt(E_{b-2}(h_b)).
\end{aligned}$$

By Lemma 5.2, K_1 is not greater than E_{b-2} .

- (f) When $i_2 = i_1 - i_0 + 1$ and $j_2 > j_1 - j_0 + 1$ and $k < n_b$, the proof is similar to the proof of case (3e).
- (g) When $i_2 = i_1 - i_0 + 1$ and $j_2 = j_1 - j_0 + 1$, the terms with $k < n_b$ are the terms in Equation (7.18), such that $i + j - s - t \leq n_b - k$. In this case, the lower $\bar{\partial}$ -lists of the terms are

$$\begin{aligned}
K_1 = \bar{\partial}^k(\underbrace{u'_{h_b+1}, \dots, u'_{i_2}, u'_{\sigma_1(i_2-1)}, \dots, u'_{\sigma_1(s+1)}, u'_{\sigma_1(s)}, \dots, u'_{\sigma_1(1)}}_{|v'_{\sigma_2(1)}, \dots, v'_{\sigma_2(t)}, v'_{\sigma_2(t+1)}, \dots, v'_{\sigma_2(j_2-1)}, v'_{j_2}, \dots, v'_{h_b+1}})
\end{aligned}$$

The underlined u and v in K_1 can be any underlined u and v in Equation (7.18). Let

$$K'_1 = \bar{\partial}^k(u'_{k_{h_b}}, \dots, u'_{k_{i_2}}, u'_{\sigma(i_2-1)}, \dots, u'_{\sigma(1)} | v'_{\sigma_2(1)}, \dots, v'_{\sigma_2(j_2-1)}, v'_{l_{j_2}}, \dots, v'_{l_{h_b}}).$$

Here $i_2 \leq k_{i_2} < k_{i_2+1} < \dots < k_{h_b} \leq h_a$ and $j_2 \leq l_{j_2} < l_{j_2+1} < \dots < l_{h_b} \leq h_a$. By Lemma 5.1, there is some K'_1 such that

$$(7.22) \quad R(E_{b-2}, K_1) \geq R(E_{b-2}, K'_1) - (j_2 - 1 - t) - (j - j_1);$$

$$(7.23) \quad L(E_{b-2}, K_1) \geq L(E_{b-2}, K'_1) - (i_2 - 1 - s) - (i - i_1).$$

since in K_1 the number of v_l^b with $v_l^b > v'_{j_2}$ is at most $j - j_1$ and the number of u_l^b with $u_l^b > u'_{i_2}$ is at most $i - i_1$. By Corollary 5.6,

$$(7.24) \quad R(E_{b-2}, K'_1) \geq R(E_{b-2}, ||E_a(h_b)||), \quad L(E_{b-2}, K'_1) \geq L(E_{b-2}, ||E_a(h_b)||).$$

By Equations (7.22), (7.23), and (7.24),

$$\begin{aligned}
L(E_{b-2}, K_1) + R(E_{b-2}, K_1) &\geq L(E_{b-2}, ||E_a(h_b)||) + R(E_{b-2}, ||E_a(h_b)||) \\
&\quad + s + t - (i - i_0) - (j - j_0) \\
(\text{by Corollary 5.5}) &\geq wt(E_a(h_b)) - wt(E_{b-2}(h_b)) + k - n_b + i_0 + j_0 \\
(\text{by Equation 7.2}) &> k - wt(E_{b-2}(h_b)).
\end{aligned}$$

By Lemma 5.2, K_1 is not greater than E_{b-2} .

□

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