



The Characteristic Gluing Problem for the Einstein Vacuum Equations: Linear and Nonlinear Analysis

Stefanos Aretakis, Stefan Czimek  and Igor Rodnianski

Abstract. This is the second paper in a series of papers addressing the characteristic gluing problem for the Einstein vacuum equations. We solve the codimension-10 characteristic gluing problem for characteristic data which are close to the Minkowski data. We derive an infinite-dimensional space of gauge-dependent charges and a 10-dimensional space of gauge-invariant charges that are conserved by the linearized null constraint equations and act as obstructions to the gluing problem. The gauge-dependent charges can be matched by applying angular and transversal gauge transformations of the characteristic data. By making use of a special hierarchy of radial weights of the null constraint equations, we construct the null lapse function and the conformal geometry of the characteristic hypersurface, and we show that the aforementioned charges are in fact the only obstructions to the gluing problem. Modulo the gauge-invariant charges, the resulting solution of the null constraint equations is C^{m+2} for any specified integer $m \geq 0$ in the tangential directions and C^2 in the transversal directions to the characteristic hypersurface. We also show that higher-order (in all directions) gluing is possible along bifurcated characteristic hypersurfaces (modulo the gauge-invariant charges).

Contents

1. Introduction	3083
1.1. Introduction to the Characteristic Gluing Problem and Overview of Results	3084
1.2. Previous Gluing Constructions	3089
1.2.1. Gluing Constructions in General Relativity	3089
1.2.2. Characteristic Gluing for the Wave Equation	3089
1.3. Double Null Coordinates	3090
1.4. Null Structure Equations	3092

1.5.	The Characteristic Initial Value Problem and the Gluing Problem	3093
1.6.	First Statement of Main Theorem	3094
1.7.	Linearized Characteristic Gluing	3096
1.7.1.	Linearized Equations, Characteristic Gluing and Conserved Charges	3096
1.7.2.	Linearized Perturbations of Sphere Data and Matching of Charges	3097
1.7.3.	Hierarchical Structure of Radial Weights in the Characteristic Gluing Problem	3098
1.8.	Solution of the Nonlinear Characteristic Gluing Problem	3099
1.9.	Codimension-10 Bifurcate Characteristic Gluing	3100
1.10.	Overview of the Paper	3101
2.	Notation, Definitions and Preliminaries	3102
2.1.	Null Geometry	3102
2.2.	Null Structure Equations and Null Bianchi Equations	3105
2.3.	Null Geometry of Minkowski and Schwarzschild Spacetimes	3107
2.4.	Tensor Spaces and Calculus Estimate	3109
2.5.	Sphere Data, Null Data and Norms	3110
2.6.	Charges $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$ and Matching Map $\mathfrak{M}$	3113
2.7.	Nilpotent Character of Null Structure Equations	3118
2.7.1.	Definition of Free Data and Derivation of Hierarchy	3118
2.7.2.	Linearized Constraint Functions at Minkowski	3122
2.8.	Perturbations of Sphere Data	3123
2.9.	Implicit Function Theorem	3127
2.10.	Notation for Characteristic Gluing of Higher-Order Derivatives	3128
3.	Statement of Main Results	3130
4.	Linearized Characteristic Gluing at Minkowski	3135
4.1.	Conserved Charges $\mathcal{Q}_i$ for the Linearized Equations	3136
4.2.	Gauge Dependence of the Conserved Charges $\mathcal{Q}_i$	3137
4.3.	Representation Formulas and Estimates	3142
4.3.1.	Analysis of $\dot{\phi}$ and $\dot{g}_c$	3142
4.3.2.	Analysis of $(\Omega \text{tr} \chi)$ and $\dot{\eta}$	3143
4.3.3.	Analysis of $\Omega \text{tr} \underline{\chi}$ and $\dot{\hat{\chi}}$	3144
4.3.4.	Analysis of $\dot{\underline{\omega}}, \dot{\underline{\alpha}}$ and $\underline{D}\dot{\underline{\omega}}$	3146
4.4.	Solution of the Linearized Characteristic Gluing Problem	3151
4.4.1.	Matching of Gauge-Dependent Charges	3151
4.4.2.	Integral Conditions on $\hat{\Omega}$ and $\hat{\chi}$	3153
4.4.3.	Construction of Solution and Estimates	3157
5.	Proof of Main Theorem	3159
5.1.	Setup of Framework for the Proof	3159
5.2.	Construction of Solution to the Null Constraint Equations	3161
5.3.	Proof of the Charge Perturbation Estimate (3.6)	3162
5.4.	Proof of the Charge Transport Estimate (3.7)	3163

5.5. Outline of the Proof of Theorem 3.2	3167
6. Bifurcate Characteristic Gluing	3168
6.1. Linearized Null Constraint Equations	3168
6.2. Preliminary Analysis of Charges	3170
6.3. Linearized Bifurcate Characteristic Gluing	3173
6.4. Proof of Proposition 3.4	3179
Acknowledgements	3180
Appendix A: Perturbations of Sphere Data	3180
A.1. Explicit Formulas for Transversal Sphere Perturbations	3180
A.1.1. Null Geometry	3181
A.1.2. Definition of u on $\tilde{\mathcal{H}}_2$ and Analysis of Foliation Geometry	3182
A.1.3. Analysis of Ricci Coefficients on $\tilde{\mathcal{H}}_2$	3183
A.1.4. Calculation of Null Curvature Components on $\tilde{\mathcal{H}}_2$	3187
A.2. Proof of Proposition 2.21	3187
Appendix B: Derivation of Null Transport Equations	3189
B.1. Derivation of Null Transport Equation for $D\omega$	3189
B.2. Derivation of Transport Equations for $\dot{\omega}$, $\dot{\alpha}$ and $D\dot{\omega}$	3189
Appendix C: Linearization at Schwarzschild	3192
C.1. Linearizations $\dot{\mathcal{P}}_f^M$ and $\dot{\mathcal{P}}_q^M$ at Schwarzschild of Mass $M \geq 0$	3192
C.2. Linearized Constraint Functions at Schwarzschild of Mass $M \geq 0$	3194
C.3. Linearized Transport Equations for $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$ at Schwarzschild	3195
Appendix D: Hodge Systems and Fourier Theory on 2-Spheres	3197
D.1. Hodge Systems on Riemannian 2-Spheres	3197
D.2. Tensor Spherical Harmonics	3198
D.3. Spectral Analysis of Differential Operators	3200
References	3203

1. Introduction

The *gluing problem* in general relativity asks to connect two given spacetimes across a gluing region. Technically speaking, one aims at solving the constraint equations with two prescribed initial data sets. The obstructions to gluing provide insights into the intrinsic rigidity of the Einstein equations. In their groundbreaking work, Corvino [24] and Corvino–Schoen [25] pioneered the study of the Riemannian gluing problem (for spacelike initial data sets). In particular, their gluing construction shed light on the importance of the interplay between the rigidity and the flexibility of the geometric character of the Einstein equations.

In [11], we initiated the study of *the gluing problem for characteristic initial data* for the Einstein vacuum equations. The characteristic gluing problem exhibits various novel features. For example, gluing along characteristic

hypersurfaces is based on solving the so-called *null constraint equations* which are of transport character, whereas the previously studied gluing problem for spacelike initial data requires to analyze the elliptic Riemannian constraint equations. Moreover, in the characteristic gluing construction, the null lapse function and the conformal geometry of the characteristic hypersurface can be freely prescribed.

The present paper provides the full details on the gluing of characteristic data which are close to the Minkowski data. Working close to Minkowski spacetime is natural in the sense that by rescaling, it corresponds to gluing near spacelike infinity in an asymptotically flat spacetime, see [11, 12]. We identify all obstructions to gluing (at the level of C^2 -gluing for the metric components) and show that they are stemming from *conservation laws of the linearized null constraint equations*. We show that these conservation laws determine a 10-dimensional space of so-called *gauge-invariant charges* and an infinite-dimensional space of so-called *gauge-dependent charges*. We prove that the gauge-dependent charges can be matched by applying transversal perturbations and gauge transformations to the characteristic data. In particular, gauge transformations alone are not sufficient to match all gauge-dependent charges. In [11, 12], we geometrically interpret the remaining 10-dimensional space of gauge-invariant charges by relating them to the ADM energy, linear momentum, angular momentum and center-of-mass, and we use this identification to glue asymptotically flat spacetimes to a member of the Kerr family.

This introduction is structured as follows: In Sect. 1.1, we introduce the characteristic gluing problem and present our main results in non-technical terms. In Sect. 1.2, we provide an overview of the literature for the gluing problem. In Sect. 1.3, we set up the null geometry framework and the characteristic initial value problem for the Einstein vacuum equations, and in Sect. 1.6, we present a more formal version of our main theorem. In Sects. 1.7 and 1.8, we provide the main ideas of our methods, and in Sect. 1.9, we discuss characteristic gluing along two null hypersurfaces bifurcating from an auxiliary sphere.

1.1. Introduction to the Characteristic Gluing Problem and Overview of Results

In this section, we introduce in a colloquial, non-technical way the characteristic gluing problem and the main results of this paper.

Consider the null hypersurfaces $(\mathcal{H}_1, \underline{\mathcal{H}}_1)$ and $(\mathcal{H}_2, \underline{\mathcal{H}}_2)$ emanating from two spheres S_1 and S_2 , respectively, in two vacuum spacetimes $\mathcal{M}_1$ and $\mathcal{M}_2$ (Fig. 1).

A first formulation of the characteristic gluing problem asks if there exists a characteristic hypersurface $\mathcal{H}$ that satisfies the null constraint equations whose characteristic data agree on its past boundary with the data on $\mathcal{H}_1$, and on its future boundary with the data on $\mathcal{H}_2$? If not, what are the obstructions to the existence of such a hypersurface? (Fig. 2).

Let us denote by x_1 and x_2 the restriction of the metric components, the Christoffel symbols and the Riemann curvature components of the spacetime metrics of $\mathcal{M}_1$ and $\mathcal{M}_2$ to the spheres S_1 and S_2 , respectively (with respect to

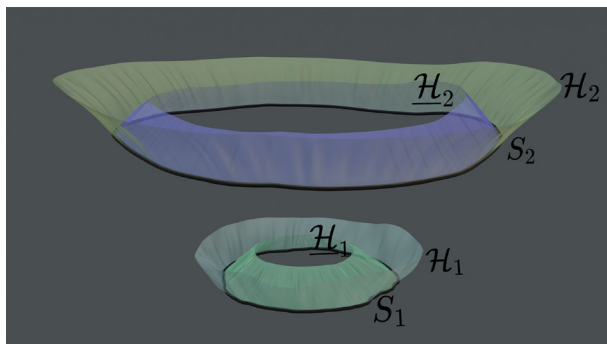


FIGURE 1. The null hypersurfaces $(\mathcal{H}_1, \underline{\mathcal{H}}_1)$ and $(\mathcal{H}_2, \underline{\mathcal{H}}_2)$ emanating from two spheres S_1 and S_2 , in the vacuum spacetimes $\mathcal{M}_1$ and $\mathcal{M}_2$

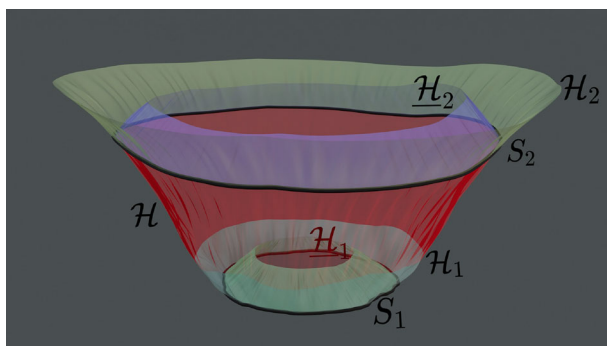


FIGURE 2. The red hypersurface glues the characteristic initial data for the two vacuum spacetimes $\mathcal{M}_1$ and $\mathcal{M}_2$

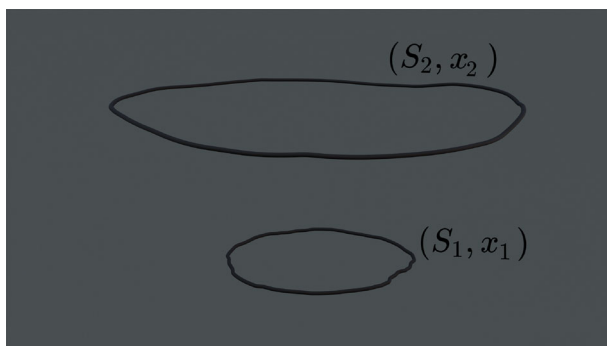


FIGURE 3. On each sphere S_1 and S_2 , we consider the restrictions x_1 and x_2 of the metric components, the Christoffel symbols and the Riemann curvature components of the metrics of the ambient spacetimes $\mathcal{M}_1$ and $\mathcal{M}_2$

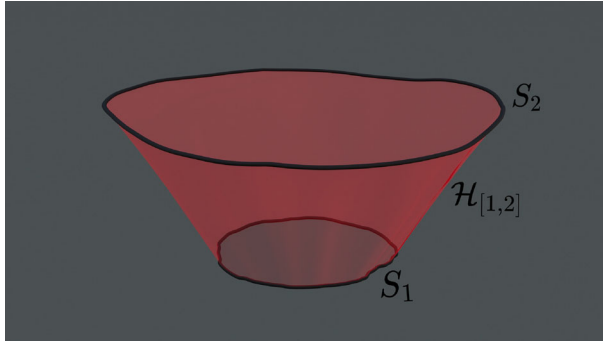


FIGURE 4. The characteristic gluing problem for sphere data x_1 and x_2

local double null coordinate systems). We will refer to x_1 and x_2 as the *sphere data* on S_1 and S_2 , respectively (Fig. 3).

A reduction of the gluing problem can be formulated as follows: Given two spheres S_1 and S_2 equipped with sphere data x_1 and x_2 , respectively, construct a solution to the null constraint equations along a null hypersurface $\mathcal{H}_{[1,2]}$ whose boundary sections admit the sphere data x_1 and x_2 . Such characteristic gluing of sphere data is at the level of C^2 -gluing for the metric components, meaning that the all metric components and their derivatives up to order 2 are glued (Fig. 4).

The characteristic gluing problem as stated above cannot always be solved. To start, one obvious obstruction is imposed by the monotonicity property of the Raychaudhuri equation along null hypersurfaces. A more subtle obstruction is imposed by the existence of an infinite-dimensional space of conservation laws for the linearized null constraint equations at Minkowski spacetime.

To address this hindrance, we need to take into account the change of sphere data under *sphere perturbations* and *sphere diffeomorphisms*. We define *sphere perturbations* of the sphere data x_2 on S_2 in the vacuum spacetime $\mathcal{M}_2$ as follows. Consider the null hypersurface $\mathcal{H}_2$ in $\mathcal{M}_2$ through S_2 that is conjugate (that is, transversal) to the null hypersurface $\mathcal{H}_1$ which the gluing hypersurface should attach to. Then, the sphere data x'_2 on a section S'_2 of $\mathcal{H}_2$ is called a sphere perturbation of x_2 on S_2 (Fig. 5). Sphere diffeomorphisms of sphere data are defined by pulling back the sphere data under a diffeomorphism of the sphere.

We arrive at the following reformulation of the characteristic gluing problem (Fig. 6).

Given sphere data x_1 and x_2 on two spheres S_1 and S_2 in vacuum spacetimes $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, construct:

- (1) a sphere perturbation S'_2 of S_2 with sphere data x'_2 (subject also to a sphere diffeomorphism),
- (2) a solution to the null constraint equations along a null hypersurface $\mathcal{H}'_{[1,2]}$ whose boundary sections admit the sphere data x_1 and x'_2 .

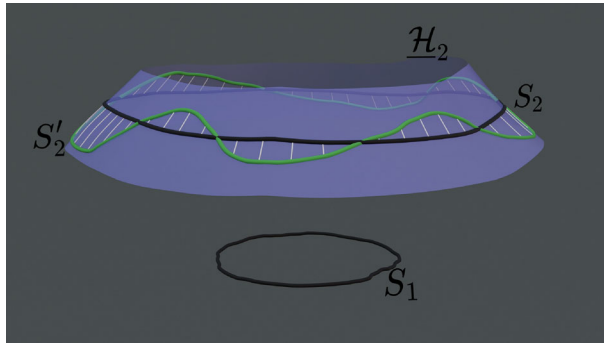


FIGURE 5. Transversal perturbations of the sphere data x_2 on S_2 in $\mathcal{M}_2$

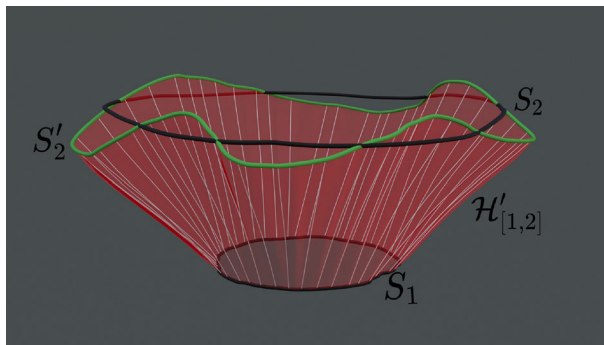


FIGURE 6. The characteristic gluing problem after taking into account transversal gauge transformations of the sphere data

We can now present a first version of the main result of this paper.

Theorem (Codimension-10 perturbative characteristic gluing, version 1). *Consider sphere data x_1 and x_2 on two spheres S_1 and S_2 which are sufficiently close to the sphere data on the round spheres of radius 1 and 2 in Minkowski spacetime, respectively. Then, modulo a 10-dimensional space of charges, characteristic gluing in the above sense is possible. In other words, there is a sphere perturbation S'_2 of the sphere S_2 with sphere data x'_2 (subject also to a sphere diffeomorphism) and a solution to the null constraint equations along a null hypersurface $\mathcal{H}'_{[1,2]}$ (connecting the spheres S_1 and S'_2) such that its restriction to S_1 admits precisely the sphere data x_1 , and its restriction to S'_2 admits the sphere data x'_2 up to 10 explicitly defined charges at S'_2 .*

Remark. In the above theorem, it is equivalently possible to perturb the sphere S_1 instead of the sphere S_2 . Moreover, higher-order derivatives tangential to the gluing hypersurface can be glued without further obstructions.

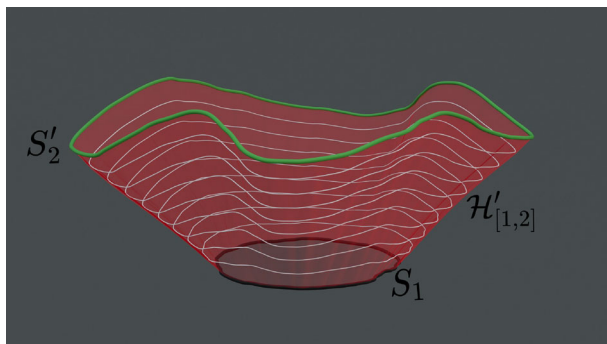


FIGURE 7. There is an infinite-dimensional space of charges defined on the sections of the characteristic hypersurface $\mathcal{H}'_{[1,2]}$. These charges are conserved by the linearized null constraint equations at Minkowski

It is important to underline that the 10 charges correspond to 10 constants and not to 10 functions on the sphere S'_2 . In fact, in this paper, we identify an infinite-dimensional space of charges determined by the sphere data $x|_S$ of the sections S of the null hypersurface $\mathcal{H}'_{[1,2]}$ and we show that it splits into a 10-dimensional space of gauge-invariant charges and an infinite-dimensional space of gauge-dependent charges (Fig. 7).

We show that the gauge-dependent charges transform linearly under sphere perturbations along $\mathcal{H}_2$ and sphere diffeomorphisms and can moreover be *matched* by an appropriate choice thereof. On the other hand, the gauge-invariant charges can in general not be adjusted by such sphere data perturbations, as they change quadratically under perturbations, which is not enough leeway to adjust them. Matching of the gauge-invariant charges is achieved in [11, 12].

Two remarks regarding the above theorem are in order:

- (1) **Sphere perturbations and sphere diffeomorphisms.** We glue from S_1 to a transversal perturbation S'_2 of S_2 , not to S_2 itself, and the sphere data on S'_2 are also subject to a sphere diffeomorphism. Sphere diffeomorphisms are gauge transformations intrinsic to the given sphere data. On the other hand, sphere perturbations are extrinsic to the given sphere data but they are intrinsic gauge transformations of the ambient spacetime; see, for example, the linearized pure gauge solutions in [27].
- (2) **Transversal regularity.** Higher-order derivatives of the sphere data which are transversal to the gluing null hypersurface are not glued at S'_2 . This is due to the existence of additional higher-order conserved charges which involve these transversal derivatives.

Our next theorem resolves both of these issues by gluing along two null hypersurfaces bifurcating from an auxiliary sphere S_{aux} . The advantage of such an approach is that we can first glue S_1 to the auxiliary sphere S_{aux} in the

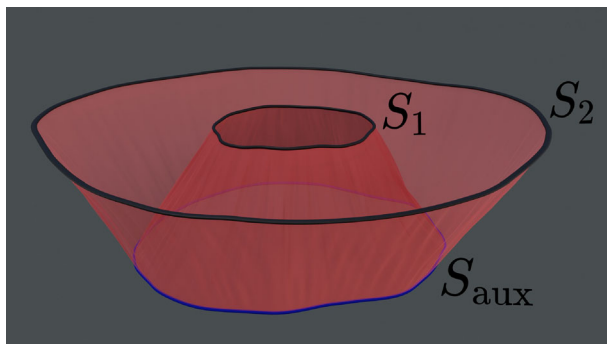


FIGURE 8. First illustration

“ingoing” direction and then glue S_{aux} to the sphere S_2 in the “outgoing” direction (Fig. 8).

Theorem (Codimension-10 bifurcate characteristic gluing, version 1). *Let $m \geq 0$ be an integer. The problem of characteristic gluing along two null hypersurfaces bifurcating from an auxiliary sphere can be solved close to Minkowski for m^{th} -order derivatives in all directions (up to the 10-dimensional space of gauge-invariant charges) without perturbing either of the spheres S_1 and S_2 .*

1.2. Previous Gluing Constructions

1.2.1. Gluing Constructions in General Relativity. Gluing constructions in general relativity are, up to now, mainly focused on the gluing of *spacelike* initial data satisfying the elliptic constraint equations.

Gluing constructions based on the gluing of connected sums (see the works [29, 40] on codimension-3 surgery for manifolds of positive scalar curvature) were studied by Chruściel–Isenberg–Pollack [19, 20], Chruściel–Mazzeo [21], Isenberg–Maxwell–Pollack [32], Isenberg–Mazzeo–Pollack [33, 34].

On the other hand, in the groundbreaking work of Corvino [24] and Corvino–Schoen [25], the *geometric under-determinedness* of the spacelike constraint equations is used to study the (codimension-1) gluing problem. In particular, they showed that asymptotically flat spacelike initial data can be glued across a compact region to exactly Kerr spacelike initial data. Further constructions and refinements based on this approach were proved by Chruściel–Delay [17, 18], Chruściel–Pollack [22], Cortier [23], Hintz [31]. Another milestone was the work by Carlotto–Schoen [14] which showed that it is possible to glue spacelike initial data—along a non-compact cone—to spacelike initial data for Minkowski.

1.2.2. Characteristic Gluing for the Wave Equation. The characteristic gluing problem was studied before by the first author [6] in the much simpler setting of the linear homogeneous wave equation in general (but fixed) Lorentzian manifolds. Similarly to the present paper, [6] determined that the only obstructions

to characteristic gluing are conservation laws along null hypersurfaces. Moreover, it was shown that a necessary and sufficient condition for the existence of such conservation laws is that the kernel of an elliptic operator defined on the null hypersurface [7] is non-trivial. Hence, for the linear wave equation, [6] derived a geometric characterization of all obstructions to characteristic gluing along a general null hypersurface. Specific examples of null hypersurfaces which admit conservation laws for the wave equation are

- (1) the standard cones in Minkowski spacetime,
- (2) the null infinity of asymptotically flat spacetimes and
- (3) the event horizon of extremal black holes.

The conserved charges in cases (2) and (3) above have important applications in the study of the evolution of scalar perturbations on black hole spacetimes. Specifically, the charges along null infinity (also known as the Newman–Penrose constants) are related to the leading-order coefficients of the late-time asymptotics of solution to the wave equation on Schwarzschild [1, 4] and Kerr [2] spacetimes. Similar results were recently obtained for the Dirac equation on Schwarzschild in [37]. On the other hand, the conservation laws on extremal horizons are the source of the horizon instability of extremal black holes [8–10]. It is worth noting that even though the latter charges are defined on sections of the extremal event horizon, they can be computed by far-away observers at null infinity and hence serve as potential observational signatures of extremal black holes [3, 5, 13].

1.3. Double Null Coordinates

In this section, we outline the geometric framework of this paper to provide a first version of our main theorem in Sect. 1.6.

Let S be a spacelike 2-sphere in a spacetime $(\mathcal{M}, \mathbf{g})$, and let u_0 and v_0 with $v_0 > u_0$ be two real numbers. Let u and v be two optical functions of $(\mathcal{M}, \mathbf{g})$ such that $S = \{u = u_0, v = v_0\}$, and for real numbers u_1 and v_1 , the hypersurfaces

$$\mathcal{H}_{u_1} := \{u = u_1\}, \quad \underline{\mathcal{H}}_{v_1} := \{v = v_1\},$$

are outgoing and ingoing null hypersurfaces, respectively. The union of these null hypersurfaces forms a so-called *double null foliation* of $(\mathcal{M}, \mathbf{g})$ (Fig. 9).

On the sphere

$$S_{u_0, v_0} := \{u = u_0, v = v_0\}, \tag{1.1}$$

we define local angular coordinates (θ^1, θ^2) and extend them everywhere by propagating them first along the null generators of $\mathcal{H}_{u_0}$ and then of $\underline{\mathcal{H}}_{v_0}$ for all v , as in Fig. 10. The resulting coordinate system $(u, v, \theta^1, \theta^2)$ is called a *double null coordinate system* (Fig. 10).

With respect to double null coordinates, it holds that

$$\mathbf{g} = -4\Omega^2 du dv + g_{AB} (d\theta^A - b^A dv) (d\theta^B - b^B dv), \tag{1.2}$$

where

- the scalar function Ω is the so-called *null lapse*,

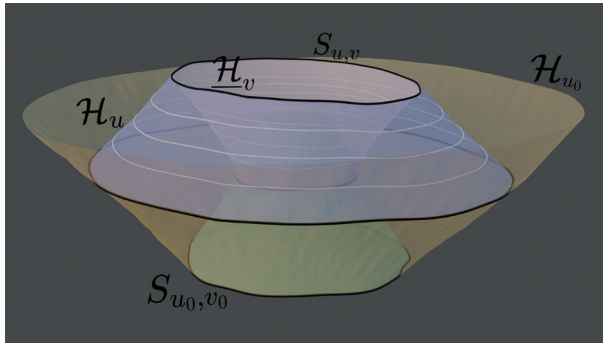


FIGURE 9. The double null foliation formed by the level sets of the optical functions u and v

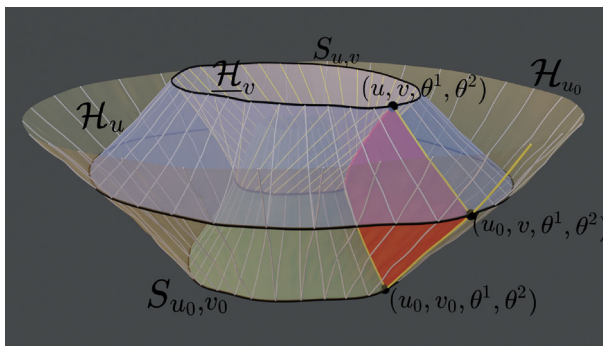


FIGURE 10. The construction of double null coordinates $(u, v, \theta^1, \theta^2)$ on $(\mathcal{M}, \mathbf{g})$

- $\mathcal{g}_{AB}$ is the induced Riemannian metric on the 2-spheres $S_{u,v}$ of constant (u, v) ,
- the $S_{u,v}$ -tangent vectorfield b is the so-called *shift vector*. By construction, it holds that b vanishes on $u = u_0$.

The induced metric $\mathcal{g}$ can be expressed as

$$\mathcal{g} = \phi^2 \mathcal{g}_c,$$

where with respect to the coordinates (θ^1, θ^2) ,

$$\phi^2 := \frac{\sqrt{\mathcal{g}}}{\sqrt{\gamma}}, \quad \mathcal{g}_c := \phi^{-2} \mathcal{g}, \quad \gamma := (d\theta^1)^2 + \sin^2 \theta^1 (d\theta^2)^2.$$

Define the null vectors

$$L := \partial_v + b, \quad \underline{L} := \partial_u, \quad \widehat{L} := \Omega^{-1} L, \quad \widehat{\underline{L}} := \Omega^{-1} \underline{L},$$

and let for $A = 1, 2$, $\partial_A := \partial_{\theta^A}$. Then for $A, B = 1, 2$, the *Ricci coefficients* are defined by

$$\begin{aligned}\chi_{AB} &:= \mathbf{g}(\mathbf{D}_A \widehat{L}, \partial_B), & \underline{\chi}_{AB} &:= \mathbf{g}(\mathbf{D}_A \widehat{\underline{L}}, \partial_B), & \zeta_A &:= \frac{1}{2} \mathbf{g}(\mathbf{D}_A \widehat{L}, \widehat{\underline{L}}), \\ \eta &:= \zeta + \not{d} \log \Omega, & \omega &:= \mathbf{D}_L \log \Omega, & \underline{\omega} &:= \mathbf{D}_{\underline{L}} \log \Omega,\end{aligned}\quad (1.3)$$

where $\mathbf{D}$ denotes the covariant derivative on $(\mathcal{M}, \mathbf{g})$, and define the *null curvature components* by

$$\begin{aligned}\alpha_{AB} &:= \mathbf{R}(\partial_A, \widehat{L}, \partial_B, \widehat{L}), & \beta_A &:= \frac{1}{2} \mathbf{R}(\partial_A, \widehat{L}, \widehat{\underline{L}}, \widehat{L}), \\ \rho &:= \frac{1}{4} \mathbf{R}(\widehat{\underline{L}}, \widehat{L}, \widehat{\underline{L}}, \widehat{L}), & \sigma \cdot \not{\epsilon}_{AB} &:= \frac{1}{2} \mathbf{R}(\partial_A, \partial_B, \widehat{\underline{L}}, \widehat{L}), \\ \beta_A &:= \frac{1}{2} \mathbf{R}(\partial_A, \widehat{\underline{L}}, \widehat{\underline{L}}, \widehat{L}), & \underline{\alpha}_{AB} &:= \mathbf{R}(\partial_A, \widehat{\underline{L}}, \partial_B, \widehat{\underline{L}}),\end{aligned}\quad (1.4)$$

where $\not{\epsilon}_{AB}$ denotes the volume form on $(S_{u,v}, \not{g})$; see (2.8) and (2.10) for details. We split χ and $\underline{\chi}$ into tracefree and trace parts as follows

$$\chi = \widehat{\chi} + \frac{1}{2} \text{tr} \chi \not{g}, \quad \underline{\chi} = \widehat{\underline{\chi}} + \frac{1}{2} \text{tr} \underline{\chi} \not{g}.$$

Moreover, for a $S_{u,v}$ -tangential tensor W , denote by

$$DW = \not{\mathcal{L}}_L W, \quad \underline{D}W = \not{\mathcal{L}}_{\underline{L}} W, \quad (1.5)$$

where $\not{\mathcal{L}}$ denotes the projection of the Lie-derivative onto $S_{u,v}$.

1.4. Null Structure Equations

A Lorentzian 4-manifold $(\mathcal{M}, \mathbf{g})$ is called a *vacuum spacetime* if it satisfies the Einstein vacuum equations

$$\mathbf{Ric} = 0, \quad (1.6)$$

where $\mathbf{Ric}$ denotes the Ricci tensor of $\mathbf{g}$. The Einstein Eqs. (1.6) together with the embedding equations for a double null foliation in a vacuum spacetime stipulate that the metric components, Ricci coefficients and null curvature components satisfy the so-called *null structure equations*. These equations are of *transport-elliptic* character, and they are either tangential to the “outgoing” or the “ingoing” null hypersurfaces. For example, they include the following (see Sect. 2.2 for the full set of equations),

$$\begin{aligned}D\phi &= \frac{\Omega \text{tr} \chi \phi}{2}, \\ D\not{g} &= 2\Omega \chi, \\ D\text{tr} \chi + \frac{\Omega}{2} (\text{tr} \chi)^2 - \omega \text{tr} \chi &= -\Omega |\widehat{\chi}|_{\not{g}}^2, \\ D\eta &= \Omega (\chi \cdot \underline{\eta} - \beta), \\ \text{div} \widehat{\chi} - \frac{1}{2} \not{d} \text{tr} \chi + \widehat{\chi} \cdot \zeta - \frac{1}{2} \text{tr} \chi \zeta &= -\beta,\end{aligned}\quad (1.7)$$

where $(\text{div} \widehat{\chi})_A := \nabla^C \widehat{\chi}_{AC}$, and ∇ denotes the covariant derivative and $\not{d}$ the exterior derivative on $S_{u,v}$.

Moreover, the so-called *null Bianchi equations* hold for the null curvature components. The null Bianchi equations for $D\beta$ and $D\rho$ are as follows.

$$\begin{aligned} D\beta + \frac{3}{2}\Omega\mathrm{tr}\chi\beta - \Omega\widehat{\chi} \cdot \beta - \omega\beta - \Omega(\mathrm{div}\alpha + (\underline{\eta} + 2\zeta) \cdot \alpha) &= 0, \\ D\rho + \frac{3}{2}\Omega\mathrm{tr}\chi\rho - \Omega\left(\mathrm{div}\beta + (2\underline{\eta} + \zeta, \beta) - \frac{1}{2}(\widehat{\chi}, \alpha)\right) &= 0, \end{aligned} \quad (1.8)$$

where $\mathrm{div}\beta := \nabla_A \beta^A$. We refer to Sect. 2.2 for the complete set of null structure equations and null Bianchi equations.

1.5. The Characteristic Initial Value Problem and the Gluing Problem

It is well known that the Einstein Eqs. (1.6) are hyperbolic and admit a well-posed initial value formulation. In the context of this paper, in particular the *characteristic initial value problem* where *characteristic initial data* are posed on two transversely intersecting null hypersurfaces is relevant.

Characteristic initial data for the Einstein equations consist of a pair of hypersurfaces $\mathcal{H}$ and $\underline{\mathcal{H}}$ intersecting at a 2-dimensional surface S together with the (free) specification of

$$(\mathcal{g}, \mathcal{g}_c) \text{ on } \mathcal{H} \cup \underline{\mathcal{H}} \text{ and } (\mathcal{g}, \mathrm{tr}\chi, \mathrm{tr}\underline{\chi}, \eta) \text{ on } S, \quad (1.9)$$

such that $\mathcal{g}_c$ is conformal to $\mathcal{g}$ on S .

The local well-posedness for the characteristic initial value problem was first obtained by Rendall [39]. Specifically, Rendall proved that for sufficiently regular characteristic initial data there exists a unique solution to the Einstein equations in a neighborhood of the surface S . Luk [35] subsequently extended the above result to appropriate neighborhoods of the initial hypersurfaces $\mathcal{H}$ and $\underline{\mathcal{H}}$.

In particular, by virtue of the null structure equations, characteristic initial data determine on the sphere $S = \mathcal{H} \cap \underline{\mathcal{H}}$ the following tuple of S -tangential tensors on S ,

$$\begin{aligned} &(\Omega, \phi, \mathcal{g}_c, \chi, \underline{\chi}, \zeta, \eta, \omega, \underline{\omega}, \alpha, \beta, \rho, \sigma, \underline{\beta}, \underline{\alpha}, \\ &D\phi, D\mathcal{g}_c, D\chi, D\underline{\chi}, D\zeta, D\eta, D\omega, D\underline{\omega}, \\ &\underline{D}\phi, \underline{D}\mathcal{g}_c, \underline{D}\chi, \underline{D}\underline{\chi}, \underline{D}\zeta, \underline{D}\eta, \underline{D}\omega, \underline{D}\underline{\omega})|_S. \end{aligned}$$

We note that the above tuple specifies all derivatives of the spacetime metric up to order 2 on S . By the null structure equations, some quantities are redundant, and we can reduce the above tuple to the following,

$$x := (\Omega, \phi, \mathcal{g}_c, \Omega\mathrm{tr}\chi, \widehat{\chi}, \Omega\mathrm{tr}\underline{\chi}, \widehat{\underline{\chi}}, \eta, \omega, D\omega, \underline{\omega}, \underline{D}\omega, \alpha, \underline{\alpha})|_S.$$

We call this tuple of tensors *C^2 -sphere data* (see also Definition 2.4). Higher-order sphere data, suitable for the solution of a higher regularity gluing problem, require inclusion of the higher-order tangential and transversal derivatives (see also Sect. 2.10). It is essential for this paper that sphere data are affected by gauge transformations (i.e., sphere perturbations and sphere diffeomorphisms).

We are now in position to state the problem of *characteristic gluing*. In the following, we suppress the indices u as we state the problem along a fixed (abstract) 3-dimensional hypersurface $\mathcal{H}_{[1,2]} := [1, 2] \times \mathbb{S}^2$ connecting the (abstract) 2-spheres S_1 and S_2 .

Characteristic gluing problem. *Given sphere data x_1 and x_2 on two space-like 2-spheres S_1 and S_2 , respectively, does there exist a family of sphere data $(x'_v)_{1 \leq v \leq 2}$ on the null hypersurface $\mathcal{H}_{[1,2]} = \cup_{1 \leq v \leq 2} S_v$ solving the null constraint equations on $\mathcal{H}_{[1,2]}$ such that*

$$x'_1 = x_1 \text{ and } x'_2 = x_2?$$

We underline that in the above problem, the sphere data x_1 and x_2 are not required to be induced from an actual vacuum spacetime, but can be seen as abstract tensor tuples, see also Definition 2.4.

The degrees of freedom in the characteristic gluing problem are the free prescription of

$$(\Omega, g_c) \text{ along } \mathcal{H}_{[1,2]}.$$

Characteristic gluing in the above generality is not always feasible. Indeed, for example, the monotonicity of $\text{tr} \chi$ due to the Raychaudhuri Eq. (2.13) forms an obstacle to characteristic gluing. In this paper, we analyze the characteristic gluing problem close to Minkowski. In fact, by applying the implicit function theorem, we can reduce to a study of the *linearized characteristic gluing problem* at Minkowski.

1.6. First Statement of Main Theorem

In the following, we state a first version of our main theorem. First, on a sphere S equipped with a round metric $\overset{\circ}{\gamma}$ we define the projections of functions f on S onto the (normalized) spherical harmonics of mode $l = 0$ and $l = 1$ as follows,

$$f^{(0)} := \int_S f \cdot Y^{(00)} d\mu_{\overset{\circ}{\gamma}}, \quad f^{(1m)} := \int_S f \cdot Y^{(1m)} d\mu_{\overset{\circ}{\gamma}} \text{ for } m = -1, 0, 1.$$

Moreover, we define the projections of vectorfields X on S onto the electric $E^{(1m)}$ and magnetic $H^{(1m)}$ vector spherical harmonics of mode $l = 1$ (defined in Appendix D.2) as follows,

$$X_E^{(1m)} := \int_S \overset{\circ}{\gamma} \left(X, E^{(1m)} \right) d\mu_{\overset{\circ}{\gamma}}, \quad X_H^{(1m)} := \int_S \overset{\circ}{\gamma} \left(X, H^{(1m)} \right) d\mu_{\overset{\circ}{\gamma}} \text{ for } m = -1, 0, 1.$$

Definition 1.1 (*Charges*). Let x be given sphere data on a sphere S . We additionally assume that S is equipped with a round metric $\overset{\circ}{\gamma}$. For $m = -1, 0, 1$, we define the charges

$$\begin{aligned}
\mathbf{E} &:= -\frac{1}{8\pi} \sqrt{4\pi} \left(r^3 (\rho + r \operatorname{div} \beta) \right)^{(0)}, \\
\mathbf{P}^m &:= -\frac{1}{8\pi} \sqrt{\frac{4\pi}{3}} \left(r^3 (\rho + r \operatorname{div} \beta) \right)^{(1m)}, \\
\mathbf{L}^m &:= \frac{1}{16\pi} \sqrt{\frac{8\pi}{3}} \left(r^3 (\not{d} \operatorname{tr} \chi + \operatorname{tr} \chi (\eta - \not{d} \log \Omega)) \right)_H^{(1m)}, \\
\mathbf{G}^m &:= \frac{1}{16\pi} \sqrt{\frac{8\pi}{3}} \left(r^3 (\not{d} \operatorname{tr} \chi + \operatorname{tr} \chi (\eta - \not{d} \log \Omega)) \right)_E^{(1m)},
\end{aligned}$$

where $r = r(x)$ denotes the area radius of the sphere S with sphere data x , and ρ and β are calculated from the sphere data by the null structure equations (more specifically, the Gauss and Gauss–Codazzi equations).

The following is the main result of this paper, see Theorem 3.1 for a precise statement.

Theorem 1.2 (Codimension-10 perturbative characteristic gluing, version 1). *Let x_1 and $x_{0,2}$ be sphere data on two spheres S_1 and $S_{0,2}$, close to sphere data on the round spheres of radius 1 and 2 in Minkowski, respectively. For a real number $\delta > 0$, let $\mathcal{H}_2 = \cup_{-\delta \leq u \leq \delta} S_{u,2}$ be an ingoing null hypersurface passing through $S_{0,2}$, equipped with a family of sphere data $(x_{u,2})_{-\delta \leq u \leq \delta}$ that is close to the respective sphere data in Minkowski and solves the null constraint equations. Then, there are*

- a family of sphere data $(x'_v)_{1 \leq v \leq 2}$ along a hypersurface $\mathcal{H}'_{[1,2]} = \cup_{1 \leq v \leq 2} S'_v$ solving the null constraint equations,
- sphere data $x'_{0,2}$ on a sphere $S'_{0,2}$ stemming from a perturbation of $S_{0,2}$ in $\mathcal{H}_2$ (and subject to a sphere diffeomorphism),

such that on S_1 we have the matching

$$x'_1 = x_1, \quad (1.10)$$

and on S'_2 we have matching of x'_2 and $x'_{0,2}$ up to the charges $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$; that is, if it holds that

$$(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})(x'_2) = (\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})(x'_{0,2}),$$

then it holds that

$$x'_2 = x'_{0,2}.$$

There is no additional obstruction to gluing higher-order tangential derivatives along $\mathcal{H}'_{[1,2]}$.

Remarks on Theorem 1.2.

- (1) The proof of Theorem 1.2 is based on the implicit function theorem and a solution of the *linearized characteristic gluing problem* transversal to the obstruction space consisting of $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$, see Sect. 4.
- (2) The gluing of Theorem 1.2 is at the level of C^2 for the metric components, meaning that we consider the matching of metric components and their derivatives up to order 2. The characteristic gluing of higher-order

tangential derivatives is stated in more detail in Theorem 3.2. For higher-order gluing of derivatives in *all* directions, see the *bifurcate characteristic gluing* in Sect. 1.9.

1.7. Linearized Characteristic Gluing

By the implicit function theorem, the study of the characteristic gluing problem in vicinity of Minkowski can be reduced to the study of the linearized characteristic gluing problem at Minkowski. In this section, we discuss the linearized null constraint equations and null Bianchi equations at Minkowski, and the corresponding linearized gluing problem.

1.7.1. Linearized Equations, Characteristic Gluing and Conserved Charges.

The linearized null constraint equations on $\mathcal{H}_{[1,2]} = \cup_{1 \leq v \leq 2} S_v$ can be derived by varying through a family of sphere data $(x_v^\varepsilon)_{1 \leq v \leq 2}$ solving the null constraint equations around Minkowski. We formally denote its expansion in the parameter ε as follows,

$$x_v^\varepsilon = \left(1, v, \overset{\circ}{\gamma}, \frac{2}{v}, 0, -\frac{2}{v}, 0, 0, 0, 0, 0, 0, 0 \right) \Big|_{S_v} + \varepsilon \cdot \left(\dot{\Omega}, \dot{\phi}, \dot{g}_c, (\Omega \dot{\text{tr}} \chi), \dot{\hat{\chi}}, (\Omega \dot{\text{tr}} \underline{\chi}), \dot{\hat{\chi}}, \dot{\eta}, \dot{\omega}, \dot{D}\omega, \dot{\underline{\omega}}, \dot{D}\underline{\omega}, \dot{\alpha}, \dot{\underline{\alpha}} \right) \Big|_{S_v} + \mathcal{O}(\varepsilon^2).$$

The resulting linearized null constraint equations and linearized null Bianchi equations are equations for the *linearized sphere data*

$$\left(\dot{\Omega}, \dot{\phi}, \dot{g}_c, (\Omega \dot{\text{tr}} \chi), \dot{\hat{\chi}}, (\Omega \dot{\text{tr}} \underline{\chi}), \dot{\hat{\chi}}, \dot{\eta}, \dot{\omega}, \dot{D}\omega, \dot{\underline{\omega}}, \dot{D}\underline{\omega}, \dot{\alpha}, \dot{\underline{\alpha}} \right).$$

For example, linearizing the null constraint Eqs. (1.7) and null Bianchi Eqs. (1.8) yields (where we have that $r = v$ and $D = \not{D}_{\partial_v}$ in Minkowski space, and denote $\overset{\circ}{\text{div}} = \text{div}_{\overset{\circ}{g}}$),

$$D \left(\frac{\dot{\phi}}{r} \right) = \frac{(\Omega \dot{\text{tr}} \chi)}{2}, \quad D \dot{g}_c = \frac{2}{r^2} \dot{\hat{\chi}}, \quad D \left(D\dot{\phi} - 2\dot{\Omega} \right) = 0, \quad (1.11)$$

and

$$D \left(r^2 \dot{\eta} \right) + \frac{r^2}{2} \not{D} \left((\Omega \dot{\text{tr}} \chi) - \frac{4}{r} \dot{\Omega} \right) - 2r \not{D} \dot{\Omega} - \overset{\circ}{\text{div}} \dot{\hat{\chi}} = 0. \quad (1.12)$$

We also note the linearized null Bianchi equations

$$D \left(r^3 \dot{\beta} \right) - r \overset{\circ}{\text{div}} \dot{\alpha} = 0, \quad D \left(r^3 \dot{\rho} \right) - r \overset{\circ}{\text{div}} \dot{\beta} = 0. \quad (1.13)$$

The *linearized characteristic gluing problem* can be stated as follows.

Linearized characteristic gluing problem. *Given linearized sphere data $\dot{x}_1$ and $\dot{x}_2$ on two spheres S_1 and S_2 , respectively, does there exist a null hypersurface $\mathcal{H}_{[1,2]} = \cup_{1 \leq v \leq 2} S_v$ equipped with a family of linearized sphere data $(\dot{x}'_v)_{1 \leq v \leq 2}$ solving the linearized constraint equations such that*

$$\dot{x}'_1 = \dot{x}_1 \text{ and } \dot{x}'_2 = \dot{x}_2?$$

The degrees of freedom in the linearized characteristic gluing problem are given by prescribing $\dot{\Omega}$ and $\dot{g}_c$ on $\mathcal{H}_{[1,2]}$. By the linearized first variation equation in (1.11), that is,

$$D\dot{g}_c = \frac{2}{r^2}\dot{\chi},$$

this is equivalent to the free prescription of $\dot{\Omega}$ and $\dot{\chi}$ on $\mathcal{H}_{[1,2]}$, which is the point of view we choose in this paper.

In the following, we analyze the obstacles to the linearized gluing problem. Combining the linearized constraint Eqs. (1.11) and (1.12), we get that

$$\begin{aligned} D\left(r^2\dot{\eta} + \frac{r^3}{2}\not{d}\left((\Omega\text{tr}\chi) - \frac{4}{r}\dot{\Omega}\right)\right) &= \text{div}\dot{\chi}, \\ D\left(\frac{r}{2}\left((\Omega\text{tr}\chi) - \frac{4}{r}\dot{\Omega}\right) + \frac{\dot{\phi}}{r}\right) &= 0. \end{aligned} \quad (1.14)$$

Importantly, projecting the second equation onto the vector spherical harmonics of mode $l = 1$ (see Appendix D.2) and using that, in general, the mode $l = 1$ of the divergence of a symmetric tracefree 2-tensor vanishes, we can read off (1.14) that the quantities

$$\begin{aligned} \mathcal{Q}_0 &:= \left(r^2\dot{\eta} + \frac{r^3}{2}\not{d}\left((\Omega\text{tr}\chi) - \frac{4}{r}\dot{\Omega}\right)\right)^{[1]}, \\ \mathcal{Q}_1 &:= \frac{r}{2}\left((\Omega\text{tr}\chi) - \frac{4}{r}\dot{\Omega}\right) + \frac{\dot{\phi}}{r}, \end{aligned} \quad (1.15)$$

are *conserved* along $\mathcal{H}_{[1,2]}$ under the linearized null constraint equations. We note that $\mathcal{Q}_0$ corresponds to 6 numbers, while $\mathcal{Q}_1$ accounts to one functional degree on the sphere.

The charges $\mathcal{Q}_0$ and $\mathcal{Q}_1$ are examples of the larger set of conserved *charges*

$$\mathcal{Q}_i \text{ for } 0 \leq i \leq 7,$$

identified in Sect. 4.1 of this paper. These charges are of fundamental importance for the characteristic gluing problem, as they form *obstructions to gluing*. In particular, the linearizations of the charges $\mathbf{E}, \mathbf{P}, \mathbf{L}$ and $\mathbf{G}$ (see Definition 1.1) form part of the set of conserved charges. In fact, $\mathcal{Q}_0$ is directly related to the linearizations of $\mathbf{L}$ and $\mathbf{G}$, see (4.6).

We remark that we alternatively could have derived $\mathcal{Q}_0$ from the linearized null Bianchi Eq. (1.13) for $D\dot{\beta}$ by relating $\dot{\beta}$ to $\dot{\eta}$ via the linearized Gauss–Codazzi equation. Similarly, the conservation law for the linearizations of $\mathbf{E}$ and $\mathbf{P}$ can be derived by the linearized null Bianchi Eq. (1.13) for $D\dot{\rho}$ by means of the linearized Gauss and Gauss–Codazzi equations.

1.7.2. Linearized Perturbations of Sphere Data and Matching of Charges. In context of the linearized characteristic gluing problem, we also analyze the linearizations of sphere perturbations and sphere diffeomorphisms of sphere

data. We remark that formally, the perturbation S'_2 of S_2 in the ingoing null hypersurface $\mathcal{H}_2 = \cup_{-\delta \leq u \leq \delta} S_{u,2}$ is defined as the level set

$$\{f(u) = 0\} \subset \mathcal{H}_2,$$

for a small *perturbation function* f defined on $\mathcal{H}_2$. We parametrize the sphere diffeomorphisms by a pair $q = (q_1, q_2)$ of scalar functions.

The linearizations of the sphere data in the perturbation functions f and q , evaluated at Minkowski and $f = q = 0$, can be explicitly calculated by the transformation formulas for sphere data (see Lemmas 2.22 and 2.23, and Appendix A). For example, the linearization on S_2 of Ω , $\text{tr}\chi$, ϕ and η under the transversal perturbation function f , evaluated at Minkowski and $f = 0$, is given by

$$\dot{\Omega} = \frac{1}{2} \partial_u \dot{f}, \quad (\Omega \dot{\text{tr}}\chi) = \frac{1}{2} \left(\overset{\circ}{\Delta} + 1 \right) \dot{f}, \quad \dot{\phi} = -\dot{f}, \quad \dot{\eta} = \not{d} \left(\partial_u \dot{f} + \frac{\dot{f}}{2} \right).$$

Plugging the above into the charge expressions (1.15), we see that the charges $\mathcal{Q}_0$ and $\mathcal{Q}_1$ on S_2 of a linearized sphere perturbations are given by the following,

$$\mathcal{Q}_0 = 0, \quad \mathcal{Q}_1 = \frac{1}{2} \overset{\circ}{\Delta} \dot{f} - \partial_u \dot{f}. \quad (1.16)$$

The identity (1.16) reflects the following essential observation: The set of conserved charges $\mathcal{Q}_i$, $0 \leq i \leq 7$, splits into two categories:

- (1) **Gauge-invariant charges.** A 10-dimensional space of *gauge-invariant charges* which are not changing under linearized sphere perturbations. This space is spanned precisely by the linearizations of $\mathbf{E}, \mathbf{P}, \mathbf{L}$ and $\mathbf{G}$.
- (2) **Gauge-dependent charges.** An infinite-dimensional space of *gauge-dependent charges* which can be adjusted in a surjective manner by a carefully chosen linearized sphere perturbations and sphere diffeomorphisms. Of all charges $\mathcal{Q}_i$, $0 \leq i \leq 7$ (see Sect. 4.1), only $\mathbf{E}, \mathbf{P}, \mathbf{L}$ and $\mathbf{G}$ do not fall into this category.

Therefore, in the linearized characteristic gluing problem, we can match all charges at S_2 —except for $\mathbf{E}, \mathbf{P}, \mathbf{L}$ and $\mathbf{G}$ —by adding a linearized sphere perturbation and a linearized sphere diffeomorphism at S_2 .

In [11, 12], we show that the charges $\mathbf{E}, \mathbf{P}, \mathbf{L}$ and $\mathbf{G}$ are related to the ADM energy, linear momentum, angular momentum and center of mass of an asymptotically flat spacetime.

1.7.3. Hierarchical Structure of Radial Weights in the Characteristic Gluing Problem. In the previous section, we showed that matching of all gauge-dependent charges is possible by adding a linearized sphere perturbation to S_2 . In this section, we explain how to prescribe, in addition to the matching of the gauge-dependent charges, the linearized free data $\dot{\Omega}$ and $\hat{\chi}$ on $\mathcal{H}_{[1,2]}$ such that on S_2 we have matching of the full linearized sphere data up to the 10-dimensional space of gauge-invariant charges.

By integrating the linearized null constraint equations and using their nilpotent character (see Sect. 2.7), we can derive representation formulas for

each linearized quantity. For example, integrating (1.11) and (1.14) from $v' = 1$ to $v' = v$, we get the following representation formulas for $\dot{\phi}$, $\dot{g}_c$ and $\dot{\eta}$,

$$\begin{aligned}\dot{\phi}(v) - v\dot{\phi}(1) - \frac{v-1}{2} \left((\Omega \text{tr} \chi)(1) - 4\dot{\Omega}(1) \right) &= 2 \int_1^v \dot{\Omega} dv', & r\text{-weight for } \dot{\Omega} : 1, \\ \dot{g}_c(v) - \dot{g}_c(1) &= 2 \int_1^v \frac{1}{r^2} \hat{\chi} dv', & r\text{-weight for } \hat{\chi} : \frac{1}{r^2} \\ \left[v'^2 \dot{\eta} + \frac{v'^3}{2} \left((\Omega \text{tr} \chi) - \frac{4}{v'} \dot{\Omega} \right) \right]_1^v &= d\dot{W} \left(\int_1^v \hat{\chi} dv' \right), & r\text{-weight for } \hat{\chi} : 1.\end{aligned}$$

Importantly, the representation formulas display a special *hierarchical structure of radial weights* where the integrals on the right-hand sides over the freely prescribed data $\dot{\Omega}$ and $\hat{\chi}$ contain *different r -weights*. Thereby, the integrals are *linearly independent* and it is possible, by prescribing the value of the weighted integrals of $\dot{\Omega}$ and $\hat{\chi}$ over the interval $v = 1$ to $v = 2$, to choose $\dot{\Omega}$ and $\hat{\chi}$ on $\mathcal{H}_{[1,2]}$ such that the corresponding solution to the linearized null constraint equations matches with the prescribed data on S_1 and S_2 .

The existence of conservation laws is connected to the presence of similar r -weights as follows: If the representation formulas for linearized quantities include only integrals of $\dot{\Omega}$ and $\hat{\chi}$ of the same r -weight, then a conserved charge can be constructed from them.

Using the above principle, we can prescribe the free data along $\mathcal{H}_{[1,2]}$ to glue transversely to the space of charges. As we matched the gauge-dependent charges by a linearized sphere perturbation in Sect. 1.7.2, it follows that we glued the linearized sphere data on S_2 up to the 10-dimensional space of gauge-invariant charges.

1.8. Solution of the Nonlinear Characteristic Gluing Problem

The proof of Theorem 1.2 is based on the implicit function theorem and our analysis of the linearized characteristic gluing problem in Sect. 1.7.

The setup for the implicit function theorem is as follows. Consider

- sphere data x_1 on a sphere S_1 ,
- a family of sphere data $(x_{u,2})_{-\delta \leq u \leq \delta}$ on the ingoing null hypersurface $\mathcal{H}_2 = \cup_{-\delta \leq u \leq \delta} S_{u,2}$,
- a family of sphere data $(x'_v)_{1 \leq v \leq 2}$ on the outgoing null hypersurface $\mathcal{H}_{[1,2]} = \cup_{1 \leq v \leq 2} S_v$,
- a sphere perturbation function f and sphere diffeomorphism function q .

Then, we define the mapping $\mathcal{F}$ as follows,

$$\begin{aligned}\mathcal{F} : (x_1, (x_{u,2})_{-\delta \leq u \leq \delta}, (x'_v)_{1 \leq v \leq 2}, f, q) \\ \mapsto \left(x'_1 - x_1, \mathfrak{M}(x'_2) - \mathfrak{M}(\mathcal{P}_{f,q}((x_{u,2})_{-\delta \leq u \leq \delta})), \mathcal{C}(x'_v)_{1 \leq v \leq 2} \right),\end{aligned}$$

where

- $\mathcal{C}$ denotes the null constraint functions (as defined in Sect. 2.7)

- $\mathfrak{M}(x)$ denotes the projection of sphere data x on a space of codimension 10 which accounts for the charges $\mathbf{E}, \mathbf{P}, \mathbf{L}$ and $\mathbf{G}$ (see Definition 2.11 and Lemma 2.12),
- $\mathcal{P}_{f,q}$ denotes the application of sphere data perturbations corresponding to f and q .

By the definition of $\mathcal{F}$, it holds that if

$$\mathcal{F}(x_1, (x_{u,2})_{-\delta \leq u \leq \delta}, (x'_v)_{1 \leq v \leq 2}, f, q) = (0, 0, 0), \quad (1.17)$$

then the family of sphere data $(x'_v)_{1 \leq v \leq 2}$ solves the null constraint equations on $\mathcal{H}_{[1,2]}$, agrees with x_1 on S_1 and matches—up to $\mathbf{E}, \mathbf{P}, \mathbf{L}$ and $\mathbf{G}$ —with a sphere perturbation and sphere diffeomorphism of x_2 on S_2 . This corresponds to solving the characteristic gluing problem as outlined in Theorem 1.2. In the following, we use the implicit function theorem to construct f, j and $(x'_v)_{1 \leq v \leq 2}$ for given x_1 and $(x_{u,2})_{-\delta \leq u \leq \delta}$ such that (1.17) holds.

The implicit function theorem implies that if the linearization of $\mathcal{F}$ in f, j and $(x'_v)_{1 \leq v \leq 2}$, evaluated at $f = 0$ and Minkowski reference data, is *surjective*, then there exists a mapping $\mathcal{G}$,

$$\mathcal{G} : (x_1, (x_{u,2})_{-\delta \leq u \leq \delta}) \mapsto ((x'_v)_{1 \leq v \leq 2}, f, q),$$

well defined close to Minkowski values, such that $\mathcal{G}(x_1, x_2)$ solves the gluing problem (1.17); that is, it holds that

$$\mathcal{F}(x_1, (x_{u,2})_{-\delta \leq u \leq \delta}, \mathcal{G}(x_1, (x_{u,2})_{-\delta \leq u \leq \delta})) = (0, 0, 0).$$

By construction, the surjectivity of the linearization of $\mathcal{F}$ is equivalent to the solvability of the linearized characteristic gluing problem for the *inhomogeneous* linearized null constraint equations. The latter can be shown by a slight generalization of the analysis of the homogeneous linearized equations in Sect. 1.7. We remark that our derived estimates for solutions to the (inhomogeneous) linearized null constraint equations follow a specific *regularity hierarchy* which is also reflected in our definition of function spaces.

More generally, the above implicit function argument applies to the study of the characteristic gluing problem near Schwarzschild of small mass $M \geq 0$. This is essential for our study of characteristic gluing to Kerr in [11, 12].

1.9. Codimension-10 Bifurcate Characteristic Gluing

In our solution to the characteristic gluing problem along $\mathcal{H}_{[1,2]}$, the gluing of higher-order tangential derivatives is in fact without obstacles. However, higher-order *transversal* derivatives cannot be glued in general as they are related to higher-order conserved charges along $\mathcal{H}_{[1,2]}$ of the linearized null constraint equations.

We show in this paper that it is possible to circumvent these conservation laws and glue derivatives of any direction and any order by gluing along two null hypersurfaces $\underline{\mathcal{H}}_{[-1,0],1}$ and $\mathcal{H}_{-1,[1,2]}$ bifurcating from an auxiliary spacelike sphere $S_{-1,1}$, see Fig. 11.

Our result can be summarized as follows, see Theorem 3.3 for a precise statement.

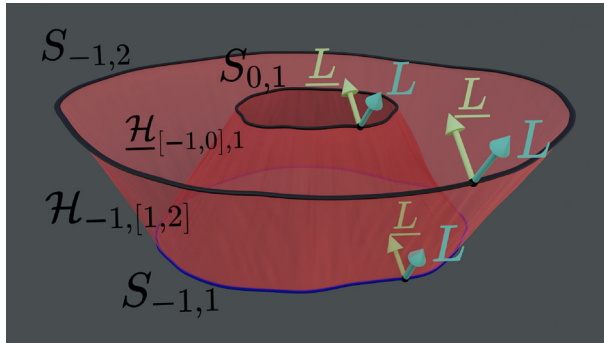


FIGURE 11. The null hypersurfaces $\mathcal{H}_{[-1,0],1}$ and $\mathcal{H}_{-1,[1,2]}$ bifurcating from $S_{-1,1}$

Theorem 1.3 (Codimension-10 bifurcate characteristic gluing, version 1). *Let $m \geq 0$ be an integer. Let $x_{0,1}$ and $x_{-1,2}$ be sphere data on two spheres $S_{0,1}$ and $S_{-1,2}$ together with derivatives in all directions up to order m . If the prescribed data on $S_{0,1}$ and $S_{-1,2}$ are sufficiently close to the respective Minkowski data, then there are families of sphere data*

$$\begin{aligned} (x'_{u,1})_{-1 \leq u \leq 0} \text{ on } \mathcal{H}_{[-1,0],1} &= \bigcup_{-1 \leq u \leq 0} S_{u,0}, \\ (x'_{-1,v})_{1 \leq v \leq 2} \text{ on } \mathcal{H}_{-1,[1,2]} &= \bigcup_{1 \leq v \leq 2} S_{-1,v}, \end{aligned}$$

solving the null constraint equations and matching to m^{th} -order at the bifurcate auxiliary sphere $S_{-1,1}$, such that we have

- m^{th} -order matching on $S_{0,1}$,
- m^{th} -order matching up to the charges $\mathbf{E}, \mathbf{P}, \mathbf{L}$ and $\mathbf{G}$ on $S_{-1,2}$.

The proof of Theorem 1.3 is based on the implicit function theorem and a study of the *linearized bifurcate characteristic gluing problem*. It is important to remark that in Theorem 1.3 we are not applying any sphere perturbations to $S_{0,1}$ and $S_{-1,2}$. The key insight is that the gauge-dependent charges along $\mathcal{H}_{-1,[1,2]}$ can be matched by adjusting the *free data* on $\mathcal{H}_{[-1,0],1}$, and vice versa, the gauge-dependent charges on $\mathcal{H}_{[-1,0],1}$ can be matched by the *free data* on $\mathcal{H}_{-1,[1,2]}$. Moreover, the spaces of gauge-invariant charges along $\mathcal{H}_{-1,[1,2]}$ and $\mathcal{H}_{[-1,0],1}$ agree. We refer to Sect. 6 for a detailed discussion.

Theorem 1.3 is applied in [11, 12] to glue *spacelike* initial data for the Einstein equations to spacelike initial data for a Kerr black hole spacetime.

1.10. Overview of the Paper

The paper is structured as follows.

- In Sect. 2, we introduce the notation and the geometric setup of this paper.
- In Sect. 3, we precisely state the main results of this paper.

- In Sect. 4, we solve the linearized codimension-10 characteristic gluing problem.
- In Sect. 5, we prove the codimension-10 perturbative characteristic gluing, Theorem 1.2.
- In Sect. 6, we prove the codimension-10 bifurcate characteristic gluing, see Theorem 1.3.
- In Appendix A, we rigorously define and estimate nonlinear perturbations of sphere data.
- In Appendix B, we derive and linearize null transport equations along $\mathcal{H}$.
- In Appendix C, we study linearized null transport equations at Schwarzschild of small mass $M \geq 0$.
- In Appendix D, we recall the theory of Hodge systems on 2-spheres and tensor spherical harmonics and provide a spectral analysis of differential operators studied in Sect. 4.

2. Notation, Definitions and Preliminaries

In this section, we introduce the notation, definitions and preliminaries of this paper. We follow the notation of [15]. For two real numbers A and B , the inequality $A \lesssim B$ means that there is a universal constant $C > 0$ such that $A \leq CB$. Greek indices range over $\alpha = 0, 1, 2, 3$, lowercase Latin indices over $a = 1, 2, 3$ and uppercase Latin indices over $A = 1, 2$.

2.1. Null Geometry

In this section, we recapitulate the well-known construction of local double null coordinates in spacetimes, see, for example, Chapter 1 of [15].

Given a spacetime $(\mathcal{M}, \mathbf{g})$, let $\mathbf{D}$ denote the covariant derivative and $\mathbf{R}$ the Riemann curvature tensor on $(\mathcal{M}, \mathbf{g})$. Let $S \subset \mathcal{M}$ be a spacelike 2-sphere, and let L' on S be an outgoing future-pointing null vectorfield normal to S . Given a scalar function Ω on S , the so-called *null lapse*, let $\underline{L}'$ denote the unique ingoing future-pointing null vectorfield normal to S such that

$$\mathbf{g}(L', \underline{L}') = -2\Omega^{-2}. \quad (2.1)$$

Extend L' and $\underline{L}'$ from S as null geodesic vectorfields in $(\mathcal{M}, \mathbf{g})$, and denote the resulting outgoing and ingoing null geodesic congruences by $\mathcal{H}$ and $\underline{\mathcal{H}}$, respectively.

Given a null lapse Ω on $\mathcal{H}$ and $\underline{\mathcal{H}}$ which extends the null lapse Ω on S , define the vectorfields

$$\hat{L} := \Omega L', \quad L := \Omega^2 L' \text{ on } \mathcal{H}, \quad \hat{\underline{L}} := \Omega \underline{L}', \quad \underline{L} := \Omega^2 \underline{L}' \text{ on } \underline{\mathcal{H}}. \quad (2.2)$$

Define on $\mathcal{H}$ the scalar function v by

$$L(v) = 1 \text{ on } \mathcal{H}, \quad v|_S = 1,$$

and define on $\underline{\mathcal{H}}$ the scalar function u by

$$\underline{L}(u) = 1 \text{ on } \underline{\mathcal{H}}, \quad u|_S = 0.$$

Let $S_{0,v} \subset \mathcal{H}$ and $S_{u,1} \subset \underline{\mathcal{H}}$ denote the level sets of v and u , respectively. On each $S_{0,v}$, define $\underline{L}'$ as the unique ingoing future-pointing null vectorfield normal to $S_{0,v}$ such that (2.1) holds. Similarly, on each $S_{u,1}$, define L' as the unique outgoing future-pointing null vectorfield normal to $S_{u,1}$ such that (2.1) holds. Extend $\underline{L}'$ from $\mathcal{H}$ and L' from $\underline{\mathcal{H}}$, respectively, as null geodesic vectorfields onto $(\mathcal{M}, \mathbf{g})$.

Subsequently, define Ω in $(\mathcal{M}, \mathbf{g})$ by (2.1) and define $(\hat{L}, \hat{\underline{L}})$ and $(L, \underline{L})$ in $\mathcal{M}$ by (2.2). Furthermore, define v and u in $\mathcal{M}$ by

$$\underline{L}(v) = 0, \quad L(u) = 0,$$

with initial values given by the constructed v on $\mathcal{H}$ and u on $\underline{\mathcal{H}}$, respectively. Denote the level sets of u and v in $\mathcal{M}$ by $\mathcal{H}_u$ and $\underline{\mathcal{H}}_v$, respectively, and let

$$\begin{aligned} S_{u,v} &:= \mathcal{H}_u \cap \underline{\mathcal{H}}_v, \quad \mathcal{H}_{u,[v_1,v_2]} := \mathcal{H}_u \cap \left(\bigcup_{v_1 \leq v \leq v_2} \underline{\mathcal{H}}_v \right), \quad \underline{\mathcal{H}}_{[u_1,u_2],v} \\ &:= \left(\bigcup_{u_1 \leq u \leq u_2} \mathcal{H}_u \right) \cap \underline{\mathcal{H}}_v, \end{aligned}$$

and let $\mathcal{g}$ denote the induced Riemannian metric on $S_{u,v}$ and ∇ the induced covariant derivative.

We are now in position to define the so-called *double null coordinates* $(u, v, \theta^1, \theta^2)$ on $\mathcal{M}$. First, define local coordinates (θ^1, θ^2) on each $S_{0,v} \subset \mathcal{H}$ by transporting local coordinates (θ^1, θ^2) on $S = S_{0,1}$ along $\mathcal{H}$ according to

$$L(\theta^1) = L(\theta^2) = 0 \text{ on } \mathcal{H},$$

and then define the local coordinates (θ^1, θ^2) on $\mathcal{M}$ by transporting (θ^1, θ^2) according to

$$\underline{L}(\theta^1) = \underline{L}(\theta^2) = 0 \text{ on } \mathcal{M},$$

with given initial values on $\mathcal{H} = \mathcal{H}_0$.

The following is shown in Chapter 1 of [15],

- The functions u and v are local optical functions on $\mathcal{M}$; that is, they satisfy the Eikonal equations

$$|\mathbf{D}u|^2 = 0, \quad |\mathbf{D}v|^2 = 0,$$

and it holds that

$$L' = -2\mathbf{D}u, \quad \underline{L}' = -2\mathbf{D}v.$$

- In double null coordinates $(u, v, \theta^1, \theta^2)$, the Lorentzian metric $\mathbf{g}$ takes the form

$$\mathbf{g} = -4\Omega^2 dudv + \mathcal{g}_{AB} (d\theta^A - b^A dv) (d\theta^B - b^B dv), \quad (2.3)$$

where the $S_{u,v}$ -tangential vectorfield $b = b^A \partial_A$ is called *shift vector* and satisfies by construction

$$b = 0 \text{ on } \mathcal{H}_0.$$

- It holds that

$$L = \partial_v + b, \quad \underline{L} = \partial_u. \quad (2.4)$$

Introduce furthermore the following notation (following [15]).

- On a sphere $(S_{u,v}, g)$, denote by r the *area radius* defined by

$$4\pi r^2 = \text{area}_g(S_{u,v}).$$

- By the coordinates (θ^1, θ^2) on $S_{u,v}$, we can equip each $S_{u,v}$ with the unit round metric

$$\overset{\circ}{\gamma} := (d\theta^1)^2 + \sin^2 \theta^1 (d\theta^2)^2, \quad (2.5)$$

and define $\gamma := (v-u)^2 \overset{\circ}{\gamma}$ on $S_{u,v}$ for $v > u$. By construction, $\overset{\circ}{\gamma}$ is invariant under the flow of L on $\mathcal{H}$ and under the flow of $\underline{L}$ on $\mathcal{M}$. Denote the volume forms of g and $\overset{\circ}{\gamma}$ on S_v by $\sqrt{g}$ and $\sqrt{\overset{\circ}{\gamma}}$, respectively.

- We decompose the metric g on $S_{u,v}$ into

$$g = \phi^2 g_c, \quad (2.6)$$

where ϕ is a scalar function and g_c is a Riemannian metric on $S_{u,v}$ given by

$$\phi^2 := \frac{\sqrt{g}}{\sqrt{\overset{\circ}{\gamma}}}, \quad g_c := \phi^{-2} g.$$

By definition, it holds that $\sqrt{g_c} = \sqrt{\overset{\circ}{\gamma}}$.

- On each $S_{u,v}$, define with respect to (θ^1, θ^2)
 - the standard (real) spherical harmonics $Y^{(lm)}$ for $l \geq 0$ and $-l \leq m \leq l$,
 - the vector spherical harmonics $E^{(lm)}$ and $H^{(lm)}$ for $l \geq 1$ and $-l \leq m \leq l$,
 - the tensor spherical harmonics $\phi^{(lm)}$ and $\psi^{(lm)}$ for $l \geq 2$ and $-l \leq m \leq l$.

We refer to Appendix D for details and properties of spherical harmonics.

- For a general $S_{u,v}$ -tangent tensorfield W , introduce the notation

$$DW := \mathcal{L}_L W, \quad \underline{D}W := \mathcal{L}_{\underline{L}} W, \quad (2.7)$$

where $\mathcal{L}$ denotes the projection of the Lie derivative on $(\mathcal{M}, g)$ onto the tangent space of $S_{u,v}$.

- For $S_{u,v}$ -tangent vectorfields X and Y , define the *Ricci coefficients* by

$$\begin{aligned} \chi(X, Y) &:= g(\mathbf{D}_X \widehat{L}, Y), & \underline{\chi}(X, Y) &:= g(\mathbf{D}_X \widehat{\underline{L}}, Y), \\ \zeta(X) &:= \frac{1}{2} g(\mathbf{D}_X \widehat{L}, \widehat{L}), & \underline{\zeta}(X) &:= \frac{1}{2} g(\mathbf{D}_X \widehat{\underline{L}}, \widehat{\underline{L}}), \\ \eta &:= \zeta + \not{d} \log \Omega, & \underline{\eta} &:= -\zeta + \not{d} \log \Omega, \\ \omega &:= D \log \Omega, & \underline{\omega} &:= \underline{D} \log \Omega, \end{aligned} \quad (2.8)$$

where $\not{d}$ denotes the extrinsic derivative of $S_{u,v}$. It holds that

$$\underline{\zeta} = -\zeta, \quad \underline{\eta} = -\eta + 2\not{d} \log \Omega. \quad (2.9)$$

- For $S_{u,v}$ -tangent vectorfields X and Y , define the *null curvature components* by

$$\begin{aligned} \alpha(X, Y) &:= \mathbf{R}(X, \widehat{L}, Y, \widehat{L}), & \beta(X) &:= \frac{1}{2} \mathbf{R}(X, \widehat{L}, \widehat{L}, \widehat{L}), \\ \rho &:= \frac{1}{4} \mathbf{R}(\widehat{L}, \widehat{L}, \widehat{L}, \widehat{L}), & \sigma \not{\lrcorner}(X, Y) &:= \frac{1}{2} \mathbf{R}(X, Y, \widehat{L}, \widehat{L}), \\ \underline{\beta}(X) &:= \frac{1}{2} \mathbf{R}(X, \widehat{L}, \widehat{L}, \widehat{L}), & \underline{\alpha}(X, Y) &:= \mathbf{R}(X, \widehat{L}, Y, \widehat{L}), \end{aligned} \quad (2.10)$$

where $\not{\lrcorner}_{AB}$ denotes the volume form on $(S_{u,v}, \not{g})$.

2.2. Null Structure Equations and Null Bianchi Equations

By the null geometry setup in Sect. 2.1 and the Einstein equations, the metric components (2.3), Ricci coefficients (2.8) and null curvature components (2.10) satisfy the so-called *null structure equations*. Before stating them, we introduce the following notation, following Chapter 1 of [15].

- For two $S_{u,v}$ -tangential 1-forms X and Y ,

$$\begin{aligned} (X, Y) &:= \not{g}(X, Y), & (*X)_A &:= \not{\lrcorner}_{AB} X^B, \\ (X \widehat{\otimes} Y)_{AB} &:= X_A Y_B + X_B Y_A - (X \cdot Y) \not{g}_{AB}, & \text{div} X &:= \nabla^A X_A, \\ (\nabla \widehat{\otimes} Y)_{AB} &:= \nabla_A Y_B + \nabla_B Y_A - (\text{div} Y) \not{g}_{AB}, & \text{curl} X &:= \not{\lrcorner}^{AB} \nabla_A X_B, \end{aligned}$$

where $\not{\lrcorner}$ denotes the area 2-form of $S_{u,v}$.

- For two symmetric $S_{u,v}$ -tangential 2-tensors V and W ,

$$\text{tr} V := \not{g}^{AB} V_{AB}, \quad \widehat{V} := V - \frac{1}{2} \text{tr} V \not{g}, \quad V \wedge W := \not{\lrcorner}^{AB} V_{AC} W^C_B.$$

- For a symmetric $S_{u,v}$ -tangential 2-tensor V and a 1-form X ,

$$(V \cdot X)_A := V_{AB} X^B.$$

- For a symmetric $S_{u,v}$ -tangential 2-tensor V ,

$$\text{div} V_A := \nabla^B V_{BA}.$$

- For a symmetric $S_{u,v}$ -tangential tensor W , let $\widehat{DW}$ and $\underline{DW}$ denote the tracefree parts of DW and $\underline{DW}$, respectively, with respect to $\not{g}$.

In this paper, we also use the operators $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_1^*$ and $\mathcal{D}_2^*$ which are introduced in Appendix D.

We are now in position to state the *null structure equations* of a space-time. We have the first variation equations,

$$D\not{g} = 2\Omega\chi, \quad \underline{D}\not{g} = 2\Omega\underline{\chi}, \quad (2.11)$$

which imply specifically that

$$D\phi = \frac{\Omega \text{tr} \chi \phi}{2}, \quad \underline{D}\phi = \frac{\Omega \text{tr} \underline{\chi} \phi}{2}, \quad (2.12)$$

the Raychaudhuri equations,

$$D\mathrm{tr}\chi + \frac{\Omega}{2}(\mathrm{tr}\chi)^2 - \omega\mathrm{tr}\chi = -\Omega|\widehat{\chi}|_{\mathcal{G}}^2, \quad \underline{D}\mathrm{tr}\underline{\chi} + \frac{\Omega}{2}(\mathrm{tr}\underline{\chi})^2 - \underline{\omega}\mathrm{tr}\underline{\chi} = -\Omega|\widehat{\underline{\chi}}|_{\mathcal{G}}^2, \quad (2.13)$$

and

$$\begin{aligned} D\widehat{\chi} &= \Omega|\widehat{\chi}|^2\mathcal{G} + \omega\widehat{\chi} - \Omega\alpha, & \underline{D}\widehat{\underline{\chi}} &= \Omega|\widehat{\underline{\chi}}|^2\mathcal{G} + \omega\widehat{\underline{\chi}} - \Omega\underline{\alpha}, \\ D\eta &= \Omega(\chi \cdot \underline{\eta} - \beta), & \underline{D}\underline{\eta} &= \Omega(\underline{\chi} \cdot \eta + \underline{\beta}), \\ D\underline{\omega} &= \Omega^2(2(\eta, \underline{\eta}) - |\eta|^2 - \rho), & \underline{D}\omega &= \Omega^2(2(\eta, \underline{\eta}) - |\underline{\eta}|^2 - \rho), \\ \mathrm{curl}\eta &= -\frac{1}{2}\widehat{\chi} \wedge \widehat{\underline{\chi}} - \sigma, & \mathrm{curl}\underline{\eta} &= -\mathrm{curl}\eta = -\mathrm{curl}\zeta, \\ D\underline{\eta} &= -\Omega(\chi \cdot \underline{\eta} - \beta) + 2\mathcal{G}\omega, & \underline{D}\eta &= -\Omega(\underline{\chi} \cdot \eta + \underline{\beta}) + 2\mathcal{G}\underline{\omega}. \end{aligned} \quad (2.14)$$

Further, we have the Gauss equation,

$$K + \frac{1}{4}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} - \frac{1}{2}(\widehat{\chi}, \widehat{\underline{\chi}}) = -\rho, \quad (2.15)$$

where K denotes the Gauss curvature of $S_{u,v}$, the Gauss–Codazzi equations

$$\begin{aligned} \mathrm{div}\widehat{\chi} - \frac{1}{2}\mathcal{G}\mathrm{tr}\chi + \widehat{\chi} \cdot \zeta - \frac{1}{2}\mathrm{tr}\chi\zeta &= -\beta, \\ \mathrm{div}\widehat{\underline{\chi}} - \frac{1}{2}\mathcal{G}\mathrm{tr}\underline{\chi} - \widehat{\underline{\chi}} \cdot \zeta + \frac{1}{2}\mathrm{tr}\underline{\chi}\zeta &= \underline{\beta}, \end{aligned} \quad (2.16)$$

and

$$\begin{aligned} D(\Omega\mathrm{tr}\underline{\chi}) &= 2\Omega^2\mathrm{div}\underline{\eta} + 2\Omega^2|\underline{\eta}|^2 - \Omega^2(\widehat{\chi}, \widehat{\underline{\chi}}) - \frac{1}{2}\Omega^2\mathrm{tr}\chi\mathrm{tr}\underline{\chi} + 2\Omega^2\rho, \\ \underline{D}(\Omega\mathrm{tr}\chi) &= 2\Omega^2\mathrm{div}\eta + 2\Omega^2|\eta|^2 - \Omega^2(\widehat{\chi}, \widehat{\underline{\chi}}) - \frac{1}{2}\Omega^2\mathrm{tr}\chi\mathrm{tr}\underline{\chi} + 2\Omega^2\rho, \end{aligned} \quad (2.17)$$

as well as

$$\begin{aligned} D(\Omega\widehat{\underline{\chi}}) &= \Omega^2 \left((\widehat{\chi}, \widehat{\underline{\chi}})\mathcal{G} + \frac{1}{2}\mathrm{tr}\chi\widehat{\underline{\chi}} + \nabla\widehat{\otimes}\underline{\eta} + \underline{\eta}\widehat{\otimes}\underline{\eta} - \frac{1}{2}\mathrm{tr}\chi\widehat{\chi} \right), \\ \underline{D}(\Omega\widehat{\chi}) &= \Omega^2 \left((\widehat{\chi}, \widehat{\underline{\chi}})\mathcal{G} + \frac{1}{2}\mathrm{tr}\underline{\chi}\widehat{\chi} + \nabla\widehat{\otimes}\eta + \eta\widehat{\otimes}\eta - \frac{1}{2}\mathrm{tr}\chi\widehat{\underline{\chi}} \right). \end{aligned} \quad (2.18)$$

By Proposition 1.2 in [15], the following *null Bianchi equations* hold,

$$\begin{aligned}
 \widehat{D}\underline{\alpha} - \frac{1}{2}\Omega\text{tr}\chi\underline{\alpha} + 2\underline{\omega}\underline{\alpha} + \Omega(-\nabla\widehat{\otimes}\beta - (4\underline{\eta} + \zeta)\widehat{\otimes}\beta + 3\widehat{\chi}\rho + 3^*\widehat{\chi}\sigma) &= 0, \\
 \widehat{D}\underline{\alpha} - \frac{1}{2}\Omega\text{tr}\chi\underline{\alpha} + 2\underline{\omega}\underline{\alpha} + \Omega(\nabla\widehat{\otimes}\beta + (4\underline{\eta} - \zeta)\widehat{\otimes}\beta + 3\widehat{\chi}\rho - 3^*\widehat{\chi}\sigma) &= 0, \\
 D\underline{\beta} + \frac{3}{2}\Omega\text{tr}\chi\underline{\beta} - \Omega\widehat{\chi} \cdot \underline{\beta} - \underline{\omega}\underline{\beta} - \Omega(\text{div}\alpha + (\underline{\eta} + 2\underline{\zeta}) \cdot \underline{\alpha}) &= 0, \\
 \underline{D}\underline{\beta} + \frac{3}{2}\Omega\text{tr}\chi\underline{\beta} - \Omega\widehat{\chi} \cdot \underline{\beta} - \underline{\omega}\underline{\beta} + \Omega(\text{div}\underline{\alpha} + (\underline{\eta} - 2\underline{\zeta}) \cdot \underline{\alpha}) &= 0, \\
 \underline{D}\underline{\beta} + \frac{1}{2}\Omega\text{tr}\chi\underline{\beta} - \Omega\widehat{\chi} \cdot \underline{\beta} + \underline{\omega}\underline{\beta} - \Omega(\not{D}\rho + {}^*\not{D}\sigma + 3\underline{\eta}\rho + 3^*\underline{\eta}\sigma + 2\widehat{\chi} \cdot \underline{\beta}) &= 0, \\
 D\underline{\beta} + \frac{1}{2}\Omega\text{tr}\chi\underline{\beta} - \Omega\widehat{\chi} \cdot \underline{\beta} + \underline{\omega}\underline{\beta} + \Omega(\not{D}\rho - {}^*\not{D}\sigma + 3\underline{\eta}\rho - 3^*\underline{\eta}\sigma - 2\widehat{\chi} \cdot \underline{\beta}) &= 0, \\
 D\rho + \frac{3}{2}\Omega\text{tr}\chi\rho - \Omega\left(\text{div}\beta + (2\underline{\eta} + \zeta, \beta) - \frac{1}{2}(\widehat{\chi}, \alpha)\right) &= 0, \\
 \underline{D}\rho + \frac{3}{2}\Omega\text{tr}\chi\rho + \Omega\left(\text{div}\underline{\beta} + (2\underline{\eta} - \zeta, \underline{\beta}) + \frac{1}{2}(\widehat{\chi}, \underline{\alpha})\right) &= 0, \\
 D\sigma + \frac{3}{2}\Omega\text{tr}\chi\sigma + \Omega\left(\text{curl}\beta + (2\underline{\eta} + \zeta, {}^*\beta) - \frac{1}{2}\widehat{\chi} \wedge \alpha\right) &= 0, \\
 \underline{D}\sigma + \frac{3}{2}\Omega\text{tr}\chi\sigma + \Omega\left(\text{curl}\underline{\beta} + (2\underline{\eta} - \zeta, {}^*\underline{\beta}) + \frac{1}{2}\widehat{\chi} \wedge \underline{\alpha}\right) &= 0.
 \end{aligned} \tag{2.19}$$

In addition to the above null structure equations, the following transport equation for $\underline{D}\omega$ is derived in Appendix B.1,

$$\begin{aligned}
 D\underline{D}\omega &= -12\Omega^2(\eta - \not{D}\log\Omega, \not{D}\omega) + 2\Omega^2\underline{\omega}((\eta, -3\underline{\eta} + 4\not{D}\log\Omega) - \rho) \\
 &\quad + 4\Omega^3\chi(\eta, \not{D}\log\Omega) + \Omega^3(\underline{\beta}, 7\underline{\eta} - 3\not{D}\log\Omega) + \frac{3}{2}\Omega^3\text{tr}\chi\rho + \Omega^3\text{div}\underline{\beta} \\
 &\quad + \frac{\Omega^3}{2}(\widehat{\chi}, \underline{\alpha}).
 \end{aligned} \tag{2.20}$$

Similar equations for higher derivatives can be derived by commuting the above equations with $D, \underline{D}, \nabla$.

2.3. Null Geometry of Minkowski and Schwarzschild Spacetimes

In this section, we discuss the null geometry of the Minkowski spacetime and the Schwarzschild family of spacetimes.

Minkowski spacetime. Minkowski spacetime, the trivial solution to the Einstein vacuum Eqs. (1.6), is given by

$$\mathcal{M} = \mathbb{R}^{1+3}, \quad \mathbf{g} = \mathbf{m} := \text{diag}(-1, 1, 1, 1).$$

From the Cartesian coordinates (t, x^1, x^2, x^3) on $\mathbb{R}^{1+3}$, the double null coordinates $(u, v, \theta^1, \theta^2)$ in Minkowski are defined by

$$(u, v, \theta^1, \theta^2) := \left(\frac{1}{2}(t - r), \frac{1}{2}(t + r), \theta^1, \theta^2 \right), \quad (2.21)$$

where $r := \sqrt{\sum_{i=1}^3 (x^i)^2}$, and with respect to which

$$\mathbf{m} = -4dudv + (v - u)^2 \overset{\circ}{\gamma}_{AB} d\theta^A d\theta^B.$$

We note that the area radius of the sphere $S_{u,v}$ is given by $r = v - u$.

In coordinates (2.21), the metric components, Ricci coefficients and null curvature components are given on $S_{u,v}$ by, with $r = v - u$ and $\overset{\circ}{\gamma}$ as in (2.5),

$$\begin{aligned} \Omega &= 1, & \not{g} &= r^2 \overset{\circ}{\gamma}, \\ \text{tr} \chi &= \frac{2}{r}, & \text{tr} \underline{\chi} &= -\frac{2}{r}, & \widehat{\chi} &= 0, & \widehat{\underline{\chi}} &= 0, \\ \eta &= 0, & \underline{\eta} &= 0, & \zeta &= 0, & \underline{\zeta} &= 0, \\ \omega &= 0, & D\omega &= 0, & \underline{\omega} &= 0, & \underline{D}\omega &= 0, \\ \alpha &= 0, & \beta &= 0, & \underline{\beta} &= 0, & \underline{\alpha} &= 0, & \rho &= 0, & \sigma &= 0. \end{aligned} \quad (2.22)$$

Schwarzschild family of spacetimes. For real numbers $M \in \mathbb{R}$, the family of Schwarzschild metrics is given in Schwarzschild coordinates $(t, r, \theta^1, \theta^2)$ by (see, for example, [30])

$$\mathbf{g} = - \left(1 - \frac{2M}{r} \right) dt^2 + \left(1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (2.23)$$

Setting $M = 0$ leads back to Minkowski spacetime, while $M > 0$ is interpreted as a black hole solution with event horizon at $\{r = 2M\}$. The so-called exterior region $\{r > 2M\}$ is covered by Eddington–Finkelstein double null coordinates $(u, v, \theta^1, \theta^2)$ in which the metric takes the form

$$\mathbf{g} = -4 \left(1 - \frac{2M}{r_M(u, v)} \right) dudv + r_M(u, v)^2 \overset{\circ}{\gamma}_{CD} d\theta^C d\theta^D, \quad (2.24)$$

where the area radius $r_M(u, v)$ is defined by (see, for example, (98) in [27])

$$\frac{v - u}{2M} = \frac{r_M(u, v)}{2M} + \log \left(\frac{r_M(u, v)}{2M} - 1 \right). \quad (2.25)$$

The area radius function $r_M(u, v)$ is smooth in M away from $M = 0$, and continuous in M at $M = 0$. The corresponding null lapse Ω_M is determined by (2.24) to be

$$\Omega_M^2 = 1 - \frac{2M}{r_M}, \quad (2.26)$$

where we note the following standard identities,

$$\partial_v \Omega_M = \frac{\Omega_M M}{r_M^2}, \quad \partial_u \Omega_M = -\frac{\Omega_M M}{r_M^2}, \quad \partial_v r_M = \Omega_M^2, \quad \partial_u r_M = -\Omega_M^2. \quad (2.27)$$

Using (2.26), (2.27) and that by (2.2) and (2.4),

$$\widehat{L} = \Omega^{-1} \partial_v, \quad \widehat{L} = \Omega^{-1} \partial_u,$$

we calculate that in Eddington–Finkelstein double null coordinates $(u, v, \theta^1, \theta^2)$, the metric components, Ricci coefficients and null curvature components are given on $S_{u,v}$ by

$$\begin{aligned} \Omega_M &= \sqrt{1 - \frac{2M}{r_M}}, & \not{g} &= r_M^2 \overset{\circ}{\gamma}, \\ \text{tr} \chi &= \frac{2\Omega_M}{r_M}, & \text{tr} \underline{\chi} &= -\frac{2\Omega_M}{r_M}, & \widehat{\chi} &= 0, & \widehat{\underline{\chi}} &= 0, \\ \eta &= 0, & \underline{\eta} &= 0, & \zeta &= 0, & \underline{\zeta} &= 0, \\ \omega &= \frac{M}{r_M^2}, & D\omega &= -\frac{2M}{r_M^3} \Omega_M^2, & \underline{\omega} &= -\frac{M}{r_M^2}, & \underline{D}\omega &= -\frac{2M}{r_M^3} \Omega_M^2, \\ \alpha &= 0, & \beta &= 0, & \underline{\beta} &= 0, & \underline{\alpha} &= 0, \\ \rho &= -\frac{2M}{r_M^3}, & \sigma &= 0, \end{aligned} \quad (2.28)$$

for $v > u$ such that $r_M = r_M(u, v) > 2M$.

2.4. Tensor Spaces and Calculus Estimate

In this section, we define the basic function spaces of this paper. We remark that the 2-spheres and 3-dimensional hypersurfaces in this section are not assumed to be lying in a vacuum spacetimes. The used double null notation $S_{u,v}$ and $\mathcal{H}_{u_0, [v_1, v_2]}$ is only employed to fix the correct Minkowski reference background sphere data and the corresponding norms.

Definition 2.1 (*Tensor spaces on Riemannian 2-spheres $S_{u,v}$*). For two real numbers $v \geq u$, let $S_{u,v}$ be a 2-sphere equipped with a round unit metric $\overset{\circ}{\gamma}$. For integers $m \geq 0$ and tensors T on $S_{u,v}$, define

$$\|T\|_{H^m(S_{u,v})}^2 := \sum_{i=0}^m \|\nabla^i T\|_{L^2(S_{u,v})}^2,$$

where the covariant derivative ∇ and the measure in $L^2(S_{u,v})$ are with respect to the round metric $\gamma = (v - u)^2 \overset{\circ}{\gamma}$. Moreover, let

$$H^m(S_{u,v}) := \{T : \|T\|_{H^m(S_{u,v})} < \infty\}.$$

Definition 2.2 (*Tensor spaces on null hypersurfaces*). Let $m \geq 0$ and $l \geq 0$ be two integers. In the following, let D and $\underline{D}$ be defined as in (2.7) for null hypersurfaces in Minkowski.

- (1) For real numbers $u_0 < v_1 < v_2$ and $S_{u_0, v}$ -tangential tensors T on $\mathcal{H}_{u_0, [v_1, v_2]}$, define

$$\|T\|_{H_l^m(\mathcal{H}_{u_0, [v_1, v_2]})}^2 := \int_{v_1}^{v_2} \sum_{1 \leq j \leq l} \|D^j T\|_{H^m(S_{u_0, v})}^2 dv,$$

and let

$$H_l^m(\mathcal{H}_{u_0, [v_1, v_2]}) := \{F : \|F\|_{H_l^m(\mathcal{H}_{u_0, [v_1, v_2]})} < \infty\}.$$

- (2) For real numbers $u_1 < u_2 < v_0$ and S_{u, v_0} -tangential tensors T on $\underline{\mathcal{H}}_{[u_1, u_2], v_0}$, define

$$\|T\|_{H_l^m(\underline{\mathcal{H}}_{[u_1, u_2], v_0})}^2 := \int_{u_1}^{u_2} \sum_{1 \leq j \leq l} \|\underline{D}^j T\|_{H^m(S_{u, v_0})}^2 du,$$

and let

$$H_l^m(\underline{\mathcal{H}}_{[u_1, u_2], v_0}) := \{F : \|F\|_{H_l^m(\underline{\mathcal{H}}_{[u_1, u_2], v_0})} < \infty\}.$$

The following standard calculus estimates are applied tacitly throughout this paper. They follow, for example, from Corollaries 3.3 and 3.4, and Lemma 3.20 in [41], and the results of Chapter 13 in [42].

Lemma 2.3 (Calculus estimates). *Let $u_0 < v_1 < v_2$ be real numbers. The following holds.*

- (1) **Trace estimate.** *For any $S_{u_0, v}$ -tangent tensor T on $\mathcal{H}_{u_0, [v_1, v_2]}$, we have that for $v_1 \leq v \leq v_2$,*

$$\|T\|_{H^0(S_{u_0, v})} \leq C_{u_0, v_1, v_2} \cdot \|T\|_{H_1^0(\mathcal{H}_{u_0, [v_1, v_2]})},$$

where the constant $C_{u_0, v_1, v_2} > 0$ depends on u_0, v_1 and v_2 .

- (2) **L^∞ -estimate.** *For any $S_{u_0, v}$ -tangent tensor T on $\mathcal{H}_{u_0, [v_1, v_2]}$, we have that*

$$\|T\|_{L^\infty(\mathcal{H}_{u_0, [v_1, v_2]})} \leq C_{u_0, v_1, v_2} \cdot \|T\|_{H_1^2(\mathcal{H}_{u_0, [v_1, v_2]})},$$

where the constant $C_{u_0, v_1, v_2} > 0$ depends on u_0, v_1 and v_2 .

- (3) **Product estimate.** *Let $m_1, m_2 \geq 2$ and $l_1, l_2 \geq 1$ be integers, and further let $T \in H_{l_1}^{m_1}(\mathcal{H}_{u_0, [v_1, v_2]})$ and $T' \in H_{l_2}^{m_2}(\mathcal{H}_{u_0, [v_1, v_2]})$ be two $S_{u_0, v}$ -tangent tensors. Then, it holds that for integers $0 \leq m \leq \min(m_1, m_2)$ and $0 \leq l \leq \min(l_1, l_2)$,*

$$\begin{aligned} \|T \cdot T'\|_{H_l^m(\mathcal{H}_{u_0, [v_1, v_2]})} &\leq C \cdot \|T\|_{H_l^{m_1}(\mathcal{H}_{u_0, [v_1, v_2]})} \cdot \|T'\|_{H_{l_2}^{m_2}(\mathcal{H}_{u_0, [v_1, v_2]})} \\ &\quad + C \cdot \|T\|_{H_{l_1}^{m_1}(\mathcal{H}_{u_0, [v_1, v_2]})} \cdot \|T'\|_{H_l^{m_2}(\mathcal{H}_{u_0, [v_1, v_2]})}, \end{aligned}$$

where the constant $C > 0$ depends on m, m_1, m_2, l, l_1 and l_2 .

2.5. Sphere Data, Null Data and Norms

In this section, we set up the essential definitions for the characteristic gluing problem.

Definition 2.4 (C^2 -sphere data). For two real numbers $v \geq u$, C^2 -sphere data x consist of a 2-sphere S equipped with a round metric $\overset{\circ}{\gamma}$, see (2.5), and the following tuple of tensors on S ,

$$x = (\Omega, \mathfrak{g}, \Omega \text{tr} \chi, \widehat{\chi}, \Omega \text{tr} \underline{\chi}, \widehat{\underline{\chi}}, \eta, \omega, D\omega, \underline{\omega}, \underline{D\omega}, \alpha, \underline{\alpha}),$$

where

- $\Omega > 0$ is a positive scalar function and g is a Riemannian metric,
- $\Omega \text{tr} \chi, \Omega \text{tr} \underline{\chi}, \omega, D\Omega, \underline{\omega}, \underline{D\omega}$ are scalar functions,
- η is a vectorfield,
- $\hat{\chi}, \hat{\underline{\chi}}, \alpha$ and $\underline{\alpha}$ are symmetric g -tracefree 2-tensors.

Remarks on Definition 2.4.

- (1) We call the above tuple of tensors “ C^2 -sphere data” because by stipulating the null structure equations and null Bianchi equations of Sect. 2.2 on S , sphere data x fully define all metric components and their derivatives up to order 2 on S . Indeed, from sphere data we can calculate the Ricci coefficients and null curvature components

$$(\underline{\eta}, \underline{\zeta}, \underline{\zeta}), (\beta, \rho, \sigma, \underline{\beta}),$$

as well as the derivatives

$$(D\eta, D\underline{\eta}, D\zeta, D\underline{\chi}, D\underline{\omega}), (\underline{D}\eta, \underline{D}\underline{\eta}, \underline{D}\zeta, \underline{D}\chi, \underline{D}\omega), \\ (D\beta, D\rho, D\sigma, D\underline{\beta}, D\underline{\alpha}), (\underline{D}\beta, \underline{D}\sigma, \underline{D}\rho, \underline{D}\beta, \underline{D}\alpha).$$

We omit the explicit verification that sphere data together with the above indeed determine all derivatives of the metric components up to order 2.

- (2) By the specification of $\overset{\circ}{\gamma}$, it follows that sphere data are coordinate-dependent. More generally, the sphere data induced on a spacelike 2-sphere in a spacetime are gauge-dependent; see also Sect. 2.8.

Notation. Let $v > u$ be two real numbers. In this paper, we denote the *reference Minkowski sphere data* $\mathfrak{m}_{u,v}$ on a sphere, correspondingly denoted by $S_{u,v}$, by

$$\mathfrak{m}_{u,v} = \left(1, r^2 \overset{\circ}{\gamma}, \frac{2}{r}, 0, -\frac{2}{r}, 0, 0, 0, 0, 0, 0, 0\right),$$

where $r = v - u$, see (2.22). For real numbers M , we similarly denote the *reference Schwarzschild sphere data* by

$$\mathfrak{m}_{u,v}^M = \left(\Omega_M, r_M^2 \overset{\circ}{\gamma}, \frac{2\Omega_M}{r_M}, 0, -\frac{2\Omega_M}{r_M}, 0, 0, \frac{M}{r_M^2}, -\frac{2M\Omega_M^2}{r_M^3}, -\frac{M}{r_M^2}, -\frac{2M\Omega_M^2}{r_M^3}, 0, 0\right),$$

where $r_M = r_M(u, v)$ is defined in (2.25) and $\Omega_M = (1 - \frac{2M}{r_M})^{1/2}$ in (2.26), see (2.28).

Definition 2.5 (*Norm for sphere data*). Let $v > u$ be two real numbers. Let $x_{u,v}$ be sphere data on a sphere denoted by $S_{u,v}$. Define

$$\|x_{u,v}\|_{\mathcal{X}(S_{u,v})} := \|\Omega\|_{H^6(S_{u,v})} + \|g\|_{H^6(S_{u,v})} + \|\Omega \text{tr} \chi\|_{H^6(S_{u,v})} + \|\hat{\chi}\|_{H^6(S_{u,v})} \\ + \|\Omega \text{tr} \underline{\chi}\|_{H^4(S_{u,v})} + \|\hat{\underline{\chi}}\|_{H^4(S_{u,v})} + \|\eta\|_{H^5(S_{u,v})} \\ + \|\omega\|_{H^6(S_{u,v})} + \|D\omega\|_{H^6(S_{u,v})} + \|\underline{\omega}\|_{H^4(S_{u,v})} + \|\underline{D\omega}\|_{H^2(S_{u,v})} \\ + \|\alpha\|_{H^6(S_{u,v})} + \|\underline{\alpha}\|_{H^2(S_{u,v})},$$

where the norms are with respect to the round metric $\gamma = (v - u)^2 \bar{\gamma}$ on $S_{u,v}$. Let

$$\mathcal{X}(S_{u,v}) := \{x_{u,v} : \|x_{u,v}\|_{\mathcal{X}(S_{u,v})} < \infty\}. \quad (2.29)$$

Remarks on Definition 2.5.

- Definition 2.5 reflects the regularity hierarchy of the null structure equations along the L -direction.

Definition 2.6 (*Null data*). We define the following.

- (1) For real numbers $u_0 < v_1 < v_2$, *outgoing null data* on the abstract manifold $\mathcal{H}_{u_0, [v_1, v_2]} := [v_1, v_2] \times \mathbb{S}^2 := \cup_{v_1 \leq v \leq v_2} S_{u_0, v}$ (where S_{u_0, v_0} are 2-spheres) are given by a tuple of $S_{u_0, v}$ -tangent tensors

$$x = (\Omega, \not{g}, \Omega \text{tr} \chi, \widehat{\chi}, \Omega \text{tr} \underline{\chi}, \widehat{\underline{\chi}}, \eta, \omega, D\omega, \underline{\omega}, \underline{D}\omega, \alpha, \underline{\alpha}), \quad (2.30)$$

such that $x_{u_0, v} := x|_{S_{u_0, v}}$ is sphere data on each $S_{u_0, v} \subset \mathcal{H}_{u_0, [v_1, v_2]}$.

- (2) For real numbers $u_1 < u_2 < v_0$, *ingoing null data* on the abstract manifold $\underline{\mathcal{H}}_{[u_1, u_2], v_0} = [u_1, u_2] \times \mathbb{S}^2 := \cup_{u_1 \leq u \leq u_2} S_{u, v_0}$ are given by a tuple of S_{u, v_0} -tangent tensors

$$\underline{x} = (\Omega, \not{g}, \Omega \text{tr} \chi, \widehat{\chi}, \Omega \text{tr} \underline{\chi}, \widehat{\underline{\chi}}, \eta, \omega, D\omega, \underline{\omega}, \underline{D}\omega, \alpha, \underline{\alpha}), \quad (2.31)$$

such that $\underline{x}_{u, v_0} := \underline{x}|_{S_{u, v_0}}$ is sphere data on each $S_{u, v_0} \subset \underline{\mathcal{H}}_{[u_1, u_2], v_0}$.

Notation. The reference outgoing and ingoing null data of Minkowski are denoted by $\mathbf{m}$ and $\underline{\mathbf{m}}$, respectively; see (2.22). The reference outgoing and ingoing null data of Schwarzschild of mass M are denoted by $\mathbf{m}^M$ and $\underline{\mathbf{m}}^M$, respectively; see (2.28).

The following norm for null data respects the *regularity hierarchy* of the null structure equations.

Definition 2.7 (*Norm for null data*). Let x be null data on $\mathcal{H} := \mathcal{H}_{u_0, [v_1, v_2]}$. Define

$$\begin{aligned} \|x\|_{\mathcal{X}(\mathcal{H})} := & \|\Omega\|_{H_3^6(\mathcal{H})} + \|\not{g}\|_{H_3^6(\mathcal{H})} + \|\Omega \text{tr} \chi\|_{H_3^6(\mathcal{H})} + \|\widehat{\chi}\|_{H_2^6(\mathcal{H})} \\ & + \|\Omega \text{tr} \underline{\chi}\|_{H_2^4(\mathcal{H})} + \|\widehat{\underline{\chi}}\|_{H_3^4(\mathcal{H})} + \|\eta\|_{H_2^5(\mathcal{H})} \\ & + \|\omega\|_{H_2^6(\mathcal{H})} + \|D\omega\|_{H_1^6(\mathcal{H})} + \|\underline{\omega}\|_{H_3^4(\mathcal{H})} + \|\underline{D}\omega\|_{H_3^2(\mathcal{H})} \\ & + \|\alpha\|_{H_1^6(\mathcal{H})} + \|\underline{\alpha}\|_{H_3^2(\mathcal{H})}. \end{aligned}$$

Let $\underline{x}$ be null data on $\underline{\mathcal{H}} := \underline{\mathcal{H}}_{[u_0, u_1], v_0}$. Define

$$\begin{aligned} \|\underline{x}\|_{\mathcal{X}(\underline{\mathcal{H}})} := & \|\Omega\|_{H_3^6(\underline{\mathcal{H}})} + \|\not{g}\|_{H_3^6(\underline{\mathcal{H}})} + \|\Omega \text{tr} \underline{\chi}\|_{H_3^6(\underline{\mathcal{H}})} + \|\widehat{\underline{\chi}}\|_{H_2^6(\underline{\mathcal{H}})} \\ & + \|\Omega \text{tr} \chi\|_{H_2^4(\underline{\mathcal{H}})} + \|\widehat{\chi}\|_{H_3^4(\underline{\mathcal{H}})} + \|\underline{\eta}\|_{H_2^5(\underline{\mathcal{H}})} \\ & + \|\underline{\omega}\|_{H_2^6(\underline{\mathcal{H}})} + \|\underline{D}\omega\|_{H_1^6(\underline{\mathcal{H}})} + \|\omega\|_{H_3^4(\underline{\mathcal{H}})} + \|D\omega\|_{H_3^2(\underline{\mathcal{H}})} \\ & + \|\underline{\alpha}\|_{H_1^6(\underline{\mathcal{H}})} + \|\alpha\|_{H_3^2(\underline{\mathcal{H}})}. \end{aligned}$$

Moreover, let

$$\mathcal{X}(\mathcal{H}) := \{x : \|x\|_{\mathcal{X}(\mathcal{H})} < \infty\}, \quad \mathcal{X}(\underline{\mathcal{H}}) := \{\underline{x} : \|\underline{x}\|_{\mathcal{X}(\underline{\mathcal{H}})} < \infty\}.$$

Remark 2.8. For given null data x on $\mathcal{H} = \mathcal{H}_{u_0, [v_1, v_2]}$, the null structure Eqs. (2.14), (2.15) and (2.16) determine the null curvature components $(\beta, \rho, \sigma, \underline{\beta})$. By standard calculus estimates on $S_{u_0, v}$ (see, for example, Lemma 2.3), it follows that for null data x close to Minkowski, that is,

$$\|x - \mathbf{m}\|_{\mathcal{X}(\mathcal{H})} \leq \varepsilon,$$

for sufficiently small $\varepsilon > 0$, they are bounded by

$$\|\beta\|_{H_2^5(\mathcal{H})} + \left\| \rho + \frac{2M}{r_M^3} \right\|_{H_2^4(\mathcal{H})} + \|\sigma\|_{H_2^4(\mathcal{H})} + \|\underline{\beta}\|_{H_2^3(\mathcal{H})} \lesssim \|x - \mathbf{m}^M\|_{\mathcal{X}(\mathcal{H})}.$$

Analogously, for null data $\underline{x}$ on $\underline{\mathcal{H}}_{[u_1, u_2], v_0} := \underline{\mathcal{H}}$,

$$\|\underline{\beta}\|_{H_2^5(\underline{\mathcal{H}})} + \left\| \rho + \frac{2M}{r_M^3} \right\|_{H_2^4(\underline{\mathcal{H}})} + \|\sigma\|_{H_2^4(\underline{\mathcal{H}})} + \|\underline{\beta}\|_{H_2^3(\underline{\mathcal{H}})} \lesssim \|\underline{x} - \mathbf{m}^M\|_{\mathcal{X}(\underline{\mathcal{H}})}.$$

In the context of sphere perturbations, we work with ingoing null data of higher regularity, see Sect. 2.8 and specifically Proposition 2.21. The corresponding norm for the higher regularity ingoing null data is denoted by $\mathcal{X}^+$. Similarly to $\mathcal{X}$ above, $\mathcal{X}^+$ respects the regularity hierarchy of the null structure equations.

Definition 2.9 (*Higher regularity norm for ingoing null data*). For ingoing null data $\underline{x}$ on $\underline{\mathcal{H}} := \underline{\mathcal{H}}_{[u_1, u_2], v_0}$, define

$$\begin{aligned} \|\underline{x}\|_{\mathcal{X}^+(\underline{\mathcal{H}})} := & \|\Omega\|_{H_3^{12}(\underline{\mathcal{H}})} + \|\not{g}\|_{H_9^{12}(\underline{\mathcal{H}})} + \|\Omega \text{tr} \chi\|_{H_9^{12}(\underline{\mathcal{H}})} + \|\widehat{\chi}\|_{H_8^{12}(\underline{\mathcal{H}})} \\ & + \|\Omega \text{tr} \chi\|_{H_8^{10}(\underline{\mathcal{H}})} + \|\widehat{\chi}\|_{H_9^{10}(\underline{\mathcal{H}})} + \|\underline{\eta}\|_{H_8^{11}(\underline{\mathcal{H}})} \\ & + \|\omega\|_{H_8^{12}(\underline{\mathcal{H}})} + \|\underline{D}\omega\|_{H_7^{12}(\underline{\mathcal{H}})} + \|\omega\|_{H_9^{10}(\underline{\mathcal{H}})} + \|D\omega\|_{H_9^8(\underline{\mathcal{H}})} \\ & + \|\underline{\alpha}\|_{H_7^{12}(\underline{\mathcal{H}})} + \|\alpha\|_{H_9^8(\underline{\mathcal{H}})}. \end{aligned}$$

Further, let

$$\mathcal{X}^+(\underline{\mathcal{H}}) := \{\underline{x} : \|\underline{x}\|_{\mathcal{X}^+(\underline{\mathcal{H}})} < \infty\}.$$

2.6. Charges $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$ and Matching Map $\mathfrak{M}$

In this section, we define the charges $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$ which are of fundamental importance for the characteristic gluing problem, see Theorem 3.1, and the matching map $\mathfrak{M}$ which is used to solve the characteristic gluing problem transversally to the charges.

Definition 2.10 (*Charges*). Let $x_{u,v}$ be sphere data. For $m = -1, 0, 1$, define

$$\begin{aligned} \mathbf{E} &:= -\frac{1}{8\pi} \sqrt{4\pi} \left(r^3 (\rho + r \text{div} \beta) \right)^{(0)}, \\ \mathbf{P}^m &:= -\frac{1}{8\pi} \sqrt{\frac{4\pi}{3}} \left(r^3 (\rho + r \text{div} \beta) \right)^{(1m)}, \\ \mathbf{L}^m &:= \frac{1}{16\pi} \sqrt{\frac{8\pi}{3}} \left(r^3 (\not{d} \text{tr} \chi + \text{tr} \chi (\eta - \not{d} \log \Omega)) \right)_H^{(1m)}, \\ \mathbf{G}^m &:= \frac{1}{16\pi} \sqrt{\frac{8\pi}{3}} \left(r^3 (\not{d} \text{tr} \chi + \text{tr} \chi (\eta - \not{d} \log \Omega)) \right)_E^{(1m)}, \end{aligned}$$

where $r = r(x_{u,v})$ denotes the area radius of $(S_{u,v}, g)$ and the spherical harmonics projections are defined with respect to the unit round metric $\hat{\gamma}$ on $S_{u,v}$, see Appendix D. Here, the null curvature components ρ and β are calculated from $x_{u,v}$ by (2.15) and (2.16).

Remarks on Definition 2.10.

- (1) The linearizations of $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$ at Minkowski satisfy conservation laws along $\mathcal{H}$, see Sect. 4 and (4.6). In [11, 12], we show that in asymptotically flat spacetimes, these conservation laws are related to the *conservation of energy, linear momentum, angular momentum* and the *equation of motion for the center of mass*.
- (2) It holds that on the sphere $S_{u,v}$,

$$(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})(\mathbf{m}^M) = (M, 0, 0, 0).$$

- (3) The charges $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$ play a major role in the characteristic gluing problem because in general they *cannot* be matched on S_2 by our methods, see the statement of Theorem 3.1. This stems from the fact that at the linear level, they satisfy conservation laws and are *invariant* under the linearized sphere perturbations introduced in Sect. 2.8.
- (4) For sphere data $x_{u,v} \in \mathcal{X}(S_{u,v})$, the charges are well defined. Indeed, first, from (2.15) and (2.16), it is straightforward to show that for sufficiently small real numbers $\varepsilon > 0$ and M , and sphere data $x_{u,v}$ with

$$\|x_{u,v} - \mathbf{m}^M\|_{\mathcal{X}(S_{u,v})} \leq \varepsilon,$$

we have that

$$\begin{aligned} & \|\beta\|_{H^5(S_{u,v})} + \left\| \rho + \frac{2M}{r_M^3} \right\|_{H^4(S_{u,v})} + \|\sigma\|_{H^4(S_{u,v})} + \|\underline{\beta}\|_{H^3(S_{u,v})} \\ & \lesssim C_{u,v} \|x_{u,v} - \mathbf{m}^M\|_{\mathcal{X}(S_{u,v})}, \end{aligned}$$

where the constant $C_{u,v} > 0$ depends on u and v . Consequently, by Definition 2.10 together with standard estimates (see, for example, Lemma 2.3), the charges are bounded by

$$|\mathbf{E} - M| + |\mathbf{P}| + |\mathbf{L}| + |\mathbf{G}| \lesssim C_{u,v} \|x_{u,v} - \mathbf{m}^M\|_{\mathcal{X}(S_{u,v})}.$$

As remarked above, the charges $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$ cannot be glued with our methods. To study the characteristic gluing problem modulo the charges, we introduce the following matching map.

Definition 2.11 (*Matching map* $\mathfrak{M}$). Let $x_{u,v}$ be sphere data on $S_{u,v}$. Define

$$\mathfrak{M}(x_{u,v}) :=$$

$$\left(\Omega, \phi, g_c, \Omega \text{tr} \chi, \hat{\chi}, (\Omega \text{tr} \underline{\chi})^{[\geq 2]}, \hat{\underline{\chi}}, \eta^{[\geq 2]}, \omega, D\omega, \underline{\omega}^{[\geq 2]}, \underline{D}\omega^{[\geq 2]}, \tilde{Q}_5, \tilde{Q}_6, \alpha, \underline{\alpha} \right),$$

where ϕ and g_c are defined by (2.6), and the superscript $[\geq 2]$ denotes projection onto the (tensor) spherical harmonics of modes $l \geq 2$, see (D.3). Moreover, in the above, $\tilde{Q}_5$ and $\tilde{Q}_6$ are defined with $r = v - u$ by

$$\begin{aligned}
\tilde{Q}_5 &:= \underline{\omega}^{[\leq 1]} + \frac{1}{4} (\Omega \text{tr} \chi)^{[\leq 1]} - \frac{1}{6r} \mathring{\text{div}} \eta^{[1]} \\
&\quad - \frac{1}{12r^3} (\mathring{\Delta} + 3) \left(\Omega \text{tr} \chi - \frac{4}{r} \Omega \right)^{[\leq 1]} - \frac{1}{2r^2} (\mathring{\Delta} + 2) \phi^{[\leq 1]}, \\
\tilde{Q}_6 &:= (D\omega)^{[\leq 1]} - \frac{1}{6} (\mathring{\Delta} - 3) \left(\frac{1}{r} \Omega \text{tr} \chi - \frac{2}{r^3} (\mathring{\Delta} + 2) \phi \right)^{[\leq 1]} \\
&\quad + \frac{1}{6r} \left(\mathring{\Delta} \mathring{\Delta} + \mathring{\Delta} - 3 \right) \left(\Omega \text{tr} \chi - \frac{4}{r} \Omega \right)^{[\leq 1]} - \frac{2}{3r^2} \mathring{\text{div}} \eta^{[1]},
\end{aligned}$$

where $\mathring{\text{div}}$ and $\mathring{\Delta}$ are the divergence and Laplace–Beltrami operator with respect to the standard unit round metric $\mathring{\gamma}$ on $S_{u,v}$. We also call $\mathfrak{M}_{u,v} := \mathfrak{M}(x_{u,v})$ the matching data at $S_{u,v}$.

Remarks on Definition 2.11.

- (1) In the proof of our main theorem, we show that we are able to glue the matching data on $S_{0,2}$.
- (2) The linearizations of $\tilde{Q}_5$ and $\tilde{Q}_6$ at Minkowski equal the gauge-dependent charges $\mathcal{Q}_5$ and $\mathcal{Q}_6$ of the linearized null constraint equations at Minkowski, see (4.5) and Lemma 4.16 in Sect. 4.3.4.

The following lemma shows that the range of the matching map $\mathfrak{M}$ is the complement to the charges $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$.

Lemma 2.12 (Matching map and charges). *Let $x_{u,v}$ and $x'_{u,v}$ be sphere data on $S_{u,v}$ such that for a real number $\varepsilon > 0$,*

$$\|x_{u,v} - \mathfrak{m}\|_{\mathcal{X}(S_{u,v})} + \|x'_{u,v} - \mathfrak{m}\|_{\mathcal{X}(S_{u,v})} \leq \varepsilon, \quad (2.32)$$

and satisfying

$$\mathfrak{M}(x_{u,v}) = \mathfrak{M}(x'_{u,v}). \quad (2.33)$$

For $\varepsilon > 0$ sufficiently small, the following holds: If, in addition to (2.33),

$$(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})(x_{u,v}) = (\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})(x'_{u,v}), \quad (2.34)$$

then

$$x_{u,v} = x'_{u,v}. \quad (2.35)$$

Proof of Lemma 2.12. First, by Definition 2.10 we rewrite the matching of $\mathbf{L}$ and $\mathbf{G}$ in (2.34) as

$$\begin{aligned}
&(\not\partial \text{tr} \chi(x_{u,v}) + \text{tr} \chi(x_{u,v})(\eta(x_{u,v}) - \not\partial \log \Omega(x_{u,v})))^{[1]} \\
&= (\not\partial \text{tr} \chi(x'_{u,v}) + \text{tr} \chi(x'_{u,v})(\eta(x'_{u,v}) - \not\partial \log \Omega(x'_{u,v})))^{[1]}.
\end{aligned} \quad (2.36)$$

By (2.33) for $\text{tr}\chi$, Ω and $\eta^{[\geq 2]}$, we can rewrite (2.36) as

$$\begin{aligned}
 0 &= (\text{tr}\chi(x'_{u,v}) (\eta(x_{u,v}) - \eta(x'_{u,v})))^{[1]} \\
 &= \left(\text{tr}\chi(x'_{u,v}) (\eta(x_{u,v}) - \eta(x'_{u,v}))^{[1]} \right)^{[1]} \\
 &= \frac{2}{r} (\eta(x_{u,v}) - \eta(x'_{u,v}))^{[1]} \\
 &\quad + \left(\left(\text{tr}\chi(x'_{u,v}) - \frac{2}{r} \right) (\eta(x_{u,v}) - \eta(x'_{u,v}))^{[1]} \right)^{[1]} \\
 &= \frac{2}{r} X + \left(\left(\text{tr}\chi(x'_{u,v}) - \frac{2}{r} \right) X \right)^{[1]},
 \end{aligned} \tag{2.37}$$

where we denoted $X = (\eta(x_{u,v}) - \eta(x'_{u,v}))^{[1]}$. We can rewrite (2.37) as

$$X = -\frac{r}{2} \left(\left(\text{tr}\chi(x'_{u,v}) - \frac{2}{r} \right) X \right)^{[1]}. \tag{2.38}$$

Note that by (2.32),

$$\left\| \text{tr}\chi(x'_{u,v}) - \frac{2}{r} \right\|_{H^6(S_{u,v})} \lesssim \varepsilon,$$

so that (for fixed $r > 0$) from (2.38) we get that

$$|X| \lesssim \varepsilon |X|. \tag{2.39}$$

The inequality (2.39) implies for $\varepsilon > 0$ sufficiently small that $X = 0$, which means, by definition of X above, that

$$\eta(x_{u,v})^{[1]} = \eta(x'_{u,v})^{[1]}. \tag{2.40}$$

By (2.33) for $\eta^{[\geq 2]}$, this implies that $\eta(x_{u,v}) = \eta(x'_{u,v})$ at $S_{u,v}$. By (2.33) and the Gauss–Codazzi equation (2.16), this further implies that $\beta(x_{u,v}) = \beta(x'_{u,v})$ at $S_{u,v}$.

Second, by (2.33), the Gauss equation (2.15) and the above, the matching of $\mathbf{E}$ and $\mathbf{P}$ in (2.34) can be written as

$$\begin{aligned}
 &\left(K(x_{u,v}) + \frac{1}{4} \text{tr}\chi(x_{u,v}) \text{tr}\underline{\chi}(x_{u,v}) - \frac{1}{2} (\widehat{\chi}(x_{u,v}), \widehat{\underline{\chi}}(x_{u,v})) \right)^{[\leq 1]} \\
 &= \left(K(x'_{u,v}) + \frac{1}{4} \text{tr}\chi(x'_{u,v}) \text{tr}\underline{\chi}(x'_{u,v}) - \frac{1}{2} (\widehat{\chi}(x'_{u,v}), \widehat{\underline{\chi}}(x'_{u,v})) \right)^{[\leq 1]}.
 \end{aligned} \tag{2.41}$$

By (2.33) for $\not\chi$, $\widehat{\chi}$, $\widehat{\chi}$, $\text{tr}\chi$ and $\text{tr}\chi^{[\geq 2]}$, we can rewrite (2.41) as

$$\begin{aligned}
 0 &= (\text{tr}\chi(x'_{u,v}) (\text{tr}\chi(x_{u,v}) - \text{tr}\chi(x'_{u,v})))^{[\leq 1]} \\
 &= \left(\text{tr}\chi(x'_{u,v}) (\text{tr}\chi(x_{u,v}) - \text{tr}\chi(x'_{u,v}))^{[\leq 1]} \right)^{[\leq 1]} \\
 &= \frac{2}{r} \left(\text{tr}\chi(x_{u,v})^{[\leq 1]} - \text{tr}\chi(x'_{u,v})^{[\leq 1]} \right) \\
 &\quad + \left(\left(\text{tr}\chi(x'_{u,v}) - \frac{2}{r} \right) (\text{tr}\chi(x_{u,v}) - \text{tr}\chi(x'_{u,v}))^{[\leq 1]} \right)^{[\leq 1]} \\
 &= \frac{2}{r} Y + \left(\left(\text{tr}\chi(x'_{u,v}) - \frac{2}{r} \right) Y \right)^{[\leq 1]},
 \end{aligned} \tag{2.42}$$

where we denoted $Y = (\text{tr}\chi(x_{u,v})^{[\leq 1]} - \text{tr}\chi(x'_{u,v})^{[\leq 1]})$. We can rewrite (2.42) as

$$Y = -\frac{r}{2} \left(\left(\text{tr}\chi(x'_{u,v}) - \frac{2}{r} \right) Y \right)^{[\leq 1]}. \tag{2.43}$$

Using that by (2.32),

$$\left\| \text{tr}\chi(x'_{u,v}) - \frac{2}{r} \right\|_{H^6(S_{u,v})} \lesssim \varepsilon,$$

the relation (2.43) implies (for $r > 0$ fixed) that

$$|Y| \lesssim \varepsilon |Y|. \tag{2.44}$$

For $\varepsilon > 0$ sufficiently small, (2.44) implies that $Y = 0$, which means, by the above definition of Y , that

$$\text{tr}\chi(x_{u,v})^{[\leq 1]} = \text{tr}\chi(x'_{u,v})^{[\leq 1]}. \tag{2.45}$$

From (2.45) and (2.33), we deduce that $\text{tr}\chi(x_{u,v}) = \text{tr}\chi(x'_{u,v})$ at $S_{u,v}$.

Third, it remains to show (2.35) for $\underline{\omega}$ and $\underline{D\omega}$. For $\underline{\omega}^{[\geq 2]}$ and $\underline{D\omega}^{[\geq 2]}$ this follows from (2.33), see Definition 2.11. Moreover, (2.33) implies (2.35) for the quantities

$$\begin{aligned}
 \tilde{Q}_5 &:= \underline{\omega}^{[\leq 1]} + \frac{1}{4} (\Omega \text{tr}\chi)^{[\leq 1]} - \frac{1}{6r} \mathring{\text{d}}\not\chi \eta^{[1]} \\
 &\quad - \frac{1}{12r^3} (\mathring{\Delta} + 3) \left(\Omega \text{tr}\chi - \frac{4}{r} \Omega \right)^{[\leq 1]} - \frac{1}{2r^2} (\mathring{\Delta} + 2) \phi^{[\leq 1]}, \\
 \tilde{Q}_6 &:= (\underline{D\omega})^{[\leq 1]} - \frac{1}{6} (\mathring{\Delta} - 3) \left(\frac{1}{r} \Omega \text{tr}\chi - \frac{2}{r^3} (\mathring{\Delta} + 2) \phi \right)^{[\leq 1]} \\
 &\quad + \frac{1}{6r} \left(\mathring{\Delta} \mathring{\Delta} + \mathring{\Delta} - 3 \right) \left(\Omega \text{tr}\chi - \frac{4}{r} \Omega \right)^{[\leq 1]} - \frac{2}{3r^2} \mathring{\text{d}}\not\chi \eta^{[1]},
 \end{aligned}$$

at $S_{u,v}$. Thus from (2.33) and the above, it follows that also $\underline{\omega}^{[\leq 1]}(x_{u,v}) = \underline{\omega}^{[\leq 1]}(x'_{u,v})$ and $\underline{D\omega}^{[\leq 1]}(x_{u,v}) = \underline{D\omega}^{[\leq 1]}(x'_{u,v})$ on $S_{u,v}$. This finishes the proof of Lemma 2.12. $\square$

The following lemma follows directly by Definition 2.11. Its proof is omitted.

Lemma 2.13 (Smoothness of $\mathfrak{M}$). *Let $v > u$ be two real numbers. The matching map $\mathfrak{M}$ is a smooth mapping from an open neighborhood of $\mathfrak{m}$ in $\mathcal{X}(S_{u,v})$ to $\mathcal{Z}_{\mathfrak{M}}(S_{u,v})$, where (with all spaces over the sphere $S_{u,v}$)*

$$\begin{aligned} \mathcal{Z}_{\mathfrak{M}}(S_{u,v}) = & H^6 \times H^6 \times H^6 \times H^6 \times H^6 \times H^4 \times H^4 \times H^5 \\ & \times H^6 \times H^6 \times H^4 \times H^2 \times H^4 \times H^2 \times H^6 \times H^2, \end{aligned}$$

and we have the estimate

$$\|\mathfrak{M}(x_{u,v}) - \mathfrak{M}(\mathfrak{m})\|_{\mathcal{Z}_{\mathfrak{M}}(S_{u,v})} \lesssim C_{u,v} \|x_{u,v} - \mathfrak{m}\|_{\mathcal{X}(S_{u,v})},$$

where the constant $C_{u,v} > 0$ depends on u and v .

2.7. Nilpotent Character of Null Structure Equations

It is well known that solutions to the null structure equations can be constructed from *free data* which is not subject to any constraint equations, see, for example, [15] and [36]. This is due to the nilpotent character of the null structure equations which reduces the problem to solving a *hierarchy of null transport equations* which can be solved subsequently from the free data. We proceed as follows.

- In Sect. 2.7.1, we define the free data and derive the hierarchy of transport equations, denoted by

$$\mathcal{C}_i = 0 \text{ for } 1 \leq i \leq 10, \quad (2.46)$$

where the maps $(\mathcal{C}_i)_{1 \leq i \leq 10}$ are called *constraint functions*. The equations (2.46) are called the *null constraint equations*.

- In Sect. 2.7.2, we calculate the linearization of the constraint functions at Schwarzschild of mass M .

2.7.1. Definition of Free Data and Derivation of Hierarchy. The following definition of free data is the starting point for the construction of solutions to the null structure equations. For explicitness, we define free data on the null hypersurface $\mathcal{H}_{0,[1,\infty)}$.

Definition 2.14 (*Free data*). On $\mathcal{H}_{0,[1,\infty)}$ prescribe

- the conformal class $\text{conf}(\mathcal{g})$ of induced Riemannian metrics $\mathcal{g}$ on $S_{0,v}$,
- a scalar function Ω , called the null lapse.

On $S_{0,1}$ prescribe

- the induced Riemannian metric $\mathcal{g}$ (compatible with the conformal class on $S_{0,1}$),
- the scalar functions $\text{tr}\chi$, $\text{tr}\underline{\chi}$, $\underline{\omega}$, $\underline{D\omega}$,
- an $S_{0,1}$ -tangential vectorfield η ,
- two $\mathcal{g}$ -tracefree $S_{0,1}$ -tangential symmetric 2-tensors $\widehat{\chi}$ and $\underline{\alpha}$.

Remark on Definition 2.14.

- (1) The above definition is compatible with the initial data for the characteristic initial value problem discussed in Sect. 1.5 where free data (1.9) are prescribed on two transversely intersecting null hypersurfaces $\underline{\mathcal{H}}$ and $\mathcal{H}$. Indeed, in that situation, the components $\text{tr}\chi$, ω , $D\omega$, $\widehat{\chi}$ and $\underline{\alpha}$ of the free data on the intersection sphere $S_{0,1}$ as defined in Definition 2.14 are determined from the free data (1.9) along $\underline{\mathcal{H}}$ through the null constraint equations along $\underline{\mathcal{H}}$.

Before constructing a solution to the null structure equations with the above free data, we introduce the following objects.

- (1) Given the conformal class $\text{conf}(\mathcal{g})$ on $S_{0,v}$, let $\mathcal{g}_c$ be the unique representative such that

$$\sqrt{\det \mathcal{g}_c(v, \theta^1, \theta^2)} = \sqrt{\det \overset{\circ}{\gamma}(\theta^1, \theta^2)}.$$

- (2) Let $\mathcal{g}$ denote the induced metric on $S_{0,v}$ of the solution of the null constraint equations to be constructed. Define $\phi > 0$ to be the conformal factor such that

$$\mathcal{g} = \phi^2 \mathcal{g}_c, \quad (2.47)$$

that is,

$$\phi^2 := \frac{\sqrt{\det \mathcal{g}}}{\sqrt{\det \overset{\circ}{\gamma}}}.$$

- (3) It is straightforward to verify that the *shear* e , defined by

$$e := |\widehat{\chi}|_{\mathcal{g}}^2,$$

is conformally invariant in the sense that e does not change under the conformal transformation $\mathcal{g} \mapsto f^2 \mathcal{g}$ for a scalar function f along $\mathcal{H}_{0,[1,\infty]}$. Hence, as $\mathcal{g}$ is conformally related to $\mathcal{g}_c$, e can be explicitly calculated from $\mathcal{g}_c$ on $\mathcal{H}_{0,[1,\infty)}$.

We are now in position to construct a solution to the null structure equations from the free data. In the following, we derive a hierarchy of null transport equations, called the *constraint functions* $\mathcal{C}_i$ for $1 \leq i \leq 10$, which can be solved based on the free data.

Equation for ϕ . By combining (2.12) and (2.13), that is,

$$D\phi = \frac{\Omega \text{tr}\chi \phi}{2}, \quad D\text{tr}\chi + \frac{\Omega}{2}(\text{tr}\chi)^2 - \omega \text{tr}\chi = -\Omega |\widehat{\chi}|^2,$$

and using (2.8), we get that ϕ satisfies the following *linear* transport equation,

$$\mathcal{C}_1 := D^2\phi - \omega \Omega \text{tr}\chi \phi + \frac{1}{2} \Omega^2 |\widehat{\chi}|^2 \phi = 0.$$

We note that ϕ together with $\mathcal{g}_c$ fully determines $\mathcal{g}$ on each sphere.

Equation for χ . By (2.11), χ satisfies

$$D\mathcal{g} - 2\Omega\chi = 0.$$

Splitting (2.11) into a trace and a tracefree part and using the decomposition (2.47), we get the constraint equations

$$\begin{aligned}\mathcal{C}_2 &:= 2\phi D\phi + \frac{\phi^2}{2}\mathrm{tr}_{\mathcal{J}_c} D\mathcal{J}_c - \Omega\mathrm{tr}\chi\phi^2 = 0, \\ \mathcal{C}_3 &:= -2\Omega\widehat{\chi} + \phi^2 \left(D\mathcal{J}_c - \frac{1}{2}(\mathrm{tr}_{\mathcal{J}_c} D\mathcal{J}_c)\mathcal{J}_c \right) = 0.\end{aligned}$$

Equation for η . By combining (2.14) and (2.16), that is,

$$D\eta = \Omega(\chi \cdot \underline{\eta} - \beta), \quad -\beta = \mathrm{div}\widehat{\chi} - \frac{1}{2}\not{D}\mathrm{tr}\chi + \widehat{\chi} \cdot \zeta - \frac{1}{2}\mathrm{tr}\chi\zeta,$$

and using that by (2.8),

$$\underline{\eta} = -\eta + 2\not{D}\log\Omega, \quad \zeta = \eta - \not{D}\log\Omega,$$

we get that η satisfies the following transport equation,

$$\mathcal{C}_4 := D\eta + \Omega\mathrm{tr}\chi\eta - \Omega \left(\mathrm{div}\widehat{\chi} - \frac{1}{2}\not{D}\mathrm{tr}\chi + \widehat{\chi}\not{D}\log\Omega + \frac{3}{2}\mathrm{tr}\chi\not{D}\log\Omega \right) = 0.$$

Equation for $\Omega\mathrm{tr}\underline{\chi}$. By combining (2.8), (2.15) and (2.17), we get that

$$\begin{aligned}\mathcal{C}_5 &:= D(\Omega\mathrm{tr}\underline{\chi}) + \Omega\mathrm{tr}\chi(\Omega\mathrm{tr}\underline{\chi}) + 2\Omega^2\mathrm{div}(\eta - 2\not{D}\log\Omega) \\ &\quad - 2\Omega^2|\eta - 2\not{D}\log\Omega|^2 + 2\Omega^2K \\ &= 0,\end{aligned}$$

where K denotes the Gauss curvature of $(S_{0,v}, \mathcal{J})$.

Equation for $\widehat{\chi}$. By (2.8) and (2.18), it follows that $\widehat{\chi}$ satisfies

$$\begin{aligned}\mathcal{C}_6 &:= D(\Omega\widehat{\chi}) - (\Omega\widehat{\chi}, \Omega\widehat{\chi})\mathcal{J} - \frac{1}{2}\Omega\mathrm{tr}\chi\Omega\widehat{\chi} \\ &\quad - \Omega^2 \left(\nabla \widehat{\otimes} (2\not{D}\log\Omega - \eta) + (2\not{D}\log\Omega - \eta) \widehat{\otimes} (2\not{D}\log\Omega - \eta) - \frac{1}{2}\mathrm{tr}\chi\widehat{\chi} \right) \\ &= 0.\end{aligned}$$

Equation for $\underline{\omega}$. By (2.8), (2.14) and (2.15), it follows that

$$\mathcal{C}_7 := D\underline{\omega} - \Omega^2 \left(4(\eta, \not{D}\log\Omega) - 3|\eta|^2 + K + \frac{1}{4}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} - \frac{1}{2}(\widehat{\chi}, \widehat{\chi}) \right) = 0.$$

Equation for $\underline{\alpha}$. By (2.8), (2.15), (2.16) and (2.19), it follows that $\underline{\alpha}$ satisfies the following transport equation,

$$\begin{aligned} \mathcal{C}_8 &:= \widehat{D}\underline{\alpha} - \frac{1}{2}\Omega\mathrm{tr}\chi\underline{\alpha} + 2\omega\underline{\alpha} \\ &\quad + \Omega\nabla\widehat{\otimes}\left(\mathrm{div}\widehat{\chi} - \frac{1}{2}\not\partial\mathrm{tr}\chi - \widehat{\chi}\cdot(\eta - \not\partial\log\Omega) + \frac{1}{2}\mathrm{tr}\chi(\eta - \not\partial\log\Omega)\right) \\ &\quad + \Omega(9\not\partial\log\Omega - 5\eta)\widehat{\otimes} \\ &\quad \left(\mathrm{div}\widehat{\chi} - \frac{1}{2}\not\partial\mathrm{tr}\chi - \widehat{\chi}\cdot(\eta - \not\partial\log\Omega) + \frac{1}{2}\mathrm{tr}\chi(\eta - \not\partial\log\Omega)\right) \\ &\quad - 3\Omega\widehat{\chi}\left(K + \frac{1}{4}\mathrm{tr}\chi\mathrm{tr}\chi - \frac{1}{2}(\widehat{\chi}, \widehat{\chi})\right) + 3\Omega^*\widehat{\chi}\left(\mathrm{curl}\eta + \frac{1}{2}\widehat{\chi}\wedge\widehat{\chi}\right) \\ &= 0. \end{aligned}$$

Equation for $\underline{D\omega}$. By (2.8), (2.9), (2.15), (2.16) and (2.20), it follows that

$$\begin{aligned} \mathcal{C}_9 &:= D\underline{D\omega} + 12\Omega^2(\eta - \not\partial\log\Omega, \not\partial\underline{\omega}) - 2\Omega^2\underline{\omega}\cdot(\eta, -3\eta + 4\not\partial\log\Omega) \\ &\quad - \left(2\Omega^2\underline{\omega} - \frac{3}{2}\Omega^3\mathrm{tr}\chi\right)\left(K + \frac{1}{4}\mathrm{tr}\chi\mathrm{tr}\chi - \frac{1}{2}(\widehat{\chi}, \widehat{\chi})\right) - 4\Omega^3\chi(\eta, \not\partial\log\Omega) \\ &\quad - \frac{\Omega^3}{2}(\widehat{\chi}, \underline{\alpha}) \\ &\quad - \Omega^3\left(\mathrm{div}\widehat{\chi} - \frac{1}{2}\not\partial\mathrm{tr}\chi - \widehat{\chi}\cdot(\eta - \not\partial\log\Omega) + \frac{1}{2}\mathrm{tr}\chi(\eta - \not\partial\log\Omega), 7\eta - 3\not\partial\log\Omega\right) \\ &\quad - \Omega^3\mathrm{div}\left(\mathrm{div}\widehat{\chi} - \frac{1}{2}\not\partial\mathrm{tr}\chi - \widehat{\chi}\cdot(\eta - \not\partial\log\Omega) + \frac{1}{2}\mathrm{tr}\chi(\eta - \not\partial\log\Omega)\right) \\ &= 0. \end{aligned}$$

Equation for α . By (2.14), α satisfies

$$\mathcal{C}_{10} := \Omega\alpha + D\widehat{\chi} - \Omega|\widehat{\chi}|^2\mathcal{G} - \omega\widehat{\chi} = 0.$$

Once we have solved the above ten *null constraint equations* for the quantities

$$(\mathcal{G}, \Omega\mathrm{tr}\chi, \widehat{\chi}, \Omega\mathrm{tr}\chi, \widehat{\chi}, \eta, \omega, \underline{D\omega}, \alpha, \underline{\alpha}),$$

the remaining Ricci coefficients, null curvature components and their derivatives can be computed on each sphere using the (sphere-tangential, elliptic equations among the) null structure equations and the null Bianchi equations; this is exactly similar to the situation with sphere data, see Remark (2) after Definition 2.4. In particular, we see that one of the central points of characteristic gluing is to control solutions to the null constraint equations by suitably prescribing characteristic seeds such that their restrictions to S_1 and S_2 admit sphere data that agree with given, prescribed sphere data on S_1 and S_2 as much as possible.

The following lemma shows that the constraint functions are a smooth mapping. Its proof is straightforward and omitted.

Lemma 2.15 (Smoothness of constraint functions). *Consider null data on $\mathcal{H}_{0,[1,2]}$,*

$$x = (\Omega, \not{g}, \Omega \text{tr} \chi, \widehat{\chi}, \Omega \text{tr} \underline{\chi}, \underline{\widehat{\chi}}, \eta, \omega, D\omega, \underline{\omega}, \underline{D\omega}, \alpha, \underline{\alpha}).$$

The constraints map $\mathcal{C}$,

$$\mathcal{C} : x \mapsto (\mathcal{C}_i(x))_{1 \leq i \leq 10},$$

is a smooth mapping from an open neighborhood of $\mathbf{m}$ in $\mathcal{X}(\mathcal{H}_{0,[1,2]})$ to $\mathcal{Z}_{\mathcal{C}}$, where

$$\mathcal{Z}_{\mathcal{C}} := H_2^6 \times H_3^6 \times H_2^6 \times H_1^5 \times H_1^4 \times H_2^5 \times H_2^4 \times H_2^2 \times H_2^2 \times H_1^6,$$

where each space is over $\mathcal{H}_{0,[1,2]}$. Moreover, we have the estimates

$$\|(\mathcal{C}_i(x))_{1 \leq i \leq 10}\|_{\mathcal{Z}_{\mathcal{C}}} \lesssim \|x - \mathbf{m}\|_{\mathcal{X}(\mathcal{H}_{0,[1,2]})}.$$

2.7.2. Linearized Constraint Functions at Minkowski. In this section, we linearize the constraint functions $(\mathcal{C}_i(x))_{1 \leq i \leq 10}$ at Minkowski, that is, at $x = \mathbf{m}$. The linearization procedure is adapted from [27]: We expand the sphere data

$$\begin{aligned} x &= (\Omega, \not{g}, \Omega \text{tr} \chi, \widehat{\chi}, \Omega \text{tr} \underline{\chi}, \underline{\widehat{\chi}}, \eta, \omega, D\omega, \underline{\omega}, \underline{D\omega}, \alpha, \underline{\alpha}) \\ &= \left(1, r^2 \overset{\circ}{\gamma}, \frac{2}{r}, 0, -\frac{2}{r}, 0, 0, 0, 0, 0, 0, 0\right) \\ &\quad + \varepsilon \cdot \left(\dot{\Omega}, \dot{\not{g}}, (\Omega \dot{\text{tr}} \chi), \dot{\widehat{\chi}}, (\Omega \dot{\text{tr}} \underline{\chi}), \dot{\underline{\widehat{\chi}}}, \dot{\eta}, \dot{\omega}, D\dot{\omega}, \dot{\underline{\omega}}, \underline{D\dot{\omega}}, \dot{\alpha}, \dot{\underline{\alpha}}\right) + \mathcal{O}(\varepsilon^2), \end{aligned}$$

and differentiate in ε at $\varepsilon = 0$. Here, we recall that the Minkowski value for r is given by $r = v - u$. The proof of the next lemma follows by explicit calculation.

Lemma 2.16 (Linearization of constraint functions at Minkowski). *Let $(\dot{\mathcal{C}}_i)_{1 \leq i \leq 10}$ denote the linearization of the constraint functions $(\mathcal{C}_i)_{1 \leq i \leq 10}$ at Minkowski. Then, it holds that*

$$\begin{aligned} \dot{\mathcal{C}}_1 &= D^2 \dot{\phi} - 2\dot{\omega} = D(D\dot{\phi} - 2\dot{\Omega}), & \dot{\mathcal{C}}_2 &= r^2 \left(2D \left(\frac{\dot{\phi}}{r} \right) - (\Omega \text{tr} \chi) \right), \\ \dot{\mathcal{C}}_3 &= r^2 D \dot{\not{g}}_c - 2\dot{\widehat{\chi}}, & \dot{\mathcal{C}}_4 &= \frac{1}{r^2} D(r^2 \dot{\eta}) - \frac{4}{r} \not{d} \dot{\Omega} - \frac{1}{r^2} \text{d}\overset{\circ}{\not{w}} \dot{\widehat{\chi}} + \frac{1}{2} \not{d} (\Omega \text{tr} \chi), \end{aligned}$$

and moreover,

$$\begin{aligned}
 \dot{C}_5 &= \frac{1}{r^2} D \left(r^2 (\Omega \dot{\text{tr}} \underline{\chi}) \right) - \frac{2}{r} (\Omega \dot{\text{tr}} \chi) + \frac{2}{r^2} \mathring{\text{div}} \left(\dot{\eta} - 2\mathring{\mathcal{A}} \dot{\Omega} \right) + 2\dot{K} + \frac{4}{r^2} \dot{\Omega}, \\
 \dot{C}_6 &= rD \left(\frac{\dot{\underline{\chi}}}{r} \right) - 2\mathcal{P}_2^* \left(\dot{\eta} - 2\mathring{\mathcal{A}} \dot{\Omega} \right) - \frac{1}{r} \dot{\hat{\chi}}, \\
 \dot{C}_7 &= D\dot{\underline{\omega}} - \dot{K} - \frac{1}{2r} (\Omega \dot{\text{tr}} \underline{\chi}) + \frac{1}{2r} (\Omega \dot{\text{tr}} \chi) - \frac{2}{r^2} \dot{\Omega}, \\
 \dot{C}_8 &= rD \left(\frac{\dot{\underline{\omega}}}{r} \right) - 2\mathcal{P}_2^* \left(\frac{1}{r^2} \mathring{\text{div}} \dot{\underline{\chi}} - \frac{1}{2} \mathring{\mathcal{A}} (\Omega \dot{\text{tr}} \underline{\chi}) - \frac{1}{r} \dot{\eta} \right), \\
 \dot{C}_9 &= D \left(\frac{D\dot{\underline{\omega}}}{r} \right) - \frac{3}{r} \left(\dot{K} + \frac{1}{2r} (\Omega \dot{\text{tr}} \underline{\chi}) - \frac{1}{2r} (\Omega \dot{\text{tr}} \chi) + \frac{2}{r^2} \dot{\Omega} \right) \\
 &\quad - \frac{1}{r^2} \mathring{\text{div}} \left(\frac{1}{r^2} \mathring{\text{div}} \dot{\underline{\chi}} - \frac{1}{2} \mathring{\mathcal{A}} (\Omega \dot{\text{tr}} \underline{\chi}) - \frac{1}{r} \dot{\eta} \right). \\
 \dot{C}_{10} &= \dot{\alpha} + D\dot{\hat{\chi}}.
 \end{aligned}$$

Remark 2.17. In addition to the above, we have by (2.6) and (2.8) that

$$\dot{g} = 2r\dot{\phi}\dot{\gamma} + r^2\dot{g}_c, \quad \dot{\omega} = D\dot{\Omega}, \quad \dot{\underline{\omega}} = \underline{D}\dot{\Omega}, \quad \dot{\underline{\eta}} = -\dot{\eta} + 2\mathring{\mathcal{A}}\dot{\Omega}. \quad (2.48)$$

Moreover, by (242) in [27] the linearization of the Gauss curvature $\dot{K}$ is given by

$$\dot{K} = \frac{1}{2r^2} \mathring{\text{div}} \mathring{\text{div}} \dot{g}_c - \frac{1}{r^3} (\mathring{\Delta} + 2)\dot{\phi}. \quad (2.49)$$

Moreover, using that the area radius r is defined by

$$r^2 = \frac{1}{4\pi} \int_{S_{u,v}} \phi^2 d\mu_{\dot{\gamma}},$$

and that for Minkowski sphere data, $\phi = r$, we have that $\dot{r}^{[\geq 1]} = 0$ and

$$\dot{r}^{(0)} = \dot{\phi}^{(0)}. \quad (2.50)$$

2.8. Perturbations of Sphere Data

In this section, we introduce the perturbation mapping $\mathcal{P}_{f,q}$ of sphere data. It is constructed from subsequent application of first *transversal perturbations of the sphere* and then *angular perturbations by sphere diffeomorphisms*.

Remark 2.18. At the *linear level* (see Lemmas 2.22 and 2.23), the perturbation mapping $\mathcal{P}_{f,q}$ corresponds directly to specific *linear gauge solutions* in [27]. However, the regularity control of $\mathcal{P}_{f,q}$ at the nonlinear level loses regularity compared to the linear level and thus needs separate discussion, see Proposition 2.21 and its proof in Appendix A.

(1) Introduction of transversal perturbations. In the following, we introduce transversal perturbations. In words, the idea is as follows. Given a spacelike 2-sphere $\tilde{S}$ in a vacuum spacetime $(\mathcal{M}, \mathbf{g})$ and a scalar function f on $\tilde{S}$, we

perturb $\tilde{S}$ along the ingoing null direction by an amount f . The resulting sphere is denoted by S and its sphere data by x .

In the following, we sketch the formal definition of transversal perturbations; we refer to Appendix A for full details. Let $(\tilde{u}, \tilde{v}, \tilde{\theta}^1, \tilde{\theta}^2)$ be a local double null coordinate system around $\tilde{S}$ such that

$$\mathbf{g} = -4\tilde{\Omega}^2 d\tilde{u}d\tilde{v} + \tilde{g}_{CD}(d\tilde{\theta}^C - \tilde{l}^C d\tilde{v})(d\tilde{\theta}^D - \tilde{l}^D d\tilde{v}),$$

and

$$\tilde{S} = \tilde{S}_{0,2} := \{\tilde{u} = 0, \tilde{v} = 2\}.$$

Denote by $\tilde{x}_{0,2}$ the sphere data on $\tilde{S}_{0,2}$ with respect to $(\tilde{u}, \tilde{v}, \tilde{\theta}^1, \tilde{\theta}^2)$.

We define new coordinates (u, θ^1, θ^2) on $\tilde{\mathcal{H}}_2 := \{\tilde{v} = 2\}$ as follows. For a given smooth scalar function $f(u, \theta^1, \theta^2)$, define (u, θ^1, θ^2) on $\tilde{\mathcal{H}}_2$ by

$$\tilde{u} = u + f(u, \theta^1, \theta^2), \quad \tilde{\theta}^1 = \theta^1, \quad \tilde{\theta}^2 = \theta^2. \quad (2.51)$$

For f sufficiently small, (u, θ^1, θ^2) indeed forms a local coordinate system on $\tilde{\mathcal{H}}_2$. Define the sphere $S' \subset \tilde{\mathcal{H}}_2$ by

$$S' := \{u = 0\} = \{\tilde{u} = f(0, \theta^1, \theta^2), \tilde{v} = 2\},$$

where we used (2.51). Let $(u, v, \theta^1, \theta^2)$ be the local double null coordinate system on $\mathcal{M}$ such that $v = \tilde{v}$ on $\mathcal{M}$ and (u, θ^1, θ^2) agree with the constructed (u, θ^1, θ^2) on $\tilde{\mathcal{H}}_2$. Let $x_{0,2}$ be the sphere data of $S_{0,2} = S'$ with respect to $(u, v, \theta^1, \theta^2)$. An explicit calculation of $x_{0,2}$ is provided in Appendix A.

The sphere data $x_{0,2}$ depend not only on f and $\tilde{x}_{0,2}$ but also on the ingoing null data $\tilde{x}$ of $(\mathcal{M}, \mathbf{g})$ on $\tilde{\mathcal{H}}_2$ (with respect to $(\tilde{u}, \tilde{v}, \tilde{\theta}^1, \tilde{\theta}^2)$). At this point, we denote the perturbation mapping $\mathcal{P}_{f,q=0}$ by

$$x_{0,2} := \mathcal{P}_{f,q=0}(\tilde{x}).$$

Remark 2.19. In Appendix A, it is shown that the sphere data $x_{0,2}$ on $S_{0,2}$ depend on f only via the four scalar functions

$$(f(0, \theta^1, \theta^2), \partial_u f(0, \theta^1, \theta^2), \partial_u^2 f(0, \theta^1, \theta^2), \partial_u^3 f(0, \theta^1, \theta^2)).$$

In the rest of the paper, we abuse notation and denote this tuple of scalar functions simply by

$$f := (f(0), \partial_u f(0), \partial_u^2 f(0), \partial_u^3 f(0)). \quad (2.52)$$

(2) Introduction of angular perturbations. In the following, we introduce angular perturbations. The setup is a generalization of the linearized pure gauge solutions in Sect. 6 of [27].

Let $v > u$ be two real numbers. Consider a 2-sphere $S_{u,v}$ with sphere data $\tilde{x}_{u,v}$, see Definition 2.4. For a given pair

$$q = (q_1, q_2) \in H^8(S_{u,v}) \times H^8(S_{u,v}), \quad (2.53)$$

on $S_{u,v}$, we define the vectorfield j on $S_{u,v}$ by

$$j := -r^2 \mathcal{P}_1^*(q_1, q_2), \quad (2.54)$$

where $r := v - u$ and the operator $\mathcal{D}_1^*$ (see Definition D.1) is with respect to the round unit metric $\overset{\circ}{\gamma}$ on $S_{u,v}$. There exists an $\varepsilon_0 > 0$ such that for $|t| < \varepsilon_0$, the flow of j , denoted by

$$\Phi_t(q) : S_{u,v} \rightarrow S_{u,v}, \quad (2.55)$$

is well defined until time t for all $|t| < \varepsilon_0$ and a diffeomorphism of $S_{u,v}$ into itself. Clearly, for q_1, q_2 sufficiently small in $H^8(S_{u,v})$, $\Phi_t(q)$ is well defined and a diffeomorphism up to and including $t = 1$; see also Appendix A.

For sufficiently small $q = (q_1, q_2) \in H^8(S_{u,v}) \times H^8(S_{u,v})$ we define the new sphere data

$$x_{u,v} := \mathcal{P}_{0,q}(\tilde{x}_{u,v})$$

to be the pullback of the sphere data $\tilde{x}_{u,v}$ under the diffeomorphism $\Phi_1(q)$.

Notation. Analogously to (2.52), in the rest of this paper we continue to abuse notation by referring to the pair of scalar functions (q_1, q_2) as perturbation function q .

For given perturbation functions f and q as in (2.52) and (2.53), respectively, the perturbation mapping $\mathcal{P}_{f,q}$ is defined as

$$\mathcal{P}_{f,q}(\tilde{x}) := \mathcal{P}_{0,q}(\mathcal{P}_{f,0}(\tilde{x})). \quad (2.56)$$

We introduce the following norms for the perturbation functions f and q .

Definition 2.20 (*Norms for perturbation functions*). We introduce the following.

- (1) For a perturbation function f on $\mathbb{S}^2$ as in (2.52) given by

$$f := (f(0), \partial_u f(0), \partial_u^2 f(0), \partial_u^3 f(0)),$$

define

$$\|f\|_{\mathcal{Y}_f} := \|f(0)\|_{H^8(\mathbb{S}^2)} + \|\partial_u f(0)\|_{H^6(\mathbb{S}^2)} + \|\partial_u^2 f(0)\|_{H^4(\mathbb{S}^2)} + \|\partial_u^3 f(0)\|_{H^2(\mathbb{S}^2)},$$

where the norms are with respect to the round unit metric $\overset{\circ}{\gamma}$. Let $\mathcal{Y}_f := \{f : \|f\|_{\mathcal{Y}_f} < \infty\}$.

- (2) For a perturbation function q as in (2.53) on $\mathbb{S}^2$ given by

$$q := (q_1, q_2),$$

define

$$\|q\|_{\mathcal{Y}_q} := \|q_1\|_{H^8(\mathbb{S}^2)} + \|q_2\|_{H^8(\mathbb{S}^2)},$$

where the norms are with respect to the round unit metric $\overset{\circ}{\gamma}$. Let $\mathcal{Y}_q := \{q : \|q\|_{\mathcal{Y}_q} < \infty\}$.

The following proposition is proved in Appendix A. We note that the regularity analysis of $\mathcal{P}_{f,q}$ is different than the analysis of its linearizations $\dot{\mathcal{P}}_f$ and $\dot{\mathcal{P}}_q$ introduced below, see Remark 2.18.

Proposition 2.21 (Smoothness of $\mathcal{P}_{f,q}$). *Let $\delta > 0$ be a real number. The mapping*

$$\begin{aligned} \mathcal{P} : \mathcal{X}^+(\tilde{\mathcal{H}}_{[-\delta,\delta],2}) \times \mathcal{Y}_f \times \mathcal{Y}_q &\rightarrow \mathcal{X}(S_{0,2}), \\ (\tilde{x}, f, q) &\mapsto x_{0,2} := \mathcal{P}_{f,q}(\tilde{x}), \end{aligned}$$

is well defined and smooth in an open neighborhood of $(\tilde{x}, f, q) = (\underline{\mathbf{m}}, 0, 0)$ and satisfies the estimate

$$\|\mathcal{P}_{f,q}(\tilde{x}) - \tilde{x}_{0,2}\|_{\mathcal{X}(S_{0,2})} \lesssim \|f\|_{\mathcal{Y}_f} + \|q\|_{\mathcal{Y}_q} + \|\tilde{x} - \underline{\mathbf{m}}\|_{\mathcal{X}^+(\tilde{\mathcal{H}}_{[-\delta,\delta],2})},$$

where we denoted $\tilde{x}_{0,2} := \tilde{x}|_{S_{0,2}}$.

In the following, we state the linearizations $\dot{\mathcal{P}}_f$ and $\dot{\mathcal{P}}_q$ of $\mathcal{P}_{f,q}$ at Minkowski and $f = q = 0$. For a proof, we refer to Lemmas C.1 and C.2 in Appendix C where, more generally, the linearizations at Schwarzschild of mass $M \geq 0$ (denoted by $\dot{\mathcal{P}}_f^M$ and $\dot{\mathcal{P}}_q^M$, respectively) are calculated. The linearizations are, by construction, closely related to the *linearized pure gauge solutions* of [27].

First, we have the following.

Lemma 2.22 (Linearization $\dot{\mathcal{P}}_f$ of $\mathcal{P}_{f,q}$). *Let $\dot{\mathcal{P}}_f$ denote the linearization of $\mathcal{P}_{f,q}$ in f at $f = 0$, $q = 0$, and Minkowski. For a given linearized perturbation function $\dot{f}$,*

$$\dot{f} := \left(\dot{f}(0, \theta^1, \theta^2), \partial_u \dot{f}(0, \theta^1, \theta^2), \partial_u^2 \dot{f}(0, \theta^1, \theta^2), \partial_u^3 \dot{f}(0, \theta^1, \theta^2) \right),$$

the non-trivial components of $\dot{\mathcal{P}}_f(\dot{f})$ are given by

$$\begin{aligned} \dot{\Omega} &= \frac{1}{2} \partial_u \left(\dot{f} \right), & \dot{\phi} &= -\dot{f}, & \dot{\eta} &= r \not{d} \left(\partial_u \left(\frac{\dot{f}}{r} \right) \right), \\ \dot{\hat{\chi}} &= -2 \mathcal{P}_2^* \not{d} \dot{f}, & (\Omega \text{tr} \underline{\chi}) &= -2 \partial_u \left(\frac{\dot{f}}{r} \right), & (\Omega \text{tr} \chi) &= \frac{2}{r^2} \left(\overset{\circ}{\Delta} + 1 \right) \dot{f}, \end{aligned}$$

and

$$\underline{\dot{\omega}} = \partial_u \left(\frac{1}{2} \partial_u \dot{f} \right), \quad \underline{D\dot{\omega}} = \partial_u^2 \left(\frac{1}{2} \partial_u \left(\dot{f} \right) \right),$$

where we tacitly evaluated at $u = 0$.

Second, we have the following lemma. Using that clearly the flow (2.55) satisfies $\Phi_1(\varepsilon \dot{j}) = \Phi_\varepsilon(\dot{j})$, it is a corollary of Lemma 6.1.3 in [27], where we note that the proof in [27] at Schwarzschild (see also Lemma C.2 in Appendix C) also goes through at Minkowski, and our notation connects to [27] as follows,

$$\hat{\mathcal{g}} = r_M^2 \dot{\mathcal{g}}_c, \quad \frac{\sqrt{\det \dot{\mathcal{g}}}}{\sqrt{\det \mathcal{g}}} = \frac{2\dot{\phi}}{r_M}.$$

Lemma 2.23 (Linearization $\dot{\mathcal{P}}_q$ of $\mathcal{P}_{f,q}$). *Let $\dot{\mathcal{P}}_q$ denote the linearization of $\mathcal{P}_{f,q}$ in q at $f = 0$, $q = 0$, and Minkowski. The non-trivial components of $\dot{\mathcal{P}}_q(\dot{q})$ are given by, with $\dot{q} = (\dot{q}_1, \dot{q}_2)$,*

$$\dot{\phi} = \frac{r}{2} \overset{\circ}{\Delta} \dot{q}_1, \quad \dot{g}_c = 2\mathcal{D}_2^* \mathcal{D}_1^*(\dot{q}_1, \dot{q}_2).$$

From Lemmas 2.22 and 2.23, we directly conclude the following.

Lemma 2.24 (Boundedness of linearized perturbations of sphere data). *For real numbers $M \geq 0$ sufficiently small, it holds that the linearizations $\dot{\mathcal{P}}_f^M$ and $\dot{\mathcal{P}}_j^M$ are bounded,*

$$\|\dot{\mathcal{P}}_f^M(\dot{f})\|_{\mathcal{X}(S_{0,2})} \lesssim \|\dot{f}\|_{\mathcal{Y}_f}, \quad \|\dot{\mathcal{P}}_q^M(\dot{q})\|_{\mathcal{X}(S_{0,2})} \lesssim \|\dot{q}\|_{\mathcal{Y}_q}.$$

2.9. Implicit Function Theorem

The proof of the main result of this paper is based on the standard implicit function theorem. In the following, we recall its statement and provide further estimates which are applied in Sect. 5.2.

In the following, for Hilbert spaces X and Z and integers $r \geq 0$, let $C^r(X; Z)$ denote the space of r -times continuously differentiable maps from X to Z . Denote the standard norm of this space by $\|\cdot\|_{C^r(X; Z)}$.

The standard implicit function theorem is as follows, see, for example, Theorem 2.5.7 in [38].

Theorem 2.25 (Implicit function theorem). *Let X, Y and Z be Hilbert spaces. Let $U \subset X$ and $V \subset Y$ be open subsets, and let $\mathcal{F} : U \times V \rightarrow Z$ be a C^r -mapping for some integer $r \geq 1$. Assume that for some $x_0 \in U$ and $y_0 \in V$, the linearization*

$$D_x \mathcal{F}|_{(x_0, y_0)} : X \rightarrow Z$$

is an isomorphism. Then, there exist open neighborhoods $V_0 \subset V$ of y_0 and $W_0 \subset Z$ of $\mathcal{F}(x_0, y_0)$ as well as a unique C^r -mapping $\mathcal{G} : V_0 \times W_0 \rightarrow U$ such that for $(y, z) \in V_0 \times W_0$,

$$\mathcal{F}(\mathcal{G}(y, z), y) = z.$$

We further state the following standard calculus estimate.

Lemma 2.26 (Calculus estimate). *Let X, Y and Z be Hilbert spaces. Let $U \subset X$ and $V \subset Y$ be open subsets around $x_0 \in X$ and $y_0 \in Y$, respectively. Let $\mathcal{F} : U \times V \rightarrow Z$ be a C^r -mapping for an integer $r \geq 1$. Then,*

$$\|\mathcal{F}(x, y) - \mathcal{F}(x_0, y_0)\|_Z \lesssim \|x - x_0\|_X + \|y - y_0\|_Y,$$

where the constant depends on $\mathcal{F}$.

Proof of Lemma 2.26. By the fundamental theorem of calculus,

$$\begin{aligned}\mathcal{F}(x, y) - \mathcal{F}(x_0, y_0) &= (\mathcal{F}(x, y) - \mathcal{F}(x_0, y)) - (\mathcal{F}(x_0, y_0) - \mathcal{F}(x_0, y)) \\ &= \int_0^1 D_x \mathcal{F}|_{(xt+(1-t)x_0, y)} (x - x_0) dt \\ &\quad + \int_0^1 D_y \mathcal{F}|_{(x_0, yt+(1-t)y_0)} (y - y_0) dt.\end{aligned}$$

Taking the norm of the above shows that

$$\begin{aligned}\|\mathcal{F}(x, y) - \mathcal{F}(x_0, y_0)\|_Z &\lesssim \|D_x \mathcal{F}\|_{C^0(U \times V; Z)} \|x - x_0\|_X \\ &\quad + \|D_y \mathcal{F}\|_{C^0(U \times V; Z)} \|y - y_0\|_Y \\ &\lesssim \|x - x_0\|_X + \|y - y_0\|_Y.\end{aligned}$$

This finishes the proof of Lemma 2.26. $\square$

2.10. Notation for Characteristic Gluing of Higher-Order Derivatives

As remarked in the introduction, our main characteristic gluing result, Theorem 3.1, can be generalized to glue higher-order derivatives, see Theorems 3.2 and 3.3 in Sect. 3. In this section, we introduce the necessary notation to precisely state these results.

First, we define higher-order sphere data (for C^{m+2} -gluing of metric components) as generalization of Definition 2.4.

Definition 2.27 (*Higher-order sphere data*). Let $m \geq 1$ be an integer, and let $x_{u,v}$ be C^2 -sphere data on a 2-sphere $S_{u,v}$. We define

- *higher-order L -derivatives sphere data of order $m \geq 1$* to be the pair

$$(x_{u,v}, \mathcal{D}_{u,v}^{L,m}),$$

where $\mathcal{D}_{u,v}^{L,m}$ is the following tuple of tensors,

$$\mathcal{D}_{u,v}^{L,m} = \left(\widehat{D}\alpha, \dots, \widehat{D}^m \alpha, D^2 \omega, \dots, D^{m+1} \omega \right), \quad (2.57)$$

where $\widehat{D}^j \alpha$, $1 \leq j \leq m$, are $\mathcal{g}$ -tracefree symmetric 2-tensors on $S_{u,v}$ and $D^j \omega$, $2 \leq j \leq m+1$, are scalar functions on $S_{u,v}$.

- *higher-order $\underline{L}$ -derivatives sphere data of order $m \geq 1$* to be the pair

$$(x_{u,v}, \mathcal{D}_{u,v}^{\underline{L},m}),$$

where $\mathcal{D}_{u,v}^{\underline{L},m}$ is a tuple of tensors,

$$\mathcal{D}_{u,v}^{\underline{L},m} = \left(\widehat{\underline{D}}\alpha, \dots, \widehat{\underline{D}}^m \alpha, \underline{D}^2 \omega, \dots, \underline{D}^{m+1} \omega \right), \quad (2.58)$$

where $\widehat{\underline{D}}^j \alpha$, $1 \leq j \leq m$, are $\mathcal{g}$ -tracefree symmetric 2-tensors on $S_{u,v}$ and $\underline{D}^j \omega$, $2 \leq j \leq m+1$, are scalar functions on $S_{u,v}$.

- *higher-order sphere data of order $m \geq 1$* to be the triplet

$$(x_{u,v}, \mathcal{D}_{u,v}^{L,m}, \mathcal{D}_{u,v}^{\underline{L},m}),$$

where $\mathcal{D}_{u,v}^{L,m}$ and $\mathcal{D}_{u,v}^{\underline{L},m}$ are tuples as in (2.57) and (2.58).

Remarks on Definition 2.27.

- By the null structure equations, higher-order L -derivatives sphere data determine higher-order L -derivatives of the metric components, Ricci coefficients and null curvature components on the sphere. In other words, by straightforward commutation applied to the null structure equations, higher L -derivatives of metric components, Ricci coefficients and null curvature components can be rewritten as higher L -derivatives of α and Ω plus angular derivatives of lower-order L -derivatives. The same holds for higher-order $\underline{L}$ -derivative sphere data. In this sense, higher-order sphere data as defined in Definition 2.27 are appropriate for the higher-order characteristic gluing problem (at the level of C^{m+2} for metric components).
- For integers $m \geq 0$, $\widehat{D}^m \alpha$ and $D^{m+1} \omega$ are at the same level of derivatives of the metric components, as α is a null curvature component, while ω is a Ricci coefficient.

Consider outgoing null data x on $\mathcal{H}_{0,[1,2]}$ (see Definition 2.6). By applying D -derivatives, the null data x determine the following tuple,

$$(x, \mathcal{D}^{L,m}) \text{ on } \mathcal{H}_{0,[1,2]},$$

where $\mathcal{D}^{L,m}$ is as in (2.58). Similarly, ingoing null data $\underline{x}$ on $\underline{\mathcal{H}}_{[-\delta,\delta],2}$ (see Definition 2.6) determine the tuple

$$(\underline{x}, \underline{\mathcal{D}}^{\underline{L},m}) \text{ on } \underline{\mathcal{H}}_{[-\delta,\delta],2},$$

where $\underline{\mathcal{D}}^{\underline{L},m}$ denotes the tuple of derivatives on the right-hand side of (2.58).

In the following, we define higher-order null data on $\mathcal{H}_{0,[1,2]}$ and $\underline{\mathcal{H}}_{[-\delta,\delta],2}$.

Definition 2.28 (*Higher-order null data*). Let $m \geq 1$ be an integer. We define

- *higher-order outgoing null data of order m* to be the triple

$$(x, \mathcal{D}^{L,m}, \mathcal{D}^{\underline{L},m}) \text{ on } \mathcal{H}_{0,[1,2]}$$

such that for each $S_{0,v} \subset \mathcal{H}_{0,[1,2]}$, $(x, \mathcal{D}^{L,m}, \mathcal{D}^{\underline{L},m})_{0,v}$ is higher-order sphere data of order m .

- *higher-order ingoing null data of order m* to be the triple

$$(\underline{x}, \underline{\mathcal{D}}^{\underline{L},m}, \underline{\mathcal{D}}^{L,m}) \text{ on } \underline{\mathcal{H}}_{[-\delta,\delta],2}$$

such that for each $S_{u,2} \subset \underline{\mathcal{H}}_{[-\delta,\delta],2}$, $(\underline{x}, \underline{\mathcal{D}}^{\underline{L},m}, \underline{\mathcal{D}}^{L,m})_{u,2}$ is higher-order sphere data of order m . Here, $\underline{\mathcal{D}}^{L,m}$ denotes the tuple of tensors on the right-hand side of (2.57).

In addition to the constraints equations

$$\mathcal{C}_i(x) = 0 \text{ for } 1 \leq i \leq 10,$$

for x on $\mathcal{H}_{0,[1,2]}$ and their direct implications for $\mathcal{D}^{L,m}$, the Einstein equations (1.6) also imply null transport equations for $\mathcal{D}^{\underline{L},m}$ along $\mathcal{H}_{0,[1,2]}$; see, for example, the null transport equation (2.20) for $\underline{D}\omega$ along $\mathcal{H}_{0,[1,2]}$. We call these null transport equations the *higher-order null constraint equations* and formally denote them by

$$\mathcal{C}(x, \mathcal{D}^{L,m}, \mathcal{D}^{\underline{L},m}) = 0 \text{ on } \mathcal{H}_{0,[1,2]}.$$

Similarly, we denote the higher-order null constraint equations on $\mathcal{H}_{[-\delta,\delta],2}$ by

$$\underline{\mathcal{C}}(\underline{x}, \underline{\mathcal{D}}^{L,m}, \underline{\mathcal{D}}^{\underline{L},m}) = 0 \text{ on } \mathcal{H}_{[-\delta,\delta],2}.$$

The higher-order null constraint equations are relevant for higher-order conservation laws which act as obstructions to higher-order characteristic gluing, see also Theorem 3.3.

Remark 2.29 (Higher-order sphere data perturbations). In the context of higher-order sphere data, the sphere data perturbations $\mathcal{P}_{f,q}$ can be straightforwardly generalized to smooth mappings of higher-order incoming null data $(\underline{x}, \underline{\mathcal{D}}^{L,m}, \underline{\mathcal{D}}^{\underline{L},m})$ on $\mathcal{H}_{[-\delta,\delta],2}$ to higher-order L -derivatives sphere data $(x_{0,2}, \mathcal{D}_{0,2}^{L,m})$ on $S_{0,2}$. It is worthwhile to note that linearly at Minkowski, $\mathcal{D}^{L,m}$ is invariant under sphere variations, that is, for all $\dot{f}$ and $\dot{q}$,

$$\dot{\mathcal{D}}_L \left(\dot{\mathcal{P}}_f(\dot{f}) + \dot{\mathcal{P}}_q(\dot{q}) \right) = 0.$$

Indeed, this follows by the direct relation between our $\dot{\mathcal{P}}_f$ and $\dot{\mathcal{P}}_q$ and Lemmas 6.1.2 and 6.1.3 in [27] (as in the proof of Lemma 2.22), where in the latter it is shown that

$$\dot{\alpha} \equiv 0, \quad \dot{\omega} \equiv 0,$$

which implies that all higher D -derivatives vanish on $S_{0,2}$.

3. Statement of Main Results

The following is the main theorem of this paper.

Theorem 3.1 (Codimension-10 perturbative characteristic gluing, version 2). *Let $\delta > 0$ be a real number. Consider sphere data $x_{0,1}$ on $S_{0,1}$, and sphere data $\tilde{x}_{0,2}$ on $\tilde{S}_{0,2}$ contained in ingoing null data $\tilde{x}$ on $\tilde{\mathcal{H}}_{[-\delta,\delta],2}$ satisfying the null constraint equations. Assume that for some real number $\varepsilon > 0$,*

$$\|x_{0,1} - \mathbf{m}^M\|_{\mathcal{X}(S_{0,1})} + \|\tilde{x} - \underline{\mathbf{m}}^M\|_{\mathcal{X}^+(\tilde{\mathcal{H}}_{[-\delta,\delta],2})} \leq \varepsilon. \quad (3.1)$$

There exist universal real numbers $M_0 > 0$ and $\varepsilon_0 > 0$ such that for all real numbers $0 \leq M < M_0$ and $0 < \varepsilon < \varepsilon_0$ sufficiently small, there are

- *a solution x to the null constraint equations on $\mathcal{H}_{0,[1,2]}$,*

- sphere data $x_{0,2}$ on a sphere $S_{0,2} \subset \tilde{\mathcal{H}}_{[-\delta,\delta],2}$ stemming from a perturbation of $\tilde{S}_{0,2}$; that is, there are perturbation functions f and q such that

$$x_{0,2} = \mathcal{P}_{f,q}(\tilde{x}),$$

such that on $S_{0,1}$ we have matching of sphere data,

$$x|_{S_{0,1}} = x_{0,1}, \quad (3.2)$$

and on $S_{0,2}$ we have matching up to the charges $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$; that is, if

$$(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})(x|_{S_{0,2}}) = (\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})(x_{0,2}), \quad (3.3)$$

then it holds that

$$x|_{S_{0,2}} = x_{0,2}. \quad (3.4)$$

Moreover, the following bounds hold,

$$\begin{aligned} \|x - \mathbf{m}^M\|_{\mathcal{X}(\mathcal{H}_{0,[1,2]})} + \|x_{0,2} - \tilde{x}_{0,2}\|_{\mathcal{X}(S_{0,2})} &\lesssim \varepsilon, \\ \|f\|_{\mathcal{Y}_f} + \|q\|_{\mathcal{Y}_q} &\lesssim \varepsilon, \end{aligned} \quad (3.5)$$

where we denoted $\tilde{x}_{0,2} := \tilde{x}|_{S_{0,2}}$. Furthermore, we have the perturbation estimate

$$|(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})(x_{0,2}) - (\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})(\tilde{x}_{0,2})| \lesssim \varepsilon M + \varepsilon^2, \quad (3.6)$$

and the transport estimate

$$|(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})(x|_{S_{0,2}}) - (\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})(x|_{S_{0,1}})| \lesssim \varepsilon M + \varepsilon^2. \quad (3.7)$$

Remarks on Theorem 3.1.

- (1) The matching on $S_{0,2}$ can be described more precisely as follows. The solution x constructed in Theorem 3.1 is such that on $S_{0,2}$,

$$\mathfrak{M}(x|_{S_{0,2}}) = \mathfrak{M}(x_{0,2}), \quad (3.8)$$

where $\mathfrak{M}$ denotes the matching map introduced in Definition 2.11. Lemma 2.12 implies then that if in addition to (3.8) we have the matching of charges (3.3), then we have the sphere data matching (3.4).

- (2) A straightforward inspection of the proof of Theorem 3.1 shows that the angular regularity of the characteristic gluing can be increased without change to the proof.
- (3) Theorem 3.1 can be equivalently stated with ingoing null data on $\tilde{\mathcal{H}}_{[-\delta,\delta],0}$ and sphere data on $S_{0,2}$. This alternative formulation of Theorem 3.1 is used in [11, 12].
- (4) The constructed solution $x \in \mathcal{X}(\mathcal{H}_{0,[1,2]})$ is sufficiently regular for the application of local existence results for the characteristic initial value problem; see [11, 12] for further discussion. The gluing of Theorem 3.1 is at the level of C^2 -gluing for metric components.
- (5) The construction of the solution x in Theorem 3.1 is based on the implicit function theorem and solving the *linearized characteristic gluing problem at Minkowski* in Sect. 4. The perturbation and transport estimates (3.6) and (3.7) for the charges $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$ require further an analysis of the linearized sphere perturbations, angular perturbations and null transport

equations for $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$ at Schwarzschild of mass $M \geq 0$ provided in Appendix C.

In addition to the gluing of higher-order angular derivatives, the proof of Theorem 3.1 accommodates also the characteristic gluing of higher-order L -derivatives in a straightforward way. The precise statement is as follows; for ease of presentation, we state it with smooth sphere data.

Theorem 3.2 (Codimension-10 perturbative characteristic gluing of higher-order L -derivatives). *Let $\delta > 0$ be a real number and let $m \geq 1$ be an integer. Consider smooth higher-order L -derivatives sphere data $(x_{0,1}, \mathcal{D}_{0,1}^{L,m})$ on $S_{0,1}$ and smooth higher-order ingoing null data $(\tilde{x}, \tilde{\underline{\mathcal{D}}}^{\tilde{L},m}, \tilde{\underline{\mathcal{D}}}^{\tilde{L},m})$ solving the higher-order null constraint equations on $\tilde{\mathcal{H}}_{[-\delta,\delta],2}$. For $(x_{0,1}, \mathcal{D}_{0,1}^{L,m})$ and $(\tilde{x}, \tilde{\underline{\mathcal{D}}}^{\tilde{L},m}, \tilde{\underline{\mathcal{D}}}^{\tilde{L},m})$ sufficiently close to their respective reference values in a Schwarzschild spacetime of sufficiently small mass $M \geq 0$, there are*

- a smooth solution $(x, \mathcal{D}^{L,m})$ to the null constraint equations on $\mathcal{H}_{0,[1,2]}$,
- smooth higher-order L -derivatives sphere data $(x_{0,2}, \mathcal{D}_{0,2}^{L,m})$ stemming from a perturbation of $\tilde{S}_{0,2}$ in $\tilde{\mathcal{H}}_{[-\delta,\delta],2}$; that is, there are perturbation functions f and q such that

$$(x_{0,2}, \mathcal{D}^{L,m}) = \mathcal{P}_{f,q} \left(\tilde{x}, \tilde{\underline{\mathcal{D}}}^{\tilde{L},m}, \tilde{\underline{\mathcal{D}}}^{\tilde{L},m} \right),$$

such that on $S_{0,1}$ we have the matching

$$(x, \mathcal{D}^{L,m})|_{S_{0,1}} = (x_{0,1}, \mathcal{D}_{0,1}^{L,m}), \quad (3.9)$$

and on $S_{0,2}$ we have matching up to the charges $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$; that is, if

$$(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G}) (x|_{S_{0,2}}) = (\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G}) (x_{0,2}),$$

then it holds that

$$(x, \mathcal{D}^{L,m})|_{S_{0,2}} = (x_{0,2}, \mathcal{D}_{0,2}^{L,m}). \quad (3.10)$$

Moreover, we have charge estimates analogous to Theorem 3.1.

Remarks on Theorem 3.2.

- (1) The proof of Theorem 3.2 is a direct generalization of the proof of Theorem 3.1. We indicate the necessary generalizations in Sect. 5.5 and Sect. 4.
- (2) The matching on $S_{0,2}$ in Theorem 3.2 is more precisely given by

$$\mathfrak{M}(x|_{S_{0,2}}) = \mathfrak{M}(x_{0,2}), \mathcal{D}^{L,m}|_{S_{0,2}} = \mathcal{D}_{0,2}^{L,m}. \quad (3.11)$$

In particular, as mentioned before, the gluing of

$$\mathcal{D}^{L,m} := \left(\hat{D}\alpha, \dots, \hat{D}^m\alpha, D^2\omega, \dots, D^{m+1}\omega \right),$$

is without obstacles.

- (3) To deduce from (3.11) the gluing of higher-order L -derivatives of metric components, Ricci coefficients and null curvature components, the full matching (3.10) is needed to apply the higher-order null structure equations on $S_{0,2}$.

Theorem 3.2 shows that $\mathcal{D}^{L,m}$ can be characteristically glued without obstruction, while $\mathcal{D}^{\underline{L},m}$ is subject to higher-order conservation laws along $\mathcal{H}_{0,[1,2]}$. In the following theorem, we show that by gluing along *two null hypersurfaces bifurcating from an auxiliary sphere*, namely along $\mathcal{H}_{[-1,0],1}$ and $\mathcal{H}_{0,[1,2]}$, it is possible to glue higher-order L - and $\underline{L}$ -derivatives.

Theorem 3.3 (Codimension-10 bifurcate characteristic gluing, version 2). *Let $m \geq 0$ be an integer. Consider smooth higher-order sphere data*

$$(x_{0,1}, \mathcal{D}_{0,1}^{L,m}, \mathcal{D}_{0,1}^{\underline{L},m}) \text{ on } S_{0,1} \text{ and } (x_{-1,2}, \mathcal{D}_{-1,2}^{L,m}, \mathcal{D}_{-1,2}^{\underline{L},m}) \text{ on } S_{-1,2}.$$

For $(x_{0,1}, \mathcal{D}_{0,1}^{L,m}, \mathcal{D}_{0,1}^{\underline{L},m})$ and $(x_{-1,2}, \mathcal{D}_{-1,2}^{L,m}, \mathcal{D}_{-1,2}^{\underline{L},m})$ sufficiently close to their respective reference values in a Schwarzschild spacetime of sufficiently small mass $M \geq 0$, there are

- *a smooth solution $(\underline{x}, \underline{\mathcal{D}}^{L,m}, \underline{\mathcal{D}}^{\underline{L},m})$ to the higher-order null constraint equations on $\mathcal{H}_{[-1,0],1}$, satisfying higher-order sphere data matching on $S_{0,1}$,*

$$(\underline{x}, \underline{\mathcal{D}}^{L,m}, \underline{\mathcal{D}}^{\underline{L},m}) \Big|_{S_{0,1}} = (x_{0,1}, \mathcal{D}_{0,1}^{L,m}, \mathcal{D}_{0,1}^{\underline{L},m}),$$

- *a smooth solution $(x, \mathcal{D}^{L,m}, \mathcal{D}^{\underline{L},m})$ to the higher-order null constraint equations on $\mathcal{H}_{-1,[1,2]}$, matching with $(\underline{x}, \underline{\mathcal{D}})$ on $S_{-1,1}$,*

$$(\underline{x}, \underline{\mathcal{D}}^{L,m}, \underline{\mathcal{D}}^{\underline{L},m}) \Big|_{S_{-1,1}} = (x, \mathcal{D}^{L,m}, \mathcal{D}^{\underline{L},m}) \Big|_{S_{-1,1}}$$

such that $(x, \mathcal{D}^{L,m}, \mathcal{D}^{\underline{L},m})$ matches $(x_{-1,2}, \mathcal{D}_{-1,2}^{L,m}, \mathcal{D}_{-1,2}^{\underline{L},m})$ up to the charges $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$ on $S_{-1,2}$; that is, if it holds that

$$(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})(x|_{S_{-1,2}}) = (\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})(x_{-1,2}),$$

then

$$(x, \mathcal{D}^{L,m}, \mathcal{D}^{\underline{L},m}) \Big|_{S_{-1,2}} = (x_{-1,2}, \mathcal{D}_{-1,2}^{L,m}, \mathcal{D}_{-1,2}^{\underline{L},m}).$$

Moreover, we have charge estimates analogous to (3.7) in Theorem 3.1 for

$$\left| (\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})(x|_{S_{-1,2}}) - (\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G} - 2\mathbf{P})(x_{0,1}) \right|.$$

Remarks on Theorem 3.3.

- (1) Theorem 3.3 shows that for bifurcate characteristic gluing, the obstruction space consists entirely of the 10-dimensional space $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$.
- (2) The proof of Theorem 3.3 is based on the methods of Theorems 3.1 and 3.2, and given in Sect. 6.
- (3) In [11, 12], Theorem 3.3 is applied to glue spacelike initial data to a Kerr black hole spacetime.
- (4) It is possible to generalize the above to an ∞ -order bifurcate characteristic gluing result as stated in the next theorem. We omit details and postpone the proof to future work.

Theorem. For integers $m \geq 1$, there are higher-order norms $\mathcal{X}^m(S_1)$ on S_1 , $\mathcal{X}^m(S_2)$ on S_2 , and $\mathcal{X}^m(\mathcal{H} \cup \underline{\mathcal{H}})$ on $\mathcal{H} \cup \underline{\mathcal{H}}$, and there is a real number $m_0 \geq 1$ such that the following holds. Given smooth ∞ -order sphere data x_1^∞ on S_1 and x_2^∞ on S_2 , being sufficiently close to Minkowski reference data in $\mathcal{X}^{m_0}(S_1)$ on S_1 and $\mathcal{X}^{m_0}(S_2)$ on S_2 , respectively, there exists a smooth solution x^∞ along $\mathcal{H} \cup \underline{\mathcal{H}}$ to ∞ -order bifurcate characteristic gluing such that for each $m \geq 1$,

$$\|x^\infty - \mathbf{m}\|_{\mathcal{X}^m(\mathcal{H} \cup \underline{\mathcal{H}})} \lesssim \|x_1^\infty - \mathbf{m}\|_{\mathcal{X}^m(S_1)} + \|x_2^\infty - \mathbf{m}\|_{\mathcal{X}^m(S_2)}.$$

The above results concern codimension-10 characteristic gluing. However, by adding to the sphere data on $S_{-1,2}$ a sphere data perturbation W which adjusts the charges $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$ (this implies that W is *not* coming from a sphere perturbation or sphere diffeomorphism), it is straightforward to extend the above results to full characteristic gluing of (higher-order) sphere data. The sphere data perturbation W can be chosen to be supported on an arbitrary angular region K on $S_{-1,2}$. An explicit such W with advantageous properties is used in the gluing problems studied in [11]. The following result is an extension of Theorem 3.3; a variant can also be stated for characteristic gluing along one null hypersurface of Theorem 3.1.

Proposition 3.4 (Bifurcate characteristic gluing with localized sphere data perturbation W). *Let $m \geq 0$ be an integer. Let $K \subset S_{-1,2}$ be an angular region. Consider smooth higher-order sphere data*

$$(x_{0,1}, \mathcal{D}_{0,1}^{L,m}, \underline{\mathcal{D}}_{0,1}^{L,m}) \text{ on } S_{0,1} \text{ and } (x_{-1,2}, \mathcal{D}_{-1,2}^{L,m}, \underline{\mathcal{D}}_{-1,2}^{L,m}) \text{ on } S_{-1,2}.$$

For $(x_{0,1}, \mathcal{D}_{0,1}^{L,m}, \underline{\mathcal{D}}_{0,1}^{L,m})$ and $(x_{-1,2}, \mathcal{D}_{-1,2}^{L,m}, \underline{\mathcal{D}}_{-1,2}^{L,m})$ sufficiently close to their respective reference values in a Schwarzschild spacetime of sufficiently small mass $M \geq 0$, there are

- *a smooth solution $(\underline{x}, \underline{\mathcal{D}}^{L,m}, \underline{\mathcal{D}}^{L,m})$ to the higher-order null constraint equations on $\underline{\mathcal{H}}_{[-1,0],1}$, satisfying higher-order sphere data matching on $S_{0,1}$,*

$$(\underline{x}, \underline{\mathcal{D}}^{L,m}, \underline{\mathcal{D}}^{L,m}) \Big|_{S_{0,1}} = (x_{0,1}, \mathcal{D}_{0,1}^{L,m}, \underline{\mathcal{D}}_{0,1}^{L,m}),$$

- *a smooth solution $(x, \mathcal{D}^{L,m}, \underline{\mathcal{D}}^{L,m})$ to the higher-order null constraint equations on $\mathcal{H}_{-1,[1,2]}$, matching with $(\underline{x}, \underline{\mathcal{D}})$ on $S_{-1,1}$,*

$$(\underline{x}, \underline{\mathcal{D}}^{L,m}, \underline{\mathcal{D}}^{L,m}) \Big|_{S_{-1,1}} = (x, \mathcal{D}^{L,m}, \underline{\mathcal{D}}^{L,m}) \Big|_{S_{-1,1}},$$

- *a smooth higher-order sphere data perturbation $(W, 0, 0)$, compactly supported in K ,*

such that

$$(x, \mathcal{D}^{L,m}, \underline{\mathcal{D}}^{L,m})|_{S_{-1,2}} = (x_{-1,2}, \mathcal{D}_{-1,2}^{L,m}, \underline{\mathcal{D}}_{-1,2}^{L,m}) + (W, 0, 0),$$

Moreover, we have appropriate bounds for W on $S_{-1,2}$, $(x, \mathcal{D}^{L,m}, \underline{\mathcal{D}}^{L,m})$ on $\mathcal{H}_{-1,[1,2]}$, and $(x, \mathcal{D}^{L,m}, \underline{\mathcal{D}}^{L,m})$ on $\mathcal{H}_{-1,[1,2]}$.

The proof of Proposition 3.4 is a slight generalization of the proof of Theorem 3.3 based on the additional localized sphere perturbation W , see Sect. 6.4.

4. Linearized Characteristic Gluing at Minkowski

The following is the main result of this section. It shows that the linearized characteristic gluing problem at Minkowski is solvable up to a 10-dimensional space and forms the basis for the proof of Theorem 3.1 in Sect. 5.

Theorem 4.1 (Codimension-10 linearized characteristic gluing at Minkowski). *Given*

- linearized sphere data $\dot{\mathfrak{X}}_{0,1} \in \mathcal{X}(S_{0,1})$ at $S_{0,1}$,
- linearized matching data $\dot{\mathfrak{M}}_{0,2} \in \mathcal{Z}_{\mathfrak{M}}(S_{0,2})$ at $S_{0,2}$,
- linearized source terms $(\dot{\mathfrak{c}}_i)_{1 \leq i \leq 10} \in \mathcal{Z}_{\mathcal{C}}$ on $\mathcal{H}_{0,[1,2]}$,

there exist

- linearized null data $\dot{x} \in \mathcal{X}(\mathcal{H}_{0,[1,2]})$ on $\mathcal{H}_{0,[1,2]}$,
- linearized perturbation functions $\dot{f}$ and $\dot{q}$ at $S_{0,2}$

such that

$$\dot{\mathcal{F}}^0(\dot{x}, \dot{f}, \dot{q}) = \left(\dot{\mathfrak{X}}_{0,1}, \dot{\mathfrak{M}}_{0,2}, (\dot{\mathfrak{c}}_i)_{1 \leq i \leq 10} \right), \quad (4.1)$$

where $\dot{\mathcal{F}}^0$ denotes the linearization of $\mathcal{F}(x, f, q)$ in

$$(x_{0,1}, \tilde{x}, x, f, q) = (\mathfrak{m}, \underline{\mathfrak{m}}, \mathfrak{m}, 0, 0).$$

In other words, (4.1) states that

$$\dot{\mathcal{C}}_i(\dot{x}) = \dot{\mathfrak{c}}_i \text{ on } \mathcal{H}_{0,[1,2]} \text{ for } 1 \leq i \leq 10, \quad (4.2)$$

$$\dot{x}|_{S_{0,1}} = \dot{\mathfrak{X}}_{0,1}, \quad \mathfrak{M} \left(\dot{x}|_{S_{0,2}} - \dot{\mathcal{P}}_q^0(\dot{q}) - \dot{\mathcal{P}}_f^0(\dot{f}) \right) = \dot{\mathfrak{M}}_{0,2}, \quad (4.3)$$

where we remark that $\mathfrak{M}$ is a linear map and we thus write $\mathfrak{M}$ instead of $\dot{\mathfrak{M}}$. Moreover, the following estimate holds,

$$\begin{aligned} & \|\dot{x}\|_{\mathcal{X}(\mathcal{H}_{0,[1,2]})} + \|\dot{f}\|_{\mathcal{Y}_f} + \|\dot{q}\|_{\mathcal{Y}_q} + \left\| \dot{\mathcal{P}}_f^0(\dot{f}) \right\|_{\mathcal{X}(S_{0,2})} + \left\| \dot{\mathcal{P}}_q^0(\dot{q}) \right\|_{\mathcal{X}(S_{0,2})} \\ & \lesssim \|\dot{\mathfrak{X}}_{0,1}\|_{\mathcal{X}(S_{0,1})} + \|\dot{\mathfrak{M}}_{0,2}\|_{\mathcal{Z}_{\mathfrak{M}}(S_{0,2})} + \|(\dot{\mathfrak{c}}_i)_{1 \leq i \leq 10}\|_{\mathcal{Z}_{\mathcal{C}}}. \end{aligned} \quad (4.4)$$

We proceed as follows.

- (1) In Sect. 4.1, we define the charges $\mathcal{Q}_i$ for $0 \leq i \leq 7$ which satisfy conservation laws along $\mathcal{H}$ by the linearized null constraint equations at Minkowski.
- (2) In Sect. 4.2, we analyze how the charges $\mathcal{Q}_i$ split into gauge-invariant and gauge-dependent charges.
- (3) In Sect. 4.3, we analyze the system (4.2). We integrate the transport equations (4.2) to get representation formulas for metric coefficients, Ricci coefficients and null curvature components.
- (4) In Sect. 4.4, we prove Theorem 4.1 by solving the linearized characteristic gluing problem.

Remark 4.2. While in the nonlinear setting we interpreted Ω and $\mathcal{J}_c$ as free conformal data, see Definition 2.14, in the linearized setting we choose $\dot{\Omega}$ and $\dot{\hat{\chi}}$ as degrees of freedom. Indeed, by the linearized equation (see Lemma 2.16)

$$D\mathcal{J}_c = \frac{2}{r^2}\dot{\hat{\chi}} + \frac{1}{r^2}\dot{\mathfrak{c}}_3$$

the two approaches are equivalent at the linear level. Consequently, in our approach to the linearized gluing problem, the gluing of the following quantities is trivial,

$$\left(\dot{\Omega}, \dot{\omega}, D\dot{\omega}, \dot{\hat{\chi}}, D\dot{\hat{\chi}}\right).$$

Remark 4.3 (Linearized characteristic gluing of higher-order L -derivatives I). Let $m \geq 1$ be an integer. In the following, we also show that if one additionally prescribes the tuple

$$\dot{\mathcal{D}}^{L,m} := \left(\widehat{D}\dot{\alpha}, \dots, \widehat{D}^m\dot{\alpha}, D^2\dot{\omega}, \dots, D^{m+1}\dot{\omega}\right)$$

on $S_{0,1}$ and $S_{0,2}$, denoted by $\dot{\mathcal{D}}_{0,1}^{L,m}$ and $\dot{\mathcal{D}}_{0,2}^{L,m}$, respectively, then we can construct $\dot{x}$ such that it satisfies, in addition to (4.3),

$$\dot{\mathcal{D}}^{L,m}(\dot{x})|_{S_{0,1}} = \dot{\mathcal{D}}_{0,1}^{L,m}, \quad \dot{\mathcal{D}}^{L,m}(\dot{x})|_{S_{0,2}} = \dot{\mathcal{D}}_{0,2}^{L,m}.$$

In this setting, the right-hand side of (4.4) is replaced by

$$\begin{aligned} & \|\dot{\mathfrak{X}}_1\|_{\mathcal{X}(S_{0,1})} + \|\mathfrak{M}_2\|_{\mathcal{Z}_{\mathfrak{M}}(S_{0,2})} + \|(\dot{\mathfrak{c}}_i)_{1 \leq i \leq 10}\|_{\mathcal{Z}_c} \\ & + \sum_{0 \leq i \leq m} \left(\|\widehat{D}^i\dot{\alpha}\|_{H^6(S_{0,1})} + \|\widehat{D}^i\dot{\alpha}\|_{H^6(S_{0,2})} \right) \\ & + \sum_{0 \leq i \leq m+1} \left(\|D^i\dot{\omega}\|_{H^6(S_{0,1})} + \|D^i\dot{\omega}\|_{H^6(S_{0,2})} \right). \end{aligned}$$

4.1. Conserved Charges $\mathcal{Q}_i$ for the Linearized Equations

The following *charges* $\mathcal{Q}_i$, $0 \leq i \leq 7$, play an essential role in the characteristic gluing problem. In Sect. 4.3, we prove that the linearized null constraint equations at Minkowski (4.2) (see also Lemma 2.16) imply *conservation laws* for the following *charges*, see Lemmas 4.8, 4.11 and 4.16.

$$\begin{aligned} \mathcal{Q}_0 &:= r^2\dot{\eta}^{[1]} + \frac{r^3}{2}\not{d} \left((\Omega \text{tr} \chi)^{[1]} - \frac{4}{r}\dot{\Omega}^{[1]} \right), \\ \mathcal{Q}_1 &:= \frac{r}{2} \left((\Omega \text{tr} \chi) - \frac{4}{r}\dot{\Omega} \right) + \frac{\dot{\phi}}{r}, \\ \mathcal{Q}_2 &:= r^2(\Omega \text{tr} \underline{\chi}) - \frac{2}{r}\mathring{\text{d}}\not{v} \left(r^2\dot{\eta} + \frac{r^3}{2}\not{d} \left((\Omega \text{tr} \chi) - \frac{4}{r}\dot{\Omega} \right) \right) \\ &\quad - r^2 \left((\Omega \text{tr} \chi) - \frac{4}{r}\dot{\Omega} \right) + 2r^3\dot{K}, \\ \mathcal{Q}_3 &:= \frac{\dot{\hat{\chi}}}{r} - \frac{1}{2} \left(\mathcal{P}_2^*\mathring{\text{d}}\not{v} + 1 \right) \dot{\mathcal{J}}_c + \mathcal{P}_2^* \left(\dot{\eta} + \frac{r}{2}\not{d} \left((\Omega \text{tr} \chi) - \frac{4}{r}\dot{\Omega} \right) \right) \end{aligned}$$

$$\begin{aligned}
& -r\mathcal{P}_2^* \mathring{d} \left((\Omega \text{tr} \chi) - \frac{4}{r} \dot{\Omega} \right), \\
\mathcal{Q}_4 &:= \frac{\dot{\alpha}_\psi}{r} + 2\mathcal{P}_2^* \left(\frac{1}{r^2} \text{div} \mathring{\chi} - \frac{1}{r} \dot{\eta} - \frac{1}{2} \mathring{d} (\Omega \text{tr} \chi) + \mathcal{P}_1^* (\dot{\omega}, 0) \right)_\psi, \\
\mathcal{Q}_5 &:= \dot{\omega}^{[\leq 1]} + \frac{1}{4r^2} \mathcal{Q}_2^{[\leq 1]} + \frac{1}{3r^3} \text{div} \mathring{\mathcal{Q}}_0, \\
\mathcal{Q}_6 &:= \underline{D} \dot{\omega}^{[\leq 1]} - \frac{1}{6r^3} (\mathring{\Delta} - 3) \mathcal{Q}_2^{[\leq 1]} + \frac{1}{r^4} \text{div} \mathring{\mathcal{Q}}_0, \\
\mathcal{Q}_7 &:= \underline{D} \dot{\omega}^{[2]} + \frac{3}{2r^3} \mathcal{Q}_2^{[2]} + \frac{1}{2r^2} \text{div} \mathring{\mathcal{Q}}_3^{[2]} - \frac{12}{r^2} \mathcal{Q}_1^{[2]} \\
& \quad + \frac{3}{2r^2} \text{div} \mathring{\mathcal{Q}} \left(\eta + \frac{r}{2} \mathring{d} \left((\Omega \text{tr} \chi) - \frac{4}{r} \dot{\Omega} \right) \right)^{[2]} - \frac{3}{4r^2} \text{div} \mathring{\mathcal{Q}} \mathring{j}_c^{[2]}, \quad (4.5)
\end{aligned}$$

where ψ denotes the electric part of a tracefree symmetric 2-tensor, see Appendix D.

Remarks on the charges $\mathcal{Q}_i$, $0 \leq i \leq 7$, in (4.5).

- (1) By explicit calculation (see (C.4) and (C.5)), the linearizations $(\dot{\mathbf{E}}, \dot{\mathbf{P}}, \dot{\mathbf{L}}, \dot{\mathbf{G}})$ of $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$ at Minkowski are related to the above charges for $m = -1, 0, 1$ by

$$\begin{aligned}
-\frac{8\pi}{\sqrt{4\pi}} \dot{\mathbf{E}} &= -\frac{1}{2} \mathcal{Q}_2^{(0)}, & -\frac{8\pi}{\sqrt{\frac{4\pi}{3}}} \dot{\mathbf{P}}^m &= -\frac{1}{2} \mathcal{Q}_2^{(1m)}, \\
\frac{16\pi}{\sqrt{\frac{8\pi}{3}}} \dot{\mathbf{L}}^m &= 2(\mathcal{Q}_0)_H^{(1m)}, & \frac{16\pi}{\sqrt{\frac{8\pi}{3}}} \dot{\mathbf{G}}^m &= 2(\mathcal{Q}_0)_E^{(1m)}. \quad (4.6)
\end{aligned}$$

- (2) The $\mathcal{Q}_5$ and $\mathcal{Q}_6$ in (4.5) are equal to the linearizations at Minkowski of $\tilde{\mathcal{Q}}_5$ and $\tilde{\mathcal{Q}}_6$ defined in context of the matching map $\mathfrak{M}$, see Definition 2.11.

4.2. Gauge Dependence of the Conserved Charges $\mathcal{Q}_i$

In this section, we show that the charges $\mathcal{Q}_i$, $0 \leq i \leq 7$, of Sect. 4.1 split into *gauge-invariant* and *gauge-dependent* charges. The following is the main result of this section.

Proposition 4.4. *The following holds.*

- (1) **Gauge-invariant charges.** *For any linearized perturbation function $\dot{f}$ and perturbation vectorfield $\dot{j}$ on S_2 , it holds that*

$$\begin{aligned}
\mathcal{Q}_0 \left(\dot{\mathcal{P}}_f(\dot{f}) + \dot{\mathcal{P}}_q(\dot{q}) \right) &= 0, \\
\mathcal{Q}_2^{[\leq 1]} \left(\dot{\mathcal{P}}_f(\dot{f}) + \dot{\mathcal{P}}_q(\dot{q}) \right) &= 0;
\end{aligned}$$

that is, the 10-dimensional space of charges $\mathcal{Q}_0$ and $\mathcal{Q}_2^{[\leq 1]}$ is invariant under linearized perturbations $\dot{\mathcal{P}}_f$ and $\dot{\mathcal{P}}_j$ of sphere data.

(2) **Gauge-dependent charges.** Let $(\mathcal{Q}_1)_0, (\mathcal{Q}_2)_0, (\mathcal{Q}_5)_0, (\mathcal{Q}_6)_0$ and $(\mathcal{Q}_7)_0$ be scalar functions on S_2 such that

$$(\mathcal{Q}_2)_0^{[\leq 1]} = 0, (\mathcal{Q}_5)_0^{[\geq 2]} = 0, (\mathcal{Q}_6)_0^{[\geq 2]} = 0, (\mathcal{Q}_7)_0 = (\mathcal{Q}_7)_0^{[2]}, \quad (4.7)$$

and let $(\mathcal{Q}_3)_0$ and $(\mathcal{Q}_4)_0$ be symmetric tracefree 2-tensors on S_2 such that

$$(\mathcal{Q}_4)_0 = ((\mathcal{Q}_4)_0)_\psi, \quad (4.8)$$

where ψ denotes the electric part of a symmetric tracefree 2-tensor. Assume that

$$\begin{aligned} \|(\mathcal{Q}_1)_0\|_{H^6(S_2)} + \|(\mathcal{Q}_2)_0\|_{H^4(S_2)} + \|(\mathcal{Q}_3)_0\|_{H^4(S_2)} + \|(\mathcal{Q}_4)_0\|_{H^2(S_2)} &< \infty, \\ \|(\mathcal{Q}_5)_0\|_{H^4(S_2)} + \|(\mathcal{Q}_6)_0\|_{H^2(S_2)} + \|(\mathcal{Q}_7)_0\|_{H^2(S_2)} &< \infty. \end{aligned}$$

Then, there exist a linearized perturbation function $\dot{f}$ and linearized perturbation vectorfield $\dot{q}$ at S_2 such that

$$\mathcal{Q}_i \left(\dot{\mathcal{P}}_f(\dot{f}) + \dot{\mathcal{P}}_q(\dot{q}) \right) = (\mathcal{Q}_i)_0 \quad \text{for } 1 \leq i \leq 7, \quad (4.9)$$

and satisfying

$$\begin{aligned} &\|\dot{f}\|_{\mathcal{Y}_f} + \|\dot{q}\|_{\mathcal{Y}_q} + \|\dot{\mathcal{P}}_f(\dot{f})\|_{\mathcal{X}(S_2)} + \|\dot{\mathcal{P}}_q(\dot{q})\|_{\mathcal{X}(S_2)} \\ &\lesssim \|(\mathcal{Q}_1)_0\|_{H^6(S_2)} + \|(\mathcal{Q}_2)_0\|_{H^4(S_2)} + \|(\mathcal{Q}_3)_0\|_{H^4(S_2)} + \|(\mathcal{Q}_4)_0\|_{H^2(S_2)} \\ &\quad + \|(\mathcal{Q}_5)_0\|_{H^4(S_2)} + \|(\mathcal{Q}_6)_0\|_{H^2(S_2)} + \|(\mathcal{Q}_7)_0\|_{H^2(S_2)}. \end{aligned} \quad (4.10)$$

The rest of this section is concerned with the proof of Proposition 4.4. First, for linearized transversal perturbations, see Lemma 2.22, the charges on S_2 (where $r = 2$, as we work in the linearized setting at Minkowski) are calculated to be

$$\begin{aligned} \mathcal{Q}_0 &= 0, & \mathcal{Q}_1 &= \frac{1}{2} \overset{\circ}{\Delta} \dot{f} - \partial_u \dot{f}, \\ \mathcal{Q}_2 &= -2 \overset{\circ}{\Delta} (\overset{\circ}{\Delta} + 2) \dot{f}, & \mathcal{Q}_3 &= \mathcal{D}_2^* \not{d} \left(2 \partial_u \dot{f} \right) - \frac{1}{2} \mathcal{D}_2^* \not{d} \left(\overset{\circ}{\Delta} \dot{f} \right), \\ \mathcal{Q}_4 &= \mathcal{D}_2^* \mathcal{D}_1^* (\partial_u^2 \dot{f}, 0), & \mathcal{Q}_5 &= \frac{1}{2} \partial_u^2 \dot{f}^{[\leq 1]}, \\ \mathcal{Q}_6 &= \frac{1}{2} \partial_u^3 \dot{f}^{[\leq 1]}, & \mathcal{Q}_7 &= \frac{1}{2} \partial_u^3 \dot{f}^{[2]} - 3 \dot{f}^{[2]}, \end{aligned}$$

where we used the formula for $\dot{K}$ in (2.49).

Second, for linearized angular perturbations, see Lemma 2.23, the charges on S_2 are calculated to be, with $\dot{q} = (\dot{q}_1, \dot{q}_2)$,

$$\begin{aligned} \mathcal{Q}_0 &= 0, & \mathcal{Q}_1 &= \frac{1}{2} \overset{\circ}{\Delta} \dot{q}_1, \\ \mathcal{Q}_2 &= 0, & \mathcal{Q}_3 &= - \left(\mathcal{D}_2^* \text{div} + 1 \right) \left(\mathcal{D}_2^* \mathcal{D}_1^* (\dot{q}_1, \dot{q}_2) \right), \\ \mathcal{Q}_4 &= 0, & \mathcal{Q}_5 &= 0, \\ \mathcal{Q}_6 &= 0, & \mathcal{Q}_7 &= 0. \end{aligned}$$

Summing up the above two yields

$$\mathcal{Q}_0 = 0, \quad (4.11a)$$

$$\mathcal{Q}_1 = \frac{1}{2} \overset{\circ}{\Delta} \dot{f} - \partial_u \dot{f} + \frac{1}{2} \overset{\circ}{\Delta} \dot{q}_1, \quad (4.11b)$$

$$\mathcal{Q}_2 = -2 \overset{\circ}{\Delta} (\overset{\circ}{\Delta} + 2) \dot{f}, \quad (4.11c)$$

$$\mathcal{Q}_3 = \mathcal{P}_2^* \mathcal{A} \left(2 \partial_u \dot{f} \right) - \frac{1}{2} \mathcal{P}_2^* \mathcal{A} \left(\overset{\circ}{\Delta} \dot{f} \right) - \left(\mathcal{P}_2^* \mathcal{D} \dot{V} + 1 \right) \left(\mathcal{P}_2^* \mathcal{P}_1^* (\dot{q}_1, \dot{q}_2) \right), \quad (4.11d)$$

$$\mathcal{Q}_4 = \mathcal{D}_2^* \mathcal{P}_1^* (\partial_u^2 \dot{f}, 0), \quad (4.11e)$$

$$\mathcal{Q}_5 = \frac{1}{2} \partial_u^2 \dot{f}^{[\leq 1]}, \quad (4.11f)$$

$$\mathcal{Q}_6 = \frac{1}{2} \partial_u^3 \dot{f}^{[\leq 1]}, \quad (4.11g)$$

$$\mathcal{Q}_7 = \frac{1}{2} \partial_u^3 \dot{f}^{[2]} - 3 \dot{f}^{[2]}. \quad (4.11h)$$

The right-hand side of (4.11c) has vanishing projection on the modes $l = 0$ and $l = 1$, and hence, (4.11a) and (4.11c) imply (1) of Proposition 4.4.

In the following, we prove (2) of Proposition 4.4 by determining $\dot{f}$, $\partial_u \dot{f}$, $\partial_u^2 \dot{f}$, $\partial_u^3 \dot{f}$ and $\dot{q}_1$ and $\dot{q}_2$ from (4.11b)-(4.11h) such that (4.9) is satisfied.

(1) Definition of $\dot{f}$ on S_2 . To solve (4.11c), define the scalar function $\dot{f} = \dot{f}(0)$ on S_2 as solution to

$$-2 \overset{\circ}{\Delta} (\overset{\circ}{\Delta} + 2) \dot{f} = (\mathcal{Q}_2)_0. \quad (4.12)$$

with the additional condition that $\dot{f}^{[\leq 1]} = 0$. In Fourier space, (4.12) is equivalent for $l \geq 2$, $m = -l, \dots, l$ to

$$-2(-l(l+1))(-l(l+1)+2) \dot{f}^{(lm)} = (\mathcal{Q}_2)_0^{(lm)},$$

which yields

$$\dot{f}^{(lm)} = \frac{1}{2l(l+1)(-l(l+1)+2)} (\mathcal{Q}_2)_0^{(lm)}.$$

Hence, $\dot{f}$ is well defined and bounded by

$$\|\dot{f}\|_{H^8(S_2)} \lesssim \|(\mathcal{Q}_2)_0\|_{H^4(S_2)}. \quad (4.13)$$

(2) Definition of $\partial_u \dot{f}$, $\dot{q}_1$ and $\dot{q}_2$ on S_2 . To solve (4.11b) and (4.11d), the scalar functions $\partial_u \dot{f}$, $\dot{q}_1$ and $\dot{q}_2$ on S_2 have to solve

$$\begin{aligned} -\partial_u \dot{f} + \frac{1}{2} \overset{\circ}{\Delta} \dot{q}_1 &= (\mathcal{Q}_1)_0 - \frac{1}{2} \overset{\circ}{\Delta} \dot{f}, \\ \mathcal{P}_2^* \mathcal{A} \left(2 \partial_u \dot{f} \right) - \left(\mathcal{P}_2^* \mathcal{D} \dot{V} + 1 \right) \left(\mathcal{P}_2^* \mathcal{P}_1^* (\dot{q}_1, \dot{q}_2) \right) &= (\mathcal{Q}_3)_0 + \frac{1}{2} \mathcal{P}_2^* \mathcal{A} \left(\overset{\circ}{\Delta} \dot{f} \right). \end{aligned} \quad (4.14)$$

First, while the second of (4.14) has no $l = 0$ and $l = 1$ mode (simply because it is a 2-tensor equation), the $l = 0$ and $l = 1$ mode of the first of (4.14) can be solved by prescribing, for $l = 0, 1$,

$$\dot{q}_1^{[l]} = 0, \quad \partial_u \dot{f}^{[l]} = - \left((\mathcal{Q}_1)_0 - \frac{1}{2} \overset{\circ}{\Delta} \dot{f} \right)^{[l]} = -(\mathcal{Q}_1)_0^{[l]},$$

where we used that $\dot{f} = \dot{f}^{[\geq 2]}$. This implies that

$$\|\partial_u \dot{f}^{[\leq 1]}\|_{H^6(S_2)} \lesssim \|(\mathcal{Q}_1)_0\|_{H^6(S_2)}. \quad (4.15)$$

Second, considering the electric part of (4.14) in Fourier space, we get that for $l \geq 2$,

$$\begin{aligned} -(\partial_u \dot{f})^{[lm]} - \frac{l(l+1)}{2} \dot{q}_1^{[lm]} &= \left((\mathcal{Q}_1)_0 - \frac{1}{2} \overset{\circ}{\Delta} \dot{f} \right)^{[lm]}, \\ -2\partial_u \dot{f}^{[lm]} - \frac{l(l+1)}{2} \dot{q}_1^{[lm]} &= \frac{\left((\mathcal{Q}_3)_0 + \frac{1}{2} \mathcal{P}_2^* \overset{\circ}{\Delta} \left(\overset{\circ}{\Delta} \dot{f} \right) \right)^{[lm]}_{\psi}}{\sqrt{l(l+1)} \sqrt{\frac{1}{2}l(l+1) - 1}}. \end{aligned} \quad (4.16)$$

The coefficient matrix on the left-hand side of (4.16) is

$$\begin{bmatrix} -1 & -\frac{l(l+1)}{2} \\ -2 & -\frac{l(l+1)}{2} \end{bmatrix}.$$

with determinant

$$\det = -\frac{l(l+1)}{2} \neq 0,$$

and matrix inverse

$$\frac{1}{\det} \begin{bmatrix} -\frac{l(l+1)}{2} & \frac{l(l+1)}{2} \\ \frac{l(l+1)}{2} & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -\frac{4}{l(l+1)} & \frac{2}{l(l+1)} \end{bmatrix}.$$

Therefore, the solution to (4.16) is given, for $l \geq 2$, by

$$\begin{aligned} \partial_u \dot{f}^{[lm]} &= \left((\mathcal{Q}_1)_0 - \frac{1}{2} \overset{\circ}{\Delta} \dot{f} \right)^{[lm]} - \frac{\left((\mathcal{Q}_3)_0 + \frac{1}{2} \mathcal{P}_2^* \overset{\circ}{\Delta} \left(\overset{\circ}{\Delta} \dot{f} \right) \right)^{[lm]}_{\psi}}{\sqrt{l(l+1)} \sqrt{\frac{1}{2}l(l+1) - 1}}, \\ \dot{q}_1^{[lm]} &= -\frac{4}{l(l+1)} \left((\mathcal{Q}_1)_0 - \frac{1}{2} \overset{\circ}{\Delta} \dot{f} \right)^{[lm]} \\ &\quad + \frac{2}{l(l+1)} \frac{\left((\mathcal{Q}_3)_0 + \frac{1}{2} \mathcal{P}_2^* \overset{\circ}{\Delta} \left(\overset{\circ}{\Delta} \dot{f} \right) \right)^{[lm]}_{\psi}}{\sqrt{l(l+1)} \sqrt{\frac{1}{2}l(l+1) - 1}}, \end{aligned}$$

from which we can derive with (4.13) the estimates

$$\begin{aligned} \|\partial_u \dot{f}^{[\geq 2]}\|_{H^6(S_2)} &\lesssim \|(\mathcal{Q}_1)_0\|_{H^6(S_2)} + \|\dot{f}\|_{H^8(S_2)} + \|(\mathcal{Q}_3)_0\|_{H^4(S_2)}, \\ &\lesssim \|(\mathcal{Q}_1)_0\|_{H^6(S_2)} + \|(\mathcal{Q}_2)_0\|_{H^4(S_2)} + \|(\mathcal{Q}_3)_0\|_{H^4(S_2)}, \\ \|\dot{q}_1^{[\geq 2]}\|_{H^8(S_2)} &\lesssim \|(\mathcal{Q}_1)_0\|_{H^6(S_2)} + \|\dot{f}\|_{H^8(S_2)} + \|(\mathcal{Q}_3)_0\|_{H^4(S_2)}, \\ &\lesssim \|(\mathcal{Q}_1)_0\|_{H^6(S_2)} + \|(\mathcal{Q}_2)_0\|_{H^4(S_2)} + \|(\mathcal{Q}_3)_0\|_{H^4(S_2)}. \end{aligned} \quad (4.17)$$

Third, we consider the magnetic part of (4.14), that is,

$$-\left(\mathcal{D}_2^* \overset{\circ}{\mathcal{D}} + 1\right) (\mathcal{D}_2^* \mathcal{D}_1^* (0, \dot{q}_2)) = (\mathcal{Q}_3)_0. \quad (4.18)$$

Going into Fourier space, (4.18) is equivalent to, for $l \geq 2$,

$$-\frac{1}{2}l(l+1)\sqrt{\frac{1}{2}l(l+1)-1}\sqrt{l(l+1)}\dot{q}_2^{[lm]} = ((\mathcal{Q}_3)_0)_\phi^{[lm]}.$$

Hence, the solution $q_2 = q_2^{[\geq 2]}$ is well defined and

$$\|\dot{q}_2^{[\geq 2]}\|_{H^8(S_2)} \lesssim \|(\mathcal{Q}_3)_0\|_{H^4(S_2)}. \quad (4.19)$$

(3) Definition of $\partial_u^2 \dot{f}$ on S_2 . To solve (4.11e), define the modes $l \geq 2$ of $\partial_u^2 \dot{f}$ by

$$(\mathcal{Q}_4)_0 = \mathcal{D}_2^* \mathcal{D}_1^* (\partial_u^2 \dot{f}, 0). \quad (4.20)$$

By (4.8), (4.20) is well defined and is in Fourier space given by, for $l \geq 2$,

$$\left(\partial_u^2 \dot{f}\right)^{(lm)} = \frac{1}{\sqrt{\frac{1}{2}l(l+1)-1}\sqrt{l(l+1)}} ((\mathcal{Q}_4)_0)_\psi^{(lm)}. \quad (4.21)$$

To solve (4.11f), define the modes $l \leq 1$ of $\partial_u^2 \dot{f}$ by

$$(\mathcal{Q}_5)_0 = \frac{1}{2} \partial_u^2 \dot{f}^{[\leq 1]},$$

which in Fourier modes equals, for $l \leq 1$,

$$\partial_u^2 \dot{f}^{(lm)} = 2(\mathcal{Q}_5)_0^{(lm)}. \quad (4.22)$$

From (4.21) and (4.22), we directly get that

$$\left\| \partial_u^2 \dot{f} \right\|_{H^4(S_2)} \lesssim \|(\mathcal{Q}_4)_0\|_{H^2(S_2)}. \quad (4.23)$$

(4) Definition of $\partial_u^3 \dot{f}$ on S_2 . To solve (4.11g) and (4.11h), define $\partial_u^3 \dot{f}^{[\leq 1]}$ by

$$(\mathcal{Q}_6)_0 = \frac{1}{2} \partial_u^3 \dot{f}^{[\leq 1]}.$$

and $\partial_u^3 \dot{f}^{[2]}$ by

$$(\mathcal{Q}_7)_0 = \frac{1}{2} \partial_u^3 \dot{f}^{[2]} - 3\dot{f}^{[2]},$$

and let $\partial_u^3 \dot{f}^{[\geq 3]} = 0$. By (4.7), $\partial_u^3 \dot{f}$ is well defined and it holds that

$$\|\partial_u^3 \dot{f}\|_{H^2(S_2)} \lesssim \|(\mathcal{Q}_6)_0\|_{H^2(S_2)} + \|(\mathcal{Q}_7)_0\|_{H^2(S_2)} + \|\dot{f}\|_{H^8(S_2)}. \quad (4.24)$$

To summarize the above, we constructed a linearized perturbation vectorfield and function

$$\dot{q} = (\dot{q}_1, \dot{q}_2) \text{ and } \dot{f} = (\dot{f}, \partial_u \dot{f}, \partial_u^2 \dot{f}, \partial_u^3 \dot{f})$$

such that (4.9) is satisfied. Further, from (4.13), (4.15), (4.17), (4.23) and (4.24), it follows that

$$\begin{aligned} \|\dot{f}\|_{\mathcal{Y}_f} + \|\dot{q}\|_{\mathcal{Y}_q} &\lesssim \|(\mathcal{Q}_1)_0\|_{H^6(S_2)} + \|(\mathcal{Q}_2)_0\|_{H^4(S_2)} \\ &\quad + \|(\mathcal{Q}_3)_0\|_{H^4(S_2)} + \|(\mathcal{Q}_4)_0\|_{H^2(S_2)} \\ &\quad + \|(\mathcal{Q}_5)_0\|_{H^4(S_2)} + \|(\mathcal{Q}_6)_0\|_{H^2(S_2)} + \|(\mathcal{Q}_7)_0\|_{H^2(S_2)}. \end{aligned}$$

The estimate (4.10) follows from the above by Lemma 2.24. This finishes the proof of Proposition 4.4.

4.3. Representation Formulas and Estimates

In this section, we rewrite the linearized null constraint equations (4.2) into a set of transport equations and integrate them to derive representation formulas and estimates.

- In Sect. 4.3.1, we consider $\dot{\phi}$ and $\dot{g}_c$.
- In Sect. 4.3.2, we consider $(\Omega \text{tr} \chi)$ and $\dot{\eta}$.
- In Sect. 4.3.3, we consider $(\Omega \text{tr} \underline{\chi})$ and $\hat{\underline{\chi}}$.
- In Sect. 4.3.4, we consider $\dot{\underline{\alpha}}$, $\dot{\underline{\omega}}$ and $\underline{D}\dot{\underline{\omega}}$.

Notation. In this section, we ease presentation by leaving away the trivial u index on spheres and sphere data, denoting $\mathcal{H} = \mathcal{H}_{0,[1,2]}$, and writing v instead of r on $\mathcal{H}$.

4.3.1. Analysis of $\dot{\phi}$ and $\dot{g}_c$. The linearized null constraint equations for $\dot{\phi}$ and $\dot{g}_c$ in (4.2), that is,

$$D \left(D\dot{\phi} - 2\dot{\Omega} \right) = \dot{\mathfrak{c}}_1, \quad r^2 D\dot{g}_c - 2\dot{\hat{\chi}} = \dot{\mathfrak{c}}_3,$$

are equivalent to

$$DD\dot{\phi} = 2D\dot{\Omega} + \dot{\mathfrak{c}}_1, \quad D\dot{g}_c = \frac{2}{r^2}\dot{\hat{\chi}} + \frac{1}{r^2}\dot{\mathfrak{c}}_3. \quad (4.25)$$

By integration of (4.25), we directly get the following lemma.

Lemma 4.5 (Representation formulas and estimates for $\dot{\phi}$ and $\dot{g}_c$). *Consider sphere data $\hat{\mathfrak{X}}_1$ on S_1 and linearized conformal data $\hat{\chi}$ and $\dot{\Omega}$ on $\mathcal{H}$. Integrating the transport equations (4.25) and using $\dot{\mathfrak{C}}_2 = \dot{\mathfrak{c}}_2$ (see Lemma 2.16) yield the following representation formulas.*

(1) *It holds that*

$$\dot{\phi} = 2 \int_1^v \dot{\Omega} dv' + \int_1^v \int_1^{v'} \dot{\mathfrak{c}}_1 dv'' dv' + v\dot{\phi}(1) + \frac{v-1}{2} \left((\Omega \text{tr} \chi)(1) - 4\dot{\Omega}(1) + \dot{\mathfrak{c}}_2(1) \right), \quad (4.26)$$

which yields the estimate

$$\|\dot{\phi}\|_{H_4^6(\mathcal{H})} \lesssim \|\dot{\mathfrak{X}}_1\|_{\mathcal{X}(S_1)} + \|\dot{\Omega}\|_{H_3^6(\mathcal{H})} + \|(\dot{\mathfrak{c}}_i)_{1 \leq i \leq 10}\|_{\mathcal{Z}_C}.$$

(2) It holds that

$$\dot{g}_c = 2 \int_1^v \frac{1}{v'^2} \dot{\chi} dv' + \int_1^v \frac{1}{v'^2} \dot{\mathfrak{c}}_3 dv' + \dot{g}_c(1), \quad (4.27)$$

which yields the estimate

$$\|\dot{g}_c\|_{H_3^6(\mathcal{H})} \lesssim \|\dot{\mathfrak{X}}_1\|_{\mathcal{X}(S_1)} + \|\dot{\chi}\|_{H_2^6(\mathcal{H})} + \|(\dot{\mathfrak{c}}_i)_{1 \leq i \leq 10}\|_{\mathcal{Z}_C}.$$

4.3.2. Analysis of $(\Omega \text{tr} \chi)$ and $\dot{\eta}$. Recall that the linearized null constraint equations for $(\Omega \text{tr} \chi)$ and $\dot{\eta}$ in (4.2) are given by

$$D \left(\frac{\dot{\phi}}{r} \right) - \frac{(\Omega \text{tr} \chi)}{2} = \frac{1}{2r^2} \dot{\mathfrak{c}}_2, \quad (4.28a)$$

$$D(r^2 \dot{\eta}) + \frac{r^2}{2} \not{D} \left((\Omega \text{tr} \chi) - \frac{4}{r} \dot{\Omega} \right) - 2r \not{D} \dot{\Omega} - \text{div} \dot{\chi} = r^2 \dot{\mathfrak{c}}_4. \quad (4.28b)$$

In the following, we rewrite (4.28a) and (4.28b) to get useful bounds and representation formulas for $(\Omega \text{tr} \chi)$ and $\dot{\eta}$.

On the one hand, using (4.25), (4.28a) can be rewritten as

$$D \left(r^2 \left((\Omega \text{tr} \chi) - \frac{4}{r} \dot{\Omega} \right) + \dot{\mathfrak{c}}_2 \right) = -4\dot{\Omega} + 2r\dot{\mathfrak{c}}_1. \quad (4.29)$$

Straightforward integration of (4.28b) and (4.29) yields the following lemma.

Lemma 4.6 (Bounds for $(\Omega \text{tr} \chi)$ and $\dot{\eta}$). *Consider given sphere data $\dot{\mathfrak{X}}_1$ on S_1 and given $\dot{\chi}$ and $\dot{\Omega}$ on $\mathcal{H}$. Integrating subsequently the transport equations (4.29) for $(\Omega \text{tr} \chi)$ and (4.28b) for $\dot{\eta}$ yields*

$$\begin{aligned} \|(\Omega \text{tr} \chi)\|_{H_3^6(\mathcal{H})} &\lesssim \|\dot{\mathfrak{X}}_1\|_{\mathcal{X}(S_1)} + \|\dot{\Omega}\|_{H_3^6(\mathcal{H})} + \|(\dot{\mathfrak{c}}_i)_{1 \leq i \leq 10}\|_{\mathcal{Z}_C}, \\ \|\dot{\eta}\|_{H_2^5(\mathcal{H})} &\lesssim \|\dot{\mathfrak{X}}_1\|_{\mathcal{X}(S_1)} + \|\dot{\Omega}\|_{H_3^6(\mathcal{H})} + \|\dot{\chi}\|_{H_2^6(\mathcal{H})} + \|(\dot{\mathfrak{c}}_i)_{1 \leq i \leq 10}\|_{\mathcal{Z}_C}. \end{aligned}$$

On the other hand, using (4.25), (4.28b) and (4.29) can be rewritten as

$$D \left(r^2 \dot{\eta} + \frac{r^3}{2} \not{D} \left((\Omega \text{tr} \chi) - \frac{4}{r} \dot{\Omega} \right) + \frac{r}{2} \not{D} \dot{\mathfrak{c}}_2 \right) = \text{div} \dot{\chi} + r^2 \dot{\mathfrak{c}}_4 + r^2 \not{D} \dot{\mathfrak{c}}_1 + \frac{1}{2} \not{D} \dot{\mathfrak{c}}_2, \quad (4.30a)$$

$$D \left(\frac{r}{2} \left((\Omega \text{tr} \chi) - \frac{4}{r} \dot{\Omega} \right) + \frac{\dot{\phi}}{r} + \frac{1}{2r} \dot{\mathfrak{c}}_2 \right) = \dot{\mathfrak{c}}_1. \quad (4.30b)$$

Integrating (4.30a), we get the following representation formula for $\dot{\eta}$.

Lemma 4.7 (Representation formulas for $\dot{\eta}$). *Consider given sphere data $\dot{\mathfrak{X}}_1$ on S_1 and given $\hat{\chi}$ and $\dot{\Omega}$ on $\mathcal{H}$. Integrating (4.30a) yields that*

$$\begin{aligned} & \left[v^2 \dot{\eta} + \frac{v^3}{2} \not{d} \left((\Omega \text{tr} \chi) - \frac{4}{v} \dot{\Omega} \right) + \frac{v}{2} \not{d} \dot{\mathfrak{c}}_2 \right]_1^v \\ &= \text{djv} \left(\int_1^v \hat{\chi} dv' \right) + \int_1^v \left(v'^2 \dot{\mathfrak{c}}_4 + v'^2 \not{d} \dot{\mathfrak{c}}_1 + \frac{1}{2} \not{d} \dot{\mathfrak{c}}_2 \right) dv'. \end{aligned}$$

Recall from (4.5) that $\mathcal{Q}_0$ and $\mathcal{Q}_1$ are defined by

$$\mathcal{Q}_0 := r^2 \dot{\eta}^{[1]} + \frac{r^3}{2} \not{d} \left((\Omega \text{tr} \chi)^{[1]} - \frac{4}{r} \dot{\Omega}^{[1]} \right), \quad \mathcal{Q}_1 := \frac{r}{2} \left((\Omega \text{tr} \chi) - \frac{4}{r} \dot{\Omega} \right) + \frac{\dot{\phi}}{r}.$$

The next lemma follows directly from (4.30a) and (4.30b).

Lemma 4.8 (Conservation laws I). *It holds that*

$$\begin{aligned} D \left(\mathcal{Q}_0 + \frac{r}{2} \not{d} \dot{\mathfrak{c}}_2^{[1]} \right) &= r^2 \dot{\mathfrak{c}}_4^{[1]} + r^2 \not{d} \dot{\mathfrak{c}}_1^{[1]} + \frac{1}{2} \not{d} \dot{\mathfrak{c}}_2^{[1]}, \\ D \left(\mathcal{Q}_1 + \frac{1}{2r} \dot{\mathfrak{c}}_2 \right) &= \dot{\mathfrak{c}}_1. \end{aligned}$$

The proof of the following lemma is omitted.

Lemma 4.9 (Properties of charges I). *The following holds.*

(1) *Let $\dot{x}_v$ be sphere data on a sphere $S_v \subset \mathcal{H}$. Then,*

$$\|\mathcal{Q}_0\|_{H^5(S_v)} + \|\mathcal{Q}_1\|_{H^6(S_v)} \lesssim \|\dot{x}_v\|_{\mathcal{X}(S_v)}.$$

(2) *For given sphere data $\dot{\mathfrak{X}}_1$ on S_1 and source terms $(\dot{\mathfrak{c}}_i)_{1 \leq i \leq 10}$ on $\mathcal{H}$, define $^{(1)}\mathcal{Q}_0$ and $^{(1)}\mathcal{Q}_1$ as solution to the transport equations of Lemma 4.8 on $\mathcal{H}$ with initial values given by $\mathcal{Q}_0$ and $\mathcal{Q}_1$ calculated from $\dot{\mathfrak{X}}_1$. Then, it holds that*

$$\begin{aligned} & \|^{(1)}\mathcal{Q}_0\|_{H_2^5(\mathcal{H})} + \|^{(1)}\mathcal{Q}_1\|_{H_3^6(\mathcal{H})} \\ & \lesssim \|\mathcal{Q}_0(\dot{\mathfrak{X}}_1)\|_{H^5(S_1)} + \|\mathcal{Q}_1(\dot{\mathfrak{X}}_1)\|_{H^6(S_1)} + \|(\dot{\mathfrak{c}}_i)_{1 \leq i \leq 10}\|_{\mathcal{Z}_C}. \end{aligned}$$

4.3.3. Analysis of $\Omega \text{tr} \underline{\chi}$ and $\dot{\underline{\chi}}$. The linearized null constraint equations (4.2) for $(\Omega \text{tr} \underline{\chi})$ and $\dot{\underline{\chi}}$ are given by

$$\begin{aligned} D \left(v^2 (\Omega \text{tr} \underline{\chi}) \right) - 2v (\Omega \text{tr} \chi) + 2 \text{djv} \left(\dot{\eta} - 2 \not{d} \dot{\Omega} \right) + 2v^2 \dot{K} + 4\dot{\Omega} &= v^2 \dot{\mathfrak{c}}_5, \\ D \left(\frac{\dot{\underline{\chi}}}{v} \right) - \frac{2}{v} \mathcal{P}_2^* \left(\dot{\eta} - 2 \not{d} \dot{\Omega} \right) - \frac{1}{v^2} \dot{\chi} &= \frac{1}{v} \dot{\mathfrak{c}}_6. \end{aligned} \tag{4.31}$$

The following lemma follows from (4.31) and Lemmas 4.5 and 4.6.

Lemma 4.10 (Bounds for $(\Omega \text{tr} \underline{\chi})$ and $\dot{\hat{\chi}}$). *Consider given sphere data $\dot{\mathfrak{X}}_1$ on S_1 and given $\hat{\chi}$ and $\dot{\Omega}$ on $\mathcal{H}$. Integrating the transport equations (4.31) for $(\Omega \text{tr} \underline{\chi})$ and $\dot{\hat{\chi}}$ yields*

$$\begin{aligned} \|(\Omega \text{tr} \underline{\chi})\|_{H_2^4(\mathcal{H})} &\lesssim \|\dot{\mathfrak{X}}_1\|_{\mathcal{X}(S_1)} + \|\dot{\Omega}\|_{H_3^8(\mathcal{H})} + \|\dot{\hat{\chi}}\|_{H_2^8(\mathcal{H})} + \|(\dot{\mathfrak{c}}_i)_{1 \leq i \leq 10}\|_{\mathcal{Z}_C}, \\ \|\dot{\hat{\chi}}\|_{H_3^4(\mathcal{H})} &\lesssim \|\dot{\mathfrak{X}}_1\|_{\mathcal{X}(S_1)} + \|\dot{\Omega}\|_{H_3^8(\mathcal{H})} + \|\dot{\hat{\chi}}\|_{H_2^8(\mathcal{H})} + \|(\dot{\mathfrak{c}}_i)_{1 \leq i \leq 10}\|_{\mathcal{Z}_C}. \end{aligned}$$

Recall from (4.5) that $\mathcal{Q}_2$ and $\mathcal{Q}_3$ are defined as

$$\begin{aligned} \mathcal{Q}_2 &:= r^2(\Omega \text{tr} \underline{\chi}) - \frac{2}{r} \mathring{\text{d}}\dot{v} \left(r^2 \dot{\eta} + \frac{r^3}{2} \not{d} \left((\Omega \text{tr} \underline{\chi}) - \frac{4}{r} \dot{\Omega} \right) \right) \\ &\quad - r^2 \left((\Omega \text{tr} \underline{\chi}) - \frac{4}{r} \dot{\Omega} \right) + 2r^3 \dot{K}, \\ \mathcal{Q}_3 &:= \frac{\dot{\hat{\chi}}}{r} - \frac{1}{2} \left(\mathcal{P}_2^* \mathring{\text{d}}\dot{v} + 1 \right) \dot{g}_c + \mathcal{P}_2^* \left(\dot{\eta} + \frac{r}{2} \not{d} \left((\Omega \text{tr} \underline{\chi}) - \frac{4}{r} \dot{\Omega} \right) \right) \\ &\quad - r \mathcal{P}_2^* \not{d} \left((\Omega \text{tr} \underline{\chi}) - \frac{4}{r} \dot{\Omega} \right). \end{aligned}$$

The following lemma shows that they are indeed subject to conservation laws.

Lemma 4.11 (Conservation laws II). *The linearized null constraint equations (4.2) imply the following transport equations for $\mathcal{Q}_2$ and $\mathcal{Q}_3$,*

$$\begin{aligned} D \left(\mathcal{Q}_2 - (\not{d} + 1) \dot{\mathfrak{c}}_2 \right) &= v^2 \dot{\mathfrak{c}}_5 - 2v \mathring{\text{d}}\dot{v} \dot{\mathfrak{c}}_4 - 2v (\not{d} + 1) \dot{\mathfrak{c}}_1 \\ &\quad - \frac{1}{v} (\not{d} + 2) \dot{\mathfrak{c}}_2 + \frac{1}{v} \mathring{\text{d}}\dot{v} \mathring{\text{d}}\dot{v} \dot{\mathfrak{c}}_3, \\ D \left(\mathcal{Q}_3 - \frac{1}{2v} \mathcal{P}_2^* \not{d} \dot{\mathfrak{c}}_2 \right) &= \frac{1}{v} \dot{\mathfrak{c}}_6 - \frac{1}{2v^2} \left(\mathcal{P}_2^* \mathring{\text{d}}\dot{v} + 1 \right) \dot{\mathfrak{c}}_3 \\ &\quad + \mathcal{P}_2^* \dot{\mathfrak{c}}_4 - \mathcal{P}_2^* \not{d} \dot{\mathfrak{c}}_1 + \frac{1}{2v^2} \mathcal{P}_2^* \not{d} \dot{\mathfrak{c}}_2. \end{aligned}$$

Proof of Lemma 4.11. First, by (2.49), (4.5), (4.30a), (4.30b) and (4.31), we have that

$$\begin{aligned} D\mathcal{Q}_2 &= v^2 \dot{\mathfrak{c}}_5 + 2v(\Omega \text{tr} \underline{\chi}) - 2\mathring{\text{d}}\dot{v} \left(\dot{\eta} - 2\not{d} \dot{\Omega} \right) - 2v^2 \dot{K} - 4\dot{\Omega} \\ &\quad - 2D \left(\frac{1}{v} \mathring{\text{d}}\dot{v} \left(v^2 \dot{\eta} + \frac{v^3}{2} \not{d} \left((\Omega \text{tr} \underline{\chi}) - \frac{4}{v} \dot{\Omega} \right) \right) \right) \\ &\quad - D \left(v^2 \left((\Omega \text{tr} \underline{\chi}) - \frac{4}{v} \dot{\Omega} \right) \right) + 2v^2 \dot{K} + 2vD \left(v^2 \dot{K} \right) \\ &= v^2 \dot{\mathfrak{c}}_5 + 2v(\Omega \text{tr} \underline{\chi}) - 2\mathring{\text{d}}\dot{v} \left(\dot{\eta} - 2\not{d} \dot{\Omega} \right) - 4\dot{\Omega} + \frac{2}{v^2} \mathring{\text{d}}\dot{v} \left(v^2 \dot{\eta} + \frac{v^3}{2} \not{d} \left((\Omega \text{tr} \underline{\chi}) - \frac{4}{v} \dot{\Omega} \right) \right) \\ &\quad - \frac{2}{v} \mathring{\text{d}}\dot{v} \left(\mathring{\text{d}}\dot{v} \dot{\hat{\chi}} + v^2 \dot{\mathfrak{c}}_4 + v^2 \not{d} \dot{\mathfrak{c}}_1 + \frac{1}{2} \not{d} \dot{\mathfrak{c}}_2 - D \left(\frac{v}{2} \not{d} \dot{\mathfrak{c}}_2 \right) \right) \\ &\quad + 4\dot{\Omega} - 2v \dot{\mathfrak{c}}_1 + D \dot{\mathfrak{c}}_2 + v \mathring{\text{d}}\dot{v} \mathring{\text{d}}\dot{v} \left(\frac{2}{v^2} \dot{\hat{\chi}} + \frac{1}{v^2} \dot{\mathfrak{c}}_3 \right) - 2v (\not{d} + 2) \left(\frac{(\Omega \text{tr} \underline{\chi})}{2} + \frac{1}{2v^2} \dot{\mathfrak{c}}_2 \right) \end{aligned}$$

$$= v^2 \dot{\mathbf{c}}_5 - 2v \mathring{\mathrm{d}}\dot{\mathbf{v}} \dot{\mathbf{c}}_4 - 2v(\mathring{\Delta} + 1) \dot{\mathbf{c}}_1 + D \left((\mathring{\Delta} + 1) \dot{\mathbf{c}}_2 \right) - \frac{1}{v} (\mathring{\Delta} + 2) \dot{\mathbf{c}}_2 + \frac{1}{v} \mathring{\mathrm{d}}\dot{\mathbf{v}} \mathring{\mathrm{d}}\dot{\mathbf{v}} \dot{\mathbf{c}}_3.$$

Second, by (4.5), (4.30a), (4.30b) and (4.31), we have that

$$\begin{aligned} D\mathcal{Q}_3 &= \frac{1}{v} \dot{\mathbf{c}}_6 + \frac{2}{v} \mathcal{P}_2^* \left(\dot{\eta} - 2\mathcal{P} \dot{\Omega} \right) + \frac{1}{v^2} \dot{\chi} - \frac{1}{2} \left(\mathcal{P}_2^* \mathring{\mathrm{d}}\dot{\mathbf{v}} + 1 \right) \left(\frac{2}{v^2} \dot{\chi} + \frac{1}{v^2} \dot{\mathbf{c}}_3 \right) \\ &\quad - \frac{2}{v} \mathcal{P}_2^* \left(\dot{\eta} + \frac{v}{2} \mathcal{P} \left((\Omega \mathrm{tr} \chi) - \frac{4}{v} \dot{\Omega} \right) \right) \\ &\quad + \frac{1}{v^2} \mathcal{P}_2^* \left(\mathring{\mathrm{d}}\dot{\mathbf{v}} \dot{\chi} + v^2 \dot{\mathbf{c}}_4 + v^2 \mathcal{P} \dot{\mathbf{c}}_1 + \frac{1}{2} \mathcal{P} \dot{\mathbf{c}}_2 - D \left(\frac{v}{2} \mathcal{P} \dot{\mathbf{c}}_2 \right) \right) \\ &\quad + \mathcal{P}_2^* \mathcal{P} \left((\Omega \mathrm{tr} \chi) - \frac{4}{v} \dot{\Omega} \right) - \frac{1}{v} \mathcal{P}_2^* \mathcal{P} \left(-4\dot{\Omega} + 2v\dot{\mathbf{c}}_1 - D\dot{\mathbf{c}}_2 \right) \\ &= \frac{1}{v} \dot{\mathbf{c}}_6 - \frac{1}{2v^2} \left(\mathcal{P}_2^* \mathring{\mathrm{d}}\dot{\mathbf{v}} + 1 \right) \dot{\mathbf{c}}_3 + \mathcal{P}_2^* \dot{\mathbf{c}}_4 - \mathcal{P}_2^* \mathcal{P} \dot{\mathbf{c}}_1 + D \left(\frac{1}{2v} \mathcal{P}_2^* \mathcal{P} \dot{\mathbf{c}}_2 \right) + \frac{1}{2v^2} \mathcal{P}_2^* \mathcal{P} \dot{\mathbf{c}}_2. \end{aligned}$$

This finishes the proof of Lemma 4.11. $\square$

The proof of the following lemma is omitted.

Lemma 4.12 (Properties of charges II). *The following holds.*

(1) *Let $\dot{x}_v$ be sphere data on a sphere $S_v \subset \mathcal{H}$. Then,*

$$\|\mathcal{Q}_2\|_{H^4(S_v)} + \|\mathcal{Q}_3\|_{H^4(S_v)} \lesssim \|\dot{x}_v\|_{\mathcal{X}(S_v)}.$$

(2) *For given sphere data $\dot{\mathfrak{X}}_1$ on S_1 and source terms $(\dot{\mathbf{c}}_i)_{1 \leq i \leq 10}$ on $\mathcal{H}$, define $^{(1)}\mathcal{Q}_i$, $2 \leq i \leq 3$, as solution to the transport equations of Lemma 4.11 on $\mathcal{H}$ with initial values given by $\mathcal{Q}_i$ calculated from $\dot{\mathfrak{X}}_1$. Then, it holds that*

$$\begin{aligned} \|^{(1)}\mathcal{Q}_2\|_{H_2^4(\mathcal{H})} + \|^{(1)}\mathcal{Q}_3\|_{H_2^4(\mathcal{H})} &\lesssim \|\mathcal{Q}_2(\dot{\mathfrak{X}}_1)\|_{H^4(S_1)} \\ &\quad + \|\mathcal{Q}_3(\dot{\mathfrak{X}}_1)\|_{H^6(S_1)} + \|(\dot{\mathbf{c}}_i)_{1 \leq i \leq 10}\|_{\mathcal{Z}_C}. \end{aligned}$$

4.3.4. Analysis of $\dot{\omega}$, $\dot{\alpha}$ and $D\dot{\omega}$. The linearized null constraint equations (4.2) for $\dot{\omega}$, $\dot{\alpha}$ and $D\dot{\omega}$ are

$$\begin{aligned} D\dot{\omega} - \dot{K} - \frac{1}{2v}(\Omega \mathrm{tr} \chi) + \frac{1}{2v}(\Omega \mathrm{tr} \chi) - \frac{2}{v^2} \dot{\Omega} &= \dot{\mathbf{c}}_7, \\ D \left(\frac{\dot{\alpha}}{v} \right) - \frac{2}{v} \mathcal{P}_2^* \left(\frac{1}{v^2} \mathring{\mathrm{d}}\dot{\mathbf{v}} \dot{\chi} - \frac{1}{2} \mathcal{P} (\Omega \mathrm{tr} \chi) - \frac{1}{v} \dot{\eta} \right) &= \frac{1}{v} \dot{\mathbf{c}}_8, \end{aligned} \tag{4.32}$$

and

$$\begin{aligned} DD\dot{\omega} - \frac{3}{v} \left(\dot{K} + \frac{1}{2v}(\Omega \mathrm{tr} \chi) - \frac{1}{2v}(\Omega \mathrm{tr} \chi) + \frac{2}{v^2} \dot{\Omega} \right) \\ = \frac{1}{v^2} \mathring{\mathrm{d}}\dot{\mathbf{v}} \left(\frac{1}{v^2} \mathring{\mathrm{d}}\dot{\mathbf{v}} \dot{\chi} - \frac{1}{v} \dot{\eta} - \frac{1}{2} \mathcal{P} (\Omega \mathrm{tr} \chi) \right) + \dot{\mathbf{c}}_9. \end{aligned} \tag{4.33}$$

Integrating (4.32) and (4.33) and applying Lemmas 4.5, 4.6 and 4.10 yield the following lemma.

Lemma 4.13 (Bounds for $\dot{\underline{\omega}}$, $\dot{\underline{\alpha}}$ and $D\dot{\underline{\omega}}$). *Consider given sphere data $\dot{\mathfrak{X}}_1$ on S_1 and given $\hat{\chi}$ and $\hat{\Omega}$ on $\mathcal{H}$. Integrating the transport equations (4.31) for $(\Omega \text{tr} \chi)$ and $\hat{\chi}$ yields*

$$\begin{aligned} & \|\dot{\underline{\omega}}\|_{H^3_3(\mathcal{H})} + \|\dot{\underline{\alpha}}\|_{H^2_3(\mathcal{H})} + \|D\dot{\underline{\omega}}\|_{H^2_3(\mathcal{H})} \\ & \lesssim \|\dot{\mathfrak{X}}_1\|_{\mathcal{X}(S_1)} + \|\hat{\Omega}\|_{H^2_3(\mathcal{H})} + \|\hat{\chi}\|_{H^2_2(\mathcal{H})} + \|(\dot{\mathfrak{c}}_i)_{1 \leq i \leq 10}\|_{\mathcal{Z}_C}. \end{aligned}$$

In Appendix B.2, it is proved that the linearized null constraint Eqs. (4.32) and (4.33) can be rewritten as follows.

Lemma 4.14. *The linearized null constraint Eqs. (4.2) imply the following transport equations.*

$$\begin{aligned} D \left(\dot{\underline{\omega}} + \frac{1}{4v^2} \mathcal{Q}_2 + \frac{1}{3v} \text{d}\dot{\mathfrak{v}} \left(\dot{\eta} + \frac{v}{2} \not{d} \left((\Omega \text{tr} \chi) - \frac{4}{v} \dot{\Omega} \right) \right) - \frac{1}{12v^2} (\dot{\Delta} + 3) \dot{\mathfrak{c}}_2 \right) \\ = \frac{1}{3v^3} \text{d}\dot{\mathfrak{v}} \text{d}\dot{\mathfrak{v}} \hat{\chi} + h_{\dot{\underline{\omega}}}, \end{aligned} \quad (4.34)$$

with source term

$$h_{\dot{\underline{\omega}}} := \dot{\mathfrak{c}}_7 + \frac{1}{4} \dot{\mathfrak{c}}_5 + \frac{1}{4v^3} \text{d}\dot{\mathfrak{v}} \text{d}\dot{\mathfrak{v}} \dot{\mathfrak{c}}_3 - \frac{1}{6v} \text{d}\dot{\mathfrak{v}} \dot{\mathfrak{c}}_4 - \frac{1}{6v} (\dot{\Delta} + 3) \dot{\mathfrak{c}}_1 - \frac{1}{12v^3} \dot{\Delta} \dot{\mathfrak{c}}_2, \quad (4.35)$$

and

$$\begin{aligned} D \left(\frac{\dot{\alpha}}{v} + \frac{2}{v} \mathcal{P}_2^* \text{d}\dot{\mathfrak{v}} \mathcal{Q}_3 - \frac{1}{2v^2} \mathcal{P}_2^* \not{d} \mathcal{Q}_2 - \frac{2}{v} \mathcal{P}_2^* \not{d} \left(\dot{\Delta} + 2 \right) \mathcal{Q}_1 \right) \\ - D \left(\frac{2}{3v} \mathcal{P}_2^* \left(\text{d}\dot{\mathfrak{v}} \mathcal{P}_2^* + 1 + \not{d} \text{d}\dot{\mathfrak{v}} \right) \left(\dot{\eta} + \frac{v}{2} \not{d} \left((\Omega \text{tr} \chi) - \frac{4}{v} \dot{\Omega} \right) \right) \right) \\ + D \left(\frac{1}{v} \mathcal{P}_2^* \left(\text{d}\dot{\mathfrak{v}} \mathcal{P}_2^* + 1 + \not{d} \text{d}\dot{\mathfrak{v}} \right) \text{d}\dot{\mathfrak{v}} \dot{\mathfrak{g}}_c - \frac{1}{3v^2} \mathcal{P}_2^* \left(\text{d}\dot{\mathfrak{v}} \mathcal{P}_2^* + 1 + \not{d} \text{d}\dot{\mathfrak{v}} \right) \text{d}\dot{\mathfrak{v}} \not{d} \dot{\mathfrak{c}}_2 \right) \\ = \frac{4}{3v^3} \mathcal{P}_2^* \left(\text{d}\dot{\mathfrak{v}} \mathcal{P}_2^* + 1 + \not{d} \text{d}\dot{\mathfrak{v}} \right) \text{d}\dot{\mathfrak{v}} \hat{\chi} + h_{\dot{\underline{\alpha}}}, \end{aligned} \quad (4.36)$$

with source term

$$\begin{aligned} h_{\dot{\underline{\alpha}}} &:= \frac{1}{v} \dot{\mathfrak{c}}_8 + \frac{2}{v} \mathcal{P}_2^* \text{d}\dot{\mathfrak{v}} (D\mathcal{Q}_3) - \frac{1}{2v^2} \mathcal{P}_2^* \not{d} (D\mathcal{Q}_2) \\ &- \frac{2}{v} \mathcal{P}_2^* \not{d} \left(\dot{\Delta} + 2 \right) (D\mathcal{Q}_1) + \frac{1}{v^3} \mathcal{P}_2^* \left(\text{d}\dot{\mathfrak{v}} \mathcal{P}_2^* + 1 + \not{d} \text{d}\dot{\mathfrak{v}} \right) \text{d}\dot{\mathfrak{v}} \dot{\mathfrak{c}}_3 \\ &- \frac{2}{3v^3} \mathcal{P}_2^* \left(\text{d}\dot{\mathfrak{v}} \mathcal{P}_2^* + 1 + \not{d} \text{d}\dot{\mathfrak{v}} \right) \text{d}\dot{\mathfrak{v}} (v^2 \dot{\mathfrak{c}}_4 + v^2 \not{d} \dot{\mathfrak{c}}_1 - \not{d} \dot{\mathfrak{c}}_2), \end{aligned}$$

and

$$\begin{aligned}
& D \left(\underline{D\dot{\omega}} - \frac{1}{6v^3} \left(\overset{\circ}{\Delta} - 3 \right) \mathcal{Q}_2 + \frac{1}{2v^2} \dot{\text{d}}\dot{\text{v}} \dot{\text{d}}\dot{\text{v}} \mathcal{Q}_3 + \frac{1}{v^2} \dot{\text{d}}\dot{\text{v}} \dot{\text{d}}\dot{\text{v}} \mathcal{P}_2^* \not\Delta \mathcal{Q}_1 \right) \\
& - D \left(\frac{1}{4v^2} \dot{\text{d}}\dot{\text{v}} \left(\not\Delta \dot{\text{d}}\dot{\text{v}} - 2 + \dot{\text{d}}\dot{\text{v}} \mathcal{P}_2^* \right) \left(\eta + \frac{v}{2} \not\Delta \left((\Omega \text{tr} \chi) - \frac{4}{v} \dot{\Omega} \right) \right) \right) \\
& + D \left(\frac{1}{8v^2} \dot{\text{d}}\dot{\text{v}} \not\Delta \dot{\text{d}}\dot{\text{v}} \dot{\text{d}}\dot{\text{v}} \dot{\text{g}}_c + \frac{1}{2v^3} \left(\frac{1}{12} \overset{\circ}{\Delta} \overset{\circ}{\Delta} - \frac{1}{6} \overset{\circ}{\Delta} + \frac{1}{4} \dot{\text{d}}\dot{\text{v}} \dot{\text{d}}\dot{\text{v}} \mathcal{P}_2^* \not\Delta - 1 \right) \dot{\text{c}}_2 \right) \\
& = \frac{1}{4v^4} \dot{\text{d}}\dot{\text{v}} \left(2 - \dot{\text{d}}\dot{\text{v}} \mathcal{P}_2^* \right) \dot{\text{d}}\dot{\text{v}} \hat{\chi} + h_{\underline{D\dot{\omega}}},
\end{aligned} \tag{4.37}$$

with source term

$$\begin{aligned}
h_{\underline{D\dot{\omega}}} &:= \dot{\text{c}}_8 + \frac{1}{2v^3} \dot{\text{d}}\dot{\text{v}} \dot{\text{d}}\dot{\text{v}} \dot{\text{c}}_6 - \frac{1}{6v} (\overset{\circ}{\Delta} - 3) \dot{\text{c}}_5 \\
&+ \frac{1}{v^2} \left(\frac{1}{12} \overset{\circ}{\Delta} \overset{\circ}{\Delta} - \frac{1}{6} \overset{\circ}{\Delta} - 1 + \frac{1}{4} \dot{\text{d}}\dot{\text{v}} \dot{\text{d}}\dot{\text{v}} \mathcal{P}_2^* \not\Delta \right) \dot{\text{c}}_1 \\
&+ \frac{1}{v^4} \left(-\frac{1}{8} \dot{\text{d}}\dot{\text{v}} \dot{\text{d}}\dot{\text{v}} \mathcal{P}_2^* \not\Delta + \frac{1}{24} \overset{\circ}{\Delta} \overset{\circ}{\Delta} + \frac{1}{12} \overset{\circ}{\Delta} + \frac{1}{2} \right) \dot{\text{c}}_2 \\
&+ \frac{1}{v^4} \dot{\text{d}}\dot{\text{v}} \left(-\frac{1}{24} \not\Delta \dot{\text{d}}\dot{\text{v}} + \frac{1}{4} - \frac{1}{4} \dot{\text{d}}\dot{\text{v}} \mathcal{P}_2^* \right) \dot{\text{d}}\dot{\text{v}} \dot{\text{c}}_3 \\
&+ \frac{1}{v^2} \left(\frac{1}{12} (\overset{\circ}{\Delta} - 6) \dot{\text{d}}\dot{\text{v}} + \frac{1}{4} \dot{\text{d}}\dot{\text{v}} \dot{\text{d}}\dot{\text{v}} \mathcal{P}_2^* \right) \dot{\text{c}}_4.
\end{aligned} \tag{4.38}$$

As consequence of Lemma 4.14, we get the following useful representation formulas for $\underline{\dot{\omega}}$, $\underline{\dot{\alpha}}$ and $\underline{D\dot{\omega}}$.

Lemma 4.15 (Representation formulas for $\underline{\dot{\omega}}$, $\underline{\dot{\alpha}}$ and $\underline{D\dot{\omega}}$). *Consider sphere data $\check{\mathfrak{X}}_1$ on S_1 and $\hat{\chi}$ and $\dot{\Omega}$ on $\mathcal{H}$. Integrating the transport equations of Lemma 4.14 yields the following representation formulas for $\underline{\dot{\omega}}$, $\underline{\dot{\alpha}}$ and $\underline{D\dot{\omega}}$.*

(1) *It holds that*

$$\begin{aligned}
& \left[\underline{\dot{\omega}} + \frac{1}{4v^2} \mathcal{Q}_2 + \frac{1}{3v} \dot{\text{d}}\dot{\text{v}} \left(\eta + \frac{v}{2} \not\Delta \left((\Omega \text{tr} \chi) - \frac{4}{v} \dot{\Omega} \right) \right) - \frac{1}{12v^2} (\overset{\circ}{\Delta} + 3) \dot{\text{c}}_2 \right]_1^v \\
&= \frac{1}{3} \dot{\text{d}}\dot{\text{v}} \dot{\text{d}}\dot{\text{v}} \left(\int_1^v \frac{1}{v'^3} \hat{\chi} dv' \right) + \int_1^v h_{\underline{\dot{\omega}}} dv',
\end{aligned}$$

(2) *It holds that*

$$\begin{aligned}
& \left[\frac{\dot{\alpha}}{v} + \frac{2}{v} \mathcal{P}_2^* \dot{\text{d}}\dot{\text{v}} \mathcal{Q}_3 - \frac{1}{2v^2} \mathcal{P}_2^* \not\Delta \mathcal{Q}_2 - \frac{2}{v} \mathcal{P}_2^* \not\Delta \left(\overset{\circ}{\Delta} + 2 \right) \mathcal{Q}_1 \right]_1^v \\
& - \left[\frac{2}{3v} \mathcal{P}_2^* \left(\dot{\text{d}}\dot{\text{v}} \mathcal{P}_2^* + 1 + \not\Delta \dot{\text{d}}\dot{\text{v}} \right) \left(\eta + \frac{v}{2} \not\Delta \left((\Omega \text{tr} \chi) - \frac{4}{v} \dot{\Omega} \right) \right) \right]_1^v \\
& + \left[\frac{1}{v} \mathcal{P}_2^* \left(\dot{\text{d}}\dot{\text{v}} \mathcal{P}_2^* + 1 + \not\Delta \dot{\text{d}}\dot{\text{v}} \right) \dot{\text{d}}\dot{\text{v}} \dot{\text{g}}_c - \frac{1}{3v^2} \mathcal{P}_2^* \left(\dot{\text{d}}\dot{\text{v}} \mathcal{P}_2^* + 1 + \not\Delta \dot{\text{d}}\dot{\text{v}} \right) \dot{\text{d}}\dot{\text{v}} \not\Delta \dot{\text{c}}_2 \right]_1^v
\end{aligned}$$

$$= \frac{4}{3} \mathcal{P}_2^* \left(\mathring{\text{div}} \mathcal{P}_2^* + 1 + \mathring{\mathcal{A}} \mathring{\text{div}} \right) \mathring{\text{div}} \left(\int_1^v \frac{1}{v'^3} \dot{\hat{\chi}} dv' \right) + \int_1^v h_{\hat{\mathcal{A}}} dv', \quad (4.39)$$

(3) It holds that

$$\begin{aligned} & \left[\underline{D\dot{\omega}} - \frac{1}{6v^3} \left(\mathring{\mathcal{A}} - 3 \right) \mathcal{Q}_2 + \frac{1}{2v^2} \mathring{\text{div}} \mathring{\text{div}} \mathcal{Q}_3 + \frac{1}{v^2} \mathring{\text{div}} \mathring{\text{div}} \mathcal{P}_2^* \mathring{\mathcal{A}} \mathcal{Q}_1 \right]_1^v \\ & - \left[\frac{1}{4v^2} \mathring{\text{div}} \left(\mathring{\mathcal{A}} \mathring{\text{div}} - 2 + \mathring{\text{div}} \mathcal{P}_2^* \right) \left(\eta + \frac{v}{2} \mathring{\mathcal{A}} \left((\Omega \text{tr} \chi) - \frac{4}{v} \dot{\Omega} \right) \right) \right]_1^v \\ & + \left[\frac{1}{8v^2} \mathring{\text{div}} \mathring{\mathcal{A}} \mathring{\text{div}} \mathring{\text{div}} \mathring{\mathcal{A}} \mathring{\mathcal{A}} + \frac{1}{2v^3} \left(\frac{1}{12} \mathring{\mathcal{A}} \mathring{\mathcal{A}} - \frac{1}{6} \mathring{\mathcal{A}} + \frac{1}{4} \mathring{\text{div}} \mathring{\text{div}} \mathcal{P}_2^* \mathring{\mathcal{A}} - 1 \right) \dot{\mathcal{C}}_2 \right]_1^v \\ & = \frac{1}{4} \mathring{\text{div}} \left(2 - \mathring{\text{div}} \mathcal{P}_2^* \right) \mathring{\text{div}} \left(\int_1^v \frac{1}{v'^4} \dot{\hat{\chi}} dv' \right) + \int_1^v h_{\underline{D\dot{\omega}}} dv'. \end{aligned} \quad (4.40)$$

In the transport equations of Lemma 4.14, we observe that the $\hat{\chi}$ -terms appearing on the right-hand sides of (4.34) and (4.36) have the same weight in v , which indicates a conservation law involving $\dot{\omega}^{[\geq 2]}$ and $\dot{\mathcal{A}}$. Moreover, the modes $l \leq 1$ of the right-hand side of (4.34) and the modes $l \leq 2$ of the right-hand side of (4.37) do not contain $\dot{\Omega}$ or $\hat{\chi}$ (see also (D.8) in Appendix D.3), which also indicates conservation laws.

Recall that $\mathcal{Q}_4, \mathcal{Q}_5, \mathcal{Q}_6$ and $\mathcal{Q}_7$ are defined in (4.5) as

$$\begin{aligned} \mathcal{Q}_4 &:= \frac{\dot{\mathcal{A}}_\psi}{r} + 2\mathcal{P}_2^* \left(\frac{1}{r^2} \mathring{\text{div}} \dot{\hat{\chi}} - \frac{1}{r} \dot{\eta} - \frac{1}{2} \mathring{\mathcal{A}} (\Omega \text{tr} \chi) + \mathcal{P}_1^* (\dot{\omega}, 0) \right)_\psi, \\ \mathcal{Q}_5 &:= \underline{\dot{\omega}}^{[\leq 1]} + \frac{1}{4r^2} \mathcal{Q}_2^{[\leq 1]} + \frac{1}{3r^3} \mathring{\text{div}} \mathcal{Q}_0, \\ \mathcal{Q}_6 &:= \underline{D\dot{\omega}}^{[\leq 1]} - \frac{1}{6r^3} (\mathring{\mathcal{A}} - 3) \mathcal{Q}_2^{[\leq 1]} + \frac{1}{r^4} \mathring{\text{div}} \mathcal{Q}_0, \\ \mathcal{Q}_7 &:= \underline{D\dot{\omega}}^{[2]} + \frac{3}{2r^3} \mathcal{Q}_2^{[2]} + \frac{1}{2r^2} \mathring{\text{div}} \mathring{\text{div}} \mathcal{Q}_3^{[2]} - \frac{12}{r^2} \mathcal{Q}_1^{[2]} \\ & \quad + \frac{3}{2r^2} \mathring{\text{div}} \left(\dot{\eta} + \frac{r}{2} \mathring{\mathcal{A}} \left((\Omega \text{tr} \chi) - \frac{4}{r} \dot{\Omega} \right) \right)^{[2]} - \frac{3}{4r^2} \mathring{\text{div}} \mathring{\text{div}} \mathring{\mathcal{A}} \mathring{\mathcal{A}}^{[2]}. \end{aligned}$$

From Lemma 4.14, we get the following conservation laws.

Lemma 4.16 (Conservation laws III). *It holds that*

$$D\mathcal{Q}_4 = \frac{1}{v} (\dot{\mathcal{C}}_8)_\psi + \frac{2}{v^2} \left(\mathcal{P}_2^* \mathring{\text{div}} \dot{\mathcal{C}}_6 \right)_\psi - \frac{2}{v} (\mathcal{P}_2^* \dot{\mathcal{C}}_4)_\psi - (\mathcal{P}_2^* \mathring{\mathcal{A}} \dot{\mathcal{C}}_5)_\psi - 2 (\mathcal{P}_2^* \mathring{\mathcal{A}} \dot{\mathcal{C}}_7)_\psi,$$

and

$$D \left(\mathcal{Q}_5 - \frac{1}{12v^2} (\mathring{\mathcal{A}} + 3) \dot{\mathcal{C}}_2^{[\leq 1]} \right) = h_{\underline{\dot{\omega}}}^{[\leq 1]},$$

where $h_{\underline{\dot{\omega}}}$ is defined in (4.35) and

$$D \left(\mathcal{Q}_6 + \frac{1}{2v^3} \left(\frac{1}{12} \overset{\circ}{\mathcal{A}} \overset{\circ}{\mathcal{A}} - \frac{1}{6} \overset{\circ}{\mathcal{A}} - 1 \right) \dot{\mathbf{c}}_2^{[\leq 1]} \right) = h_{\underline{\dot{\omega}}}^{[\leq 1]},$$

and

$$D \left(\mathcal{Q}_7 - \frac{1}{v^3} \dot{\mathbf{c}}_2^{[2]} \right) = h_{\underline{\dot{\omega}}}^{[2]},$$

where $h_{\underline{\dot{\omega}}}$ is defined in (4.38).

Proof of Lemma 4.16. By the linearized null constraint equations (4.2), we have that (with $r = v$)

$$\begin{aligned} D\mathcal{Q}_4 &= D \left(\frac{\dot{\mathbf{c}}_8}{v} + 2\mathcal{P}_2^* \left(\frac{1}{v} \overset{\circ}{\text{div}} \left(\frac{\dot{\chi}}{v} \right) - \frac{1}{v^3} (v^2 \dot{\eta}) - \frac{1}{2v^2} \mathcal{A} (v^2 (\Omega \text{tr} \chi)) - \mathcal{A} \dot{\omega} \right) \right)_{\psi} \\ &= \frac{1}{v} (\dot{\mathbf{c}}_8)_{\psi} + \frac{2}{v} \mathcal{P}_2^* \left(\frac{1}{v^2} \overset{\circ}{\text{div}} \dot{\chi} - \frac{1}{2} \mathcal{A} (\Omega \text{tr} \chi) - \frac{1}{v} \dot{\eta} \right)_{\psi} \\ &\quad + 2\mathcal{P}_2^* \left(-\frac{1}{v^2} \overset{\circ}{\text{div}} \left(\frac{\dot{\chi}}{v} \right) + \frac{3}{v^4} (v^2 \dot{\eta}) + \frac{1}{v^3} \mathcal{A} (v^2 (\Omega \text{tr} \chi)) \right)_{\psi} \\ &\quad + \frac{2}{v} \mathcal{P}_2^* \overset{\circ}{\text{div}} \left(\frac{1}{v} \dot{\mathbf{c}}_6 + \frac{2}{v} \mathcal{P}_2^* (\dot{\eta} - 2\mathcal{A} \dot{\Omega}) + \frac{1}{v^2} \dot{\chi} \right)_{\psi} \\ &\quad - \frac{2}{v^3} \mathcal{P}_2^* \left(v^2 \dot{\mathbf{c}}_4 + 4v \mathcal{A} \dot{\Omega} + \overset{\circ}{\text{div}} \dot{\chi} - \frac{v^2}{2} \mathcal{A} (\Omega \text{tr} \chi) \right)_{\psi} \\ &\quad - \frac{1}{v^2} \mathcal{P}_2^* \mathcal{A} \left(v^2 \dot{\mathbf{c}}_5 + 2v (\Omega \text{tr} \chi) - 2 \overset{\circ}{\text{div}} (\dot{\eta} - 2\mathcal{A} \dot{\Omega}) - 2v^2 \dot{K} - 4\dot{\Omega} \right)_{\psi} \\ &\quad - 2\mathcal{P}_2^* \mathcal{A} \left(\dot{K} + \frac{1}{2v} (\Omega \text{tr} \chi) - \frac{1}{2v} (\Omega \text{tr} \chi) + \frac{2}{v^2} \dot{\Omega} + \dot{\mathbf{c}}_7 \right)_{\psi}. \end{aligned} \tag{4.41}$$

Summing up the terms on the right-hand side of (4.41) and using that by (D.7) in Appendix D, for all S_v -tangential vectorfields X ,

$$\left(1 + \overset{\circ}{\text{div}} \mathcal{P}_2^* + \frac{1}{2} \mathcal{A} \overset{\circ}{\text{div}} \right) X = 0,$$

it follows that

$$D\mathcal{Q}_4 = \frac{1}{v} (\dot{\mathbf{c}}_8)_{\psi} + \frac{2}{v^2} \left(\mathcal{P}_2^* \overset{\circ}{\text{div}} \dot{\mathbf{c}}_6 \right)_{\psi} - \frac{2}{v} (\mathcal{P}_2^* \dot{\mathbf{c}}_4)_{\psi} - (\mathcal{P}_2^* \mathcal{A} \dot{\mathbf{c}}_5)_{\psi} - 2 (\mathcal{P}_2^* \mathcal{A} \dot{\mathbf{c}}_7)_{\psi}.$$

The conservation laws for $\mathcal{Q}_5$, $\mathcal{Q}_6$ and $\mathcal{Q}_7$ follow directly by projecting the transport equations for $\underline{\dot{\omega}}$ and $\underline{D\dot{\omega}}$ of Lemma 4.14 onto the modes $l \leq 1$ and $l = 2$. This finishes the proof of Lemma 4.16. $\square$

The proof of the following lemma is omitted.

Lemma 4.17 (Properties of charges III). *The following holds.*

(1) Let $\dot{x}_v$ be sphere data on a sphere $S_v \subset \mathcal{H}$. Then,

$$\|\mathcal{Q}_4\|_{H^4(S_v)} + \|\mathcal{Q}_5\|_{H^4(S_v)} + \|\mathcal{Q}_6\|_{H^2(S_v)} + \|\mathcal{Q}_7\|_{H^2(S_v)} \lesssim \|\dot{x}_v\|_{\mathcal{X}(S_v)}.$$

(2) For given sphere data $\dot{\mathfrak{X}}_1$ on S_1 , define $^{(1)}\mathcal{Q}_i$, $4 \leq i \leq 6$, as solution to the transport equation of Lemma 4.16 on $\mathcal{H}$ with initial values given by $\mathcal{Q}_i$ calculated from $\dot{\mathfrak{X}}_1$. Then, it holds that

$$\begin{aligned} & \|^{(1)}\mathcal{Q}_4\|_{H^4_2(\mathcal{H})} + \|^{(1)}\mathcal{Q}_5\|_{H^4_2(\mathcal{H})} + \|^{(1)}\mathcal{Q}_6\|_{H^2_2(\mathcal{H})} + \|^{(1)}\mathcal{Q}_7\|_{H^2_2(\mathcal{H})} \\ & \lesssim \|\mathcal{Q}_4(\dot{\mathfrak{X}}_1)\|_{H^4(S_1)} + \|\mathcal{Q}_5(\dot{\mathfrak{X}}_1)\|_{H^4(S_1)} + \|\mathcal{Q}_6(\dot{\mathfrak{X}}_1)\|_{H^2(S_1)} \\ & \quad + \|\mathcal{Q}_7(\dot{\mathfrak{X}}_1)\|_{H^2(S_1)} + \|(\dot{\mathfrak{c}}_i)_{1 \leq i \leq 10}\|_{\mathcal{Z}_C}. \end{aligned}$$

4.4. Solution of the Linearized Characteristic Gluing Problem

In this section, we prove Theorem 4.1; that is, we solve the linearized characteristic gluing problem. Consider given

- source terms $(\dot{\mathfrak{c}}_i)_{1 \leq i \leq 10} \in \mathcal{Z}_C$,
- sphere data $\dot{\mathfrak{X}}_1 \in \mathcal{X}(S_1)$,
- matching data $\dot{\mathfrak{M}}_2 \in \mathcal{Z}_{\mathfrak{M}}(S_2)$.

In the following, we use the charges, representation formulas and estimates of Sects. 4.1, 4.2 and 4.3 to construct a solution $\dot{x}$ on $\mathcal{H}$ satisfying (4.2) with matching conditions (4.3) and bounds (4.4). We proceed as follows.

- In Sect. 4.4.1, we apply Proposition 4.4 to add linearized perturbations of sphere data on S_2 to match the gauge-dependent charges on S_2 with the gauge-dependent charges coming from S_1 .
- In Sect. 4.4.2, we derive conditions on the free conformal data $\dot{\Omega}$ and $\dot{\hat{\chi}}$ on $\mathcal{H}$ such that the constructed solution $\dot{x}$, given through the representation formulas in Sect. 4.3, satisfies (4.3).
- In Sect. 4.4.3, we prove the estimate (4.4) for the constructed solution and the linearized perturbation function and vectorfield.

4.4.1. Matching of Gauge-Dependent Charges. In this section, we apply Proposition 4.4 to add linearized perturbations of sphere data to S_2 to match the gauge-dependent charges

$$\left(\mathcal{Q}_1, \mathcal{Q}_2^{[\geq 2]}, \mathcal{Q}_3, \mathcal{Q}_4, \mathcal{Q}_5, \mathcal{Q}_6, \mathcal{Q}_7 \right).$$

On the one hand, for given sphere data $\dot{\mathfrak{X}}_1 \in \mathcal{X}(S_1)$ on S_1 and matching condition on S_1

$$\dot{x}|_{S_{0,1}} = \dot{\mathfrak{X}}_1,$$

define $^{(1)}\mathcal{Q}_i$, $0 \leq i \leq 7$ on $\mathcal{H}$ to be the solutions to the transport equations of Lemmas 4.8, 4.11 and 4.16 with initial values $^{(1)}\mathcal{Q}_i$ on S_1 calculated from $\dot{\mathfrak{X}}_1$. We underline that the charges $^{(1)}\mathcal{Q}_i$ on $\mathcal{H}$ depend only on $\dot{\mathfrak{X}}_1$ and $(\dot{\mathfrak{c}}_i)_{1 \leq i \leq 10}$ and are independent of the solution $\dot{x}$ to the linearized null constraint equations

to be constructed on $\mathcal{H}$. Lemmas 4.9, 4.12 and 4.17 imply that

$$\begin{aligned} & \|^{(1)}\mathcal{Q}_0\|_{H^5(S_2)} + \|^{(1)}\mathcal{Q}_1\|_{H^6(S_2)} + \|^{(1)}\mathcal{Q}_2\|_{H^4(S_2)} + \|^{(1)}\mathcal{Q}_3\|_{H^4(S_2)} \\ & + \|^{(1)}\mathcal{Q}_4\|_{H^4(S_2)} + \|^{(1)}\mathcal{Q}_5\|_{H^4(S_2)} + \|^{(1)}\mathcal{Q}_6\|_{H^2(S_2)} + \|^{(1)}\mathcal{Q}_7\|_{H^2(S_2)} \\ & \lesssim \|\dot{\mathfrak{X}}_1\|_{\mathcal{X}(S_1)} + \|(\dot{\mathfrak{c}}_i)_{1 \leq i \leq 10}\|_{\mathcal{Z}_c}. \end{aligned} \quad (4.42)$$

On the other hand, for given linearized matching data $\dot{\mathfrak{M}}_2 \in \mathcal{Z}_{\mathfrak{M}}(S_2)$ at S_2 , $\dot{\mathfrak{M}}_2 = (\dot{\Omega}, \dot{\phi}, \dot{g}_c, (\Omega \text{tr} \chi), \dot{\hat{\chi}}, (\Omega \text{tr} \underline{\chi})^{[\geq 2]}, \dot{\hat{\chi}}, \dot{\eta}^{[\geq 2]}, \dot{\omega}, D\dot{\omega}, \dot{\underline{\omega}}^{[\geq 2]}, D\dot{\underline{\omega}}^{[\geq 2]}, \mathcal{Q}_5, \mathcal{Q}_6, \dot{\alpha}, \dot{\underline{\alpha}})$,

define on S_2 the charges

$$\left({}^{(2)}\mathcal{Q}_1, {}^{(2)}\mathcal{Q}_2^{[\geq 2]}, {}^{(2)}\mathcal{Q}_3, {}^{(2)}\mathcal{Q}_4, {}^{(2)}\mathcal{Q}_5, {}^{(2)}\mathcal{Q}_6, {}^{(2)}\mathcal{Q}_7 \right).$$

By definition of $\mathfrak{M}$ in Definition 2.11, all charges are well defined, and we have the bounds

$$\begin{aligned} & \|^{(2)}\mathcal{Q}_1\|_{H^6(S_2)} + \|^{(2)}\mathcal{Q}_2^{[\geq 2]}\|_{H^4(S_2)} + \|^{(2)}\mathcal{Q}_3\|_{H^4(S_2)} \\ & + \|^{(2)}\mathcal{Q}_4\|_{H^4(S_2)} + \|^{(2)}\mathcal{Q}_5\|_{H^4(S_2)} + \|^{(2)}\mathcal{Q}_6\|_{H^2(S_2)} + \|^{(2)}\mathcal{Q}_7\|_{H^2(S_2)} \\ & \lesssim \|\dot{\mathfrak{M}}_2\|_{\mathcal{Z}_{\mathfrak{M}}}. \end{aligned} \quad (4.43)$$

Applying Proposition 4.4 with

$$\begin{aligned} (\mathcal{Q}_1)_0 &:= {}^{(1)}\mathcal{Q}_1 - {}^{(2)}\mathcal{Q}_1, & (\mathcal{Q}_2)_0 &:= {}^{(1)}\mathcal{Q}_2^{[\geq 2]} - {}^{(2)}\mathcal{Q}_2^{[\geq 2]}, & (\mathcal{Q}_3)_0 &:= {}^{(1)}\mathcal{Q}_3 - {}^{(2)}\mathcal{Q}_3, \\ (\mathcal{Q}_4)_0 &:= {}^{(1)}\mathcal{Q}_4 - {}^{(2)}\mathcal{Q}_4, & (\mathcal{Q}_5)_0 &:= {}^{(1)}\mathcal{Q}_5 - {}^{(2)}\mathcal{Q}_5, & (\mathcal{Q}_6)_0 &:= {}^{(1)}\mathcal{Q}_6 - {}^{(2)}\mathcal{Q}_6, \\ (\mathcal{Q}_7)_0 &:= {}^{(1)}\mathcal{Q}_7 - {}^{(2)}\mathcal{Q}_7, \end{aligned} \quad (4.44)$$

it follows that there exist linearized perturbation functions $\dot{f}$ and $\dot{q}$ at S_2 such that the gauge-dependent charges of

$$\dot{x}|_{S_2} + \dot{\mathcal{P}}_f(\dot{f}) + \dot{\mathcal{P}}_q(\dot{q})$$

match with the gauge-dependent charges ${}^{(1)}\mathcal{Q}_i$, that is, for $i = 1, 3, 4, 5, 6, 7$,

$$\begin{aligned} \mathcal{Q}_i \left(\dot{x}|_{S_2} + \dot{\mathcal{P}}_f(\dot{f}) + \dot{\mathcal{P}}_q(\dot{q}) \right) &= {}^{(2)}\mathcal{Q}_i + \mathcal{Q}_i \left(\dot{\mathcal{P}}_f(\dot{f}) + \dot{\mathcal{P}}_q(\dot{q}) \right) \\ &= {}^{(2)}\mathcal{Q}_i + (\mathcal{Q}_i)_0 \\ &= {}^{(1)}\mathcal{Q}_i, \end{aligned} \quad (4.45)$$

and

$$\begin{aligned} \mathcal{Q}_2^{[\geq 2]} \left(\dot{x}|_{S_2} + \dot{\mathcal{P}}_f(\dot{f}) + \dot{\mathcal{P}}_q(\dot{q}) \right) &= {}^{(2)}\mathcal{Q}_2^{[\geq 2]} + \mathcal{Q}_2^{[\geq 2]} \left(\dot{\mathcal{P}}_f(\dot{f}) + \dot{\mathcal{P}}_q(\dot{q}) \right) \\ &= {}^{(2)}\mathcal{Q}_2^{[\geq 2]} + (\mathcal{Q}_2)_0 \\ &= {}^{(1)}\mathcal{Q}_2^{[\geq 2]}. \end{aligned} \quad (4.46)$$

Moreover, by Proposition 4.4, (4.42), (4.43) and (4.44), we have the estimate

$$\begin{aligned}
 & \|\dot{f}\|_{\mathcal{Y}_f} + \|\dot{q}\|_{\mathcal{Y}_q} + \|\dot{\mathcal{P}}_f(\dot{f})\|_{\mathcal{X}(S_2)} + \|\dot{\mathcal{P}}_q(\dot{q})\|_{\mathcal{X}(S_2)} \\
 & \lesssim \|(\mathcal{Q}_1)_0\|_{H^6(S_2)} + \|(\mathcal{Q}_2)_0\|_{H^4(S_2)} + \|(\mathcal{Q}_3)_0\|_{H^4(S_2)} + \|(\mathcal{Q}_4)_0\|_{H^2(S_2)} \\
 & \quad + \|(\mathcal{Q}_5)_0\|_{H^4(S_2)} + \|(\mathcal{Q}_6)_0\|_{H^2(S_2)} + \|(\mathcal{Q}_7)_0\|_{H^2(S_2)} \\
 & \lesssim \|\dot{\mathfrak{X}}_1\|_{\mathcal{X}(S_1)} + \|\dot{\mathfrak{M}}_2\|_{\mathcal{Z}_{\mathfrak{M}}(S_2)} + \|(\dot{\mathfrak{c}}_i)_{1 \leq i \leq 10}\|_{\mathcal{Z}_C}.
 \end{aligned} \tag{4.47}$$

This finishes our matching of the gauge-dependent charges on S_2 .

4.4.2. Integral Conditions on $\dot{\Omega}$ and $\dot{\hat{\chi}}$. In the previous section, we constructed linearized perturbation functions $\dot{f}$ and $\dot{q}$ such that the gauge-dependent charges match on S_2 , see (4.45) and (4.46). In this section, we use the matching of gauge-dependent charges together with the representation formulas for $\dot{\phi}$, $\dot{g}_c$, $\dot{\eta}^{[\geq 2]}$, $\dot{\alpha}$ and $\underline{D}\dot{\omega}$ and our freedom of prescribing $\dot{\Omega}$ and $\dot{\hat{\chi}}$ along $\mathcal{H}$ to construct a solution $\dot{x}$ of the linearized null constraint equations (4.2) satisfying

$$\begin{aligned}
 \dot{x}|_{S_1} &= \dot{\mathfrak{X}}_1, \\
 \mathfrak{M}(\dot{x}|_{S_2} + \dot{\mathcal{P}}_f(\dot{f}) + \dot{\mathcal{P}}_q(\dot{q})) &= \dot{\mathfrak{M}}_2 \text{ on } S_2,
 \end{aligned} \tag{4.48}$$

where we recall from Definition 2.11 that

$$\begin{aligned}
 \mathfrak{M}(\dot{x}_2) &:= \left(\dot{\Omega}, \dot{\phi}, \dot{g}_c, (\Omega \text{tr} \chi), \dot{\hat{\chi}}, (\Omega \text{tr} \underline{\chi})^{[\geq 2]}, \dot{\underline{\chi}}, \dot{\eta}^{[\geq 2]}, \dot{\omega}, D\dot{\omega}, \dot{\omega}^{[\geq 2]}, \right. \\
 & \quad \left. \underline{D}\dot{\omega}^{[\geq 2]}, \mathcal{Q}_5, \mathcal{Q}_6, \dot{\alpha}, \dot{\alpha} \right),
 \end{aligned}$$

and we recall that $\mathfrak{M}$ is a linear map, and hence, we denote $\dot{\mathfrak{M}}$ also $\mathfrak{M}$.

Specifically, the additional conditions derived below on $\dot{\Omega}$ and $\dot{\hat{\chi}}$ are independent of the boundary values of $\dot{\Omega}$ and $\dot{\hat{\chi}}$ on S_1 and S_2 which are already determined by (4.48). In Sect. 4.4.3, we show that there exist $\dot{\Omega}$ and $\dot{\hat{\chi}}$ satisfying all derived conditions and derive estimates.

(1) Gluing of $\dot{\phi}$. By the representation formula (4.26) for $\dot{\phi}$, we have

$$\dot{\phi} = 2 \int_1^v \dot{\Omega} dv' + \int_1^v \int_1^{v'} \dot{\mathfrak{c}}_1 dv'' dv' + v \dot{\phi}(1) + \frac{v-1}{2} \left((\Omega \text{tr} \chi)(1) - 4\dot{\Omega}(1) + \dot{\mathfrak{c}}_2(1) \right).$$

To match $\dot{\phi}$ according to (4.48), prescribe $\dot{\Omega}$ on $\mathcal{H}$ such that

$$2 \int_1^2 \dot{\Omega} dv' = \dot{\phi}(2) - 2\dot{\phi}(1) - \frac{1}{2} \left((\Omega \text{tr} \chi)(1) - 4\dot{\Omega}(1) + \dot{\mathfrak{c}}_2(1) \right) - \int_1^2 \int_1^{v'} \dot{\mathfrak{c}}_1 dv'' dv'. \tag{4.49}$$

In particular, it holds that

$$\left\| \int_1^2 \dot{\Omega} dv' \right\|_{H^6(S_1)} \lesssim \|\dot{\mathfrak{X}}_1\|_{\mathcal{X}(S_1)} + \|\dot{\mathfrak{M}}_2\|_{\mathcal{Z}_{\mathfrak{M}}(S_2)} + \|(\dot{\mathfrak{c}}_i)_{1 \leq i \leq 10}\|_{\mathcal{Z}_C}. \tag{4.50}$$

(2) Gluing of $\dot{g}_c$. By the representation formula (4.27) for $\dot{g}_c$, we have that

$$\dot{g}_c(v) = 2 \int_1^v \frac{1}{v'^2} \dot{\chi} dv' + \dot{g}_c(1) + \int_1^v \frac{1}{v'^2} \dot{\mathfrak{c}}_3 dv'.$$

To match $\dot{g}_c$ according to (4.48), prescribe $\hat{\chi}$ on $\mathcal{H}$ such that

$$\int_1^2 \frac{1}{v'^2} \dot{\chi} dv' = \frac{1}{2} \left(\dot{g}_c(2) - \dot{g}_c(1) - \int_1^2 \frac{1}{v'^2} \dot{\mathfrak{c}}_3 dv' \right). \quad (4.51)$$

In particular, it holds that

$$\left\| \int_1^2 \frac{1}{v'^2} \dot{\chi} dv' \right\|_{H^5(S_1)} \lesssim \|\dot{\mathfrak{X}}_1\|_{\mathcal{X}(S_1)} + \|\dot{\mathfrak{M}}_2\|_{\mathcal{Z}_{\mathfrak{M}}(S_2)} + \|(\dot{\mathfrak{c}}_i)_{1 \leq i \leq 10}\|_{\mathcal{Z}_C}. \quad (4.52)$$

(3) Gluing of $(\Omega \text{tr} \chi)$. By the above gluing of $\dot{\Omega}$, $\dot{\phi}$ and the matching of

$$\mathcal{Q}_1 := \frac{v}{2} \left((\Omega \text{tr} \chi) - \frac{4}{v} \dot{\Omega} \right) + \frac{\dot{\phi}}{v},$$

at S_2 in (4.45), it follows that $(\Omega \text{tr} \chi)^{[\geq 0]}$ is matched according to (4.48).

(4) Gluing of $\eta^{[\geq 2]}$. By the representation formula for η in Lemma 4.7, we have that

$$\begin{aligned} & \left[v'^2 \dot{\eta} + \frac{v'^3}{2} \not{d} \left((\Omega \text{tr} \chi) - \frac{4}{v'} \dot{\Omega} \right) + \frac{v'}{2} \not{d} \dot{\mathfrak{c}}_2 \right]_1^v \\ &= \mathring{\text{div}} \left(\int_1^v \dot{\chi} dv' \right) + \int_1^v \left(v'^2 \dot{\mathfrak{c}}_4 + v'^2 \not{d} \dot{\mathfrak{c}}_1 + \frac{1}{2} \not{d} \dot{\mathfrak{c}}_2 \right) dv'. \end{aligned}$$

By the above gluing of $(\Omega \text{tr} \chi)$ and $\dot{\Omega}$, to glue $\eta^{[\geq 2]}$ according to (4.48) it suffices to choose $\hat{\chi}$ such that

$$\begin{aligned} \mathring{\text{div}} \left(\int_1^2 \dot{\chi} dv' \right) &= \left[v^2 \dot{\eta} + \frac{v^3}{2} \not{d} \left((\Omega \text{tr} \chi) - \frac{4}{v} \dot{\Omega} \right) + \frac{v}{2} \not{d} \dot{\mathfrak{c}}_2 \right]_1^2 \\ &\quad - \int_1^2 \left(v'^2 \dot{\mathfrak{c}}_4 + v'^2 \not{d} \dot{\mathfrak{c}}_1 + \frac{1}{2} \not{d} \dot{\mathfrak{c}}_2 \right) dv' \end{aligned} \quad (4.53)$$

By elliptic estimates for the operator $\mathring{\text{div}}$, see Appendix D, we have that

$$\left\| \int_1^2 \dot{\chi} dv' \right\|_{H^5(S_1)} \lesssim \|\dot{\mathfrak{X}}_1\|_{\mathcal{X}(S_1)} + \|\dot{\mathfrak{M}}_2\|_{\mathcal{Z}_{\mathfrak{M}}(S_2)} + \|(\dot{\mathfrak{c}}_i)_{1 \leq i \leq 10}\|_{\mathcal{Z}_C}. \quad (4.54)$$

(5) Gluing of $(\Omega \dot{\text{tr}} \chi)^{[\geq 2]}$. By the matching of

$$\mathcal{Q}_2 := v^2(\Omega \dot{\text{tr}} \chi) - \frac{2}{v} \dot{\text{d}}\dot{\text{v}} \left(v^2 \dot{\eta} + \frac{v^3}{2} \not{\partial} \left((\Omega \dot{\text{tr}} \chi) - \frac{4}{v} \dot{\Omega} \right) \right) - v^2 \left((\Omega \dot{\text{tr}} \chi) - \frac{4}{v} \dot{\Omega} \right) + 2v^3 \dot{K},$$

for modes $l \geq 2$ in (4.46) and the relation (2.49), that is,

$$\dot{K} = \frac{1}{2r^2} \dot{\text{d}}\dot{\text{v}} \dot{\text{d}}\dot{\text{v}} \dot{g}_c - \frac{1}{r^3} (\dot{\Delta} + 2) \dot{\phi},$$

it follows with the above gluing of $\dot{\Omega}$, $\dot{g}_c$, $\dot{\phi}$, $(\Omega \dot{\text{tr}} \chi)$ and $\dot{\eta}^{[\geq 2]}$ that $(\Omega \dot{\text{tr}} \chi)^{[\geq 2]}$ is glued at S_2 according to (4.48).

(6) Gluing of $\dot{\chi}$. By the matching of

$$\mathcal{Q}_3 := \frac{\dot{\chi}}{v} - \frac{1}{2} \left(\mathcal{P}_2^* \dot{\text{d}}\dot{\text{v}} + 1 \right) \dot{g}_c + \mathcal{P}_2^* \left(\dot{\eta} + \frac{v}{2} \not{\partial} \left((\Omega \dot{\text{tr}} \chi) - \frac{4}{v} \dot{\Omega} \right) \right) - v \mathcal{P}_2^* \not{\partial} \left((\Omega \dot{\text{tr}} \chi) - \frac{4}{v} \dot{\Omega} \right),$$

at S_2 in (4.45), together with the above gluing of $\dot{\Omega}$, $\dot{g}_c$, $(\Omega \dot{\text{tr}} \chi)$ and $\dot{\eta}^{[\geq 2]}$, it follows that $\dot{\chi}$ is glued at S_2 according to (4.48).

(7) Gluing of $\dot{\alpha}$. We have by the linearized null constraint Eqs. (4.2) that on $\mathcal{H}$

$$\dot{\alpha} + D\dot{\chi} = \dot{\mathbf{c}}_{10}. \quad (4.55)$$

Hence, we glue $\dot{\alpha}$ at S_2 according to (4.48) by prescribing,

$$D\dot{\chi}(1) = \dot{\mathbf{c}}_{10}(1) - \dot{\alpha}(1), \quad D\dot{\chi}(2) = \dot{\mathbf{c}}_{10}(2) - \dot{\alpha}(2). \quad (4.56)$$

This implies that

$$\begin{aligned} & \|D\dot{\chi}(1)\|_{H^6(S_1)} + \|D\dot{\chi}(2)\|_{H^6(S_2)} \\ & \lesssim \|\dot{\mathbf{c}}_{10}\|_{H^6(S_1)} + \|\dot{\mathbf{c}}_{10}\|_{H^6(S_2)} + \|\dot{\mathfrak{X}}_1\|_{\mathcal{X}(S_1)} + \|\dot{\mathfrak{M}}_2\|_{\mathcal{Z}_{\mathfrak{M}}(S_2)}. \end{aligned} \quad (4.57)$$

(8) Gluing of $\dot{\omega}$ and $D\dot{\omega}$. By the relation $\dot{\omega} = D\dot{\Omega}$, the gluing of $\dot{\omega}$ and $D\dot{\omega}$ at S_2 according to (4.48) is satisfied if

$$\begin{aligned} \dot{\omega}(1) &= D\dot{\Omega}(1), & \dot{\omega}(2) &= D\dot{\Omega}(2), \\ D\dot{\omega}(1) &= D^2\dot{\Omega}(1), & D\dot{\omega}(2) &= D^2\dot{\Omega}(2). \end{aligned} \quad (4.58)$$

In particular, we have the bound

$$\begin{aligned} & \|D\dot{\Omega}(1)\|_{H^6(S_1)} + \|D\dot{\Omega}(2)\|_{H^6(S_2)} + \|D^2\dot{\Omega}(1)\|_{H^6(S_1)} + \|D^2\dot{\Omega}(2)\|_{H^6(S_2)} \\ & \lesssim \|\dot{\mathfrak{X}}_1\|_{\mathcal{X}(S_1)} + \|\dot{\mathfrak{M}}_2\|_{\mathcal{Z}_{\mathfrak{M}}}. \end{aligned} \quad (4.59)$$

(9) Gluing of $\dot{\alpha}$. Using the representation formula (4.39) for $\dot{\alpha}$, to glue $\dot{\alpha}$ at S_2 according to (4.48), we can pick $\dot{\chi}$ such that

$$\begin{aligned} & \left[\frac{\dot{\alpha}}{v} + \frac{2}{v} \mathcal{P}_2^* \dot{\text{d}}\dot{\text{v}} \mathcal{Q}_3 - \frac{1}{2v^2} \mathcal{P}_2^* \not{\partial} \mathcal{Q}_2 - \frac{2}{v} \mathcal{P}_2^* \not{\partial} \left(\dot{\Delta} + 2 \right) \mathcal{Q}_1 \right]_1^2 \\ & - \left[\frac{2}{3v} \mathcal{P}_2^* \left(\dot{\text{d}}\dot{\text{v}} \mathcal{P}_2^* + 1 + \not{\partial} \dot{\text{d}}\dot{\text{v}} \right) \left(\dot{\eta} + \frac{v}{2} \not{\partial} \left((\Omega \dot{\text{tr}} \chi) - \frac{4}{v} \dot{\Omega} \right) \right) \right]_1^2 \end{aligned}$$

$$\begin{aligned}
& + \left[\frac{1}{v} \mathcal{P}_2^* \left(\mathring{\mathrm{d}}\dot{\mathcal{V}} \mathcal{P}_2^* + 1 + \not\!d \mathring{\mathrm{d}}\dot{\mathcal{V}} \right) \mathring{\mathrm{d}}\dot{\mathcal{V}} \dot{\mathcal{J}}_c - \frac{1}{3v^2} \mathcal{P}_2^* \left(\mathring{\mathrm{d}}\dot{\mathcal{V}} \mathcal{P}_2^* + 1 + \not\!d \mathring{\mathrm{d}}\dot{\mathcal{V}} \right) \mathring{\mathrm{d}}\dot{\mathcal{V}} \not\!d \dot{\mathcal{I}}_2 \right]_1^2 \\
& = \frac{4}{3} \mathcal{P}_2^* \left(\mathring{\mathrm{d}}\dot{\mathcal{V}} \mathcal{P}_2^* + 1 + \not\!d \mathring{\mathrm{d}}\dot{\mathcal{V}} \right) \mathring{\mathrm{d}}\dot{\mathcal{V}} \left(\int_1^2 \frac{1}{v'^3} \hat{\chi} dv' \right) + \int_1^2 h_{\underline{\alpha}} dv'. \quad (4.60)
\end{aligned}$$

By the elliptic estimate (D.6) in Appendix D for the operator

$$\mathcal{P}_2^* \left(\mathring{\mathrm{d}}\dot{\mathcal{V}} \mathcal{P}_2^* + 1 + \not\!d \mathring{\mathrm{d}}\dot{\mathcal{V}} \right) \mathring{\mathrm{d}}\dot{\mathcal{V}}$$

it follows that the integral is well defined and

$$\left\| \int_1^2 \frac{1}{v'^3} \hat{\chi} dv' \right\|_{H^6(S_1)} \lesssim \|\dot{\mathcal{X}}_1\|_{\mathcal{X}(S_1)} + \|\dot{\mathcal{M}}_2\|_{\mathcal{ZM}(S_2)} + \|(\dot{\mathcal{I}}_i)_{1 \leq i \leq 10}\|_{\mathcal{ZC}}. \quad (4.61)$$

(10) Gluing of $\underline{\dot{\omega}}^{[\geq 2]}$. By the matching of

$$\mathcal{Q}_4 := \frac{\dot{\alpha}_\psi}{v} + 2\mathcal{P}_2^* \left(\frac{1}{v^2} \mathring{\mathrm{d}}\dot{\mathcal{V}} \dot{\underline{\chi}} - \frac{1}{v} \dot{\eta} - \frac{1}{2} \not\!d (\Omega \mathrm{tr} \chi) + \mathcal{P}_1^* (\underline{\dot{\omega}}, 0) \right)_\psi,$$

in (4.45), the above gluing of $\underline{\dot{\alpha}}$, $\underline{\dot{\chi}}$, $\dot{\eta}^{[\geq 2]}$ and $(\Omega \mathrm{tr} \chi)$, and the fact that the operator

$$\mathcal{P}_2^* (\mathcal{P}_1^* (\underline{\dot{\omega}}, 0))_\psi$$

has trivial kernel and is elliptic (see Appendix D), it follows that $\underline{\dot{\omega}}^{[l \geq 2]}$ is glued at S_2 according to (4.48).

(11) Gluing of $D\underline{\dot{\omega}}^{[\geq 2]}$. On the one hand, by the representation formula for $D\underline{\dot{\omega}}$ of Lemma 4.15, we have that

$$\begin{aligned}
& \left[D\underline{\dot{\omega}} - \frac{1}{6v^3} \left(\mathring{\Delta} - 3 \right) \mathcal{Q}_2 + \frac{1}{2v^2} \mathring{\mathrm{d}}\dot{\mathcal{V}} \mathring{\mathrm{d}}\dot{\mathcal{V}} \mathcal{Q}_3 + \frac{1}{v^2} \mathring{\mathrm{d}}\dot{\mathcal{V}} \mathring{\mathrm{d}}\dot{\mathcal{V}} \mathcal{P}_2^* \not\!d \mathcal{Q}_1 \right]_1^2 \\
& - \left[\frac{1}{4v^2} \mathring{\mathrm{d}}\dot{\mathcal{V}} \left(\not\!d \mathring{\mathrm{d}}\dot{\mathcal{V}} - 2 + \mathring{\mathrm{d}}\dot{\mathcal{V}} \mathcal{P}_2^* \right) \left(\eta + \frac{v}{2} \not\!d \left((\Omega \mathrm{tr} \chi) - \frac{4}{v} \dot{\Omega} \right) \right) \right]_1^2 \\
& + \left[\frac{1}{8v^2} \mathring{\mathrm{d}}\dot{\mathcal{V}} \not\!d \mathring{\mathrm{d}}\dot{\mathcal{V}} \mathring{\mathrm{d}}\dot{\mathcal{V}} \dot{\mathcal{J}}_c + \frac{1}{2v^3} \left(\frac{1}{12} \mathring{\Delta} \mathring{\Delta} - \frac{1}{6} \mathring{\Delta} + \frac{1}{4} \mathring{\mathrm{d}}\dot{\mathcal{V}} \mathring{\mathrm{d}}\dot{\mathcal{V}} \mathcal{P}_2^* \not\!d - 1 \right) \dot{\mathcal{I}}_2 \right]_1^2 \\
& = \frac{1}{4} \mathring{\mathrm{d}}\dot{\mathcal{V}} \left(2 - \mathring{\mathrm{d}}\dot{\mathcal{V}} \mathcal{P}_2^* \right) \mathring{\mathrm{d}}\dot{\mathcal{V}} \left(\int_1^2 \frac{1}{v'^4} \hat{\chi} dv' \right) + \int_1^2 h_{D\underline{\dot{\omega}}} dv'. \quad (4.62)
\end{aligned}$$

On modes $l \geq 3$, the operator on the right-hand side of (4.62)

$$\mathring{\mathrm{d}}\dot{\mathcal{V}} \left(2 - \mathring{\mathrm{d}}\dot{\mathcal{V}} \mathcal{P}_2^* \right) \mathring{\mathrm{d}}\dot{\mathcal{V}},$$

has trivial kernel and is elliptic, see Appendix D. Hence, projecting the above representation formula onto modes $l \geq 3$, we can pick the $\hat{\chi}$ -integral such

that $\underline{D\dot{\omega}}^{[\geq 3]}$ is glued at S_2 according to (4.48). Picking the projection of the $\hat{\chi}$ -integral onto the mode $l = 2$ to vanish, we get the estimate

$$\left\| \int_1^2 \frac{1}{v'^4} \hat{\chi} dv' \right\|_{H^6(S_1)} \lesssim \|\dot{\mathfrak{X}}_1\|_{\mathcal{X}(S_1)} + \|\mathfrak{M}_2\|_{\mathcal{Z}_{\mathfrak{M}}(S_2)} + \|(\dot{\mathfrak{c}}_i)_{1 \leq i \leq 10}\|_{\mathcal{Z}_c}. \quad (4.63)$$

On the other hand, by the matching of $\mathcal{Q}_1$, $\mathcal{Q}_2^{[2]}$, $\mathcal{Q}_3$ and

$$\begin{aligned} \mathcal{Q}_7 := & \underline{D\dot{\omega}}^{[2]} + \frac{3}{2v^3} \mathcal{Q}_2^{[2]} + \frac{1}{2v^2} \mathring{\mathrm{d}}\dot{v} \mathring{\mathrm{d}}\dot{v} \mathcal{Q}_3^{[2]} - \frac{12}{v^2} \mathcal{Q}_1^{[2]} \\ & + \frac{3}{2v^2} \mathring{\mathrm{d}}\dot{v} \left(\eta + \frac{v}{2} \not{d} \left((\Omega \mathrm{tr} \chi) - \frac{4}{v} \dot{\Omega} \right) \right)^{[2]} - \frac{3}{4v^2} \mathring{\mathrm{d}}\dot{v} \mathring{\mathrm{d}}\dot{v} \dot{\mathfrak{g}}_c^{[2]}, \end{aligned}$$

in (4.46) and the above gluing of $\dot{\Omega}$, $\dot{\mathfrak{g}}_c$, $(\Omega \mathrm{tr} \chi)$, $\dot{\eta}^{[\geq 2]}$ it follows that $\underline{D\dot{\omega}}^{[2]}$ is glued at S_2 according to (4.48).

To summarize the above, we derived the integral conditions (4.49), (4.51), (4.53), (4.56), (4.58), (4.60) and (4.62) on $\dot{\Omega}$ and $\hat{\chi}$ along $\mathcal{H}$ which, if satisfied, imply the matching of *matching data* (4.48) on S_2 , that is,

$$\mathfrak{M} \left(\dot{x}|_{S_2} + \dot{\mathcal{P}}_f(\dot{f}) + \dot{\mathcal{P}}_j(\dot{j}) \right) = \mathfrak{M}_2 \text{ on } S_2,$$

with

$$\mathfrak{M}(x_2) := \left(\dot{\Omega}, \dot{\phi}, \dot{\mathfrak{g}}_c, (\Omega \mathrm{tr} \chi), \hat{\chi}, (\Omega \mathrm{tr} \underline{\chi})^{[\geq 2]}, \hat{\underline{\chi}}, \dot{\eta}^{[\geq 2]}, \dot{\omega}, D\dot{\omega}, \underline{\dot{\omega}}^{[\geq 2]}, \underline{D\dot{\omega}}^{[\geq 2]}, \mathcal{Q}_5, \mathcal{Q}_6, \dot{\alpha}, \dot{\underline{\alpha}} \right).$$

In the next section, we show that $\dot{\Omega}$ and $\hat{\chi}$ satisfying these conditions can be constructed in a regular fashion and prove estimates for the constructed solution $\dot{x}$.

4.4.3. Construction of Solution and Estimates. In this section, we pick $\dot{\Omega}$ and $\hat{\chi}$ subject to the conditions (4.48), (4.49), (4.51), (4.53), (4.56), (4.58), (4.60) and (4.62) and subsequently prove the estimate (4.4) for the constructed solution $\dot{x}$.

Choice of $\hat{\chi}$ and $\dot{\Omega}$ and estimates. The proof of the following technical lemma follows from a straightforward orthogonality construction and is omitted.

Lemma 4.18 (Technical lemma). *Consider scalar functions h_i , $1 \leq i \leq 7$, on the round unit sphere S_1 . Then, there exists a scalar function $\dot{\Omega}$ on $\mathcal{H}$ such that*

$$\begin{aligned} \dot{\Omega}(1) &= h_1, & D\dot{\Omega}(1) &= h_2, & D^2\dot{\Omega}(1) &= h_3, \\ \dot{\Omega}(2) &= h_4, & D\dot{\Omega}(2) &= h_5, & D^2\dot{\Omega}(2) &= h_6, \end{aligned} \quad (4.64)$$

and

$$\int_1^2 \dot{\Omega} dv' = h_7,$$

and the following bound holds,

$$\|\dot{\Omega}\|_{H_2^g(\mathcal{H})} \lesssim \sum_{1 \leq i \leq 7} \|h_i\|_{H^6(S_1)}. \quad (4.65)$$

Further, consider tracefree symmetric 2-tensors W_i , $1 \leq i \leq 8$ on S_1 . Then, there exists a tracefree symmetric S_v -tangent 2-tensor $\hat{\chi}$ on $\mathcal{H}$ such that

$$\begin{aligned} \hat{\chi}(1) &= W_1, & D\hat{\chi}(1) &= W_3, \\ \hat{\chi}(2) &= W_2, & D\hat{\chi}(2) &= W_4, \end{aligned} \quad (4.66)$$

and

$$\int_1^2 \frac{1}{v'^2} \hat{\chi} dv' = W_5, \quad \int_1^2 \hat{\chi} dv' = W_6, \quad \int_1^2 \frac{1}{v'^3} \hat{\chi} dv' = W_7, \quad \int_1^2 \frac{1}{v'^4} \hat{\chi} dv' = W_8.$$

and the following estimates hold,

$$\|\hat{\chi}\|_{H_2^g(\mathcal{H})} \lesssim \sum_{1 \leq i \leq 8} \|W_i\|_{H^6(S_1)}. \quad (4.67)$$

Remark 4.19. (Linearized characteristic gluing of higher-order L -derivatives II) Let $m \geq 0$ be an integer. By the linearized null constraint equations

$$D\hat{\chi} = -\dot{\alpha} + \dot{\mathcal{C}}_{10}, \quad \dot{\omega} = D\dot{\Omega},$$

Lemma 4.18 extends in a straightforward way to the higher-order boundary conditions given by (4.64), (4.66) and the additional

$$D^i \dot{\omega}(1) = V_{1,i}, \quad D^i \dot{\omega}(2) = V_{2,i}, \quad \widehat{D}^i \dot{\alpha}(1) = V_{3,i}, \quad \widehat{D}^i \dot{\alpha}(2) = V_{4,i} \text{ for } 1 \leq i \leq m.$$

In this setting, the right-hand sides of (4.65) and (4.67) get the following additional terms, respectively,

$$\begin{aligned} & \sum_{0 \leq i \leq m} (\|h_i\|_{H^6(S_1)} + \|V_{1,i}\|_{H^6(S_2)} + \|V_{2,i}\|_{H^6(S_2)}) \\ & \text{and} \quad \sum_{0 \leq i \leq m} (\|W_i\|_{H^6(S_1)} + \|V_{3,i}\|_{H^6(S_2)} + \|V_{4,i}\|_{H^6(S_2)}). \end{aligned}$$

Let $\dot{\Omega}$ and $\hat{\chi}$ be the quantities constructed in Lemma 4.18 subject to the gluing conditions

- (4.49) for $\dot{\Omega}$,
- (4.51), (4.53), (4.60) and (4.62) for $\hat{\chi}$,

and the prescribed boundary values given by $\dot{\mathfrak{X}}_1$ on S_1 , $\dot{\mathfrak{M}}_2$ on S_2 and (4.56) and (4.58).

By Lemma 4.18 together with the estimates (4.50), (4.52), (4.54), (4.57), (4.59), (4.61) and (4.63), the constructed $\dot{\Omega}$ and $\hat{\chi}$ satisfy

$$\|\dot{\Omega}\|_{H_3^g(\mathcal{H})} + \|\hat{\chi}\|_{H_2^g(\mathcal{H})} \lesssim \|\dot{\mathfrak{X}}_1\|_{\mathcal{X}(S_1)} + \|\dot{\mathfrak{M}}_2\|_{\mathcal{Z}_{\mathfrak{M}}(S_2)} + \|(\dot{\mathfrak{c}}_i)_{1 \leq i \leq 10}\|_{\mathcal{Z}_C}. \quad (4.68)$$

Estimates for remaining quantities. In the following, we prove the next bound,

$$\begin{aligned} \|\dot{x}\|_{\mathcal{X}(\mathcal{H})} + \|\dot{f}\|_{\mathcal{Y}_f} + \|\dot{q}\|_{\mathcal{Y}_q} + \|\mathcal{P}_f(\dot{f})\|_{\mathcal{X}(S_2)} + \|\mathcal{P}_q(\dot{q})\|_{\mathcal{X}(S_2)} \\ \lesssim \|\dot{\mathfrak{X}}_1\|_{\mathcal{X}(S_1)} + \|\mathfrak{M}_2\|_{\mathcal{Z}_{\mathfrak{M}}(S_2)} + \|(\dot{\mathfrak{C}}_i)_{1 \leq i \leq 10}\|_{\mathcal{Z}_C}. \end{aligned} \quad (4.69)$$

First, by Lemmas 4.5, 4.6, 4.10 and 4.13 and (4.68), we have that

$$\|\dot{x}\|_{\mathcal{X}(\mathcal{H})} \lesssim \|\dot{\mathfrak{X}}_1\|_{\mathcal{X}(S_1)} + \|\dot{\Omega}\|_{H_3^0(\mathcal{H})} + \|\dot{\hat{\chi}}\|_{H_2^0(\mathcal{H})} + \|(\dot{\mathfrak{C}}_i)_{1 \leq i \leq 10}\|_{\mathcal{Z}_C}.$$

Second, we have by (4.47) in Sect. 4.4.1 that

$$\begin{aligned} \|\dot{f}\|_{\mathcal{Y}_f} + \|\dot{q}\|_{\mathcal{Y}_q} + \|\mathcal{P}_f(\dot{f})\|_{\mathcal{X}(S_2)} + \|\mathcal{P}_q(\dot{q})\|_{\mathcal{X}(S_2)} \\ \lesssim \|\dot{\mathfrak{X}}_1\|_{\mathcal{X}(S_1)} + \|\mathfrak{M}_2\|_{\mathcal{Z}_{\mathfrak{M}}(S_2)} + \|(\dot{\mathfrak{C}}_i)_{1 \leq i \leq 10}\|_{\mathcal{Z}_C}. \end{aligned}$$

This finishes the proof of (4.69) and hence of Theorem 4.1.

5. Proof of Main Theorem

In this section, we prove Theorem 3.1. We proceed as follows.

- In Sect. 5.1, we set up the framework for the implicit function theorem.
- In Sect. 5.2, we use the implicit function theorem and the solution to the linearized characteristic gluing problem at Minkowski (see Sect. 4) to construct the solution x to the null constraint equations on $\mathcal{H}_{0,[1,2]}$.
- In Sects. 5.3 and 5.4, we prove the additional charge estimates (3.6) and (3.7), respectively. These estimates for $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$ are based on the construction in Sect. 5.2 as well as the analysis of the linearizations of the sphere perturbations, angular perturbations and null transport equations for charges *at Schwarzschild of mass* $M \geq 0$ provided in Appendix C.
- In Sect. 5.5, we give an outline of the proof of Theorem 3.2, that is, the characteristic gluing of higher-order L -derivatives.

5.1. Setup of Framework for the Proof

The proof of Theorem 3.1 is based on the application of the implicit function theorem to the following mapping $\mathcal{F}$.

Definition 5.1 (*Definition of $\mathcal{F}$*). Let

- $x_{0,1} \in \mathcal{X}(S_{0,1})$ be sphere data,
- $\tilde{x} \in \mathcal{X}^+(\tilde{\mathcal{H}}_{[-\delta,\delta],2})$ be ingoing null data on $\tilde{\mathcal{H}}_{[-\delta,\delta],2}$,
- $x \in \mathcal{X}(\mathcal{H}_{0,[1,2]})$ be null data on $\mathcal{H}_{0,[1,2]}$,
- $f \in \mathcal{Y}_f$ and $q \in \mathcal{Y}_q$ be two perturbation functions,

where the spaces $\mathcal{X}(S_{0,1})$, $\mathcal{X}^+(\tilde{\mathcal{H}}_{[-\delta,\delta],2})$, $\mathcal{X}(\mathcal{H}_{0,[1,2]})$, $\mathcal{Y}_f$ and $\mathcal{Y}_q$ are introduced in Definitions 2.5, 2.9, 2.7, 2.20, respectively. Define the mapping $\mathcal{F}$ by

$$\mathcal{F}(x_{0,1}, \tilde{x}, x, f, q) := \left(x|_{S_{0,1}} - x_{0,1}, \mathfrak{M}(x|_{S_{0,2}}) - \mathfrak{M}(\mathcal{P}_{f,q}(\tilde{x})), (\mathcal{C}_i(x))_{1 \leq i \leq 10} \right), \quad (5.1)$$

where

- $\mathfrak{M}$ is the matching map of Definition 2.11,
- $(\mathcal{C}_i)_{1 \leq i \leq 10}$ are the constraint functions defined in Sect. 2.7,

- $\mathcal{P}_{f,q}$ is the perturbation of sphere data defined in Sect. 2.8.

From Definition 5.1, Lemma 2.15 and Proposition 2.21, we make the following observations concerning $\mathcal{F}$.

- (1) For each real number $M \geq 0$,

$$\mathcal{F}(\mathbf{m}^M, \underline{\mathbf{m}}^M, \mathbf{m}^M, 0, 0) = (0, 0, 0), \quad (5.2)$$

where $\mathbf{m}^M$ denotes the Schwarzschild sphere data.

- (2) The mapping $\mathcal{F}$ is well defined and smooth as mapping between the spaces

$$\begin{aligned} \mathcal{F} : \mathcal{X}(S_{0,1}) \times \mathcal{X}^+(\tilde{\mathcal{H}}_{[-\delta, \delta], 2}) \times \mathcal{X}(\mathcal{H}_{0, [1, 2]}) \times \mathcal{Y}_f \times \mathcal{Y}_q \\ \rightarrow \mathcal{X}(S_{0,1}) \times \mathcal{Z}_{\mathfrak{M}}(S_{0,2}) \times \mathcal{Z}_{\mathcal{C}} \end{aligned}$$

in an open neighborhood of

$$(x_{0,1}, \tilde{x}, x, f, q) = (\mathbf{m}, \underline{\mathbf{m}}, \mathbf{m}, 0, 0).$$

Indeed, this follows from the explicit definition of $\mathcal{F}$, see (5.1), together with Proposition 2.21 and Lemma 2.15.

- (3) For real numbers $M \geq 0$ sufficiently small, the linearization $\dot{\mathcal{F}}^M$ of $\mathcal{F}$ in (x, f, q) at

$$(x_{0,1}, \tilde{x}, x, f, q) = (\mathbf{m}^M, \underline{\mathbf{m}}^M, \mathbf{m}^M, 0, 0),$$

is a well-defined, bounded linear operator between the spaces

$$\dot{\mathcal{F}}^M : \mathcal{X}(\mathcal{H}_{0, [1, 2]}) \times \mathcal{Y}_f \times \mathcal{Y}_q \rightarrow \mathcal{X}(S_{0,1}) \times \mathcal{Z}_{\mathfrak{M}}(S_{0,2}) \times \mathcal{Z}_{\mathcal{C}},$$

and explicitly given by

$$\dot{\mathcal{F}}^M(\dot{x}, \dot{f}, \dot{q}) = \left(\dot{x}|_{S_{0,1}}, \mathfrak{M} \left(\dot{x}|_{S_{0,2}} - \dot{\mathcal{P}}_q^M(\dot{q}) - \dot{\mathcal{P}}_f^M(\dot{f}) \right), \left(\dot{c}_i^M \right)_{1 \leq i \leq 10} \right), \quad (5.3)$$

where the linearized constraint functions $\dot{c}_i^M$, $1 \leq i \leq 10$, are given in Sect. 2.7 and the linearized perturbations $\dot{\mathcal{P}}_f^M$ and $\dot{\mathcal{P}}_q^M$ are given in Sect. 2.8.

Importantly, the linearization $\dot{\mathcal{F}}^0$ at Minkowski, given in (5.3), is in accordance with the setup of the linearized characteristic gluing problem at Minkowski in Sect. 4, so that Theorem 4.1 implies that the linearization $\dot{\mathcal{F}}^0$ is surjective. This constitutes the central ingredient for the proof of Theorem 3.1.

As $\dot{\mathcal{F}}^0$ is a bounded linear mapping between Hilbert spaces, its kernel $\ker(\dot{\mathcal{F}}^0)$ is a closed subspace and the following splitting holds,

$$\mathcal{X}(\mathcal{H}_{0, [1, 2]}) \times \mathcal{Y}_f \times \mathcal{Y}_q = \ker(\dot{\mathcal{F}}^0) \oplus \left(\ker(\dot{\mathcal{F}}^0) \right)^\perp.$$

In the following, we consider only the restriction $\overline{\mathcal{F}}$ of $\mathcal{F}$ to

$$\overline{\mathcal{F}} : \mathcal{X}(S_{0,1}) \times \mathcal{X}^+(\tilde{\mathcal{H}}_{[-\delta, \delta], 2}) \times \left(\ker(\dot{\mathcal{F}}^0) \right)^\perp \rightarrow \mathcal{X}(S_{0,1}) \times \mathcal{Z}_{\mathfrak{M}}(S_{0,2}) \times \mathcal{Z}_{\mathcal{C}}.$$

In this setting, the linearization $\dot{\overline{\mathcal{F}}}^0$ is a *bijection* between Hilbert spaces.

By the continuity in $M \geq 0$ of the family of linearizations

$$\dot{\mathcal{F}}^M : \left(\ker(\dot{\mathcal{F}}^0) \right)^\perp \rightarrow \mathcal{X}(S_{0,1}) \times \mathcal{Z}_{\mathfrak{M}}(S_{0,2}) \times \mathcal{Z}_C,$$

and the classical functional analysis result that *bijectivity is an open property of bounded linear operators*, we have the following corollary.

Corollary 5.2 (Bijectivity of $\dot{\mathcal{F}}^M$). *For real numbers $M \geq 0$ sufficiently small, the linearization $\dot{\mathcal{F}}^M$ is a bijection, and the solution $(\dot{x}, \dot{f}, \dot{q})$ of*

$$\dot{\mathcal{F}}^M(\dot{x}, \dot{f}, \dot{q}) = \left(\dot{\mathfrak{x}}_{0,1}, \dot{\mathfrak{m}}_{0,2}, (\dot{\mathfrak{c}}_i)_{1 \leq i \leq 10} \right)$$

satisfies the following estimate,

$$\begin{aligned} & \|\dot{x}\|_{\mathcal{X}(\mathcal{H}_{0,[1,2]})} + \|\dot{f}\|_{\mathcal{Y}_f} + \|\dot{q}\|_{\mathcal{Y}_q} + \left\| \dot{\mathcal{P}}_q^M(\dot{q}) \right\|_{\mathcal{X}(S_{0,2})} + \left\| \dot{\mathcal{P}}_f^M(\dot{f}) \right\|_{\mathcal{X}(S_{0,2})} \\ & \lesssim \|\dot{\mathfrak{x}}_{0,1}\|_{\mathcal{X}(S_{0,1})} + \|\dot{\mathfrak{m}}_{0,2}\|_{\mathcal{Z}_{\mathfrak{M}}(S_{0,2})} + \|(\dot{\mathfrak{c}}_i)_{1 \leq i \leq 10}\|_{\mathcal{Z}_C}. \end{aligned}$$

5.2. Construction of Solution to the Null Constraint Equations

In this section, we apply the implicit function theorem to $\overline{\mathcal{F}}$ to construct solutions x to the null constraint equations satisfying matching conditions.

With view on applying the implicit function theorem (see Theorem 2.25) to $\overline{\mathcal{F}}$ at Schwarzschild of small mass $M \geq 0$, we recall the following properties from Sect. 5.1.

- (1) For $M \geq 0$ sufficiently small, the mapping

$$\overline{\mathcal{F}} : \mathcal{X}(S_{0,1}) \times \mathcal{X}^+(\tilde{\mathcal{H}}_{[-\delta,\delta],2}) \times \left(\ker(\dot{\mathcal{F}}^0) \right)^\perp \rightarrow \mathcal{X}(S_{0,1}) \times \mathcal{Z}_{\mathfrak{M}}(S_{0,2}) \times \mathcal{Z}_C$$

is a well-defined and smooth mapping between Hilbert spaces in an open neighborhood of

$$(x_{0,1}, \underline{x}, x, f, q) = (\mathfrak{m}^M, \underline{\mathfrak{m}}^M, \mathfrak{m}^M, 0, 0), \quad (5.4)$$

where the size of the neighborhood is independent of M .

- (2) By (5.2), it holds that

$$\overline{\mathcal{F}}(\mathfrak{m}^M, \underline{\mathfrak{m}}^M, \mathfrak{m}^M, 0, 0) = (0, 0, 0).$$

- (3) For $M \geq 0$ sufficiently small, the linearization $\dot{\mathcal{F}}^M$ of $\overline{\mathcal{F}}$ in (x, f, q) evaluated at (5.4) is a bijection.

By the above, for $M \geq 0$ sufficiently small, we can apply the implicit function theorem to $\overline{\mathcal{F}}$ at (5.4). We conclude that there are a universal radius $r_0 > 0$ and a smooth mapping

$$\mathcal{G}^M : B((\mathfrak{m}^M, \underline{\mathfrak{m}}^M), r_0) \rightarrow \left(\ker(\dot{\mathcal{F}}^0) \right)^\perp \subset \mathcal{X}(\mathcal{H}_{0,[1,2]}) \times \mathcal{Y}_f \times \mathcal{Y}_j,$$

where $B((\mathfrak{m}^M, \underline{\mathfrak{m}}^M), r_0)$ denotes the open ball of radius $r_0 > 0$ centered at $(\mathfrak{m}^M, \underline{\mathfrak{m}}^M)$,

$$B((\mathfrak{m}^M, \underline{\mathfrak{m}}^M), r_0) \subset \mathcal{X}(S_{0,1}) \times \mathcal{X}^+(\mathcal{H}_{[-\delta,\delta],2}),$$

such that for all $(x_{0,1}, \tilde{x}) \in B((\mathbf{m}^M, \underline{\mathbf{m}}^M), r_0)$,

$$\overline{\mathcal{F}}(x_{0,1}, \tilde{x}, \mathcal{G}^M(x_{0,1}, \tilde{x})) = (0, 0, 0). \quad (5.5)$$

Defining for given $(x_{0,1}, \tilde{x}) \in B((\mathbf{m}^M, \underline{\mathbf{m}}^M), r_0)$,

$$(x, f, q) := \mathcal{G}^M(x_{0,1}, \tilde{x}),$$

we have by (5.5) and the definition of $\overline{\mathcal{F}}$ as restriction of the mapping $\mathcal{F}$ introduced in (5.1) that

$$\begin{aligned} \mathcal{C}_i(x) &= 0 \quad \text{on } \mathcal{H}_{0,[1,2]} \text{ for } 1 \leq i \leq 10, \\ x|_{S_{0,1}} &= x_{0,1}, \quad \mathfrak{M}(x|_{S_{0,2}}) = \mathfrak{M}(\mathcal{P}_{f,q}(\tilde{x})). \end{aligned} \quad (5.6)$$

This proves (3.2). The matching (3.4) under the charge matching condition (3.3) follows directly from Lemma 2.12.

We turn to the proof of (3.5). Applying Lemma 2.26 to the smooth map $\mathcal{G}^M$, we get that

$$\begin{aligned} \|x - \mathbf{m}^M\|_{\mathcal{X}(\mathcal{H}_{0,[1,2]})} + \|f\|_{\mathcal{Y}_f} + \|q\|_{\mathcal{Y}_q} \\ \lesssim \|x_{0,1} - \mathbf{m}_{0,1}^M\|_{\mathcal{X}(S_{0,1})} + \|\tilde{x}_{[-\delta,\delta],2} - \underline{\mathbf{m}}^M\|_{\tilde{\mathcal{X}}^+(\underline{\mathcal{H}}_{[-\delta,\delta],2})}. \end{aligned} \quad (5.7)$$

By (5.7) and Proposition 2.21, it further follows that for $\varepsilon > 0$ sufficiently small,

$$\begin{aligned} \|\mathcal{P}_{f,q}(\tilde{x}) - \tilde{x}_{0,2}\|_{\mathcal{X}(S_{0,2})} &\lesssim \|q\|_{\mathcal{Y}_q} + \|f\|_{\mathcal{Y}_f} + \|\tilde{x}_{[-\delta,\delta],2} - \underline{\mathbf{m}}^M\|_{\mathcal{X}^+(\underline{\mathcal{H}}_{[-\delta,\delta],2})} \\ &\lesssim \|x_{0,1} - \mathbf{m}^M\|_{\mathcal{X}(S_{0,1})} + \|\tilde{x}_{[-\delta,\delta],2} - \underline{\mathbf{m}}^M\|_{\mathcal{X}^+(\underline{\mathcal{H}}_{[-\delta,\delta],2})}. \end{aligned}$$

We underline that the radius $r_0 > 0$ and the constants in the above estimates are universal for small $M \geq 0$. This follows from the smoothness of $\overline{\mathcal{F}}$ and the continuity in $M \geq 0$ of the Schwarzschild data (5.4); see also, for example, Proposition 2.5.6 in [38]. This finishes the proof of the estimates (3.5).

5.3. Proof of the Charge Perturbation Estimate (3.6)

In this section, we prove (3.6); that is, for $M \geq 0$ and $\varepsilon > 0$ sufficiently small, the following estimate holds,

$$|(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})(x_{0,2}) - (\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})(\tilde{x}_{0,2})| \lesssim \varepsilon M + \varepsilon^2, \quad (5.8)$$

where $x_{0,2} := \mathcal{P}_j \mathcal{P}_f(\tilde{x})$.

Consider first (5.8) for the charge $\mathbf{E}$. For this section, we introduce the map

$$\begin{aligned} \mathbf{E} : \mathcal{X}^+(\underline{\mathcal{H}}_{[-\delta,\delta],2}) \times \mathcal{Y}_f \times \mathcal{Y}_q &\rightarrow H^4(S_{0,2}), \\ (\tilde{x}, f, q) &\mapsto \mathbf{E}(\tilde{x}, f, q) := \mathbf{E}(\mathcal{P}_{f,q}(\tilde{x})). \end{aligned} \quad (5.9)$$

From the smoothness of the perturbation $\mathcal{P}_{f,q}$ and definition of $\mathbf{E}$ in Definition 2.10, it follows that $\mathbf{E}$ in (5.9) is a smooth map in an open neighborhood of

$$(\tilde{x}, f, q) = (\underline{\mathbf{m}}, 0, 0).$$

Indeed, let x be the constructed solution to the null constraint equations on $\mathcal{H}_{0,[1,2]}$. By the fundamental theorem of calculus,

$$\mathbf{E}(x)|_{S_{0,2}} - \mathbf{E}(x)|_{S_{0,1}} = \int_1^2 D\mathbf{E}(x)|_{S_{0,v}} dv. \quad (5.17)$$

In the following, we analyze $D\mathbf{E}(x)|_{S_{0,v}}$ for $1 \leq v \leq 2$. Using that for Schwarzschild reference data $\mathbf{m}^M$, it holds that

$$\mathbf{E}(\mathbf{m}^M) = M, \quad D\mathbf{E}(\mathbf{m}^M) = 0 \quad \text{on } \mathcal{H}_{0,[1,2]},$$

we can express the integrand $D\mathbf{E}(x)$ on the right-hand side of (5.17) by

$$D\mathbf{E}(x) = D\mathbf{E}(x) - D\mathbf{E}(\mathbf{m}^M). \quad (5.18)$$

We make two observations. First, the map

$$\begin{aligned} D\mathbf{E} : \mathcal{X}(\mathcal{H}_{0,[1,2]}) &\rightarrow H_1^4(\mathcal{H}_{0,[1,2]}), \\ x &\mapsto D\mathbf{E}(x), \end{aligned}$$

is smooth in an open neighborhood of $x = \mathbf{m}$. Second, at the end of this section (see Lemma 5.3) we show that by the implicit function theorem construction of our solution x , there is a smooth family

$$(x_s)_{0 \leq s \leq 1} \subset \mathcal{X}(\mathcal{H}_{0,[1,2]}) \quad (5.19)$$

of solutions to the null constraint equations on $\mathcal{H}_{0,[1,2]}$ such that

$$x_{s=0} = \mathbf{m}^M \quad \text{and} \quad x_{s=1} = x \quad \text{on } \mathcal{H}_{0,[1,2]}, \quad (5.20)$$

and satisfying the estimate

$$\sup_{0 \leq s \leq 1} \|x_s - \mathbf{m}^M\|_{\mathcal{X}(\mathcal{H})} \lesssim \varepsilon, \quad \sup_{0 \leq s \leq 1} \|\dot{x}_s\|_{\mathcal{X}(\mathcal{H}_{0,[1,2]})} \lesssim \varepsilon, \quad (5.21)$$

where $\dot{x}_s$ denotes the variation through the family (5.19). Hence, we can rewrite (5.18) by the fundamental theorem of calculus as

$$D\mathbf{E}(x) = D\mathbf{E}(x) - D\mathbf{E}(\mathbf{m}^M) = \int_0^1 \frac{d}{ds} (D\mathbf{E}(x_s)) ds = \int_0^1 D\dot{\mathbf{E}}|_{x_s}(\dot{x}_s) ds, \quad (5.22)$$

where $D\dot{\mathbf{E}}|_{x_s}$ denotes the linearization of $D\mathbf{E}(x)$ in x evaluated at x_s .

By construction, see (5.20), $\dot{x}_{s=0}$ is a solution to the *homogeneous linearized null constraint equations at Schwarzschild*, that is, for $\dot{x} = \dot{x}_{s=0}$,

$$\dot{\mathcal{C}}^M(\dot{x}) = 0. \quad (5.23)$$

In Appendix C, see (C.7) it is shown that for all solutions $\dot{x}$ to (5.23),

$$\|D\dot{\mathbf{E}}|_{\mathbf{m}^M}(\dot{x})\|_{H_1^4(\mathcal{H}_{0,[1,2]})} \lesssim M \|\dot{x}\|_{\mathcal{X}(\mathcal{H}_{0,[1,2]})}. \quad (5.24)$$

Using the estimate (5.24), we can estimate the integrand $D\dot{\mathbf{E}}|_{x_s}(\dot{x}_s)$ on the right-hand side of (5.22) as follows. We write

$$D\dot{\mathbf{E}}|_{x_s}(\dot{x}_s) = \underbrace{\left(D\dot{\mathbf{E}}|_{x_s} - D\dot{\mathbf{E}}|_{\mathbf{m}^M}\right)}_{:=\mathcal{I}_1}(\dot{x}_s) + \underbrace{D\dot{\mathbf{E}}|_{\mathbf{m}^M}}_{:=\mathcal{I}_2}(\dot{x}_s).$$

The term $\mathcal{I}_1$ is estimated by the smoothness of $D\mathbf{E}$ and (5.21) as follows,

$$\begin{aligned} |\mathcal{I}_1| &\lesssim \|x_s - \mathbf{m}^M\|_{\mathcal{X}(\mathcal{H}_{0,[1,2]})} \cdot \|\dot{x}_s\|_{\mathcal{X}(\mathcal{H}_{0,[1,2]})} \\ &\lesssim \varepsilon^2. \end{aligned}$$

The term $\mathcal{I}_2$ can be analyzed by decomposing

$$\dot{x}_s = (\dot{x}_s)^\perp + (\dot{x}_s)^\top, \quad (5.25)$$

where $(\dot{x}_s)^\perp \in (\ker \dot{\mathcal{C}}^M)^\perp$ is defined as solution to

$$\begin{aligned} \dot{\mathcal{C}}^M((\dot{x}_s)^\perp) &= \dot{\mathcal{C}}^M(\dot{x}_s), \quad \dot{\Omega}((\dot{x}_s)^\perp) = 0, \quad \dot{\chi}((\dot{x}_s)^\perp) = 0 \text{ on } \mathcal{H}_{0,[1,2]}, \\ &(\dot{x}_s)^\perp|_{S_{0,1}} = 0. \end{aligned} \quad (5.26)$$

and $(\dot{x}_s)^\top \in \ker \dot{\mathcal{C}}^M$ is defined as

$$(\dot{x}_s)^\top := \dot{x}_s - (\dot{x}_s)^\perp.$$

By bounds for the system (5.26) analogous to Theorem 4.1 and Corollary 5.2 (see Lemmas 4.5, 4.6, 4.10, 4.13) together with the estimate

$$\begin{aligned} \|\dot{\mathcal{C}}^M(\dot{x}_s)\|_{\mathcal{Z}_C} &= \|(\dot{\mathcal{C}}^M - \dot{\mathcal{C}}|_{x_s})(\dot{x}_s)\|_{\mathcal{Z}_C} \\ &\lesssim \|x_s - \mathbf{m}^M\|_{\mathcal{X}(\mathcal{H}_{0,[1,2]})} \cdot \|\dot{x}_s\|_{\mathcal{X}(\mathcal{H}_{0,[1,2]})} \\ &\lesssim \varepsilon^2, \end{aligned}$$

where we used (5.21) and that $\dot{\mathcal{C}}|_{x_s}(\dot{x}_s) = 0$ by definition of x_s in (5.19), it follows that

$$\|(\dot{x}_s)^\perp\|_{\mathcal{X}(\mathcal{H}_{0,[1,2]})} \lesssim \varepsilon^2. \quad (5.27)$$

This further implies that

$$\begin{aligned} \|(\dot{x}_s)^\top\|_{\mathcal{X}(\mathcal{H}_{0,[1,2]})} &= \|\dot{x}_s - (\dot{x}_s)^\perp\|_{\mathcal{X}(\mathcal{H}_{0,[1,2]})} \\ &\lesssim \varepsilon + \varepsilon^2 \\ &\lesssim \varepsilon. \end{aligned} \quad (5.28)$$

By (5.24), (5.25), (5.27) and (5.28), and using that $D\mathbf{E}|_{\mathbf{m}^M}$ is a bounded operator, we have

$$\begin{aligned} \mathcal{I}_2 &= D\dot{\mathbf{E}}|_{\mathbf{m}^M}((\dot{x}_s)^\top) + D\dot{\mathbf{E}}|_{\mathbf{m}^M}((\dot{x}_s)^\perp) \\ &\lesssim M\|(\dot{x}_s)^\top\|_{\mathcal{X}(\mathcal{H}_{0,[1,2]})} + \|(\dot{x}_s)^\perp\|_{\mathcal{X}(\mathcal{H}_{0,[1,2]})} \\ &\lesssim M\varepsilon + \varepsilon^2. \end{aligned}$$

To summarize the above, we conclude that for $0 \leq s \leq 1$,

$$\|D\dot{\mathbf{E}}|_{x_s}(\dot{x}_s)\|_{H_1^4(\mathcal{H}_{0,[1,2]})} \lesssim M\varepsilon + \varepsilon^2,$$

which, plugged into (5.22), yields that

$$\|D\mathbf{E}(x)\|_{H_1^4(\mathcal{H}_{0,[1,2]})} \lesssim M\varepsilon + \varepsilon^2,$$

and, consequently, by plugging into (5.17), proves the charge estimate (5.16) for $\mathbf{E}$.

We claim that the charge estimates for $\mathbf{P}$, $\mathbf{L}$ and $\mathbf{G}$ are proved similarly. Indeed, in Appendix C, see (C.7), in addition to (5.24) it is shown that for solutions $\dot{x}$ to

$$\dot{\mathcal{C}}^M(\dot{x}) = 0,$$

it holds that

$$\begin{aligned} & \|D\dot{\mathbf{P}}|_{\mathfrak{m}^M}(\dot{x})\|_{H_1^4(\mathcal{H}_{0,[1,2]})} + \|D\dot{\mathbf{L}}|_{\mathfrak{m}^M}(\dot{x})\|_{H_1^5(\mathcal{H}_{0,[1,2]})} + \|D\dot{\mathbf{G}}|_{\mathfrak{m}^M}(\dot{x})\|_{H_1^5(\mathcal{H}_{0,[1,2]})} \\ & \lesssim M\|\dot{x}\|_{\mathcal{X}(\mathcal{H}_{0,[1,2]})}. \end{aligned} \quad (5.29)$$

Thus, the remaining charge estimates in (5.15) are proved by following the same argument as above.

It remains to prove (5.19), that is, the existence of the smooth family

$$(x_s)_{0 \leq s \leq 1} \subset \mathcal{X}(\mathcal{H}_{0,[1,2]})$$

of solutions to the null constraint equations satisfying (5.20) and (5.21), that is,

$$x_{s=0} = \mathfrak{m}^M \text{ and } x_{s=1} = x \text{ on } \mathcal{H}_{0,[1,2]},$$

and

$$\sup_{0 \leq s \leq 1} \|x_s - \mathfrak{m}^M\|_{\mathcal{X}(\mathcal{H})} \lesssim \varepsilon, \quad \sup_{0 \leq s \leq 1} \|\dot{x}_s\|_{\mathcal{X}(\mathcal{H}_{0,[1,2]})} \lesssim \varepsilon.$$

Indeed, using the smooth map $\mathcal{G}^M$ constructed in Sect. 5.2, this follows directly from the following lemma. We remark here that the linearization $\dot{\mathcal{G}}^M$ is by construction bijective and uniformly bounded for $M \geq 0$ sufficiently small, see Sect. 5.2.

Lemma 5.3 (Existence of smooth family of data). *Let $M \geq 0$ and $\varepsilon > 0$ be sufficiently small. There are smooth families of*

- *sphere data $(x_{0,1})_s \in \mathcal{X}(S_{0,1})$ for $0 \leq s \leq 1$,*
- *ingoing null data $\tilde{x}_s \in \mathcal{X}^+(\tilde{\mathcal{H}}_{[-\delta,\delta],2})$ for $0 \leq s \leq 1$, solving the null constraint equations on $\tilde{\mathcal{H}}_{[-\delta,\delta],2}$,*

such that

$$((x_{0,1})_s, \underline{x}_s)|_{s=0} = (\mathfrak{m}^M, \underline{\mathfrak{m}}^M), \quad ((x_{0,1})_s, \tilde{x}_s)|_{s=1} = (x_{0,1}, \tilde{x}),$$

and satisfying

$$\begin{aligned} & \| (x_{0,1})_s - \mathfrak{m}^M \|_{\mathcal{X}(S_{0,1})} + \| \dot{(x_{0,1})}_s \|_{\mathcal{X}(S_{0,1})} \lesssim \varepsilon, \\ & \| \underline{x}_s - \underline{\mathfrak{m}}^M \|_{\mathcal{X}^+(\tilde{\mathcal{H}}_{[-\delta,\delta],2})} + \| \dot{\underline{x}}_s \|_{\mathcal{X}^+(\tilde{\mathcal{H}}_{[-\delta,\delta],2})} \lesssim \varepsilon, \end{aligned} \quad (5.30)$$

where $\dot{(x_{0,1})}_s$ and $\dot{\underline{x}}_s$ denote the variations of $(x_{0,1})_s$ and $\underline{x}_s$, respectively.

First, the smooth family of sphere data $(x_{0,1})_s$ can be defined by

$$(x_{0,1})_s := (x_{0,1} - \mathfrak{m}^M) \cdot s + \mathfrak{m}^M. \quad (5.31)$$

Remark 5.4. In (5.31), we abuse notation, as by Definition 2.4 the tensors $\widehat{\chi}$, $\widehat{\underline{\chi}}$, α and $\underline{\alpha}$ are required to be symmetric $\mathcal{g}$ -tracefree 2-tensors which is a constraint and not compatible with the linear operation depicted in (5.31). In (5.31), we interpret the prescription of $\widehat{\chi}$, $\widehat{\underline{\chi}}$, α and $\underline{\alpha}$ in the sense that two tensor components are freely prescribable (these are added on the right-hand side of (5.31)), and the other two tensor components are fully determined by the condition to be symmetric and tracefree with respect to $\mathcal{g}$. Same goes for the prescription of the symmetric tensor $\mathcal{g}$. In this sense, the prescription of sphere data is without constraint, and (5.31) is well defined.

Second, the family of ingoing null data $\tilde{x}_s$ is given by constructing solutions to the null constraint equations (as proved in Sect. 5.2 but at the higher level of regularity $\mathcal{X}^+$) on $\tilde{\mathcal{H}}_{[-\delta, \delta], 2}$ from sphere data $(\tilde{x}_{0,2})_s$ on $\tilde{S}_{0,2}$ given for $0 \leq s \leq 1$ by (see Remark 5.4)

$$(\tilde{x}_{0,2})_s := (\tilde{x}|_{S_{0,2}} - \mathfrak{m}^M) \cdot s + \mathfrak{m}^M, \quad (5.32)$$

and free data Ω_s and $\text{conf}(\mathcal{g})_s$ on $\tilde{\mathcal{H}}_{[-\delta, \delta], 2}$ given for $0 \leq s \leq 1$ by

$$\Omega_s := (\Omega - \Omega_M) \cdot s + \Omega_M, \quad \text{conf}(\mathcal{g})_s := (\text{conf}(\mathcal{g}) - \text{conf}(\mathfrak{m}^M)) \cdot s + \text{conf}(\mathfrak{m}^M). \quad (5.33)$$

The estimate (5.30) follows by the general estimates proved for the construction of solutions to the null constraint equations (see Sect. 5.2) and the explicit prescriptions (5.32) and (5.33). This finishes the proof of Lemma 5.3 and hence of the charge estimate (3.7).

5.5. Outline of the Proof of Theorem 3.2

In this section, we indicate how the proof of Theorem 3.2, that is, the characterizing gluing with higher-order L -derivatives, is based on the proof of Theorem 3.1.

- (1) The proof of Theorem 3.2 is, similarly as in Sects. 5.1 and 5.2, reduced to studying the *linearized* characteristic gluing problem with higher-order L -derivatives by the implicit function theorem.
- (2) At the linear level at Minkowski, the sphere data perturbations $\dot{\mathcal{P}}_f$ and $\dot{\mathcal{P}}_q$ leave $\mathcal{D}^{L,m}$ invariant; see Remark 2.29. Hence, the linearized gluing of $\mathcal{D}^{L,m}$ solely depends on the prescription of the free data $\dot{\Omega}$ and $\dot{\hat{\chi}}$ on $\mathcal{H}_{0,[1,2]}$.
- (3) By the linearized null constraint equations in Sect. 4, $\dot{\mathcal{D}}^{L,m}$ can directly be calculated from $\dot{\Omega}$ and $\dot{\hat{\chi}}$ on $\mathcal{H}_{0,[1,2]}$. In particular, there is no obstruction to matching $\dot{\mathcal{D}}^{L,m}$ on $S_{0,1}$ and $S_{0,2}$ by adjusting $\dot{\Omega}$ and $\dot{\hat{\chi}}$ on $\mathcal{H}_{0,[1,2]}$, see Remark 4.19. This shows that the linearized characteristic gluing problem for higher-order L -derivatives is solvable.

This finishes our discussion of Theorem 3.2.

6. Bifurcate Characteristic Gluing

In this section, we prove Theorem 3.3, that is, the codimension-10 characteristic gluing of higher-order sphere data along two null hypersurfaces bifurcating from an auxiliary sphere. Analogously to the perturbative characteristic gluing of Theorem 3.1, the two main ingredients are

- (1) solving the *linearized* characteristic gluing problem,
- (2) applying the implicit function theorem.

As the application of the implicit function theorem is similar as in Sects. 5.1 and 5.2, we focus in this section on the new ideas necessary for (1). We proceed as follows.

- In Sect. 6.1, we discuss the linearized null constraint equations and the conserved charges along $\underline{\mathcal{H}}_{[-1,0],1}$ and $\mathcal{H}_{-1,[1,2]}$.
- In Sect. 6.2, we derive relations between charges on $\underline{\mathcal{H}}_{[-1,0],1}$ and $\mathcal{H}_{-1,[1,2]}$.
- In Sect. 6.3, we state and prove the linearized characteristic gluing along two transversely intersecting null hypersurfaces.

In Sect. 6.4, we prove Proposition 3.4.

6.1. Linearized Null Constraint Equations

In this section, we analyze the linearized null constraint equations at Minkowski along $\underline{\mathcal{H}}_{[-1,0],1}$ and $\mathcal{H}_{-1,[1,2]}$, and discuss the corresponding conserved charges.

Remark 6.1. In the following, we study the *homogeneous* linearized null constraint equations; that is, we follow the formalism of [27] and linearize through a family of solutions to the Einstein equations. The analysis of the *inhomogeneous* linearized null constraint equations along $\underline{\mathcal{H}}_{[-1,0],1}$ with source terms $(\underline{c}_i)_{1 \leq i \leq 10}$ (which is necessary for the application of the implicit function theorem) is then analogous to Sect. 2.7.2.

Linearized null constraint equations along $\mathcal{H}_{-1,[1,2]}$. In Sect. 2.7.2, we linearized the null constraint equations along $\mathcal{H}_{0,[1,2]}$, see Lemma C.3. Setting $M = 0$ in Lemma C.3 yields the linearized equations at Minkowski. This linearization and the resulting linearized equations clearly apply analogously to the null hypersurface $\mathcal{H}_{-1,[1,2]}$ considered in this section.

We recall that in Sect. 4.3 we identified *charges*

$$\mathcal{Q}_i \text{ for } 0 \leq i \leq 7, \quad (6.1)$$

which satisfy conservation laws along $\mathcal{H}_{-1,[1,2]}$,

$$D\mathcal{Q}_i = 0 \text{ for } 0 \leq i \leq 7.$$

We refer to (4.5) for the precise definitions of these charges.

Linearized null constraint equations in $\underline{L}$ -direction. Similar to Lemma C.3, by linearization at Minkowski of the null constraint equations along $\underline{\mathcal{H}}_{[-1,0],1}$ we get the following, see also [27]. The linearized first variation equation,

$$\underline{D} \left(\frac{\dot{\phi}}{r} \right) = \frac{(\Omega \dot{\text{tr}} \underline{\chi})}{2}, \quad \underline{D} \dot{g}_c = \frac{2}{r^2} \dot{\underline{\chi}}, \quad (6.2)$$

the linearized Raychaudhuri equation,

$$\underline{D} \left(r^2 \left((\Omega \dot{\text{tr}} \underline{\chi}) + \frac{4}{r} \dot{\Omega} \right) \right) = -4\dot{\Omega}, \quad (6.3)$$

as well as

$$\begin{aligned} \underline{D} (r^2 \dot{\eta}) &= \dot{\text{div}} \dot{\underline{\chi}} - \frac{r^2}{2} \not{d} \left((\Omega \dot{\text{tr}} \underline{\chi}) + \frac{4}{r} \dot{\Omega} \right) - 2r \not{d} \dot{\Omega}, \\ \underline{D} \left(r^2 (\Omega \dot{\text{tr}} \underline{\chi}) \right) &= 2\dot{\text{div}} \left(-\dot{\eta} + 2\not{d} \dot{\Omega} \right) - \dot{\text{div}} \dot{\text{div}} \dot{g}_c + \frac{2}{r} \left(\dot{\Delta} + 2 \right) \dot{\phi} \\ &\quad - 2r(\Omega \dot{\text{tr}} \underline{\chi}) - 4\dot{\Omega}. \end{aligned} \quad (6.4)$$

As in Sect. 4.3, we can derive null transport equations along $\mathcal{H}_{[-1,0],1}$ from the linearized null constraint Eqs. (6.2), (6.3) and (6.4); the explicit proof of the following lemma is omitted.

Lemma 6.2 (Null transport equations in $\underline{L}$ -direction). *The linearized null constraint equations imply the following null transport equations,*

$$\begin{aligned} \underline{D} \left(\frac{r}{2} \left((\Omega \dot{\text{tr}} \underline{\chi}) + \frac{4}{r} \dot{\Omega} \right) - \frac{\dot{\phi}}{r} \right) &= 0, \\ \underline{D} \left(r^2 \dot{\eta} - \frac{r^3}{2} \not{d} \left((\Omega \dot{\text{tr}} \underline{\chi}) + \frac{4}{r} \dot{\Omega} \right) \right) &= -\dot{\text{div}} \dot{\underline{\chi}}, \end{aligned}$$

and

$$\underline{D} \left(r^2 (\Omega \dot{\text{tr}} \underline{\chi}) + \frac{2}{r} \dot{\text{div}} \left(r^2 \dot{\eta} - \frac{r^3}{2} \not{d} \left((\Omega \dot{\text{tr}} \underline{\chi}) + \frac{4}{r} \dot{\Omega} \right) \right) - 2r^3 \dot{K} - r^2 \left((\Omega \dot{\text{tr}} \underline{\chi}) + \frac{4}{r} \dot{\Omega} \right) \right) = 0.$$

Remarks on Lemma 6.2.

- (1) By Lemma 6.2, the charges $\underline{\mathcal{Q}}_0, \underline{\mathcal{Q}}_1$ and $\underline{\mathcal{Q}}_2$ defined on the sphere $S_{u,v}$ with $r = v - u > 0$ by

$$\begin{aligned} \underline{\mathcal{Q}}_0 &:= \left(r^2 \dot{\eta} - \frac{r^3}{2} \not{d} \left((\Omega \dot{\text{tr}} \underline{\chi}) + \frac{4}{r} \dot{\Omega} \right) \right)^{[1]}, \\ \underline{\mathcal{Q}}_1 &:= \frac{r}{2} \left((\Omega \dot{\text{tr}} \underline{\chi}) + \frac{4}{r} \dot{\Omega} \right) - \frac{\dot{\phi}}{r}, \\ \underline{\mathcal{Q}}_2 &:= r^2 (\Omega \dot{\text{tr}} \underline{\chi}) + \frac{2}{r} \dot{\text{div}} \left(r^2 \dot{\eta} - \frac{r^3}{2} \not{d} \left((\Omega \dot{\text{tr}} \underline{\chi}) + \frac{4}{r} \dot{\Omega} \right) \right) \\ &\quad - r^2 \left((\Omega \dot{\text{tr}} \underline{\chi}) + \frac{4}{r} \dot{\Omega} \right) - 2r^3 \dot{K}, \end{aligned} \quad (6.5)$$

satisfy the following conservation laws along $\mathcal{H}_{[-1,0],1}$,

$$\underline{D} \underline{\mathcal{Q}}_0 = 0, \quad \underline{D} \underline{\mathcal{Q}}_1 = 0, \quad \underline{D} \underline{\mathcal{Q}}_2 = 0.$$

- (2) In analogy to the analysis in Sect. 4.3.2, the null constraint Eqs. (6.2), (6.3), (6.4) and the null transport equations of Lemma 6.2 imply that the quantities

$$\dot{\phi}, \dot{g}_c, \left(r^2 \dot{\eta} - \frac{r^3}{2} \not{d} \left((\Omega \text{tr} \chi) + \frac{4}{r} \dot{\Omega} \right) \right)^{[\geq 2]} \quad (6.6)$$

can be glued without obstacles along $\mathcal{H}_{[0,-1],1}$ by using the degrees of freedom $\dot{\Omega}$ and $\dot{\chi}$ on $\mathcal{H}_{[-1,0],1}$.

- (3) As for the linearized null constraint equations on $\mathcal{H}_{-1,[1,2]}$, there are further charges $\underline{\mathcal{Q}}_i$, $4 \leq i \leq 7$, and higher-order charges along $\mathcal{H}_{[-1,0],1}$. However, for the purposes of this paper, the explicit expressions for $\underline{\mathcal{Q}}_1$, $\underline{\mathcal{Q}}_2$ and $\underline{\mathcal{Q}}_3$ are sufficient.

6.2. Preliminary Analysis of Charges

In this section, we derive relations between the charges

$$\mathcal{Q}_0, \mathcal{Q}_1, \mathcal{Q}_2 \text{ and } \underline{\mathcal{Q}}_0, \underline{\mathcal{Q}}_1, \underline{\mathcal{Q}}_2.$$

The following lemma is the main result of this section.

Lemma 6.3 (Charge identities). *Consider linearized sphere data $\dot{x}_{u,v}$ on a sphere $S_{u,v}$. Let $\mathcal{Q}_i$ and $\underline{\mathcal{Q}}_i$, $0 \leq i \leq 2$, denote the associated charges on $S_{u,v}$. Then, it holds that, with $r = v - u$,*

$$\begin{aligned} \mathcal{Q}_1 - \frac{1}{2r} \mathcal{Q}_2 - \underline{\mathcal{Q}}_1 &= -2\dot{\Omega} + \frac{1}{2} \overset{\circ}{\text{d}}\overset{\circ}{v} \overset{\circ}{\text{d}}\overset{\circ}{v} \dot{g}_c \\ &\quad - \frac{1}{r^2} \overset{\circ}{\text{d}}\overset{\circ}{v} \left(r^2 \dot{\eta} - \frac{r^3}{2} \not{d} \left((\Omega \text{tr} \chi) + \frac{4}{r} \dot{\Omega} \right) \right) - \frac{1}{r} \overset{\circ}{\Delta} \dot{\phi}, \end{aligned}$$

and

$$\begin{aligned} & - \frac{1}{4r} (\mathcal{Q}_2 + \underline{\mathcal{Q}}_2) - \frac{1}{2} \overset{\circ}{\Delta} (\mathcal{Q}_1 - \underline{\mathcal{Q}}_1) \\ &= \overset{\circ}{\Delta} \dot{\Omega} - \frac{1}{r^2} \overset{\circ}{\text{d}}\overset{\circ}{v} \left(r^2 \dot{\eta} - \frac{r^3}{2} \not{d} \left((\Omega \text{tr} \chi) + \frac{4}{r} \dot{\Omega} \right) \right) - \frac{1}{r} \overset{\circ}{\Delta} \dot{\phi}. \end{aligned}$$

and moreover,

$$(\mathcal{Q}_0)_H = -(\underline{\mathcal{Q}}_0)_H, \quad (\mathcal{Q}_0)_E = (\underline{\mathcal{Q}}_0)_E + \frac{r}{2} \not{d} \underline{\mathcal{Q}}_2^{[1]}, \quad \mathcal{Q}_2^{[0]} = -\underline{\mathcal{Q}}_2^{[0]}, \quad \mathcal{Q}_2^{[1]} = \underline{\mathcal{Q}}_2^{[1]}. \quad (6.7)$$

Remarks on Lemma 6.3.

- (1) The significance of the first two identities of Lemma 6.3 is that by (6.6) the terms on the right-hand side are freely glueable along $\mathcal{H}_{[-1,0],1}$.
- (2) The relations (6.7) show that $\underline{\mathcal{Q}}_0$ and $\underline{\mathcal{Q}}_2^{[\leq 1]}$ fully determine $\mathcal{Q}_0$ and $\mathcal{Q}_2^{[\leq 1]}$ on $S_{u,v}$, and vice versa.

The rest of this section contains the proof of Lemma 6.3. In the following, we use the definition of $\underline{\mathcal{Q}}_1$, $\underline{\mathcal{Q}}_2$ and $\underline{\mathcal{Q}}_3$, see (6.5), as well as (2.48), (2.49) and (4.5), that is,

$$\dot{\eta} = -\dot{\eta} + 2\dot{\phi} \dot{\Omega}, \quad \dot{K} = \frac{1}{2r^2} \overset{\circ}{\text{d}}\overset{\circ}{v} \overset{\circ}{\text{d}}\overset{\circ}{v} \dot{g}_c - \frac{1}{r^3} (\overset{\circ}{\Delta} + 2) \dot{\phi},$$

and

$$\begin{aligned}\mathcal{Q}_0 &:= \left(r^2 \dot{\eta} + \frac{r^3}{2} \not{d} \left((\Omega \operatorname{tr} \chi) - \frac{4}{r} \dot{\Omega} \right) \right)^{[1]}, \\ \mathcal{Q}_1 &:= \frac{r}{2} \left((\Omega \operatorname{tr} \chi) - \frac{4}{r} \dot{\Omega} \right) + \frac{\dot{\phi}}{r}, \\ \mathcal{Q}_2 &:= r^2 (\Omega \operatorname{tr} \underline{\chi}) - \frac{2}{r} \operatorname{div} \left(r^2 \dot{\eta} + \frac{r^3}{2} \not{d} \left((\Omega \operatorname{tr} \chi) - \frac{4}{r} \dot{\Omega} \right) \right) \\ &\quad - r^2 \left((\Omega \operatorname{tr} \chi) - \frac{4}{r} \dot{\Omega} \right) + 2r^3 \dot{K}\end{aligned}$$

Analysis of $\mathcal{Q}_2$. We have that

$$\begin{aligned}\mathcal{Q}_2 + \underline{\mathcal{Q}}_2 &= -\frac{2}{r} \operatorname{div} \left(r^2 \dot{\eta} + \frac{r^3}{2} \not{d} \left((\Omega \operatorname{tr} \chi) - \frac{4}{r} \dot{\Omega} \right) \right) \\ &\quad + \frac{2}{r} \operatorname{div} \left(r^2 \dot{\eta} - \frac{r^3}{2} \not{d} \left((\Omega \operatorname{tr} \underline{\chi}) + \frac{4}{r} \dot{\Omega} \right) \right) \\ &= \frac{4}{r} \operatorname{div} \left(r^2 \dot{\eta} - \frac{r^3}{2} \not{d} \left((\Omega \operatorname{tr} \underline{\chi}) + \frac{4}{r} \dot{\Omega} \right) \right) \\ &\quad - \frac{2}{r} \operatorname{div} \left(\frac{r^3}{2} \not{d} (\Omega \operatorname{tr} \chi) - \frac{r^3}{2} \not{d} (\Omega \operatorname{tr} \underline{\chi}) - 2r^2 \dot{\Omega} \right) \\ &= \frac{4}{r} \operatorname{div} \left(r^2 \dot{\eta} - \frac{r^3}{2} \not{d} \left((\Omega \operatorname{tr} \underline{\chi}) + \frac{4}{r} \dot{\Omega} \right) \right) \\ &\quad - \not{\Delta} \left(r^2 (\Omega \operatorname{tr} \chi) - r^2 (\Omega \operatorname{tr} \underline{\chi}) - 4r \dot{\Omega} \right) \\ &= \frac{4}{r} \operatorname{div} \left(r^2 \dot{\eta} - \frac{r^3}{2} \not{d} \left((\Omega \operatorname{tr} \underline{\chi}) + \frac{4}{r} \dot{\Omega} \right) \right) \\ &\quad - \not{\Delta} \left(\mathcal{Q}_2 - \frac{2}{r} \operatorname{div} \left(r^2 \dot{\eta} - \frac{r^3}{2} \not{d} \left((\Omega \operatorname{tr} \underline{\chi}) + \frac{4}{r} \dot{\Omega} \right) \right) + 2r^3 \dot{K} \right) \\ &= \frac{2}{r} \left(\not{\Delta} + 2 \right) \operatorname{div} \left(r^2 \dot{\eta} - \frac{r^3}{2} \not{d} \left((\Omega \operatorname{tr} \underline{\chi}) + \frac{4}{r} \dot{\Omega} \right) \right) \\ &\quad - \not{\Delta} \underline{\mathcal{Q}}_2 - r \not{\Delta} \operatorname{div} \operatorname{div} \not{g}_c + 2 \not{\Delta} \left(\not{\Delta} + 2 \right) \dot{\phi},\end{aligned}$$

which can be rewritten as

$$\begin{aligned}& -\frac{1}{2r} \left(\mathcal{Q}_2 + (\not{\Delta} + 1) \underline{\mathcal{Q}}_2 \right) \\ &= \not{\Delta} \left(-\frac{1}{r^2} \operatorname{div} \left(r^2 \dot{\eta} - \frac{r^3}{2} \not{d} \left((\Omega \operatorname{tr} \underline{\chi}) + \frac{4}{r} \dot{\Omega} \right) \right) + \frac{1}{2} \operatorname{div} \operatorname{div} \not{g}_c - \frac{1}{r} \not{\Delta} \dot{\phi} \right) \\ &\quad - \frac{2}{r^2} \operatorname{div} \left(r^2 \dot{\eta} - \frac{r^3}{2} \not{d} \left((\Omega \operatorname{tr} \underline{\chi}) + \frac{4}{r} \dot{\Omega} \right) \right) - \frac{2}{r} \not{\Delta} \dot{\phi}.\end{aligned}\tag{6.8}$$

Projecting (6.8) onto the modes $l = 0$ and $l = 1$ proves the last two of (6.7),

$$\mathcal{Q}_2^{[0]} = -\underline{\mathcal{Q}}_2^{[0]}, \quad \mathcal{Q}_2^{[1]} = \underline{\mathcal{Q}}_2^{[1]}.$$

Analysis of $\mathcal{Q}_1$. We have that

$$\begin{aligned} \mathcal{Q}_1 &= \frac{1}{2r} \left(r^2 (\Omega \operatorname{tr} \chi) \right) - 2\dot{\Omega} + \frac{\dot{\phi}}{r} \\ &= \frac{1}{2r} \left(\underline{\mathcal{Q}}_2 - \frac{2}{r} \overset{\circ}{\operatorname{div}} \left(r^2 \underline{\dot{\eta}} - \frac{r^3}{2} \not{d} \left((\Omega \operatorname{tr} \chi) + \frac{4}{r} \dot{\Omega} \right) \right) \right) \\ &\quad + \frac{1}{2r} \left(r^2 \left((\Omega \operatorname{tr} \chi) + \frac{4}{r} \dot{\Omega} \right) + 2r^3 \dot{K} \right) - 2\dot{\Omega} + \frac{\dot{\phi}}{r} \\ &= \frac{1}{2r} \underline{\mathcal{Q}}_2 - \frac{1}{r^2} \overset{\circ}{\operatorname{div}} \left(r^2 \underline{\dot{\eta}} - \frac{r^3}{2} \not{d} \left((\Omega \operatorname{tr} \chi) + \frac{4}{r} \dot{\Omega} \right) \right) \\ &\quad + \left(\underline{\mathcal{Q}}_1 + \frac{\dot{\phi}}{r} \right) + r^2 \dot{K} - 2\dot{\Omega} + \frac{\dot{\phi}}{r} \\ &= \frac{1}{2r} \underline{\mathcal{Q}}_2 - \frac{1}{r^2} \overset{\circ}{\operatorname{div}} \left(r^2 \underline{\dot{\eta}} - \frac{r^3}{2} \not{d} \left((\Omega \operatorname{tr} \chi) + \frac{4}{r} \dot{\Omega} \right) \right) \\ &\quad + \underline{\mathcal{Q}}_1 - 2\dot{\Omega} + \frac{1}{2} \overset{\circ}{\operatorname{div}} \overset{\circ}{\operatorname{div}} \dot{g}_c - \frac{1}{r} \overset{\circ}{\Delta} \dot{\phi}, \end{aligned}$$

which proves the first equation of Lemma 6.3, that is,

$$\begin{aligned} \mathcal{Q}_1 - \underline{\mathcal{Q}}_1 - \frac{1}{2r} \underline{\mathcal{Q}}_2 &= -2\dot{\Omega} + \frac{1}{2} \overset{\circ}{\operatorname{div}} \overset{\circ}{\operatorname{div}} \dot{g}_c \\ &\quad - \frac{1}{r^2} \overset{\circ}{\operatorname{div}} \left(r^2 \underline{\dot{\eta}} - \frac{r^3}{2} \not{d} \left((\Omega \operatorname{tr} \chi) + \frac{4}{r} \dot{\Omega} \right) \right) - \frac{1}{r} \overset{\circ}{\Delta} \dot{\phi}. \end{aligned} \tag{6.9}$$

Moreover, plugging (6.9) into (6.8), we get

$$\begin{aligned} & - \frac{1}{2r} \left(\mathcal{Q}_2 + (\overset{\circ}{\Delta} + 1) \underline{\mathcal{Q}}_2 \right) \\ &= \overset{\circ}{\Delta} \left(\mathcal{Q}_1 - \underline{\mathcal{Q}}_1 - \frac{1}{2r} \underline{\mathcal{Q}}_2 + 2\dot{\Omega} \right) \\ &\quad - \frac{2}{r^2} \overset{\circ}{\operatorname{div}} \left(r^2 \underline{\dot{\eta}} - \frac{r^3}{2} \not{d} \left((\Omega \operatorname{tr} \chi) + \frac{4}{r} \dot{\Omega} \right) \right) - \frac{2}{r} \overset{\circ}{\Delta} \dot{\phi}, \end{aligned}$$

which proves the second equation of Lemma 6.3, that is,

$$\begin{aligned} & - \frac{1}{4r} (\mathcal{Q}_2 + \underline{\mathcal{Q}}_2) - \frac{1}{2} \overset{\circ}{\Delta} (\mathcal{Q}_1 - \underline{\mathcal{Q}}_1) \\ &= \overset{\circ}{\Delta} \dot{\Omega} - \frac{1}{r^2} \overset{\circ}{\operatorname{div}} \left(r^2 \underline{\dot{\eta}} - \frac{r^3}{2} \not{d} \left((\Omega \operatorname{tr} \chi) + \frac{4}{r} \dot{\Omega} \right) \right) - \frac{1}{r} \overset{\circ}{\Delta} \dot{\phi}. \end{aligned}$$

Analysis of $\mathcal{Q}_0$. We have that

$$\begin{aligned}\mathcal{Q}_0 + \underline{\mathcal{Q}}_0 &= \left(r^2 \dot{\eta} + \frac{r^3}{2} \not{d} \left((\Omega \text{tr} \chi) - \frac{4}{r} \dot{\Omega} \right) \right)^{[1]} + \left(-r^2 \dot{\eta} - \frac{r^3}{2} \not{d} (\Omega \text{tr} \underline{\chi}) \right)^{[1]} \\ &= r^2 \not{d} \left(\frac{r}{2} (\Omega \text{tr} \chi) - \frac{r}{2} (\Omega \text{tr} \underline{\chi}) - 2 \not{d} \dot{\Omega} \right)^{[1]} \\ &= r^2 \not{d} \left(\frac{r}{2} \left((\Omega \text{tr} \chi) - \frac{4}{r} \dot{\Omega} \right) + \frac{\dot{\phi}}{r} - \left(\frac{r}{2} \left((\Omega \text{tr} \underline{\chi}) + \frac{4}{r} \dot{\Omega} \right) - \frac{\dot{\phi}}{r} \right) + 2\dot{\Omega} - \frac{2\dot{\phi}}{r} \right)^{[1]} \\ &= r^2 \not{d} \left(\mathcal{Q}_1 - \underline{\mathcal{Q}}_1 + 2\dot{\Omega} - \frac{2\dot{\phi}}{r} \right)^{[1]}.\end{aligned}$$

On the one hand, this shows the first of (6.7), that is,

$$(\mathcal{Q}_0)_H = -(\underline{\mathcal{Q}}_0)_H.$$

On the other hand, together with (6.9) we get that

$$\begin{aligned}(\mathcal{Q}_0)_E &= -(\underline{\mathcal{Q}}_0)_E + r^2 \not{d} \left(\mathcal{Q}_1 - \underline{\mathcal{Q}}_1 + 2\dot{\Omega} - \frac{2\dot{\phi}}{r} \right)_E^{[1]} \\ &= -(\underline{\mathcal{Q}}_0)_E + r^2 \not{d} \left(\frac{1}{2r} \mathcal{Q}_2 - \frac{1}{r^2} \mathring{\text{div}} \underline{\mathcal{Q}}_0 \right)_E^{[1]} \\ &= (\underline{\mathcal{Q}}_0)_E + \frac{r}{2} \not{d} \underline{\mathcal{Q}}_2^{[1]},\end{aligned}$$

where we used that $\left(\not{d} \mathring{\text{div}} \underline{\mathcal{Q}}_0 \right)_E = -2(\underline{\mathcal{Q}}_0)_E$ due to $\underline{\mathcal{Q}}_0 = \underline{\mathcal{Q}}_0^{[1]}$ and (D.1).

This proves the second of (6.7). This finishes the proof of Lemma 6.3.

6.3. Linearized Bifurcate Characteristic Gluing

In this section, we state and solve the linearized codimension-10 bifurcate characteristic gluing problem.

Notation. To ease presentation, we do not explicitly state the corresponding higher regularity norms and assume to work in a smooth setting. In the following, let $m \geq 0$ be an integer.

Theorem 6.4 (Linearized bifurcate characteristic gluing). *Consider on spheres $S_{0,1}$ and $S_{-1,2}$, respectively, the following smooth linearized higher-order sphere data,*

$$(\dot{x}_{0,1}, \dot{\mathcal{D}}_{0,1}^{L,m}, \dot{\mathcal{D}}_{0,1}^{\underline{L},m}) \text{ and } (\dot{x}_{-1,2}, \dot{\mathcal{D}}_{-1,2}^{L,m}, \dot{\mathcal{D}}_{-1,2}^{\underline{L},m}).$$

There exist

- a smooth solution $(\dot{x}, \dot{\mathcal{D}}^{L,m}, \dot{\mathcal{D}}^{\underline{L},m})$ to the higher-order null constraint equations on $\mathcal{H}_{-1,[1,2]}$
- a smooth solution $(\dot{\underline{x}}, \dot{\underline{\mathcal{D}}}^{L,m}, \dot{\underline{\mathcal{D}}}^{\underline{L},m})$ to the higher-order null constraint equations on $\underline{\mathcal{H}}_{[-1,0],1}$,
- smooth higher-order sphere data $(\dot{x}_{-1,1}, \dot{\mathcal{D}}_{-1,1}^{L,m}, \dot{\mathcal{D}}_{-1,1}^{\underline{L},m})$ on $S_{-1,1}$,

fully matching on $S_{-1,1}$,

$$(\dot{x}, \dot{\mathcal{D}}^{L,m}, \dot{\mathcal{D}}^{\underline{L},m})|_{S_{-1,1}} = (\dot{x}, \dot{\underline{\mathcal{D}}}^{L,m}, \dot{\underline{\mathcal{D}}}^{\underline{L},m})|_{S_{-1,1}} = (\dot{x}_{-1,1}, \dot{\mathcal{D}}_{-1,1}^{L,m}, \dot{\mathcal{D}}_{-1,1}^{\underline{L},m}),$$

such that we have higher-order matching on $S_{0,1}$,

$$(\dot{x}, \dot{\mathcal{D}}^{L,m}, \dot{\mathcal{D}}^{\underline{L},m})|_{S_{0,1}} = (\dot{x}_{0,1}, \dot{\mathcal{D}}_{0,1}^{L,m}, \dot{\mathcal{D}}_{0,1}^{\underline{L},m}),$$

and higher-order matching up to the charges $\mathcal{Q}_0$ and $\mathcal{Q}_2^{[\leq 1]}$ on $S_{-1,2}$, that is,

$$\mathfrak{M}(\dot{x}|_{S_{-1,2}}) = \mathfrak{M}(\dot{x}_{-1,2}), \quad \dot{\mathcal{D}}^{L,m}|_{S_{-1,2}} = \dot{\mathcal{D}}_{-1,2}^{L,m}, \quad (6.10)$$

where $\mathfrak{M}$ denotes the matching map defined in Definition 2.11 applied to linearized sphere data, and

$$\begin{aligned} & \left(\mathcal{Q}_1, \mathcal{Q}_2^{[\geq 2]}, \mathcal{Q}_3, \dots, \mathcal{Q}_{i(m)} \right) \left((\dot{x}, \dot{\mathcal{D}}^{L,m}, \dot{\mathcal{D}}^{\underline{L},m})|_{S_{-1,2}} \right) \\ &= \left(\mathcal{Q}_1, \mathcal{Q}_2^{[\geq 2]}, \mathcal{Q}_3, \dots, \mathcal{Q}_{i(m)} \right) \left(\dot{x}_{-1,2}, \dot{\mathcal{D}}_{-1,2}^{L,m}, \dot{\mathcal{D}}_{-1,2}^{\underline{L},m} \right). \end{aligned} \quad (6.11)$$

Remarks on Theorem 6.4.

- (1) The matching (6.10) and (6.11) equals higher-order matching up to charges $\mathcal{Q}_0$ and $\mathcal{Q}_2^{[\leq 1]}$ in the following sense. Analogous to Lemma 2.12, if it holds on $S_{-1,2}$ in addition to (6.10) that

$$\left(\mathcal{Q}_0, \mathcal{Q}_2^{[\leq 1]} \right) (\dot{x}|_{S_{-1,2}}) = \left(\mathcal{Q}_0, \mathcal{Q}_2^{[\leq 1]} \right) (\dot{x}_{-1,2}),$$

then we have that

$$\dot{x}|_{S_{-1,2}} = \dot{x}_{-1,2},$$

by which we can subsequently deduce from (6.10) and (6.11) the full higher-order matching

$$(\dot{x}, \dot{\mathcal{D}}^{L,m}, \dot{\mathcal{D}}^{\underline{L},m})|_{S_{-1,2}} = (\dot{x}_{-1,2}, \dot{\mathcal{D}}_{-1,2}^{L,m}, \dot{\mathcal{D}}_{-1,2}^{\underline{L},m}).$$

In the following, we outline the proof of Theorem 6.4. We first make some remarks.

- In accordance with the definition of free data, see Sect. 2.7.1 and Remark 4.2, our degrees of freedom in the linearized gluing problem are the prescriptions of

$$\dot{\hat{\chi}} \text{ and } \dot{\hat{\Omega}} \text{ on } \underline{\mathcal{H}}_{[-1,0],1}, \quad \dot{\hat{\chi}} \text{ and } \dot{\hat{\Omega}} \text{ on } \mathcal{H}_{-1,[1,2]}. \quad (6.12)$$

- The gluing of

$$\dot{\Omega}, \dot{\omega}, \dot{\hat{\chi}}, \dot{\alpha}, \mathcal{D}^{L,m} \text{ along } \mathcal{H}_{-1,[1,2]} \quad (6.13)$$

and of

$$\dot{\Omega}, \dot{\omega}, \dot{\hat{\chi}}, \dot{\alpha}, \mathcal{D}^{L,m} \text{ along } \underline{\mathcal{H}}_{[-1,0],1} \quad (6.14)$$

follows without obstructions from the degrees of freedom (6.12). Indeed, see the discussions concerning the linearized higher-order L -gluing along $\mathcal{H}_{-1,[1,2]}$ in Remark 4.3, which also generalizes to higher-order $\underline{L}$ -gluing along $\underline{\mathcal{H}}_{[-1,0],1}$.

- Given linearized higher-order sphere data $(\dot{x}_{0,1}, \dot{\mathcal{D}}_{0,1}^{L,m}, \dot{\mathcal{D}}_{0,1}^{\underline{L},m})$ on $S_{0,1}$, the conservation laws along $\mathcal{H}_{[-1,0],1}$ determine the charges $\mathcal{Q}_i$, $0 \leq i \leq i(m)$ on $S_{-1,1}$. Similarly, given linearized higher-order sphere data $(\dot{x}_{-1,2}, \dot{\mathcal{D}}_{-1,2}^{L,m}, \dot{\mathcal{D}}_{-1,2}^{\underline{L},m})$ on $S_{-1,2}$, the conservation laws along $\mathcal{H}_{-1,[1,2]}$ determine the charges $\mathcal{Q}_i$, $0 \leq i \leq i(m)$, on $S_{-1,1}$. The latter are denoted in the following by

$$(\mathcal{Q}_i)_0 \text{ for } 0 \leq i \leq i(m). \quad (6.15)$$

In the following, we prove Theorem 6.4 in two steps.

- (1) We construct a solution $(\dot{x}, \dot{\mathcal{D}}^{L,m}, \dot{\mathcal{D}}^{\underline{L},m})$ to the linearized higher-order null constraint equations on $\mathcal{H}_{[-1,0],1}$ such that on $S_{0,1}$ it fully matches the given higher-order sphere data, and on $S_{-1,1}$ we have the charge matching

$$\begin{aligned} & \left(\mathcal{Q}_1, \mathcal{Q}_2^{[\geq 2]}, \mathcal{Q}_3, \dots, \mathcal{Q}_{i(m)} \right) (\dot{x}, \dot{\mathcal{D}}^{L,m}, \dot{\mathcal{D}}^{\underline{L},m})|_{S_{-1,1}} \\ &= \left((\mathcal{Q}_1)_0, \left(\mathcal{Q}_2^{[\geq 2]} \right)_0, (\mathcal{Q}_3)_0, \dots, (\mathcal{Q}_{i(m)})_0 \right), \end{aligned} \quad (6.16)$$

where the right-hand side charges are defined in (6.15). This is the content of Proposition 6.5.

- (2) We construct a solution $(\dot{x}, \dot{\mathcal{D}}^{L,m}, \dot{\mathcal{D}}^{\underline{L},m})$ to the linearized higher-order null constraint equations on $\mathcal{H}_{-1,[1,2]}$ such that on $S_{-1,1}$ it fully matches the given higher-order sphere data, and on $S_{-1,2}$ we have the matching

$$\mathfrak{M}(\dot{x}|_{S_{-1,2}}) = \mathfrak{M}(\dot{x}_{-1,2}), \quad \dot{\mathcal{D}}^{L,m}|_{S_{-1,2}} = \dot{\mathcal{D}}_{-1,2}^{L,m}.$$

By the charge matching (6.16) and conservation laws along $\mathcal{H}_{-1,[1,2]}$, it follows that the condition (6.11) is satisfied on $S_{-1,2}$.

The above two steps can be seen as generalization of Sects. 4.4.1 and 4.4.2, respectively.

Proposition 6.5 (Characteristic gluing along $\mathcal{H}_{[-1,0],1}$ with charge matching on $S_{-1,1}$). *Let $(\dot{x}_{0,1}, \dot{\mathcal{D}}_{0,1}^{L,m}, \dot{\mathcal{D}}_{0,1}^{\underline{L},m})$ be given linearized higher-order sphere data on $S_{0,1}$, and let*

$$\left((\mathcal{Q}_1)_0, \left(\mathcal{Q}_2^{[\geq 2]} \right)_0, (\mathcal{Q}_3)_0, \dots, (\mathcal{Q}_{i(m)})_0 \right) \quad (6.17)$$

be a given tuple of charge values on $S_{-1,1}$. There exists higher-order ingoing null data $(\dot{x}, \dot{\mathcal{D}}^{L,m}, \dot{\mathcal{D}}^{\underline{L},m})$ on $\mathcal{H}_{[-1,0],1}$ solving the linearized higher-order null constraint equations such that

$$(\dot{x}, \dot{\mathcal{D}}^{L,m}, \dot{\mathcal{D}}^{\underline{L},m})|_{S_{0,1}} = (\dot{x}_{0,1}, \dot{\mathcal{D}}_{0,1}^{L,m}, \dot{\mathcal{D}}_{0,1}^{\underline{L},m}),$$

and

$$\begin{aligned} & \left(\mathcal{Q}_1, \mathcal{Q}_2^{[\geq 2]}, \mathcal{Q}_3, \dots, \mathcal{Q}_{i(m)} \right) ((\dot{x}, \dot{\mathcal{D}}^{L,m}, \dot{\mathcal{D}}^{\underline{L},m})|_{S_{-1,1}}) \\ &= \left((\mathcal{Q}_1)_0, \left(\mathcal{Q}_2^{[\geq 2]} \right)_0, (\mathcal{Q}_3)_0, \dots, (\mathcal{Q}_{i(m)})_0 \right). \end{aligned}$$

Remarks on Proposition 6.5.

- (1) The charges $\mathcal{Q}_0$ and $\mathcal{Q}_2^{[\leq 1]}$ can in general not be matched on $S_{-1,1}$ because by Lemma 6.3, they are determined from $\underline{\mathcal{Q}}_0$ and $\underline{\mathcal{Q}}_2^{[\leq 1]}$ which are in turn determined from $\dot{x}_{0,1}$ by the conservation laws on $\mathcal{H}_{[-1,0],1}$.
- (2) The sphere data $\dot{x}|_{S_{-1,1}}$ are a priori not fully determined by the matching conditions of Proposition 6.5, and thus, our construction admits some freedom of choice.

Proof of Proposition 6.5. First consider the matching

$$\left(\mathcal{Q}_1, \mathcal{Q}_2^{[\geq 2]}\right)(\dot{x}|_{S_{-1,1}}) = \left((\mathcal{Q}_1)_0, \left(\mathcal{Q}_2^{[\geq 2]}\right)_0\right). \quad (6.18)$$

By Lemma 6.3, we have to solve the following system on $S_{-1,1}$,

$$\begin{aligned} (\mathcal{Q}_1)_0 - \frac{1}{2r}\mathcal{Q}_2 - \underline{\mathcal{Q}}_1 &= -2\dot{\Omega} + \frac{1}{2}\mathring{\mathrm{d}}\dot{\chi}\mathring{\mathrm{d}}\dot{\chi}\dot{g}_c \\ &\quad - \frac{1}{r^2}\mathring{\mathrm{d}}\dot{\chi}\left(r^2\dot{\eta} - \frac{r^3}{2}\not\!d\left((\Omega\mathrm{tr}\underline{\chi}) + \frac{4}{r}\dot{\Omega}\right)\right) - \frac{1}{r}\mathring{\Delta}\dot{\phi}, \end{aligned} \quad (6.19)$$

and

$$\begin{aligned} &-\frac{1}{4r}\left(\left(\mathcal{Q}_2^{[\geq 2]}\right)_0 + \underline{\mathcal{Q}}_2^{[\geq 2]}\right) - \frac{1}{2}\mathring{\Delta}\left((\mathcal{Q}_1)_0 - \underline{\mathcal{Q}}_1\right)^{[\geq 2]} \\ &= \mathring{\Delta}\dot{\Omega}^{[\geq 2]} - \frac{1}{r^2}\mathring{\mathrm{d}}\dot{\chi}\left(r^2\dot{\eta} - \frac{r^3}{2}\not\!d\left((\Omega\mathrm{tr}\underline{\chi}) + \frac{4}{r}\dot{\Omega}\right)\right)^{[\geq 2]} - \frac{1}{r}\mathring{\Delta}\dot{\phi}^{[\geq 2]}, \end{aligned} \quad (6.20)$$

where the charges $\underline{\mathcal{Q}}_1$ and $\underline{\mathcal{Q}}_2$ are determined on $S_{-1,1}$ from the sphere data $\dot{x}_{0,1}$ on $S_{0,1}$.

We recall from (6.6) that the quantities

$$\dot{\Omega}, \dot{\phi}, \dot{g}_c, \left(r^2\dot{\eta} - \frac{r^3}{2}\not\!d\left((\Omega\mathrm{tr}\underline{\chi}) + \frac{4}{r}\dot{\Omega}\right)\right)^{[\geq 2]}$$

on the right-hand side of (6.19) and (6.20) are glueable along $\mathcal{H}_{[-1,0],1}$ and can thus be freely prescribed on $S_{-1,1}$. In the following, we show in detail how to prescribe them such that (6.19) and (6.20) are satisfied.

Matching of modes $l = 0$. By projecting (6.19) onto the modes $l = 0$, we get

$$(\mathcal{Q}_1)_0^{[0]} = \frac{1}{2r}\mathcal{Q}_2^{[0]} + \underline{\mathcal{Q}}_1^{[0]} - 2\dot{\Omega}^{[0]}.$$

Hence, we prescribe on $S_{-1,1}$,

$$\dot{\Omega}^{[0]} = \frac{1}{2}\left(-\mathcal{Q}_1^{[0]} + \frac{1}{2r}\mathcal{Q}_2^{[0]} + \underline{\mathcal{Q}}_1^{[0]}\right), \quad \dot{\phi}^{[0]} = 0.$$

Matching of modes $l = 1$. By projecting (6.19) onto the modes $l = 1$, we get

$$\begin{aligned} (\mathcal{Q}_1)_0^{[1]} &= \frac{1}{2r}\mathcal{Q}_2^{[1]} - \frac{1}{r^2}\mathring{\mathrm{d}}\dot{\chi}\left(r^2\dot{\eta} - \frac{r^3}{2}\not\!d\left((\Omega\mathrm{tr}\underline{\chi}) + \frac{4}{r}\dot{\Omega}\right)\right)^{[1]} \\ &\quad + \underline{\mathcal{Q}}_1^{[1]} - 2\dot{\Omega}^{[1]} + \frac{2}{r}\dot{\phi}^{[1]} \\ &= \frac{1}{2r}\mathcal{Q}_2^{[1]} - \frac{1}{r^2}\mathring{\mathrm{d}}\dot{\chi}\underline{\mathcal{Q}}_0 + \underline{\mathcal{Q}}_1^{[1]} - 2\dot{\Omega}^{[1]} + \frac{2}{r}\dot{\phi}^{[1]}. \end{aligned}$$

Hence, we prescribe on $S_{-1,1}$,

$$\dot{\Omega}^{[1]} = 0, \quad \dot{\phi}^{[1]} = \frac{r}{2} \left((\mathcal{Q}_1)_0^{[1]} - \frac{1}{2r} \underline{\mathcal{Q}}_2^{[1]} + \frac{1}{r^2} \mathring{\text{div}} \underline{\mathcal{Q}}_0 - \underline{\mathcal{Q}}_1^{[1]} \right).$$

Matching of modes $l \geq 2$. By projecting (6.19) and (6.20) onto the modes $l \geq 2$, we get

$$\begin{aligned} & (\mathcal{Q}_1)_0^{[\geq 2]} - \frac{1}{2r} \underline{\mathcal{Q}}_2^{[\geq 2]} - \underline{\mathcal{Q}}_1^{[\geq 2]} \\ &= -2\dot{\Omega}^{[\geq 2]} + \frac{1}{2} \mathring{\text{div}} \mathring{\text{div}} \dot{\mathcal{J}}_c^{[\geq 2]} \\ &\quad - \frac{1}{r^2} \mathring{\text{div}} \left(r^2 \dot{\eta} - \frac{r^3}{2} \not\partial \left((\Omega \text{tr} \chi) + \frac{4}{r} \dot{\Omega} \right) \right)^{[\geq 2]} - \frac{1}{r} \mathring{\Delta} \dot{\phi}^{[\geq 2]}, \end{aligned}$$

and

$$\begin{aligned} & -\frac{1}{4r} \left((\underline{\mathcal{Q}}_2^{[\geq 2]})_0 + \underline{\mathcal{Q}}_2^{[\geq 2]} \right) - \frac{1}{2} \mathring{\Delta} ((\mathcal{Q}_1)_0 - \underline{\mathcal{Q}}_1)^{[\geq 2]} \\ &= \mathring{\Delta} \dot{\Omega}^{[\geq 2]} - \frac{1}{r^2} \mathring{\text{div}} \left(r^2 \dot{\eta} - \frac{r^3}{2} \not\partial \left((\Omega \text{tr} \chi) + \frac{4}{r} \dot{\Omega} \right) \right)^{[\geq 2]} - \frac{1}{r} \mathring{\Delta} \dot{\phi}^{[\geq 2]}, \end{aligned}$$

It is straightforward to check that the following prescription on $S_{-1,1}$ is a solution of the above system,

$$\dot{\Omega}^{[\geq 2]} = 0, \quad \left(r^2 \dot{\eta} - \frac{r^3}{2} \not\partial \left((\Omega \text{tr} \chi) + \frac{4}{r} \dot{\Omega} \right) \right)^{[\geq 2]} = 0,$$

with $\dot{\phi}^{[\geq 2]}$ and $\dot{\mathcal{J}}_c^{[\geq 2]}$ defined subsequently on $S_{-1,1}$ as solutions to

$$\begin{aligned} & -\frac{1}{r} \mathring{\Delta} \dot{\phi}^{[\geq 2]} = -\frac{1}{4r} \left((\underline{\mathcal{Q}}_2^{[\geq 2]})_0 + \underline{\mathcal{Q}}_2^{[\geq 2]} \right) - \frac{1}{2} \mathring{\Delta} ((\mathcal{Q}_1)_0 - \underline{\mathcal{Q}}_1)^{[\geq 2]}, \\ & \frac{1}{2} \mathring{\text{div}} \mathring{\text{div}} \dot{\mathcal{J}}_c^{[\geq 2]} = (\mathcal{Q}_1)_0^{[\geq 2]} - \frac{1}{2r} \underline{\mathcal{Q}}_2^{[\geq 2]} - \underline{\mathcal{Q}}_1^{[\geq 2]} + \frac{1}{r} \mathring{\Delta} \dot{\phi}^{[\geq 2]}, \\ & (\mathring{\text{div}} \dot{\mathcal{J}}_c)_H = 0. \end{aligned}$$

Using that the Laplacian and div-curl are elliptic Hodge systems, see Appendix D, it is straightforward to prove regularity estimates for $\dot{\phi}$ and $\dot{\mathcal{J}}_c$ which show that the above construction is consistent with the regularity hierarchy of the linearized null constraint equations. This proves the matching (6.18) of $\mathcal{Q}_1$ and $\mathcal{Q}_2^{[\geq 2]}$.

It remains to realize the matching

$$(\mathcal{Q}_3, \dots, \mathcal{Q}_{i(m)}) (\dot{x}|_{S_{-1,1}}) = ((\mathcal{Q}_3)_0, \dots, (\mathcal{Q}_{i(m)})_0). \quad (6.21)$$

First, from (4.5) it follows that the matching condition

$$(\mathcal{Q}_3, \dots, \mathcal{Q}_7) (\dot{x}|_{S_{-1,1}}) = ((\mathcal{Q}_3)_0, \dots, (\mathcal{Q}_7)_0), \quad (6.22)$$

can be realized by an appropriate choice of

$$\hat{\chi}, \dot{\alpha}, \dot{\omega}, \underline{D}\dot{\omega} \text{ on } S_{-1,1}. \quad (6.23)$$

By (6.14), the quantities in (6.23) can be glued without obstructions along $\mathcal{H}_{[-1,0],1}$ and can thus be freely prescribed and realized on $S_{-1,1}$. This proves the matching (6.22).

Second, the matching of higher-order charges

$$(\mathcal{Q}_8, \dots, \mathcal{Q}_{i(m)}) (\dot{x}_{|S_{-1,1}}) = ((\mathcal{Q}_8)_0, \dots, (\mathcal{Q}_{i(m)})_0) \quad (6.24)$$

follows analogously to the matching (6.22) by the free prescription of higher $\underline{D}$ -derivatives of $\dot{\underline{\alpha}}$ and $\underline{D}\dot{\underline{\omega}}$ on $S_{-1,1}$, in other words, the prescription of $\mathcal{D}^{\underline{L},m}$ on $S_{-1,1}$. Indeed, it is straightforward to check that the charge expressions contain top-order $\underline{D}$ -derivatives of $\dot{\underline{\alpha}}$ and $\underline{D}\dot{\underline{\omega}}$. This is due to the formulas being derived by commuting the null structure equations with $\underline{D}$ -derivatives, where it is well known that the commutator $[D, \underline{D}]$ is sphere-tangent (i.e., not causing D -derivatives) and higher-order $\underline{D}$ -derivatives of metric coefficients, Ricci coefficients and null curvature components can be expressed as higher-order $\underline{D}$ -derivatives of $\dot{\underline{\alpha}}$ and $\underline{D}\dot{\underline{\omega}}$ plus angular derivatives of lower-order $\underline{D}$ -derivatives by the null structure equations. To conclude, the ability to glue along $\mathcal{H}_{[-1,0],1}$ to such prescribed $\mathcal{D}^{\underline{L},m}$ on $S_{-1,1}$ follows directly from (6.14). This finishes the proof of Proposition 6.5. $\square$

We are now in position to conclude the proof of Theorem 6.4 by step (2) outlined before. We recall that from Proposition 6.5 we have a solution

$$(\dot{x}, \dot{\underline{D}}^{\underline{L},m}, \dot{\underline{D}}^{\underline{L},m}) \text{ on } \mathcal{H}_{[-1,0],1}$$

to the linearized higher-order null constraint equations such that we have full higher-order matching on $S_{0,1}$ and the following charge matching on $S_{-1,1}$,

$$\begin{aligned} & \left(\mathcal{Q}_1, \mathcal{Q}_2^{[\geq 2]}, \mathcal{Q}_3, \dots, \mathcal{Q}_{i(m)} \right) \left((\dot{x}, \dot{\underline{D}}^{\underline{L},m}, \dot{\underline{D}}^{\underline{L},m})|_{S_{-1,1}} \right) \\ &= \left((\mathcal{Q}_1)_0, (\mathcal{Q}_2^{[\geq 2]})_0, (\mathcal{Q}_3)_0, \dots, (\mathcal{Q}_{i(m)})_0 \right), \end{aligned}$$

where the right-hand side are the conserved charges determined by conservation laws along $\mathcal{H}_{-1,[1,2]}$ from the higher-order sphere data on $S_{-1,2}$,

$$(\dot{x}_{-1,2}, \dot{\mathcal{D}}_{-1,2}^{\underline{L},m}, \dot{\mathcal{D}}_{-1,2}^{\underline{L},m}).$$

By applying the characteristic gluing of Sect. 4 along $\mathcal{H}_{-1,[1,2]}$, we construct a solution

$$(\dot{x}, \dot{\mathcal{D}}^{\underline{L},m}, \dot{\mathcal{D}}^{\underline{L},m}) \text{ on } \mathcal{H}_{-1,[1,2]}$$

to the linearized higher-order null constraint equations such that we have full higher-order matching on $S_{-1,1}$ and we have the following higher-order matching on $S_{-1,2}$,

$$\mathfrak{M}(\dot{x}_{|S_{-1,2}}) = \mathfrak{M}(\dot{x}_{-1,2}), \quad \dot{\mathcal{D}}^{\underline{L},m}|_{S_{-1,2}} = \dot{\mathcal{D}}_{-1,2}^{\underline{L},m},$$

and

$$\begin{aligned} & \left(\mathcal{Q}_1, \mathcal{Q}_2^{[\geq 2]}, \mathcal{Q}_3, \dots, \mathcal{Q}_{i(m)} \right) \left((\dot{x}, \dot{\mathcal{D}}^{\underline{L},m}, \dot{\mathcal{D}}^{\underline{L},m})|_{S_{-1,2}} \right) \\ &= \left(\mathcal{Q}_1, \mathcal{Q}_2^{[\geq 2]}, \mathcal{Q}_3, \dots, \mathcal{Q}_{i(m)} \right) \left(\dot{x}_{-1,2}, \dot{\mathcal{D}}_{-1,2}^{\underline{L},m}, \dot{\mathcal{D}}_{-1,2}^{\underline{L},m} \right). \end{aligned}$$

The following remark on the linearized charges finishes the proof of Theorem 6.4.

Remark 6.6 (Relations for $\mathcal{Q}_0$ and $\mathcal{Q}_2^{[\leq 1]}$). By Lemma 6.3, that is, the relations

$$(\mathcal{Q}_0)_H = -(\underline{\mathcal{Q}}_0)_H, \quad (\mathcal{Q}_0)_E = (\underline{\mathcal{Q}}_0)_E + \frac{r}{2} \not{d} \underline{\mathcal{Q}}_2^{[1]}, \quad \mathcal{Q}_2^{[0]} = -\underline{\mathcal{Q}}_2^{[0]}, \quad \mathcal{Q}_2^{[1]} = \underline{\mathcal{Q}}_2^{[1]}. \quad (6.25)$$

and the conservation laws along $\mathcal{H}_{[-1,0],1}$ and $\mathcal{H}_{-1,[1,2]}$, we have that

$$((\mathcal{Q}_0)_H, \mathcal{Q}_2^{[0]}, \mathcal{Q}_2^{[1]})(\dot{x}|_{S_{-1,2}}) = ((\mathcal{Q}_0)_H, \mathcal{Q}_2^{[0]}, \mathcal{Q}_2^{[1]})(\dot{x}_{0,1}).$$

Moreover, from (6.25) and the conservation laws we deduce that

$$\begin{aligned} (\mathcal{Q}_0)_E(\dot{x}|_{S_{-1,2}}) &= (\mathcal{Q}_0)_E(\dot{x}|_{S_{-1,1}}) \\ &= (\underline{\mathcal{Q}}_0)_E(\dot{x}|_{S_{-1,1}}) + \frac{2}{2} \not{d} \underline{\mathcal{Q}}_2^{[1]}(\dot{x}|_{S_{-1,1}}) \\ &= (\underline{\mathcal{Q}}_0)_E(\dot{x}|_{S_{-1,1}}) + \frac{2}{2} \not{d} \underline{\mathcal{Q}}_2^{[1]}(\dot{x}|_{S_{-1,1}}) \\ &= (\underline{\mathcal{Q}}_0)_E(\dot{x}|_{S_{0,1}}) + \frac{2}{2} \not{d} \underline{\mathcal{Q}}_2^{[1]}(\dot{x}|_{S_{0,1}}) \\ &= \left((\underline{\mathcal{Q}}_0)_E(\dot{x}|_{S_{0,1}}) + \frac{1}{2} \not{d} \underline{\mathcal{Q}}_2^{[1]}(\dot{x}|_{S_{0,1}}) \right) + \frac{1}{2} \not{d} \underline{\mathcal{Q}}_2^{[1]}(\dot{x}|_{S_{0,1}}) \\ &= (\mathcal{Q}_0)_E(\dot{x}_{0,1}) + \frac{1}{2} \not{d} \mathcal{Q}_2^{[1]}(\dot{x}_{0,1}), \end{aligned}$$

where we used that by construction, $\dot{x}|_{S_{0,1}} = \dot{x}_{0,1}$. Using (D.1) we can rewrite the above as, for $m = -1, 0, 1$,

$$(\mathcal{Q}_0)_E^{(1m)}(\dot{x}|_{S_{-1,2}}) = (\mathcal{Q}_0)_E^{(1m)}(\dot{x}_{0,1}) - \frac{\sqrt{2}}{2} \mathcal{Q}_2^{(1m)}(\dot{x}_{0,1}).$$

In terms of $\dot{\mathbf{P}}$ and $\dot{\mathbf{G}}$ (see Remark (4.6)), this can be written as

$$\dot{\mathbf{G}}^m(\dot{x}|_{S_{-1,2}}) = \dot{\mathbf{G}}^m(\dot{x}|_{S_{0,1}}) - 2\dot{\mathbf{P}}^m(\dot{x}|_{S_{0,1}}).$$

6.4. Proof of Proposition 3.4

In this section, we prove Proposition 3.4, that is, the bifurcate characteristic gluing with localized sphere data perturbation W .

The proof is a slight generalization of the proof of Theorem 3.3. As before, by the implicit function theorem, the proof can be reduced to solving the linearized problem. In this case, the linearized characteristic gluing problem admits the additional freedom of adding a linearized localized sphere data perturbation $\dot{W}$ to the linearized sphere data on $S_{-1,2}$.

Given the explicit formulas in Sect. 6.1, it is straightforward to construct a smooth sphere data perturbation $\dot{W}$, compactly supported in the prescribed angular region $K \subset S_{-1,2}$ (see Proposition 3.4), with prescribed values for $\mathcal{Q}_0(\dot{W})$ and $\mathcal{Q}_2^{[\leq 1]}(\dot{W})$, and derive appropriate bounds; see, for example, the explicit choice in [11]. This allows to match $\mathcal{Q}_0$ and $\mathcal{Q}_2^{[\leq 1]}$ on $S_{-1,2}$.

Adding $\dot{W}$ to the linearized sphere data on $S_{-1,2}$ changes the gauge-dependent charges on $S_{-1,2}$ (by a well-controlled amount). Using the results

of Sect. 6.3 for the linearized characteristic gluing along two transversely intersecting null hypersurfaces, we can match the solution to the null constraint equations to these new values of gauge-dependent charges on $S_{-1,2}$. This solves the linearized characteristic gluing problem for Proposition 3.4.

We remark that for controlling the support of W in the application of the implicit function theorem, W is bounded in an L^2 -based Sobolev space with weights (in particular, this space is Hilbert) which ensure its smooth vanishing toward the boundary of the angular region K , see, for example, [24, 25].

Acknowledgements

S.A. acknowledges support through the NSERC grant 502581 and the Ontario Early Researcher Award. S.C. acknowledges support through the NSF grant DMS-1439786 of the Institute for Computational and Experimental Research in Mathematics (ICERM). I.R. acknowledges support through NSF grants DMS-2005464, DMS-1709270 and a Simons Investigator Award.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.

Appendix A: Perturbations of Sphere Data

In this section, we prove Proposition 2.21; that is, we show that

$$\begin{aligned} \mathcal{P} : \mathcal{X}^+(\tilde{\mathcal{H}}_{[-\delta, \delta], 2}) \times \mathcal{Y}_f \times \mathcal{Y}_q &\rightarrow \mathcal{X}(S_{0,2}), \\ (\tilde{x}, f, q) &\mapsto x_{0,2} := \mathcal{P}_{f,q}(\tilde{x}), \end{aligned}$$

is well defined and smooth in an open neighborhood of $(\tilde{x}, f, q) = (\underline{\mathbf{m}}, 0, 0)$, and satisfies the estimate

$$\|\mathcal{P}_{f,q}(\tilde{x}) - \tilde{x}_{0,2}\|_{\mathcal{X}(S_{0,2})} \lesssim \|f\|_{\mathcal{Y}_f} + \|q\|_{\mathcal{Y}_q} + \|\tilde{x} - \underline{\mathbf{m}}\|_{\mathcal{X}^+(\tilde{\mathcal{H}}_{[-\delta, \delta], 2})},$$

where we denoted $\tilde{x}_{0,2} := \tilde{x}|_{S_{0,2}}$.

In Sect. A.1, we derive explicit expressions for the sphere data $\mathcal{P}_{f,0}(\tilde{x})$. In Sect. A.2, we prove Proposition 2.21; that is, we analyze $\mathcal{P}_{f,q}(\tilde{x})$.

A.1. Explicit Formulas for Transversal Sphere Perturbations

In the following, we rigorously set up *transversal perturbations* $\mathcal{P}_{f,0}$ and write out explicit formulas for the resulting sphere data. In Sect. A.1.1, we recapitulate the null geometry setting. In Sect. A.1.2, we define sphere perturbations and analyze metric coefficients. In Sects. A.1.3 and A.1.4, we analyze Ricci coefficients and null curvature components, respectively.

A.1.1. Null Geometry. First we recall the null geometry setup. Let $\tilde{S}$ be a spacelike 2-sphere in a spacetime $(\mathcal{M}, \mathbf{g})$. Let $(\tilde{u}, \tilde{v}, \tilde{\theta}^1, \tilde{\theta}^2)$ be a local double null coordinate system around $\tilde{S}$, that is,

$$\mathbf{g} = -4\tilde{\Omega}^2 d\tilde{u}d\tilde{v} + \tilde{g}_{CD}(d\tilde{\theta}^C - \tilde{b}^C d\tilde{v})(d\tilde{\theta}^D - \tilde{b}^D d\tilde{v}), \quad (\text{A.1})$$

such that $\tilde{S} = \tilde{S}_{0,2} := \{\tilde{u} = 0, \tilde{v} = 2\}$. We recall the following standard notation, see, for example, Sect. 1 of [15].

- The *geodesic null vectorfields* are defined by

$$\tilde{L}' := -2\mathbf{D}\tilde{v}, \quad \tilde{L}' := -2\mathbf{D}\tilde{u}, \quad (\text{A.2})$$

where $\mathbf{D}$ denotes the covariant derivative on $(\mathcal{M}, \mathbf{g})$.

- The *normalized null vectorfields* are defined by

$$\hat{\tilde{L}} := \Omega \tilde{L}', \quad \hat{\underline{\tilde{L}}} := \Omega \underline{\tilde{L}}'.$$

- The *equivariant null vectorfields* are defined by

$$\tilde{L} := \tilde{\Omega}^2 L', \quad \underline{\tilde{L}} := \tilde{\Omega}^2 \underline{L}'. \quad (\text{A.3})$$

- The Ricci coefficients are defined with respect to the above vectorfields as follows,

$$\begin{aligned} \tilde{\chi}_{AB} &:= \mathbf{g}(\mathbf{D}_{\tilde{A}} \hat{\tilde{L}}, \partial_{\tilde{B}}), & \tilde{\chi}_{AB} &:= \mathbf{g}(\mathbf{D}_{\tilde{A}} \hat{\underline{\tilde{L}}}, \partial_{\tilde{B}}), & \tilde{\zeta}_A &:= \frac{1}{2} \mathbf{g}(\mathbf{D}_{\tilde{A}} \hat{\tilde{L}}, \hat{\underline{\tilde{L}}}), \\ \tilde{\eta} &:= \tilde{\zeta} + \tilde{\mathcal{L}} \log \tilde{\Omega}, & \tilde{\omega} &:= \tilde{L} \log \tilde{\Omega}, & \tilde{\omega} &:= \underline{\tilde{L}} \log \tilde{\Omega}, \end{aligned} \quad (\text{A.4})$$

where $\tilde{\mathcal{L}}$ denotes the exterior derivative on spheres $\tilde{S}_{u,v}$.

We have the following practical lemma, see, for example, [15].

Lemma A.1 (Properties of double null coordinates). *The following holds.*

- (1) The inverse $\mathbf{g}^{-1}$ of (A.1) is given by

$$\mathbf{g}^{-1} = -\frac{1}{2\tilde{\Omega}^2} (\partial_{\tilde{u}} \otimes \partial_{\tilde{v}} + \partial_{\tilde{v}} \otimes \partial_{\tilde{u}}) - \frac{\tilde{b}^C}{2\tilde{\Omega}^2} (\partial_{\tilde{u}} \otimes \partial_{\tilde{C}} + \partial_{\tilde{C}} \otimes \partial_{\tilde{u}}) + \tilde{g}^{AB} \partial_{\tilde{A}} \otimes \partial_{\tilde{B}}. \quad (\text{A.5})$$

Specifically,

$$\mathbf{g}^{\tilde{v}\tilde{v}} = \mathbf{g}^{\tilde{v}\tilde{A}} = 0. \quad (\text{A.6})$$

- (2) It holds that $\mathbf{g}(L', \underline{L}') = -2\tilde{\Omega}^{-2}$, and

$$\tilde{L} = \partial_{\tilde{v}} + \tilde{b}^A \partial_{\tilde{\theta}^A}, \quad \underline{\tilde{L}} = \partial_{\tilde{u}}. \quad (\text{A.7})$$

- (3) It holds that for $A = 1, 2$,

$$\partial_{\tilde{u}} \tilde{b}^A = 4\tilde{\Omega}^2 \tilde{\zeta}^A. \quad (\text{A.8})$$

- (4) It holds that

$$\Gamma_{\tilde{u}\tilde{u}}^{\tilde{v}} = \Gamma_{\tilde{u}\tilde{v}}^{\tilde{v}} = \Gamma_{\tilde{u}\tilde{A}}^{\tilde{v}} = 0, \quad \Gamma_{\tilde{A}\tilde{v}}^{\tilde{v}} = \partial_{\tilde{A}} \log \tilde{\Omega} - \tilde{\zeta}_A - \frac{1}{2\tilde{\Omega}} \tilde{\chi}_{AB} \tilde{b}^B, \quad \Gamma_{\tilde{A}\tilde{B}}^{\tilde{v}} = \frac{1}{2\tilde{\Omega}} \tilde{\chi}_{AB}, \quad (\text{A.9})$$

where the Christoffel symbols are defined by $\Gamma_{\mu\nu}^{\gamma} := \frac{1}{2}\mathbf{g}^{\gamma\alpha}(\partial_{\mu}\mathbf{g}_{\alpha\nu} + \partial_{\nu}\mathbf{g}_{\alpha\mu} - \partial_{\alpha}\mathbf{g}_{\mu\nu})$.

A.1.2. Definition of u on $\tilde{\mathcal{H}}_2$ and Analysis of Foliation Geometry. In the following, we change $\tilde{u}$ to u on $\tilde{\mathcal{H}}_2 := \{\tilde{v} = 2\}$ and analyze how the foliation geometry of the resulting local double null coordinates $(u, v, \theta^1, \theta^2)$ (with $v = \tilde{v}$ on $\mathcal{M}$) relates to the foliation geometry of the local double null coordinates $(\tilde{u}, \tilde{v}, \tilde{\theta}^1, \tilde{\theta}^2)$.

For a given scalar function $f = f(u, \theta^1, \theta^2)$, define (u, θ^1, θ^2) on $\tilde{\mathcal{H}}_2$ by

$$\tilde{u} = u + f(u, \theta^1, \theta^2), \quad \tilde{\theta}^1 = \theta^1, \quad \tilde{\theta}^2 = \theta^2. \quad (\text{A.10})$$

For f sufficiently small, (u, θ^1, θ^2) are a coordinate system on $\tilde{\mathcal{H}}_2$ and we have that

$$\partial_u = (1 + \partial_u f) \partial_{\tilde{u}}, \quad \partial_{\theta^A} = \partial_{\tilde{\theta}^A} + (\partial_{\tilde{\theta}^A} f) \partial_{\tilde{u}}, \quad \partial_{\tilde{\theta}^A} f = (1 + \partial_u f) \partial_{\tilde{\theta}^A} f. \quad (\text{A.11})$$

In accordance with (A.2) and (A.7), define on $\tilde{\mathcal{H}}_2$

$$\underline{L} := \partial_u, \quad \underline{L}' := -2\mathbf{D}\tilde{v} = \tilde{L}', \quad (\text{A.12})$$

and define in accordance with (A.3) the null lapse Ω on $\tilde{\mathcal{H}}_2$ through the relation

$$\underline{L} = \Omega^2 \underline{L}'. \quad (\text{A.13})$$

We can relate the foliation geometry of (u, θ^1, θ^2) to the geometry of $(\tilde{u}, \tilde{\theta}^1, \tilde{\theta}^2)$ as follows.

- (1) We explicitly calculate Ω on $\tilde{\mathcal{H}}_2$ as follows. Using (A.3), (A.10), (A.11) and (A.12), it holds that on $\{\tilde{v} = 2\}$,

$$\underline{L} = (1 + \partial_u f) \tilde{L} = (1 + \partial_u f) \tilde{\Omega}^2 \tilde{L}' = (1 + \partial_u f) \tilde{\Omega}^2 \underline{L}', \quad (\text{A.14})$$

from which we conclude by (A.13) that on $\tilde{\mathcal{H}}_2$,

$$\Omega^2 = \tilde{\Omega}^2 (1 + \partial_u f). \quad (\text{A.15})$$

- (2) By (A.11), it follows that the induced metric $\mathcal{g}$ on level sets of u on $\tilde{\mathcal{H}}_2$ is given for $A, B = 1, 2$ by

$$\mathcal{g}_{AB} := \mathbf{g}(\partial_A, \partial_B) = \mathbf{g}(\partial_{\tilde{A}}, \partial_{\tilde{B}}) = \tilde{\mathcal{g}}_{AB}. \quad (\text{A.16})$$

This implies further that

$$\mathcal{g}^{AB} = \tilde{\mathcal{g}}^{AB}. \quad (\text{A.17})$$

We remark that in explicit notation, (A.15) and (A.16) are

$$\begin{aligned} \Omega^2(u, \theta^1, \theta^2) &= (1 + (\partial_u f)(u, \theta^1, \theta^2)) \tilde{\Omega}^2(u + f(u, \theta^1, \theta^2), \theta^1, \theta^2), \\ \mathcal{g}_{AB}(u, \theta^1, \theta^2) &= \tilde{\mathcal{g}}_{AB}(u + f(u, \theta^1, \theta^2), \theta^1, \theta^2). \end{aligned}$$

- (3) The vectorfield $\underline{L}$ and the scalar function Ω uniquely determine the null vectorfield L on $\{\bar{v} = 2\}$ defined by

$$\mathbf{g}(L, \underline{L}) = -2\Omega^2, \quad \mathbf{g}(L, \partial_{\theta^1}) = \mathbf{g}(L, \partial_{\theta^2}) = 0. \quad (\text{A.18})$$

An explicit calculation shows that L is given by

$$L = \left(\tilde{\Omega}^2 |\nabla f|_{\tilde{g}}^2 \right) \tilde{L} + \tilde{L} + \left(2\tilde{\Omega}^2 \tilde{g}^{AC} \partial_C f \right) \partial_{\bar{\theta}^A}. \quad (\text{A.19})$$

where $|\nabla f|_{\tilde{g}}^2 := \tilde{g}^{AB} \partial_A f \partial_B f$. We define further $\hat{L} := \Omega^{-1} L$.

A.1.3. Analysis of Ricci Coefficients on $\tilde{\mathcal{H}}_2$. The Ricci coefficients with respect to $(\hat{L}, \hat{\underline{L}})$ are defined as follows,

$$\begin{aligned} \chi_{AB} &:= \mathbf{g}(\mathbf{D}_A \hat{L}, \partial_B), & \underline{\chi}_{AB} &:= \mathbf{g}(\mathbf{D}_A \hat{\underline{L}}, \partial_B), & \zeta_A &:= \frac{1}{2} \mathbf{g}(\mathbf{D}_A \hat{L}, \hat{\underline{L}}), \\ \eta &:= \zeta + \not{d} \log \Omega, & \omega &:= L \log \Omega, & \underline{\omega} &:= \underline{L} \log \Omega. \end{aligned}$$

We analyze the Ricci coefficients in the order $(\underline{\omega}, \underline{\chi}, \omega, \zeta, \eta, \chi, \underline{D\omega}, D\omega)$.

Analysis of $\underline{\omega}$. On the one hand, we have by (A.12) that

$$\underline{\omega} := \underline{L} \log \Omega = \Omega^{-1} \partial_u \Omega = \frac{1}{2\Omega^2} \partial_u (\Omega^2).$$

On the other hand, we have by (A.11) and (A.15) that

$$\partial_u (\Omega^2) = \partial_u \left(\tilde{\Omega}^2 (1 + \partial_u f) \right) = 2\tilde{\Omega} \partial_{\bar{u}} \tilde{\Omega} (1 + \partial_u f)^2 + \tilde{\Omega}^2 \partial_u^2 f.$$

Combining the above two and using (A.4), it follows that

$$\underline{\omega} = \frac{1}{2\Omega^2} \left(2\tilde{\Omega} \partial_{\bar{u}} \tilde{\Omega} (1 + \partial_u f)^2 + \tilde{\Omega}^2 \partial_u^2 f \right) = \tilde{\omega} (1 + \partial_u f) + \frac{1}{2} \frac{\tilde{\Omega}^2}{\Omega^2} \partial_u^2 f. \quad (\text{A.20})$$

Analysis of $\underline{\chi}$. By explicit computation, we have that

$$\underline{\chi}_{AB} := \mathbf{g}(\mathbf{D}_A \hat{\underline{L}}, \partial_B) = \Omega^{-1} (1 + \partial_u f) \tilde{\Omega} \tilde{\chi}_{AB}, \quad (\text{A.21})$$

where we used that

$$\mathbf{g}(\mathbf{D}_{\partial_{\bar{u}}} \partial_{\bar{B}}, \partial_{\bar{B}}) = \tilde{\Omega}^4 \mathbf{g}(\mathbf{D}_{\underline{L}'} \tilde{\underline{L}}', \partial_{\bar{B}}) = 0.$$

We can separate (A.21) into

$$\Omega \text{tr} \underline{\chi} = (1 + \partial_u f) \tilde{\Omega} \text{tr} \tilde{\chi}, \quad \underline{\chi}_{AB} = \frac{\tilde{\Omega}}{\Omega} (1 + \partial_u f) \tilde{\chi}_{AB}, \quad (\text{A.22})$$

that is, in explicit notation,

$$\begin{aligned} (\Omega \text{tr} \underline{\chi})(u, \theta^1, \theta^2) &= (1 + \partial_u f(u, \theta^1, \theta^2)) \tilde{\Omega} \text{tr} \tilde{\chi}(u + f(u, \theta^1, \theta^2), \theta^1, \theta^2), \\ \underline{\chi}_{AB}(u, \theta^1, \theta^2) &= \frac{\tilde{\Omega}}{\Omega} (u + f(u, \theta^1, \theta^2), \theta^1, \theta^2) (1 + \partial_u f(u, \theta^1, \theta^2)) \\ &\quad \tilde{\chi}_{AB}(u + f(u, \theta^1, \theta^2), \theta^1, \theta^2). \end{aligned}$$

Analysis of ω . We calculate

$$\omega := L \log \Omega$$

as follows. First, by construction of the double coordinates $(u, \tilde{v}, \theta^1, \theta^2)$, see (A.18) and (A.12), Ω is defined on $\mathcal{M}$ through

$$L'(\tilde{v}) = \Omega^{-2}. \quad (\text{A.23})$$

In particular, this implies with the geodesic equation satisfied by L' that

$$L'(\Omega^{-2}) = L'(L'(\tilde{v})) = \mathbf{D}_{L'}(\mathbf{D}_{L'}\tilde{v}) = \underbrace{\mathbf{D}_{\mathbf{D}_{L'}L'}\tilde{v}}_{=0} + \mathbf{D}_{L'}\mathbf{D}_{L'}\tilde{v} = \Omega^{-4}\mathbf{D}_L\mathbf{D}_L\tilde{v}, \quad (\text{A.24})$$

where we note that $\mathbf{D}\mathbf{D}\tilde{v}$ is the covariant Hessian.

Second, we have the algebraic relation

$$L'(\Omega^{-2}) = -\frac{2}{\Omega^3}L'(\Omega). \quad (\text{A.25})$$

By (A.24) and (A.25), we get that

$$\omega = L(\log \Omega) = \Omega L'(\Omega) = -\frac{\Omega^4}{2}L'(\Omega^{-2}) = -\frac{1}{2}\mathbf{D}_L\mathbf{D}_L\tilde{v}. \quad (\text{A.26})$$

Plugging (A.19) into (A.26) and using (A.3), (A.7), we get that

$$\begin{aligned} \omega = & -\frac{1}{2}\left(\tilde{\Omega}^2|\nabla f|_{\tilde{g}}^2\right)^2 \underbrace{\mathbf{D}_{\tilde{u}}\mathbf{D}_{\tilde{u}}\tilde{v} - \frac{1}{2}\mathbf{D}_{\tilde{L}}\mathbf{D}_{\tilde{L}}\tilde{v}}_{=\tilde{\omega}} - 2\tilde{\Omega}^4\tilde{g}^{AB}\tilde{g}^{CD}(\partial_B f)(\partial_C f)\mathbf{D}_{\tilde{A}}\mathbf{D}_{\tilde{D}}\tilde{v} \\ & - \left(\tilde{\Omega}^2|\nabla f|_{\tilde{g}}^2\right)\mathbf{D}_{\tilde{u}}\mathbf{D}_{\partial_{\tilde{v}}+\tilde{b}^C\partial_{\tilde{C}}}\tilde{v} - \left(\tilde{\Omega}^2|\nabla f|_{\tilde{g}}^2\right)\left(2\tilde{\Omega}^2\tilde{g}^{AB}\partial_B f\right)\mathbf{D}_{\tilde{u}}\mathbf{D}_{\tilde{A}}\tilde{v} \\ & - \left(2\tilde{\Omega}^2\tilde{g}^{AB}\partial_B f\right)\mathbf{D}_{\partial_{\tilde{v}}+\tilde{b}^C\partial_{\tilde{C}}}\mathbf{D}_{\tilde{A}}\tilde{v}. \end{aligned} \quad (\text{A.27})$$

Here, the Hessian $\mathbf{D}\mathbf{D}\tilde{v}$ is given in coordinates $\mu, \nu \in \{\tilde{u}, \tilde{v}, \tilde{\theta}^1, \tilde{\theta}^2\}$ by

$$\mathbf{D}_\mu\mathbf{D}_\nu\tilde{v} = \partial_\mu\partial_\nu\tilde{v} - \Gamma_{\mu\nu}^\lambda\partial_\lambda\tilde{v} = -\Gamma_{\mu\nu}^{\tilde{v}}. \quad (\text{A.28})$$

From (A.9) and (A.28), we conclude that $\mathbf{D}_{\tilde{u}}\mathbf{D}_{\tilde{u}}\tilde{v} = \mathbf{D}_{\tilde{u}}\mathbf{D}_{\tilde{v}}\tilde{v} = \mathbf{D}_{\tilde{u}}\mathbf{D}_{\tilde{A}}\tilde{v} = 0$ and

$$\mathbf{D}_{\tilde{A}}\mathbf{D}_{\tilde{v}}\tilde{v} = -\partial_{\tilde{A}}\log\tilde{\Omega} + \tilde{\zeta}_A + \frac{1}{2\tilde{\Omega}}\tilde{\chi}_{AB}\tilde{b}^B, \quad \mathbf{D}_{\tilde{A}}\mathbf{D}_{\tilde{B}}\tilde{v} = -\frac{1}{2\tilde{\Omega}}\tilde{\chi}_{AB}. \quad (\text{A.29})$$

Plugging (A.29) into (A.27), we get that

$$\omega = \tilde{\omega} + \tilde{\Omega}^3\tilde{\chi}^{AB}(\partial_A f)(\partial_B f) + \left(2\tilde{\Omega}^2\tilde{g}^{AB}\partial_B f\right)\left(\partial_{\tilde{A}}\log\tilde{\Omega} - \tilde{\zeta}_A\right). \quad (\text{A.30})$$

Analysis of ζ and η . Using (A.15), we have by explicit computation that

$$\begin{aligned} \zeta_A &:= \frac{1}{2}\mathbf{g}(\mathbf{D}_A\hat{L}, \hat{L}) \\ &= \frac{\tilde{\Omega}^2}{\Omega^2}\partial_A\partial_u f - \frac{1}{2\Omega^2}(1 + \partial_u f)\mathbf{g}(L, \mathbf{D}_A\partial_{\tilde{u}}) - \partial_A\log\Omega. \end{aligned} \quad (\text{A.31})$$

By (A.11), (A.14), (A.19) and the geodesic equation for $\tilde{\underline{L}}'$, we have that

$$\mathbf{g}(L, \mathbf{D}_A \partial_{\tilde{u}}) = -2\tilde{\Omega}^2 \left(\tilde{\zeta}_A + \partial_{\tilde{A}} \log \tilde{\Omega} + 2(\partial_A f) \tilde{\omega} - \tilde{\Omega} \tilde{g}^{BC} (\partial_B f) \tilde{\chi}_{AC} \right). \quad (\text{A.32})$$

Plugging (A.32) into (A.31), we get that

$$\begin{aligned} \zeta_A &= -\partial_A \log \Omega + \frac{\tilde{\Omega}^2}{\Omega^2} \partial_A \partial_u f \\ &\quad + \frac{\tilde{\Omega}^2}{\Omega^2} (1 + \partial_u f) \left(\tilde{\zeta}_A + \partial_{\tilde{A}} \log \tilde{\Omega} + 2(\partial_A f) \tilde{\omega} - \tilde{\Omega} \tilde{g}^{BC} (\partial_B f) \tilde{\chi}_{AC} \right) \end{aligned} \quad (\text{A.33})$$

We conclude from the above that

$$\begin{aligned} \eta_A &:= \zeta_A + \partial_A \log \Omega \\ &= \frac{\tilde{\Omega}^2}{\Omega^2} \partial_A \partial_u f + \frac{\tilde{\Omega}^2}{\Omega^2} (1 + \partial_u f) \left(\tilde{\zeta}_A + \partial_{\tilde{A}} \log \tilde{\Omega} + 2(\partial_A f) \tilde{\omega} - \tilde{\Omega} \tilde{g}^{BC} (\partial_B f) \tilde{\chi}_{AC} \right) \\ &\quad - \frac{\tilde{\Omega}^3}{2\Omega^2} (1 + \partial_u f) (\partial_A f) \text{tr} \tilde{\chi}. \end{aligned} \quad (\text{A.34})$$

Analysis of χ . By (A.11), (A.19) and (A.21), we have by explicit computation that

$$\begin{aligned} \chi_{AB} &:= \mathbf{g}(\mathbf{D}_A \tilde{L}, \partial_B) = \frac{\tilde{\Omega}^3}{\Omega} |\nabla f|^2 \tilde{\chi}_{AB} + \frac{\tilde{\Omega}}{\Omega} \tilde{\chi}_{AB} + \Omega^{-1} (\partial_A f) \mathbf{g}(\mathbf{D}_{\tilde{L}} \tilde{L}, \partial_{\tilde{B}}) \\ &\quad + \Omega^{-1} (\partial_A f) (\partial_B f) \mathbf{g}(\mathbf{D}_{\tilde{L}} \tilde{L}, \tilde{\underline{L}}) + \Omega^{-1} (\partial_B f) \mathbf{g}(\mathbf{D}_{\tilde{A}} \tilde{L}, \tilde{\underline{L}}) \\ &\quad + \Omega^{-1} \mathbf{g}(\mathbf{D}_A (2\tilde{\Omega}^2 \tilde{g}^{CD} \partial_C f) \partial_{\tilde{D}}, \partial_B). \end{aligned} \quad (\text{A.35})$$

By (A.7), (A.11) and (A.21), we have

$$\mathbf{g}(\mathbf{D}_{\tilde{L}} \tilde{L}, \partial_{\tilde{B}}) = 2\tilde{\Omega}^2 \tilde{\eta}_B, \quad \mathbf{g}(\mathbf{D}_{\tilde{A}} \tilde{L}, \tilde{\underline{L}}) = 2\tilde{\Omega}^2 (\tilde{\eta}_A - 2\partial_{\tilde{A}} \log \tilde{\Omega}),$$

as well as

$$\mathbf{g}(\mathbf{D}_{\tilde{L}} \tilde{L}, \tilde{\underline{L}}) = \tilde{\Omega}^2 \mathbf{g}(\mathbf{D}_{\tilde{L}} \tilde{L}, \tilde{\underline{L}}') = -\tilde{\Omega}^2 \mathbf{g}(\tilde{L}, \mathbf{D}_{\tilde{L}} \tilde{\underline{L}}') = -\tilde{\Omega}^4 \mathbf{g}(\tilde{L}, \underbrace{\mathbf{D}_{\tilde{L}} \tilde{\underline{L}}'}_{=0}) = 0,$$

and

$$\begin{aligned} \mathbf{g}(\mathbf{D}_A (2\tilde{\Omega} \tilde{g}^{CD} (\partial_C f) \partial_{\tilde{D}}), \partial_B) &= \partial_A (2\tilde{\Omega}^2) \partial_B f \\ &\quad + (2\tilde{\Omega}^2) \left((\partial_A \partial_B f) + (\partial_C f) \mathbf{g}(\mathbf{D}_A (\tilde{g}^{CD} \partial_{\tilde{D}}), \partial_B) \right), \end{aligned}$$

where on the right-hand side we can rewrite with (A.11)

$$\mathbf{g}(\mathbf{D}_A (\tilde{g}^{CD} \partial_{\tilde{D}}), \partial_B) = -\tilde{\Gamma}_{AB}^C - (\partial_A f) \tilde{\Omega} \tilde{\chi}_{BD} \tilde{g}^{CD} - (\partial_B f) \tilde{g}^{CD} \tilde{\Omega} \tilde{\chi}_{AD},$$

yielding that

$$\begin{aligned} & \mathbf{g} \left(\mathbf{D}_A \left(2\tilde{\Omega} \tilde{g}^{CD} (\partial_C f) \partial_{\tilde{D}} \right), \partial_B \right) \\ &= \partial_A \left(2\tilde{\Omega}^2 \right) \partial_B f + \left(2\tilde{\Omega}^2 \right) \left((\partial_A \partial_B f) - (\partial_C f) \tilde{\Gamma}_{AB}^C \right) \\ &\quad - \left(2\tilde{\Omega}^2 \right) \left((\partial_A f) \tilde{\Omega} \tilde{\chi}_{BD} \tilde{g}^{CD} (\partial_C f) + (\partial_B f) \tilde{\Omega} \tilde{\chi}_{AD} \tilde{g}^{CD} (\partial_C f) \right). \end{aligned}$$

Plugging the above into (A.35), we have that

$$\begin{aligned} \chi_{AB} &= \frac{\tilde{\Omega}^3}{\Omega} |\nabla f|^2_{\tilde{g}} \tilde{\chi}_{AB} + \frac{\tilde{\Omega}}{\Omega} \tilde{\chi}_{AB} \\ &\quad + \frac{2\tilde{\Omega}^2}{\Omega} \left((\partial_A f) \tilde{\eta}_B + (\partial_B f) \tilde{\eta}_A \right) + \frac{2\tilde{\Omega}^2}{\Omega} \left(\partial_A \partial_B f - \tilde{\Gamma}_{AB}^C \partial_C f \right) \\ &\quad - \frac{2\tilde{\Omega}^2}{\Omega} \left((\partial_A f) \tilde{\Omega} \tilde{\chi}_{BD} \tilde{g}^{CD} (\partial_C f) + (\partial_B f) \tilde{\Omega} \tilde{\chi}_{AD} \tilde{g}^{CD} (\partial_C f) \right). \end{aligned} \quad (\text{A.36})$$

Analysis of $\underline{D\omega}$. We have by explicit computation that

$$\underline{D\omega} := \partial_u (\partial_u \log \Omega) = \frac{\partial_u^3 f}{2(1 + \partial_u f)} - \frac{(\partial_u^2 f)^2}{2(1 + \partial_u f)^2} + \tilde{D}\tilde{\omega} (1 + \partial_u f)^2 + \tilde{\omega} \partial_u^2 f. \quad (\text{A.37})$$

Analysis of $D\omega$. Using that (see also (A.23))

$$L'(\Omega) = -\frac{\Omega^3}{2} L'(\Omega^{-2}) = -\frac{\Omega^3}{2} L'(L'(\tilde{v})),$$

we have that

$$\begin{aligned} D\omega &= L(L \log \Omega) = \Omega^2 L'(\Omega L'(\Omega)) = \Omega^2 L' \left(-\frac{\Omega^4}{2} L'(L'(\tilde{v})) \right) \\ &= 4\omega^2 - \frac{\Omega^6}{2} L'(L'(L'(\tilde{v}))). \end{aligned}$$

Using that $\mathbf{D}_{L'} L' = 0$, it follows further that

$$D\omega = 4\omega^2 - \frac{\Omega^6}{2} \mathbf{D}_{L'} \mathbf{D}_{L'} \mathbf{D}_{L'} \tilde{v} = 4\omega^2 - \frac{1}{2} \mathbf{D}_L \mathbf{D}_L \mathbf{D}_L \tilde{v}.$$

By (A.19), we can furthermore expand $\mathbf{D}_L \mathbf{D}_L \mathbf{D}_L \tilde{v}$ (explicit calculation omitted here), to conclude that $D\omega$ can be written as a sum of products of first angular derivatives of f and (the following all with tilde) null curvature components, first derivatives of Ricci coefficients and second derivatives of metric coefficients.

Remark A.2. The only linear terms in f in the expression for $D\omega$ are

$$2 \left(2\tilde{\Omega}^2 \tilde{g}^{AB} (\partial_A f) \right) \mathbf{D}_{\tilde{L}} \mathbf{D}_{\tilde{L}} \mathbf{D}_{\tilde{B}} \tilde{v} \text{ and } 2\tilde{\Omega}^2 \tilde{g}^{EF} (\partial_E f) \mathbf{D}_{\tilde{F}} \mathbf{D}_{\tilde{L}} \mathbf{D}_{\tilde{L}} \tilde{v},$$

and we note that at Minkowski,

$$\mathbf{D}_{\tilde{L}} \mathbf{D}_{\tilde{L}} \mathbf{D}_{\tilde{L}} \tilde{v} = \mathbf{D}_{\tilde{L}} \mathbf{D}_{\tilde{L}} \mathbf{D}_{\tilde{B}} \tilde{v} = \mathbf{D}_{\tilde{F}} \mathbf{D}_{\tilde{L}} \mathbf{D}_{\tilde{L}} \tilde{v} = 0.$$

Hence, the linearization of $D\omega$ vanishes at Minkowski.

A.1.4. Calculation of Null Curvature Components on $\tilde{\mathcal{H}}_2$. We recall from (2.10) the definition of the null curvature components,

$$\begin{aligned} \alpha_{AB} &:= \mathbf{R}(\partial_A, \hat{L}, \partial_B, \hat{L}), & \beta_A &:= \frac{1}{2} \mathbf{R}(\partial_A, \hat{L}, \hat{L}, \hat{L}), & \rho &:= \frac{1}{4} \mathbf{R}(\hat{L}, \hat{L}, \hat{L}, \hat{L}), \\ \sigma \not\in_{AB} &:= \frac{1}{2} \mathbf{R}(\partial_A, \partial_B, \hat{L}, \hat{L}), & \underline{\beta}_A &:= \frac{1}{2} \mathbf{R}(\partial_A, \hat{L}, \hat{L}, \hat{L}), & \underline{\alpha}_{AB} &:= \mathbf{R}(\partial_A, \hat{L}, \partial_B, \hat{L}). \end{aligned} \quad (\text{A.38})$$

Plugging (A.11), (A.14), (A.15) and (A.19), that is,

$$\begin{aligned} \partial_{\theta^A} &= \partial_{\tilde{\theta}^A} + (\partial_{\theta^A} f) \partial_{\tilde{u}}, & \Omega^2 &= \tilde{\Omega}^2 (1 + \partial_u f), \\ \underline{L} &= (1 + \partial_u f) \tilde{L}, & L &= \left(\tilde{\Omega}^2 |\nabla f|_{\tilde{g}}^2 \right) \tilde{L} + \tilde{L} + \left(2\tilde{\Omega}^2 \tilde{g}^{AC} \partial_C f \right) \partial_{\tilde{\theta}^A}, \end{aligned}$$

into (A.38), it follows that the null curvature components $(\alpha, \beta, \rho, \sigma, \underline{\beta}, \underline{\alpha})$ can be expressed as sum of products of $(\tilde{\alpha}, \tilde{\beta}, \tilde{\rho}, \tilde{\sigma}, \tilde{\underline{\beta}}, \tilde{\underline{\alpha}})$ and $f, \partial_A f$, $A = 1, 2$, and $\partial_u f$.

A.2. Proof of Proposition 2.21

In this section, we prove Proposition 2.21; that is, we discuss the mapping $\mathcal{P}_{f,q}(\tilde{x})$ and prove estimates.

First, recall from (2.56) that

$$\mathcal{P}_{f,q}(\tilde{x}) := \mathcal{P}_{0,q} \mathcal{P}_{f,0}(\tilde{x}),$$

and that in Sects. A.1.1-A.1.4 we discussed the explicit formulas for $\mathcal{P}_{f,0}(\tilde{x})$.

Second, recall that $q \in \mathcal{Y}_q = H^8(S_{0,2}) \times H^8(S_{0,2})$ and that $H^{\tilde{m}}(S_{0,2})$ is an algebra for integers $\tilde{m} \geq 2$, the following basic estimate holds, see, for example, [28]. There is a real number $\varepsilon_0 > 0$ such that for all q satisfying

$$\|q\|_{\mathcal{Y}_q} \leq \varepsilon_0,$$

it holds that for a tensor $T \in H^m(S_{0,2})$ on $S_{0,2}$ (with $0 \leq m \leq 6$ an integer), its pullback $\Phi_1(q)^*(T)$ under $\Phi_1(q)$ is well defined and bounded by

$$\begin{aligned} \|\Phi_1(q)^*(T) - T\|_{H^m(S_{0,2})} &\leq C \|T\|_{H^{m+1}(S_{0,2})} \|q\|_{\mathcal{Y}_q}, \\ \|\Phi_1(q)^*(T)\|_{H^m(S_{0,2})} &\leq (1 + \|q\|_{\mathcal{Y}_q}) \|T\|_{H^m(S_{0,2})}. \end{aligned} \quad (\text{A.39})$$

We emphasize that the first of (A.39) as stated loses derivatives in T but the second estimate does not.

We omit the proof that the pullback under $\Phi_1(q)$ with $q \in \mathcal{Y}_q$ is a *smooth* mapping from tensors in $H^m(S_{0,2})$ to tensors in $H^m(S_{0,2})$ for integers $0 \leq m \leq 6$.

We are now in position to prove Proposition 2.21. The important step is to prove that $\mathcal{P}_{f,q}$ maps into $\mathcal{X}(S_{0,2})$. Then, the property that $\mathcal{P}_{f,q}$ is well defined and smooth near $(\tilde{x}, f, q) = (\underline{\mathbf{m}}, 0, 0)$ follows in a straightforward fashion. Hence, it remains to bound the sphere data

$$x_{0,2} := \mathcal{P}_{f,q}(\tilde{x}) = \mathcal{P}_{0,q} \mathcal{P}_{f,0}(\tilde{x}).$$

In the following, we prove that

$$\|\mathcal{P}_{f,q}(\tilde{x}) - \tilde{x}_{0,2}\|_{\mathcal{X}(S_{0,2})} \lesssim \|f\|_{\mathcal{Y}_f} + \|q\|_{\mathcal{Y}_q} + \|\tilde{x} - \underline{\mathbf{m}}\|_{\mathcal{X}^+(\tilde{\mathcal{H}}_{[-\delta, \delta], 2})}, \quad (\text{A.40})$$

where we recall from Definition 2.5 the sphere data norm

$$\begin{aligned} \|x_{0,2}\|_{\mathcal{X}(S_{0,2})} &:= \|\Omega\|_{H^6(S_{0,2})} + \|\mathcal{G}\|_{H^6(S_{0,2})} + \|\Omega \operatorname{tr} \chi\|_{H^6(S_{0,2})} + \|\widehat{\chi}\|_{H^6(S_{0,2})} \\ &\quad + \|\Omega \operatorname{tr} \underline{\chi}\|_{H^4(S_{0,2})} + \|\widehat{\underline{\chi}}\|_{H^4(S_{0,2})} + \|\eta\|_{H^5(S_{0,2})} \\ &\quad + \|\omega\|_{H^6(S_{0,2})} + \|D\omega\|_{H^6(S_{0,2})} + \|\underline{\omega}\|_{H^4(S_{0,2})} + \|\underline{D}\omega\|_{H^2(S_{0,2})} \\ &\quad + \|\alpha\|_{H^6(S_{0,2})} + \|\underline{\alpha}\|_{H^2(S_{0,2})}, \end{aligned}$$

and from Definition 2.20 the sphere perturbation function norms

$$\begin{aligned} \|f\|_{\mathcal{Y}_f} &:= \|f(0)\|_{H^8(\mathbb{S}^2)} + \|\partial_u f(0)\|_{H^6(\mathbb{S}^2)} + \|\partial_u^2 f(0)\|_{H^4(\mathbb{S}^2)} + \|\partial_u^3 f(0)\|_{H^2(\mathbb{S}^2)}, \\ \|q\|_{\mathcal{Y}_q} &:= \|q_1\|_{H^8(\mathbb{S}^2)} + \|q_2\|_{H^8(\mathbb{S}^2)}. \end{aligned}$$

Indeed, the proof is based on three ingredients. First, we work with a *higher regularity* $\tilde{x}$ along $\tilde{\mathcal{H}}_{[-\delta, \delta, 2]}$ and thus higher derivatives falling on $\tilde{x}$ can still be bounded; in other words, loss of derivatives here is acceptable. Second, there are no higher derivatives that fall onto f ; this is already visible from the explicit formulas of Sects. A.1.1–A.1.4. Third, the terms which need to be estimated using the first estimate of (A.39) (which loses derivatives in T) are in fact—due to ingredient (1) above—of higher regularity, and thus, this loss can be tolerated. This can again be verified by inspection of the explicit formulas of Sects. A.1.1–A.1.4.

In other words, there is a loss of derivative in the sphere perturbation mapping but it does not involve the functions f and q , and hence, our choice of function spaces (in particular, the higher regularity $\tilde{x}$) allows to use the implicit function theorem setup around the sphere perturbation mapping nevertheless.

Let us illustrate the third ingredient by an example. Using (A.37), that is,

$$\begin{aligned} \underline{D}\omega(u, \theta^1, \theta^2) &= \left(\frac{\partial_u^3 f}{2(1 + \partial_u f)} - \frac{(\partial_u^2 f)^2}{2(1 + \partial_u f)^2} + \tilde{D}\tilde{\omega}(1 + \partial_u f)^2 + \tilde{\omega}\partial_u^2 f \right) \\ &\quad (u + f(u, \theta^1, \theta^2), \theta^1, \theta^2), \end{aligned}$$

and that by definition of $\mathcal{P}_{f,q}$,

$$\underline{D}\omega := \underline{D}\omega(\mathcal{P}_{f,0}(\tilde{x})) \circ \Phi_1(q),$$

where $\underline{D}\omega(\mathcal{P}_{f,0}(\tilde{x}))$ denotes the component $\underline{D}\omega$ of $\mathcal{P}_{f,0}(\tilde{x})$, we estimate

$$\begin{aligned} &\|\underline{D}\omega - \tilde{D}\tilde{\omega}\|_{H^2(S_{0,2})} \\ &\leq \|\tilde{D}\tilde{\omega} \circ \Phi_1(q) - \tilde{D}\tilde{\omega}\|_{H^2(S_{0,2})} \\ &\quad + \left\| \left(\frac{\partial_u^3 f}{2(1 + \partial_u f)} - \frac{(\partial_u^2 f)^2}{2(1 + \partial_u f)^2} + (2\partial_u f + (\partial_u f)^2)\tilde{D}\tilde{\omega} + \tilde{\omega}\partial_u^2 f \right) \circ \Phi_1(q) \right\|_{H^2(S_{0,2})} \\ &\lesssim \|\tilde{x} - \underline{m}\|_{\mathcal{X} + (\tilde{\mathcal{H}}_{[-\delta, \delta], 2})} + \|q\|_{\mathcal{Y}_q} + \|f\|_{\mathcal{Y}_f} \\ &\quad + (1 + \|q\|_{\mathcal{Y}_q}) \left(\|f\|_{\mathcal{Y}_f} + \|\tilde{x} - \underline{m}\|_{\mathcal{X} + (\tilde{\mathcal{H}}_{[-\delta, \delta], 2})} \right), \end{aligned}$$

where we applied the first and second of (A.39) to the first and second line after the first equality, respectively.

This finishes the proof of Proposition 2.21.

Appendix B: Derivation of Null Transport Equations

In this section, we prove null transport equations used in this paper. In Sect. B.1, we prove the nonlinear null transport equation (2.20) for $\underline{D}\omega$ along $\mathcal{H}$. In Sect. B.2, we derive the linearized null transport equations of Lemma 4.14 for $\dot{\omega}$, $\underline{D}\dot{\omega}$ and $\dot{\alpha}$.

B.1. Derivation of Null Transport Equation for $\underline{D}\omega$

In this section, we prove the transport Eq. (2.20) for $\underline{D}\omega$ along $\mathcal{H}$. We remark that in case of a geodesic foliation on $\mathcal{H} = \mathcal{H}_{0,[1,2]}$, that is, $\Omega \equiv 1$ on $\mathcal{H}$, this transport equation is readily available in [15]. We first have the following commutator identities, see Chapter 1 in [15].

Lemma B.1 (Commutator identity). *Let W be an S_v -tangent tensorfield. Then,*

$$\underline{D}D\underline{W} - D\underline{D}W = \mathcal{L}_{4\Omega^2\zeta}W.$$

We are now in position to derive the null transport equation for $\underline{D}\omega$. From the null structure equations (2.14), we have that

$$D\underline{\omega} = \Omega^2 (2(\eta, \underline{\eta}) - |\eta|^2 - \rho). \quad (\text{B.1})$$

Applying the $\underline{D}$ -derivative to (B.1) and using (2.11), (2.14), (2.19) and Lemma B.1 with $W = \omega$, we have that

$$\begin{aligned} \underline{D}D\underline{\omega} &= -4\Omega^2\zeta(\omega) + \underline{D}D\underline{\omega} \\ &= -4\Omega^2\zeta(\underline{\omega}) + \underline{D}(\Omega^2(2(\eta, \underline{\eta}) - |\eta|^2 - \rho)) \\ &= -4\Omega^2\zeta(\underline{\omega}) + 2\Omega^2\underline{\omega}(2(\eta, \underline{\eta}) - |\eta|^2 - \rho) \\ &\quad + \Omega^2(4\Omega\underline{\chi}(\eta, \underline{\eta}) + 2(-\Omega(\underline{\chi} \cdot \eta + \underline{\beta}) + 2\mathcal{L}\omega, \underline{\eta}) + 2(\eta, \Omega(\underline{\chi} \cdot \eta + \underline{\beta}))) \\ &\quad + \Omega^2(-2\Omega\underline{\chi}(\eta, \eta) - 2(-\Omega(\underline{\chi} \cdot \eta + \underline{\beta}) + 2\mathcal{L}\omega, \eta)) \\ &\quad + \Omega^2\left(\frac{3}{2}\Omega\text{tr}\underline{\chi}\rho + \Omega\left(\text{div}\underline{\beta} + (2\eta - \zeta, \underline{\beta}) + \frac{1}{2}(\widehat{\chi}, \underline{\alpha})\right)\right) \\ &= -12\Omega^2(\eta - \mathcal{L}\log\Omega, \mathcal{L}\omega) + 2\Omega^2\underline{\omega}((\eta, -3\eta + 4\mathcal{L}\log\Omega) - \rho) \\ &\quad + 4\Omega^3\underline{\chi}(\eta, \mathcal{L}\log\Omega) + \Omega^3(\underline{\beta}, 7\eta - 3\mathcal{L}\log\Omega) \\ &\quad + \frac{3}{2}\Omega^3\text{tr}\underline{\chi}\rho + \Omega^3\text{div}\underline{\beta} + \frac{\Omega^3}{2}(\widehat{\chi}, \underline{\alpha}). \end{aligned}$$

where we used (2.8) and (2.9). This finishes the proof of (2.20).

B.2. Derivation of Transport Equations for $\dot{\omega}$, $\dot{\alpha}$ and $\underline{D}\dot{\omega}$

In this section, we prove Lemma 4.14. To simplify notation, we use that in Minkowski on $\mathcal{H} = \mathcal{H}_{0,[1,2]}$ it holds that $r = v$. First we recall from (4.5) that

$$\begin{aligned} \mathcal{Q}_1 &:= \frac{v}{2} \left((\Omega\text{tr}\chi) - \frac{4}{v}\dot{\Omega} \right) + \frac{\dot{\phi}}{v}, \\ \mathcal{Q}_2 &:= v^2(\Omega\text{tr}\underline{\chi}) - \frac{2}{v}\text{div}\left(v^2\dot{\eta} + \frac{v^3}{2}\mathcal{L}\left((\Omega\text{tr}\chi) - \frac{4}{v}\dot{\Omega}\right)\right) - v^2\left((\Omega\text{tr}\chi) - \frac{4}{v}\dot{\Omega}\right) + 2v^3\dot{K}, \\ \mathcal{Q}_3 &:= \frac{\dot{\chi}}{v} - \frac{1}{2}\left(\mathcal{P}_2^*\text{div} + 1\right)\mathcal{G}_c + \mathcal{P}_2^*\left(\dot{\eta} + \frac{v}{2}\mathcal{L}\left((\Omega\text{tr}\chi) - \frac{4}{v}\dot{\Omega}\right)\right) - v\mathcal{P}_2^*\mathcal{L}\left((\Omega\text{tr}\chi) - \frac{4}{v}\dot{\Omega}\right). \end{aligned}$$

and that by Lemmas 4.8 and 4.11,

$$\begin{aligned}
 D\mathcal{Q}_1 &= \dot{\mathbf{c}}_1 - D\left(\frac{1}{2v}\dot{\mathbf{c}}_2\right), \\
 D\mathcal{Q}_2 &= v^2\dot{\mathbf{c}}_5 - 2v\mathring{\mathbf{d}}\dot{\mathbf{v}}\dot{\mathbf{c}}_4 - 2v(\mathring{\mathcal{A}} + 1)\dot{\mathbf{c}}_1 + D\left((\mathring{\mathcal{A}} + 1)\dot{\mathbf{c}}_2\right) \\
 &\quad - \frac{1}{v}(\mathring{\mathcal{A}} + 2)\dot{\mathbf{c}}_2 + \frac{1}{v}\mathring{\mathbf{d}}\dot{\mathbf{v}}\mathring{\mathbf{d}}\dot{\mathbf{v}}\dot{\mathbf{c}}_3, \\
 D\mathcal{Q}_3 &= \frac{1}{v}\dot{\mathbf{c}}_6 - \frac{1}{2v^2}\left(\mathcal{P}_2^*\mathring{\mathbf{d}}\dot{\mathbf{v}} + 1\right)\dot{\mathbf{c}}_3 + \mathcal{P}_2^*\dot{\mathbf{c}}_4 - \mathcal{P}_2^*\mathring{\mathbf{d}}\dot{\mathbf{c}}_1 + D\left(\frac{1}{2v}\mathcal{P}_2^*\mathring{\mathbf{d}}\dot{\mathbf{c}}_2\right) \\
 &\quad + \frac{1}{2v^2}\mathcal{P}_2^*\mathring{\mathbf{d}}\dot{\mathbf{c}}_2.
 \end{aligned}$$

Transport equation for $\dot{\underline{\omega}}$. We have by using (2.16) that

$$\begin{aligned}
 D\left(\dot{\underline{\omega}} + \frac{1}{4v^2}\mathcal{Q}_2 + \frac{1}{3v}\mathring{\mathbf{d}}\dot{\mathbf{v}}\left(\dot{\eta} + \frac{v}{2}\mathring{\mathbf{d}}\left((\Omega\mathrm{tr}\chi) - \frac{4}{v}\dot{\Omega}\right)\right)\right) \\
 &= \dot{\mathbf{c}}_7 + \dot{K} + \frac{1}{2v}(\Omega\mathrm{tr}\underline{\chi}) - \frac{1}{2v}(\Omega\mathrm{tr}\chi) + \frac{2}{v^2}\dot{\Omega} + \frac{1}{4v^2}D\mathcal{Q}_2 \\
 &\quad - \frac{1}{2v^3}\left(v^2(\Omega\mathrm{tr}\underline{\chi}) - \frac{2}{v}\mathring{\mathbf{d}}\dot{\mathbf{v}}\left(v^2\dot{\eta} + \frac{v^3}{2}\mathring{\mathbf{d}}\left((\Omega\mathrm{tr}\chi) - \frac{4}{v}\dot{\Omega}\right)\right) - v^2\left((\Omega\mathrm{tr}\chi) - \frac{4}{v}\dot{\Omega}\right) + 2v^3\dot{K}\right) \\
 &\quad - \frac{1}{v^2}\mathring{\mathbf{d}}\dot{\mathbf{v}}\left(\dot{\eta} + \frac{v}{2}\mathring{\mathbf{d}}\left((\Omega\mathrm{tr}\chi) - \frac{4}{v}\dot{\Omega}\right)\right) \\
 &\quad + \frac{1}{3v^3}\mathring{\mathbf{d}}\dot{\mathbf{v}}\left(\mathring{\mathbf{d}}\dot{\mathbf{v}}\dot{\chi} + v^2\dot{\mathbf{c}}_4 + v^2\mathring{\mathbf{d}}\dot{\mathbf{c}}_1 + \frac{1}{2}\mathring{\mathbf{d}}\dot{\mathbf{c}}_2 - D\left(\frac{v}{2}\mathring{\mathbf{d}}\dot{\mathbf{c}}_2\right)\right) \\
 &= \dot{\mathbf{c}}_7 + \frac{1}{4v^2}D\mathcal{Q}_2 + \frac{1}{3v^3}\mathring{\mathbf{d}}\dot{\mathbf{v}}\left(\mathring{\mathbf{d}}\dot{\mathbf{v}}\dot{\chi} + v^2\dot{\mathbf{c}}_4 + v^2\mathring{\mathbf{d}}\dot{\mathbf{c}}_1 + \frac{1}{2}\mathring{\mathbf{d}}\dot{\mathbf{c}}_2 - D\left(\frac{v}{2}\mathring{\mathbf{d}}\dot{\mathbf{c}}_2\right)\right) \\
 &= \dot{\mathbf{c}}_7 + \frac{1}{4v^2}D\mathcal{Q}_2 + \frac{1}{3v^3}\mathring{\mathbf{d}}\dot{\mathbf{v}}\left(\mathring{\mathbf{d}}\dot{\mathbf{v}}\dot{\chi} + v^2\dot{\mathbf{c}}_4 + v^2\mathring{\mathbf{d}}\dot{\mathbf{c}}_1 - \mathring{\mathbf{d}}\dot{\mathbf{c}}_2\right) - D\left(\frac{1}{6v^2}\mathring{\mathcal{A}}\dot{\mathbf{c}}_2\right) \\
 &= \dot{\mathbf{c}}_7 + \frac{1}{4}\dot{\mathbf{c}}_5 + \frac{1}{4v^3}\mathring{\mathbf{d}}\dot{\mathbf{v}}\mathring{\mathbf{d}}\dot{\mathbf{v}}\dot{\mathbf{c}}_3 - \frac{1}{6v}\mathring{\mathbf{d}}\dot{\mathbf{v}}\dot{\mathbf{c}}_4 - \frac{1}{6v}(\mathring{\mathcal{A}} + 3)\dot{\mathbf{c}}_1 - \frac{1}{12v^3}\mathring{\mathcal{A}}\dot{\mathbf{c}}_2 + \frac{1}{3v^3}\mathring{\mathbf{d}}\dot{\mathbf{v}}\mathring{\mathbf{d}}\dot{\mathbf{v}}\dot{\chi}.
 \end{aligned}$$

Transport equation for $\dot{\underline{\alpha}}$. Using the above definition of $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_3$, we have by the system (4.2),

$$\begin{aligned}
 D\left(\frac{\dot{\underline{\alpha}}}{v} + \frac{2}{v}\mathcal{P}_2^*\mathring{\mathbf{d}}\dot{\mathbf{v}}\mathcal{Q}_3 - \frac{1}{2v^2}\mathcal{P}_2^*\mathring{\mathbf{d}}\mathcal{Q}_2 - \frac{2}{v}\mathcal{P}_2^*\mathring{\mathbf{d}}\left(\mathring{\mathcal{A}} + 2\right)\mathcal{Q}_1\right) \\
 &= \frac{1}{v}\dot{\mathbf{c}}_8 + \frac{2}{v}\mathcal{P}_2^*\left(\frac{1}{v^2}\mathring{\mathbf{d}}\dot{\mathbf{v}}\dot{\chi} - \frac{1}{2}\mathring{\mathbf{d}}(\Omega\mathrm{tr}\underline{\chi}) - \frac{1}{v}\dot{\eta}\right) - \frac{2}{v^2}\mathcal{P}_2^*\mathring{\mathbf{d}}\dot{\mathbf{v}}\mathcal{Q}_3 + \frac{2}{v}\mathcal{P}_2^*\mathring{\mathbf{d}}\dot{\mathbf{v}}(D\mathcal{Q}_3) \\
 &\quad + \frac{1}{v^3}\mathcal{P}_2^*\mathring{\mathbf{d}}\mathcal{Q}_2 - \frac{1}{2v^2}\mathcal{P}_2^*\mathring{\mathbf{d}}(D\mathcal{Q}_2) + \frac{2}{v^2}\mathcal{P}_2^*\mathring{\mathbf{d}}\left(\mathring{\mathcal{A}} + 2\right)\mathcal{Q}_1 - \frac{2}{v}\mathcal{P}_2^*\mathring{\mathbf{d}}\left(\mathring{\mathcal{A}} + 2\right)(D\mathcal{Q}_1) \\
 &= \frac{1}{v}\dot{\mathbf{c}}_8 + \frac{2}{v}\mathcal{P}_2^*\left(\frac{1}{v^2}\mathring{\mathbf{d}}\dot{\mathbf{v}}\dot{\chi} - \frac{1}{2}\mathring{\mathbf{d}}(\Omega\mathrm{tr}\underline{\chi}) - \frac{1}{v}\dot{\eta}\right) \\
 &\quad + \frac{2}{v}\mathcal{P}_2^*\mathring{\mathbf{d}}\dot{\mathbf{v}}(D\mathcal{Q}_3) - \frac{1}{2v^2}\mathcal{P}_2^*\mathring{\mathbf{d}}(D\mathcal{Q}_2) - \frac{2}{v}\mathcal{P}_2^*\mathring{\mathbf{d}}\left(\mathring{\mathcal{A}} + 2\right)(D\mathcal{Q}_1) \\
 &\quad - \frac{2}{v^2}\mathcal{P}_2^*\mathring{\mathbf{d}}\dot{\mathbf{v}}\left(\frac{\dot{\chi}}{v} - \frac{1}{2}\left(\mathcal{P}_2^*\mathring{\mathbf{d}}\dot{\mathbf{v}} + 1\right)\dot{g}_c + \mathcal{P}_2^*\left(\dot{\eta} + \frac{v}{2}\mathring{\mathbf{d}}\left((\Omega\mathrm{tr}\chi) - \frac{4}{v}\dot{\Omega}\right)\right)\right) \\
 &\quad - v\mathcal{P}_2^*\mathring{\mathbf{d}}\left((\Omega\mathrm{tr}\chi) - \frac{4}{v}\dot{\Omega}\right)
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{v^3} \mathcal{P}_2^* \mathcal{A} \left(v^2 (\Omega \text{tr} \chi) - \frac{2}{v} \mathring{\text{d}}\dot{\chi} \left(v^2 \dot{\eta} + \frac{v^3}{2} \mathcal{A} \left((\Omega \text{tr} \chi) - \frac{4}{v} \dot{\Omega} \right) \right) \right. \\
& - v^2 \left((\Omega \text{tr} \chi) - \frac{4}{v} \dot{\Omega} \right) + 2v^3 \dot{K} \left. \right) \\
& + \frac{2}{v^2} \mathcal{P}_2^* \mathcal{A} \left(\mathring{\mathcal{A}} + 2 \right) \left(\frac{v}{2} \left((\Omega \text{tr} \chi) - \frac{4}{v} \dot{\Omega} \right) + \frac{\dot{\phi}}{v} \right),
\end{aligned}$$

and

$$\begin{aligned}
& - \frac{2}{3} \mathcal{P}_2^* \left(\mathring{\text{d}}\dot{\chi} \mathcal{P}_2^* + 1 + \mathcal{A} \mathring{\text{d}}\dot{\chi} \right) D \left(\frac{1}{v^3} \left(v^2 \dot{\eta} + \frac{v^3}{2} \mathcal{A} \left((\Omega \text{tr} \chi) - \frac{4}{v} \dot{\Omega} \right) \right) \right) \\
& = \frac{2}{v^2} \mathcal{P}_2^* \left(\mathring{\text{d}}\dot{\chi} \mathcal{P}_2^* + 1 + \mathcal{A} \mathring{\text{d}}\dot{\chi} \right) \left(\dot{\eta} + \frac{v}{2} \mathcal{A} \left((\Omega \text{tr} \chi) - \frac{4}{v} \dot{\Omega} \right) \right) \\
& - \frac{2}{3v^3} \mathcal{P}_2^* \left(\mathring{\text{d}}\dot{\chi} \mathcal{P}_2^* + 1 + \mathcal{A} \mathring{\text{d}}\dot{\chi} \right) \left(\mathring{\text{d}}\dot{\chi} \hat{\chi} + v^2 \mathring{\mathbf{c}}_4 + v^2 \mathcal{A} \mathring{\mathbf{c}}_1 + \frac{1}{2} \mathcal{A} \mathring{\mathbf{c}}_2 - D \left(\frac{v}{2} \mathcal{A} \mathring{\mathbf{c}}_2 \right) \right).
\end{aligned}$$

and

$$\begin{aligned}
& \mathcal{P}_2^* \left(\mathring{\text{d}}\dot{\chi} \mathcal{P}_2^* + 1 + \mathcal{A} \mathring{\text{d}}\dot{\chi} \right) \mathring{\text{d}}\dot{\chi} D \left(\frac{1}{v} \mathring{g}_c \right) \\
& = - \frac{1}{v^2} \mathcal{P}_2^* \left(\mathring{\text{d}}\dot{\chi} \mathcal{P}_2^* + 1 + \mathcal{A} \mathring{\text{d}}\dot{\chi} \right) \mathring{\text{d}}\dot{\chi} \mathring{g}_c + \frac{1}{v} \mathcal{P}_2^* \left(\mathring{\text{d}}\dot{\chi} \mathcal{P}_2^* + 1 + \mathcal{A} \mathring{\text{d}}\dot{\chi} \right) \mathring{\text{d}}\dot{\chi} \left(\frac{2}{v^2} \hat{\chi} + \frac{1}{v^2} \mathring{\mathbf{c}}_3 \right).
\end{aligned}$$

Summing up the above and using that, see Sect. D.3,

$$2 \mathring{\text{d}}\dot{\chi} \mathcal{P}_2^* \mathcal{A} + \mathcal{A} (\mathring{\mathcal{A}} + 2) = 2 \left(\mathring{\text{d}}\dot{\chi} \mathcal{P}_2^* + \frac{1}{2} \mathcal{A} \mathring{\text{d}}\dot{\chi} + 1 \right) \mathcal{A} = 0$$

yield the transport equation for $\dot{\chi}$ in Lemma 4.14.

Transport equation for $D\dot{\omega}$. We have that

$$\begin{aligned}
& D \left(D\dot{\omega} - \frac{1}{6v^3} \left(\mathring{\mathcal{A}} - 3 \right) \mathcal{Q}_2 + \frac{1}{2v^2} \mathring{\text{d}}\dot{\chi} \mathring{\text{d}}\dot{\chi} \mathcal{Q}_3 + \frac{1}{v^2} \mathring{\text{d}}\dot{\chi} \mathring{\text{d}}\dot{\chi} \mathcal{P}_2^* \mathcal{A} \mathcal{Q}_1 \right) \\
& = \mathring{\mathbf{c}}_9 + \frac{3}{v} \left(\dot{K} + \frac{1}{2v} (\Omega \text{tr} \chi) - \frac{1}{2v} (\Omega \text{tr} \chi) + \frac{2}{v^2} \dot{\Omega} \right) \\
& + \frac{1}{v^2} \mathring{\text{d}}\dot{\chi} \left(\frac{1}{v^2} \mathring{\text{d}}\dot{\chi} \hat{\chi} - \frac{1}{v} \dot{\eta} - \frac{1}{2} \mathcal{A} (\Omega \text{tr} \chi) \right) \\
& - \frac{1}{6v^3} (\mathring{\mathcal{A}} - 3) D \mathcal{Q}_2 + \frac{1}{2v^2} \mathring{\text{d}}\dot{\chi} \mathring{\text{d}}\dot{\chi} D \mathcal{Q}_3 + \frac{1}{v^2} \mathring{\text{d}}\dot{\chi} \mathring{\text{d}}\dot{\chi} \mathcal{P}_2^* \mathcal{A} D \mathcal{Q}_1 \\
& + \frac{1}{2v^4} (\mathring{\mathcal{A}} - 3) \left(v^2 (\Omega \text{tr} \chi) - \frac{2}{v} \mathring{\text{d}}\dot{\chi} \left(v^2 \dot{\eta} + \frac{v^3}{2} \mathcal{A} \left((\Omega \text{tr} \chi) - \frac{4}{v} \dot{\Omega} \right) \right) \right. \\
& \left. - v^2 \left((\Omega \text{tr} \chi) - \frac{4}{v} \dot{\Omega} \right) + 2v^3 \dot{K} \right) \\
& - \frac{1}{v^3} \mathring{\text{d}}\dot{\chi} \mathring{\text{d}}\dot{\chi} \left(\frac{\hat{\chi}}{v} - \frac{1}{2} \left(\mathcal{P}_2^* \mathring{\text{d}}\dot{\chi} + 1 \right) \mathring{g}_c \right) \\
& + \mathcal{P}_2^* \left(\dot{\eta} + \frac{v}{2} \mathcal{A} \left((\Omega \text{tr} \chi) - \frac{4}{v} \dot{\Omega} \right) \right) - v \mathcal{P}_2^* \mathcal{A} \left((\Omega \text{tr} \chi) - \frac{4}{v} \dot{\Omega} \right) \\
& - \frac{2}{v^3} \mathring{\text{d}}\dot{\chi} \mathring{\text{d}}\dot{\chi} \mathcal{P}_2^* \mathcal{A} \left(\frac{v}{2} \left((\Omega \text{tr} \chi) - \frac{4}{v} \dot{\Omega} \right) + \frac{\dot{\phi}}{v} \right),
\end{aligned}$$

as well as

$$\begin{aligned} & -D \left(\frac{1}{4v^2} \left(\left(\overset{\circ}{\Delta} - 2 \right) \dot{\text{d}}\dot{v} + \dot{\text{d}}\dot{v} \dot{\text{d}}\dot{v} \mathcal{P}_2^* \right) \left(\dot{\eta} + \frac{v}{2} \not{d} \left((\Omega \text{tr} \chi) - \frac{4}{v} \dot{\Omega} \right) \right) \right) \\ & = \frac{1}{v^3} \left(\left(\overset{\circ}{\Delta} - 2 \right) \dot{\text{d}}\dot{v} + \dot{\text{d}}\dot{v} \dot{\text{d}}\dot{v} \mathcal{P}_2^* \right) \left(\dot{\eta} + \frac{v}{2} \not{d} \left((\Omega \text{tr} \chi) - \frac{4}{v} \dot{\Omega} \right) \right) \\ & \quad - \frac{1}{4v^4} \left(\left(\overset{\circ}{\Delta} - 2 \right) \dot{\text{d}}\dot{v} + \dot{\text{d}}\dot{v} \dot{\text{d}}\dot{v} \mathcal{P}_2^* \right) \left(\dot{\text{d}}\dot{v} \dot{\chi} + v^2 \dot{\epsilon}_4 + v^2 \not{d} \dot{\epsilon}_1 + \frac{1}{2} \not{d} \dot{\epsilon}_2 - D \left(\frac{v}{2} \not{d} \dot{\epsilon}_2 \right) \right), \end{aligned}$$

and

$$\begin{aligned} D \left(\frac{1}{8v^2} \dot{\text{d}}\dot{v} \not{d} \dot{\text{d}}\dot{v} \dot{\text{d}}\dot{v} \dot{g}_c \right) & = -\frac{1}{4v^3} \dot{\text{d}}\dot{v} \not{d} \dot{\text{d}}\dot{v} \dot{\text{d}}\dot{v} \dot{g}_c \\ & \quad + \frac{1}{8v^2} \dot{\text{d}}\dot{v} \not{d} \dot{\text{d}}\dot{v} \dot{\text{d}}\dot{v} \left(\frac{2}{v^2} \dot{\chi} + \frac{1}{v^2} \dot{\epsilon}_3 \right). \end{aligned}$$

Summing up the above and using Lemma 4.11 yield the transport equation for $\underline{D}\dot{\omega}$ and thus finish the proof of Lemma 4.14.

Appendix C: Linearization at Schwarzschild

In this section, we first derive the linearizations of $\mathcal{P}_f$ and $\mathcal{P}_j$ at Schwarzschild of mass $M \geq 0$, see Sect. C.1. The linearizations are used in (5.13) in Sect. 5.3 for proving the perturbation estimate (3.6) for $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$. Then in Sects. C.2 and C.3, we calculate the linearizations of the constraint functions and the null transport equations for the linearizations of $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$ at Schwarzschild of mass $M \geq 0$, respectively. The latter are used for (5.24) and (5.29) in Sect. 5.4 to prove the transport estimate (3.7) for $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$.

C.1. Linearizations $\dot{\mathcal{P}}_f^M$ and $\dot{\mathcal{P}}_q^M$ at Schwarzschild of Mass $M \geq 0$

In this section, we define the linearization $\dot{\mathcal{P}}_f^M$ and $\dot{\mathcal{P}}_q^M$. As visible in the proofs of Lemmas C.1 and C.2, their linearization is closely connected to the more general *linearized pure gauge solutions* of [27].

First, we have the following lemma for $\dot{\mathcal{P}}_f^M$.

Lemma C.1 (Linearization $\dot{\mathcal{P}}_f^M$ of $\mathcal{P}_{f,q}$). *Let $\dot{\mathcal{P}}_f^M$ denote the linearization of $\mathcal{P}_{f,q}$ in f at $f = 0$, $q = 0$, and Schwarzschild of mass $M \geq 0$. For a given linearized perturbation function $\dot{f}$,*

$$\dot{f} := \left(\dot{f}(0, \theta^1, \theta^2), \partial_u \dot{f}(0, \theta^1, \theta^2), \partial_u^2 \dot{f}(0, \theta^1, \theta^2), \partial_u^3 \dot{f}(0, \theta^1, \theta^2) \right),$$

the non-trivial components of $\dot{\mathcal{P}}_f^M(\dot{f})$ are given by

$$\begin{aligned} \dot{\Omega} &= \frac{1}{2\Omega_M} \partial_u \left(\dot{f} \Omega_M^2 \right), & \dot{\phi} &= -\Omega_M^2 \dot{f}, & \dot{\eta} &= \frac{r_M}{\Omega_M^2} \not{d} \left(\partial_u \left(\frac{\Omega_M^2}{r_M} \dot{f} \right) \right), \\ \dot{\chi} &= -2\Omega_M \mathcal{P}_2^* \not{d} \dot{f}, & (\Omega \text{tr} \chi) &= -2\partial_u \left(\frac{\dot{f} \Omega_M^2}{r_M} \right), & (\Omega \text{tr} \chi) &= \frac{2\Omega_M^2}{r_M^2} \left(\overset{\circ}{\Delta} \dot{f} - \dot{f} (1 - 2\Omega_M^2) \right), \end{aligned}$$

and

$$\underline{\dot{\omega}} = \partial_u \left(\frac{1}{2\Omega_M^2} \partial_u \left(\dot{f} \Omega_M^2 \right) \right), \quad \underline{D}\dot{\omega} = \partial_u^2 \left(\frac{1}{2\Omega_M^2} \partial_u \left(\dot{f} \Omega_M^2 \right) \right),$$

and

$$\dot{\beta} = -\frac{6M\Omega_M}{r_M^3} \not\!d f, \quad \dot{\rho} = -\frac{6M\Omega_M^2}{r_M^4} f.$$

where we tacitly evaluated at $u = 0$.

Proof of Lemma C.1. The direct way to prove Lemma C.1 is to linearize $\mathcal{P}_{f,0}$ by hand, using the explicit formulas of Appendix A. In the following, we argue that $\dot{\mathcal{P}}_f^M$ is readily calculated in [27].

Indeed, in [27] the following mapping is studied. Let $(\tilde{u}, \tilde{v}, \tilde{\theta}^1, \tilde{\theta}^2)$ be Eddington–Finkelstein coordinates on the exterior region of a Schwarzschild spacetime of small mass $M > 0$, see (2.24). Consider the sphere

$$\tilde{S} = \tilde{S}_{0,2} := \{\tilde{u} = 0, \tilde{v} = 2\}.$$

Given a smooth and sufficiently small scalar function $f = f(u, \theta^1, \theta^2)$, define new local coordinates $(u, v, \theta^1, \theta^2)$

$$\tilde{u} = u + f(u, \theta^1, \theta^2), \quad \tilde{v} = v, \quad \tilde{\theta}^1 = \theta^1, \quad \tilde{\theta}^2 = \theta^2.$$

The coordinate system $(u, v, \theta^1, \theta^2)$ is not double null. However, it is shown in (173) in [27] that $(u, v, \theta^1, \theta^2)$ is double null *to first order in f* .

Hence, to first order in f , the sphere data calculated with respect to $(u, v, \theta^1, \theta^2)$ on the sphere

$$S = S_{0,2} := \{u = 0, v = 2\},$$

agree with the sphere data $x_{0,2} := \mathcal{P}_{f,0}(\underline{\mathbf{m}}^M)$ constructed by our mapping $\mathcal{P}_{f,q}$. Consequently, their linearizations in f (evaluated at $f = 0$, $q = 0$, and Schwarzschild of mass $M > 0$) agree. This linearization is calculated in Lemma 6.1.1 in [27] and agrees with our expressions in Lemma 2.22. We note that the expression for $\underline{\dot{\omega}}$ follows from (2.48).

We remark that in [27] the linearization is calculated at Schwarzschild of mass $M > 0$, but a straightforward inspection shows that the calculation goes through for $M \geq 0$. This finishes the proof of Lemma C.1. $\square$

Second, we have the following lemma for $\dot{\mathcal{P}}_q$. It is a corollary of Lemma 6.1.3 in [27], where we note that our notation connects to [27] as follows,

$$\hat{\not\!d} = r_M^2 \not\!d_c, \quad \frac{\sqrt{\det \not\!d}}{\sqrt{\det \not\!d}} = \frac{2\dot{\phi}}{r_M}.$$

Lemma C.2 (Linearization $\dot{\mathcal{P}}_q$ of $\mathcal{P}_{f,q}$). *Let $\dot{\mathcal{P}}_q^M$ denote the linearization of $\mathcal{P}_{f,q}$ in q at $f = 0$, $q = 0$, and Schwarzschild sphere data $\mathbf{m}^M$. The non-trivial components of $\dot{\mathcal{P}}_q^M(\dot{q})$ are given by*

$$\dot{\phi} = \frac{r_M}{2} \not\!d_c \dot{q}_1, \quad \not\!d_c = 2\mathcal{P}_2^* \mathcal{P}_1^*(\dot{q}_1, \dot{q}_2).$$

C.2. Linearized Constraint Functions at Schwarzschild of Mass $M \geq 0$

In this section, we linearize the constraint functions $(\mathcal{C}_i(x))_{1 \leq i \leq 10}$ at Schwarzschild of mass $M \geq 0$, that is, at $x = \mathbf{m}^M$. The linearization procedure is adapted from [27]: We expand the sphere data

$$\begin{aligned} x &= (\Omega, \not{g}, \Omega \text{tr} \chi, \hat{\chi}, \Omega \text{tr} \underline{\chi}, \hat{\underline{\chi}}, \eta, \omega, D\omega, \underline{\omega}, \underline{D\omega}, \alpha, \underline{\alpha}) \\ &= \left(\Omega_M, r_M^2 \overset{\circ}{\gamma}, \frac{2\Omega_M}{r}, 0, -\frac{2\Omega_M}{r}, 0, 0, \frac{M}{r_M^2}, -\frac{2M\Omega_M^2}{r_M^3}, -\frac{M}{r_M^2}, -\frac{2M\Omega_M^2}{r_M^3}, 0, 0 \right) \\ &\quad + \varepsilon \cdot \left(\dot{\Omega}, \dot{\not{g}}, (\Omega \text{tr} \chi), \dot{\hat{\chi}}, (\Omega \text{tr} \underline{\chi}), \dot{\hat{\underline{\chi}}}, \dot{\eta}, \dot{\omega}, D\dot{\omega}, \dot{\underline{\omega}}, \underline{D\dot{\omega}}, \dot{\alpha}, \dot{\underline{\alpha}} \right) + \mathcal{O}(\varepsilon^2), \end{aligned}$$

and differentiate in ε at $\varepsilon = 0$. Here, the Schwarzschild quantities $r_M = r_M(u, v)$ and $\Omega_M = \Omega_M(u, v)$ are defined in (2.25) and (2.26), respectively.

The proof of the next lemma follows by explicit calculation, see also Sect. 5 in [27].

Lemma C.3 (Linearization of constraint functions at Schwarzschild). *Let $M \geq 0$ be a real number, and let $(\dot{\mathcal{C}}_i^M)_{1 \leq i \leq 10}$ denote the linearization of the constraint functions $(\mathcal{C}_i)_{1 \leq i \leq 10}$ at Schwarzschild of mass M . Then, it holds that*

$$\begin{aligned} \dot{\mathcal{C}}_1^M &= D^2 \dot{\phi} - 2\Omega_M^2 \dot{\omega} - \frac{M}{r_M} (\Omega \text{tr} \chi) - \frac{2M\Omega_M^2}{r_M^3} \dot{\phi}, \\ \dot{\mathcal{C}}_2^M &= r_M^2 \left(2D \left(\frac{\dot{\phi}}{r_M} \right) - (\Omega \text{tr} \chi) \right), \\ \dot{\mathcal{C}}_3^M &= r_M^2 D \dot{g}_c - 2\Omega_M \dot{\chi}, \\ \dot{\mathcal{C}}_4^M &= \frac{1}{r_M^2} D (r_M^2 \dot{\eta}) - \Omega_M \left(\frac{1}{r_M^2} \text{div} \dot{\hat{\chi}} - \frac{1}{2\Omega_M} \not{d} (\Omega \text{tr} \chi) + \frac{4}{r_M} \not{d} \dot{\Omega} \right), \\ \dot{\mathcal{C}}_5^M &= \frac{1}{r_M^2} D (r_M^2 (\Omega \text{tr} \underline{\chi})) - \frac{2\Omega_M^2}{r_M} (\Omega \text{tr} \chi) + \frac{2\Omega_M^2}{r_M^2} \text{div} \dot{\hat{\underline{\chi}}} \left(\dot{\eta} - \frac{2}{\Omega_M} \not{d} \dot{\Omega} \right) + \frac{4\Omega_0}{r_M^2} \dot{\Omega} + 2\Omega_M^2 \dot{K}, \end{aligned}$$

and moreover

$$\begin{aligned} \dot{\mathcal{C}}_6^M &= \Omega_M r_M D \left(\frac{\dot{\hat{\chi}}}{r_M} \right) + \dot{\hat{\chi}} \frac{\Omega_M M}{r_M^2} - 2\Omega_M^2 \mathcal{P}_2^* \left(\dot{\eta} - \frac{2}{\Omega_M} \not{d} \dot{\Omega} \right) - \frac{\Omega_M^3}{r_M} \dot{\hat{\chi}}, \\ \dot{\mathcal{C}}_7^M &= D \dot{\underline{\omega}} - \frac{2\Omega_M}{r_M^2} \dot{\Omega} - \Omega_M^2 \dot{K} + \frac{\Omega_M^2}{2r_M} \left((\Omega \text{tr} \chi) - (\Omega \text{tr} \underline{\chi}) \right), \\ \dot{\mathcal{C}}_8^M &= D \dot{\underline{\alpha}} - \frac{\Omega_M^2}{r_M} \dot{\alpha} + \frac{2M}{r_M^2} \dot{\alpha} - 2\mathcal{P}_2^* \left(\frac{\Omega_M}{r_M^2} \text{div} \dot{\hat{\chi}} - \frac{1}{2} \not{d} (\Omega \text{tr} \underline{\chi}) - \frac{\Omega_M^2}{r_M} \dot{\eta} \right) - \frac{6M\Omega_M}{r_M^3} \dot{\hat{\chi}}, \\ \dot{\mathcal{C}}_9^M &= D (\underline{D\dot{\omega}}) - \frac{2M}{r_M^3} \left(2\Omega_M^2 \dot{\omega} - \frac{3}{2} \Omega_M^2 (\Omega \text{tr} \underline{\chi}) - \left(-\frac{6}{r_M} + \frac{16M}{r_M^2} \right) \Omega_M \dot{\Omega} \right) \\ &\quad - \Omega_M^2 \left(\frac{3}{r_M} - \frac{8M}{r_M^2} \right) \left(\dot{K} + \frac{1}{2r_M} \left((\Omega \text{tr} \underline{\chi}) - (\Omega \text{tr} \chi) \right) + \frac{2\Omega_M \dot{\Omega}}{r_M^2} \right) \\ &\quad - \frac{\Omega_M^3}{r_M^2} \text{div} \dot{\hat{\chi}} \left(\frac{1}{r_M^2} \text{div} \dot{\hat{\chi}} - \frac{1}{2\Omega_M} \not{d} (\Omega \text{tr} \underline{\chi}) - \frac{\Omega_M}{r_M} \dot{\eta} \right). \\ \dot{\mathcal{C}}_{10}^M &= \Omega_M \dot{\alpha} + D \dot{\hat{\chi}} - \frac{M}{r_M^2} \dot{\hat{\chi}}. \end{aligned}$$

Remark C.4. In addition to the above, we have by (2.6) and (2.8) that

$$\dot{g} = 2r_M \dot{\phi} \dot{\gamma} + r_M^2 \dot{g}_c, \quad \dot{\omega} = D \left(\frac{\dot{\Omega}}{\Omega_M} \right), \quad \dot{\omega} = \underline{D} \left(\frac{\dot{\Omega}}{\Omega_M} \right), \quad \dot{\eta} = -\dot{\eta} + \frac{2}{\Omega_M} \not{d} \dot{\Omega}. \quad (\text{C.1})$$

Moreover, by (242) in [27] the linearization of the Gauss curvature $\dot{K}$ is given by

$$\dot{K} = \frac{1}{2r_M^2} \text{d}\not{v} \text{d}\not{v} \dot{g}_c - \frac{1}{r_M^3} (\not{d} \dot{\Omega} + 2) \dot{\phi}. \quad (\text{C.2})$$

Moreover, using that for Schwarzschild sphere data, $\phi = r_M$ and $r = r_M$, we have that $\dot{r}^{[\geq 1]} = 0$ and

$$\dot{r}^{(0)} = \dot{\phi}^{(0)}. \quad (\text{C.3})$$

C.3. Linearized Transport Equations for (E, P, L, G) at Schwarzschild

In this section, we linearize the charges $(\dot{\mathbf{E}}, \dot{\mathbf{P}}, \dot{\mathbf{L}}, \dot{\mathbf{G}})$ at Schwarzschild of mass $M \geq 0$ and analyze their transport equations along $\mathcal{H}$.

First, from Definition 2.10 and (2.15) and (2.16), we recall that

$$\begin{aligned} -\frac{8\pi}{\sqrt{4\pi}} \mathbf{E} &:= (r^3 (\rho + r \text{d}\not{v} \beta))^{(0)} \\ &= - \left(r^3 \left(K + \frac{1}{4} \text{tr} \chi \text{tr} \underline{\chi} - \frac{1}{2} (\widehat{\chi}, \widehat{\chi}) \right) \right)^{(0)} \\ &\quad - \left(r^4 \left(\text{d}\not{v} \text{d}\not{v} \widehat{\chi} - \frac{1}{2} \not{d} \text{tr} \chi + \text{d}\not{v} \left(\widehat{\chi} \cdot \zeta - \frac{1}{2} \text{tr} \chi \zeta \right) \right) \right)^{(0)}, \end{aligned} \quad (\text{C.4})$$

and

$$\begin{aligned} -\frac{8\pi}{\sqrt{\frac{4\pi}{3}}} \mathbf{P}^m &:= (r^3 (\rho + r \text{d}\not{v} \beta))^{(1m)} \\ &= - \left(r^3 \left(K + \frac{1}{4} \text{tr} \chi \text{tr} \underline{\chi} - \frac{1}{2} (\widehat{\chi}, \widehat{\chi}) \right) \right)^{(1m)} \\ &\quad - \left(r^4 \left(\text{d}\not{v} \text{d}\not{v} \widehat{\chi} - \frac{1}{2} \not{d} \text{tr} \chi + \text{d}\not{v} \left(\widehat{\chi} \cdot \zeta - \frac{1}{2} \text{tr} \chi \zeta \right) \right) \right)^{(1m)}, \quad (\text{C.5}) \\ \frac{16\pi}{\sqrt{\frac{8\pi}{3}}} \mathbf{L}^m &:= (r^3 (\not{d} \text{tr} \chi + \text{tr} \chi (\eta - \not{d} \log \Omega)))_H^{(1m)}, \\ \frac{16\pi}{\sqrt{\frac{8\pi}{3}}} \mathbf{G}^m &:= (r^3 (\not{d} \text{tr} \chi + \text{tr} \chi (\eta - \not{d} \log \Omega)))_E^{(1m)}. \end{aligned}$$

Linearizing these expressions at Schwarzschild of mass $M \geq 0$, see (2.28), and using (2.49) and (2.50), we get the explicit expressions

$$\begin{aligned}
 -\frac{8\pi}{\sqrt{4\pi}}\dot{\mathbf{E}} &= -\frac{6M\dot{\phi}^{(0)}}{r_M} + 2\dot{\phi}^{(0)} - 2r_M\Omega_M\dot{\Omega}^{(0)} + \frac{r_M^2}{2}(\Omega\dot{\text{tr}}\chi)^{(0)} - \frac{r_M^2}{2}(\Omega\dot{\text{tr}}\underline{\chi})^{(0)}, \\
 -\frac{8\pi}{\sqrt{\frac{4\pi}{3}}}\dot{\mathbf{P}}^m &= 2r_M(2 - \Omega_M)\dot{\Omega}^{(1m)} + \frac{r_M^2}{2}\left(1 - \frac{2}{\Omega_M}\right)(\Omega\dot{\text{tr}}\chi)^{(1m)} \\
 &\quad - \frac{r_M^2}{2}(\Omega\dot{\text{tr}}\underline{\chi})^{(1m)} + r_M\Omega_M\overset{\circ}{\text{d}}\dot{\chi}\dot{\eta}^{(1m)}, \\
 \frac{16\pi}{\sqrt{\frac{8\pi}{3}}}\dot{\mathbf{L}}^m &= 2r_M^2\Omega_M\dot{\eta}_H^{(1m)}, \\
 \frac{16\pi}{\sqrt{\frac{8\pi}{3}}}\dot{\mathbf{G}}^m &= \frac{r_M^3}{\Omega_M}\not{d}(\Omega\dot{\text{tr}}\chi)_E^{(1m)} + 2r_M^2\Omega_M\dot{\eta}_E^{(1m)} - 4r_M^2\not{d}\dot{\Omega}_E^{(1m)},
 \end{aligned}$$

where we used that for scalar functions f , $(\overset{\circ}{\Delta} f)^{[1]} = -2f^{[1]}$, see Appendix D.

By applying the homogeneous linearized null constraint equations at Schwarzschild, see Lemma C.3, together with (C.2) and (C.3), it is straightforward to derive transport equations for these linearized charges at Schwarzschild. The resulting equations are summarized in the following lemma.

Lemma C.5 (Linearized transport equations for charges at Schwarzschild). *The following hold for $M \geq 0$ and $m = -1, 0, 1$,*

$$\begin{aligned}
 -\frac{8\pi}{\sqrt{4\pi}}D\left(\dot{\mathbf{E}}\right) &= M(\Omega\dot{\text{tr}}\chi)^{(0)}, \\
 -\frac{8\pi}{\sqrt{\frac{4\pi}{3}}}D\left(\dot{\mathbf{P}}^m\right) &= -2\left(2(1 - \Omega_M) - \frac{6M}{r_M}\right)\dot{\Omega}^{(1m)} \\
 &\quad - \left(M\left(\frac{1}{\Omega_M} - 3\right) + r_M(1 - \Omega_M)\right)(\Omega\dot{\text{tr}}\chi)^{(1m)} \\
 &\quad - \left(\frac{M}{r_M}(2 - 3\Omega_M) + (\Omega_M - 1)\right)(\overset{\circ}{\text{d}}\dot{\chi}\dot{\eta})^{(1m)}, \\
 \frac{16\pi}{\sqrt{\frac{8\pi}{3}}}D\left(\dot{\mathbf{L}}^m\right) &= 2\Omega_MM\dot{\eta}_H^{(1m)}, \\
 \frac{16\pi}{\sqrt{\frac{8\pi}{3}}}D\left(\dot{\mathbf{G}}^m\right) &= -\frac{Mr_M}{\Omega_M}(\not{d}(\Omega\dot{\text{tr}}\chi))_E^{(1m)} + 2\Omega_MM\dot{\eta}_E^{(1m)} \\
 &\quad - \frac{4M\Omega_M}{r_M}(\not{d}\dot{\phi})_E^{(1m)} - 4M\left(\not{d}\dot{\Omega}\right)_E^{(1m)}
 \end{aligned}$$

By definition of Ω_M in (2.28) it holds that for M small,

$$|\Omega_M - 1| \lesssim M, \tag{C.6}$$

so that we can write the equations of Lemma C.5 for $M \geq 0$ small as

$$\begin{aligned}
 D\left(\dot{\mathbf{E}}\right) &= \mathcal{O}(M)(\Omega \dot{\text{tr}} \chi)^{(0)}, \\
 D\left(\dot{\mathbf{P}}^m\right) &= \mathcal{O}(M) \dot{\Omega}^{(1m)} + \mathcal{O}(M)(\Omega \dot{\text{tr}} \chi)^{(1m)} + \mathcal{O}(M)(\text{d}\dot{\nabla} \dot{\eta})^{(1m)}, \\
 D\left(\dot{\mathbf{L}}^m\right) &= \mathcal{O}(M) \dot{\eta}_H^{(1m)}, \\
 D\left(\dot{\mathbf{G}}^m\right) &= \mathcal{O}(M)(\not{d}(\Omega \dot{\text{tr}} \chi))_E^{(1m)} + \mathcal{O}(M) \dot{\eta}_E^{(1m)} \\
 &\quad + \mathcal{O}(M)(\not{d} \dot{\phi})_E^{(1m)} + \mathcal{O}(M)\left(\not{d} \dot{\Omega}\right)_E^{(1m)},
 \end{aligned} \tag{C.7}$$

where $\mathcal{O}(M)$ denotes terms that are bounded by M .

Appendix D: Hodge Systems and Fourier Theory on 2-Spheres

In this Sect. D.1, we recall the theory of 2-dimensional Hodge systems, see also [16]. In Sect. D.2, we recapitulate the definition and properties of tensor spherical harmonics, following the notation of [26]. In Sect. D.3, we use tensor spherical harmonics to analyze differential operators which appear in this paper.

D.1. Hodge Systems on Riemannian 2-Spheres

Definition D.1 (*Hodge operators*). Let (S, g) be a Riemannian 2-sphere. Define

(1) for a 1-form X_A ,

$$\mathcal{D}_1(X) := (\text{d}\nabla X, \text{curl} X).$$

(2) for a 2-tensor W_{AB} ,

$$\mathcal{D}_2(W)_C := (\text{d}\nabla W)_C.$$

(3) for a pair of functions (f_1, f_2) ,

$$\mathcal{D}_1^*(f_1, f_2) := -\not{d} f_1 + * \not{d} f_2.$$

(4) for a 1-form X_A ,

$$\mathcal{D}_2^*(X)_{AB} := -\frac{1}{2}(\nabla_A X_B + \nabla_B X_A - (\text{d}\nabla X)g_{AB}).$$

Throughout the paper, we abuse notation by denoting $\mathcal{D}_2$ as $\text{d}\nabla$. In the following, we use on the round sphere $(S_{u,v}, (v-u)^2 \overset{\circ}{g})$ the notation

$$\overset{\circ}{\mathcal{D}}_1 := (v-u)^2 \mathcal{D}_1, \quad \overset{\circ}{\mathcal{D}}_2 := (v-u)^2 \mathcal{D}_2.$$

The following lemma is a paraphrase of the material in [16].

Lemma D.2. *The following holds.*

- (1) *The kernels of $\mathcal{D}_1$ and $\mathcal{D}_2$ are trivial.*
- (2) *The kernel of $\mathcal{D}_1^*$ consists of pairs of constant functions $(f_1, f_2) = (c_1, c_2)$.*
- (3) *The kernel of $\mathcal{D}_2^*$ consists of the set of conformal Killing vectorfields (a 6-dimensional space on the round sphere).*

- (4) The L^2 -range of $\mathcal{D}_1$ consists of all pairs of functions (f_1, f_2) on S with vanishing mean.
- (5) The L^2 -range of $\mathcal{D}_2$ consists of all L^2 -integrable 1-forms on S which are orthogonal to the conformal Killing vectorfields.
- (6) The operators $\mathcal{D}_1^*$ and $\mathcal{D}_2^*$ are conformally invariant.

D.2. Tensor Spherical Harmonics

Tensor spherical harmonics are defined on the standard round unit sphere as follows.

Definition D.3 (*Tensor spherical harmonics*). Introduce the following spherical harmonics functions, vectorfields and tracefree symmetric 2-tensors.

- (1) For integers $l \geq 0$, $-l \leq m \leq l$, let $Y^{(lm)}$ be the standard (real-valued) spherical harmonics on the round unit sphere S_1 .
- (2) For $l \geq 1$, $-l \leq m \leq l$, define the vectorfields

$$E^{(lm)} := \frac{1}{\sqrt{l(l+1)}} \mathcal{D}_1^*(Y^{(lm)}, 0), \quad H^{(lm)} := \frac{1}{\sqrt{l(l+1)}} \mathcal{D}_1^*(0, Y^{(lm)}).$$

The vectorfields $E^{(lm)}$ and $H^{(lm)}$ are called *electric* and *magnetic*, respectively.

- (3) For $l \geq 2$, $-l \leq m \leq l$, define the tracefree symmetric 2-tensors

$$\psi^{(lm)} := \frac{1}{\sqrt{\frac{1}{2}l(l+1) - 1}} \mathcal{D}_2^*(E^{(lm)}), \quad \phi^{(lm)} := \frac{1}{\sqrt{\frac{1}{2}l(l+1) - 1}} \mathcal{D}_2^*(H^{(lm)}).$$

The tensors $\psi^{(lm)}$ and $\phi^{(lm)}$ are called *electric* and *magnetic*, respectively.

The following lemma is a summary of properties of spherical harmonics, see, for example, [26] for more details and proofs.

Lemma D.4. *The following holds.*

- (1) On the round unit sphere S_1 , L^2 -integrable functions f , vectorfields X and tracefree symmetric 2-tensors V can be decomposed as follows,

$$\begin{aligned} f &= \sum_{l \geq 0} \sum_{-l \leq m \leq l} f^{lm} Y^{(lm)}, \\ X &= \sum_{l \geq 1} \sum_{-l \leq m \leq l} X_E^{lm} E^{(lm)} + X_H^{lm} H^{(lm)}, \\ V &= \sum_{l \geq 2} \sum_{-l \leq m \leq l} V_\psi^{lm} \psi^{(lm)} + V_\phi^{lm} \phi^{(lm)}, \end{aligned}$$

where

$$\begin{aligned} f^{(lm)} &:= \int_{S_1} f Y^{(lm)} d\mu_{\dot{\gamma}}, \\ X_E^{(lm)} &:= \int_{S_1} X \cdot E^{(lm)} d\mu_{\dot{\gamma}}, \quad X_H^{(lm)} := \int_{S_1} X \cdot H^{(lm)} d\mu_{\dot{\gamma}}, \\ V_\psi^{(lm)} &:= \int_{S_1} V \cdot \psi^{(lm)} d\mu_{\dot{\gamma}}, \quad V_\phi^{(lm)} := \int_{S_1} V \cdot \phi^{(lm)} d\mu_{\dot{\gamma}}, \end{aligned}$$

where $d\mu_{\dot{\gamma}}$ denotes the volume element of the standard round unit metric on S_1 and $\cdot$ denotes the product with respect to $\dot{\gamma}$.

(2) It holds that for $l \geq 1$,

$$\begin{aligned} (\not{d}f)_E^{(lm)} &= -\sqrt{l(l+1)}f^{(lm)}, & (\not{d}f)_H^{(lm)} &= 0, \\ (\mathcal{P}_1^*(0, f))_E^{(lm)} &= 0, & (\mathcal{P}_1^*(0, f))_H^{(lm)} &= \sqrt{l(l+1)}f^{(lm)}, \quad (\text{D.1}) \\ (\text{div}^\circ X)^{(lm)} &= \sqrt{l(l+1)}X_E^{(lm)}, \end{aligned}$$

and for $l \geq 2$,

$$\begin{aligned} \mathcal{P}_2^*(X)_\psi^{(lm)} &= \sqrt{\frac{1}{2}l(l+1) - 1}X_E^{(lm)}, \quad \mathcal{P}_2^*(X)_\phi^{(lm)} = \sqrt{\frac{1}{2}l(l+1) - 1}X_H^{(lm)}, \\ (\text{div}^\circ V)_E^{(lm)} &= \sqrt{\frac{1}{2}l(l+1) - 1}V_\psi^{(lm)}, \quad (\text{div}^\circ V)_H^{(lm)} = \sqrt{\frac{1}{2}l(l+1) - 1}V_\phi^{(lm)}. \end{aligned} \quad (\text{D.2})$$

(3) The operator $\mathcal{P}_1$ is a bijection between vectorfields and pairs of functions (f, g) with vanishing means,

$$\mathcal{P}_1 : X^{[l \geq 1]} \rightarrow (Y^{[l \geq 1]}, Y^{[l \geq 1]}).$$

Moreover, the following restrictions are bijections:

$$\begin{aligned} \mathcal{P}_1 : E^{[l \geq 1]} &\rightarrow (Y^{[l \geq 1]}, 0), \\ \mathcal{P}_1 : H^{[l \geq 1]} &\rightarrow (0, Y^{[l \geq 1]}). \end{aligned}$$

The spherical harmonics vectorfields of mode $l = 1$ form the space of conformal Killing vectorfields of the unit round sphere.

(4) The operator $\mathcal{P}_2$ is a bijection between tracefree symmetric 2-tensors and vectorfields of modes $l \geq 2$,

$$\mathcal{P}_2 : V^{[l \geq 2]} \rightarrow X^{[l \geq 2]}.$$

Moreover, the following mappings are bijections:

$$\mathcal{P}_2 : \psi^{[l \geq 2]} \rightarrow E^{[l \geq 2]}, \quad \mathcal{P}_2 : \phi^{[l \geq 2]} \rightarrow H^{[l \geq 2]}.$$

(5) Let $k \geq 0$ be an integer. There exists a constant $C_k > 0$, depending only on k , such that for scalar functions f , vectorfields X and symmetric tracefree 2-tensors V on S_1 , we have the following equivalence of norms,

$$\begin{aligned} \sum_{0 \leq k' \leq k} \|\nabla^{k'} f\|_{L^2(S_1)}^2 &\sim \sum_{l \geq 0} \sum_{-l \leq m \leq l} (l+1)^{2k} \left(f^{(lm)} \right)^2, \\ \sum_{0 \leq k' \leq k} \|\nabla^{k'} X\|_{L^2(S_1)}^2 &\sim \sum_{l \geq 1} \sum_{-l \leq m \leq l} (l+1)^{2k} \left(\left(X_E^{(lm)} \right)^2 + \left(X_H^{(lm)} \right)^2 \right), \\ \sum_{0 \leq k' \leq k} \|\nabla^{k'} V\|_{L^2(S_1)}^2 &\sim \sum_{l \geq 2} \sum_{-l \leq m \leq l} (l+1)^{2k} \left(\left(V_\psi^{(lm)} \right)^2 + \left(V_\phi^{(lm)} \right)^2 \right). \end{aligned}$$

Notation. Given a scalar function

$$f = \sum_{l \geq 0} \sum_{-l \leq m \leq l} f^{lm} Y^{(lm)}, \quad (\text{D.3})$$

we denote, for integers $l' \geq 0$,

$$\begin{aligned} f^{[l']} &= \sum_{l=l'} \sum_{-l \leq m \leq l} f^{lm} Y^{(lm)}, \quad f^{[\geq l']} = \sum_{l \geq l'} \sum_{-l \leq m \leq l} f^{lm} Y^{(lm)}, \quad f^{[\leq l']} \\ &= \sum_{0 \leq l \leq l'} \sum_{-l \leq m \leq l} f^{lm} Y^{(lm)}, \end{aligned}$$

similarly for vectorfields X and symmetric tracefree 2-tensors V . Moreover, denote the electric part and the magnetic part of a vectorfield X by X_E and X_H , respectively, and similarly for symmetric tracefree 2-tensors V by V_ψ and V_ϕ , respectively.

D.3. Spectral Analysis of Differential Operators

In this section, we discuss the differential operators that appeared in Sect. 4.

Analysis of $\mathcal{D}_2^* \overset{\circ}{\text{d}\nabla} + 1$. Let V be a tracefree symmetric 2-tensor,

$$V = \sum_{l \geq 2} \sum_{-l \leq m \leq l} V_\psi^{(lm)} \psi^{lm} + V_\phi^{(lm)} \phi^{lm}.$$

Then,

$$(\mathcal{D}_2^* \overset{\circ}{\text{d}\nabla} + 1)V = \sum_{l \geq 2} \sum_{-l \leq m \leq l} \underbrace{\left(\frac{1}{2} l(l+1) - 1 + 1 \right)}_{>0 \text{ for } l \geq 2} \left(V_\psi^{(lm)} \psi^{lm} + V_\phi^{(lm)} \phi^{lm} \right).$$

Hence, the operator has no kernel and we have the following elliptic estimate. Let W be a given tracefree symmetric 2-tensor. Then, there exists a unique solution V to

$$\left(\mathcal{D}_2^* \overset{\circ}{\text{d}\nabla} + 1 \right) V = W,$$

and for integers $k \geq 0$ we have the estimate,

$$\|V\|_{H^{k+2}(S_1)} \lesssim \|W\|_{H^k(S_1)}. \quad (\text{D.4})$$

Analysis of $(\mathring{\text{div}} \mathcal{P}_2^* + 1 + \mathring{\mathcal{A}} \mathring{\text{div}})$. Let X be a vectorfield,

$$X = \sum_{l \geq 1} \sum_{-l \leq m \leq l} X_E^{(lm)} E^{(lm)} + X_H^{(lm)} H^{(lm)}.$$

Then, it holds that

$$\mathring{\text{div}} \mathcal{P}_2^* X = \sum_{l \geq 1} \left(\frac{1}{2} l(l+1) - 1 \right) \left(X_E^{(lm)} E^{(lm)} + X_H^{(lm)} H^{(lm)} \right),$$

and

$$\mathring{\mathcal{A}} \mathring{\text{div}} X = \sum_{l \geq 1} \sum_{-l \leq m \leq l} (-l(l+1)) X_E^{(lm)} E^{(lm)}.$$

Therefore,

$$\begin{aligned} & \left(\mathring{\text{div}} \mathcal{P}_2^* + 1 + \mathring{\mathcal{A}} \mathring{\text{div}} \right) X \\ &= \sum_{l \geq 1} \sum_{-l \leq m \leq l} X_E^{(lm)} E^{(lm)} \left(\left(\frac{1}{2} l(l+1) - 1 \right) + 1 - l(l+1) \right) \\ & \quad + \sum_{l \geq 1} \sum_{-l \leq m \leq l} X_H^{(lm)} H^{lm} \left(\left(\frac{1}{2} l(l+1) - 1 \right) + 1 \right) \\ &= \sum_{l \geq 1} \sum_{-l \leq m \leq l} X_E^{(lm)} E^{lm} \underbrace{\left(-\frac{1}{2} l(l+1) \right)}_{<0 \text{ for } l \geq 1} + \sum_{l \geq 1} X_H^{(lm)} H^{lm} \underbrace{\left(\frac{1}{2} l(l+1) \right)}_{>0 \text{ for } l \geq 1}. \end{aligned}$$

Hence, the operator has no kernel and we have the following elliptic estimate. Let Y be a given vectorfield. Then, there exists a unique solution X to

$$(\mathring{\text{div}} \mathcal{P}_2^* + 1 + \mathring{\mathcal{A}} \mathring{\text{div}}) X = Y,$$

and for integers $k \geq 0$ we have the estimate,

$$\|X\|_{H^{k+2}(S_1)} \lesssim \|Y\|_{H^k(S_1)}. \quad (\text{D.5})$$

By (D.2) and (D.5), it follows in particular that the operator

$$\mathcal{P}_2^* (\mathring{\text{div}} \mathcal{P}_2^* + 1 + \mathring{\mathcal{A}} \mathring{\text{div}}) \mathring{\text{div}}$$

has no kernel and admits the following estimate. For any given symmetric tracefree 2-tensor W , there exists a unique solution V to

$$\mathcal{P}_2^* (\mathring{\text{div}} \mathcal{P}_2^* + 1 + \mathring{\mathcal{A}} \mathring{\text{div}}) \mathring{\text{div}} V = W$$

satisfying

$$\|V\|_{H^{k+4}(S_1)} \lesssim \|W\|_{H^k(S_1)}. \quad (\text{D.6})$$

Analysis of $(\mathring{\text{div}} \mathcal{P}_2^* + 1 + \frac{1}{2} \mathring{\mathcal{A}} \mathring{\text{div}}) \mathring{\mathcal{A}}$. Let f be a scalar function,

$$f = \sum_{l \geq 0} \sum_{-l \leq m \leq l} f^{(lm)} Y^{lm}.$$

Then,

$$\mathring{\mathrm{d}}\!\!\!\!\!\mathrm{iv} \mathcal{P}_2^* \not\! \mathrm{d} f = \sum_{l \geq 0} \sum_{-l \leq m \leq l} \left(\frac{1}{2} l(l+1) - 1 \right) \left(-\sqrt{l(l+1)} \right) E^{(lm)},$$

and

$$\not\! \mathrm{d} \mathring{\mathrm{d}}\!\!\!\!\!\mathrm{iv} \not\! \mathrm{d} f = \sum_{l \geq 0} \sum_{-l \leq m \leq l} \left(-\sqrt{l(l+1)} \right) (-l(l+1)) f^{(lm)} E^{(lm)}.$$

Therefore,

$$\begin{aligned} & \left(\mathring{\mathrm{d}}\!\!\!\!\!\mathrm{iv} \mathcal{P}_2^* + 1 + \frac{1}{2} \not\! \mathrm{d} \mathring{\mathrm{d}}\!\!\!\!\!\mathrm{iv} \right) \not\! \mathrm{d} f \\ &= \sum_{l \geq 0} \sum_{-l \leq m \leq l} \underbrace{\left(\left(\frac{1}{2} l(l+1) - 1 \right) + 1 + \frac{1}{2} (-l(l+1)) \right)}_{=0} \left(-\sqrt{l(l+1)} \right) E^{lm}. \end{aligned}$$

This shows that

$$\left(\mathring{\mathrm{d}}\!\!\!\!\!\mathrm{iv} \mathcal{P}_2^* + 1 + \frac{1}{2} \not\! \mathrm{d} \mathring{\mathrm{d}}\!\!\!\!\!\mathrm{iv} \right) \not\! \mathrm{d} f = 0. \quad (\text{D.7})$$

Analysis of the operator $\left(2 - \mathring{\mathrm{d}}\!\!\!\!\!\mathrm{iv} \mathcal{P}_2^* \right)$. Let X be a vectorfield,

$$X = \sum_{l \geq 1} \sum_{-l \leq m \leq l} X_E^{(lm)} E^{(lm)} + X_H^{(lm)} H^{(lm)}.$$

Then, it holds that

$$\begin{aligned} (2 - \mathring{\mathrm{d}}\!\!\!\!\!\mathrm{iv} \mathcal{P}_2^*) X &= \sum_{l \geq 1} \left(2 - \left(\frac{1}{2} l(l+1) - 1 \right) \right) \left(X_E^{(lm)} E^{(lm)} + X_H^{(lm)} H^{(lm)} \right) \\ &= \sum_{l \geq 1} \frac{1}{2} (6 - l(l+1)) \left(X_E^{(lm)} E^{(lm)} + X_H^{(lm)} H^{(lm)} \right) \end{aligned}$$

We conclude that the kernel of the operator $\left(2 - \mathring{\mathrm{d}}\!\!\!\!\!\mathrm{iv} \mathcal{P}_2^* \right)$ is given by the set of vectorfields

$$\{X : X = X^{[2]}\}. \quad (\text{D.8})$$

Further, let Y be a vectorfield such that $Y = Y^{[\geq 3]}$. Then, there exists a unique vectorfield X such that $X = X^{[\geq 3]}$ and

$$(2 - \mathring{\mathrm{d}}\!\!\!\!\!\mathrm{iv} \mathcal{P}_2^*) X = Y,$$

with the estimate

$$\|X\|_{H^{k+2}(S_1)} \lesssim \|Y\|_{H^k(S_1)}.$$

In particular, it follows moreover that for any given function f with $f = f^{[\geq 3]}$, there is a unique solution V with $V = V^{[\geq 3]}$ of

$$\mathring{\mathrm{d}}\!\!\!\!\!\mathrm{iv} \left(2 - \mathring{\mathrm{d}}\!\!\!\!\!\mathrm{iv} \mathcal{P}_2^* \right) \mathring{\mathrm{d}}\!\!\!\!\!\mathrm{iv} V = f.$$

with the estimate

$$\|X\|_{H^{k+4}(S_1)} \lesssim \|f\|_{H^k(S_1)}. \quad (\text{D.9})$$

References

- [1] Angelopoulos, Y., Aretakis, S., Gajic, D.: Price’s Law and Precise Late-Time Asymptotics for Subextremal Reissner–Nordström Black Holes, p. 65. [arXiv:2102.11888](#)
- [2] Angelopoulos, Y., Aretakis, S., Gajic, D.: Late-Time Tails and Mode Coupling of Linear Waves on Kerr Spacetimes, p. 100. [arXiv:2102.11884](#)
- [3] Angelopoulos, Y., Aretakis, S., Gajic, D.: Late-time asymptotics for the wave equation on extremal Reissner–Nordström backgrounds. *Adv. Math.* **375**, 107363, 139 (2020)
- [4] Angelopoulos, Y., Aretakis, S., Gajic, D.: Late-time asymptotics for the wave equation on spherically symmetric, stationary spacetimes. *Adv. Math.* **323**, 529–621 (2018)
- [5] Angelopoulos, Y., Aretakis, S., Gajic, D.: Horizon hair of extremal black holes and measurements at null infinity. *Phys. Rev. Lett.* **121**(13), 131102 (2018)
- [6] Aretakis, S.: The characteristic gluing problem and conservation laws for the wave equation on null hypersurfaces. *Ann. PDE* **3**(1), 3–56 (2017)
- [7] Aretakis, S.: On a foliation-covariant elliptic operator on null hypersurface. *Int. Math. Res. Not.* **15**, 6433–6469 (2015)
- [8] Aretakis, S.: Stability and instability of extreme Reissner–Nordström black hole spacetimes for linear scalar perturbations I. *Commun. Math. Phys.* **307**(1), 17–63 (2011)
- [9] Aretakis, S.: Stability and instability of extreme Reissner–Nordström black hole spacetimes for linear scalar perturbations II. *Ann. Henri Poincaré* **12**(8), 1491–1538 (2011)
- [10] Aretakis, S.: Horizon instability of extremal black holes. *Adv. Theor. Math. Phys.* **19**(3), 507–530 (2015)
- [11] Aretakis, S., Czimek, S., Rodnianski, I.: Characteristic gluing for the Einstein equations and applications, p. 31
- [12] Aretakis, S., Czimek, S., Rodnianski, I.: Characteristic Gluing to the Kerr Family and Application to Spacelike Gluing, p. 88
- [13] Burko, L., Khanna, G., Sabharwal, S.: Scalar and gravitational hair for extreme Kerr black holes. *Phys. Rev. D* **103**, 021502 (2021)
- [14] Carlotto, A., Schoen, R.: Localizing solutions of the Einstein constraint equations. *Invent. Math.* **205**(3), 559–615 (2016)
- [15] Christodoulou, D.: The formation of black holes in general relativity. European Mathematical Society (EMS), Zürich, x+589 pp (2009)
- [16] Christodoulou, D., Klainerman, S.: The Global Nonlinear Stability of the Minkowski Space. Princeton Mathematical Series, 41. Princeton University Press, Princeton, NJ, x+514 (1993)
- [17] Chruściel, P., Delay, E.: Existence of non-trivial, vacuum, asymptotically simple spacetimes. *Class. Quant. Grav.* **19**(9), L71 (2002)

- [18] Chruściel, P., Delay, E.: On mapping properties of the general relativistic constraints operator in weighted function spaces, with applications. *Mém. Soc. Math. Fr. (N.S.)* **94**, vi+103 (2003)
- [19] Chruściel, P., Isenberg, J., Pollack, D.: Gluing initial data sets for general relativity. *Phys. Rev. Lett.* **93**(8), 081101 (2004)
- [20] Chruściel, P., Isenberg, J., Pollack, D.: Initial data engineering. *Commun. Math. Phys.* **257**(1), 29–42 (2005)
- [21] Chruściel, P., Mazzeo, R.: On ‘many-black-hole’ vacuum spacetimes. *Class. Quant. Grav.* **20**(4), 729–754 (2003)
- [22] Chruściel, P., Pollack, D.: Singular Yamabe metrics and initial data with exactly Kottler–Schwarzschild–de Sitter ends. *Ann. Henri Poincaré* **9**(4), 639–654 (2008)
- [23] Cortier, J.: Gluing construction of initial data with Kerr-de Sitter ends. *Ann. Henri Poincaré* **14**(5), 1109–1134 (2013)
- [24] Corvino, J.: Scalar curvature deformation and a gluing construction for the Einstein constraint equations. *Commun. Math. Phys.* **214**(1), 137–189 (2000)
- [25] Corvino, J., Schoen, R.: On the asymptotics for the vacuum Einstein constraint equations. *J. Differ. Geom.* **73**(2), 185–217 (2006)
- [26] Czimek, S.: An extension procedure for the constraint equations. *Ann. PDE* **4**(1), 130 (2018)
- [27] Dafermos, M., Holzegel, G., Rodnianski, I.: The linear stability of the Schwarzschild solution to gravitational perturbations. *Acta Math.* **222**(1), 1–214 (2019)
- [28] Ebin, D., Marsden, J.: Groups of Diffeomorphisms and the Motion of an Incompressible Fluid. *Ann. Math. Second Ser.* **92**(1), 102–163 (1970)
- [29] Gromov, M., Lawson, H.: The classification of simply connected manifolds of positive scalar curvature. *Ann. Math. (2)* **111**(3), 423–434 (1980)
- [30] Hawking, S., Ellis, G.: The large scale structure of space-time. *Cambridge Monographs on Mathematical Physics*, No. 1, xi+391. Cambridge University Press, London (1973).
- [31] Hintz, P.: Black hole gluing in de Sitter space. *Commun. Partial Differ. Equ.* (2021). <https://doi.org/10.1080/03605302.2020.1871368>
- [32] Isenberg, J., Maxwell, D., Pollack, D.: A gluing construction for non-vacuum solutions of the Einstein-constraint equations. *Adv. Theor. Math. Phys.* **9**(1), 129–172 (2005)
- [33] Isenberg, J., Mazzeo, R., Pollack, D.: Gluing and wormholes for the Einstein constraint equations. *Commun. Math. Phys.* **231**(3), 529–568 (2002)
- [34] Isenberg, J., Mazzeo, R., Pollack, D.: On the topology of vacuum spacetimes. *Ann. Henri Poincaré* **4**, 369–383 (2003)
- [35] Luk, J.: On the local existence for the characteristic initial value problem in general relativity. *Int. Math. Res. Not. IMRN* **20**, 4625–4678 (2012)
- [36] Luk, J., Rodnianski, I.: Local propagation of impulsive gravitational waves. *Commun. Pure Appl. Math.* **68**(4), 511–624 (2015)
- [37] Ma, S., Zhang, L.: Sharp decay estimates for massless Dirac fields on a Schwarzschild background. (2020). [arXiv:2008.11429](https://arxiv.org/abs/2008.11429)

- [38] Ratiu, T., Abraham, R., Marsden, J. E.: Manifolds, tensor analysis, and applications. Third edition. Applied Mathematical Sciences, 75, p. x+654. Springer, New York (1988)
- [39] Rendall, A.: Reduction of the characteristic initial value problem to the Cauchy problem and its applications to the Einstein equations. Proc. R. Soc. Lond. Ser. A **427**(1872), 221–239 (1990)
- [40] Schoen, R., Yau, S.: On the structure of manifolds with positive scalar curvature. Manuscripta Math. **28**(1–3), 159–183 (1979)
- [41] Szeftel, J.: Parametrix for wave equations on a rough background III: space-time regularity of the phase. Astérisque **401**, viii+321 (2018)
- [42] Taylor, M.: Partial Differential Equations III: Nonlinear equations. Applied Mathematical Sciences, 117, xxii+608. Springer, New York (1997)

Stefanos Aretakis
Department of Mathematics
University of Toronto
40 St George Street
Toronto ON
Canada
e-mail: aretakis@math.toronto.edu

Stefan Czimek
Mathematisches Institut
Universität Leipzig
Augustusplatz 10
04109 Leipzig
Germany
e-mail: stefan.czimek@uni-leipzig.de

Igor Rodnianski
Department of Mathematics
Princeton University
Fine Hall, Washington Road
Princeton NJ 08544
USA
e-mail: irod@math.princeton.edu

Communicated by Mihalis Dafermos.

Received: September 22, 2021.

Accepted: November 10, 2023.