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Characteristic Gluing to the Kerr Family and Application to Spacelike Gluing

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Abstract: This is the third paper in a series of papers adressing the characteristic gluing problem for the Einstein vacuum equations. We provide full details of our characteristic gluing (including the 10 charges) of strongly asymptotically flat data to the data of a suitably chosen Kerr spacetime. The choice of the Kerr spacetime crucially relies on relating the 10 charges to the ADM energy, linear momentum, angular momentum and the center-of-mass. As a corollary, we obtain an alternative proof of the Corvino-Schoen spacelike gluing construction for strongly asymptotically flat spacelike initial data.

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1. Introduction

The gluing problem in general relativity investigates whether it is possible to join two given vacuum spacetimes. Concretely, one can approach this problem by attempting to construct a solution to the constraint equations which agrees inside a bounded domain with specified initial data, and on the complement of a large ball with other specified initial data. The geometric obstructions to solving the gluing problem provide insights into the rigidity properties of the Einstein equations.

In [11] we initiated the study of the characteristic gluing problem for initial data for the Einstein vacuum equations. This problem amounts to connecting two initial data sets along a truncated null hypersurface by solving the null constraint equations. There are several reasons for considering the characteristic gluing problem: (1) the null constraint equations are of transport character (in contrast to the previously studied gluing problem for spacelike initial data which requires to analyze the elliptic Riemannian constraint equations), (2) the null lapse function and the conformal geometry of the characteristic hypersurface can be freely prescribed, (3) characteristic gluing of spacetimes implies spacelike gluing of the spacetimes.

In [11,12] we explicitly derived a 10-dimensional space of gauge-invariant charges on sections of null hypersurfaces that act as obstructions to the characteristic gluing problem and we showed that, modulo this 10-dimensional space, characteristic gluing is always possible for data sets that are close to the Minkowski data. In this paper, we prove that characteristic initial data that are close to the Minkowski data can be fully glued (including the 10 charges) to the characteristic data of a suitable Kerr spacetime. By rescaling we show that strongly asymptotically flat data can also be characteristically glued to the data of some Kerr spacetime. As a corollary, we obtain an alternative proof of the Corvino–Schoen gluing construction (for strongly asymptotically flat spacelike initial data) that relies on solving the null constraint equations instead of the Riemannian constraint equations. Our approach crucially relies on relating the 10 charges to the ADM energy, linear momentum, angular momentum and center-of-mass.

In Sect. 1.1 we discuss the characteristic gluing problem. In Sect. 1.2 we outline the main results. In Sect. 1.3 we give an overview of the main ideas of the proofs.

1.1. The characteristic gluing problem. In this section we discuss the *codimension*-10 *characteristic gluing* for the Einstein vacuum equations introduced in [11]. Before stating the main results of that paper, we introduce the following notation. Let $(\mathcal{M}_1, \mathbf{g}_1)$ and $(\mathcal{M}_2, \mathbf{g}_2)$ be two vacuum spacetimes. Let S_1 and S_2 be two spacelike 2-spheres in $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, and assume (without loss of generality) they are each intersection spheres of local double null coordinate systems, respectively. We define *sphere data* x_1 on S_1 and x_2 on S_2 to be given by the respective restriction of the metric components, Ricci coefficients and components of the Riemann curvature tensor of the spacetimes to the respective spheres (see also Sect. 1.3.2) with respect to the respective double null coordinate system.

One of the main insights of [11,12] is the derivation of a family of charges on the sections of null hypersurfaces that act as obstructions to the characteristic gluing problem. The charges arise from conservation laws for the linearized constraint equations. They split into two classes: An infinite-dimensional space of *gauge-dependent charges* and a 10-dimensional space of *gauge-invariant charges*. The former charges can always be overcome by gauge perturbations. We will refer to the gauge-invariant charges as simply the charges. For further discussion, see Sect. 1.2.1. For precise definition of the charges, see Sects. 1.3.2 and 2.5.

The main result of [11,12] can be summarized as follows; see Theorem 2.20 in Sect. 2.9 for the precise statement.

Perturbative codimension-10 **characteristic gluing** [11,12]. *Let on two spheres S_1 and S_2 be given the sphere data x_1 and x_2, sufficiently close to the respective sphere data on the round spheres of radius* 1 *and* 2 *in Minkowski spacetime, respectively. Then there is a null hypersurface $\mathcal{H}'_{[1,2]}$, connecting the sphere data x_1 on S_1 to a transversal perturbation S'_2 of the sphere S_2 with sphere data x'_2, solving the null constraint equations, and such that all derivatives tangential to $\mathcal{H}'_{[1,2]}$ of the sphere data x_1 and x'_2 are – up to a* 10*-dimensional space of charges explicitly defined at S'_2 – smoothly glued.*

Sphere data determines all derivatives of the metric components up to order 2, hence *perturbative gluing is gluing at the level of C^2* of the metric components (up to the 10 charges).

In [11,12] we also consider characteristic gluing along two null hypersurfaces bifurcating from an auxiliary sphere, and prove the following result, see also Fig. 1; see Theorem 2.21 in Sect. 2.9 for the precise citation from [12].

Bifurcate codimension-10 **characteristic gluing** [11,12]. *Let $m \geq 2$ be an integer. Consider two spheres S_1 and S_2 equipped with sphere data x_1 and x_2 as well as prescribed m^{th}-order derivatives in all directions, respectively. If this m^{th}-order data on S_1 and S_2 is sufficiently close to the respective m^{th}-order data on the round spheres of radius* 1

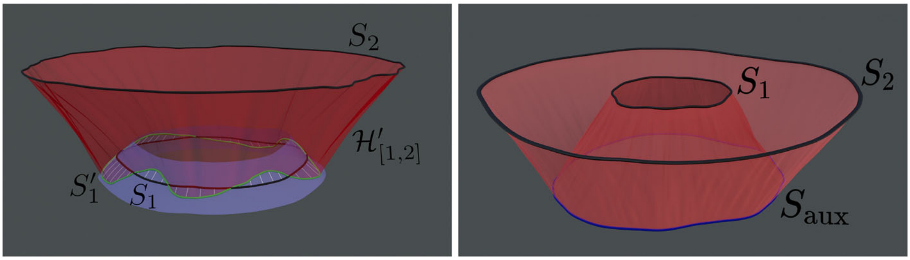

Fig. 1. Perturbative codimension-10 characteristic gluing (on the left) and higher-order codimension-10 characteristic gluing along two null hypersurfaces bifurcating from an auxiliary sphere (on the right)

and 2 *in Minkowski spacetime, then it is possible to characteristically glue – up to a* 10*-dimensional space of charges – the* m^{th}*-order data of* S_1 *and* S_2 *along two null hypersurfaces bifurcating from an auxiliary sphere* S_{aux}*.*

We note that in the above result the spheres S_1 and S_2 are not perturbed. Moreover, *bifurcate gluing is higher-regularity gluing*, that is, we can glue any order $m \geq 2$ of derivatives of the metric components (up to the 10-dimensional space of charges).

We note that the characteristic gluing problem was previously studied by the first author [8,9] in the much simpler setting of the linear homogeneous wave equation on general (but fixed) Lorentzian manifolds. Similarly to the present paper, [8] determined that the only obstructions to solving the characteristic gluing problem are conservation laws along null hypersurfaces. In the following it was shown that these conservation laws have important applications in the study of the evolution of scalar perturbations on both sub-extremal [1,2,5,39] and extremal [3,4,6,7,10,16] black hole spacetimes.

1.2. Main results on the characteristic gluing to the Kerr family. In this section we outline the main results of this paper on the characteristic gluing to Kerr.

1.2.1. Geometric interpretation of charges As discussed above, the characteristic gluing of [11,12] holds up to a 10-dimensional space of charges. These charges are calculated as integrals over spacelike 2-spheres and are denoted by the real number **E** and the 3-dimensional vectors **P**, **L** and **G**. At the linear level, the charges **E** and **P** are proportional to the modes $l = 0$ and $l = 1$ of $\rho + r\, \mathrm{d}\!\!/\mathrm{iv} \beta$, while **L** and **G** are proportional to the magnetic and electric parts of the mode $l = 1$ of β, see Sect. 2.5 for precise definitions.

Theorem 1.1. *Given a strongly asymptotically flat family of sphere data on spheres* S_R*, as defined in Sect. 2.7, the charges* $(\mathbf{E}_R, \mathbf{P}_R, \mathbf{L}_R, \mathbf{G}_R)$ *have a limit* $(\mathbf{E}_\infty, \mathbf{P}_\infty, \mathbf{L}_\infty, \mathbf{G}_\infty)$*, called the asymptotic charges. In case the spheres* S_R *lie in a strongly asymptotically flat spacelike hypersurface, the asymptotic charges are related to the ADM asymptotic invariants of the spacelike hypersurface by*

$$(\mathbf{E}_\infty, \mathbf{P}_\infty, \mathbf{L}_\infty, \mathbf{G}_\infty) = (\mathbf{E}_{\text{ADM}}, \mathbf{P}_{\text{ADM}}, \mathbf{L}_{\text{ADM}}, \mathbf{C}_{\text{ADM}}),$$

where $\mathbf{E}_{\text{ADM}}$ *denotes the energy (often called mass),* $\mathbf{P}_{\text{ADM}}$ *the linear momentum,* $\mathbf{L}_{\text{ADM}}$ *the angular momentum, and* $\mathbf{C}_{\text{ADM}}$ *the center-of-mass.*

Our definitions of the charges are, to leading order, consistent with previous definitions in general relativity of mass, linear and angular momentum in terms of integrals over spheres; see [35,36,44].

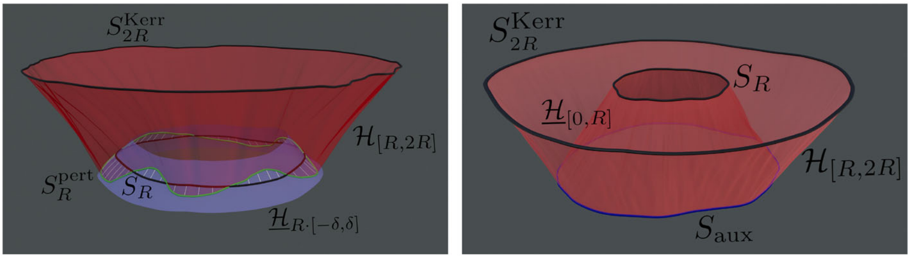

Fig. 2. Perturbative (left) and bifurcate (right) characteristic gluing to the Kerr family

1.2.2. Perturbative characteristic gluing to Kerr The following is a first version of our main result for characteristic gluing to Kerr along one null hypersurface, see Theorem 3.1 for a precise version.

Theorem 1.2. *Consider a strongly asymptotically flat family of sphere data x_R on spheres S_R. For $R \geq 1$ sufficiently large, there exist (1) a perturbation S_R^{pert} of S_R along the ingoing null hypersurface $\underline{\mathcal{H}}_{R\cdot[-\delta,\delta]}$, (2) a sphere S_{2R}^{Kerr} in some Kerr spacetime, and (3) a null hypersurface $\mathcal{H}_{[R,2R]}$, solving the constraint equations, and connecting S_R^{pert} and S_{2R}^{Kerr} and their respective sphere data.*

The above perturbative characteristic gluing to Kerr is C^2-gluing for the metric components. In Theorem 1.2 we glue to a reference sphere in Kerr. We could alternatively glue to a perturbation of the reference sphere in Kerr to avoid perturbing S_R to S_R^{pert}. In Theorem 1.2 it is not necessary to have a family of sphere data data. Indeed, one can replace this family with one fixed sphere datum with sufficiently strong bounds.

1.2.3. Bifurcate characteristic gluing to Kerr We can also characteristically glue m^{th}-order derivatives in all directions, for any integer $m \geq 2$, (without perturbing any of the spheres) to Kerr by applying the *bifurcate* characteristic gluing of [11,12], see the discussion above and Theorem 2.21 below. This yields higher-regularity gluing of metric components. We refer to Theorem 3.2 for a precise version of the following; see also Fig. 2 below.

Theorem 1.3. *Let $m \geq 2$ be an integer. On spheres S_R let x_R be a strongly asymptotically flat family of sphere data together with prescribed m^{th}-order derivatives. For $R \geq 1$ sufficiently large, we can characteristically glue, to m^{th}-order, along two null hypersurfaces bifurcating from an auxiliary sphere, the sphere S_R to a sphere S_{2R}^{Kerr} in some Kerr spacetime.*

In Theorem 1.3 it is not necessary to have a strongly asymptotic family of sphere data. Indeed, one can replace this family with one fixed sphere data with sufficiently strong bounds.

1.2.4. Spacelike gluing to Kerr As corollary of Theorem 1.3 we can deduce spacelike gluing to Kerr for strongly asymptotically flat spacelike initial data, see Corollary 3.3 for a precise version, and Fig. 3 below.

Corollary 1.4. *Let $m \geq 0$ be an integer. Let (Σ, g, k) be smooth strongly asymptotically flat spacelike initial data with asymptotic invariants $(\mathbf{E}_{\mathrm{ADM}}, \mathbf{P}_{\mathrm{ADM}}, \mathbf{L}_{\mathrm{ADM}}, \mathbf{C}_{\mathrm{ADM}})$ such that $(\mathbf{E}_{\mathrm{ADM}})^2 > |\mathbf{P}_{\mathrm{ADM}}|^2$. Then, sufficiently far out, (g, k) can be glued in C^m-regularity across a compact region to spacelike initial data for some Kerr spacetime.*

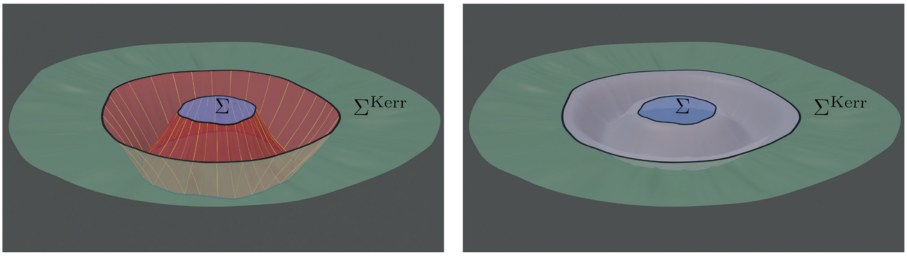

Fig. 3. Application of bifurcate characteristic gluing in the proof of smooth spacelike gluing to Kerr

The assumption of strong asymptotic flatness of the spacelike initial data corresponds to working in the center-of-mass frame of the isolated gravitational system, see [19]. The proof of Corollary 1.4 is by combining the bifurcate gluing to Kerr with local existence [37,38] for the characteristic initial value problem. In particular, the ∞^{th}-order version of bifurcate codimension-10 characteristic gluing (see remarks above) should yield a smooth spacelike gluing to Kerr; however, we will not provide details here.

1.3. Overview of the main ideas.

1.3.1. Main steps In this section we will outline the main steps of the proofs of Theorems 1.2 and 1.3. We refer the reader to Fig. 4 below for an illustration of the relevant spheres and charges.

(1) We setup the problem by distinguishing two cases:
  (a) We are given a strongly asympotically flat spacelike initial data set (Σ, g, k) foliated by 2-spheres S_R.
  (b) We are given a strongly asymptotically flat family of spheres S_R with sphere data (x_R) which are not lying in strongly asymptotically flat spacelike initial data.

(2) We apply the perturbative codimension-10 characteristic gluing of [11,12] to glue a perturbation $S_R^{\text{pert}(\lambda_R)}$ of S_R to a sphere $S_{2R}^{\lambda_R}$ in some Kerr spacetime λ_R along a null hypersurface $\mathcal{H}_{[R,2R]}$. Here the vector λ_R parametrizes the Kerr spacetimes through asymptotic invariants, see Step (4) below. The gluing holds up to the 10-dimensional space of charges. We denote the associated charges on S_R and $S_R^{\text{pert}(\lambda_R)}$ by Q_R and $Q_R^{\text{pert}(\lambda_R)}$, respectively. Moreover, we denote by $Q_{2R}^{\lambda_R}$ the charges on $S_{2R}^{\lambda_R}$ calculated from Kerr, and by $Q_{2R}^{\text{glue}(\lambda_R)}$ the charges on the same sphere calculated from the gluing solution on $\mathcal{H}_{[R,2R]}$. We consider the charge difference

$$(\Delta Q)(\lambda_R) := Q_{2R}^{\lambda_R} - Q_{2R}^{\text{glue}(\lambda_R)}. \tag{1.1}$$

Our goal is to determine a Kerr parameter λ_R for which $(\Delta Q)(\lambda_R) = 0$.

(3) We derive asymptotic expansions for the charges Q_R for large $R \geq 1$ (and denote the limits by Q_∞). In case (a) of a spacelike hypersurface Σ, we show that these are related to the asymptotic invariants $\text{AI}(\Sigma)$ of (Σ, g, k), thus yielding a geometric interpretation of the charges. See Sect. 1.3.3.

(4) We make use of the 10-dimensional parametrization $\lambda \in \mathbb{R}^{10}$ of Kerr spacelike initial data $(\Sigma^\lambda, g^\lambda, k^\lambda)$ by its asymptotic invariants $\text{AI}(\Sigma^\lambda)$ developed in Chruściel–Delay [21], and consider spheres S_{2R}^λ with sphere data x_{2R}^λ lying in these spacelike initial

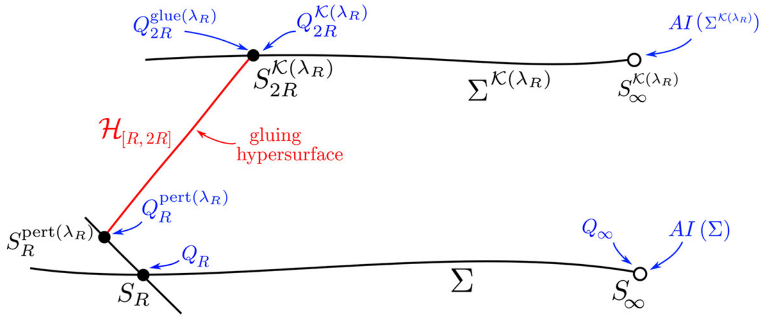

Fig. 4. The relevant spheres and charges for the characteristic gluing to the Kerr family

data. In case (a) it suffices to consider the parameter λ of Kerr spacelike initial data to lie in a ball B of finite radius in $\mathbb{R}^{10}$. In case (b) we need to consider a larger set of parameters $\lambda = \lambda_R$, namely, an ellipsoid $\mathcal{E}_R$ with semi-major axis proportional to $R^{1/2}$. See Sect. 1.3.4.

(5) We derive a homotopy between the charge difference $(\Delta Q)(\lambda_R)$ and an appropriate difference between the asymptotic charges Q_∞ and the asymptotic invariants $\mathrm{AI}(\Sigma^{\lambda_R})$ of the Kerr initial data Σ^{λ_R}. The latter difference is shown to always admit a root λ'_R. We prove uniform estimates for the homotopy that allow us to conclude by a topological degree argument that the charge difference $(\Delta Q)(\lambda_R)$ also admits a root. See Sect. 1.3.4.

1.3.2. Sphere data and charges For a given sphere S, the sphere data x on S is given by the following geometric components on S,

$$x = (\Omega, \not{g}, \mathrm{tr}\chi, \widehat{\chi}, \mathrm{tr}\underline{\chi}, \underline{\widehat{\chi}}, \eta, \omega, D\omega, \underline{\omega}, \underline{D\omega}, \alpha, \beta, \rho, \sigma, \underline{\beta}, \underline{\alpha}).$$

The above components are expressed in a null frame in the context of a double null coordinate system. For the precise definitions we refer to Sect. 2.1. The data x determines all derivatives of the spacetime time metric up to order 2 (see also Sect. 2.4). Furthermore, given sphere data on S we introduce the associated charges $Q = (\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$ on S to be the integrals (2.21) over S.

We define *strongly asymptotically flat sphere data* on a family of spheres S_R in accordance with the decay towards *spacelike* infinity in the works [19] and [34]. Indeed, we show by explicit construction that each strongly asymptotically flat spacelike initial data admits strongly asymptotically flat sphere data on families of spheres (but in general we do not assume that sphere data stems from spacelike initial data). Our construction is such that the special case of Schwarzschild spacelike data (expressed in isotropic coordinates so that strong asymptotic flatness holds) leads precisely to the family of Schwarzschild reference sphere data with respect to Eddington–Finkelstein double null coordinates. To achieve this for general spacelike initial data, we rescale the coordinate sphere S_r to S_1, apply the coordinate change from isotropic coordinates to Schwarzschild coordinates, make appropriate gauge choices for $\widehat{L}$, $\underline{\widehat{L}}$ and Ω, and use the definitions of Ricci coefficients and null curvature components. We show that the rescaled quantities are well-defined and derive estimates and then we rescale back up. See Sects. 7.1 and 7.2.

1.3.3. Relation of charges to asymptotic invariants For strongly asymptotically flat families of sphere data x_R on spheres S_R, the following asymptotic expansions hold for large $R \geq 1$,

$$\mathbf{E}\,(S_R) = \mathbf{E}_\infty + \mathcal{O}(R^{-1/2}),\ \mathbf{P}\,(S_R) = \mathcal{O}(R^{-1/2}),$$
$$\mathbf{L}\,(S_R) = \mathbf{L}_\infty + \mathcal{O}(1),\ \mathbf{G}\,(S_R) = \mathbf{G}_\infty + \mathcal{O}(1).$$

where $(\mathbf{E}_\infty, \mathbf{P}_\infty = 0, \mathbf{L}_\infty, \mathbf{G}_\infty)$ are defined as the limits of $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$ on S_R as $R \to \infty$. These asymptotic charges can also be defined for more general families of sphere data (not necessarily strongly asymptotically flat), in which case we expect $\mathbf{P}_\infty \neq 0$ and $\mathbf{G}_\infty = +\infty$. On the other hand, if the family of sphere data lies in strongly asymptotically flat spacelike initial data, then we have the following stronger decay rates for $\mathbf{E}$ and $\mathbf{P}$,

$$\mathbf{E}(S_R) = \mathbf{E}_\infty + \mathcal{O}(R^{-1}),\ \ \mathbf{P}(S_R) = \mathcal{O}(R^{-3/2}). \tag{1.2}$$

The charges $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$ have a well-defined connection to the ADM asymptotic invariants. Indeed, for spheres S_R in asymptotically flat spacelike initial data with well-defined ADM asymptotic invariants, we can relate the charges $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$ to the local integrals $\mathbf{E}^{\mathrm{loc}}_{\mathrm{ADM}}$ of the ADM *energy*, $\mathbf{P}^{\mathrm{loc}}_{\mathrm{ADM}}$ of the ADM *linear momentum*, $\mathbf{L}^{\mathrm{loc}}_{\mathrm{ADM}}$ of the ADM *angular momentum*, and $\mathbf{C}^{\mathrm{loc}}_{\mathrm{ADM}}$ of the ADM *center-of-mass* as follows,

$$\begin{aligned} \mathbf{E}(S_R) =&\mathbf{E}^{\mathrm{loc}}_{\mathrm{ADM}}(S_R) + \mathcal{O}(1), \quad \mathbf{P}(S_R) =\mathbf{P}^{\mathrm{loc}}_{\mathrm{ADM}}(S_R) + \mathcal{O}(1),\\ \mathbf{L}(S_R) =&\mathbf{L}^{\mathrm{loc}}_{\mathrm{ADM}}(S_R) + \mathcal{O}(1), \quad \mathbf{G}(S_R) =\mathbf{C}^{\mathrm{loc}}_{\mathrm{ADM}}(S_R) - R\cdot \mathbf{P}^{\mathrm{loc}}_{\mathrm{ADM}}(S_R) + \mathcal{O}(1). \end{aligned} \tag{1.3}$$

Hence in that case,

$$\mathbf{E}_\infty = \mathbf{E}_{\mathrm{ADM}},\ \ \mathbf{P}_\infty = \mathbf{P}_{\mathrm{ADM}},\ \ \mathbf{L}_\infty = \mathbf{L}_{\mathrm{ADM}}.$$

For families of sphere data in an asymptotically flat spacelike hypersurface with non-vanishing total ADM linear momentum $\mathbf{P}_{\mathrm{ADM}} \neq 0$ (such hypersurfaces are not strongly asymptotically flat), (1.3) shows that $\mathbf{P}_\infty \neq 0$, and subsequently, $|\mathbf{G}_\infty| = +\infty$. Importantly, the Kerr spacelike initial data satisfies in general $\mathbf{P}_{\mathrm{ADM}} \neq 0$ which has significant repercussions for our analysis of the gluing problem to the Kerr family. In particular, it forces us to consider and prove delicate estimates for spacelike initial data with very large center-of-mass $\mathbf{C}_{\mathrm{ADM}}$, see also the discussion in Sect. 1.3.4.

On the other hand, for families of sphere data in strongly asymptotically flat spacelike hypersurfaces (in which case $\mathbf{P}_{\mathrm{ADM}} = 0$ and $\mathbf{P}^{\mathrm{loc}}_{\mathrm{ADM}}(S_R) = \mathcal{O}(R^{-3/2})$), we have by (1.3) that $\mathbf{G}_\infty = \mathbf{C}_{\mathrm{ADM}}$ is well-defined.

1.3.4. Choice of Kerr to glue to The goal is to prove that for sufficiently large $R \geq 1$ we can characteristically glue to a Kerr sphere S^λ_{2R}. Ideally, one would like to consider a fixed set of Kerr parameters λ such that the spheres S^λ_{2R} in the corresponding spacelike initial data have asymptotic charges $(\mathbf{E}^{\mathrm{Kerr}}_\infty, \mathbf{P}^{\mathrm{Kerr}}_\infty, \mathbf{L}^{\mathrm{Kerr}}_\infty, \mathbf{G}^{\mathrm{Kerr}}_\infty)$ close to the given $(\mathbf{E}_\infty, \mathbf{P}_\infty, \mathbf{L}_\infty, \mathbf{G}_\infty)$, and subsequently argue that there exists a λ in that set which solves the gluing problem.

However, for each fixed λ with $\mathbf{P}^{\mathrm{Kerr}}_{\mathrm{ADM}} \neq 0$ we have by (1.3) that as $R \to \infty$,

$$\mathbf{G}^{\mathrm{Kerr}}(S^\lambda_{2R}) = \underbrace{\mathbf{C}^{\mathrm{loc}}_{\mathrm{ADM}}(S^\lambda_{2R})}_{\to \mathbf{C}^{\mathrm{Kerr}}_{\mathrm{ADM}}} - (2R)\cdot \underbrace{\mathbf{P}^{\mathrm{loc}}_{\mathrm{ADM}}(S^\lambda_{2R})}_{\to \mathbf{P}^{\mathrm{Kerr}}_{\mathrm{ADM}}} + \mathcal{O}(1) \to \infty, \tag{1.4}$$

which in particular shows that $\mathbf{G}_\infty^{\mathrm{Kerr}}$ is far away from matching the finite $\mathbf{G}_\infty$. Thus we have to change our approach and consider an R-*dependent* set of λ which accommodates bounded $\mathbf{G}^{\mathrm{Kerr}}(S_{2R}^{\lambda})$ by allowing for growing center-of-mass $\mathrm{C}_{\mathrm{ADM}}^{\mathrm{Kerr}} = \mathcal{O}(R^{1/2})$ and small linear momentum $\mathrm{P}_{\mathrm{ADM}}^{\mathrm{Kerr}} = \mathcal{O}(R^{-1/2})$ to cancel to top order. Namely, we consider the ellipsoid $\mathcal{E}_R(\mathbf{E}_\infty)$ defined by

$$\left(R^{1/2}|\mathbf{E}(\lambda)-\mathbf{E}_\infty|\right)^2+\left(R^{1/2}|\mathbf{P}(\lambda)|\right)^2+\left(R^{-1/4}|\mathbf{L}(\lambda)|\right)^2+\left(R^{-1/2}|\mathbf{C}(\lambda)|\right)^2\leq(\mathbf{E}_\infty)^2.$$

In the simpler case where the spheres S_R lie in strongly asymptotically flat spacelike hypersurfaces, we have the stronger decay $\mathbf{P}(S_R) = \mathcal{O}(R^{-3/2})$ which implies in the matching process that $\mathbf{G}^{\mathrm{Kerr}}(S_{2R}^{\lambda})$ remains finite as $R\to\infty$ (unlike in (1.4)).

To determine the Kerr parameter λ_R which makes the charge difference $(\Delta Q)(\lambda_R) = 0$ (see (1.1) for definition), we use the asymptotic expansions of $\mathcal{Q}_R$ and $\mathcal{Q}_{2R}^{\lambda_R}$ (discussed in Sect. 1.3.3 above) to construct a homotopy from $F_1(\lambda_R) := (\Delta Q)(\lambda_R)$ to the mapping $F_0(\lambda_R)$ defined by

$$(\mathbf{E}_\infty,\mathbf{P}_\infty,\mathbf{L}_\infty,\mathbf{C}_\infty)-\left(\mathbf{E}_{\mathrm{ADM}}^{\mathrm{Kerr}},\mathbf{P}_{\mathrm{ADM}}^{\mathrm{Kerr}},\mathbf{L}_{\mathrm{ADM}}^{\mathrm{Kerr}},\mathbf{C}_{\mathrm{ADM}}^{\mathrm{Kerr}}-2R\cdot\mathbf{P}_{\mathrm{ADM}}^{\mathrm{Kerr}}\right).$$

The mapping $F_0(\lambda_R)$ has a unique zero in the interior of $\mathcal{E}_R(\mathbf{E}_\infty)$ for large $R\geq 1$. Moreover, as indicated in the previous section, the asymptotic expansions for $\mathcal{Q}_{2R}^{\lambda_R}$ hold *uniformly* for large $R\geq 1$ and $\lambda\in\mathcal{E}_R(\mathbf{E}_\infty)$, so that in particular we have uniform estimates for the constructed homotopy. Therefore we conclude by a topological degree argument that the charge difference $(\Delta Q)(\lambda_R)$ must have a zero.

1.4. Overview of the paper. The paper is structured as follows.

- In Sect. 2 we introduce the notation and state the definitions and preliminaries.
- In Sect. 3 we precisely state the main results of this paper.
- In Sect. 4 we prove the main theorem of this paper, Theorem 3.1.
- In Sect. 5 we prove Corollary 3.3, the gluing of spacelike initial data to Kerr.
- In Sect. 6 we recapitulate spacelike initial data and asymptotic invariants.
- In Sect. 7 we construct strongly asymptotically flat families of sphere data from strongly asymptotically flat spacelike initial data, and relate the charges $(\mathbf{E},\mathbf{P},\mathbf{L},\mathbf{G})$ to the integrals of the ADM asymptotic invariants.

2. Notation, Definitions and Preliminaries

For two real numbers A and B, the inequality $A\lesssim B$ means that there is a universal constant $C>0$ such that $A\leq C\,B$. Greek indices range over $\alpha=0,1,2,3$, lowercase Latin indices over $a=1,2,3$ and uppercase Latin indices over $A=1,2$. For a real number $r>0$ and a point x in a metric space X, denote by $B(x,r)$ the open ball in X of radius r centered at x. For real numbers $\varepsilon>0$ and $\alpha\geq 0$, let $\mathcal{O}(\varepsilon^\alpha)$ denote terms such that $\mathcal{O}(\varepsilon^\alpha)/\varepsilon^\alpha$ remains bounded as $\varepsilon\to 0$, and $o(\varepsilon^\alpha)$ denotes terms such that

$$\lim_{\varepsilon\to 0}\frac{\mathcal{O}(\varepsilon^\alpha)}{\varepsilon^\alpha}=0.$$

For given Cartesian coordinates (x^1,x^2,x^3), define the spherical coordinates (r,θ^1,θ^2) by

$$x^1=r\sin\theta^1\cos\theta^2,\;\; x^2=r\sin\theta^1\sin\theta^2,\;\; x^3=r\cos\theta^1. \tag{2.1}$$

2.1. Double null coordinates. In this section we summarize the standard setup of double null coordinates, Ricci coefficients and null curvature components. We refer to Section 2.1 in [12] for full details. Let $(\mathcal{M}, \mathbf{g})$ be a vacuum spacetime, and denote by $\mathbf{D}$ its covariant derivative and by $\mathbf{R}$ its Riemann curvature tensor. Let u and v be two local optical functions on $\mathcal{M}$, and denote

$$\mathcal{H}_{u_0} = \{u = u_0\}, \ \underline{\mathcal{H}}_{v_0} = \{v = v_0\}, \ S_{u_0,v_0} = \mathcal{H}_{u_0} \cap \underline{\mathcal{H}}_{v_0},$$

where we assume the optical functions u and v are such that the $S_{u,v}$ are spacelike 2-spheres. Let $\not{g}$ denote the induced metric on $S_{u,v}$, and $\not{\nabla}$ the induced covariant derivative. Let $r(u, v)$ denote the *area radius* of $(S_{u,v}, \not{g})$, defined by $\text{area}_{\not{g}}\left(S_{u,v}\right) = 4\pi r^2$. Define the *geodesic null vectorfields* L' and $\underline{L}'$, the *null lapse* Ω, and the *normalized null vectorfields* $\widehat{L}$ and $\widehat{\underline{L}}$ by

$$L' := -2\mathbf{D}u, \ \ \underline{L}' := -2\mathbf{D}v, \ \ \Omega^{-2} := -\frac{1}{2}\mathbf{g}\left(L', \underline{L}'\right), \ \ \widehat{L} := \Omega L', \ \ \widehat{\underline{L}} := \Omega \underline{L}'. \quad (2.2)$$

Let (θ^1, θ^2) be local coordinates on S_{u_0,v_0}, for some given real numbers $v_0 > u_0$. We extend (θ^1, θ^2) to $\mathcal{M}$ by first transporting them along the null generators L' of $\mathcal{H}_{u_0}$ and then onto $\mathcal{M}$ along the generators $\underline{L}'$ of the null hypersurfaces $\underline{\mathcal{H}}_v$. The coordinates $(u, v, \theta^1, \theta^2)$ are called *double null coordinates*. In particular, it holds in double null coordinates the null vectorfields $L := \Omega\widehat{L}$ and $\underline{L} := \Omega\widehat{\underline{L}}$ can be expressed as $L = \partial_v + b$ and $\underline{L} = \partial_u$, and the spacetime metric $\mathbf{g}$ can be written as $\mathbf{g} = -4\Omega^2 du dv + \not{g}_{AB}\left(d\theta^A + b^A dv\right)\left(d\theta^B + b^B dv\right)$, where the *shift vector* $b = b^A\partial_A$ is an $S_{u,v}$-tangential vectorfield, satisfying $b = 0$ on $\mathcal{H}_{u_0}$. Through the coordinates (θ^1, θ^2), we define on each $S_{u,v}$ the unit round metric

$$\overset{\circ}{\gamma} := \left(d\theta^1\right)^2 + \sin^2\theta^1\left(d\theta^2\right)^2. \quad (2.3)$$

We also define standard vector spherical harmonics (and the associated projections) with respect to $\overset{\circ}{\gamma}$ on $S_{u,v}$; see [11,12,24] for a detailed setup. We decompose the induced metric $\not{g}$ into

$$\not{g} = \phi^2 \not{g}_c \text{ where } \phi^2 := \sqrt{\not{g}}\sqrt{\overset{\circ}{\gamma}}^{-1}, \ \ \not{g}_c := \phi^{-2}\not{g}, \quad (2.4)$$

where $\sqrt{\not{g}}$ and $\sqrt{\overset{\circ}{\gamma}}$ denote the volume forms of $\not{g}$ and $\overset{\circ}{\gamma}$ with respect to (θ^1, θ^2), respectively.

The proof of the following calculus lemma is straight-forward and omitted.

Lemma 2.1 (Calculus lemma). *Let $(S, \not{g})$ be a Riemannian 2-sphere equipped with a round metric $\overset{\circ}{\gamma}$ as defined in (2.3), and consider the associated spherical harmonics projections. Let X be a 1-form and W a $\not{g}$-tracefree symmetric 2-tensor on S, and let K denote the Gauss curvature of $\not{g}$. Assume that for a real number $\varepsilon > 0$, it holds that*

$$\|\not{g} - \overset{\circ}{\gamma}\|_{H^6(S)} \leq \varepsilon.$$

There exists a universal real number $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$,

$$\left|(\not{\text{div}}_{\not{g}} X)^{[0]}\right| \lesssim \|\not{g} - \overset{\circ}{\gamma}\|_{H^6(S)} \cdot \|X\|_{H^2(S)}, \quad \left|(\not{\text{div}}_{\not{g}} W)^{[1]}\right| \lesssim \|\not{g} - \overset{\circ}{\gamma}\|_{H^6(S)} \cdot \|W\|_{H^2(S)}.$$

and for $m = -1, 0, 1$, $\left|K^{(1m)}\right| \lesssim \left(\|\not{g} - \overset{\circ}{\gamma}\|_{H^6(S)}\right)^2.$

We define the *Ricci coefficients* as follows. For $S_{u,v}$-tangent vectorfields X and Y, let

$$\begin{aligned}
&\chi(X,Y) := \mathbf{g}(\mathbf{D}_X \widehat{L}, Y), \quad \underline{\chi}(X,Y) := \mathbf{g}(\mathbf{D}_X \underline{\widehat{L}}, Y), \quad \zeta(X) := \frac{1}{2}\mathbf{g}(\mathbf{D}_X \widehat{L}, \underline{\widehat{L}}),\\
&\underline{\zeta}(X) := \frac{1}{2}\mathbf{g}(\mathbf{D}_X \underline{\widehat{L}}, \widehat{L}), \quad \eta := \zeta + \not{d}\log\Omega, \quad \underline{\eta} := -\zeta + \not{d}\log\Omega,\\
&\omega := D\log\Omega, \quad \underline{\omega} := \underline{D}\log\Omega,
\end{aligned} \tag{2.5}$$

where $\not{d}$ is the extrinsic derivative of $S_{u,v}$, and for an $S_{u,v}$-tangent tensor W on $\mathcal{M}$, we define $DW := \not{\mathcal{L}}_L W$ and $\underline{D}W := \not{\mathcal{L}}_{\underline{L}} W$, where $\not{\mathcal{L}}$ denotes the projection of the Lie derivative on $\mathcal{M}$ onto the tangent space of $S_{u,v}$. We remark that $\zeta = -\underline{\zeta}$ and $\underline{\eta} = -\eta + 2\not{d}\log\Omega$.

We define the *null curvature components* as follows. For $S_{u,v}$-tangent vectorfields X and Y, let

$$\begin{aligned}
&\alpha(X,Y) := \mathbf{R}(X, \widehat{L}, Y, \widehat{L}), \quad \beta(X) := \frac{1}{2}\mathbf{R}(X, \widehat{L}, \underline{\widehat{L}}, \widehat{L}),\\
&\rho := \frac{1}{4}\mathbf{R}(\underline{\widehat{L}}, \widehat{L}, \underline{\widehat{L}}, \widehat{L}), \quad \sigma \not\in(X,Y) := \frac{1}{2}\mathbf{R}(X, Y, \underline{\widehat{L}}, \widehat{L}),\\
&\underline{\beta}(X) := \frac{1}{2}\mathbf{R}(X, \underline{\widehat{L}}, \underline{\widehat{L}}, \widehat{L}), \quad \underline{\alpha}(X,Y) := \mathbf{R}(X, \underline{\widehat{L}}, Y, \underline{\widehat{L}}).
\end{aligned} \tag{2.6}$$

2.2. Null structure equations. The geometric setting and the Einstein equations imply relations between the metric components, Ricci coefficients and null curvature components, the so-called *null structure equations*. Before stating them, we introduce the following notation from Chapter 1 of [18]. For two $S_{u,v}$-tangential 1-forms X and Y, let

$$\begin{aligned}
&(X,Y) := \not{g}(X,Y), \ ({}^*X)_A := \in_{AB} X^B, \ \not{\text{div}} X := \not\nabla^A X_A, \ \not{\text{curl}} X := \in^{AB} \not\nabla_A X_B,\\
&(X\widehat{\otimes}Y)_{AB} := X_A Y_B + X_B Y_A - (X\cdot Y)\not{g}_{AB}, \ (\not\nabla\widehat{\otimes}Y)_{AB} := \not\nabla_A Y_B + \not\nabla_B Y_A - (\not{\text{div}} Y)\not{g}_{AB},
\end{aligned}$$

where $\in$ denotes the area 2-form of $S_{u,v}$. For two symmetric $S_{u,v}$-tangential 2-tensors V and W, and a 1-form X let

$$\begin{aligned}
&\text{tr}V := \not{g}^{AB} V_{AB}, \ \widehat{V} := V - \frac{1}{2}\text{tr}V\not{g}, \ V\wedge W := \not\in^{AB} V_{AC} W^C{}_B,\\
&(V\cdot X)_A := V_{AB}X^B, \ \not{\text{div}} V_A := \not\nabla^B V_{BA}.
\end{aligned}$$

For a symmetric $S_{u,v}$-tangential tensor W, let $\widehat{D}W$ denote the tracefree part of DW with respect to $\not{g}$, and $\underline{\widehat{D}}W$ the tracefree part of $\underline{D}W$ with respect to $\not{g}$.
We are now in position to discuss the *null structure equations*. We have the first variation equations,

$$D\not{g} = 2\Omega\chi, \quad \underline{D}\not{g} = 2\Omega\underline{\chi}, \quad D\phi = \frac{\Omega\text{tr}\chi\phi}{2}, \tag{2.7}$$

the Raychauduri equations,

$$D\text{tr}\chi + \frac{\Omega}{2}(\text{tr}\chi)^2 - \omega\text{tr}\chi = -\Omega|\widehat{\chi}|^2_{\not{g}}, \quad \underline{D}\text{tr}\underline{\chi} + \frac{\Omega}{2}(\text{tr}\underline{\chi})^2 - \underline{\omega}\text{tr}\underline{\chi} = -\Omega|\underline{\widehat{\chi}}|^2_{\not{g}}, \tag{2.8}$$

and further

$$\begin{aligned} D\widehat{\chi} &= \Omega|\widehat{\chi}|^2 \mathit{g\!\!/} + \omega\widehat{\chi} - \Omega\alpha, & \underline{D}\widehat{\underline{\chi}} &= \Omega|\widehat{\underline{\chi}}|^2 \mathit{g\!\!/} + \underline{\omega}\widehat{\underline{\chi}} - \Omega\underline{\alpha}, \\ D\eta &= \Omega(\chi\cdot\underline{\eta} - \beta), & \underline{D}\underline{\eta} &= \Omega(\underline{\chi}\cdot\eta + \underline{\beta}), \\ D\underline{\omega} &= \Omega^2(2(\eta,\underline{\eta}) - |\eta|^2 - \rho), & \underline{D}\omega &= \Omega^2(2(\eta,\underline{\eta}) - |\underline{\eta}|^2 - \rho), \\ \mathrm{cu\!\!/rl}\,\eta &= -\frac{1}{2}\widehat{\chi}\wedge\widehat{\underline{\chi}} - \sigma, & \mathrm{cu\!\!/rl}\,\underline{\eta} &= -\,\mathrm{cu\!\!/rl}\,\eta = -\,\mathrm{cu\!\!/rl}\,\zeta, \\ D(\underline{\eta}) &= -\,\Omega(\chi\cdot\underline{\eta} - \beta) + 2\mathrm{d\!\!/}\,\omega, & \underline{D}(\eta) &= -\,\Omega(\underline{\chi}\cdot\eta + \underline{\beta}) + 2\mathrm{d\!\!/}\,\underline{\omega}. \end{aligned} \tag{2.9}$$

Moreover, we have the Gauss equation,

$$K + \frac{1}{4}\mathrm{tr}\chi\,\mathrm{tr}\underline{\chi} - \frac{1}{2}(\widehat{\chi},\widehat{\underline{\chi}}) = -\rho, \tag{2.10}$$

where K denotes the Gauss curvature of $S_{u,v}$, the Gauss–Codazzi equations

$$\mathrm{di\!\!/v}\widehat{\chi} - \frac{1}{2}\mathrm{d\!\!/}\,\mathrm{tr}\chi + \widehat{\chi}\cdot\zeta - \frac{1}{2}\mathrm{tr}\chi\zeta = -\beta,\ \mathrm{di\!\!/v}\widehat{\underline{\chi}} - \frac{1}{2}\mathrm{d\!\!/}\,\mathrm{tr}\underline{\chi} - \widehat{\underline{\chi}}\cdot\zeta + \frac{1}{2}\mathrm{tr}\underline{\chi}\zeta = \underline{\beta}, \tag{2.11}$$

While there are more null structure equations, the above suffice for the explicit calculations in this paper. We refer to [12] for a complete list of the equations.

2.3. Minkowski, Schwarzschild and Kerr spacetimes. In this section we discuss the geometry of Minkowski, Schwarzschild and Kerr spacetimes.

Minkowski spacetime. The trivial solution to the Einstein equations is Minkowski spacetime $(\mathbb{R}^4, \mathbf{m})$ where $\mathbf{m} = \mathrm{diag}(-1, 1, 1, 1)$. Defining standard spherical coordinates on $\mathbb{R}^3$ by (2.1), the reference double null coordinates on Minkowski are given by

$$(u, v, \theta^1, \theta^2) = \left(\frac{1}{2}(t-r), \frac{1}{2}(t+r), \theta^1, \theta^2\right), \tag{2.12}$$

with respect to which $\mathbf{m} = -4dudv + (v-u)^2 \overset{\circ}{\gamma}_{AB} d\theta^A d\theta^B$, where $\overset{\circ}{\gamma}$ is defined in (2.3). We note that the area radius of the sphere $S_{u,v}$ is given by $r = v - u$. Explicitly, with respect to the coordinates (2.12), the non-trivial Minkowski metric components, Ricci coefficients and null curvature components on $S_{u,v}$ are given by (with $r = v - u$)

$$\Omega = 1, \quad \mathit{g\!\!/} = r^2\overset{\circ}{\gamma}, \quad \mathrm{tr}\chi = \frac{2}{r}, \quad \mathrm{tr}\underline{\chi} = -\frac{2}{r}. \tag{2.13}$$

The family of Schwarzschild spacetimes. For real numbers $M \geq 0$, let

$$\mathbf{g}^M = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{2.14}$$

For $M = 0$, the metric (2.14) is Minkowski, while for $M > 0$ it yields a black hole solution with event horizon at $\{r = 2M\}$. The so-called exterior region $\{r > 2M\}$ can be covered by Eddington–Finkelstein double null coordinates $(u, v, \theta^1, \theta^2)$ with respect to which

$$\mathbf{g}^M = -4\Omega_M^2 du dv + r_M(u, v)^2 \overset{\circ}{\gamma}_{CD} d\theta^C d\theta^D,$$

where $\Omega_M := \sqrt{1 - \frac{2M}{r}}$ and the area radius $r_M(u, v)$ is implicitly defined by (see (98) in [26])

$$\frac{v-u}{2M} = \frac{r_M(u, v)}{2M} + \log\left(\frac{r_M(u, v)}{2M} - 1\right). \tag{2.15}$$

By explicit computation it follows that in Eddington–Finkelstein coordinates, for real numbers $v > u$ such that $r_M(u, v) > 2M$, the non-vanishing Schwarzschild metric components, Ricci coefficients and null curvature components on $S_{u,v}$ are given by

$$\begin{aligned} \Omega_M &= \sqrt{1 - \frac{2M}{r_M}}, & \not{g} &= r_M^2 \overset{\circ}{\gamma}, & \mathrm{tr}\chi &= \frac{2\Omega_M}{r_M}, \\ \mathrm{tr}\underline{\chi} &= -\frac{2\Omega_M}{r_M}, & \rho &= -\frac{2M}{r_M^3} & \omega &= \frac{M}{r_M^2}, \\ D\omega &= -\frac{2M}{r_M^3}\Omega_M^2, & \underline{\omega} &= -\frac{M}{r_M^2}, & \underline{D\omega} &= -\frac{2M}{r_M^3}\Omega_M^2. \end{aligned} \tag{2.16}$$

Kerr spacetimes. The Kerr metric is given in Boyer–Lindquist coordinates $(t, r, \theta^1, \theta^2)$ by

$$\begin{aligned} \mathbf{g} = & -dt^2 + \Sigma\left(\frac{1}{\Delta}dr^2 + d(\theta^1)^2\right) + (r^2 + a^2)\sin^2\theta^1 d(\theta^2)^2 \\ & + \frac{2Mr}{\Sigma}\left(a \sin^2\theta^1\, d(\theta^2)^2 - dt\right)^2, \end{aligned}$$

where $\Delta = r^2 - 2Mr + a^2$ and $\Sigma = r^2 + a^2\cos^2\theta^1$. Define the set $I(0)$ of timelike 4-vectors by

$$I(0) := \{(\mathbf{E}, \mathbf{P}) \in \mathbb{R} \times \mathbb{R}^3 : \mathbf{E}^2 - |\mathbf{P}|^2 > 0\} \subset \mathbb{R}^4, \tag{2.17}$$

and define *asymptotic invariants vectors* λ to be elements of the set $\lambda \in I(0) \times \mathbb{R}^3 \times \mathbb{R}^3$. We denote the components of $\lambda \in I(0) \times \mathbb{R}^3 \times \mathbb{R}^3$ by $\lambda = (\mathbf{E}(\lambda), \mathbf{P}(\lambda), \mathbf{L}(\lambda), \mathbf{C}(\lambda))$. In Appendix F of [21] it is shown that for every $\lambda \in I(0) \times \mathbb{R}^3 \times \mathbb{R}^3$ there is a Kerr spacetime $(\mathcal{M}^\lambda, \mathbf{g}^\lambda)$ with spacelike hypersurface Σ^λ carrying induced initial data (g^λ, k^λ) satisfying

$$(\mathbf{E}_{\mathrm{ADM}}, \mathbf{P}_{\mathrm{ADM}}, \mathbf{L}_{\mathrm{ADM}}, \mathbf{C}_{\mathrm{ADM}})(g^\lambda, k^\lambda) = (\mathbf{E}(\lambda), \mathbf{P}(\lambda), \mathbf{L}(\lambda), \mathbf{C}(\lambda)).$$

2.4. Sphere data and null data. In this section we define the notions of *sphere data* and *null data*.

Definition 2.2. (Sphere data). For real numbers $v > u$, let $S_{u,v}$ be a 2-sphere equipped with a round metric $\overset{\circ}{\gamma}$ as in (2.3). Sphere data $x_{u,v}$ on $S_{u,v}$ is given by

$$x = (\Omega, g\!\!\!/, \Omega \mathrm{tr}\chi, \widehat{\chi}, \Omega \mathrm{tr}\underline{\chi}, \widehat{\underline{\chi}}, \eta, \omega, D\omega, \underline{\omega}, \underline{D\omega}, \alpha, \underline{\alpha}),$$

where

- $\Omega > 0$ is a positive scalar function and $g\!\!\!/$ is a Riemannian metric,
- $\Omega \mathrm{tr}\chi$, $\Omega \mathrm{tr}\underline{\chi}$, ω, $D\Omega$, $\underline{\omega}$, $\underline{D\omega}$, ρ and σ are scalar functions,
- η, β and $\underline{\beta}$ are vectorfields,
- $\widehat{\chi}$, $\widehat{\underline{\chi}}$, α and $\underline{\alpha}$ are $g\!\!\!/$-tracefree symmetric 2-tensors.

Remarks on Definition 2.2.

(1) Sphere data is gauge-dependent, see [11,12].
(2) The null structure equations and null Bianchi equations of Sect. 2.2 determine from sphere data the Ricci coefficients $(\underline{\eta}, \zeta, \underline{\zeta})$ and null curvature components $(\beta, \rho, \sigma, \underline{\beta})$, as well as the derivatives

$$\left(D\eta, D\underline{\eta}, D\zeta, D\chi, D\underline{\chi}, D\underline{\omega}\right), \left(D\beta, D\rho, D\sigma, D\underline{\beta}, D\underline{\alpha}\right),$$
$$\left(\underline{D}\eta, \underline{D}\underline{\eta}, \underline{D}\zeta, \underline{D}\chi, \underline{D}\underline{\chi}, \underline{D}\omega\right), \left(\underline{D}\underline{\beta}, \underline{D}\sigma, \underline{D}\rho, \underline{D}\beta, \underline{D}\alpha\right).$$

(3) In the following we denote by $(\beta, \rho, \sigma, \underline{\beta})(x_{u,v})$ the null curvature components calculated from $x_{u,v}$ by the null structure equations (2.9), (2.10) and (2.11), and interpret them as part of sphere data.

Notation. We denote the *Minkowski reference sphere data* on $S_{u,v}$ coming from (2.13) by $\mathfrak{m}_{u,v}$, and for real numbers $M \geq 0$, we denote the *Schwarzschild reference sphere data* on $S_{u,v}$ coming from (2.16) by $\mathfrak{m}^M_{u,v}$.

In Sect. 7 we discuss how to construct a family of sphere data in asymptotically flat spacelike initial data. Applying this construction to the above Kerr spacelike initial data $(\Sigma^\lambda, g^\lambda, k^\lambda)$, we get a *family of Kerr sphere data* $x^\lambda_{-R,2R}$ (lying on spheres $S_{-R,2R}$ foliating Σ^λ). By deep inspection of the construction of [21] and applying the ideas of Sect. 7 to relate the charges $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$ to the ADM local integrals $(\mathbf{E}^{\mathrm{loc}}_{\mathrm{ADM}}, \mathbf{P}^{\mathrm{loc}}_{\mathrm{ADM}}, \mathbf{L}^{\mathrm{loc}}_{\mathrm{ADM}}, \mathbf{C}^{\mathrm{loc}}_{\mathrm{ADM}})$, it is possible to prove the following proposition. For readability of this paper, the explicit proof is omitted.

Proposition 2.3 (Convergence of charges to asymptotic invariants) *Let* $R \geq 1$ *and* $\mathbf{E}_0 > 0$ *be two real numbers. Let* $\mathcal{E}_R(\mathbf{E}_0)$ *be the set of asymptotic invariants vectors* λ *such that*

$$\left(R^{1/2}|\mathbf{E}(\lambda) - \mathbf{E}_0|\right)^2 + \left(R^{1/2}|\mathbf{P}(\lambda)|\right)^2 + \left(R^{-1/4}|\mathbf{L}(\lambda)|\right)^2 + \left(R^{-1/2}|\mathbf{C}(\lambda)|\right)^2 \leq (\mathbf{E}_0)^2. \tag{2.18}$$

Then for $R \geq 1$ *sufficiently large, for all* $\lambda \in \mathcal{E}_R(\mathbf{E}_0)$*, for* $m = -1, 0, 1$*,* $(i_{-1}, i_0, i_1) = (2, 3, 1)$*, the Kerr sphere data* $x^\lambda_{-R,2R}$ *is well-defined and*

$$\begin{aligned} \|x^\lambda_{-R,2R} - \mathfrak{m}^{\mathbf{E}_0}\|_{\mathcal{X}(S_{-R,2R})} \lesssim & R^{-1} \cdot |\mathbf{E}(\lambda) - \mathbf{E}_0| + R^{-1} \cdot |\mathbf{P}(\lambda)| \\ & + R^{-2} \cdot |\mathbf{L}(\lambda)| + R^{-2} \cdot |\mathbf{C}(\lambda)| \\ & + \left(\frac{R^{-2} \cdot |\mathbf{L}(\lambda)| + \frac{|\mathbf{P}(\lambda)|}{\mathbf{E}_0} \cdot R^{-2} \cdot |\mathbf{C}(\lambda)|}{\mathbf{E}_0/R} \right)^2, \end{aligned} \tag{2.19}$$

and

$$\mathbf{E}\left(x^{\lambda}_{-R,2R}\right) = \mathbf{E}(\lambda) + \mathcal{O}(R^{-1}), \qquad \mathbf{P}^m\left(x^{\lambda}_{-R,2R}\right) = \mathbf{P}(\lambda)^{i_m} + \mathcal{O}(R^{-3/2}),$$
$$\mathbf{L}^m\left(x^{\lambda}_{-R,2R}\right) = \mathbf{L}(\lambda)^{i_m} + \mathcal{O}(R^{-1/2}), \quad \mathbf{G}^m\left(x^{\lambda}_{-R,2R}\right) = \mathbf{C}(\lambda)^{i_m} - 3R\cdot\mathbf{P}(\lambda)^{i_m} + \mathcal{O}(R^{-1/4}).$$

Along a null hypersurface, we consider the following *null data*.

Definition 2.4 (*Ingoing and outgoing null data*). For three real numbers $u_0 < v_1 < v_2$, *outgoing null data* $x_{u_0,[v_1,v_2]}$ *on* $\mathcal{H}_{u_0,[v_1,v_2]}$ is given by a family of sphere data $\left(x_{u_0,v}\right)_{v_1\leq v\leq v_2}$ on $\mathcal{H}_{u_0,[v_1,v_2]} = \bigcup_{v_1\leq v\leq v_2} S_{u_0,v}$. Similarly, for three real numbers $u_1 < u_2 < v_0$, *ingoing null data* $x_{[u_1,u_2],v_0}$ *on* $\underline{\mathcal{H}}_{[u_1,u_2],v_0}$ is given by a family of sphere data $\left(x_{u,v_0}\right)_{u_1\leq u\leq u_2}$ on $\underline{\mathcal{H}}_{[u_1,u_2],v_0} = \bigcup_{u_1\leq u\leq u_2} S_{u,v_0}$.

In addition to the above sphere data $x_{u,v}$ on spheres $S_{u,v}$, we also consider for integers $m \geq 1$ the *higher-order sphere data* on $S_{u,v}$

$$\left(x_{u,v}, \mathcal{D}^{L,m}_{u,v}, \mathcal{D}^{\underline{L},m}_{u,v}\right), \tag{2.20}$$

where $x_{u,v}$ denotes sphere data and $\mathcal{D}^{L,m}$ and $\mathcal{D}^{\underline{L},m}$ are tuples of L- and $\underline{L}$-derivatives of sphere data up to order m; we refer to Section 2.10 in [12] for definitions and discussion. We denote the Schwarzschild reference higher-order sphere data of order m by $\left(\mathfrak{m}^M_{u,v}, \mathcal{D}^{L,m,M}_{u,v}, \mathcal{D}^{\underline{L},m,M}_{u,v}\right)$.

Importantly, the gluing of higher-order sphere data implies the higher regularity (in all directions) of the constructed gluing solution. Similarly, we consider higher-order outgoing and ingoing null data on $\mathcal{H}_{u_0,[v_1,v_2]}$ and $\underline{\mathcal{H}}_{[u_1,u_2],v_0}$, respectively. We remark that higher derivatives are subject to the *higher-order null structure equations*, see Section 2.10 in [12] for details.

2.5. Definition of charges ($\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G}$). The following charges play an essential role for the characteristic gluing problem. In [11,12] they are identified as geometric obstacles to characteristic gluing.

Definition 2.5 (*Charges*). For sphere data $x_{u,v}$ and $m = -1, 0, 1$ define the charges

$$\begin{aligned}
\mathbf{E} &:= -\frac{1}{8\pi}\sqrt{4\pi}\left(r^3\left(\rho + r\,\mathrm{d}\!\!/\mathrm{iv}\beta\right)\right)^{(0)},\\
\mathbf{P}^m &:= -\frac{1}{8\pi}\sqrt{\frac{4\pi}{3}}\left(r^3\left(\rho + r\,\mathrm{d}\!\!/\mathrm{iv}\beta\right)\right)^{(1m)},\\
\mathbf{L}^m &:= \frac{1}{16\pi}\sqrt{\frac{8\pi}{3}}\left(r^3\left(\mathrm{d}\!\!/\,\mathrm{tr}\underline{\chi} + \mathrm{tr}\underline{\chi}\left(\eta - \mathrm{d}\!\!/\log\Omega\right)\right)\right)^{(1m)}_H,\\
\mathbf{G}^m &:= \frac{1}{16\pi}\sqrt{\frac{8\pi}{3}}\left(r^3\left(\mathrm{d}\!\!/\,\mathrm{tr}\underline{\chi} + \mathrm{tr}\underline{\chi}\left(\eta - \mathrm{d}\!\!/\log\Omega\right)\right)\right)^{(1m)}_E,
\end{aligned} \tag{2.21}$$

where r denotes the area radius calculated from $x_{u,v}$, and the spherical harmonics projections are defined with respect to the unit round metric $\overset{\circ}{\gamma}$ on $S_{u,v}$.

The numerical factors in the definitions of the charges are determined by comparison to the ADM asymptotic invariants, see Sects. 7.3, 7.4, 7.5 and 7.6. By explicit calculation, for real numbers $M \geq 0$, and $v > u$, $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})\,(\mathfrak{m}^M_{u,v}) = (M, 0, 0, 0)$.

2.6. Norms on spheres and null hypersurfaces. In this section we define the norms used in this paper. They are analogous to the norms in [12], but with the difference that they contain weights in $v-u$ for suitable scaling properties (see Lemma 2.16, and also [11]).

Definition 2.6 (*Norms on 2-spheres*). Let $v > u$ be two real numbers and let $S_{u,v}$ be a 2-sphere equipped with a round metric $\overset{\circ}{\gamma}$ as in (2.3). For integers $m \geq 0$ and $S_{u,v}$-tangent k-tensors T, define

$$\|T\|^2_{H^m(S_{u,v})} := \sum_{0\leq i\leq m} (v-u)^{2(i+k-1)} \left\| \nabla\!\!\!\!/^i T \right\|^2_{L^2(S_{u,v})},$$

where the covariant derivative $\nabla\!\!\!\!/$ and the volume element of the L^2-norm are with respect to the round metric $\gamma = (v-u)^2 \overset{\circ}{\gamma}$ on $S_{u,v}$. Moreover, let $H^m(S_{u,v}) := \{T : \|T\|_{H^m(S_{u,v})} < \infty\}$.

Definition 2.7 (*Norms on null hypersurfaces*). For real numbers $u_0 < v_1 < v_2$, let T be an $S_{u_0,v}$-tangential tensor on $\mathcal{H}_{u_0,[v_1,v_2]}$. For integers $m \geq 0$ and $l \geq 0$, define

$$\|T\|^2_{H^m_l(\mathcal{H}_{u_0,[v_1,v_2]})} := \int_{v_1}^{v_2} \sum_{0\leq i\leq l} (v-u_0)^{2i-1} \left\| D^i T \right\|^2_{H^m(S_{u_0,v})} dv,$$

where the Lie derivative D is with respect to the reference Minkowski metric on $\mathcal{H}_{u_0,[v_1,v_2]}$. Let further $H^m_l\left(\mathcal{H}_{u_0,[v_1,v_2]}\right) := \{T : \|T\|_{H^m_l(\mathcal{H}_{u_0,[v_1,v_2]})} < \infty\}$.
For real numbers $u_1 < u_2 < v_0$, let T be an S_{u,v_0}-tangential tensor on $\underline{\mathcal{H}}_{[u_1,u_2],v_0}$. For integers $m \geq 0$ and $l \geq 0$, define

$$\|T\|^2_{H^m_l\left(\underline{\mathcal{H}}_{[u_1,u_2],v_0}\right)} := \int_{u_1}^{u_2} \sum_{0\leq i\leq l} (v_0-u)^{2i-1} \left\| \underline{D}^i T \right\|^2_{H^m(S_{u,v_0})} du,$$

where the Lie derivative $\underline{D}$ is with respect to the reference Minkowski metric on $\underline{\mathcal{H}}_{[u_1,u_2],v_0}$. Let further $H^m_l\left(\underline{\mathcal{H}}_{[u_1,u_2],v_0}\right) := \{T : \|T\|_{H^m_l\left(\underline{\mathcal{H}}_{[u_1,u_2],v_0}\right)} < \infty\}$.

In the following we define the norms of sphere data and null data using the above norms on spheres and null hypersurfaces. Their definition includes weights in $v-u$ to make them invariant under the scaling introduced in Sect. 2.8, see Lemma 2.17.

Definition 2.8 (*Norm for sphere data*). Let $x_{u,v}$ be sphere data on the sphere $S_{u,v}$. The norm of $x_{u,v}$ is defined by

$$\begin{aligned}
\|x_{u,v}\|_{\mathcal{X}(S_{u,v})} :=& \|\Omega\|_{H^6(S_{u,v})} + (v-u)^{-2}\|g\!\!\!/\|_{H^6(S_{u,v})} + \|\eta\|_{H^5(S_{u,v})} \\
&+ (v-u)\|\mathrm{tr}\chi\|_{H^6(S_{u,v})} + (v-u)^{-1}\|\widehat{\chi}\|_{H^6(S_{u,v})} \\
&+ (v-u)\|\mathrm{tr}\underline{\chi}\|_{H^4(S_{u,v})} + (v-u)^{-1}\|\widehat{\underline{\chi}}\|_{H^4(S_{u,v})} \\
&+ (v-u)\|\omega\|_{H^6(S_{u,v})} + (v-u)^2\|D\omega\|_{H^6(S_{u,v})} \\
&+ (v-u)\|\underline{\omega}\|_{H^4(S_{u,v})} + (v-u)^2\|\underline{D\omega}\|_{H^2(S_{u,v})} + \|\alpha\|_{H^6(S_{u,v})} \\
&+ (v-u)\|\beta\|_{H^5(S_{u,v})} + (v-u)^2\|\rho\|_{H^4(S_{u,v})} + (v-u)^2\|\sigma\|_{H^4(S_{u,v})} \\
&+ (v-u)\|\underline{\beta}\|_{H^3(S_{u,v})} + \|\underline{\alpha}\|_{H^2(S_{u,v})},
\end{aligned}$$

where the norms are with respect to $(v-u)^2\overset{\circ}{\gamma}$ on $S_{u,v}$, see Definition 2.6. Moreover, let

$$\mathcal{X}(S_{u,v}) := \{x_{u,v} : \|x_{u,v}\|_{\mathcal{X}(S_{u,v})} < \infty\}.$$

Definition 2.8 reflects the regularity hierarchy of the null structure equations in the L-direction. For sphere data $x_{u,v} \in \mathcal{X}(S_{u,v})$, the charges $\mathbf{E}, \mathbf{P}, \mathbf{L}$ and $\mathbf{G}$ introduced in Definition 2.5 are well-defined.

Definition 2.9 (*Norms for null data*). Let $R \geq 1$ be a real number. We have the following.

- Let $u_0 < v_1 < v_2$ be three real numbers. Let $x_R := x_{R\cdot u_0, R\cdot[v_1,v_2]}$ be null data on $\mathcal{H}_R := \mathcal{H}_{R\cdot u_0, R\cdot[v_1,v_2]}$. The norm of x_R on $\mathcal{H}_R$ is defined by

$$\begin{aligned}\|x_R\|_{\mathcal{X}(\mathcal{H}_R)} :=& \|\Omega\|_{H^6_3(\mathcal{H}_R)} + \|\not{g}\|_{H^6_3(\mathcal{H}_R)} + \|\eta\|_{H^5_2(\mathcal{H}_R)} + R\|\Omega\mathrm{tr}\chi\|_{H^6_3(\mathcal{H}_R)}\\ &+ R^{-1}\|\widehat{\chi}\|_{H^6_2(\mathcal{H}_R)} + R\|\Omega\mathrm{tr}\underline{\chi}\|_{H^4_2(\mathcal{H}_R)} + R^{-1}\|\widehat{\underline{\chi}}\|_{H^4_3(\mathcal{H}_R)}\\ &+ R\|\omega\|_{H^6_2(\mathcal{H}_R)} + R^2\|D\omega\|_{H^6_1(\mathcal{H}_R)} + R\|\underline{\omega}\|_{H^4_3(\mathcal{H}_R)}\\ &+ R^2\|\underline{D\omega}\|_{H^2_3(\mathcal{H}_R)} + \|\alpha\|_{H^6_1(\mathcal{H}_R)} + R\|\beta\|_{H^5_2(\mathcal{H}_R)} + R^2\|\rho\|_{H^4_2(\mathcal{H}_R)}\\ &+ R^2\|\sigma\|_{H^4_2(\mathcal{H}_R)} + R\|\underline{\beta}\|_{H^3_2(\mathcal{H}_R)} + \|\underline{\alpha}\|_{H^2_3(\mathcal{H}_R)};\end{aligned}$$

  see Definition 2.7 for norms over $\mathcal{H}_{-R,[R,2R]}$. Let $\mathcal{X}(\mathcal{H}_R) := \{x_R : \|x_R\|_{\mathcal{X}(\mathcal{H}_R)} < \infty\}$.
- Let $u_1 < u_2 < v_0$ be three real numbers. Let $x_R := x_{R\cdot[u_1,u_2], R\cdot v_0}$ be null data on $\underline{\mathcal{H}}_R := \underline{\mathcal{H}}_{R\cdot[u_1,u_2], R\cdot v_0}$. The norm of x_R on $\underline{\mathcal{H}}_R$ is defined by

$$\begin{aligned}\|x_R\|_{\mathcal{X}(\underline{\mathcal{H}}_R)} :=& \|\Omega\|_{H^6_3(\underline{\mathcal{H}}_R)} + \|\not{g}\|_{H^6_3(\underline{\mathcal{H}}_R)} + \|\eta\|_{H^5_2(\underline{\mathcal{H}}_R)} + R\|\Omega\mathrm{tr}\underline{\chi}\|_{H^6_3(\underline{\mathcal{H}}_R)}\\ &+ R^{-1}\|\widehat{\underline{\chi}}\|_{H^6_2(\underline{\mathcal{H}}_R)} + R\|\Omega\mathrm{tr}\chi\|_{H^4_2(\underline{\mathcal{H}}_R)} + R^{-1}\|\widehat{\chi}\|_{H^4_3(\underline{\mathcal{H}}_R)}\\ &+ R\|\underline{\omega}\|_{H^6_2(\underline{\mathcal{H}}_R)} + R^2\|\underline{D\omega}\|_{H^6_1(\underline{\mathcal{H}}_R)} + R\|\omega\|_{H^4_3(\underline{\mathcal{H}}_R)}\\ &+ R^2\|D\omega\|_{H^2_3(\underline{\mathcal{H}}_R)} + \|\underline{\alpha}\|_{H^6_1(\underline{\mathcal{H}}_R)} + R\|\underline{\beta}\|_{H^5_2(\mathcal{H}_R)} + R^2\|\sigma\|_{H^4_2(\mathcal{H}_R)}\\ &+ R^2\|\rho\|_{H^4_2(\mathcal{H}_R)} + R\|\beta\|_{H^3_2(\mathcal{H}_R)} + \|\alpha\|_{H^2_3(\underline{\mathcal{H}}_R)};\end{aligned}$$

  see Definition 2.7 for norms over $\underline{\mathcal{H}}_R$. Let $\mathcal{X}(\underline{\mathcal{H}}_R) := \{x_R : \|x_R\|_{\mathcal{X}(\mathcal{H}_R)} < \infty\}$.

In addition to the above norm $\mathcal{X}(\underline{\mathcal{H}})$ for ingoing null data, we define the following higher regularity norm $\mathcal{X}^+(\underline{\mathcal{H}})$. This norm is necessary for the characteristic gluing of [11,12].

Definition 2.10 (*Norm for higher-regularity ingoing null data*). Let $u_1 < u_2 < v_0$ be three real numbers. Let $x_R := x_{R\cdot[u_1,u_2], R\cdot v_0}$ be null data on $\underline{\mathcal{H}}_R := \underline{\mathcal{H}}_{R\cdot[u_1,u_2], R\cdot v_0}$. The norm of x is defined by

$$\begin{aligned}\|x_R\|_{\mathcal{X}^+(\underline{\mathcal{H}}_R)} :=& \|\Omega\|_{H^{12}_9(\underline{\mathcal{H}}_R)} + \|\not{g}\|_{H^{12}_9(\underline{\mathcal{H}}_R)} + \|\eta\|_{H^{11}_8(\underline{\mathcal{H}}_R)} + R\|\Omega\mathrm{tr}\underline{\chi}\|_{H^{12}_9(\underline{\mathcal{H}}_R)}\\ &+ R^{-1}\|\widehat{\underline{\chi}}\|_{H^{12}_8(\underline{\mathcal{H}}_R)} + R\|\Omega\mathrm{tr}\chi\|_{H^{10}_8(\underline{\mathcal{H}}_R)} + R^{-1}\|\widehat{\chi}\|_{H^{10}_9(\underline{\mathcal{H}}_R)}\\ &+ R\|\underline{\omega}\|_{H^{12}_8(\underline{\mathcal{H}}_R)} + R^2\|\underline{D\omega}\|_{H^{12}_7(\underline{\mathcal{H}}_R)} + R\|\omega\|_{H^{10}_9(\underline{\mathcal{H}}_R)}\\ &+ R^2\|D\omega\|_{H^8_9(\underline{\mathcal{H}}_R)} + \|\underline{\alpha}\|_{H^{12}_7(\underline{\mathcal{H}}_R)} + R\|\underline{\beta}\|_{H^{11}_8(\mathcal{H}_R)} + R^2\|\sigma\|_{H^{10}_8(\mathcal{H}_R)}\\ &+ R^2\|\rho\|_{H^{10}_8(\mathcal{H}_R)} + R\|\beta\|_{H^9_8(\mathcal{H}_R)} + \|\alpha\|_{H^8_9(\underline{\mathcal{H}}_R)},\end{aligned}$$

where the norms over $\underline{\mathcal{H}}_R$ are defined in Definition 2.7. Moreover, let

$$\mathcal{X}^+(\underline{\mathcal{H}}_R) := \{x : \|x\|_{\mathcal{X}^+(\underline{\mathcal{H}}_R)} < \infty\}.$$

Notation. Given sphere data $x_{u,v}$ on $S_{u,v}$ and real numbers $M \geq 0$, we write $\|x_{u,v} - \mathfrak{m}^M\|_{\mathcal{X}(S_{u,v})}$ to denote $\|x_{u,v} - \mathfrak{m}^M_{u,v}\|_{\mathcal{X}(S_{u,v})}$. Similarly for outgoing null data $x_{u_0,[v_1,v_2]}$ on $\mathcal{H}_{u_0,[v_1,v_2]}$ and ingoing null data $x_{[u_0,u_1],v_0}$ on $\underline{\mathcal{H}}_{[u_0,u_1],v_0}$.

2.7. Asymptotically flat families of sphere data and ingoing null data. In this section, we introduce *asymptotically flat families of sphere data* and *ingoing null data*, and introduce the *asymptotic charges*.

Definition 2.11 (*Strongly asymptotically flat sphere data*). Let $v > u$ be two fixed real numbers, and let $(x_{R\cdot u,R\cdot v})_{R\geq 1}$ be a family of sphere data. We say that $(x_{R\cdot u,R\cdot v})$ is a *strongly asymptotically flat family of sphere data* if there is a real number $M \geq 0$ such that

$$\|x_{R\cdot u,R\cdot v} - \mathfrak{m}^M\|_{\mathcal{X}(S_{R\cdot u,R\cdot v})} = \mathcal{O}(R^{-3/2}),\ \|\beta^{[1]}(x_{R\cdot u,R\cdot v})\|_{L^2(S_{R\cdot u,R\cdot v})} = \mathcal{O}\left(R^{-3}\right). \tag{2.22}$$

Remarks on Definition 2.11.

(1) In this paper we work with strongly asymptotically families of sphere data $(x_{0,R})$ (that is, $u = 0$ and $v = 1$), in Theorem 3.2 and the proof of Corollary 3.3, and $(x_{-R,R})$ (that is, $u = -1$ and $v = 1$) in Definition 2.14 below.

(2) By Definition 2.8, the decay (2.22) implies in particular

$$\begin{aligned} R\|\alpha\|_{L^2(S_{R\cdot u,R\cdot v})} + R\|\underline{\alpha}\|_{L^2(S_{R\cdot u,R\cdot v})} + R\|\beta\|_{L^2(S_{R\cdot u,R\cdot v})} &= \mathcal{O}\left(R^{-3/2}\right), \\ R^{3/2}\|\beta^{[1]}\|_{L^2(S_{R\cdot u,R\cdot v})} &= \mathcal{O}(R^{-3/2}). \end{aligned}$$

(3) **These decay rates are in agreement with a sequence of spheres going to *spacelike* infinity in a strongly asymptotically flat spacetime**; see Theorem 7.1 and [19].

(4) Clearly, the above definition can be generalized in a straight-forward way to m^{th}-order sphere data; we omit the explicit setup of the appropriate higher-regularity norm.

We define the following asymptotic charges.

Definition 2.12 (*Asymptotic charges*). Let $(x_{R\cdot u,R\cdot v})$ be a strongly asymptotically flat family of sphere data. Let

$$\begin{aligned} \mathbf{E}_\infty &:= \lim_{R\to\infty} \mathbf{E}(x_{R\cdot u,R\cdot v}), & \mathbf{P}_\infty &:= \lim_{R\to\infty} \mathbf{P}(x_{R\cdot u,R\cdot v}), \\ \mathbf{L}_\infty &:= \lim_{R\to\infty} \mathbf{L}(x_{R\cdot u,R\cdot v}), & \mathbf{G}_\infty &:= \lim_{R\to\infty} \mathbf{G}(x_{R\cdot u,R\cdot v}), \end{aligned}$$

where the charges $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$ are defined in Definition 2.5.

The asymptotic charges satisfy the following basic properties. We omit their proofs.

Lemma 2.13 (Properties of asymptotic charges). *Let $(x_{R\cdot u,R\cdot v})$ be a strongly asymptotically flat family of sphere data. Then its asymptotic charges are well-defined,*

$$|\mathbf{E}_\infty| + |\mathbf{P}_\infty| + |\mathbf{L}_\infty| + |\mathbf{G}_\infty| < \infty,$$

and it holds that $\mathbf{E}_\infty = M$ *and* $\mathbf{P}_\infty = 0$*, where* M *is the real number appearing in* (2.22)*, and*

$$\begin{aligned}\mathbf{E}\left(x_{R\cdot u,R\cdot v}\right) =&\mathbf{E}_\infty + \mathcal{O}(R^{-1/2}), \quad \mathbf{P}\left(x_{R\cdot u,R\cdot v}\right) =\mathcal{O}(R^{-1/2}),\\ \mathbf{L}\left(x_{R\cdot u,R\cdot v}\right) =&\mathbf{L}_\infty + \mathcal{O}(1), \qquad\quad \mathbf{G}\left(x_{R\cdot u,R\cdot v}\right) =\mathbf{G}_\infty + \mathcal{O}(1).\end{aligned}$$

The above notion of asymptotic flatness is generalized to ingoing null data as follows.

Definition 2.14 (*Strongly asymptotically flat ingoing null data*). Let $\delta > 0$ be a real number. Let $(x_{-R+R\cdot[-\delta,\delta],R})_{R\geq 1}$ be a family of ingoing null data. We say that $(x_{-R+R\cdot[-\delta,\delta],R})$ is *strongly asymptotically flat* if there is a real number $M > 0$ such that, as $R \to \infty$,

$$\begin{aligned}\|x_{-R+R\cdot[-\delta,\delta],R} - \mathfrak{m}^M\|_{\mathcal{X}^+(\underline{\mathcal{H}}_{-R+R\cdot[-\delta,\delta],R})} &= \mathcal{O}\left(R^{-3/2}\right),\\ \|\beta^{[1]}(x_{-R,R})\|_{L^2(S_{-R,R})} &= \mathcal{O}\left(R^{-3}\right),\end{aligned} \tag{2.23}$$

where the sphere data $x_{-R,R} := x|_{S_{-R,R}}$. Define the asymptotic charges $(\mathbf{E}_\infty, \mathbf{P}_\infty, \mathbf{L}_\infty, \mathbf{G}_\infty)$ of the family of ingoing null data $(x_{-R+R\cdot[-\delta,\delta],R})$ by applying Definition 2.12 to the family $(x_{-R,R})$.

In this paper, strongly asymptotically flat families of ingoing null data $(x_{-R+R\cdot[-\delta,\delta],R})$ are used in Theorem 3.1 and Theorem 7.1. Moreover, for strongly asymptotically flat families of ingoing null data $(x_{-R+R\cdot[-\delta,\delta],R})$, the sphere data $(x_{-R,R}) := (x|_{S_{-R,R}})$ forms a strongly asymptotically flat family of sphere data.

2.8. Scaling of Einstein equations. In this section we introduce the scaling used in this paper and subsequently discuss how geometric quantities change under scaling. Consider local double null coordinates $(u, v, \theta^1, \theta^2)$ in a spacetime $(\mathcal{M}, \mathbf{g})$,

$$\mathbf{g} = -4\Omega^2 du dv + \mathit{g}\!\!/_{AB}\left(d\theta^A + b^A dv\right)\left(d\theta^B + b^B dv\right).$$

The scaling is defined in two steps.

(1) For a real number $R \geq 1$, define the local coordinates $(R\cdot\tilde{u}, R\cdot\tilde{v}, \tilde{\theta}^1, \tilde{\theta}^2) = (u, v, \theta^1, \theta^2)$. Clearly it holds that $du = Rd\tilde{u}$, $dv = Rd\tilde{v}$, $d\theta^1 = d\tilde{\theta}^1$, $d\theta^2 = d\tilde{\theta}^2$, and thus

$$\begin{aligned}\mathbf{g} =& -4R^2\cdot\Omega^2 d\tilde{u}d\tilde{v} + \mathit{g}\!\!/_{AB}\left(d\tilde{\theta}^A + R\cdot b^A d\tilde{v}\right)\left(d\tilde{\theta}^B + R\cdot b^B d\tilde{v}\right)\\ =&R^2\left(-4\cdot\Omega^2 d\tilde{u}d\tilde{v} + R^{-2}\mathit{g}\!\!/_{AB}\left(d\tilde{\theta}^A + R\cdot b^A d\tilde{v}\right)\left(d\tilde{\theta}^B + R\cdot b^B d\tilde{v}\right)\right).\end{aligned}$$

(2) It is well-known that given a Lorentzian metric $\mathbf{g}$ is a solution to the Einstein equations, the conformal metric ${}^{(R)}\mathbf{g} := R^{-2}\mathbf{g}$ is also a solution.

Expressing ${}^{(R)}\mathbf{g}$ in coordinates $(\tilde{u}, \tilde{v}, \tilde{\theta}^1, \tilde{\theta}^2)$, we get the spacetime metric

$$ {}^{(R)}\mathbf{g} = -4{}^{(R)}\Omega^2 d\tilde{u}d\tilde{v} + {}^{(R)}\mathit{g}\!\!/_{AB}\left(d\tilde{\theta}^A + {}^{(R)}b^A d\tilde{v}\right)\left(d\tilde{\theta}^B + {}^{(R)}b^B d\tilde{v}\right) $$

with ${}^{(R)}\Omega(\tilde{u},\tilde{v}) := \Omega(R\tilde{u}, R\tilde{v})$, ${}^{(R)}\mathit{g}\!\!/(\tilde{u},\tilde{v}) := R^{-2}\mathit{g}\!\!/(R\tilde{u}, R\tilde{v})$, ${}^{(R)}b(\tilde{u},\tilde{v}) := R\,b(R\tilde{u}, R\tilde{v})$.

Notation. Denote in the following the scaling $\Psi_R(\tilde{u}, \tilde{v}, \tilde{\theta}^1, \tilde{\theta}^2) := (R\cdot\tilde{u}, R\cdot\tilde{v}, \tilde{\theta}^1, \tilde{\theta}^2)$. The following lemma shows how the Ricci coefficients and null curvature components change under scaling; the proof is by explicit computation and omitted.

Lemma 2.15 (Scaling of Ricci coefficients and null curvature components). *Under the above scaling, the Ricci coefficients and null curvature components transform as follows,*

$$ \begin{aligned} &{}^{(R)}\chi_{AB} = R^{-1}(\chi_{AB}\circ\Psi_R), \quad {}^{(R)}\zeta_A = \zeta_A\circ\Psi_R, {}^{(R)}\eta_A = \eta_A\circ\Psi_R, {}^{(R)}\omega = R(\omega\circ\Psi_R),\\ &{}^{(R)}\underline{\chi}_{AB} = R^{-1}\left(\underline{\chi}_{AB}\circ\Psi_R\right), {}^{(R)}\underline{\zeta}_A = \underline{\zeta}_A\circ\Psi_R, {}^{(R)}\underline{\eta}_A = \underline{\eta}_A\circ\Psi_R, {}^{(R)}\underline{\omega} = R\left(\underline{\omega}\circ\Psi_R\right),\\ &{}^{(R)}\alpha_{AB} = \alpha_{AB}\circ\Psi_R, {}^{(R)}\beta_A = R(\beta_A\circ\Psi_R), {}^{(R)}\rho = R^2(\rho\circ\Psi_R),\\ &{}^{(R)}\sigma = R^2(\sigma\circ\Psi_R), {}^{(R)}\underline{\beta}_A = R\left(\underline{\beta}_A\circ\Psi_R\right), {}^{(R)}\underline{\alpha}_{AB} = \underline{\alpha}_{AB}\circ\Psi_R.\\ &\mathrm{tr}_{{}^{(R)}\mathit{g}\!\!/}{}^{(R)}\chi = R(\mathrm{tr}\chi\circ\Psi_R), {}^{(R)}\widehat{\chi}_{AB} = R^{-1}(\widehat{\chi}_{AB}\circ\Psi_R), {}^{(R)}(D\omega) = R^2((D\omega)\circ\Psi_R),\\ &\mathrm{tr}_{{}^{(R)}\mathit{g}\!\!/}{}^{(R)}\underline{\chi} = R\left(\mathrm{tr}\underline{\chi}\circ\Psi_R\right), {}^{(R)}\widehat{\underline{\chi}}_{AB} = R^{-1}\left(\widehat{\underline{\chi}}_{AB}\circ\Psi_R\right), {}^{(R)}\underline{D\omega} = R^2\left((\underline{D\omega})\circ\Psi_R\right), \end{aligned} $$

where the tracefree parts of ${}^{(R)}\chi$, ${}^{(R)}\underline{\chi}$ *and* χ, $\underline{\chi}$ *are calculated with respect to* ${}^{(R)}\mathit{g}\!\!/$ *and* $\mathit{g}\!\!/$, *respectively. Furthermore, the area radius* r *scales as* ${}^{(R)}r = R^{-1}(r\circ\Psi_R)$.

Notation. For real numbers $R \geq 1$ and sphere data $x_{Ru,Rv}$ on $S_{Ru,Rv}$, denote the rescaled sphere data on $S_{u,v}$ following Lemma 2.15 by ${}^{(R)}x_{u,v}$.
By the invariance of the Einstein equations under the above scaling, it follows that the null structure equations and the null Bianchi equations of Sect. 2.2 are scale-invariant under the scaling of Lemma 2.15. Importantly, we have the following *scale-invariance of Schwarzschild and Kerr*. It is straight-forward to show that for real numbers $M \geq 0$, $R \geq 1$ and $v > u$, for Schwarzschild,

$$ {}^{(R)}\mathfrak{m}^M_{u,v} = \mathfrak{m}^{M/R}_{u,v}, \tag{2.24} $$

and for $\lambda \in I(0)\times\mathbb{R}^3\times\mathbb{R}^3$, for Kerr,

$$ {}^{(R)}x^{\lambda}_{-1,2} := x^{{}^{(R)}\lambda}_{-1,2}, \tag{2.25} $$

for ${}^{(R)}\lambda = (R^{-1}\mathbf{E}(\lambda), R^{-1}\mathbf{P}(\lambda), R^{-2}\mathbf{L}(\lambda), R^{-2}\mathbf{C}(\lambda))$.

The following three lemmas consider the scaling of tensor norms, data norms and charges. Their proofs are omitted.

Lemma 2.16 (Scaling of tensor norms). *Let* $p \in \mathbb{R}$ *and* $R \geq 1$ *be two real numbers, and let* $m \geq 0$ *and* $l \geq 0$ *be two integers. Let* F *be a tensor on* $S_{u,v}$ *for real numbers*

$v > u$, and define the tensor ${}^{(R)}F$ by ${}^{(R)}F := R^p \cdot (F \circ \Psi_R)$. Then it holds for integers $i, l \geq 0$ that

$$
\begin{aligned}
\|{}^{(R)}F\|_{H^i(S_{u,v})} =& R^p \cdot \|F\|_{H^i(S_{R\cdot u,R\cdot v})},\\
\|{}^{(R)}F\|_{H^i_l(\mathcal{H}_{u_0,[v_1,v_2]})} =& R^p \cdot \|F\|_{H^i_l(\mathcal{H}_{r\cdot u_0,[R\cdot v_1,R\cdot v_2]})},\\
\|{}^{(R)}F\|_{H^i_l\left(\underline{\mathcal{H}}_{[u_1,u_2],v_0}\right)} =& R^p \cdot \|F\|_{H^i_l\left(\underline{\mathcal{H}}_{R\cdot[u_1,u_2],R\cdot v_0}\right)}.
\end{aligned}
$$

Lemma 2.17 (Scale-invariance of data norms). *Let $R \geq 1$ be a real number. Then it holds that for sphere data $x_{u,v}$ on $S_{u,v}$,*

$$
\|{}^{(R)}x_{R^{-1}u,R^{-1}v}\|_{\mathcal{X}(S_{R^{-1}u,R^{-1}v})} = \|x_{u,v}\|_{\mathcal{X}(S_{u,v})},
$$

for outgoing null data $x_{u_0,[v_1,v_2]}$ on $\mathcal{H}_{u_0,[v_1,v_2]}$,

$$
\|{}^{(R)}x_{R^{-1}u_0,R^{-1}[v_1,v_2]}\|_{\mathcal{X}(\mathcal{H}_{R^{-1}u_0,R^{-1}[v_1,v_2]})} = \|x_{u_0,[v_1,v_2]}\|_{\mathcal{X}(\mathcal{H}_{u_0,[v_1,v_2]})},
$$

and for ingoing null data $x_{[u_1,u_2],v_0}$ on $\underline{\mathcal{H}}_{[u_1,u_2],v_0}$,

$$
\begin{aligned}
\|{}^{(R)}x_{R^{-1}[u_1,u_2],R^{-1}v_0}\|_{\mathcal{X}(\underline{\mathcal{H}}_{R^{-1}[u_1,u_2],R^{-1}v_0})} =& \|x_{[u_1,u_2],v_0}\|_{\mathcal{X}(\underline{\mathcal{H}}_{[u_1,u_2],v_0})},\\
\|{}^{(R)}x_{R^{-1}[u_1,u_2],R^{-1}v_0}\|_{\mathcal{X}^+(\underline{\mathcal{H}}_{R^{-1}[u_1,u_2],R^{-1}v_0})} =& \|x_{[u_1,u_2],v_0}\|_{\mathcal{X}^+(\underline{\mathcal{H}}_{[u_1,u_2],v_0})},
\end{aligned}
$$

where the norms $\mathcal{X}(S_{u,v})$, $\mathcal{X}(\mathcal{H}_{u_0,[v_1,v_2]})$, $\mathcal{X}(\underline{\mathcal{H}}_{[u_1,u_2],v_0})$ and $\mathcal{X}^+(\underline{\mathcal{H}}_{[u_1,u_2],v_0})$ are defined in Definitions 2.8, 2.9 and 2.10, respectively.

Lemma 2.18 (Scaling of charges). *Let $x_{u,v}$ be sphere data and let $R \geq 1$ be a real number. Let ${}^{(R)}x_{R^{-1}u,R^{-1}v}$ denote the rescaling of $x_{u,v}$ according to Definition 2.15. Then it holds that*

$$
\begin{aligned}
\mathbf{E}\left({}^{(R)}x_{R^{-1}u,R^{-1}v}\right) =& R^{-1} \cdot \mathbf{E}(x_{u,v}), \quad \mathbf{P}\left({}^{(R)}x_{R^{-1}u,R^{-1}v}\right) = R^{-1} \cdot \mathbf{P}(x_{u,v}),\\
\mathbf{L}\left({}^{(R)}x_{R^{-1}u,R^{-1}v}\right) =& R^{-2} \cdot \mathbf{L}(x_{u,v}), \quad \mathbf{G}\left({}^{(R)}x_{R^{-1}u,R^{-1}v}\right) = R^{-2} \cdot \mathbf{G}(x_{u,v}).
\end{aligned}
\tag{2.26}
$$

From Lemmas 2.13, 2.17 and 2.18, (2.24) and Definitions 2.5, 2.11 and 2.12 we directly get the following lemma.

Lemma 2.19 (Rescaling of strongly asymptotically flat families). *For two fixed real numbers $v > u$, let $(x_{R\cdot u,R\cdot v})$ be strongly asymptotically flat family of sphere data with asymptotic charge $\mathbf{E}_\infty$ as defined in Definition 2.12. Then it holds that*

$$
\|{}^{(R)}x_{u,v} - \mathfrak{m}^{\mathbf{E}_\infty/R}_{u,v}\|_{\mathcal{X}(S_{u,v})} = \mathcal{O}\left(R^{-3/2}\right), \quad \|\beta^{[1]}({}^{(R)}x_{u,v})\|_{L^2(S_{u,v})} = \mathcal{O}\left(R^{-2}\right).
$$

Moreover, for a fixed real number $\delta > 0$, let $(x_{-R+R\cdot[-\delta,\delta],R})$ be a strongly asymptotically flat family of ingoing null data with asymptotic charge $\mathbf{E}_\infty$ as defined in Definition 2.14. Then it holds that

$$
\begin{aligned}
\|{}^{(R)}x_{-1+[-\delta,\delta],1} - \mathfrak{m}^{\mathbf{E}_\infty/R}_{-1+[-\delta,\delta],1}\|_{\mathcal{X}^+(\underline{\mathcal{H}}_{-1+[-\delta,\delta],1})} =& \mathcal{O}\left(R^{-3/2}\right), \|\beta^{[1]}({}^{(R)}x|_{-1,1})\|_{L^2(S_{-1,1})}\\
=& \mathcal{O}\left(R^{-2}\right).
\end{aligned}
$$

*2.9. Codimension-*10 *characteristic gluing results of [12].* In this section we state the precise codimension-10 null gluing result of [12]. First, we have the following *perturbative* null gluing result.

Theorem 2.20 (Codimension-10 perturbative null gluing of [12]) *Let $\delta > 0$ be a real number. Let $x_{0,1}$ be sphere data on $S_{0,1}$, and consider sphere data $\tilde{x}_{0,2}$ on $\tilde{S}_{0,2}$ contained in ingoing null data $\underline{\tilde{x}}$ on $\underline{\tilde{\mathcal{H}}}_{[-\delta,\delta],2}$ solving the null structure equations. Assume that for some $\varepsilon > 0$ it holds that*

$$\|x_{0,1} - \mathfrak{m}^M\|_{\mathcal{X}(S_{0,1})} + \|\underline{\tilde{x}} - \underline{\mathfrak{m}}^M\|_{\mathcal{X}^+(\underline{\tilde{\mathcal{H}}}_{[-\delta,\delta],2})} \leq \varepsilon. \tag{2.27}$$

There are universal reals $M_0 > 0$ and $\varepsilon_0 > 0$ such that for all reals $0 \leq M < M_0$ and $0 < \varepsilon < \varepsilon_0$ sufficiently small, there exist

- *a solution x to the null structure equations on on $\mathcal{H}_{0,[1,2]}$,*
- *sphere data $x_{0,2}$ on the sphere $S_{0,2} \subset \underline{\tilde{\mathcal{H}}}_{[-\delta,\delta],2}$ stemming from a perturbation of $\tilde{S}_{0,2}$, that is, there exist perturbation functions f and q (see [12] for the precise setup) such that*

$$x_{0,2} = \mathcal{P}_{f,q}(\underline{\tilde{x}}),$$

such that on $S_{0,1}$ the following matching of sphere data holds,

$$x|_{S_{0,1}} = x_{0,1}, \tag{2.28}$$

and on $S_{0,2}$, matching up to the charges $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$ holds, that is, if

$$(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})\,(x|_{S_{0,2}}) = (\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})\left(x_{0,2}\right), \tag{2.29}$$

then

$$x|_{S_{0,2}} = x_{0,2}. \tag{2.30}$$

Moreover, the following estimates hold,

$$\begin{aligned} \|x - \mathfrak{m}^M\|_{\mathcal{X}(\mathcal{H}_{0,[1,2]})} + \|x_{0,2} - \underline{\tilde{x}}_{0,2}\|_{\mathcal{X}(S_{0,2})} &\lesssim \varepsilon, \\ \|f\|_{\mathcal{Y}_f} + \|q\|_{\mathcal{Y}_q} &\lesssim \varepsilon, \end{aligned} \tag{2.31}$$

where we denoted $\underline{\tilde{x}}_{0,2} := \underline{\tilde{x}}|_{S_{0,2}}$. In addition, the following perturbation estimate holds,

$$\left|(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})\left(x_{0,2}\right) - (\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})\left(\underline{\tilde{x}}_{0,2}\right)\right| \lesssim \varepsilon M + \varepsilon^2, \tag{2.32}$$

as well as the transport estimate

$$\left|(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})\left(x|_{S_{0,2}}\right) - (\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})\left(x|_{S_{0,1}}\right)\right| \lesssim \varepsilon M + \varepsilon^2. \tag{2.33}$$

Second, we have the following *bifurcate* null gluing result.

Theorem 2.21 (Codimension-10 bifurcate null gluing of [12]) *Let $m \geq 0$ be an integer. Consider given smooth higher-order sphere data*

$$(x_{0,1}, \mathcal{D}^{L,m}_{0,1}, \mathcal{D}^{\underline{L},m}_{0,1}) \text{ on } S_{0,1} \text{ and } (x_{-1,2}, \mathcal{D}^{L,m}_{-1,2}, \mathcal{D}^{\underline{L},m}_{-1,2}) \text{ on } S_{-1,2}.$$

For $(x_{0,1}, \mathcal{D}^{L,m}_{0,1}, \mathcal{D}^{\underline{L},m}_{0,1})$ and $(x_{-1,2}, \mathcal{D}^{L,m}_{-1,2}, \mathcal{D}^{\underline{L},m}_{-1,2})$ sufficiently close to the their respective reference values in a Schwarzschild spacetime of sufficiently small mass $M \geq 0$, there exist

- *a smooth solution* $(\underline{x}, \underline{\mathcal{D}}^{L,m}, \underline{\mathcal{D}}^{\underline{L},m})$ *to the higher-order null constraint equations on* $\underline{\mathcal{H}}_{[-1,0],1}$*, satisfying the following higher-order sphere data matching on* $S_{0,1}$*,*

$$(\underline{x}, \underline{\mathcal{D}}^{L,m}, \underline{\mathcal{D}}^{\underline{L},m})\Big|_{S_{0,1}} = (x_{0,1}, \mathcal{D}^{L,m}_{0,1}, \mathcal{D}^{\underline{L},m}_{0,1}),$$

- *a smooth solution* $(x, \mathcal{D}^{L,m}, \mathcal{D}^{\underline{L},m})$ *to the higher-order null structure equations on* $\mathcal{H}_{-1,[1,2]}$*, agreeing with* $(\underline{x}, \underline{\mathcal{D}})$ *on* $S_{-1,1}$*,*

$$(\underline{x}, \underline{\mathcal{D}}^{L,m}, \underline{\mathcal{D}}^{\underline{L},m})\Big|_{S_{-1,1}} = (x, \mathcal{D}^{L,m}, \mathcal{D}^{\underline{L},m})\Big|_{S_{-1,1}}$$

such that $(x, \mathcal{D}^{L,m}, \mathcal{D}^{\underline{L},m})$ *matches* $(x_{-1,2}, \mathcal{D}^{L,m}_{-1,2}, \mathcal{D}^{\underline{L},m}_{-1,2})$ *up to the charges* $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$ *on the sphere* $S_{-1,2}$*, that is, if it holds that*

$$(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G}) \left(x|_{S_{-1,2}}\right) = (\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G}) \left(x_{-1,2}\right),$$

then automatically

$$(x, \mathcal{D}^{L,m}, \mathcal{D}^{\underline{L},m})\Big|_{S_{-1,2}} = (x_{-1,2}, \mathcal{D}^{L,m}_{-1,2}, \mathcal{D}^{\underline{L},m}_{-1,2}).$$

Moreover, we have charge estimates analogous to (2.33) *in Theorem* 2.20 *for*

$$\left|(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})(x|_{S_{-1,2}}) - (\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G} - 2\mathbf{P})\,(x_{0,1})\right|.$$

3. Statement of Main Results

The following is the precise version of the main theorem of this paper.

Theorem 3.1 (Perturbative characteristic gluing to Kerr, version 2) *Let* $\delta > 0$ *be a real number, and let* $(\tilde{x}_{-R+R\cdot[-\delta,\delta],R})$ *along* $\tilde{\underline{\mathcal{H}}}_{-R+R\cdot[-\delta,\delta],R}$ *be a strongly asymptotically flat family of ingoing null data with asymptotic charges* $(\mathbf{E}_\infty, \mathbf{P}_\infty = 0, \mathbf{L}_\infty, \mathbf{G}_\infty) \in I(0) \times \mathbb{R}^3 \times \mathbb{R}^3$*. For sufficiently large* $R \geq 1$*, there exist*

- *sphere data* $x'_{-R,R}$ *on a perturbation* $S'_{-R,R}$ *of the sphere* $\tilde{S}_{-R,R}$ *along* $\tilde{\underline{\mathcal{H}}}_{-R+R\cdot[-\delta,\delta],R}$*,*
- *outgoing null data* x *on a null hypersurface* $\mathcal{H}_{-R,[R,2R]}$ *solving the null structure equations,*
- *sphere data* $x^{\mathrm{Kerr}}_{-R,2R}$ *on a spacelike* 2*-sphere* $S^{\mathrm{Kerr}}_{-R,2R}$ *in a Kerr spacetime,*

such that we have full matching of sphere data on $S_{-R,R} \subset \mathcal{H}_{-R,[R,2R]}$ *and on* $S_{-R,2R} \subset \mathcal{H}_{-R,[R,2R]}$*,*

$$x_{-R,R} = x'_{-R,R}, \;\; x_{-R,2R} = x^{\mathrm{Kerr}}_{-R,2R}, \tag{3.1}$$

and the following estimates hold,

$$\|x - \mathfrak{m}^{\mathbf{E}_\infty}\|_{\mathcal{X}(\mathcal{H}_{-R,[R,2R]})} + \|x'_{-R,R} - \mathfrak{m}^{\mathbf{E}_\infty}\|_{\mathcal{X}(S_{-R,R})} = \mathcal{O}\left(R^{-3/2}\right). \tag{3.2}$$

Moreover, the sphere $S^{\mathrm{Kerr}}_{-R,2R}$ *in Kerr lies in a spacelike hypersurface* Σ^{Kerr} *whose asymptotic invariants are bounded by*

$$\begin{aligned} \mathbf{E}_{\mathrm{ADM}} &= \mathbf{E}_\infty + \mathcal{O}(R^{-1/2}), \quad & \mathbf{P}_{\mathrm{ADM}} &= \mathcal{O}(R^{-1/2}), \\ \mathbf{L}_{\mathrm{ADM}} &= \mathbf{L}_\infty + \mathcal{O}(1), & \mathbf{C}_{\mathrm{ADM}} &= \mathbf{G}_\infty + 3R \cdot \mathbf{P}_{\mathrm{ADM}} + \mathcal{O}(1). \end{aligned} \tag{3.3}$$

Moreover, if the strongly asymptotically flat family of ingoing null data $(\tilde{x}_{-R+R\cdot[-\delta,\delta],R})$ *satisfies the stronger decay rates*

$$\mathbf{E}(\tilde{x}_{-R,R}) = \mathbf{E}_\infty + \mathcal{O}(R^{-1}), \ \mathbf{P}(\tilde{x}_{-R,R}) = \mathcal{O}(R^{-3/2}), \tag{3.4}$$

then the asymptotic invariants of the spacelike hypersurface Σ^{Kerr} *are bounded by*

$$\begin{aligned} \mathbf{E}_{\text{ADM}} &= \mathbf{E}_\infty + \mathcal{O}(R^{-1}), \ \mathbf{P}_{\text{ADM}} = \mathcal{O}(R^{-3/2}), \\ \mathbf{L}_{\text{ADM}} &= \mathbf{L}_\infty + \mathcal{O}(1), \ \mathbf{C}_{\text{ADM}} = \mathbf{G}_\infty + \mathcal{O}(1), \end{aligned} \tag{3.5}$$

so, in particular, $\mathbf{C}_{\text{ADM}}$ *is not growing in* R.

Remarks on Theorem 3.1.

(1) The key ingredients of the proof are the perturbative characteristic gluing of [11,12] (used as black box) and the geometric interpretation of the asymptotic charges $(\mathbf{E}_\infty, \mathbf{P}_\infty, \mathbf{L}_\infty, \mathbf{G}_\infty)$ in terms of the ADM asymptotic invariants *energy*, *linear momentum*, *angular momentum*, and *center-of-mass* in Sect. 7.
(2) The additional convergence condition (3.4) is satisfied by sphere data constructed from strongly asymptotically flat spacelike initial data, see Sect. 7.
(3) The smallness on the right-hand side of (3.2) is consistent with our definition of strong asymptotic flatness, see Definition 2.14.
(4) Theorem 3.1 is at the level of C^2-gluing for the metric components. It can be generalized to include higher-order derivatives *tangential to the gluing hypersurface* $\mathcal{H}_{-R,[R,2R]}$; see Theorem 3.2 in [12] for the corresponding setup. For the gluing of higher-order derivatives in *all directions*, we refer to Theorem 3.2 below.
(5) More precisely, in Theorem 3.1 we glue to a *Kerr reference sphere* $S^\lambda_{-R,2R}$ for some *asymptotic invariants vector* $\lambda \in I(0) \times \mathbb{R}^3 \times \mathbb{R}^3$.
(6) In Theorem 3.1 it is not necessary to have a family of ingoing null data data. Indeed, one can replace this family with one fixed ingoing null datum with sufficiently strong bounds.
(7) The sphere $S^{\text{Kerr}}_{-R,2R}$ in Kerr admits a future-complete outgoing null congruence and past-complete ingoing null congruence. The explicit proof of this property, based on a classical perturbation argument, is omitted here.
(8) The condition $(\mathbf{E}_\infty, \mathbf{P}_\infty = 0) \in I(0)$ (see definition in (2.17)) implies in particular that $\mathbf{E}_\infty > 0$.

The argument for the matching to Kerr in Theorem 3.1 applies similarly to the *bifurcate* characteristic gluing of [11,12] (i.e. Theorem 2.21), see Remark 4.2. The corresponding theorem is the following.

Theorem 3.2 (Bifurcate characteristic gluing to Kerr) *Let* $m \geq 1$ *be an integer. Let*

$$\left(x_{0,R}, \mathcal{D}^{L,m}_{0,R}, \mathcal{D}^{\underline{L},m}_{0,R}\right)$$

be a strongly asymptotically flat family of smooth higher-order sphere data with asymptotic charges

$$(\mathbf{E}_\infty, \mathbf{P}_\infty = 0, \mathbf{L}_\infty, \mathbf{G}_\infty) \in I(0) \times \mathbb{R}^3 \times \mathbb{R}^3.$$

For sufficiently large $R \geq 1$, *there exist*

- *smooth higher-order ingoing null data* $(x, \mathcal{D}^{L,m}, \mathcal{D}^{\underline{L},m})$ *on* $\underline{\mathcal{H}}_{[-R,0],R}$ *and outgoing higher-order null data* $(x, \mathcal{D}^{L,m}, \mathcal{D}^{\underline{L},m})$ *on* $\mathcal{H}_{-R,[R,2R]}$ *solving the higher-order null structure equations and matching to order m on* $S_{-R,R}$*,*
- *smooth higher-order sphere data* $\left(x^{\mathrm{Kerr}}_{-R,2R}, \mathcal{D}^{L,m,\mathrm{Kerr}}_{-R,2R}, \mathcal{D}^{\underline{L},m,\mathrm{Kerr}}_{-R,2R}\right)$ *on a smooth spacelike 2-sphere* $S^{\mathrm{Kerr}}_{-R,2R}$ *in a Kerr spacetime,*

such that

$$\left(x, \mathcal{D}^{L,m}, \mathcal{D}^{\underline{L},m}\right)|_{S_{0,R}} = \left(x_{0,R}, \mathcal{D}^{L,m}_{0,R}, \mathcal{D}^{\underline{L},m}_{0,R}\right),$$
$$\left(x, \mathcal{D}^{L,m}, \mathcal{D}^{\underline{L},m}\right)\big|_{S_{-R,2R}} = \left(x^{\mathrm{Kerr}}_{-R,2R}, \mathcal{D}^{L,m,\mathrm{Kerr}}_{-R,2R}, \mathcal{D}^{\underline{L},m,\mathrm{Kerr}}_{-R,2R}\right).$$

The sphere $S^{\mathrm{Kerr}}_{-R,2R}$ *in Kerr lies in a spacelike hypersurface with asymptotic invariants*

$$\mathbf{E}_{\mathrm{ADM}} = \mathbf{E}_\infty + \mathcal{O}(R^{-1/2}), \quad \mathbf{P}_{\mathrm{ADM}} = \mathcal{O}(R^{-1/2}),$$
$$\mathbf{L}_{\mathrm{ADM}} = \mathbf{L}_\infty + \mathcal{O}(1), \quad \mathbf{C}_{\mathrm{ADM}} = \mathbf{G}_\infty + 3R \cdot \mathbf{P}(\lambda) + \mathcal{O}(1).$$

amd admits a future-complete outgoing null congruence and past-complete ingoing null congruence. Moreover, if the strongly asymptotically flat family of sphere data $(x_{0,R})$ *satisfies the stronger decay condition*

$$\mathbf{E}(x_{0,R}) = \mathbf{E}_\infty + \mathcal{O}(R^{-1}), \ \mathbf{P}(x_{0,R}) = \mathcal{O}(R^{-3/2}),$$

then the asymptotic invariants of the spacelike hypersurface Σ^{Kerr} *are bounded by*

$$\mathbf{E}_{\mathrm{ADM}} = \mathbf{E}_\infty + \mathcal{O}(R^{-1}), \ \mathbf{P}_{\mathrm{ADM}} = \mathcal{O}(R^{-3/2}),$$
$$\mathbf{L}_{\mathrm{ADM}} = \mathbf{L}_\infty + \mathcal{O}(1), \ \mathbf{C}_{\mathrm{ADM}} = \mathbf{G}_\infty + \mathcal{O}(1).$$

Remarks on Theorem 3.2.

(1) The strong asymptotic flatness of the family $x_{0,R}$ is consistent with decay towards spacelike infinity. In particular, the spheres $S_{0,R}$ should be interpreted as spheres on a spacelike hypersurface with radius of size R.
(2) Theorem 3.2 is at the level of C^{m+2}-gluing for the metric components, for integers $m \geq 0$; see Section 2.10 in [12] for the precise definition of higher-order sphere data.

As corollary of Theorem 3.2, we give in Sect. 5 an alternative proof of the Corvino–Schoen [22,23] gluing to Kerr for strongly asymptotically flat spacelike initial data. We refer to Sect. 6 below for the definition of spacelike initial data, strong asymptotic flatness and asymptotic invariants $\mathbf{E}_{\mathrm{ADM}}$, $\mathbf{P}_{\mathrm{ADM}}$, $\mathbf{L}_{\mathrm{ADM}}$ and $\mathbf{C}_{\mathrm{ADM}}$.

Corollary 3.3 (Spacelike gluing to Kerr, version 2). *Let* $m \geq 0$ *be an integer. Consider smooth strongly asymptotically flat spacelike initial data* (Σ, g, k) *with asymptotic invariants*

$$(\mathbf{E}_{\mathrm{ADM}}, \mathbf{P}_{\mathrm{ADM}} = 0, \mathbf{L}_{\mathrm{ADM}}, \mathbf{C}_{\mathrm{ADM}}) \in I(0) \times \mathbb{R}^3 \times \mathbb{R}^3.$$

For real numbers $R \geq 1$ *sufficiently large, there exists a Kerr spacetime* $(\mathcal{M}^{\mathrm{Kerr}}, \mathbf{g}^{\mathrm{Kerr}})$ *and a spacelike hypersurface* Σ^{Kerr} *with asymptotic invariants* $\left(\mathbf{E}^{\mathrm{Kerr}}_{\mathrm{ADM}}, \mathbf{P}^{\mathrm{Kerr}}_{\mathrm{ADM}}, \mathbf{L}^{\mathrm{Kerr}}_{\mathrm{ADM}}, \mathbf{C}^{\mathrm{Kerr}}_{\mathrm{ADM}}\right)$ *such that the spacelike initial data* (g, k) *of* Σ *can be glued in* C^m*-regularity* *across a spacelike annulus $A_{[R,3R]}$ to the induced spacelike initial data $(g^{\mathrm{Kerr}}, k^{\mathrm{Kerr}})$ of Σ^{Kerr}. The Kerr asymptotic invariants can bounded by*

$$\mathbf{E}^{\mathrm{Kerr}}_{\mathrm{ADM}} = \mathbf{E}_{\mathrm{ADM}} + \mathcal{O}(R^{-1}),\ \mathbf{P}^{\mathrm{Kerr}}_{\mathrm{ADM}} = \mathcal{O}(R^{-3/2}),$$
$$\mathbf{L}^{\mathrm{Kerr}}_{\mathrm{ADM}} = \mathbf{L}_{\mathrm{ADM}} + \mathcal{O}(1),\ \mathbf{C}^{\mathrm{Kerr}}_{\mathrm{ADM}} = \mathbf{C}_{\mathrm{ADM}} + \mathcal{O}(1).$$

More precisely, in Corollary 3.3 we glue to Kerr spacelike initial data $(g^{\lambda}, k^{\lambda})$ for some *asymptotic invariants vector* $\lambda \in I(0) \times \mathbb{R}^3 \times \mathbb{R}^3$.

The spacelike gluing of Corollary 3.3 should also be available in the smooth category. Indeed, Theorem 2.21 (and hence Theorem 3.2) should extend to *smooth codimension-*10 *bifurcate null gluing of smooth* ∞^{th}*-order sphere data*. We will, however, not be providing details here.

4. Proof of Perturbative Characteristic Gluing to Kerr

In this section we prove Theorem 3.1. Let $\delta > 0$ be a real number and let $(\tilde{x}_{-R+R\cdot[-\delta,\delta],R})$ be a strongly asymptotically flat family of ingoing null data with asymptotic charges

$$(\mathbf{E}_\infty, \mathbf{P}_\infty = 0, \mathbf{L}_\infty, \mathbf{G}_\infty)$$

with $\mathbf{E}_\infty > 0$. We proceed as follows.

(1) In Sect. 4.1 we rescale the strongly asymptotically flat ingoing null data $(\tilde{x}_{-R+R\cdot[-\delta,\delta],R})$ to ingoing null data $({}^{(R)}\tilde{x}_{-1+[-\delta,\delta],1})$. For $R \geq 1$ sufficiently large, this rescaled ingoing null data is close to Schwarzschild of mass $\mathbf{E}_\infty/R$, see Sect. 4.1 below.
(2) In Sect. 4.2 we apply the perturbative characteristic gluing of [11,12] to glue – up to the 10-dimensional space of charges $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$ – from the rescaled ingoing null data $({}^{(R)}\tilde{x}_{-1+[-\delta,\delta],1})$ to sphere data corresponding to a sphere in a Kerr spacetime to be determined.
(3) In Sect. 4.3 we use a classical topological degree argument to prove that there exists a sphere in a Kerr spacetime such that, following Step (2) above, also the charges $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$ are glued.
(4) In Sect. 4.4 we conclude the proof of Theorem 3.1 by writing out the explicit estimates and scaling the gluing construction from $\mathcal{H}_{-1,[1,2]}$ to $\mathcal{H}_{-R,[R,2R]}$.

4.1. Rescaling to small sphere data. Using the scaling of the Einstein equations, see Definition 2.15, for $R \geq 1$ large we rescale $(\tilde{x}_{-R+R\cdot[-\delta,\delta],R})$ to ingoing null data $({}^{(R)}\tilde{x}_{-1+[-\delta,\delta],1})$. By Lemma 2.19 and (2.24), it holds that

$$\|{}^{(R)}\tilde{x}_{-1+[-\delta,\delta],1} - \mathfrak{m}^{\mathbf{E}_\infty/R}\|_{\mathcal{X}^+(\underline{\mathcal{H}}_{-1+[-\delta,\delta],1})} = \mathcal{O}(R^{-3/2}). \tag{4.1}$$

4.2. Application of perturbative characteristic gluing of [11,12]. In this section we apply the perturbative characteristic gluing of [11,12] (i.e. Theorem 2.20) to glue from the rescaled ingoing null data $({}^{(R)}\tilde{x}_{-1+[-\delta,\delta],1})$ to sphere data of a Kerr spacetime.

Consider asymptotic invariants vectors $\lambda \in \mathcal{E}_R(\mathbf{E}_\infty)$, see (2.18). Then by Proposition 2.3 we have that

$$\begin{aligned} \|{}^{(R)}x^{\lambda}_{-1,2} - \mathfrak{m}^{\mathbf{E}_\infty/R}\|_{\mathcal{X}(S_{-1,2})} \lesssim & R^{-1} \cdot |\mathbf{E}(\lambda) - \mathbf{E}_\infty| + R^{-1} \cdot |\mathbf{P}(\lambda)| \\ & + R^{-2} \cdot |\mathbf{L}(\lambda)| + R^{-2} \cdot |\mathbf{C}(\lambda)| \\ & + \left(\frac{R^{-2} \cdot |\mathbf{L}(\lambda)| + \frac{|\mathbf{P}(\lambda)|}{\mathbf{E}_\infty} \cdot R^{-2} \cdot |\mathbf{C}(\lambda)|}{\mathbf{E}_\infty/R} \right)^2 \\ & = \mathcal{O}\left(R^{-3/2}\right). \end{aligned} \tag{4.2}$$

Hence, by (4.1) and (4.2), for $R \geq 1$ sufficiently large we can apply the perturbative characteristic gluing of [11,12], i.e. Theorem 2.20, with $M = \mathbf{E}_\infty/R$ and $\varepsilon = \mathcal{O}(R^{-3/2})$ to glue from the rescaled ingoing null data ${}^{(R)}\tilde{x}_{-1+[-\delta,\delta],1}$ to ${}^{(R)}x^{\lambda}_{-1,2}$ for an asymptotic invariants vector $\lambda \in \mathcal{E}_R(\mathbf{E}_\infty)$ to be determined. That is, there are

- sphere data ${}^{(R)}x_{-1,1}$ on a sphere $S_{-1,1}$ stemming from a perturbation of $\tilde{S}_{-1,1}$ in $\underline{\tilde{\mathcal{H}}}_{-1+[-\delta,\delta],1}$,
- a solution $x \in \mathcal{X}(\mathcal{H}_{-1,[1,2]})$ to the null structure equations on $\mathcal{H}_{-1,[1,2]}$,

such that $x|_{S_{-1,1}} = {}^{(R)}x_{-1,1}$ and $x|_{S_{-1,2}}$ agrees with ${}^{(R)}x^{\lambda}_{-1,2}$ up to the 10-dimensional space of charges $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$, that is, if we have that

$$(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})\,(x|_{S_{-1,2}}) = (\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})\left({}^{(R)}x^{\lambda}_{-1,2}\right), \tag{4.3}$$

then the constructed solution x satisfies $x|_{S_{-1,2}} = {}^{(R)}x^{\lambda}_{-1,2}$.

By $M = \mathbf{E}_\infty/R$ and $\varepsilon = \mathcal{O}(R^{-3/2})$ with the estimates proved in [12], see Theorem 2.20, the following general charge perturbation estimate holds,

$$\begin{aligned} (\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})\left(x|_{S_{-1,2}}\right) - (\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})\left({}^{(R)}\tilde{x}_{-1,1}\right) &= \mathcal{O}\left(\frac{\mathbf{E}_\infty}{R} R^{-3/2}\right) + \mathcal{O}(R^{-3}) \\ &= \mathcal{O}(R^{-5/2}). \end{aligned} \tag{4.4}$$

4.3. Choice of Kerr spacetime. In this section we use a classical topological degree argument to determine an asymptotic invariants vector $\lambda' \in \mathcal{E}_R(\mathbf{E}_\infty)$ such that (4.3) holds. The idea to determine the λ' by a degree argument is similar to [23].

First, for asymptotic invariants vectors $\lambda \in \mathcal{E}_R(\mathbf{E}_\infty)$, we define the error map $f_R(\lambda)$ by

$$f_R(\lambda) := \left(R\,\mathbf{E}, R\,\mathbf{P}, R^2\,\mathbf{L}, R^2\,\mathbf{G}\right)\left(x|_{S_{-1,2}}\right) - \left(R\,\mathbf{E}, R\,\mathbf{P}, R^2\,\mathbf{L}, R^2\,\mathbf{G}\right)\left({}^{(R)}x^{\lambda}_{-1,2}\right). \tag{4.5}$$

In the following we show that for $R \geq 1$ sufficiently large, there is a $\lambda' \in \mathcal{E}_R(\mathbf{E}_\infty)$ such that

$$f_R(\lambda') = 0. \tag{4.6}$$

By definition of f_R in (4.5), the condition (4.6) is equivalent to charge matching at $S_{-1,2}$, see (4.3), which subsequently implies the full matching.

To prove the existence of λ' satisfying (4.6), we estimate $f_R(\lambda)$ and apply a topological degree argument. First, we estimate $f_R(\lambda)$ by using the following estimates for the charges $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$.

- From (4.4) we have

$$(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G}) \left(x|_{S_{-1,2}}\right) - (\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G}) \left({}^{(R)}\tilde{x}_{-1,1}\right) = \mathcal{O}(R^{-5/2}). \tag{4.7}$$

- By Lemma 2.13, the strongly asymptotically flat family of sphere data $(\tilde{x}_{-R,R})$ satisfies (with $\mathbf{P}_\infty = 0$)

$$(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G}) \left(\tilde{x}_{-R,R}\right) = (\mathbf{E}_\infty, \mathbf{P}_\infty, \mathbf{L}_\infty, \mathbf{G}_\infty) + \left(\mathcal{O}(R^{-1/2}), \mathcal{O}(R^{-1/2}), \mathcal{O}(1), \mathcal{O}(1)\right). \tag{4.8}$$

- By Proposition 2.3, for $R \geq 1$ large, the Kerr reference sphere data $x^\lambda_{-R,2R}$ satisfies for $\lambda \in \mathcal{E}_R(\mathbf{E}_\infty)$,

$$\begin{aligned} \mathbf{E}\left(x^\lambda_{-R,2R}\right) &= \mathbf{E}(\lambda) + \mathcal{O}(R^{-1}), & \mathbf{P}\left(x^\lambda_{-R,2R}\right) &= \mathbf{P}(\lambda) + \mathcal{O}(R^{-3/2}), \\ \mathbf{L}\left(x^\lambda_{-R,2R}\right) &= \mathbf{L}(\lambda) + \mathcal{O}(R^{-1/2}), & \mathbf{G}\left(x^\lambda_{-R,2R}\right) &= \mathbf{C}(\lambda) - 3R \cdot \mathbf{P}(\lambda) + \mathcal{O}(R^{-1/4}). \end{aligned} \tag{4.9}$$

Applying (4.7), (4.8) and (4.9) to (4.5), we can estimate $f_R(\lambda)$ as follows,

$$\begin{aligned} f_R(\lambda) =& \left(R\,\mathbf{E}, R\,\mathbf{P}, R^2\,\mathbf{L}, R^2\,\mathbf{G}\right)\left({}^{(R)}\tilde{x}_{-1,1}\right) - \left(R\,\mathbf{E}, R\,\mathbf{P}, R^2\,\mathbf{L}, R^2\,\mathbf{G}\right)\left({}^{(R)}x^\lambda_{-1,2}\right) \\ &+ \left(\mathcal{O}(R^{-3/2}), \mathcal{O}(R^{-3/2}), \mathcal{O}(R^{-1/2}), \mathcal{O}(R^{-1/2})\right) \\ =& (\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})\left(\tilde{x}_{-R,R}\right) - (\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})\left(x^\lambda_{-R,2R}\right) \\ &+ \left(\mathcal{O}(R^{-3/2}), \mathcal{O}(R^{-3/2}), \mathcal{O}(R^{-1/2}), \mathcal{O}(R^{-1/2})\right) \\ =& (\mathbf{E}_\infty, \mathbf{P}_\infty, \mathbf{L}_\infty, \mathbf{G}_\infty) - (\mathbf{E}(\lambda), \mathbf{P}(\lambda), \mathbf{L}(\lambda), \mathbf{C}(\lambda) - 3R \cdot \mathbf{P}(\lambda)) \\ &+ \left(\mathcal{O}(R^{-1/2}), \mathcal{O}(R^{-1/2}), \mathcal{O}(1), \mathcal{O}(1)\right), \end{aligned} \tag{4.10}$$

where we underline that the error terms also depend on $\lambda \in \mathcal{E}_R(\mathbf{E}_\infty)$.

Second, we have the following classical topological degree argument; see Chapter 1 of [41].

Lemma 4.1 (Topological degree argument). *Let $B \subset \mathbb{R}^{10}$ be the open unit ball. Let f_0 and f_1 be two continuous maps on $\overline{B}$ into $\mathbb{R}^{10}$ and assume that f_0 is a homeomorphism on $\overline{B}$ with $f_0(\lambda_0) = 0$ for a $\lambda_0 \in B$. For $0 \leq t \leq 1$, let $f(\lambda, t)$ be a homotopy on $\overline{B}$ such that $f(\lambda, 0) = f_0(\lambda)$ and $f(\lambda, 1) = f_1(\lambda)$. If for all $0 \leq t \leq 1$ it holds that $0 \notin f(\partial B, t)$, then there exists $\lambda' \in B$ such that $f_1(\lambda') = 0$.*

Remark 4.2. The proof of (4.6) below uses only the charge estimate (4.7). Given that the bifurcate characteristic gluing (see Theorem 2.21) provides analogous charge estimates, the proof applies also to the matching to Kerr in Theorem 3.2.

We are now in position to prove the existence of λ' such that (4.6) is satisfied. Based on (4.10), define for $0 \leq t \leq 1$ the homotopy $f_R(\lambda, t)$ on $\mathcal{E}_R(\mathbf{E}_\infty)$ by

$$\begin{aligned} f_R(\lambda, t) :=& (\mathbf{E}_\infty, \mathbf{P}_\infty, \mathbf{L}_\infty, \mathbf{G}_\infty) - (\mathbf{E}(\lambda), \mathbf{P}(\lambda), \mathbf{L}(\lambda), \mathbf{C}(\lambda) - 3R \cdot \mathbf{P}(\lambda)) \\ &+ t \cdot \left(\mathcal{O}(R^{-1/2}), \mathcal{O}(R^{-1/2}), \mathcal{O}(1), \mathcal{O}(1)\right), \end{aligned} \tag{4.11}$$

such that

$$\begin{aligned} f_R(\lambda, 0) =& (\mathbf{E}_\infty, \mathbf{P}_\infty, \mathbf{L}_\infty, \mathbf{G}_\infty) - (\mathbf{E}(\lambda), \mathbf{P}(\lambda), \mathbf{L}(\lambda), \mathbf{C}(\lambda) - 3R \cdot \mathbf{P}(\lambda)), \\ f_R(\lambda, 1) =& f_R(\lambda). \end{aligned} \tag{4.12}$$

We make the following three observations.

(1) From (4.12) it follows that $f_R(\lambda, 0)$ is a homeomorphism on $\mathcal{E}_R(\mathbf{E}_\infty)$.
(2) For $R \geq 1$ large, we have that $\lambda_0 := (\mathbf{E}_\infty, 0, \mathbf{L}_\infty, \mathbf{G}_\infty) \in \mathcal{E}_R(\mathbf{E}_\infty)$ and satisfies, by the definition of $f_R(\lambda, 0)$ in (4.12), $f_R(\lambda_0, 0) = 0$.
(3) For $R \geq 1$ sufficiently large and all $0 \leq t \leq 1$, it holds that

$$0 \notin f_R(\partial\mathcal{E}_R(\mathbf{E}_\infty), t). \tag{4.13}$$

Indeed, assume by contradiction that there are $\tilde{\lambda} \in \partial\mathcal{E}_R(\mathbf{E}_\infty)$ and $0 \leq \tilde{t} \leq 1$ such that

$$f_R(\tilde{\lambda}, \tilde{t}) = 0. \tag{4.14}$$

Then by definition of $f_R(\lambda, t)$ in (4.11),

$$\begin{aligned} &R^{1/2}\left(\mathbf{E}(\tilde{\lambda}) - \mathbf{E}_\infty\right) = \tilde{t} \cdot \mathcal{O}(1), \; R^{1/2} \cdot \mathbf{P}(\tilde{\lambda}) = \tilde{t} \cdot \mathcal{O}(1), \\ &R^{-1/4}\left(\mathbf{L}(\tilde{\lambda}) - \mathbf{L}_\infty\right) = \tilde{t} \cdot \mathcal{O}(R^{-1/4}), \end{aligned}$$

and

$$\begin{aligned} \frac{R^{-1/2}}{2}\left(\mathbf{C}(\tilde{\lambda}) - \mathbf{G}_\infty\right) =& \frac{R^{-1/2}}{2}\left(-3R \cdot \mathbf{P}(\tilde{\lambda}) + \tilde{t} \cdot \mathcal{O}(1)\right) \\ =& \frac{R^{-1/2}}{2}\left(-3R \cdot \tilde{t} \cdot \mathcal{O}(R^{-1/2}) + \tilde{t} \cdot \mathcal{O}(1)\right) = \tilde{t} \cdot \mathcal{O}(1). \end{aligned}$$

The above estimates imply that for $R \geq 1$ sufficiently large,

$$\begin{aligned} &\left(R^{1/2}|\mathbf{E}(\tilde{\lambda}) - \mathbf{E}_\infty|\right)^2 + \left(R^{1/2}|\mathbf{P}(\tilde{\lambda})|\right)^2 + \left(R^{-1/4}|\mathbf{L}(\tilde{\lambda})|\right)^2 + \left(R^{-1/2}|\mathbf{C}(\tilde{\lambda})|\right)^2 \\ &\lesssim \tilde{t} \cdot \mathcal{O}(1) < (\mathbf{E}_\infty)^2, \end{aligned}$$

which implies that $\tilde{\lambda} \notin \partial\mathcal{E}_R(\mathbf{E}_\infty)$ (see the definition of $\mathcal{E}_R(\mathbf{E}_\infty)$ in (2.18)). This is a contradiction and hence finishes the proof of (4.13).

By the above observations and the fact that the set $\mathcal{E}_R(\mathbf{E}_\infty) \subset \mathbb{R}^{10}$ is topologically a ball, we can apply Lemma 4.1 to the homotopy $f_R(\lambda, t)$ for $R \geq 1$ sufficiently large, and conclude the existence of a vector $\lambda' \in \mathcal{E}_R(\mathbf{E}_\infty)$ such that

$$f_R(\lambda', 1) = 0. \tag{4.15}$$

This finishes the proof of (4.6). Moreover, we deduce from (4.11) that

$$\begin{aligned} \mathbf{E}(\lambda') &= \mathbf{E}_\infty + \mathcal{O}(R^{-1/2}), & \mathbf{P}(\lambda') &= \mathcal{O}(R^{-1/2}), \\ \mathbf{L}(\lambda') &= \mathbf{L}_\infty + \mathcal{O}(1), & \mathbf{C}(\lambda') &= \mathbf{G}_\infty + 3R \cdot \mathbf{P}(\lambda') + \mathcal{O}(1). \end{aligned} \tag{4.16}$$

It remains to show that in case of the stronger decay assumption (3.4),

$$\mathbf{E}(x_{-R,R}) = \mathbf{E}_\infty + \mathcal{O}(R^{-1}), \; \mathbf{P}(x_{-R,R}) = \mathcal{O}(R^{-3/2}), \tag{4.17}$$

we have the improved estimate (3.5) for λ',

$$\begin{aligned} &\mathbf{E}(\lambda') = \mathbf{E}_\infty + \mathcal{O}(R^{-1}), \; \mathbf{P}(\lambda') = \mathcal{O}(R^{-3/2}), \\ &\mathbf{L}(\lambda') = \mathbf{L}_\infty + \mathcal{O}(1), \; \mathbf{C}(\lambda') = \mathbf{G}_\infty + \mathcal{O}(1). \end{aligned} \tag{4.18}$$

Indeed, applying (4.17) in the above derivation of (4.10), we get that the error map $f_R(\lambda)$ satisfies the improved bound

$$\begin{aligned} f_R(\lambda) := &(\mathbf{E}_\infty, \mathbf{P}_\infty, \mathbf{L}_\infty, \mathbf{G}_\infty) - (\mathbf{E}(\lambda), \mathbf{P}(\lambda), \mathbf{L}(\lambda), \mathbf{C}(\lambda) - 3R \cdot \mathbf{P}(\lambda)) \\ &+ \left(\mathcal{O}(R^{-1}), \mathcal{O}(R^{-3/2}), \mathcal{O}(1), \mathcal{O}(1)\right). \end{aligned}$$

This shows that λ' which satisfies by construction $f_R\left(\lambda'\right) = 0$, see (4.6), satisfies the improved bound (4.18).

4.4. Conclusion of proof. In this section we conclude the proof of Theorem 3.1. By (4.16) with the first of (4.2) (see also Proposition 2.3), we have the estimate

$$\begin{aligned} \|{}^{(R)}x^{\lambda'}_{-1,2} - \mathfrak{m}^{\mathbf{E}_\infty/R}\|_{\mathcal{X}(S_{-1,2})} \lesssim\; & R^{-1} \cdot |\mathbf{E}(\lambda') - \mathbf{E}_\infty| + R^{-1} \cdot |\mathbf{P}(\lambda')| + R^{-2} \cdot |\mathbf{L}(\lambda')| \\ & + R^{-2} \cdot |\mathbf{C}(\lambda')| \\ & + \left(\frac{R^{-2} \cdot |\mathbf{L}(\lambda')| + \frac{|\mathbf{P}(\lambda')|}{\mathbf{E}_\infty} \cdot R^{-2} \cdot |\mathbf{C}(\lambda')|}{\mathbf{E}_\infty/R} \right)^2 \\ =\; & \mathcal{O}\left(R^{-3/2}\right), \end{aligned}$$

which, together with (4.1), implies that the constructed solution x on $\mathcal{H}_{-1,[1,2]}$ is bounded by

$$\|x - \mathfrak{m}^{\mathbf{E}_\infty/R}\|_{\mathcal{X}(\mathcal{H}_{-1,[1,2]})} + \left\| x|_{S_{-1,1}} - {}^{(R)}\tilde{x}_{-1,1} \right\|_{\mathcal{X}(S_{-1,1})} = \mathcal{O}(R^{-3/2}). \tag{4.19}$$

Applying the scaling of Sect. 2.8 with scale factor R^{-1}, we get by (2.24), (4.19) and Lemma 2.17 that

$$\|{}^{(R^{-1})}x - \mathfrak{m}^{\mathbf{E}_\infty}\|_{\mathcal{X}(\mathcal{H}_{-R,[R,2R]})} + \|{}^{(R^{-1})}x - \tilde{x}_{-R,R}\|_{\mathcal{X}(S_{-R,R})} = \mathcal{O}(R^{-3/2}).$$

This finishes the proof of Theorem 3.1.

5. Proof of Spacelike Gluing to Kerr

In this section we prove Corollary 3.3, the gluing of spacelike initial data to Kerr. Let (Σ, g, k) be given smooth strongly asymptotically flat spacelike initial data with asymptotic invariants

$$(\mathbf{E}_{\text{ADM}}, \mathbf{P}_{\text{ADM}}, \mathbf{L}_{\text{ADM}}, \mathbf{C}_{\text{ADM}}) \in I(0) \times \mathbb{R}^3 \times \mathbb{R}^3,$$

where by the strong asymptotic flatness, $\mathbf{P}_{\text{ADM}} = 0$. We proceed in four steps.

(1) We apply the material of Sects. 7.1 and 7.2 where it is shown how to construct and estimate families of higher-order sphere data from given spacelike initial data. We work in the rescaled picture, that is, we construct smooth higher-order sphere data on a sphere $S_{0,1} \subset \Sigma$ in the *rescaled* spacelike initial data.
(2) We use the bifurcate characteristic gluing of Theorem 3.2 to glue the constructed higher-order sphere data on $S_{0,1}$ to a sphere $S^{(R)\lambda}_{-1,2}$ in a Kerr spacetime.
(3) We construct a local spacetime $(\mathcal{M}, \mathbf{g})$ by applying local existence results for the spacelike and characteristic initial value problem for the Einstein equations, and pick a spacelike hypersurface connecting $S_{0,1}$ and $S^{(R)\lambda}_{-1,2}$. We conclude the construction by rescaling.

Notation. For ease of presentation, we work in the following with smooth spacelike initial data and smooth higher-order sphere data $(x, \mathcal{D}^{L,m}, \mathcal{D}^{\underline{L},m})$ for a fixed integer $m \geq 1$.

(1) Rescaling and construction of sphere data. In this section we follow the construction of Sects. 7.1 and 7.2: We start by rescaling the given spacelike initial data by scaling factor R to $({}^{(R)}g, {}^{(R)}k)$ and constructing on the sphere $S_{0,1} := S_{r_{\mathbf{E}_{\text{ADM}}/R}(0,1)} \subset \Sigma$ the higher-order sphere data (see (2.20) and also Remark 7.2)

$$\left({}^{(R)}x_{0,1}, {}^{(R)}\mathcal{D}^{L,m}_{0,1}, {}^{(R)}\mathcal{D}^{\underline{L},m}_{0,1}\right). \tag{5.1}$$

In Sects. 7.1 and 7.2 it is shown that by the strong asymptotic flatness and the scaling of spacelike initial data (see Sect. 6.4), the constructed higher-order sphere data (5.1) is – with respect to an appropriate higher-regularity norm – $\mathcal{O}(R^{-3/2})$-close to Schwarzschild reference higher-order sphere data of order m of mass $\mathbf{E}_{\text{ADM}}/R$; we denote this by

$$\left({}^{(R)}x_{0,1}, {}^{(R)}\mathcal{D}^{L,m}_{0,1}, {}^{(R)}\mathcal{D}^{\underline{L},m}_{0,1}\right) - \left(\mathfrak{m}^{\mathbf{E}_{\text{ADM}}/R}_{0,1}, \mathcal{D}^{L,m,\mathbf{E}_{\text{ADM}}/R}_{0,1}, \mathcal{D}^{\underline{L},m,\mathbf{E}_{\text{ADM}}/R}_{0,1}\right) = \mathcal{O}(R^{-3/2}). \tag{5.2}$$

Moreover, in Theorem 7.1 it is proved that the charges $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})({}^{(R)}x_{0,1})$ can be estimated by

$$\begin{aligned} &\left(R \cdot \mathbf{E}\left({}^{(R)}x_{0,1}\right), R \cdot \mathbf{P}\left({}^{(R)}x_{0,1}\right), R^2 \cdot \mathbf{L}\left({}^{(R)}x_{0,1}\right), R^2 \cdot \mathbf{G}\left({}^{(R)}x_{0,1}\right)\right) \\ &= (\mathbf{E}_{\text{ADM}}, \mathbf{P}_{\text{ADM}}, \mathbf{L}_{\text{ADM}}, \mathbf{C}_{\text{ADM}}) + \left(\mathcal{O}(R^{-1}), \mathcal{O}(R^{-3/2}), o(1), o(1)\right). \end{aligned} \tag{5.3}$$

(2) Application of bifurcate characteristic gluing to Kerr. By (5.2) and (5.3), we can apply Theorem 3.2 (to be precise, the rescaled version thereof) to the higher-order sphere data (5.1) to get

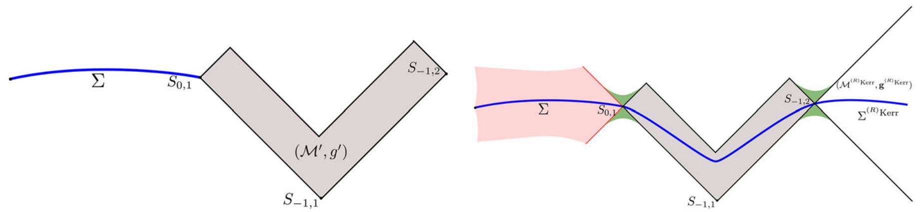

Fig. 5. The spacetime $(\mathcal{M}', \mathbf{g}')$ is denoted as shaded region, and the spacelike hypersurface Σ'' is indicated by the bold blue line

- smooth higher-order outgoing null data $({}^{(R)}x, {}^{(R)}\mathcal{D}^{L,m}, {}^{(R)}\mathcal{D}^{\underline{L},m})$ on $\mathcal{H}_{-1,[1,2]}$ and smooth higher-order ingoing null data $({}^{(R)}x, {}^{(R)}\mathcal{D}^{L,m}, {}^{(R)}\mathcal{D}^{\underline{L},m})$ on $\underline{\mathcal{H}}_{[-1,0],1}$ satisfying the higher-order null structure equations and matching on $S_{-1,1}$,
- a Kerr reference sphere $S_{-1,2}^{(R)\mathrm{Kerr}}$ in a Kerr spacetime $(\mathcal{M}^{(R)\mathrm{Kerr}}, \mathbf{g}^{(R)\mathrm{Kerr}})$ with Kerr reference higher-order sphere data

$$(x_{-1,2}^{(R)\mathrm{Kerr}}, \mathcal{D}_{-1,2}^{L,m,(R)\mathrm{Kerr}}, \mathcal{D}_{-1,2}^{\underline{L},m,(R)\mathrm{Kerr}}).$$

such that we have matching up to order m on $S_{-1,1}$, $S_{0,1}$ and $S_{-1,2}$,

$$\begin{aligned}({}^{(R)}x, {}^{(R)}\mathcal{D}^{L,m}, {}^{(R)}\mathcal{D}^{\underline{L},m})|_{S_{-1,1}} =&({}^{(R)}x, {}^{(R)}\mathcal{D}^{L,m}, {}^{(R)}\mathcal{D}^{\underline{L},m})|_{S_{-1,1}},\\ ({}^{(R)}x, {}^{(R)}\mathcal{D}^{L,m}, {}^{(R)}\mathcal{D}^{\underline{L},m})|_{S_{0,1}} =&({}^{(R)}x_{0,1}, {}^{(R)}\mathcal{D}_{0,1}^{L,m}, {}^{(R)}\mathcal{D}_{0,1}^{\underline{L},m}),\\ ({}^{(R)}x, {}^{(R)}\mathcal{D}^{L,m}, {}^{(R)}\mathcal{D}^{\underline{L},m})|_{S_{-1,2}} =&(x_{-1,2}^{(R)\mathrm{Kerr}}, \mathcal{D}_{-1,2}^{L,m,(R)\mathrm{Kerr}}, \mathcal{D}_{-1,2}^{\underline{L},m,(R)\mathrm{Kerr}}).\end{aligned}$$

In particular, it holds that

(1) the null data $({}^{(R)}x, {}^{(R)}\mathcal{D}^{L,m}, {}^{(R)}\mathcal{D}^{\underline{L},m})$ on $\mathcal{H}_{-1,[1,2]}$ and $({}^{(R)}x, {}^{(R)}\mathcal{D}^{L,m}, {}^{(R)}\mathcal{D}^{\underline{L},m})$ on $\underline{\mathcal{H}}_{[-1,0],1}$ are $\mathcal{O}(R^{-3/2})$-close to Schwarzschild reference higher-order null data of mass M/R, respectively, and
(2) the sphere $S_{-1,2}^{(R)\mathrm{Kerr}}$ lies in a Kerr reference spacelike hypersurface $\Sigma^{(R)\mathrm{Kerr}} \subset \mathcal{M}^{(R)\mathrm{Kerr}}$ with asymptotic invariants (see Sects. 2.3 and 6.4)

$$\begin{aligned}\mathbf{E}_{\mathrm{ADM}}^{(R)\mathrm{Kerr}} =&R^{-1}\cdot\mathbf{E}_{\mathrm{ADM}} + \mathcal{O}(R^{-2}), \quad \mathbf{P}_{\mathrm{ADM}}^{(R)\mathrm{Kerr}} =\mathcal{O}(R^{-5/2}),\\ \mathbf{L}_{\mathrm{ADM}}^{(R)\mathrm{Kerr}} =&R^{-2}\cdot\mathbf{L}_{\mathrm{ADM}} + \mathcal{O}(R^{-2}), \quad \mathbf{C}_{\mathrm{ADM}}^{(R)\mathrm{Kerr}} =R^{-2}\cdot\mathbf{C}_{\mathrm{ADM}} + \mathcal{O}(R^{-2}).\end{aligned} \tag{5.4}$$

(3) Construction of spacelike hypersurface. The constructed solutions to the higher-order null structure equations,

$$({}^{(R)}x, {}^{(R)}\mathcal{D}^{L,m}, {}^{(R)}\mathcal{D}^{\underline{L},m}) \text{ on } \mathcal{H}_{-1,[1,2]} \text{ and } ({}^{(R)}x, {}^{(R)}\mathcal{D}^{L,m}, {}^{(R)}\mathcal{D}^{\underline{L},m}) \text{ on } \underline{\mathcal{H}}_{[-1,0],0},$$

form *characteristic initial data* for the Einstein vacuum equations which is $\mathcal{O}(R^{-3/2})$-close to Schwarzschild of mass $\mathbf{E}_\infty/R$. By the work of Luk and Luk–Rodnianski on the characteristic initial value problem for the Einstein equations [37,38], for $R \geq 1$ sufficiently large, the associated maximal globally hyperbolic spacetime $(\mathcal{M}', \mathbf{g}')$ contains slabs of universal width along the null hypersurface $\underline{\mathcal{H}}_{[-1,0],1}$ and $\mathcal{H}_{-1,[1,2]}$; see also Remark 5.1 below.
Applying local existence for the spacelike Cauchy problem defined on Σ (the resulting region is shaded red in Fig. 5), and, subsequently, for the characteristic Cauchy problems

(the resulting regions are shaded green in Fig. 5) defined on $\partial^+\mathcal{D}(\Sigma)$ and $\partial^+\mathcal{M}'$, $\partial^-\mathcal{D}(\Sigma)$ and $\underline{\mathcal{H}}$, $\partial^+\mathcal{M}'$ and $\partial^+\mathcal{M}^{(R)\mathrm{Kerr}}$, and $\mathcal{H}$ and $\partial^-\mathcal{M}^{(R)\mathrm{Kerr}}$, we construct the spacetime $(\mathcal{M}'', \mathbf{g}'')$, see Fig. 5. Here ∂^+ and ∂^- denote the future and past boundaries, and $\mathcal{D}(\Sigma)$ the domain of dependence of Σ.

In $(\mathcal{M}'', \mathbf{g}'')$ we define a spacelike hypersurface Σ'' (see Fig. 5) such that (i) Σ'' agrees with Σ in $(\mathcal{M}, \mathbf{g})$, (ii) Σ'' is spacelike and contained in the slabs in $(\mathcal{M}', \mathbf{g}')$, (iii) Σ'' agrees with $\Sigma^{(R)\mathrm{Kerr}}$ in $(\mathcal{M}^{(R)\mathrm{Kerr}}, \mathbf{g}^{(R)\mathrm{Kerr}})$. By construction, the induced spacelike initial data on Σ'' agrees with $({}^{(R)}g, {}^{(R)}k)$ on Σ, and with Kerr reference spacelike initial data $(g^{(R)\mathrm{Kerr}}, k^{(R)\mathrm{Kerr}})$ on $\Sigma^{(R)\mathrm{Kerr}}$. In particular, it is a solution to the spacelike gluing problem from the rescaled spacelike initial data $({}^{(R)}g, {}^{(R)}k)$ to the Kerr reference spacelike initial data $(g^{(R)\mathrm{Kerr}}, k^{(R)\mathrm{Kerr}})$.

Scaling the above spacelike initial data by factor R^{-1}, and using the scale-invariance of the Kerr reference spacelike initial data (see also (2.25)), we conclude the spacelike gluing to Kerr at the level of order m sphere data.

Remark 5.1 (On the well-posedness of the characteristic Cauchy problem for the constructed initial data and the regularity of the resulting spacetime). First we recall the norm $\mathcal{X}(\mathcal{H})$ in which our constructed gluing solution lies (see Definition 2.9, where also the analogous $\mathcal{X}(\underline{\mathcal{H}})$ is defined),

$$\begin{aligned}\|x\|_{\mathcal{X}(\mathcal{H})} :=& \|\Omega\|_{H^6_3(\mathcal{H})} + \|\not{g}\|_{H^6_3(\mathcal{H})} + \|\eta\|_{H^5_2(\mathcal{H})} + \|\Omega\mathrm{tr}\chi\|_{H^6_3(\mathcal{H})} + \|\widehat{\chi}\|_{H^6_2(\mathcal{H})} \\ &+ \|\Omega\mathrm{tr}\underline{\chi}\|_{H^4_2(\mathcal{H})} + \|\widehat{\underline{\chi}}\|_{H^4_3(\mathcal{H})} + \|\omega\|_{H^6_2(\mathcal{H})} + \|D\omega\|_{H^6_1(\mathcal{H})} \\ &+ \|\underline{\omega}\|_{H^4_3(\mathcal{H})} + \|\underline{D\omega}\|_{H^2_3(\mathcal{H})} + \|\alpha\|_{H^6_1(\mathcal{H})} + \|\beta\|_{H^5_2(\mathcal{H})} + \|\rho\|_{H^4_2(\mathcal{H})} \\ &+ \|\sigma\|_{H^4_2(\mathcal{H})} + \|\underline{\beta}\|_{H^3_2(\mathcal{H})} + \|\underline{\alpha}\|_{H^2_3(\mathcal{H})},\end{aligned} \tag{5.5}$$

where we recall that $H^m_l(\mathcal{H})$ bounds m $\nabla\!\!\!/$-derivatives and l ∂_v-derivatives in $L^2(\mathcal{H})$, that is, for a tensor T

$$\|T\|^2_{H^m_l(\mathcal{H})} := \int_0^1 \sum_{0\leq i\leq l} \left\| D^i T \right\|^2_{H^m(S_{0,\underline{u}})} dv.$$

For local existence of the characteristic Cauchy problem we refer to the main theorem of the work [38] by Luk–Rodnianski which states the following. Let $\not{g}$ denote the induced metric on the spheres $S_{0,\underline{u}}$ and $S_{u,0}$ foliating the null hypersurfaces $\mathcal{H}$ and $\underline{\mathcal{H}}$, respectively. Let ψ and Ψ denote Ricci coefficients and null curvature components, respectively. Consider characteristic initial data satisfying, for two real numbers $0 < c < C$,

$$\begin{aligned}& c < |\det(\not{g})| < C, \\ & \sum_{i\leq 3} \left|\partial^i_\theta \not{g}\right| \leq C, \\ & \sum_{i\leq 3} \left(\sup_{\underline{u}} \|\nabla\!\!\!/^i \psi\|_{L^2(S_{0,\underline{u}})} + \sup_u \|\nabla\!\!\!/^i \psi\|_{L^2(S_{u,0})} \right), \\ & \sum_{i\leq 2} \left(\sum_{\Psi\in\{\beta,\rho,\sigma,\underline{\beta}\}} \sup_{\underline{u}} \|\nabla\!\!\!/^i \Psi\|_{L^2(S_{0,\underline{u}})} + \sum_{\Psi\in\{\rho,\sigma,\underline{\beta},\underline{\alpha}\}} \sup_u \|\nabla\!\!\!/^i \Psi\|_{L^2(S_{u,0})} \right),\end{aligned} \tag{5.6}$$

where ∂_θ denotes coordinate angular derivatives and $\nabla\!\!\!/$ the covariant derivative with respect to $g\!\!\!/$. For $\varepsilon > 0$ sufficiently small depending on the constants $C > 0$ and $c > 0$, *there exists a spacetime* $(\mathcal{M}, g)$ endowed with a double null foliation $u, \underline{u}$ solving the characteristic initial value problem to the vacuum Einstein equations in $0 \leq u \leq u_*$, $0 \leq \underline{u} \leq \underline{u}_*$ for $u_*, \underline{u}_* \leq \varepsilon$. *The metric is continuous* and takes the form

$$g = -2\Omega^2(du \otimes d\underline{u} + d\underline{u} \otimes du) + g\!\!\!/_{AB}(d\theta^A - b^A du)(d\theta^B - b^B du).$$

The spacetime $(\mathcal{M}, g)$ is a C^0-limit of smooth solutions to the vacuum Einstein equations and is the *unique spacetime solving the characteristic initial value problem* among all C^0 limits of smooth solutions. In $(\mathcal{M}, g)$ it holds

$$\partial_\theta g, \partial_u g \in C^0_u C^0_{\underline{u}} L^4(S), \qquad \partial^2_\theta g, \partial_u \partial_\theta g, \partial^2_u \in C^0_u C^0_{\underline{u}} L^2(S),$$
$$\partial_{\underline{u}} g, \partial_{\underline{u}} \left(g\!\!\!/^{AB} \partial_{\underline{u}} g\!\!\!/_{AB} \right) \in L^\infty_u L^\infty_{\underline{u}} L^\infty(S), \quad \partial_\theta \partial_{\underline{u}} g, \partial_u \partial_{\underline{u}} g, \partial^2_{\underline{u}} b^A \in L^\infty_u L^\infty_{\underline{u}} L^4(S).$$

In the $(u, \underline{u}, \theta^1, \theta^2)$-coordinates, the Einstein equations are satisfied in $L^\infty_{\underline{u}} L^\infty_u L^2(S)$. Furthermore, higher angular differentiability in the data results in higher angular differentiability of the solution.

In short, (5.6) asks that *3* $\nabla\!\!\!/$*-derivatives* of Ricci coefficients, and *2* $\nabla\!\!\!/$*-derivatives* of null curvature components are bounded in $L^2(S_{0,\underline{u}})$ (or $L^2(S_{u,0})$, respectively). *The norm* $\mathcal{X}(\mathcal{H})$ *defined in* (5.5) *bounds these quantities* by a standard Sobolev trace theorem (or $\mathcal{X}(\underline{\mathcal{H}})$ bounds them, respectively) and the norm is actually stronger than necessary for this local existence result.

We remark that the result of Luk–Rodnianski assumes that the characteristic initial data satisfies the *gauge-condition* $\Omega \equiv 1$ along $\mathcal{H}$, which is not the case for our constructed characteristic initial data. However, once our gluing characteristic initial data is constructed, we can apply a straight-forward gauge-change on $\mathcal{H}$ to make $\Omega \equiv 1$ (in other words: make a change of the v-foliation along $\mathcal{H}$). From the explicit transformation formulas for Ricci coefficients and null curvature components under change of v-foliation (see, for example, [25]) one can see that our constructed Ω is sufficiently regular (see the norm (5.5)) that this gauge-change does not lead to a change of regularity for the Ricci coefficients and null curvature components (i.e. we are still in $\mathcal{X}(\mathcal{H})$). We then can cite the Luk–Rodnianski local existence result.

6. Spacelike Initial Data and Asymptotic Invariants

6.1. The spacelike constraint equations and spacelike initial data. Let $(\mathcal{M}, \mathbf{g})$ be a spacetime, and denote its Riemann curvature tensor by $\mathbf{R}$. Let Σ be a spacelike hypersurface in $\mathcal{M}$ with future-directed timelike unit normal T. The *electric-magnetic decomposition* of $\mathbf{R}$ on Σ is given by

$$E_{ab} := \mathbf{R}_{TaTb}, \;\; H_{ab} := {}^*\mathbf{R}_{TaTb}, \tag{6.1}$$

where ${}^*\mathbf{R}_{\alpha\beta\gamma\delta} := \frac{1}{2} \in_{\alpha\beta\mu\nu} \mathbf{R}^{\mu\nu}{}_{\gamma\delta}$ denotes the Hodge dual of $\mathbf{R}$ with respect to the volume form $\in$ on $(\mathcal{M}, \mathbf{g})$. The 2-tensors E and H are symmetric and tracefree, and (see [19])

$$\mathbf{R}_{abcT} = - \in_{ab}{}^s H_{sc}, \;\; \mathbf{R}_{abcd} = - \in_{abs} \in_{cdl} E^{sl}, \tag{6.2}$$

where $\in_{abc} := \in_{Tabc}$ denotes the induced volume element on Σ.

In the following, let g be the induced metric and k be the second fundamental form of $\Sigma \subset \mathcal{M}$. Denote the covariant derivative of g by ∇ and the Ricci tensor of g by Ric. It holds that

$$\text{Ric}_{ij} - k_{ia}k^{a}{}_{j} + k_{ij}\text{tr}k = E_{ij}, \ \ \nabla_i k_{jm} - \nabla_j k_{im} = \in_{ij}{}^{l} H_{lm}, \tag{6.3}$$

where $\text{tr}k := g^{ab}k_{ab}$. Taking the trace of (6.3) with respect to g leads to the *spacelike constraint equations*,

$$R_{\text{scal}} = |k|^2 - (\text{tr}k)^2, \ \ \text{div}k = d(\text{tr}k), \tag{6.4}$$

where d denotes the exterior derivative on Σ and $R_{\text{scal}} := g^{ab}\text{Ric}_{ab}$.

Spacelike initial data for the Einstein equations is specified by a triple (Σ, g, k) where (Σ, g) is a Riemannian 3-manifold and k is a symmetric 2-tensor on Σ, satisfying the spacelike constraint equations (6.4). Local well-posedness of the Cauchy problem of general relativity with sufficiently regular spacelike initial data is well-known [15,28, 33].

Schwarzschild reference spacelike initial data. The Schwarzschild metric of mass $M \geq 0$ is given in Schwarzschild coordinates (t, r, θ, ϕ) by

$$\mathbf{g} = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

The induced spacelike initial data on the spacelike hypersurface $\{t = 0\} \cap \{r > 2M\}$ is given by

$$(\Sigma, g, k) = \left(\mathbb{R}^3 \setminus \overline{B(0, 2M)}, \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right), 0\right). \tag{6.5}$$

It is well-known that the induced metric g is conformally flat. Indeed, defining *isotropic coordinates* $(\tilde{r}, \tilde{\theta}, \tilde{\phi})$ the from Schwarzschild coordinates (r, θ, ϕ) by the relations

$$\frac{r}{\tilde{r}} = \left(1 + \frac{M}{2\tilde{r}}\right)^2, \ \ \tilde{\theta} = \theta, \ \ \tilde{\phi} = \phi, \tag{6.6}$$

it holds that for $r > 2M$,

$$g = \left(1 + \frac{M}{2\tilde{r}}\right)^4 \tilde{e}, \tag{6.7}$$

where $\tilde{e}_{ij}$ denotes the Euclidean metric in Cartesian isotropic coordinates $(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$ defined by (2.1) from $(\tilde{r}, \tilde{\theta}, \tilde{\phi})$.

Notation. For real numbers $M \geq 0$, we denote the metric components of the Schwarzschild reference metric g in Schwarzschild Cartesian coordinates (x^1, x^2, x^3) by g^{M}_{ij}, and in isotropic Cartesian coordinates by $\tilde{g}^{M}_{ij}$. The following *strong asymptotic flatness* corresponds to the center-of-mass frame of the isolated system under consideration, see [17,19,34].

Definition 6.1 (*Strong asymptotic flatness*). Spacelike initial data (Σ, g, k) is *strongly asymptotically flat* if there exist a real number $M \geq 0$, a compact set $K \subset \Sigma$ such that its complement $\Sigma \setminus K$ is diffeomorphic to the complement of the closed unit ball in $\mathbb{R}^3$, and a coordinate system (x^1, x^2, x^3) defined near spacelike infinity such that, as $|x| \to \infty$,

$$g_{ij}(x) = \left(1 + \frac{2M}{|x|}\right) e_{ij} + \mathcal{O}\left(|x|^{-3/2}\right), \; k_{ij}(x) = \mathcal{O}\left(|x|^{-5/2}\right). \tag{6.8}$$

We moreover require analogous conditions on successive derivatives as needed.

Remarks on Definition 6.1.

(1) In this paper, strong asymptotic flatness is used to bound the error terms when we relate the local integrals $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$ on the large sphere S_R to the limits $(\mathbf{E}_\infty, \mathbf{P}_\infty, \mathbf{L}_\infty, \mathbf{G}_\infty)$. These error terms need to be sufficiently small for the classical degree argument work.
(2) The class of strongly asymptotically flat spacelike initial data is of interest for the community (in particular, the data does not need to be Kerr outside a compact set); see, for example, the work by Dain-Friedrich [27] where a large class of spacelike initial data with the following (stronger) asymptotics is constructed,

$$g_{ij}(x) = \left(1 + \frac{2M}{|x|}\right) e_{ij} + \mathcal{O}\left(|x|^{-2}\right), \; k_{ij}(x) = \mathcal{O}\left(|x|^{-3}\right).$$

6.2. Asymptotic invariants of asymptotically flat spacelike initial data. Given asymptotically flat spacelike initial data (Σ, g, k) with Cartesian coordinates (x^1, x^2, x^3) near spacelike infinity, define standard spherical coordinates (r, θ^1, θ^2) by (2.1), and let the 2-spheres $S_r \subset \Sigma$ be defined as the level sets of r. The following *asymptotic invariants* are fundamental quantities in mathematical relativity, see [13,17,19,23].

Definition 6.2 (*Asymptotic invariants*). Let (Σ, g, k) be asymptotically flat spacelike initial data with coordinates (x^1, x^2, x^3) near spacelike infinity. For $i = 1, 2, 3$, define

$$\mathbf{E}_{\mathrm{ADM}} := \lim_{r\to\infty} \frac{1}{16\pi} \int_{S_r} \sum_{j=1,2,3} \left(\partial_j g_{jl} - \partial_l g_{jj}\right) N^l d\mu_{g},$$

$$(\mathbf{P}_{\mathrm{ADM}})^i := \lim_{r\to\infty} \frac{1}{8\pi} \int_{S_r} (k_{il} - \mathrm{tr} k \, g_{il}) N^l d\mu_{g},$$

$$(\mathbf{L}_{\mathrm{ADM}})^i := \lim_{r\to\infty} \frac{1}{8\pi} \int_{S_r} (k_{jl} - \mathrm{tr} k \, g_{jl}) \left(Y_{(i)}\right)^j N^l d\mu_{g},$$

and

$$(\mathbf{C}_{\mathrm{ADM}})^i := \lim_{r\to\infty} \int_{S_r} \left(x^i \sum_{j=1,2,3} \left(\partial_j g_{jl} - \partial_l g_{jj}\right) N^l - \sum_{j=1,2,3} \left(g_{ji} N^j - g_{jj} N^i \right) \right) d\mu_{g},$$

where N denotes the outward-pointing unit normal to S_r and $d\mu_{g}$ the induced volume element on S_r. Furthermore, $Y_{(i)}$, $i = 1, 2, 3$, are the rotation fields defined by $(Y_{(i)})_j := \in_{ilj} x^l$.

Remarks on Definition 6.2.

(1) The asymptotic invariants can be interpreted as *energy* $\mathbf{E}_{\mathrm{ADM}}$, *linear momentum* $\mathbf{P}_{\mathrm{ADM}}$, *angular momentum* $\mathbf{L}_{\mathrm{ADM}}$ and *center-of-mass* $\mathbf{C}_{\mathrm{ADM}}$ of the spacelike initial data set.
(2) The asymptotic invariants are well-defined and foliation-independent for strongly asymptotically flat spacelike initial data (as well as for more general asymptotics), see [17,23] and references therein.
(3) By the positive energy theorem [42,43,45] it holds for sufficiently regular asymptotically flat spacelike initial data that $\mathbf{E}_{\mathrm{ADM}} \geq 0$. Moreover, if equality holds, then the initial data must be isometric to initial data for Minkowski spacetime.
(4) For strongly asymptotically flat initial data, it holds that (see [19]) $\mathbf{E}_{\mathrm{ADM}} = M$ and $\mathbf{P}_{\mathrm{ADM}} = 0$, where M is the real number appearing in (6.8).

It is well-known (see [14,20,30–32,40]) that the asymptotic invariants $\mathbf{E}_{\mathrm{ADM}}$ and $\mathbf{C}_{\mathrm{ADM}}$ can be calculated in terms of the Ricci tensor as follows.

Theorem 6.3 (Alternative expressions for $\mathbf{E}_{\mathrm{ADM}}$ and $\mathbf{C}_{\mathrm{ADM}}$). *Let (Σ, g, k) be asymptotically flat spacelike initial data such that $\mathbf{E}_{\mathrm{ADM}} > 0$. Then it holds that for $i = 1, 2, 3$,*

$$\mathbf{E}_{\mathrm{ADM}} = \lim_{r\to\infty} -\frac{1}{8\pi} \int_{S_r} \left(\mathrm{Ric} - \frac{1}{2} R_{\mathrm{scal}}\, g\right)(X, N) d\mu_{\not g},$$

$$(\mathbf{C}_{\mathrm{ADM}})^i = \lim_{r\to\infty} \frac{1}{16\pi} \int_{S_r} \left(\mathrm{Ric} - \frac{1}{2} R_{\mathrm{scal}}\, g\right)(Z^{(i)}, N) d\mu_{\not g},$$

where X and $Z^{(i)}$, $i = 1, 2, 3$, are defined with respect to Cartesian coordinates (x^1, x^2, x^3) by

$$X := x^i \partial_i, \;\; Z^{(i)} := \left(|x|^2 \delta^{ij} - 2x^i x^j\right) \partial_j. \tag{6.9}$$

The vectorfields X and $Z^{(i)}$, $i = 1, 2, 3$, are conformal Killing vectorfields of Euclidean space. An explicit calculation shows that $Z^{(i)}$, $i = 1, 2, 3$, can be expressed in terms of spherical harmonics as follows, with $(m_1, m_2, m_3) := (1, -1, 0)$,

$$Z^{(i)} = -|x|^3 \sqrt{\frac{8\pi}{3}} E^{(1m_i)} - |x|^2 \left(\sqrt{\frac{4\pi}{3}} Y^{(1m_i)}\right) \partial_r. \tag{6.10}$$

Based on Definition 6.2 and Theorem 6.3 we introduce the following *local integrals*. Their relations to the charges $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$ of Definition 2.5 is studied in Sects. 7.3, 7.4, 7.5 and 7.6.

Definition 6.4 (*Local integrals*). Let (Σ, g, k) be asymptotically flat spacelike initial data such that $\mathbf{E}_{\mathrm{ADM}} > 0$, and let (x^1, x^2, x^3) be corresponding Cartesian coordinates near spacelike infinity. For real numbers $r \geq 1$ sufficiently large and $i = 1, 2, 3$, define

$$
\begin{aligned}
\mathbf{E}^{\mathrm{loc}}_{\mathrm{ADM}}(S_r, g, k) &:= -\frac{1}{8\pi}\int_{S_r}\left(\mathrm{Ric} - \frac{1}{2}R_{\mathrm{scal}}\, g\right)(X, N)d\mu_{\not g},\\
\left(\mathbf{P}^{\mathrm{loc}}_{\mathrm{ADM}}\right)^i(S_r, g, k) &:= \frac{1}{8\pi}\int_{S_r}(k_{il} - \mathrm{tr}k\, g_{il})\, N^l d\mu_{\not g},\\
\left(\mathbf{L}^{\mathrm{loc}}_{\mathrm{ADM}}\right)^i(S_r, g, k) &:= \frac{1}{8\pi}\int_{S_r}(k_{jl} - \mathrm{tr}k\, g_{jl})\left(Y_{(i)}\right)^j N^l d\mu_{\not g},\\
\left(\mathbf{C}^{\mathrm{loc}}_{\mathrm{ADM}}\right)^i(S_r, g, k) &:= \frac{1}{16\pi}\int_{S_r}\left(\mathrm{Ric} - \frac{1}{2}R_{\mathrm{scal}}\, g\right)(Z^{(i)}, N)d\mu_{\not g}.
\end{aligned}
\tag{6.11}
$$

Remark 6.5. The local integrals $\mathbf{E}^{\mathrm{loc}}_{\mathrm{ADM}}$ and $\mathbf{C}^{\mathrm{loc}}_{\mathrm{ADM}}$ are defined following Theorem 6.3. This has the advantage that $\mathbf{E}^{\mathrm{loc}}_{\mathrm{ADM}}$ and $\mathbf{C}^{\mathrm{loc}}_{\mathrm{ADM}}$ are more natural to relate to the charges $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$, see Sects. 7.3 and 7.6.

The following classical lemma analyses the convergence rates of $\mathbf{E}^{\mathrm{loc}}_{\mathrm{ADM}}$ and $\mathbf{P}^{\mathrm{loc}}_{\mathrm{ADM}}$ for strongly asymptotically flat spacelike initial data. It is applied in Sect. 7.7. Its proof is based on Stokes' theorem and the spacelike constraint equations, and is omitted here.

Lemma 6.6 (Convergence rates for $\mathbf{E}^{\mathrm{loc}}_{\mathrm{ADM}}$ and $\mathbf{P}^{\mathrm{loc}}_{\mathrm{ADM}}$ for strongly asymptotically flat initial data). *Let (Σ, g, k) be strongly asymptotically flat initial data. Then it holds that*

$$
\mathbf{E}^{\mathrm{loc}}_{\mathrm{ADM}}(S_r, g, k) = \mathbf{E}_{\mathrm{ADM}} + \mathcal{O}(r^{-1}),\ \ \mathbf{P}^{\mathrm{loc}}_{\mathrm{ADM}}(S_r, g, k) = \mathcal{O}(r^{-3/2}).
$$

6.3. Foliation geometry in spacelike initial data. In this section we set up notation for the geometry of foliations of spacelike initial data by 2-spheres. Let (Σ, g, k) be strongly asymptotically flat spacelike initial data, and let (x^1, x^2, x^3) be corresponding Cartesian coordinates near spacelike infinity. Denote by (r, θ^1, θ^2) the associated spherical coordinates, see (2.1). We have the following notation.

- Let S_r denote the level sets of r, and let $\not g$ and $\not\nabla$ denote the induced metric and covariant derivative. Let K denote the Gauss curvature of $\not g$.
- Let N denote the outward pointing unit normal to S_r. The second fundamental form Θ of S_r is defined by $\Theta_{AB} := \mathbf{D}_A N_B$, and composes into trace and tracefree part as follows,

$$
\mathrm{tr}\Theta := \not g^{AB}\Theta_{AB},\ \ \widehat{\Theta}_{AB} := \Theta_{AB} - \frac{1}{2}\mathrm{tr}\Theta \not g_{AB}.
$$

- Let $(e_A)_{A=1,2}$ denote a local orthonormal frame on S_r. We decompose the symmetric 2-tensor k into the S_r-tangent tensors

$$
k_{NN},\ \ \not k_{NA} := k_{NA},\ \ \not k_{AB} := k_{AB}. \tag{6.12}
$$

The Gauss–Codazzi equations of $S_r \subset \Sigma$ are (see Section 3.1 in [19])

$$
\mathrm{Ric}_{AN} = \not{\mathrm{div}}\widehat{\Theta}_A - \frac{1}{2}\not d\, \mathrm{tr}\theta_A,\ \ \mathrm{Ric}_{NN} - \frac{1}{2}R_{\mathrm{scal}} = -K + \frac{1}{4}(\mathrm{tr}\Theta)^2 + \frac{1}{2}|\widehat{\Theta}|^2, \tag{6.13}
$$

where $\not d$ denotes the exterior derivative on S_r, and for a symmetric 2-tenors V on S_r,

$$
(\not{\mathrm{div}} V)_A := \not\nabla^B V_{BA},\ \ |V|^2 := \not g^{AB}\not g^{CD} V_{AC} V_{BD}.
$$

6.4. Scaling of spacelike initial data and local norms. In this section we define, analogous to Sect. 2.8, the scaling of spacelike initial data, and introduce local norms.

Let (Σ, g, k) be an asymptotically flat spacelike initial data set and let (x^1, x^2, x^3) denote associated coordinates near spacelike infinity. We define the scaling of (g, k) in two steps.

(1) For a real number $R \geq 1$, define new coordinates (y^1, y^2, y^3) by

$$\Psi_R(y^1, y^2, y^3) := (R \cdot y^1, R \cdot y^2, R \cdot y^3) = (x^1, x^2, x^3). \tag{6.14}$$

(2) Based on the conformal scaling of spacetime metrics ${}^{(R)}\mathbf{g} := R^{-2}\mathbf{g}$ (see Sect. 2.8) and that k is the second fundamental form of Σ, we define $({}^{(R)}g, {}^{(R)}k)$ by

$$ {}^{(R)}g := R^{-2} g, \quad {}^{(R)}k := R^{-1} k. \tag{6.15}$$

By construction, $({}^{(R)}g, {}^{(R)}k)$ solve the spacelike constraint equations (6.4).

By (6.14) and (6.15), for all integers $l \geq 0$, we have the relations

$$\partial_y^l \left({}^{(R)}g_{ij} \right) = R^l \left(\partial_x^l g_{ij} \right) \circ \Psi_R, \ \ \partial_y^l \left({}^{(R)}k_{ij} \right) = R^{l+1} \left(\partial_x^l k_{ij} \right) \circ \Psi_R, \tag{6.16}$$

where we denote

$$ {}^{(R)}g_{ij} := {}^{(R)}g(\partial_{y^i}, \partial_{y^j}), \ \ {}^{(R)}k_{ij} := {}^{(R)}k(\partial_{y^i}, \partial_{y^j}).$$

Remarks on the scaling of spacelike initial data.

(1) Analogous to Lemma 2.18, we deduce from (6.16) that the charges scale as follows. The proof is omitted.

$$\begin{aligned}
\mathbf{E}^{\mathrm{loc}}_{\mathrm{ADM}} \left(S_{r_0}, {}^{(R)}g, {}^{(R)}k \right) &= R^{-1} \mathbf{E}^{\mathrm{loc}}_{\mathrm{ADM}} \left(S_{R \cdot r_0}, g, k \right), \\
\mathbf{P}^{\mathrm{loc}}_{\mathrm{ADM}} \left(S_{r_0}, {}^{(R)}g, {}^{(R)}k \right) &= R^{-1} \mathbf{P}^{\mathrm{loc}}_{\mathrm{ADM}} \left(S_{R \cdot r_0}, g, k \right), \\
\mathbf{L}^{\mathrm{loc}}_{\mathrm{ADM}} \left(S_{r_0}, {}^{(R)}g, {}^{(R)}k \right) &= R^{-2} \mathbf{L}^{\mathrm{loc}}_{\mathrm{ADM}} \left(S_{R \cdot r_0}, g, k \right), \\
\mathbf{C}^{\mathrm{loc}}_{\mathrm{ADM}} \left(S_{r_0}, {}^{(R)}g, {}^{(R)}k \right) &= R^{-2} \mathbf{C}^{\mathrm{loc}}_{\mathrm{ADM}} \left(S_{R \cdot r_0}, g, k \right).
\end{aligned}$$

(2) Applying the scaling to Schwarzschild reference spacelike initial data, see (6.5), we have

$$ {}^{(R)}g^M_{ij} = g^{M/R}_{ij}, \ \ {}^{(R)}\tilde{g}^M_{ij} = \tilde{g}^{M/R}_{ij} \tag{6.17}$$

(3) By (6.16), the property of strong asymptotic flatness is conserved under rescaling.

We now turn to the introduction of local norms for spacelike initial data. For ease of presentation we use C^k-spaces.

Definition 6.7 (*Norms for tenors*). Let $K \subset \mathbb{R}^3$ denote a compact set with smooth boundary, and let T be an j-tensor on K. For integers $k \geq 0$ define

$$\|T\|_{C^k(K)} := \sum_{1 \leq i_1, \cdots i_j \leq 3} \sum_{0 \leq |\alpha| \leq k} \|\partial^\alpha T_{i_1 \cdots i_j}\|_{L^\infty(K)},$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$, $\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$ and $T_{i_1\cdots i_l}$ denotes the Cartesian coordinate components of T. Define $C^k(K)$ to be the space of k-times continuously differentiable tensors T on K with $\|T\|_{C^k(K)} < \infty$. Moreover, let $C^k_{\mathrm{loc}}(\mathbb{R}^3 \setminus B(0,1))$ be the space of k-times continuously differentiable tensors T on $\mathbb{R}^3 \setminus B(0,1)$ such that $\|T\|_{C^k(K)} < \infty$ for each compact subset $K \subset \mathbb{R}^3 \setminus B(0,1)$.

Definition 6.8 (*Local norm for spacelike initial data*). Let $0 < r_1 < r_2$ be two real numbers, and let $k \geq 1$ be an integer. We define for spacelike initial data (g, k) on the annulus $A_{[r_1,r_2]} := \left\{x \in \mathbb{R}^3 : r_1 \leq |x| \leq r_2\right\}$ the norm

$$\|(g,k)\|_{C^k(A_{[r_1,r_2]})\times C^{k-1}(A_{[r_1,r_2]})} := \|g\|_{C^k(A_{[r_1,r_2]})} + \|k\|_{C^{k-1}(A_{[r_1,r_2]})}.$$

Notation. In the following we assume that the metric g is k_0-times and the second fundamental form k is k_0-times continuously differentiable, where the universal integer $k_0 \geq 8$ is determined in Sect. 7 by the condition that the ingoing null data to be constructed from spacelike initial data is sufficiently regular.
By scaling and the definition of strong asymptotic flatness, we have the following estimates for rescaled spacelike initial data. Its straight-forward proof is omitted.

Lemma 6.9 (Smallness of rescaled spacelike initial data). *Let (Σ, g, k) be strongly asymptotically flat spacelike initial data with Cartesian coordinates (x^1, x^2, x^3) near spacelike infinity. For real numbers $R \geq 1$ sufficiently large, the rescaled spacelike initial data $({}^{(R)}g_{ij}, {}^{(R)}k_{ij})$ is well-defined on $A_{[1/2,7/2]}$ and*

$$\| \left({}^{(R)}g - \tilde{g}^{M/R}, {}^{(R)}k\right) \|_{C^{k_0}(A_{[1/2,7/2]})\times C^{k_0-1}(A_{[1/2,7/2]})} = \mathcal{O}(R^{-3/2}), \qquad (6.18)$$

where M is the real number appearing in (6.8).

Moreover, we note the following lemma. Its proof follows from (6.5) and is omitted.

Lemma 6.10 *(Estimates for Schwarzschild reference metric). For real numbers $M \geq 0$ sufficiently small,*

$$\|g^M - e\|_{C^{k_0}(A_{[1/2,7/2]})} \lesssim M.$$

7. Construction of Sphere Data from Spacelike Initial Data

In this section we construct families of ingoing null data from spacelike initial data. The following theorem is the main result of this section.

Theorem 7.1 (Construction of ingoing null data from spacelike initial data). *Let (Σ, g, k) be strongly asymptotically flat spacelike initial data with asymptotic invariants*

$$(\mathbf{E}_{\mathrm{ADM}}, \mathbf{P}_{\mathrm{ADM}}, \mathbf{L}_{\mathrm{ADM}}, \mathbf{C}_{\mathrm{ADM}}),$$

where $\mathbf{P}_{\mathrm{ADM}} = 0$ by the strong asymptotic flatness. There is a real number $\delta > 0$ and a strongly asymptotically flat family of ingoing data $(x_{-R+R\cdot[-\delta,\delta],R})$, constructed on spheres in Σ, such that for $m = -1, 0, 1$ and $(i_{-1}, i_0, i_1) = (2, 3, 1)$,

$$\mathbf{E}(x_{-R,R}) = \mathbf{E}_{\mathrm{ADM}} + \mathcal{O}(R^{-1}), \qquad \mathbf{P}^m(x_{-R,R}) = (\mathbf{P}_{\mathrm{ADM}})^{i_m} + \mathcal{O}(R^{-3/2}),$$
$$\mathbf{L}^m(x_{-R,R}) = (\mathbf{L}_{\mathrm{ADM}})^{i_m} + \mathcal{O}(1), \qquad \mathbf{G}^m(x_{-R,R}) = (\mathbf{C}_{\mathrm{ADM}})^{i_m} + \mathcal{O}(1).$$

Moreover, if the spacelike initial data is smooth, then the constructed ingoing null data is smooth, along with all higher-order derivatives in all directions.

In the particular case of Schwarzschild reference spacelike initial data in isotropic coordinates, $\tilde{g}_{ij}^{\mathbf{E}_{\text{ADM}}}$, the construction of Theorem 7.1 produces the Schwarzschild reference family of sphere data $\mathfrak{m}_{-R,R}^{\mathbf{E}_{\text{ADM}}}$ in Eddington–Finkelstein coordinates, see (2.16). The proof of Theorem 7.1 is structured as follows.

- In Sect. 7.1 we rescale the strongly asymptotically flat spacelike initial data on the annulus $A_{[R/2,7R/2]}$ to spacelike initial data on $A_{[1/2,7/2]}$ and change from isotropic to Schwarzschild coordinates, to arrive at spacelike initial data on $A_{[1,3]}$ close to Schwarzschild (in Schwarzschild coordinates) of mass M/R.
- In Sect. 7.2 we construct from the spacelike initial data on $A_{[1,3]}$ ingoing null data $({}^{(R)}x_{-1+[-\delta,\delta],1})$, and prove estimates.
- In Sects. 7.3, 7.4, 7.5 and 7.6 we compare the charges $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$ of ${}^{(R)}x_{-1,1}$ with the local integrals $(\mathbf{E}^{\text{loc}}_{\text{ADM}}, \mathbf{P}^{\text{loc}}_{\text{ADM}}, \mathbf{L}^{\text{loc}}_{\text{ADM}}, \mathbf{C}^{\text{loc}}_{\text{ADM}})$ on $S_{-1,1} \subset A_{[1,3]}$ of the spacelike initial data.
- In Sect. 7.7 we conclude the proof of Theorem 7.1 by scaling the constructed ingoing null data $({}^{(R)}x_{-1+[-\delta,\delta],1})$ up to $(x_{-R+R\cdot[-\delta,\delta],R})$, and analyzing the asymptotics of $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})(x_{-R,R})$ by use of the estimates of Sects. 7.3, 7.4, 7.5 and 7.6.

7.1. Rescaling and change to Schwarzschild coordinates. Let (Σ, g, k) be strongly asymptotically flat spacelike initial data, and let (x^1, x^2, x^3) denote corresponding coordinates near spacelike infinity. In the following we first rescale to small data on an annulus $A_{[1/2,7/2]}$ and then change from isotropic coordinates to Schwarzschild coordinates, see (6.6), yielding spacelike initial data on the annulus $A_{[1,3]}$.

In the particular case of Schwarzschild reference spacelike data in isotropic coordinates of mass M, denoted by $\tilde{g}_{ij}^{M}$, the following construction maps to Schwarzschild reference spacelike initial data in Schwarzschild coordinates of mass M/R, denoted by g_{ij}^{M}/R.

First, let $({}^{(R)}g, {}^{(R)}k)$ denote the rescaled spacelike initial data. By Lemma 6.9 we have that

$$\left\| \left({}^{(R)}g - \tilde{g}^{M/R}, {}^{(R)}k \right) \right\|_{C^{k_0}(A_{[1/2,7/2]}) \times C^{k_0-1}(A_{[1/2,7/2]})} = \mathcal{O}(R^{-3/2}), \tag{7.1}$$

Second, we apply the coordinate change Φ from isotropic coordinates $(\tilde{r}, \tilde{\theta}^1, \tilde{\theta}^2)$ to Schwarzschild coordinates (r, θ^1, θ^2), see (6.6), with M/R,

$$\Phi : (\tilde{r}, \tilde{\theta}^1, \tilde{\theta}^2) \to (r, \theta^1, \theta^2) := \left(\tilde{r} \left(1 + \frac{M/R}{2\tilde{r}} \right)^2, \tilde{\theta}^1, \tilde{\theta}^2 \right),$$

On the one hand, for $R \geq 1$ sufficiently large, the Schwarzschild coordinates (r, θ^1, θ^2) range over the coordinate domain $A_{[1,3]}$. On the other hand, by (6.6) we can estimate for $R \geq 1$ sufficiently large

$$\|\mathrm{D}\Phi - \mathrm{Id}\|_{C^{k_0}(A_{[1/2,7/2]})} \leq C, \tag{7.2}$$

where $C > 0$ is a universal constant. Thus by (7.1) and (7.2),

$$\begin{aligned} \left\| \Phi^* \left({}^{(R)}g \right) - g^{M/R} \right\|_{C^{k_0}(A_{[1,3]})} &= \left\| \Phi^* \left({}^{(R)}g - \tilde{g}^{M/R} \right) \right\|_{C^{k_0}(A_{[1,3]})} \\ &\lesssim \left\| {}^{(R)}g - \tilde{g}^{M/R} \right\|_{C^{k_0}(A_{[1/2,7/2]})} = \mathcal{O}(R^{-3/2}), \end{aligned}$$

where we used that by (6.7), $\Phi^*\left(\tilde{g}^{M/R}\right) = g^{M/R}$. Furthermore, by (7.1) and (7.2) it similarly follows that

$$\left\| \Phi^*\left({}^{(R)}k\right) \right\|_{C^{k_0-1}(A_{[1,3]})} = \mathcal{O}(R^{-3/2}).$$

To summarize the above, for $R \geq 1$ sufficiently large, we constructed from strongly asymptotically flat spacelike initial data (Σ, g, k) the spacelike initial data

$$\left(\Phi^*\left({}^{(R)}g\right), \Phi^*\left({}^{(R)}k\right)\right) \text{ on } A_{[1,3]},$$

satisfying

$$\left\| \Phi^*\left({}^{(R)}g\right) - g^{M/R} \right\|_{C^{k_0}(A_{[1,3]})} + \left\| \Phi^*\left({}^{(R)}k\right) \right\|_{C^{k_0}(A_{[1,3]})} = \mathcal{O}(R^{-3/2}). \tag{7.3}$$

Notation. We use the following notation in Sects. 7.2, 7.3, 7.4, 7.5 and 7.6.

(1) We denote the R-dependent smallness on the right-hand side of (7.3) by

$$\varepsilon_R := \mathcal{O}(R^{-3/2}). \tag{7.4}$$

(2) For ease of presentation we abuse notation by denoting $\left(\Phi^*\left({}^{(R)}g\right), \Phi^*\left({}^{(R)}k\right)\right)$ by (g, k).

7.2. Construction of sphere data. Let $M \geq 0$ and $R \geq 1$ be two real numbers. Consider spacelike initial data (g, k) on $A_{[1,3]}$ such that

$$\left\| \left(g - g^{M/R}, k\right) \right\|_{C^{k_0}(A_{[1,3]}) \times C^{k_0-1}(A_{[1,3]})} \leq \varepsilon_R, \tag{7.5}$$

see (7.4) for the ε_R-notation. In this section we construct from (g, k) the ingoing null data $({}^{(R)}x_{-1+[-\delta,\delta],1})$ and prove that for $R \geq 1$ sufficiently large,

$$\|{}^{(R)}x_{-1+[-\delta,\delta],1} - \mathfrak{m}^{M/R}\|_{\mathcal{X}^+(\underline{\mathcal{H}}_{-1+[-\delta,\delta],1})} \lesssim \varepsilon_R. \tag{7.6}$$

In the particular case Schwarzschild reference data in Schwarzschild coordinates, $g^{M/R}$, the construction of this section produces the Schwarzschild reference ingoing null data in Eddington–Finkelstein coordinates $\mathfrak{m}^{M/R}_{-1+[-\delta,\delta],1}$. We remark that the universal integer $k_0 \geq 6$ is determined from the regularity in (7.6), see the notational remark after Definition 6.8.

Definition of $S_{-1,1}$ and gauge choices. Let $(\mathcal{M}, \mathbf{g})$ denote the unique maximal future globally-hyperbolic development of the spacelike initial data $(A_{[1,3]}, g, k)$. Let T denote the future-directed timelike unit vector to $A_{[1,3]}$ in $(\mathcal{M}, \mathbf{g})$, and let N denote the outward pointing unit normal to $S_r \subset \Sigma$ tangent to $A_{[1,3]}$ for $1 \leq r \leq 3$. For $R \geq 1$ sufficiently large, consider the sphere $S_{r_{M/R}(-1,1)} \subset A_{[1,3]}$ where the definition of $r_M(u, v)$ is given in (2.15). On $S_{r_{M/R}(-1,1)} \subset \Sigma$ define the renormalized null vectors $(\widehat{L}, \widehat{\underline{L}})$ by

$$\widehat{L} = T + N, \quad \widehat{\underline{L}} = T - N, \tag{7.7}$$

which satisfy by construction $\mathbf{g}(\widehat{L}, \widehat{\underline{L}}) = -2$. We can construct around $S_{r_{M/R}(-1,1)} \subset \Sigma$ a local double null coordinate system $(u, v, \theta^1, \theta^2)$ such that with respect to (u, v) we have $S_{-1,1} = S_{r_{M/R}(-1,1)}$, and moreover, the following holds on $S_{-1,1}$ (which is in agreement with (2.16))

$$\begin{aligned}\Omega^2 :=& 1 - \frac{2M/R}{r_{M/R}(-1,1)}, \qquad \omega := \frac{M/R}{\left(r_{M/R}(-1,1)\right)^2}, \qquad \underline{\omega} := -\frac{M/R}{\left(r_{M/R}(-1,1)\right)^2},\\ D\omega :=& -\frac{2\Omega^2 M/R}{\left(r_{M/R}(-1,1)\right)^3}, \quad \underline{D\omega} := -\frac{2\Omega^2 M/R}{\left(r_{M/R}(-1,1)\right)^3}.\end{aligned} \tag{7.8}$$

Definition and analysis of χ and $\underline{\chi}$. Defining χ and $\underline{\chi}$ as in Definition 2.5, we have by (6.12) and (7.7),

$$\chi_{AB} = -\not{k}_{AB} + \Theta_{AB}, \quad \underline{\chi}_{AB} = -\not{k}_{AB} - \Theta_{AB}. \tag{7.9}$$

Taking the tracefree part and trace with respect to $\not{g}$, we get

$$\mathrm{tr}\chi = -\mathrm{tr}\not{k} + \mathrm{tr}\Theta, \; \widehat{\chi}_{AB} = -\widehat{\not{k}}_{AB} + \widehat{\Theta}_{AB}, \; \mathrm{tr}\underline{\chi} = -\mathrm{tr}\not{k} - \mathrm{tr}\Theta, \; \widehat{\underline{\chi}}_{AB} = -\widehat{\not{k}}_{AB} - \widehat{\Theta}_{AB}. \tag{7.10}$$

By (2.16), (7.5) and (7.10) we have that for $R \geq 1$ sufficiently large,

$$\begin{aligned}\|\widehat{\chi}\|_{H^6(S_{-1,1})} + \|\widehat{\underline{\chi}}\|_{H^6(S_{-1,1})} \lesssim& \varepsilon_R,\\ \left\|\mathrm{tr}\chi - \frac{2\Omega_M}{r_{M/R}(-1,1)}\right\|_{H^6(S_{-1,1})} + \left\|\mathrm{tr}\underline{\chi} + \frac{2\Omega_M}{r_{M/R}(-1,1)}\right\|_{H^6(S_{-1,1})} \lesssim& \varepsilon_R.\end{aligned} \tag{7.11}$$

Definition and analysis of ζ and η. Defining ζ and η on $S_{-1,1}$ as in Definition 2.5, we have by (7.7) that

$$\begin{aligned}\zeta_A :=& \frac{1}{2}\mathbf{g}\left(\mathbf{D}_A\widehat{L}, \widehat{\underline{L}}\right) = -\frac{1}{2}\mathbf{g}\left(\mathbf{D}_A T, N\right) + \frac{1}{2}\mathbf{g}\left(\mathbf{D}_A N, T\right) = -\mathbf{g}\left(\mathbf{D}_A T, N\right) = \not{k}_A,\\ \eta_A :=& \zeta_A + \not{d}\log\Omega = \not{k}_A + \not{d}\log\Omega = \not{k}_A,\end{aligned} \tag{7.12}$$

where we used (7.7) and (7.8). Subsequently, by (7.5) and (7.12) we have that for $R \geq 1$ sufficiently large,

$$\|\zeta\|_{H^5(S_{-1,1})} + \|\eta\|_{H^5(S_{-1,1})} \lesssim \varepsilon_R. \tag{7.13}$$

Definition and analysis of α and $\underline{\alpha}$. By Definition 2.6, (6.1), (6.2), (6.3) and (7.7), we have that

$$\begin{aligned}\alpha_{AB} :=& \mathbf{R}\left(e_A, \widehat{L}, e_B, \widehat{L}\right)\\ =& \mathbf{R}_{ATBT} + \mathbf{R}_{ATBN} + \mathbf{R}_{ANBT} + \mathbf{R}_{ANBN}\\ =& E_{AB} - \in_{AN}{}^{s} H_{sB} - \in_{BN}{}^{s} H_{sA} + \in_{ANs}\in_{BNl} E^{sl}.\end{aligned} \tag{7.14}$$

Subsequently, by (7.5) and (7.14) we have that for $R \geq 1$ sufficiently large,

$$\|\alpha\|_{H^6(S_{-1,1})} \lesssim \varepsilon_R. \tag{7.15}$$

Similarly, by Definition 2.6, (6.1), (6.2), (6.3), (7.7) and (7.14) we get that for $R \geq 1$ sufficiently large,

$$\|\underline{\alpha}\|_{H^2(S_{-1,1})} \lesssim \varepsilon_R. \tag{7.16}$$

Definition and analysis of $^{(R)}x_{-1,1}$ on $S_{-1,1}$. Let $^{(R)}x_{-1,1}$ be the sphere data on $S_{-1,1}$ determined by the quantities constructed in (7.8), (7.9), (7.10), (7.12) and (7.14). From the estimates (7.11), (7.13), (7.15) and (7.16), it follows that

$$\|^{(R)}x_{-1,1} - \mathfrak{m}^{M/R}\|_{\mathcal{X}(S_{-1,1})} \lesssim \varepsilon_R. \tag{7.17}$$

Definition and analysis of $^{(R)}x_{-1+[-\delta,\delta],1}$ on $\underline{\mathcal{H}}_{-1+[-\delta,\delta],1}$. Following (7.7) and (7.8), define $\underline{L}'$ on $S_{-1,1}$ by

$$\underline{L}' := \Omega_M^{-1}\widehat{\underline{L}} = \frac{1}{\sqrt{1-\frac{2M/R}{r_{M/R}(-1,1)}}}(T-N),$$

and extend $\underline{L}'$ to the spacetime $(\mathcal{M}, \mathbf{g})$ as null geodesic vectorfield. The ingoing null hypersurface $\underline{\mathcal{H}}_1 \subset \mathcal{M}$ passing through $S_{r_{M/R}(-1,1)} \subset \Sigma$ is ruled by $\underline{L}'$. We define on $\underline{\mathcal{H}}_1$ the function u by

$$\underline{L}'(u) = \frac{1}{\sqrt{1-\frac{2M/R}{r_{M/R}(u,1)}}} \text{ on } \underline{\mathcal{H}}_1 \text{ and } u|_{S_{-1,1}} = -1.$$

The level sets $S_{u,1} \subset \underline{\mathcal{H}}_1$ of u are locally well-defined and foliate $\underline{\mathcal{H}}_1$ by construction with Schwarzschild reference null lapse.

By the smallness (7.5) together with the above gauge choices (7.7) and (7.8), by the local existence and Cauchy stability for the spacelike Cauchy problem, see [15], it follows that for $R \geq 1$ and $k_0 \geq 6$ sufficiently large, there is a universal real number $\delta > 0$, such that the foliated null hypersurface $\underline{\mathcal{H}}_{-1+[-\delta,\delta],1} := \bigcup_{-\delta\leq u\leq\delta} S_{u,1}$ is well-defined in $(\mathcal{M}, \mathbf{g})$ and the induced null data, denoted by $^{(R)}x_{-1+[-\delta,\delta],1}$, satisfies

$$\|^{(R)}x_{-1+[-\delta,\delta],1} - \mathfrak{m}^{M/R}\|_{\mathcal{X}^+(\underline{\mathcal{H}}_{-1+[-\delta,\delta],1})} \lesssim \varepsilon_R. \tag{7.18}$$

To summarise the above, we constructed ingoing null data $(^{(R)}x_{-1+[-\delta,\delta],1})$ satisfying

$$\|^{(R)}x_{-1+[-\delta,\delta],1} - \mathfrak{m}^{M/R}\|_{\mathcal{X}^+(\underline{\mathcal{H}}_{-1+[-\delta,\delta],1})} \lesssim \varepsilon_R. \tag{7.19}$$

This finishes the proof of (7.6).

Remark 7.2 In case of higher regularity, we impose gauge conditions on $D^m\omega$ and $\underline{D}^m\underline{\omega}$, for integers $m \geq 2$ on $S_{-1,1}$ in accordance with the Schwarzschild reference higher-order sphere data (2.16). Subsequently, the higher-order sphere data on $S_{-1,1}$ can be explicitly calculated and estimated by the Bianchi identities.

7.3. Comparison of $\mathbf{E}$ *and* $\mathbf{E}^{\mathrm{loc}}_{\mathrm{ADM}}$. In this section we prove that

$$\mathbf{E}({}^{(R)}x_{-1,1}) = \mathbf{E}^{\mathrm{loc}}_{\mathrm{ADM}}(S_{-1,1}, g, k) + \mathcal{O}\left(\frac{M}{R}\varepsilon_R\right) + \mathcal{O}(\varepsilon_R^2). \tag{7.20}$$

In the following we rewrite $\mathbf{E}$ into $\mathbf{E}^{\mathrm{loc}}_{\mathrm{ADM}}$, where we eased notation. Using the null structure equations (2.10) and (2.11), and the relations (7.10) and (7.12), we can write

$$\begin{aligned}\rho + r\,\mathrm{d}\!\!/\mathrm{iv}\beta = &-\left(K - \frac{1}{4}(\mathrm{tr}\Theta)^2 + \frac{1}{2}|\widehat{\Theta}|^2 + \frac{1}{4}(\mathrm{tr}k\!\!\!/)^2 + \frac{1}{2}|\widehat{k\!\!\!/}|^2\right)\\ &- r\,\mathrm{d}\!\!/\mathrm{iv}\left(\mathrm{d}\!\!/\mathrm{iv}\left(-\widehat{k\!\!\!/} + \widehat{\Theta}\right) - \frac{1}{2}d\!\!\!/\left(-\mathrm{tr}k\!\!\!/ + \mathrm{tr}\Theta\right)\right)\\ &- r\,\mathrm{d}\!\!/\mathrm{iv}\left(\left(-\widehat{k\!\!\!/} + \widehat{\Theta}\right)\cdot k\!\!\!/ - \frac{1}{2}\left(-\mathrm{tr}k\!\!\!/ + \mathrm{tr}\Theta\right)k\!\!\!/\right).\end{aligned} \tag{7.21}$$

Plugging the Gauss equation (6.13) into the right-hand side of (7.21) leads to

$$\begin{aligned}\rho + r\,\mathrm{d}\!\!/\mathrm{iv}\beta = &-\frac{1}{2}\left(R_{\mathrm{scal}} - 2\mathrm{Ric}_{NN} + \frac{1}{2}(\mathrm{tr}k\!\!\!/)^2 + |\widehat{k\!\!\!/}|^2 + |\widehat{\Theta}|^2\right)\\ &- r\,\mathrm{d}\!\!/\mathrm{iv}\left(\mathrm{d}\!\!/\mathrm{iv}\left(-\widehat{k\!\!\!/} + \widehat{\Theta}\right) - \frac{1}{2}d\!\!\!/\left(-\mathrm{tr}k\!\!\!/ + \mathrm{tr}\Theta\right)\right)\\ &- r\,\mathrm{d}\!\!/\mathrm{iv}\left(\left(-\widehat{k\!\!\!/} + \widehat{\Theta}\right)\cdot k\!\!\!/ - \frac{1}{2}\left(-\mathrm{tr}k\!\!\!/ + \mathrm{tr}\Theta\right)k\!\!\!/\right).\end{aligned}$$

Hence we get that

$$\begin{aligned}-\frac{8\pi}{\sqrt{4\pi}}\mathbf{E} = &-\left(\frac{r^3}{2}\left(R_{\mathrm{scal}} - 2\mathrm{Ric}_{NN} + \frac{1}{2}(\mathrm{tr}k\!\!\!/)^2 + |\widehat{k\!\!\!/}|^2 + |\widehat{\Theta}|^2\right)\right)^{(0)}\\ &-\left(r^4\,\mathrm{d}\!\!/\mathrm{iv}\left(\mathrm{d}\!\!/\mathrm{iv}\left(-\widehat{k\!\!\!/} + \widehat{\Theta}\right) - \frac{1}{2}d\!\!\!/\left(-\mathrm{tr}k\!\!\!/ + \mathrm{tr}\Theta\right)\right)\right)^{(0)}\\ &-\left(r^4\,\mathrm{d}\!\!/\mathrm{iv}\left(\left(-\widehat{k\!\!\!/} + \widehat{\Theta}\right)\cdot k\!\!\!/ - \frac{1}{2}\left(-\mathrm{tr}k\!\!\!/ + \mathrm{tr}\Theta\right)k\!\!\!/\right)\right)^{(0)},\end{aligned}$$

which we can estimate by (7.5) and Lemma 2.1 for $R \geq 1$ sufficiently large and (6.11) as

$$\begin{aligned}\mathbf{E} &= \frac{\sqrt{4\pi}}{8\pi}\left(\frac{r^3}{2}\left(R_{\mathrm{scal}} - 2\mathrm{Ric}_{NN}\right)\right)^{(0)} + \mathcal{O}\left(\frac{M}{R}\varepsilon_R\right) + \mathcal{O}(\varepsilon_R^2)\\ &= -\frac{1}{8\pi}\int_{S_{-1,1}}\left(\mathrm{Ric} - \frac{1}{2}R_{\mathrm{scal}}\,g\right)(rN, N)d\mu_{g\!\!\!/} + \mathcal{O}\left(\frac{M}{R}\varepsilon_R\right) + \mathcal{O}(\varepsilon_R^2)\\ &= -\frac{1}{8\pi}\int_{S_{-1,1}}\left(\mathrm{Ric} - \frac{1}{2}R_{\mathrm{scal}}\,g\right)(x^j\partial_j, N)d\mu_{g\!\!\!/} + \mathcal{O}\left(\frac{M}{R}\varepsilon_R\right) + \mathcal{O}(\varepsilon_R^2)\\ &= \mathbf{E}^{\mathrm{loc}}_{\mathrm{ADM}} + \mathcal{O}\left(\frac{M}{R}\varepsilon_R\right) + \mathcal{O}(\varepsilon_R^2),\end{aligned}$$

where we used Lemma 6.10. This finishes the proof of (7.20).

7.4. Comparison of $\mathbf{P}$ *and* $\mathbf{P}^{\mathrm{loc}}_{\mathrm{ADM}}$.. Next we prove that for $i = 1, 2, 3$, $(m_1, m_2, m_3) = (1, -1, 0)$,

$$\left(\mathbf{P}^{\mathrm{loc}}_{\mathrm{ADM}}\right)^i (S_{-1,1}, g, k) = \mathbf{P}^{m_i}({}^{(R)}x_{-1,1}) + \mathcal{O}\left(\frac{M}{R}\varepsilon_R\right) + \mathcal{O}(\varepsilon_R^2). \tag{7.22}$$

In the following we rewrite $\left(\mathbf{P}^{\mathrm{loc}}_{\mathrm{ADM}}\right)^i$ into $\mathbf{P}^{m_i}$, where we eased notation. By (7.5), for $R \geq 1$ sufficiently large, it holds that on the annulus $A_{[1,3]}$, $(\partial_i)^j - (\nabla x^i)^j = e^{ij} - g^{ij} = \mathcal{O}\left(\frac{M}{R}\right) + \mathcal{O}(\varepsilon_R)$. Hence we can write, using that N is normal to $S_{-1,1}$,

$$\begin{aligned}
&\int_{S_{-1,1}} (k_{iN} - \mathrm{tr}k\, g_{iN})\, d\mu_{\not g} \\
&= \int_{S_{-1,1}} \left(k_{jN} - \mathrm{tr}k\, g_{jN}\right) (\nabla x^i)^j d\mu_{\not g} + \mathcal{O}\left(\frac{M}{R}\varepsilon_R\right) + O\left(\varepsilon_R^2\right) \\
&= \int_{S_{-1,1}} \left(k_{jN} - \mathrm{tr}k\, g_{jN}\right) \left(N(x^i)N^j + \left(\not\nabla x^i\right)^j\right) d\mu_{\not g} + \mathcal{O}\left(\frac{M}{R}\varepsilon_R\right) + O\left(\varepsilon_R^2\right) \\
&= \int_{S_{-1,1}} \left((k_{NN} - \mathrm{tr}k)\, N(x^i) + k(N, \not\nabla x^i)\right) d\mu_{\not g} + \mathcal{O}\left(\frac{M}{R}\varepsilon_R\right) + O\left(\varepsilon_R^2\right).
\end{aligned} \tag{7.23}$$

Using that by (7.7) and (7.12), for $R \geq 1$ sufficiently large we have on $S_{-1,1}$,

$$\begin{aligned}
k_{NN} - \mathrm{tr}k &= -\mathrm{tr}\not k = \frac{1}{2}\left(\mathrm{tr}\underline{\chi} + \mathrm{tr}\chi\right), \ \ k(N, \not\nabla x^i) = \zeta(x^i), \\
N(x^i) &= \frac{x^i}{r} + \mathcal{O}\left(\frac{M}{R}\right) + \mathcal{O}\left(\varepsilon_R\right),
\end{aligned}$$

where r denotes the area radius on $(S_{-1,1}, \not g)$, we get from (7.23) that

$$\begin{aligned}
&\int_{S_{-1,1}} (k_{iN} - \mathrm{tr}k\, g_{iN})\, d\mu_{\not g} \\
&= \int_{S_{-1,1}} \left(\frac{1}{2}\left(\mathrm{tr}\underline{\chi} + \mathrm{tr}\chi\right)\frac{x^i}{r} + \zeta(x^i)\right) d\mu_{\not g} + \mathcal{O}\left(\frac{M}{R}\varepsilon_R\right) + \mathcal{O}\left(\varepsilon_R^2\right) \\
&= \int_{S_{-1,1}} \left(\frac{1}{2}\left(\mathrm{tr}\underline{\chi} - \mathrm{tr}\chi\right) + \mathrm{tr}\chi - r\, \not{\mathrm{div}}\zeta\right)\left(\sqrt{\frac{4\pi}{3}}Y^{(1m_i)}\right) d\mu_{r^2\overset{\circ}{\gamma}} \\
&\quad + \mathcal{O}\left(\frac{M}{R}\varepsilon_R\right) + \mathcal{O}\left(\varepsilon_R^2\right),
\end{aligned} \tag{7.24}$$

where we used that for $i = 1, 2, 3$, $\frac{x^i}{|x|} = \sqrt{\frac{4\pi}{3}}Y^{(1m_i)}$ with $(m_1, m_2, m_3) = (1, -1, 0)$. In the following we use two identities to rewrite the right-hand side of (7.24). First, by (2.10),

$$\frac{1}{2}\left(\mathrm{tr}\underline{\chi}-\mathrm{tr}\chi\right)=r\left(-\rho-K+\frac{1}{2}(\widehat{\chi},\widehat{\underline{\chi}})\right)-\frac{1}{r}-\frac{r}{4}\left(\mathrm{tr}\chi-\frac{2}{r}\right)\left(\mathrm{tr}\underline{\chi}+\frac{2}{r}\right), \quad (7.25)$$

Next, we can express by (2.11)

$$\begin{aligned} r\,\mathrm{d\!/\!iv}\zeta &= r^2\,\mathrm{d\!/\!iv}\beta-\frac{1}{2}\overset{\circ}{\Delta\!\!\!/}\mathrm{tr}\chi+\frac{1}{2}\left(\overset{\circ}{\Delta\!\!\!/}-r^2\Delta\!\!\!/\right)\mathrm{tr}\chi \\ &\quad+r^2\left(\mathrm{d\!/\!iv}\,\mathrm{d\!/\!iv}\widehat{\chi}+\mathrm{d\!/\!iv}\,(\widehat{\chi}\cdot\zeta)\right)-\frac{r^2}{2}\,\mathrm{d\!/\!iv}\left(\left(\mathrm{tr}\chi-\frac{2}{r}\right)\zeta\right). \end{aligned} \quad (7.26)$$

Plugging (7.25) and (7.26) into the right-hand side of (7.24), and using that for any scalar function f, for $m=-1,0,1$, $(\overset{\circ}{\Delta\!\!\!/}f)^{(1m)}=-2f^{(1m)}$, we get that for $R\geq 1$ sufficiently large,

$$\begin{aligned} &8\pi\cdot\left(\mathbf{P}^{\mathrm{loc}}_{\mathrm{ADM}}\right)^i \\ &=\int_{S_{-1,1}}\left(\frac{1}{2}\left(\mathrm{tr}\underline{\chi}-\mathrm{tr}\chi\right)+\mathrm{tr}\chi-r\,\mathrm{d\!/\!iv}\zeta\right)\sqrt{\frac{4\pi}{3}}Y^{(1m_i)}d\mu_{r^2\overset{\circ}{\gamma}}+\mathcal{O}\left(\frac{M}{R}\varepsilon_R\right)+\mathcal{O}\left(\varepsilon_R^2\right) \\ &=\int_{S_{-1,1}}\left(-r\rho-r^2\,\mathrm{d\!/\!iv}\beta+\mathrm{tr}\chi+\frac{1}{2}\overset{\circ}{\Delta\!\!\!/}\mathrm{tr}\chi\right)\sqrt{\frac{4\pi}{3}}Y^{(1m_i)}d\mu_{r^2\overset{\circ}{\gamma}}+\mathcal{O}\left(\frac{M}{R}\varepsilon_R\right)+\mathcal{O}\left(\varepsilon_R^2\right)+\mathcal{R} \\ &=8\pi\cdot\mathbf{P}^{m_i}+\mathcal{O}\left(\frac{M}{R}\varepsilon_R\right)+\mathcal{O}\left(\varepsilon_R^2\right)+\mathcal{R}, \end{aligned} \quad (7.27)$$

where the remainder term $\mathcal{R}$ is given by

$$\begin{aligned} \mathcal{R}=&\int_{S_{-1,1}}\left(-rK+\frac{r}{2}(\widehat{\chi},\widehat{\underline{\chi}})-\frac{1}{r}-\frac{r}{4}\left(\mathrm{tr}\chi-\frac{2}{r}\right)\left(\mathrm{tr}\underline{\chi}+\frac{2}{r}\right)\right)\sqrt{\frac{4\pi}{3}}Y^{(1m_i)}d\mu_{r^2\overset{\circ}{\gamma}} \\ &-\int_{S_{-1,1}}\left(\frac{1}{2}\left(\overset{\circ}{\Delta\!\!\!/}-r^2\Delta\!\!\!/\right)\left(\mathrm{tr}\chi-\frac{2}{r}\right)+\frac{r^2}{2}\,\mathrm{d\!/\!iv}\,\mathrm{d\!/\!iv}\widehat{\chi}\right)\sqrt{\frac{4\pi}{3}}Y^{(1m_i)}d\mu_{r^2\overset{\circ}{\gamma}} \\ &-\int_{S_{-1,1}}\left(r^2\,\mathrm{d\!/\!iv}\,(\widehat{\chi}\cdot\zeta)-\frac{r^2}{2}\,\mathrm{d\!/\!iv}\left(\left(\mathrm{tr}\chi-\frac{2}{r}\right)\zeta\right)\right)\sqrt{\frac{4\pi}{3}}Y^{(1m_i)}d\mu_{r^2\overset{\circ}{\gamma}}. \end{aligned}$$

To conclude (7.22), it remains to show that for $R\geq 1$ sufficiently large,

$$\mathcal{R}=\mathcal{O}(\varepsilon_R^2)+\mathcal{O}\left(\frac{M}{R}\varepsilon_R\right). \quad (7.28)$$

Indeed, (7.28) follows in a straight-forward fashion, using Lemma 2.1. This proves (7.22).

7.5. Comparison of $\mathbf{L}$ *and* $\mathbf{L}^{\mathrm{loc}}_{\mathrm{ADM}}$.. In this section we prove that for $i = 1, 2, 3$ and $(m_1, m_2, m_3) = (1, -1, 0)$,

$$\mathbf{L}^{m_i}({}^{(R)}x_{-1,1}) = \left(\mathbf{L}^{\mathrm{loc}}_{\mathrm{ADM}}\right)^i (S_{-1,1}, g, k) + \mathcal{O}\left(\frac{M}{R}\varepsilon_R\right) + \mathcal{O}(\varepsilon_R^2). \tag{7.29}$$

On the one hand we have, using that $\not{d}\,\Omega_M = 0$, and that $(\not{d}\, f)_H = 0$ for any scalar function f,

$$\begin{aligned} \mathbf{L}^{m_i} =& \frac{1}{16\pi}\sqrt{\frac{8\pi}{3}} r^3 \,(\mathrm{tr}\chi \cdot \eta)_H^{(1m_i)} \\ =& \frac{1}{8\pi}\sqrt{\frac{8\pi}{3}} r^2 \eta_H^{(1m_i)} + \frac{1}{16\pi}\sqrt{\frac{8\pi}{3}} r^3 \left(\left(\mathrm{tr}\chi - \frac{2}{r}\right)\cdot \eta\right)_H^{(1m_i)} \\ =& \frac{1}{8\pi}\sqrt{\frac{8\pi}{3}} r^2 \eta_H^{(1m_i)} + \mathcal{O}\left(\frac{M}{R}\varepsilon_R\right) + \mathcal{O}(\varepsilon_R^2), \end{aligned} \tag{7.30}$$

On the other hand,

$$\begin{aligned} \left(\mathbf{L}^{\mathrm{loc}}_{\mathrm{ADM}}\right)^i =& \frac{1}{8\pi}\int_{S_{-1,1}} k_{jN}\left(Y_{(i)}\right)^j d\mu_{\not{g}} - \frac{1}{8\pi}\int_{S_{-1,1}} \mathrm{tr}k \underbrace{g\left(Y_{(i)}, N\right)}_{=0} d\mu_{\not{g}} \\ =& \frac{1}{8\pi}\int_{S_{-1,1}} \eta_A \left(Y_{(i)}\right)^A d\mu_{r^2\overset{\circ}{\gamma}} + \mathcal{O}(\varepsilon_R^2) \\ =& \frac{1}{8\pi}\sqrt{\frac{8\pi}{3}} r^2 \eta_H^{(1m_i)} + \mathcal{O}(\varepsilon_R^2), \end{aligned} \tag{7.31}$$

where we used (7.12) and that the rotation fields $Y_{(i)}$, $i = 1, 2, 3$, are $S_{-1,1}$-tangential and related to the standard vector spherical harmonic $H^{(1m)}$, $m = -1, 0, 1$, as follows

$$Y_{(i)} = \sqrt{\frac{8\pi}{3}} |x|^2 H^{(1m_i)} \text{ with } (m_1, m_2, m_3) = (1, -1, 0).$$

Combining (7.30) and (7.31) finishes the proof of (7.29).

7.6. Expression of $\mathbf{G}$ *in terms of* $\mathbf{C}^{\mathrm{loc}}_{\mathrm{ADM}}$ *and* $\mathbf{P}^{\mathrm{loc}}_{\mathrm{ADM}}$.. In this section we prove that for $i = 1, 2, 3$ and $(m_1, m_2, m_3) = (1, -1, 0)$,

$$\begin{aligned} \mathbf{G}^m({}^{(R)}x_{-1,1}) =& \left(\mathbf{C}^{\mathrm{loc}}_{\mathrm{ADM}}\right)^{i_m} (S_{-1,1}, g, k) - r(S_{-1,1}, g, k)\cdot \left(\mathbf{P}^{\mathrm{loc}}_{\mathrm{ADM}}\right)^{i_m} (S_{-1,1}, g, k) \\ &+ \mathcal{O}\left(\frac{M}{R}\varepsilon_R\right) + \mathcal{O}\left(\varepsilon_R^2\right). \end{aligned} \tag{7.32}$$

Consider first $\mathbf{C}^{\mathrm{loc}}_{\mathrm{ADM}}$. By (6.10) and $g(E^{(1m)}, N) = 0$ we have that

$$\begin{aligned} 16\pi\left(\mathbf{C}^{\mathrm{loc}}_{\mathrm{ADM}}\right)^i =& -\sqrt{\frac{8\pi}{3}}\int_{S_{-1,1}} |x|^3 \mathrm{Ric}\left(E^{(1m_i)}, N\right) d\mu_{\not{g}} \\ & -\sqrt{\frac{4\pi}{3}}\int_{S_{-1,1}} |x|^2 \left(\mathrm{Ric} - \frac{1}{2}R_{\mathrm{scal}}\, g\right)(\partial_r, N) Y^{(1m_i)} d\mu_{\not{g}}, \end{aligned} \tag{7.33}$$

By (6.13), the second integral on the right-hand side of (7.33) can be expressed as

$$
\begin{aligned}
&\int_{S_{-1,1}} |x|^2 \left(\mathrm{Ric} - \frac{1}{2} R_{\mathrm{scal}}\, g \right) (\partial_r, N) Y^{(1m_i)} d\mu_{\not g} \\
&= \int_{S_{-1,1}} |x|^4 \left(-K + \frac{1}{4}(\mathrm{tr}\Theta)^2 + |\widehat{\Theta}|^2 \right) Y^{(1m_i)} d\mu_{\overset{\circ}{\gamma}} + \mathcal{O}\left(\frac{M}{R}\varepsilon_R\right) + \mathcal{O}(\varepsilon_R^2) \\
&= \underbrace{- \int_{S_{-1,1}} |x|^4 K \cdot Y^{(1m_i)} d\mu_{\overset{\circ}{\gamma}}}_{:=\mathcal{I}_1} + \underbrace{\frac{1}{4} \int_{S_2} |x|^4 (\mathrm{tr}\Theta)^2 \cdot Y^{(1m_i)} d\mu_{\overset{\circ}{\gamma}}}_{:=\mathcal{I}_2} + \mathcal{O}\left(\frac{M}{R}\varepsilon_R\right) + \mathcal{O}(\varepsilon_R^2),
\end{aligned}
\tag{7.34}
$$

where we used Lemma 6.10. In the following we analyse $\mathcal{I}_1$ and $\mathcal{I}_2$. First, by (7.1) and Lemma 2.1 we have that for $R \geq 1$ sufficiently large,

$$
\mathcal{I}_1 = |x|^4 K^{(1m_i)} = \mathcal{O}(\varepsilon_R^2). \tag{7.35}
$$

Second, by the relation $Y^{(1m)} = \frac{1}{\sqrt{2}} \overset{\circ}{\mathrm{d}\!\!/\mathrm{iv}} E^{(1m)}$, integration by parts, and the Gauss–Codazzi equations (6.13), we have that

$$
\begin{aligned}
\mathcal{I}_2 &= -\frac{4}{\sqrt{2}} \int_{S_{-1,1}} |x|^4 \cdot \mathrm{tr}\Theta \cdot \overset{\circ}{\gamma} \left(\frac{1}{2} \not{d}\, \mathrm{tr}\Theta, E^{(1m_i)} \right) d\mu_{\overset{\circ}{\gamma}} \\
&= \frac{8}{\sqrt{2}} \int_{S_{-1,1}} |x|^3 \cdot \not g \left(\mathrm{d}\!\!/\mathrm{iv} \widehat{\Theta} - \frac{1}{2} \not{d}\, \mathrm{tr}\Theta, E^{(1m_i)} \right) d\mu_{\not g} + \mathcal{O}\left(\frac{M}{R}\varepsilon_R\right) + \mathcal{O}(\varepsilon_R^2) \\
&= \frac{8}{\sqrt{2}} \int_{S_{-1,1}} |x|^3 \cdot \mathrm{Ric}(E^{(1m_i)}, N) d\mu_{\not g} + \mathcal{O}\left(\frac{M}{R}\varepsilon_R\right) + \mathcal{O}(\varepsilon_R^2),
\end{aligned}
\tag{7.36}
$$

where we used that $\mathrm{tr}\Theta - \frac{2}{|x|} = \mathcal{O}\left(\frac{M}{R}\right) + \mathcal{O}(\varepsilon_R)$ by (7.1) and Lemma 6.10. Plugging (7.35) and (7.36) into (7.34) and subsequently into (7.33), we get that

$$
16\pi \left(\mathbf{C}_{\mathrm{ADM}}^{\mathrm{loc}} \right)^i = -2\sqrt{\frac{8\pi}{3}} \int_{S_{-1,1}} |x|^3 \cdot \mathrm{Ric}(E^{(1m_i)}, N) d\mu_{\not g} + \mathcal{O}\left(\frac{M}{R}\varepsilon_R\right) + \mathcal{O}(\varepsilon_R^2).
\tag{7.37}
$$

Consider now $\mathbf{G}^m$. By Definition 2.5, (2.11) and (7.8), we have that

$$
8\pi\sqrt{\frac{3}{8\pi}} \cdot \mathbf{G}^m = \left(r^3 \left(\beta + \mathrm{d}\!\!/\mathrm{iv} \widehat{\chi} + \widehat{\chi} \cdot \eta \right) \right)_E^{(1m)} = r^3 \beta_E^{(1m)} + \mathcal{O}(\varepsilon_R^2), \tag{7.38}
$$

where we used (7.1), (7.10) and (7.12).

Recalling the definition of $\mathbf{P}$ from Definition 2.5 and applying (7.1), it holds that

$$
-8\pi\sqrt{\frac{3}{4\pi}} \frac{1}{r^3} \mathbf{P}^m = \rho^{(1m)} + \left(\frac{1}{r} \overset{\circ}{\mathrm{d}\!\!/\mathrm{iv}} \beta \right)^{(1m)} + \mathcal{O}(\varepsilon_R^2) = \rho^{(1m)} + \left(\frac{1}{r} \sqrt{2} \beta_E^{(1m)} \right) + \mathcal{O}(\varepsilon_R^2).
$$

In particular, $\beta_E^{(1m)}$ can expressed as $\beta_E^{(1\,m)} = -\frac{r}{\sqrt{2}}\left(8\pi\sqrt{\frac{3}{4\pi}}\frac{1}{r^3}\mathbf{P}^m + \rho^{(1\,m)}\right) + \mathcal{O}(\varepsilon_R^2)$. Plugging this into the right-hand side of (7.38) yields

$$\begin{aligned}8\pi\sqrt{\frac{3}{8\pi}}\cdot\mathbf{G}^m &= r^3\beta_E^{(1m)} + \mathcal{O}(\varepsilon_R^2) = r^3\left(-\frac{r}{\sqrt{2}}\left(8\pi\sqrt{\frac{3}{4\pi}}\frac{1}{r^3}\mathbf{P}^m + \rho^{(1m)}\right)\right) + \mathcal{O}(\varepsilon_R^2)\\ &= -8\pi\sqrt{\frac{3}{8\pi}}\cdot r\cdot\mathbf{P}^m - \frac{r^4}{\sqrt{2}}\rho^{(1m)} + \mathcal{O}(\varepsilon_R^2).\end{aligned} \tag{7.39}$$

The second term on the right-hand side of (7.39) can be rewritten by the Gauss equation (2.10), Lemma 2.1, application of (7.1) and (7.10), and use of (7.36) as follows,

$$\begin{aligned}\rho^{(1m)} &= \left(-K - \frac{1}{4}\mathrm{tr}\chi\,\mathrm{tr}\underline{\chi} + \frac{1}{2}(\widehat{\chi},\widehat{\underline{\chi}})\right)^{(1m)}\\ &= -\frac{1}{4}\left(\mathrm{tr}\chi\,\mathrm{tr}\underline{\chi}\right)^{(1m)} + \mathcal{O}(\varepsilon_R^2)\\ &= -\frac{1}{4}\left((-\mathrm{tr}\overline{k} + \mathrm{tr}\Theta)(-\mathrm{tr}\Theta - \mathrm{tr}\overline{k})\right)^{(1m)} + \mathcal{O}(\varepsilon_R^2)\\ &= \frac{1}{4}\left((\mathrm{tr}\Theta)^2\right)^{(1m)} + \mathcal{O}(\varepsilon_R^2)\\ &= \frac{2}{\sqrt{2}}\int_{S_{-1,1}}\frac{1}{|x|}\cdot\mathrm{Ric}(E^{(1m_i)}, N)d\mu_{\not g} + \mathcal{O}\left(\frac{M}{R}\varepsilon_R\right) + \mathcal{O}(\varepsilon_R^2).\end{aligned}$$

Plugging this into (7.39), and using (7.22) and (7.37), we get

$$\begin{aligned}8\pi\sqrt{\frac{3}{8\pi}}\mathbf{G}^m &= -8\pi\sqrt{\frac{3}{8\pi}}\cdot r\cdot\mathbf{P}^m - r^3\int_{S_2}\mathrm{Ric}(N, E^{(1m)})d\mu_{\not g} + \mathcal{O}\left(\frac{M}{R}\varepsilon_R\right) + \mathcal{O}\left(\varepsilon_R^2\right)\\ &= -8\pi\sqrt{\frac{3}{8\pi}}\cdot r\cdot\left(\mathbf{P}_{\mathrm{ADM}}^{\mathrm{loc}}\right)^{i_m} + 8\pi\sqrt{\frac{3}{8\pi}}\left(\mathbf{C}_{\mathrm{ADM}}^{\mathrm{loc}}\right)^{i_m} + \mathcal{O}\left(\frac{M}{R}\varepsilon_R\right) + \mathcal{O}\left(\varepsilon_R^2\right).\end{aligned}$$

This finishes the proof of (7.32).

7.7. Conclusion of proof of Theorem 7.1. From (7.6) we have that the constructed ingoing null data $({}^{(R)}x_{-1+[-\delta,\delta],1})$ satisfies, for $R \geq 1$ sufficiently large,

$$\|{}^{(R)}x_{-1+[-\delta,\delta],1} - \mathfrak{m}^{M/R}\|_{\mathcal{X}^+(\underline{\mathcal{H}}_{-1+[-\delta,\delta],1})} = \mathcal{O}\left(R^{-3/2}\right).$$

By Lemma 2.17, the rescaled ingoing null data $x_{-R+R[-\delta,\delta],R} := {}^{(R^{-1})}\left({}^{(R)}x_{-1+[-\delta,\delta],1}\right)$ satisfies

$$\|x_{-R+R[-\delta,\delta],R} - \mathfrak{m}^M\|_{\mathcal{X}^+(\underline{\mathcal{H}}_{-R+R[-\delta,\delta],R})} = \mathcal{O}\left(R^{-3/2}\right).$$

Next we show that

$$\|\beta^{[1]}(x_{-R,R})\|_{L^2(S_{-R,R})} = \mathcal{O}\left(R^{-3}\right). \tag{7.40}$$

We claim that (7.40) follows from the finiteness of the charges $\mathbf{L}_\infty$ and $\mathbf{G}_\infty$ shown below. Indeed, by Definition 2.5, (2.11), Lemma 2.18 and (7.6), we have that for $R \geq 1$ large,

$$\mathbf{L}^m(x_{-R,R}) = R^2 \cdot \mathbf{L}^m({}^{(R)}x_{-1,1}) = R^2 \cdot \left(-r^3\left(\beta + \mathrm{d\!/\!iv}\,\widehat{\chi} + \widehat{\chi}\cdot(\eta - \mathrm{d\!/}\log\Omega)\right)\right)_H^{(1m)}({}^{(R)}x_{-1,1})$$
$$\mathbf{G}^m(x_{-R,R}) = R^2 \cdot \mathbf{G}^m({}^{(R)}x_{-1,1}) = R^2 \cdot \left(-\left(r_{M/R}(-1,1)\right)^3 \beta_E^{(1m)}({}^{(R)}x_{-1,1}) + \mathcal{O}(R^{-3})\right).$$

Hence by Definition 2.12 and the finiteness of $\mathbf{L}_\infty$ and $\mathbf{G}_\infty$ (discussed below) we get that $|\beta_H^{(1\,m)}({}^{(R)}x_{-1,1})| + |\beta_E^{(1\,m)}({}^{(R)}x_{-1,1})| = \mathcal{O}(R^{-2})$. Together with the scaling of β, see Lemma 2.15, this implies that

$$\|\beta^{[1]}(x_{-R,R})\|^2_{L^2(S_{-R,R})} = R^{-2} \cdot \|\beta^{[1]}({}^{(R)}x_{-1,1})\|^2_{L^2(S_{-1,1})} = \mathcal{O}(R^{-6}).$$

This finishes the proof of (7.40). It thus only remains to analyze the asymptotics of the charges $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})(x_{-R,R})$. These follow straight-forward from Lemmas 2.18 and 6.6, Sect. 6.4 and (7.20), (7.22), (7.29) and (7.32), and are omitted. This finishes the proof of Theorem 7.1.

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Declarations

Conflict of interest There is no conflict of interest.

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