

Resilient Control of Dynamic Flow Networks Subject to Stochastic Cyber-Physical Disruptions

Yu Tang and Li Jin

Abstract—Modern network systems, such as transportation and communication systems, are prone to cyber-physical disruptions and thus suffer efficiency loss. This paper studies network resiliency, in terms of throughput, and develops resilient control to improve throughput. We consider single-commodity networks that admit congestion propagation. We also apply a Markov process to model disruption switches. For throughput analysis, we first use insights into congestion spillback to propose novel Lyapunov functions and then exploit monotone network dynamics to reduce computational costs of verifying stability conditions. For control design, we show that (i) for a network with infinite link storage space, there exists an open-loop control that attains the min-expected-cut capacity; (ii) for a network with observable disruptions that restrict maximum sending and/or receiving flows, there exists a mode-dependent control that attains the expected-min-cut capacity; (iii) for general networks, there exists a closed-loop control with throughput guarantees. We also derive lower bounds of resiliency scores for a set of numerical examples and verify resiliency improvement with our method.

Index Terms: Dynamic flow networks, cyber-physical disruptions, piecewise-deterministic Markov processes, monotone dynamical systems.

I. INTRODUCTION

A. Motivation

Dynamic flow networks are widely used to model engineering systems including transportation systems [1] and communication networks [2]. These systems are susceptible to disruptions both in physical and cyber parts. In the physical part, link flows can be disrupted by capacity-reducing events such as traffic incidents [3]. In the cyber part, unreliable state observation or faulty feedback actuation may occur, which degrades the effectiveness of feedback controllers and lead to physical losses [4]. Typically, both types of disruptions are hard to predict and thus need treatment in a stochastic manner. For instance, probabilistic models were used to evaluate freeway capacity, which is supported by field studies [5]. This modeling technique is also applicable to sensing faults [6]. Since network systems involve very large numbers of cyber-physical components, it is neither economically feasible

nor technically necessary to prevent all disruptions. Instead, a practical solution is to integrate disruptions into modeling control schemes [7]. However, limited studies have discussed such method for dynamic flow networks subject to stochastic disruptions.

In this paper, we first evaluate network throughput as a measure of resiliency against stochastic cyber-physical disruptions, arising from reliability failures¹, and then design control strategies that mitigate throughput losses. To this end, we consider single-commodity networks that have found applications in real systems, such as freeway systems [1]. Although single-commodity networks could simplify real-world cases, they are worth studying since multi-commodity networks require origin-destination (OD) demands that are hard to acquire in practice. We use a finite-state Markov process to model the occurrence and clearance of disruptions. Then we investigate stability condition of networks, which allows for resiliency quantification and resilient control configuration. Particularly, our discussion on control design, inspired by the classical max-flow min-cut theorem [8] for static flow networks, proposes *min-expected-cut capacity* (MECC) and *expected-min-cut capacity* (EMCC) and reveals their relation to throughput, network storage space, disruptions and control laws. In case of a general network whose throughput is not guaranteed with these capacities, we present a closed-loop control with lower-bounded throughput.

B. Related work

Previous work on dynamic flow networks typically considered a nominal or robust setting. In the disruption-free case, in-depth stability analysis was provided for networks characterized by various flow structures including monotone dynamics, nonlinear demand/supply constraints, and congestion spillback [9]–[13]. Besides, optimal routing control has been extensively investigated for dynamical flow networks. [9], [14]. Robust control strategies, along with resiliency analysis, were developed in the face of physical disruptions [15]–[20]. However, the above work assumes perfect sensing and actuation; the resiliency against cyber disruptions remains unclear. Besides, robust control handles disturbances with uncertainty sets [20]; it does not apply to recurrently switching disruptions.

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Y. Tang is with the Tandon School of Engineering, New York University, USA. L. Jin is with the UM Joint Institute and with the Department of Automation, Shanghai Jiao Tong University, China (emails: tangyu@nyu.edu, li.jin@sjtu.edu.cn).

¹In this paper, reliability failures refer to temporary malfunction of components of controlled dynamic flow networks, such as node/link breakdown and sensing/actuation faults. They do not include communication delay or malicious cyber attacks. Though this paper mainly considers sensing failures for cyber disruptions, our approach can be applied to address actuation faults with minor modifications.

The model we consider belongs to a subclass of piecewise-deterministic Markov processes (PDMP) [21], sometimes known as Markov jump nonlinear systems (MJNS) [22], where continuous states (traffic densities) evolve according to a set of ordinary differential equations and a discrete state (disruption mode) determines the mode of the continuous dynamics [23]. Such a formulation allows analysis of joint impacts of cyber and physical disruptions on network resiliency. Although the general theories of PDMPs [23]–[25] and MJNSs [26], [27] have been extensively investigated, the implementation is still challenging due to nonlinear flow dynamics. Typical Lyapunov functions, such as piecewise quadratic functions [27], fail to capture congestion dynamics and thus only lead to trivial stability analysis; see our discussion in Section III. It is vital to design appropriate Lyapunov functions for analyzing our model.

Our work is also related to stochastic fluid models [28], [29]. These models are applied to performance analysis or admission/priority control for servers subject to demand or service disruptions [30], [31]. Currently, limited results consider congestion spillback over stochastic flow networks, let alone the corresponding control design. Only a few results were developed for networks with special structures, such as parallel links [3], [32] and serial links [33], [34]. To our best knowledge, the general stochastic flow networks are still not well investigated.

C. Our contributions

This paper focuses on the two following questions:

- (i) How to quantify resiliency against stochastic disruptions, especially for networks with congestion propagation?
- (ii) How to attain resiliency-by-design or improve resiliency?

We define *resiliency score* as a ratio of disrupted network throughput to its nominal value. Here the disrupted (resp. nominal) throughput means the maximal inflow under which the network with (resp. without) disruptions can be stabilized, i.e. traffic densities in all links being bounded on average. The max-flow min-cut theorem states that the nominal throughput equals *min-cut capacity* [8], but we can hardly resolve the disrupted throughput in an analytical way. It is because our PDMP model allows complicated disruptions, including the physical ones creating new bottlenecks and the cyber ones inducing mismatches between control instructions and actual inter-link flows. Thus we address (i) by deriving and sharpening lower bounds of throughput. The lower bounds are obtained from a set of stability conditions built on the Foster-Lyapunov criterion [35]. To formulate the stability conditions, we use the insights into network-wide congestion propagation to propose a set of novel Lyapunov functions. We also exploit monotone network dynamics to simplify the condition verification from over unbounded sets to over only compact sets, which saves computational costs.

As indicated above, the max-flow min-cut theorem could be compromised. Thus we consider its variants for resiliency-by-design control. Meanwhile, a more practical concern is that full observation of disruption modes and network states (traffic densities) could not be available to control design.

To answer (ii), we first show that there exists an open-loop control that attains the MECC if all links have infinite storage space. Second, we show that there exists a mode-dependent control that attains the EMCC if the disruptions restrict maximum sending/receiving flows. The above results resemble the classical max-flow min-cut theorem. Third, we propose a density-dependent control, that is throughput-guaranteed, for general networks disrupted stochastically. Finally, we use numerical examples to demonstrate that our methods can enhance network resiliency.

The rest of this paper is organized as follows. Section II introduces our PDMP model for networks subject to stochastic cyber-physical disruptions. Section III analyzes the resiliency of this network. Section IV presents resilient control design. Section V summarizes the main conclusions and discusses future directions.

II. DYNAMIC FLOW NETWORK WITH CYBER-PHYSICAL DISRUPTIONS

Consider a single-origin-single-destination directed network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ and $\mathcal{E}$ denote the node set and the link set, respectively. Though single-commodity networks may have multiple origins and destinations, we can introduce one artificial origin and destination so that we obtain single-origin-single-destination networks where demands are routed from the artificial origin to real ones by proportions. For ease of presentation, we assume that the network is acyclic. Note that the proposed method can be applied to cyclic networks as well; see our discussion in Appendix B. We denote the starting and ending nodes of link e by σ_e and τ_e , respectively. We number the origin node as v_o and the destination node as v_d . The origin is subject to a constant inflow of $\alpha \in \mathbb{R}_{\geq 0}$. Without loss of generality, we assume that the flow enters the network via link e_o ; see Fig. 1. Following the convention [18], we consider

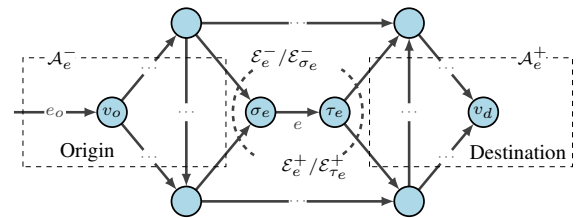


Fig. 1: A single-origin-single-destination network: we denote by $\mathcal{E}_e^-$ (resp. $\mathcal{E}_{\sigma_e}^-$) the set of incoming links of link e (resp. node σ_e), by $\mathcal{E}_e^+$ (resp. $\mathcal{E}_{\tau_e}^+$) the set of outgoing links of link e (resp. node τ_e), by $\mathcal{A}_e^-$ the set of links upstream of link e , and by $\mathcal{A}_e^+$ the set of links downstream of link e .

traffic density (mass per unit) as the network state and denote by $X_e(t)$ the density of link e at time t . We also assume that all links have the same unit length for convenience of computing densities. For link e with finite storage space, $X_e(t)$ can only take values from a closed interval $[0, x_e^{\max}]$, where $x_e^{\max} < \infty$ is called the *jam density*. For link e with infinite storage space, $X_e(t)$ can take values from $\mathbb{R}_{\geq 0}$ and we let $x_e^{\max} = \infty$. In particular, we assume that link e_o has infinite storage space; this ensures that no traffic is rejected into e_o . We use $\mathcal{X}_e$ to denote the set of $X_e(t)$, $\mathcal{X} = \prod_{e \in \mathcal{E}} \mathcal{X}_e \subseteq \mathbb{R}_{\geq 0}^{\mathcal{E}}$ to denote the

set of the state vector $X(t)$, where $\mathbb{R}_{\geq 0}^{\mathcal{E}}$ stands for a set of non-negative vectors whose components are indexed by elements of $\mathcal{E}$.

For a node $v \in \mathcal{V}$, we let $\mathcal{E}_v^- := \{e \in \mathcal{E} | \tau_e = v\}$ (resp. $\mathcal{E}_v^+ := \{e \in \mathcal{E} | \sigma_e = v\}$) denote the set of its incoming (resp. outgoing) links; for a link $e \in \mathcal{E}$, we use $\mathcal{E}_e^- := \{i \in \mathcal{E} | \tau_i = \sigma_e\}$ (resp. $\mathcal{E}_e^+ := \{j \in \mathcal{E} | \sigma_j = \tau_e\}$) to denote the set of its upstream (resp. downstream) adjacent links. Clearly, we have $\mathcal{E}_v^- = \mathcal{E}_e^-$ if $v = \sigma_e$ and $\mathcal{E}_v^+ = \mathcal{E}_e^+$ if $v = \tau_e$. We also let $\mathcal{P} := \{(e, j) | e \in \mathcal{E}, j \in \mathcal{E}_e^+\}$ denote the set of ordered pairs of adjacent links. It follows $|\mathcal{P}| = \sum_{e \in \mathcal{E}} |\mathcal{E}_e^+|$, where $|\cdot|$ denotes the cardinality of a set. We say that link e is accessible from link i , denoted by $i \rightarrow e$, if there exists a directed path starts with link i and ends at link e . For link $e \in \mathcal{E}$, let $\mathcal{A}_e^- := \{i \in \mathcal{E} | i \rightarrow e\}$ be the set of links from which link e is accessible, and let $\mathcal{A}_e^+ := \{j \in \mathcal{E} | e \rightarrow j\}$ be the set of links that are accessible from link e .

In the rest of this section, we first define flow functions, control laws, and disruption modes (Section II-A), and then specify the network's dynamics as a piecewise-deterministic Markov process (Section II-B). Finally, we define network stability and resiliency score (Section II-C).

A. Flows, control laws and disruptions

Below we introduce the essential definitions and assumptions for our network model. The illustrative examples are also provided.

1) *Sending/receiving flows*: The *sending flow* of link e is specified by $f_e : \mathcal{X}_e \rightarrow \mathbb{R}_{\geq 0}$. It stands for the maximum outflow from link e given a state x_e . The *receiving flow* of link e is specified by $r_e : \mathcal{X}_e \rightarrow \mathbb{R}_{\geq 0}$. It is recognized as the maximum inflow allowed into link e given a state x_e . We assume that these flows satisfy:

Assumption 1 (Sending/receiving flows).

- 1.1 *Sending flows*: For every $e \in \mathcal{E}$, $f_e(x_e)$ is Lipschitz continuous and non-decreasing in x_e . Furthermore, $f_e(x_e) = 0$ if $x_e = 0$ and $\sup_{x_e} f_e(x_e) < \infty$.
- 1.2 *Receiving flows*: For every $e \in \mathcal{E}$, $r_e(x_e)$ is Lipschitz continuous and non-increasing in x_e . Furthermore, for link e with finite storage space, we assume $r_e(x_e) = 0$ if $x_e = x_e^{\max}$; for link e with infinite storage space, we assume a constant receiving flow r_e independent of x_e .

Note that receiving flows are not considered in some flow networks. In that case we just let link e have infinite storage space with $r_e = \infty$.

We define *link capacity* Q_e and *critical density* x_e^c as

$$Q_e := \sup_{x_e \in \mathcal{X}_e} \min\{f_e(x_e), r_e(x_e)\}, \quad (1a)$$

$$x_e^c := \inf\{x_e \in \mathcal{X}_e | f_e(x_e) = Q_e\}. \quad (1b)$$

Practically, the capacity Q_e indicates an upper bound of sustainable outflow from link e , and the critical density x_e^c denotes an threshold where the capacity flow Q_e can be maintained with a relatively high speed Q_e/x_e^c . Link e is considered as ‘‘congested’’ when its density x_e exceeds the critical value x_e^c .

Here are two examples of sending and receiving flows. In road network, as per the cell transmission model (CTM [36]), the sending and receiving flows are given by:

$$f_e(x_e) = \min\{v_f x_e, Q_e\}, \quad (2a)$$

$$r_e(x_e) = \min\{Q_e, w_c(x_e^{\max} - x_e)\}, \quad (2b)$$

where x_e represents vehicle density of road section e , and traffic parameters v_f , Q_e , w_c and $x_e^{\max}$ are typically assumed to satisfy $Q_e/x_e^{\max} \leq v_f w_c / (v_f + w_c)$ [34]. It follows $x_e^c = Q_e/v_f$. In data networks, the sending flow can be approximated as a fluid model [18] with

$$f_e(x_e) = Q_e(1 - e^{-\rho_e x_e}), \quad (3)$$

where x_e stands for queue length on channel e , ρ_e is a positive constant, and Q_e is channel capacity. The receiving flow is not explicitly modeled in data networks and thus we assume $r_e = \infty$, which yields $x_e^c = \infty$.

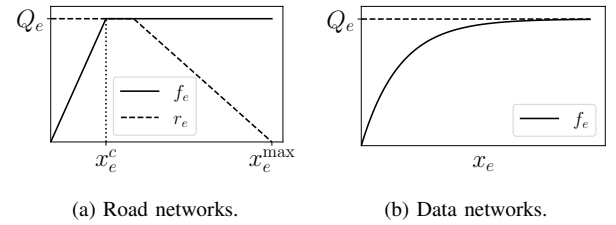


Fig. 2: Examples of sending/receiving flows.

2) *Control laws*: We consider a control law $\mu : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}^{\mathcal{P}}$ to be a concatenation of controllers $\mu_{ej} : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$, $(e, j) \in \mathcal{P}$, that regulate flows from link e to link j . In this paper we focus on locally responsive control laws defined below.

Definition 1 (Locally responsive control laws). A control law μ for a network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is *locally responsive* if $\mu_{ej}(x)$, for any $(e, j) \in \mathcal{P}$, can depend on the states of the links in $\mathcal{E}_{\tau_e}^- \cup \mathcal{E}_{\tau_e}^+$ only, where $\mathcal{E}_{\tau_e}^- \cup \mathcal{E}_{\tau_e}^+$ denotes the set of incoming and outgoing links of node τ_e .

In general, *actual inter-link flows* $q^\mu : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}^{\mathcal{P}}$ may not equal the control inputs $\mu(x)$ if the control law violates sending or receiving flows, e.g. under open-loop control. We define it as follows:

$$q_{ej}^\mu(x) := \min\{\mu_{ej}(x), f_{ej}^\mu(x), r_{ej}^\mu(x)\}, \quad (4)$$

where

$$f_{ej}^\mu(x) := \frac{\mu_{ej}(x)}{\sum_{j' \in \mathcal{E}_e^+} \mu_{ej'}(x)} f_e(x_e), \quad (5a)$$

$$r_{ej}^\mu(x) := \frac{\mu_{ej}(x)}{\sum_{e' \in \mathcal{E}_j^-} \mu_{e'j}(x)} r_j(x_j). \quad (5b)$$

This modelling approach enables us to study inappropriate control instructions induced by cyber-physical disruption introduced later. It is necessary since controllers may not have accurate knowledge of sending/receiving flows due to disruptions. The actual controlled flow $q_{ej}^\mu(x)$ represents how the network responds to the control input μ . The sending and receiving flows are allocated proportionally to $\mu_{ej}(x)$, which

is a typical modeling approach; for more discussion on this and alternative models, see [37]. Obviously, these techniques, with minor modifications, can be applied to flow allocations in multi-commodity dynamical flow networks [38]. Note that the superscript “ μ ” in (4) indicates the reliance of inter-links flows on the control law. We typically omit it unless we want to emphasize the dependence on a particular control law.

Assumption 2 (Controlled flows).

2.1 *Continuity: The controlled flow $q(x)$ is Lipschitz continuous in x_e for any $e \in \mathcal{E}$.*

2.2 *Monotonicity: For any two different links $e, e' \in \mathcal{E}$,*

- (i) $\sum_{i \in \mathcal{E}_e^-} q_{ie}(x)$ is non-decreasing in $x_{e'}$;
- (ii) $\sum_{j \in \mathcal{E}_e^+} q_{ej}(x)$ is non-increasing in $x_{e'}$.

Assumption 2.1 indicates that our approach is applicable to non-smooth control laws. Assumption 2.2 implies that the considered dynamic flow network is cooperative, which is a special class of monotone systems [39]. The following elaborates Assumption 2.2 (i) to reveal the cooperativity. If link e' locates upstream of link e and $x_{e'}$ increases, Assumption 2.2 (i) implies that the link set $\mathcal{E}_e^-$ tends to send more flows to link e in order to avoid density accumulation in the upstream. If e' does not appear upstream of link e and $e' \neq e$, the increase of $x_{e'}$ implies that somewhere (at least not link e) becomes congested and Assumption 2.2 (i) indicates that the link set $\mathcal{E}_e^-$ tends to send more flows to link e , which benefits alleviating congestion somewhere. A similar explanation applies to Assumption 2.2 (ii). Thus Assumption 2.2 indicates how the links reduce congestion cooperatively.

While dynamic flow networks are, in general, not monotone [10], the rationale behind Assumption 2.2 lies in that the monotonicity applies to a broad class of flow networks [19] and provides a tractable structure that benefits analysis and computation. The assumption of monotonicity can be relaxed in our stability analysis, with a rise of computational costs; see Propositions 1 and 2.

Below are the examples regarding routing and merging strategies satisfying Assumption 2. Consider routing control for a diverging junction with an upstream link e . The classical logit routing policy [18] gives

$$q_{ej}(x) = \min \left\{ \frac{e^{-\theta_j x_j}}{\sum_{j' \in \mathcal{E}_e^+} e^{-\theta_{j'} x_{j'}}} f_e(x_e), r_j(x_j) \right\}, \quad (6)$$

where θ_j quantifies the sensitivity of route choice to x_j . Now consider a merging junction with a minor upstream link e_{i_1} , a major upstream link e_{i_2} and a downstream link e . A typical ramp metering policy, namely the occupancy strategy [40], yields

$$q_{e_{i_1}e}(x) = \min\{u - \kappa x_{e_{i_2}}, f_{e_{i_1}}(x_{e_{i_1}}), r_e(x_e)\}, \quad (7a)$$

$$q_{e_{i_2}e}(x) = \min\{r_e(x_e) - q_{e_{i_1}e}(x), f_{e_{i_2}}(x_{e_{i_2}})\}, \quad (7b)$$

where $u, \kappa \geq 0$ are control parameters.

3) *Disruption modes:* We consider a set $\mathcal{S}$ of *modes* that capture cyber and/or physical disruptions. With a slight abuse of notation, we use $f_e : \mathcal{S} \times \mathcal{X}_e \rightarrow \mathbb{R}_{\geq 0}$, $r_e : \mathcal{S} \times \mathcal{X}_e \rightarrow \mathbb{R}_{\geq 0}$, $\mu : \mathcal{S} \times \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}^{\mathcal{P}}$ and $q : \mathcal{S} \times \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}^{\mathcal{P}}$ to denote the sending flow of link e , the receiving flow of link e , the control

law and the actual inter-link flows influenced by disruptions, respectively.

Assumption 3 (Cyber-physical disruptions).

3.1 *Nominal mode: There exists a nominal mode, denoted by $s_0 \in \mathcal{S}$, under which the network is free from cyber-physical disruptions and thus stable.*

3.2 *Disrupted sending/receiving flows: For every $s \in \mathcal{S}$ and $e \in \mathcal{E}$, $f_e(s, x_e)$ and $r_e(s, x_e)$ satisfy Assumption 1. Furthermore, $f_e(s, x_e) \leq f_e(s_0, x_e)$ and $r_e(s, x_e) \leq r_e(s_0, x_e)$.*

3.3 *Disrupted controlled flows: For every $s \in \mathcal{S}$ and $e \in \mathcal{E}$, $q(s, x)$ satisfies Assumption 2.*

Assumption 3.1 ensures one stable mode. In this paper, our primal interest is to study whether the network is destabilized given the switched modes. Assumptions 3.2 and 3.3 imply that disruptions will not fundamentally change the flow dynamics. They hold for typical reliability failures²; see the examples later. Then mode-specific link capacities and critical densities are given by

$$Q_{se} := \sup_{x_e \in \mathcal{X}_e} \min\{f_e(s, x_e), r_e(s, x_e)\}, \quad (8a)$$

$$x_{se}^c := \inf\{x_e \in \mathcal{X}_e | f_e(s, x_e) = Q_{se}\}. \quad (8b)$$

Note that Assumption 3.2 also indicates $Q_{se} \leq Q_{s_0e}$ for any $s \in \mathcal{S}$ and $e \in \mathcal{E}$. We denote by x_e^{c*} the maximum critical density over disruption modes:

$$x_e^{c*} := \max_{s \in \mathcal{S}} x_{se}^c, \quad e \in \mathcal{E}. \quad (9)$$

The following are examples of physical and cyber disruptions. Consider logit routing for a diverging junction with an upstream link e , and suppose that the outflow from link e may decrease by a certain ratio. Thus we consider $\mathcal{S} = \{s_0, s_1\}$ and $f_e(s_1, x_e) = \chi f_e(s_0, x_e)$, where $\chi \in [0, 1)$. The actual inter-link flows are given by

$$q_{ej}(s, x) = \min \left\{ \frac{e^{-\theta_j x_j}}{\sum_{j' \in \mathcal{E}_e^+} e^{-\theta_{j'} x_{j'}}} f_e(s, x_e), r_j(x_j) \right\}. \quad (10)$$

Now suppose that the observation of link $j^* \in \mathcal{E}_e^+$, denoted by $T_{j^*}(s, x)$, may be biased. Without loss of generality, we let $T_{j^*}(s_1, x) := \kappa x_{j^*}$ with some $\kappa \geq 0$ and $T_{j'}(s, x) = x_{j'}$ for any $(s, j') \neq (s_1, j^*)$. The controlled flows are given by

$$q_{ej}(s, x) = \min \left\{ \frac{e^{-\theta_j T_{j^*}(s, x)}}{\sum_{j' \in \mathcal{E}_e^+} e^{-\theta_{j'} T_{j'}(s, x)}} f_e(s, x_e), r_j(x_j) \right\}. \quad (11)$$

B. PDMP model

Now we can define the piecewise-deterministic dynamics of the controlled process $\{(S(t), X(t)); t \geq 0\}$. The discrete-state process $\{S(t); t \geq 0\}$ of the mode is a homogeneous finite-state Markov process that is independent of the continuous-state process $\{X(t); t \geq 0\}$ of the traffic densities. The state space of the discrete process is $\mathcal{S} :=$

²Security failures, which are beyond the scope of this paper, however, may violate this assumption.

$\{s_0, s_1, \dots, s_M\}$. The *transition rate* from mode s to mode s' is $\lambda_{ss'}$. Without loss of generality, we assume that $\lambda_{ss} = 0$ for all $s \in \mathcal{S}$. The discrete-state process evolves as follows:

$$\Pr\{S(t + \delta) = s' | S(t) = s\} = \lambda_{ss'}\delta + o(\delta), \quad \forall s, s' \in \mathcal{S},$$

where δ denotes an infinitesimal increment. We assume that the discrete-state process is ergodic, which means that every disruption will be resolved in finite time almost surely. It follows that $\{S(t); t \geq 0\}$ admits a unique steady-state probability vector $p := [p_{s_0}, p_{s_1}, \dots, p_{s_M}]^T \in \mathbb{R}_{\geq 0}^{\mathcal{S}}$ satisfying

$$\Lambda^T p = \mathbf{0}, \quad (12)$$

where $\mathbf{0}$ denotes the zero vector and Λ is the transition matrix given by

$$\Lambda := \begin{bmatrix} -\sum_{s' \in \mathcal{S}} \lambda_{s_0 s'} & \lambda_{s_0 s_1} & \cdots & \lambda_{s_0 s_M} \\ \lambda_{s_1 s_0} & -\sum_{s' \in \mathcal{S}} \lambda_{s_1 s'} & \cdots & \lambda_{s_1 s_M} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{s_M s_0} & \lambda_{s_M s_1} & \cdots & -\sum_{s' \in \mathcal{S}} \lambda_{s_M s'} \end{bmatrix}. \quad (13)$$

The continuous-state process $\{X(t); t \geq 0\}$ is defined as follows. We let $G_e(s, x) := (d/dt)X_e(t)$. The conservation law associated with flows implies

$$G_e(S(t), X(t)) =$$

$$\begin{cases} \alpha - \sum_{j \in \mathcal{E}_e^+} q_{ej}(S(t), X(t)), & \text{if } \tau_e = v_o \\ \sum_{i \in \mathcal{E}_e^-} q_{ie}(S(t), X(t)) \\ - \sum_{j \in \mathcal{E}_e^+} q_{ej}(S(t), X(t)), & \text{if } \tau_e \notin \{v_o, v_d\} \\ \sum_{i \in \mathcal{E}_e^-} q_{ie}(S(t), X(t)) \\ - f_e(S(t), X(t)), & \text{if } \tau_e = v_d. \end{cases} \quad (14a) \quad (14b) \quad (14c)$$

Note G_e is bounded. So is $G(s, x) := (d/dt)X(t)$.

The joint evolution of $S(t)$ and $X(t)$ is a PDMP and can be described compactly using an infinitesimal generator [35]

$$\mathcal{L}V(s, x) =$$

$$G(s, x)^T \nabla_x V(s, x) + \sum_{s' \in \mathcal{S}} \lambda_{ss'} (V(s', x) - V(s, x)) \quad (15)$$

for any differentiable function $V : \mathcal{S} \times \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$, where $\nabla_x V(s, x)$ is the gradient of V with respect to x . The bounded dynamics $G(s, x)$ indicates that $\{(S(t), X(t)); t \geq 0\}$ is a *non-explosive* Markov process, which is a prerequisite for discussing stability [35].

C. Stability and resiliency

In this subsection, we define network stability and resiliency, which are the focus of the subsequent analysis.

1) Stability and invariant set: Below we present several concepts regarding stability.

Definition 2 (Stability & Instability). The network is *stable* if there exists a scalar $Z < \infty$ such that for any initial condition $(s, x) \in \mathcal{S} \times \mathcal{X}$

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_{\tau=0}^t \mathbb{E}[|X(\tau)| | S(0) = s, X(0) = x] d\tau \leq Z, \quad (16)$$

where $|X(t)|$ denotes 1-norm of $X(t)$, namely $|X(t)| = \sum_{e \in \mathcal{E}} |X_e(t)|$. The network is *unstable* if there does not exist $Z < \infty$ such that for any initial condition $(s, x) \in \mathcal{S} \times \mathcal{X}$ such that the inequality (16) holds.

The notion of stability follows a classical definition [28], which is closely related to “first-moment stable” [22]. Practically, if the time-average traffic densities in all links are bounded, the network is stable; otherwise, it is unstable.

Verifying the stability requires to check (16) for all initial conditions $(s, x) \in \mathcal{S} \times \mathcal{X}$. This can be simplified by considering an *invariant set* $\mathcal{X}^\mu \subseteq \mathcal{X}$ [24], which is defined as follows:

Definition 3 (Invariant set). For the PDMP $\{(S(t), X(t)); t \geq 0\}$, a subset $\mathcal{X}^\mu \subseteq \mathcal{X}$ is an invariant set if it is (i) globally attracting and (ii) positively invariant:

$$(i) \quad \forall (S(0), X(0)) \in \mathcal{S} \times \mathcal{X},$$

$$\lim_{t \rightarrow \infty} \Pr\{X(t) \in \mathcal{X}^\mu | (S(0), X(0))\} = 1;$$

$$(ii) \quad \forall (S(0), X(0)) \in \mathcal{S} \times \mathcal{X}^\mu, \forall t \geq 0, X(t) \in \mathcal{X}^\mu.$$

The definition above indicates that i) for any initial condition $X(0) \in \mathcal{X}$, the state $X(t)$ enters $\mathcal{X}^\mu$ almost surely, and that ii) given any initial condition $X(0) \in \mathcal{X}^\mu$, the state $X(t)$ never leaves $\mathcal{X}^\mu$. Note that one trivial candidate set is $\mathcal{X}$ itself, but a tight invariant set depends on the control law μ and disruption modes $\mathcal{S}$. It is assumed that $\mathcal{X}^\mu$ is unbounded; otherwise, the network is naturally stable. Besides, we define

$$\underline{x}_e^\mu := \inf_{x \in \mathcal{X}^\mu} x_e, \quad \bar{x}_e^\mu := \sup_{x \in \mathcal{X}^\mu} x_e, \quad e \in \mathcal{E}.$$

If x_e does not have an invariant upper bound, we let $\bar{x}_e^\mu = \infty$. Clearly, we have $\mathcal{X}^\mu \subseteq \prod_{e \in \mathcal{E}} [\underline{x}_e^\mu, \bar{x}_e^\mu]$.

The following sets of links are induced by the invariant set $\mathcal{X}^\mu$. Clearly they also depend on the control law μ , but we omit the superscript “ μ ” for notational convenience without causing confusion. We denote by $\mathcal{E}_{\text{inf}} := \{e \in \mathcal{E} | \bar{x}_e^\mu = \infty\}$ the set of density-unbounded links. For any $e \in \mathcal{E}_{\text{inf}}$, let $\mathcal{A}_{e, \text{inf}}^- := \mathcal{A}_e^- \cap \mathcal{E}_{\text{inf}}$ (resp. $\mathcal{A}_{e, \text{inf}}^+ := \mathcal{A}_e^+ \cap \mathcal{E}_{\text{inf}}$) denote its upstream (resp. downstream) density-unbounded links. We also define the set of density-bounded links that are upstream (resp. downstream) of link $e \in \mathcal{E}_{\text{inf}}$, denoted by $\mathcal{A}_{e, \text{fin}}^- := \mathcal{A}_e^- \setminus \mathcal{A}_{e, \text{inf}}^-$ (resp. $\mathcal{A}_{e, \text{fin}}^+ := \mathcal{A}_e^+ \setminus \mathcal{A}_{e, \text{inf}}^+$). Particularly, we let

$$\mathcal{B}_e^+ := \{n \in \mathcal{A}_e^+ | \forall \ell \in \{n\} \cup (\mathcal{A}_e^+ \cap \mathcal{A}_n^-),$$

$$(i) \quad \ell \in \mathcal{A}_{e, \text{fin}}^+,$$

$$(ii) \quad \exists (s, x) \in \mathcal{S} \times \mathcal{X}^\mu, \exists i \in \mathcal{E}_\ell^-,$$

$$\min\{\mu_{i\ell}(s, x), f_{i\ell}^\mu(s, x)\} > r_{i\ell}^\mu(s, x)\} \quad (17)$$

denote the set of links downstream of link $e \in \mathcal{E}_{\text{inf}}$ such that (i) link $n \in \mathcal{B}_e^+$ is density-bounded, (ii) link $n \in \mathcal{B}_e^+$ can block the flows from its upstream link i , and (iii) any link ℓ between e and n , namely $\ell \in \mathcal{A}_e^+ \cap \mathcal{A}_n^-$ satisfies (i) and (ii), where $f_{i\ell}^\mu(s, x)$ (resp. $r_{i\ell}^\mu(s, x)$), similar to (5a) (resp. (5b)), denotes allocated sending (resp. receiving) flow between links i and ℓ . Intuitively, $\mathcal{B}_e^+$ is the set of connected bottlenecks, between link e and its downstream density-unbounded links, that induce congestion spillback and block the discharging flow of link e .

We use the classical Wheatstone bridge [41] to illustrate the definitions above; see Fig. 3. Suppose that only links e_o and e_5 have infinite storage and that the flows out of links e_2 and e_5 are decreased randomly by physical disruptions. Then the congestion builds up at links e_2 and e_5 . Due to the infinite storage, the traffic density at link e_5 may blow up. Meanwhile, link e_o could be blocked as well once the jam at link e_2 spills back to link e_o via link e_1 . It indicates $\mathcal{E}_{\text{inf}} = \{e_o, e_5\}$ with $\mathcal{B}_{e_o}^+ = \{e_1, e_2\}$ and $\mathcal{B}_{e_5}^+ = \emptyset$.

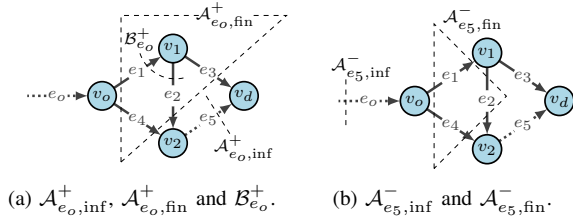


Fig. 3: Wheatstone bridge with $\mathcal{E}_{\text{inf}} = \{e_o, e_5\}$ denoted by dotted lines: $\mathcal{B}_{e_o}^+ = \{e_1, e_2\}$, $\mathcal{A}_{e_o, \text{inf}}^+ = \{e_5\}$, $\mathcal{A}_{e_o, \text{fin}}^+ = \{e_1, e_2, e_3, e_4\}$, $\mathcal{A}_{e_5, \text{inf}}^- = \{e_o\}$ and $\mathcal{A}_{e_5, \text{fin}}^- = \{e_1, e_2, e_4\}$.

2) Resiliency, throughput and min-cut capacities: The throughput α^μ of a network with control μ is defined as the maximal demand that the network can accept while maintaining stability, i.e. traffic densities in all links being bounded on average:

$$\alpha^\mu := \sup \alpha \quad \text{s.t. network is stable.} \quad (18)$$

For a network $\mathcal{G} = (\mathcal{E}, \mathcal{V})$ with link capacities $\mathcal{Q} := \{Q_e | e \in \mathcal{E}\}$, we denote by $C(\mathcal{Q}; \mathcal{G})$ its min-cut capacity (MCC). The min-cut max-flow theorem states that the nominal throughput equals the MCC [8], which is obtained by solving the maximum flow problem:

$$\begin{aligned} (\text{P}_1) \quad & \max_{\alpha \geq 0, u \in \mathbb{R}_{\geq 0}^{\mathcal{E}}} \alpha \\ \text{s.t.} \quad & \alpha = \sum_{j \in \mathcal{E}_e^+} u_{ej}, \quad \tau_e = v_o, \end{aligned} \quad (19a)$$

$$\sum_{i \in \mathcal{E}_e^-} u_{ie} = \sum_{j \in \mathcal{E}_e^+} u_{ej}, \quad \tau_e \in \mathcal{V} \setminus \{v_o, v_d\}, \quad (19b)$$

$$\sum_{i \in \mathcal{E}_e^-} u_{ie} \leq Q_e, \quad \tau_e \in \mathcal{V} \setminus \{v_o\}, \quad (19c)$$

$$\sum_{j \in \mathcal{E}_e^+} u_{ej} \leq Q_e, \quad \tau_e \in \mathcal{V} \setminus \{v_d\}, \quad (19d)$$

where the optimal α^* is equal to $C(\mathcal{Q}; \mathcal{G})$. Then we define resiliency score as

$$\eta^\mu := \alpha^\mu / C(\mathcal{Q}_{s_0}; \mathcal{G}), \quad (20)$$

where $C(\mathcal{Q}_{s_0}; \mathcal{G})$ denotes the MCC in the nominal mode s_0 . Assumption 3.2 indicates $\eta^\mu \in [0, 1]$. If $\eta^\mu = 1$, we say that the control law μ is strongly resilient against disruptions. The motivation for the resiliency score lies in that min-cut capacity is an important measurement of network capability. Meanwhile, for monotone dynamical flow networks, it is easily attainable with an open-loop control in the nominal case; see Appendix A. Thus the ratio can capture throughput losses due to disruptions.

III. RESILIENCY ANALYSIS

In this section, we analyze a network's stability under a given control law μ . The main results allow quantification of the resiliency score η^μ given by (20).

To state the results, we define

$$I_e(s, x) := \sum_{m \in \mathcal{A}_e^-} G_m(s, x) + \sum_{i \in \mathcal{E}_e^-} q_{ie}(s, x), \quad (21a)$$

$$O_e(s, x) := \sum_{j \in \mathcal{E}_e^+} q_{ej}(s, x) - \sum_{\ell \in \mathcal{B}_e^+} \rho_\ell(x_\ell) G_\ell(s, x), \quad (21b)$$

where $\rho_\ell : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is a weight function:

$$\rho_\ell(x_\ell) = \begin{cases} 0 & x_\ell < \underline{x}_\ell^\mu, \\ \frac{x_\ell - \underline{x}_\ell^\mu}{\bar{x}_\ell^\mu - \underline{x}_\ell^\mu} & \underline{x}_\ell^\mu \leq x_\ell < \bar{x}_\ell^\mu, \\ 1 & x_\ell \geq \bar{x}_\ell^\mu. \end{cases} \quad (22)$$

The weight function indicates that G_ℓ , the dynamics of link ℓ , has more impacts on (21b) as the traffic density x_ℓ increases. We interpret (21a) as the inflow from the origin v_o to link e , by considering the flow conservation but not the dynamics on the intermediate links. To see that, we first notice by (14a)-(14c) that $G_m(s, x)$ is the net flow only considering neighbor links of link m , and then recognize $\sum_{m \in \mathcal{A}_e^-} G_m(s, x)$ as the net flow of the upstream network whose inflow is the demand α and whose outflows comprise of i) those passing link e and ii) those never passing link e . Since the former outflows are canceled by $\sum_{i \in \mathcal{E}_e^-} q_{ie}(s, x)$ in (21a), it is easy to understand that (21a) denotes the demand α minus the flows never passing link e . In a similar way, we interpret (21b) as a weighted outflow of link e including those traversing the downstream bottlenecks and those not. Aware of the physical meanings of (21a)-(21b), we let

$$N_e(s, x) := I_e(s, x) - O_e(s, x) \quad (23)$$

denote the net flow of link e considered in a larger scope, from the location of demand generation to the sites of bottleneck dissipation. It captures necessary networkwide dynamics and thus helps investigate the network stability conditions.

We use the Wheatstone bridge in Fig. 3 to enhance the understandings of (21a)-(21b). First, we have $I_{e_5}(s, x) = \alpha - q_{e_1 e_3}(s, x)$, where the flow $q_{e_1 e_3}$ never enters link e_5 . If we plug $\rho_\ell \equiv 1$ into (21b), we obtain $O_{e_o}(s, x) = q_{e_o e_4}(s, x) + q_{e_1 e_3}(s, x) + q_{e_2 e_5}(s, x)$, where $q_{e_2 e_5}$ denotes the outflow of link e_o through the bottleneck and $q_{e_o e_4} + q_{e_1 e_3}$ represents the flows never entering the bottleneck. Although the physical meaning is clear given $\rho_\ell \equiv 1$, it leads to trivial stability analysis. This is because congestion spillback

is oversimplified. For example, the flows $q_{e_0 e_1}$ and $q_{e_1 e_2}$ are omitted in $O_{e_0}(s, x)$. This will happen if piecewise quadratic Lyapunov functions with $\rho_\ell \equiv 1$ are used.

A. Main results

This subsection presents two main results of stability analysis. The first stability condition emphasizes a physical intuition that any dynamic flow network is stable if the long-term net flow is negative for each density-unbounded link; the second is stronger but more abstract.

The first main result is as follows:

Theorem 1. *Consider an acyclic network satisfying Assumptions 3.1-3.3. Suppose that the network admits a demand α and an invariant set $\mathcal{X}^\mu \subseteq \prod_{e \in \mathcal{E}} [\underline{x}_e^\mu, \bar{x}_e^\mu]$ under a control law $\mu : \mathcal{S} \times \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}^P$. Then, the network is stable if*

$$\sum_{s \in \mathcal{S}} p_s \max_{x \in \mathcal{D}_e^\mu} N_e(s, x) < 0, \quad \forall e \in \mathcal{E}_{\text{inf}}, \quad (24)$$

where $\{p_s | s \in \mathcal{S}\}$ is the steady-state probability distribution of disruption modes and $\mathcal{D}_e^\mu$ is given by

$$\begin{aligned} \mathcal{D}_e^\mu &:= \{x | x_e = x_e^{c*}, x_m = \underline{x}_m^\mu, \forall m \in \mathcal{A}_e^-, \\ &\quad x_n = \bar{x}_n^\mu, \forall n \in \mathcal{E} \setminus (\mathcal{A}_e^- \cup \{e\} \cup \mathcal{B}_e^+), \\ &\quad x_\ell \in [\underline{x}_\ell^\mu, \bar{x}_\ell^\mu], \forall \ell \in \mathcal{B}_e^+\}. \end{aligned} \quad (25)$$

Theorem 1 essentially states that the network is stable if the expectation of the maximum $N_e(s, x)$ over $\mathcal{D}_e^\mu$ is negative for every link $e \in \mathcal{E}_{\text{inf}}$, where $\mathcal{D}_e^\mu$ is a refinement of $\prod_{e \in \mathcal{E}} [\underline{x}_e^\mu, \bar{x}_e^\mu]$ by using monotone network dynamics. Recalling the definition of $N_e(s, x)$ and the physical meanings of $I_e(s, x)$ and $O_e(s, x)$, we see that the lower bounds $\underline{x}_m^\mu$ and the upper bounds $\bar{x}_n^\mu$ in $\mathcal{D}_e^\mu$ make our stability verification consider as many flows as possible from the origin into link e to act as the inflow of link e . Note that x_n will take the infinite value if $\bar{x}_n^\mu = \infty$; we conclude the limit exists as $x_n \rightarrow \infty$ by noting that $N_e(s, x)$ is non-decreasing in x_n for $n \in \mathcal{E} \setminus \{\mathcal{A}_e^- \cup \{e\} \cup \mathcal{B}_e^+\}$ (see the proof in Section III-C) and that $N_e(s, x)$ is bounded.

Another observation on Theorem 1 is that we use a rectangle-like set $\prod_{e \in \mathcal{E}} [\underline{x}_e^\mu, \bar{x}_e^\mu]$, instead of $\mathcal{X}^\mu$, to simplify the computation. Though simple, such kind of set suffices to yield interesting results; see Theorems 3 and 4. In general, the invariant set $\mathcal{X}^\mu$ could be in other shapes; the stability condition can be further refined given a specific invariant set.

One can obtain a lower bound of the resiliency score by finding the supremum of those demands α that satisfy the criterion (24). It involves solving a set of maximization problems only over compact sets $\prod_{\ell \in \mathcal{B}_e^+} [\underline{x}_\ell^\mu, \bar{x}_\ell^\mu]$. Thus we are able to find the global optimal solution efficiently with searching algorithms. This is a significant refinement with respect to the general stability criteria, which essentially require search over unbounded sets [24]. Note that though the criterion (24) is a sufficient condition, it is also necessary in particular settings; see Theorems 3 and 4.

Theorem 1 is proved based on a novel Lyapunov function (32) relying on the weight function (22). We enhance it by considering more general weights for links from the sets

$\mathcal{A}_{e, \text{fin}}^-$ and $\mathcal{B}_e^+$, which is achieved by a more sophisticated Lyapunov function (41). Weighting link dynamics $G_\ell(s, x)$, $\ell \in \mathcal{A}_{e, \text{fin}}^-$, can enhance the stability analysis if there are upstream bottlenecks with congestion spillback. Note that by appropriately restricting the weights for $\mathcal{A}_{e, \text{fin}}^-$ and $\mathcal{B}_e^+$, we only need to consider $x_m = \underline{x}_m^\mu$, $m \in \mathcal{A}_{e, \text{inf}}^-$, when checking the stability, just like Theorem 1. We do not consider weighting $G_m(s, x)$, $m \in \mathcal{A}_{e, \text{inf}}^-$ for two reasons. First, link $m \in \mathcal{A}_{e, \text{inf}}^-$ has infinite storage and does not block its inflow. Second, weighting $G_m(s, x)$, $m \in \mathcal{A}_{e, \text{inf}}^-$, will destroy the monotonicity with respect to x_m , which incurs additional computational costs of the stability verification.

For $\mathcal{A}_{e, \text{fin}}^- \cup \mathcal{B}_e^+ = \{e_{i_1}, e_{i_2}, \dots, e_{i_H}\}$, we define its state vector by

$$x_e^* := [x_{e_{i_1}}, x_{e_{i_2}}, \dots, x_{e_{i_H}}]^T \in \prod_{h=1}^H \mathcal{X}_{e_{i_h}}^\mu.$$

Along with the advanced Lyapunov function (41), we let

$$\begin{aligned} N_e^*(s, x) &:= \sum_{m \in \mathcal{A}_{e, \text{inf}}^- \cup \{e\}} G_m(s, x) + 2\xi_k^T(x_e^*) B_{se} \dot{\xi}_k(x_e^*) \\ &\quad + \sum_{s' \in \mathcal{S}} \lambda_{ss'} \xi_k^T(x_e^*) (B_{s'e} - B_{se}) \xi_k(x_e^*) \end{aligned} \quad (26)$$

denote a generalization of $N_e(s, x)$, where

$$\xi_k(x_e^*) := [1, x_{e_{i_1}}, \dots, x_{e_{i_H}}, \dots, x_{e_{i_1}}^k, \dots, x_{e_{i_H}}^k]^T \in \mathbb{R}_{\geq 0}^{kH+1}$$

is a monomial basis of degree k for x_e^* ,

$$\dot{\xi}_k(x_e^*) := \frac{d}{dt} \xi_k(x_e^*(t))$$

is the derivative of $\xi_k(x_e^*)$ with respect to time t , and $B_{se} \in \mathbb{S}_+^{kH+1}$ is a symmetric positive definite matrix for $s \in \mathcal{S}$ and $e \in \mathcal{E}_{\text{inf}}$.

The second main result of this section is as follows:

Theorem 2. *Consider an acyclic network satisfying Assumptions 3.1-3.3. Suppose that the network admits a demand α and an invariant set $\mathcal{X}^\mu \subseteq \prod_{e \in \mathcal{E}} [\underline{x}_e^\mu, \bar{x}_e^\mu]$ under a control law $\mu : \mathcal{S} \times \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}^P$. Besides, there exist a set of symmetric positive definite matrices $B_{se} \in \mathbb{S}_+^{kH+1}$, $e \in \mathcal{E}_{\text{inf}}$, $s \in \mathcal{S}$, such that*

$$0 \leq 2J_{\xi_k}^T(x_e^*) B_{se} \xi_k(x_e^*) \leq 1, \quad \forall x_e^* \in \prod_{h=1}^H \mathcal{X}_{e_{i_h}}^\mu, \quad (27)$$

where

$$J_{\xi_k}(x_e^*) := \frac{d}{dx_e^*} \xi_k(x_e^*) \in \mathbb{R}^{(kH+1) \times H}$$

is a Jacobian matrix. Then, the network is stable if

$$\max_{x \in \mathcal{D}_e^{\mu*}} N_e^*(s, x) < 0, \quad \forall e \in \mathcal{E}_{\text{inf}}, s \in \mathcal{S}, \quad (28)$$

where $\mathcal{D}_e^{\mu*}$ is given by

$$\begin{aligned} \mathcal{D}_e^{\mu*} &:= \{x | x_e = x_e^{c*}, x_m = \underline{x}_m^\mu, \forall m \in \mathcal{A}_{e, \text{inf}}^-, \\ &\quad x_n = \bar{x}_n^\mu, \forall n \in \mathcal{E} \setminus (\mathcal{A}_e^- \cup \{e\} \cup \mathcal{B}_e^+), \\ &\quad x_\ell \in [\underline{x}_\ell^\mu, \bar{x}_\ell^\mu], \forall \ell \in \mathcal{A}_{e, \text{fin}}^- \cup \mathcal{B}_e^+\}. \end{aligned} \quad (29)$$

The weight constraint (27) induces each weight function of the link dynamics $G_{e_{i_h}}(s, x)$ to have a range $[0, 1]$. The non-negativity endows the weighted $G_{e_{i_h}}(s, x)$ with the correct physical meaning of net flows, and less than one makes x_m , $m \in \mathcal{A}_{e, \text{inf}}^- \subseteq \mathcal{A}_e^-$, attain the minimum value by monotone network dynamics, like Theorem 1.

The proof of Theorem 2 is based on the generalized Lyapunov function (41). Note that the extension is not unique. For example, we can further adapt the Lyapunov function if the control law is specified; see our proof of Theorem 3 in Section IV-C.

In general, Theorem 2 yields stronger criteria than Theorem 1. The throughput bounds derived from Theorem 2 can be further sharpened by increasing the degree k of the basis $\xi_k(x_e^*)$, but it requires more computational costs. We check the stability by solving the following *semi-infinite programming* (SIP [42]) with finite decision variables but infinite constraints:

$$\begin{aligned} (\text{P}_2) \quad & \min_{\gamma, B_{se}} \gamma \\ \text{s.t. } & (27), \\ & \gamma \geq N_e^*(s, x), \quad \forall (s, x) \in \mathcal{S} \times \mathcal{D}_e^{\mu*}, \forall e \in \mathcal{E}_{\text{inf}}. \end{aligned} \quad (30)$$

The programming problem has infinite constraints because (27) and (30) should hold for infinitely many states x . If the optimal $\gamma^* < 0$, we say the network is stable. Since link $\ell \in \mathcal{A}_{e, \text{fin}}^- \cup \mathcal{B}_e^+$ is density-bounded, the constraint (27) is required to hold over the compact set $\prod_{h=1}^H \mathcal{X}_{e_{i_h}}^{\mu}$ as well. Thus we can solve P_2 efficiently with the solution algorithms for SIPs, such as adaptive convexification [42].

The rest of this section is devoted to a numerical example for resiliency analysis based on the above results (Section III-B) and the proof of Theorem 1 (Section III-C) and Theorem 2 (Section III-D).

B. Numerical example

Consider the network in Fig. 3. The sending flow function of link e is given by $f_e(s, x_e) = \min\{v_f x_e, Q_{se}\}$, where $v_f > 0$ is a coefficient of free-flow speed [36] and Q_{se} is the mode-specific capacity of link e . The receiving flow functions are given by

$$r_e(x_e) = \begin{cases} \min\{w_c(x_e^{\max} - x_e), Q_e\}, & \text{if } x_e^{\max} < \infty, \\ \infty, & \text{if } x_e^{\max} = \infty, \end{cases}$$

where $w_c > 0$ is a coefficient of congestion-wave speed [36] and Q_e is the nominal capacity. In this example, we set $v_f = 1$, $w_c = 1/2$, $Q_{e_o} = Q_{e_1} = Q_{e_5} = 1$, and $Q_{e_2} = Q_{e_3} = Q_{e_4} = 1/2$. If links $e_1, \dots, e_5$ have finite storage space, we let $x_{e_1}^{\max} = x_{e_5}^{\max} = 3$, and $x_{e_2}^{\max} = x_{e_3}^{\max} = x_{e_4}^{\max} = 3/2$.

The network is subject to cyber-physical disruptions as follows. On link e_4 , the traffic state x_{e_4} may temporarily appear to be zero to the controller. On link e_5 , the sending capacity can be temporarily reduced to zero. Table I characterizes the fault mapping $T_{e_4}(s, x)$ for cyber disruptions and the mode-specific capacity Q_{se_5} for physical disruptions. The other links are not subject to disruptions; i.e. $T_e(s, x) = x_e$ for all s and all $e \neq e_4$, and $Q_{se} = Q_e$ for all s and all $e \neq e_5$. Hence, the network has at most four modes.

TABLE I: Modes of network in Figure 3.

Mode	s_0	s_1	s_2	s_3
$T_{e_4}(s, x)$	x_{e_4}	x_{e_4}	0	0
Q_{se_5}	1	0	1	0

If the network suffers the cyber and physical disruption simultaneously, the mode transition rate matrix is

$$\Lambda = \begin{bmatrix} -0.2 & 0.1 & 0.1 & 0 \\ 0.1 & -0.2 & 0 & 0.1 \\ 0.1 & 0 & -0.2 & 0.1 \\ 0 & 0.1 & 0.1 & -0.2 \end{bmatrix}.$$

The corresponding steady-state probabilities p_s can be computed for the mode $s \in \{s_0, s_1, s_2, s_3\}$: $p_{s_0} = p_{s_1} = p_{s_2} = p_{s_3} = 1/4$.

In the following we specify network dynamics. The traffic flows at the diverges are routed according to the classical logit model:

$$\begin{aligned} q_{e_o e_1}^{\text{logit}}(s, x) &= \min\left\{\frac{e^{-\nu x_{e_1}}}{e^{-\nu x_{e_1}} + e^{-\nu T_{e_4}(s, x)}} f_{e_o}(s, x_{e_o}), r_{e_1}(x_{e_1})\right\}, \\ q_{e_o e_4}^{\text{logit}}(s, x) &= \min\left\{\frac{e^{-\nu T_{e_4}(s, x)}}{e^{-\nu x_{e_1}} + e^{-\nu T_{e_4}(s, x)}} f_{e_o}(s, x_{e_o}), r_{e_4}(x_{e_4})\right\}, \\ q_{e_1 e_2}^{\text{logit}}(s, x) &= \min\left\{\frac{e^{-\nu x_{e_2}}}{e^{-\nu x_{e_2}} + e^{-\nu x_{e_3}}} f_{e_1}(s, x_{e_1}), r_{e_2}(x_{e_2})\right\}, \\ q_{e_1 e_3}^{\text{logit}}(s, x) &= \min\left\{\frac{e^{-\nu x_{e_3}}}{e^{-\nu x_{e_2}} + e^{-\nu x_{e_3}}} f_{e_1}(s, x_{e_1}), r_{e_3}(x_{e_3})\right\}. \end{aligned}$$

In this example, we select $\nu = 2$ as the sensitivity coefficient. At the merge, we assume that link e_2 is prioritized over link e_4 . Such a priority can be realized by

$$\begin{aligned} q_{e_2 e_5}^{\text{logit}}(s, x) &= \min\{f_{e_2}(s, x_{e_2}), r_{e_5}(x_{e_5})\}, \\ q_{e_4 e_5}^{\text{logit}}(s, x) &= \min\{f_{e_4}(s, x_{e_4}), r_{e_5}(x_{e_5}) - q_{e_2 e_5}^{\text{logit}}(s, x)\}. \end{aligned}$$

TABLE II: Resiliency scores in various scenarios.

Infinite space	Cyber disruptions	Physical disruptions	η_1^{logit}	η_2^{logit}	$\eta_{\text{sim}}^{\text{logit}}$
yes	no	no	1	1	1
yes	yes	no	0.783	0.984	1
yes	no	yes	0.667	0.667	0.667
yes	yes	yes	0.602	0.623	0.637
no	no	no	1	1	1
no	yes	no	0.792	0.910	0.921
no	no	yes	0.741	0.848	0.860
no	yes	yes	0.493	0.708	0.722

Table II lists resiliency scores in various scenarios. The column ‘‘Infinite space’’ indicates whether links $e_1, e_2, \dots, e_5$ have infinite storage space. Clearly, we have the nominal MCC equal to one, i.e. $C(Q_{s_0}; \mathcal{G}) = 1$. The lower bounds η_1^{logit} are derived from Theorem 1, while the bounds η_2^{logit} are obtained through Theorem 2. The details are available in the supplementary material. We also used numerical simulation to evaluate true resiliency scores, denoted by $\eta_{\text{sim}}^{\text{logit}}$. Our findings are summarized below, where the finding (iii) is a little surprising at the first glance:

- (i) It is demonstrated that Theorem 2 yields tighter lower bounds than Theorem 1. More importantly, the bounds

η_2^{logit} are close to $\eta_{\text{sim}}^{\text{logit}}$, i.e. the resiliency scores revealed by numerical simulation.

- (ii) Cyber-physical disruptions can significantly decrease the network performance. For instance, the resiliency score decreases to 0.860 when links $e_1, \dots, e_5$ only have finite space and the network suffers the physical disruption. It further declines to 0.722 when the network is also subject to the cyber disruption.
- (iii) Increasing link storage may not help improve resiliency under inappropriate localized controls. As seen in Table II, the disruptions have more significant impacts given infinite link storage space. The reason lies in that the congestion in link e_5 does not affect the upstream routing when link e_5 has infinite storage.

C. Proof of Theorem 1

The proof consists of two steps. We first prove Proposition 1, and then show that under Assumption 3.3 we only need to check over $\mathcal{D}_e^\mu$ given by (25) instead of the unbounded set $\tilde{\mathcal{D}}_e^\mu$ in Proposition 1.

Proposition 1. *Consider an acyclic network satisfying Assumptions 3.1-3.2. Suppose that the network admits a demand α and an invariant set $\mathcal{X}^\mu \subseteq \prod_{e \in \mathcal{E}} [\underline{x}_e^\mu, \bar{x}_e^\mu]$ under a control law $\mu : \mathcal{S} \times \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}^P$. Then the network is stable if*

$$\sum_{s \in \mathcal{S}} p_s \max_{x \in \tilde{\mathcal{D}}_e^\mu} N_e(s, x) < 0, \quad \forall e \in \mathcal{E}_{\text{inf}}, \quad (31)$$

where $\tilde{\mathcal{D}}_e^\mu := \{x \in \mathcal{X}^\mu | x_e \geq x_e^{c*}\}$.

We prove Proposition 1 by applying the following Foster-Lyapunov criterion adapted from Theorem 4.3 (ii) in [35]:

Foster-Lyapunov criterion. Consider a non-explosive piecewise-deterministic Markov process $\{(S(t), X(t)); t \geq 0\}$ with a state space $\mathcal{S} \times \mathcal{X}$, an infinitesimal generator $\mathcal{L}$, and a Lyapunov function $V : \mathcal{S} \times \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$. If there exist constants $c > 0$, $d < \infty$ and a function $f : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$, with $f(x) \rightarrow \infty$ as $|x| \rightarrow \infty$, such that

$$\mathcal{L}V(s, x) \leq -cf(x) + d, \quad \forall (s, x) \in \mathcal{S} \times \mathcal{X},$$

then, for each initial condition $(s, x) \in \mathcal{S} \times \mathcal{X}$,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_{\tau=0}^t \mathbb{E}[f(X(\tau)) | S(0) = s, X(0) = x] d\tau \leq d/c.$$

Consider a novel switched polynomial Lyapunov function:

$$\begin{aligned} V(s, x) := & \sum_{e \in \mathcal{E}_{\text{inf}}} a_e x_e \left[\frac{1}{2} x_e + \sum_{m \in \mathcal{A}_e^-} x_m \right. \\ & \left. + \sum_{\ell \in \mathcal{B}_e^+} \int_{\underline{x}_\ell}^{x_\ell} \rho_\ell(\zeta) d\zeta + b_{se} \right] \end{aligned} \quad (32)$$

where $a_e > 0$ and $b_{se} \geq 0$ for any $s \in \mathcal{S}$ and $e \in \mathcal{E}_{\text{inf}}$. The design of (32) is motivated by network dynamics:

- (i) For each $e \in \mathcal{E}_{\text{inf}}$, its upstream links, $\mathcal{A}_e^-$, and downstream bottlenecks, $\mathcal{B}_e^+$, are taken into account;
- (ii) The weight function (22) is introduced to better capture bottleneck dynamics. If $\rho_\ell \equiv 1$, (32) is reduced to a piecewise quadratic function.

- (iii) The parameters a_e and b_{se} are used to model the impact of mode s on link e . We will show that such a design saves finding explicit values of a_e and b_{se} .

In the following we show that if (31) holds, we must have $a_e > 0$, $b_{se} \geq 0$, $c > 0$ and $d < \infty$ such that $V(s, x)$ satisfies

$$\mathcal{L}V(s, x) \leq -c \sum_{e \in \mathcal{E}_{\text{inf}}} x_e + d, \quad \forall (s, x) \in \mathcal{S} \times \mathcal{X}^\mu. \quad (33)$$

To proceed, we apply the infinitesimal generator $\mathcal{L}$ to the Lyapunov function $V(s, x)$. By (15), we have

$$\begin{aligned} \mathcal{L}V(s, x) &= \sum_{e \in \mathcal{E}_{\text{inf}}} \left(a_e x_e \left(N_e(s, x) + \lambda_s^T (b_e - b_{se} \mathbf{1}) \right) \right. \\ &\quad \left. + a_e G_e(s, x) \left(\sum_{m \in \mathcal{A}_e^-} x_m + \sum_{\ell \in \mathcal{B}_e^+} \int_{\underline{x}_\ell}^{x_\ell} \rho(\zeta) d\zeta + b_{se} \right) \right), \end{aligned}$$

where $\mathbf{1}$ denotes the all-ones vector, $\lambda_s := [\lambda_{ss_0}, \dots, \lambda_{ss_M}]^T$ and $b_e := [b_{s_0 e}, \dots, b_{s_M e}]^T$.

Noting

$$\begin{aligned} \sum_{e \in \mathcal{E}_{\text{inf}}} G_e(s, x) \sum_{m \in \mathcal{A}_{e, \text{inf}}^-} x_m &= \sum_{\substack{e \in \mathcal{E}_{\text{inf}} \\ m \in \mathcal{E}_{\text{inf}}}} G_e(s, x) x_m \mathbb{I}_{\mathcal{A}_{e, \text{inf}}^-}(m) \\ &= \sum_{\substack{e \in \mathcal{E}_{\text{inf}} \\ n \in \mathcal{E}_{\text{inf}}}} G_n(s, x) x_e \mathbb{I}_{\mathcal{A}_{n, \text{inf}}^+}(e) \\ &= \sum_{e \in \mathcal{E}_{\text{inf}}} x_e \sum_{n \in \mathcal{A}_{e, \text{inf}}^+} G_n(s, x), \end{aligned}$$

we obtain

$$\begin{aligned} \mathcal{L}V(s, x) &= \sum_{e \in \mathcal{E}_{\text{inf}}} \left(D_e(s, x) x_e + a_e G_e(s, x) \left(\sum_{m \in \mathcal{A}_{e, \text{fin}}^-} x_m \right. \right. \\ &\quad \left. \left. + \sum_{\ell \in \mathcal{B}_e^+} \int_{\underline{x}_\ell}^{x_\ell} \rho_\ell(\zeta) d\zeta + b_{se} \right) \right), \end{aligned}$$

where $\mathbb{I}_{\mathcal{A}}(e)$ is an indicator function such that $\mathbb{I}_{\mathcal{A}}(e) = 1$ if $e \in \mathcal{A}$ and $\mathbb{I}_{\mathcal{A}}(e) = 0$ otherwise, and

$$\begin{aligned} D_e(s, x) &:= a_e \left(N_e(s, x) + \lambda_s^T (b_e - b_{se} \mathbf{1}) \right) + \sum_{n \in \mathcal{A}_{e, \text{inf}}^+} a_n G_n(s, x). \end{aligned}$$

It follows $\mathcal{L}V(s, x) \leq \sum_{e \in \mathcal{E}_{\text{inf}}} D_e(s, x) x_e + d_0$ with $d_0 < \infty$. The existence of d_0 is guaranteed by the boundedness of x_m for $m \in \mathcal{A}_{e, \text{fin}}^-$, of x_ℓ for $\ell \in \mathcal{B}_e^+$ and of $G_e(s, x)$ for $e \in \mathcal{E}$.

Next, we show, if (31) holds, there exist $a_e > 0$, $b_{se} \geq 0$, $c_e > 0$ and $d_e < \infty$ such that

$$D_e(s, x) x_e \leq -c_e x_e + d_e, \quad \forall (s, x) \in \mathcal{S} \times \mathcal{X}^\mu, \forall e \in \mathcal{E}_{\text{inf}}, \quad (34)$$

which implies that (33) is satisfied with $c := \min_{e \in \mathcal{E}_{\text{inf}}} c_e$ and $d := \sum_{e \in \mathcal{E}_{\text{inf}}} d_e + d_0$.

For any $e \in \mathcal{E}_{\text{inf}}$, we must have $d_{e,1} < \infty$ such that

$$\max_{(s, x) \in \mathcal{S} \times (\mathcal{X}^\mu \setminus \tilde{\mathcal{D}}_e^\mu)} D_e(s, x) x_e \leq d_{e,1}. \quad (35)$$

Besides, for any $(s, x) \in \mathcal{S} \times \tilde{\mathcal{D}}_e^\mu$,

$$\begin{aligned} D_e(s, x) &\leq a_e[\bar{N}_{se} + \lambda_s^\top(b_e - b_{se}\mathbf{1})] + \bar{G} \sum_{n \in \mathcal{A}_{e,\text{inf}}^+} a_n \\ &\leq a_e \sum_{s' \in \mathcal{S}} p_{s'} \bar{N}_{s'e} + \bar{G} \sum_{n \in \mathcal{A}_{e,\text{inf}}^+} a_n \end{aligned} \quad (36)$$

where $\bar{N}_{se} := \max_{x \in \tilde{\mathcal{D}}_e^\mu} N_e(s, x)$ and $\bar{G}$ denotes an upper bound of $G_e(s, x)$ for $e \in \mathcal{E}$. The last step is derived from the following lemma, which we prove in Appendix C.

Lemma 1. *The system of equations*

$$z_s + \sum_{s' \in \mathcal{S}} \lambda_{ss'}(y_{s'} - y_s) = \sum_{s' \in \mathcal{S}} p_{s'} z_{s'}, \quad s \in \mathcal{S} \quad (37)$$

has a non-negative solution for $y = [y_{s_0}, \dots, y_{s_M}]^\top$, where p_s , $\lambda_{ss'}$ and z_s are given, and p_s and $\lambda_{ss'}$ satisfy (12).

Given (31) and (36), Lemma 2 proved in Appendix D tells that there exists a_e for $e \in \mathcal{E}_{\text{inf}}$ such that $D_e(s, x) < 0$, $\forall (s, x) \in \mathcal{S} \times \tilde{\mathcal{D}}_e^\mu, \forall e \in \mathcal{E}_{\text{inf}}$. Thus we must have $c_e > 0$ for each $e \in \mathcal{E}_{\text{inf}}$ such that

$$D_e(s, x)x_e \leq -c_e x_e, \quad \forall (s, x) \in \mathcal{S} \times \tilde{\mathcal{D}}_e^\mu, \forall e \in \mathcal{E}_{\text{inf}}. \quad (38)$$

Lemma 2. *The system of equations*

$$\pi_e a_e + \bar{G} \sum_{n \in \mathcal{A}_{e,\text{inf}}^+} a_n < 0, \quad \forall e \in \mathcal{E}_{\text{inf}} \quad (39)$$

has a positive solution for $\{a_e | e \in \mathcal{E}_{\text{inf}}\}$ if $\pi_e < 0$ for any $e \in \mathcal{E}_{\text{inf}}$.

Combining (35) and (38) yields (34) with $d_e = d_{e,1} + c_e x_e^{c*}$. It indicates that we finish the proof of Proposition 1.

Noting $\mathcal{X}^\mu \subseteq \prod_{e \in \mathcal{E}} [\underline{x}_e^\mu, \bar{x}_e^\mu]$, we must have

$$\max_{x \in \prod_{e \in \mathcal{E}} [\underline{x}_e^\mu, \bar{x}_e^\mu], x_e \geq x_e^{c*}} N_e(s, x) \geq \max_{x \in \tilde{\mathcal{D}}_e^\mu} N_e(s, x).$$

Then, the rest of the proof is devoted to showing that

$$\max_{x \in \tilde{\mathcal{D}}_e^\mu} N_e(s, x) = \max_{x \in \prod_{e \in \mathcal{E}} [\underline{x}_e^\mu, \bar{x}_e^\mu], x_e \geq x_e^{c*}} N_e(s, x) \quad (40)$$

under Assumption 3.3. Using (40), we can derive (31) from (24), which finishes the proof.

The definition of localized control laws indicates

$$\max_{x_{e'}} N_e(s, x) = \max_{x_{e'}} G_{e,\tau_{e'}}(s, x) + G_{e,\sigma_{e'}}(s, x), \quad \forall e' \in \mathcal{E},$$

where $G_{e,\tau_{e'}}(s, x)$ and $G_{e,\sigma_{e'}}(s, x)$ denote the projections of $N_e(s, x)$ onto nodes $\tau_{e'}$ and $\sigma_{e'}$, respectively. The projection onto an arbitrary node $v \in \mathcal{V}$ is given by

$$\begin{aligned} G_{e,v}(s, x) &:= \sum_{i \in \mathcal{E}_v^-, j_1 \in \mathcal{E}_{e,v}^{1+}} q_{ij_1}(s, x) + \sum_{i \in \mathcal{E}_v^-, j_1 \in \mathcal{E}_{e,v}^{3+}} \rho_{j_3}(x_{j_3}) q_{ij_3}(s, x) \\ &\quad - \sum_{i_1 \in \mathcal{E}_{e,v}^{1-}, j \in \mathcal{E}_v^+} q_{i_1 j}(s, x) - \sum_{i_3 \in \mathcal{E}_{e,v}^{3-}, j \in \mathcal{E}_v^+} \rho_{i_3}(x_{i_3}) q_{ij_3}(s, x), \end{aligned}$$

where $\rho_{j_3}(x_{j_3})$ and $\rho_{i_3}(x_{i_3})$ are specified by (22), and the link sets $\mathcal{E}_{e,v}^{1+}$, $\mathcal{E}_{e,v}^{3+}$, $\mathcal{E}_{e,v}^{1-}$ and $\mathcal{E}_{e,v}^{3-}$ are illustrated in Fig. 4(a). Note that these sets could be empty. We also use the example

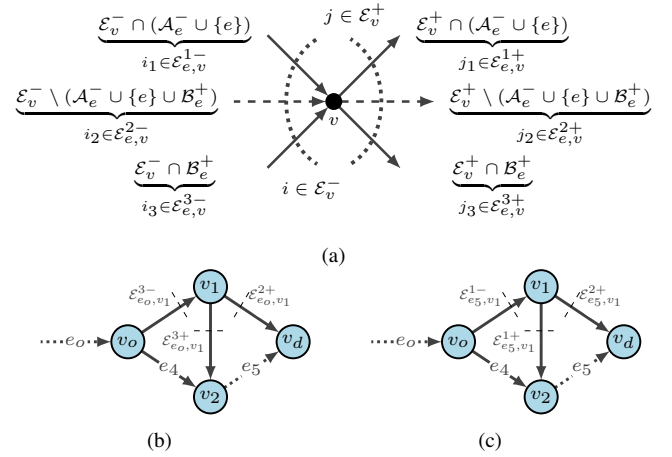


Fig. 4: Classification of incoming links $\mathcal{E}_v^-$ and outgoing links $\mathcal{E}_v^+$ for $e \in \mathcal{E}_{\text{inf}}$ and $v \in \mathcal{V}$: (a) a general case, (b) classification for link e_o and node v_1 , given $\mathcal{E}_{v_1}^- = \{e_1\}$, $\mathcal{E}_{v_1}^+ = \{e_2, e_3\}$, $\mathcal{A}_{e_o}^- = \emptyset$ and $\mathcal{B}_{e_o}^+ = \{e_1, e_2\}$, (c) classification for link e_5 and node v_1 , given $\mathcal{E}_{v_1}^- = \{e_1\}$, $\mathcal{E}_{v_1}^+ = \{e_2, e_3\}$, $\mathcal{A}_{e_5}^- = \{e_o, e_1, e_2, e_4\}$ and $\mathcal{B}_{e_5}^+ = \emptyset$.

in Fig. 3 to present two specific cases in Figs. 4(b) and (c), respectively.

The following first shows $N_e(s, x)$ is non-decreasing in x_n for $n \in \mathcal{E} \setminus (\mathcal{A}_e^- \cup \{e\} \cup \mathcal{B}_e^+)$ by proving $G_{e,\tau_n}(s, x) + G_{e,\sigma_n}(s, x)$ is non-decreasing in x_n . By Assumption 3.3, the incoming flow $\sum_{i \in \mathcal{E}_v^-} q_{ij}(s, x)$ into link $j \in \mathcal{E}_{e,v}^{1+} \cup \mathcal{E}_{e,v}^{3+}$ is non-decreasing in x_{j_2} ; the outgoing flow $\sum_{j \in \mathcal{E}_v^+} q_{ij}(s, x)$ from link $i \in \mathcal{E}_{e,v}^{1-} \cup \mathcal{E}_{e,v}^{3-}$ is non-increasing in x_{j_2} . Thus $G_{e,v}(s, x)$ is non-decreasing in x_{j_2} . Similarly, it is easy to verify that $G_{e,v}(s, x)$ is non-decreasing in x_{i_2} . This implies that both $G_{e,\sigma_n}(s, x)$ and $G_{e,\tau_n}(s, x)$ are non-decreasing in x_n for any $n \in \mathcal{E} \setminus (\mathcal{A}_e^- \cup \{e\} \cup \mathcal{B}_e^+)$.

Now we show $N_e(s, x)$ is non-increasing in x_m for $m \in \mathcal{A}_e^- \cup \{e\}$ by proving $G_{e,\tau_m}(s, x) + G_{e,\sigma_m}(s, x)$ is non-increasing in x_m . Noting $\mathcal{E}_{e,v}^{1-} \cup \mathcal{E}_{e,v}^{2-} \cup \mathcal{E}_{e,v}^{3-} = \mathcal{E}_v^-$ and $\mathcal{E}_{e,v}^{1+} \cup \mathcal{E}_{e,v}^{2+} \cup \mathcal{E}_{e,v}^{3+} = \mathcal{E}_v^+$ from Fig. 4, we rewrite $G_{e,v}(s, x)$ as

$$\begin{aligned} G_{e,v}(s, x) &= \sum_{i_2 \in \mathcal{E}_{e,v}^{2-}, j \in \mathcal{E}_v^+} q_{ij_2}(s, x) + \sum_{i_3 \in \mathcal{E}_{e,v}^{3-}, j \in \mathcal{E}_v^+} (1 - \rho_{i_3}(x_{i_3})) q_{ij_3}(s, x) \\ &\quad - \sum_{i \in \mathcal{E}_v^-, j_2 \in \mathcal{E}_{e,v}^{2+}} q_{ij_2}(s, x) - \sum_{i \in \mathcal{E}_v^-, j_3 \in \mathcal{E}_{e,v}^{3+}} (1 - \rho_{j_3}(x_{j_3})) q_{ij_3}(s, x). \end{aligned}$$

By Assumption 3.3, the incoming flow $\sum_{i \in \mathcal{E}_v^-} q_{ij}(s, x)$ into link $j \in \mathcal{E}_{e,v}^{2+} \cup \mathcal{E}_{e,v}^{3+}$ is non-decreasing in x_{j_1} ; the outgoing flow $\sum_{j \in \mathcal{E}_v^+} q_{ij}(s, x)$ from link $i \in \mathcal{E}_{e,v}^{2-} \cup \mathcal{E}_{e,v}^{3-}$ is non-increasing in x_{j_1} . Thus $G_{e,v}(s, x)$ is non-increasing in x_{j_1} . Similarly, we can show $G_{e,v}(s, x)$ is non-increasing in x_{i_1} . Thus we conclude both $G_{e,\sigma_m}(s, x)$ and $G_{e,\tau_m}(s, x)$ are non-increasing in x_m for any $m \in \mathcal{A}_e^- \cup \{e\}$.

Thus we conclude (40) by the monotonicity of $N_e(s, x)$, which completes the proof. $\square$

D. Proof of Theorem 2

The proof is similar to that of Theorem 1. The major difference lies in the following Lyapunov function:

$$V^*(s, x) := \sum_{e \in \mathcal{E}_{\text{inf}}} a_e x_e \left[\frac{1}{2} x_e + \sum_{m \in \mathcal{A}_{e, \text{inf}}^-} x_m + \xi_k^T(x_e^*) B_{se} \xi_k(x_e^*) \right]. \quad (41)$$

Applying the infinitesimal generator yields

$$\mathcal{L}V^*(s, x) \leq \sum_{e \in \mathcal{E}_{\text{inf}}} [D_e^*(s, x) x_e + d_0],$$

where $D_e^*(s, x) := a_e N_e^*(s, x) + \sum_{n \in \mathcal{A}_{e, \text{inf}}^+} a_n G_n(s, x)$. Then we obtain the following result similar to Proposition 1:

Proposition 2. Consider an acyclic network satisfying Assumptions 3.1-3.2. Suppose that the network admits a demand α and an invariant set $\mathcal{X}^\mu \subseteq \prod_{e \in \mathcal{E}} [\underline{x}_e^\mu, \bar{x}_e^\mu]$ under a control law $\mu : \mathcal{S} \times \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}^P$. Then the network is stable if

$$\max_{x \in \tilde{\mathcal{D}}_e^\mu} N_e^*(s, x) < 0, \quad \forall e \in \mathcal{E}_{\text{inf}}, \quad \forall s \in \mathcal{S}, \quad (42)$$

where $\tilde{\mathcal{D}}_e^\mu := \{x \in \mathcal{X}^\mu \mid x_e \geq x_e^{c*}\}$.

Because of $\mathcal{X}^\mu \subseteq \prod_{e \in \mathcal{E}} [\underline{x}_e^\mu, \bar{x}_e^\mu]$, we must have

$$\max_{x \in \prod_{e \in \mathcal{E}} [\underline{x}_e^\mu, \bar{x}_e^\mu], x_e \geq x_e^{c*}} N_e^*(s, x) \geq \max_{x \in \tilde{\mathcal{D}}_e^\mu} N_e^*(s, x).$$

Next, we only need to prove

$$\max_{x \in \tilde{\mathcal{D}}_e^\mu} N_e^*(s, x) = \max_{x \in \prod_{e \in \mathcal{E}} [\underline{x}_e^\mu, \bar{x}_e^\mu], x_e \geq x_e^{c*}} N_e^*(s, x). \quad (43)$$

For $N_e^*(s, x)$ given by (26), we notice

$$2\xi_k^T(x_e^*) B_{se} \dot{\xi}_k(x_e^*) = \underbrace{2\xi_k^T(x_e^*) B_{se} J_{\xi_k}(x_e^*)}_{[\rho_{e_{i_1}}^*(x), \rho_{e_{i_2}}^*(x), \dots, \rho_{e_{i_H}}^*(x)]} \begin{bmatrix} G_{e_{i_1}}(s, x) \\ G_{e_{i_2}}(s, x) \\ \vdots \\ G_{e_{i_H}}(s, x) \end{bmatrix}.$$

Clearly, (27) indicates $\rho_{e_{i_h}}^*(x) \in [0, 1]$ for $h = 1, 2, \dots, H$. We see $\rho_{e_{i_h}}^*(x)$ as a generalization of $\rho_{e_{i_h}}(x_{e_{i_h}})$ given by (22). Then we consider the link classification as shown in Fig. 5.

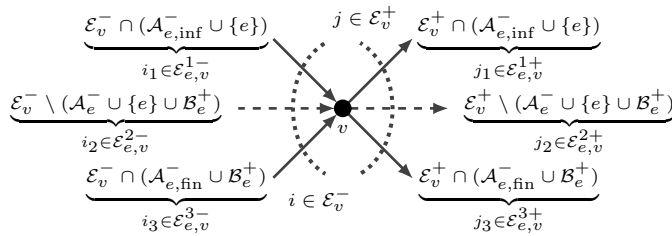


Fig. 5: Classification of incoming links $\mathcal{E}_v^-$ and outgoing links $\mathcal{E}_v^+$ $e \in \mathcal{E}_{\text{inf}}$ and $v \in \mathcal{V}$ in Theorem 2.

We can show the monotonicity of $N_e^*(s, x)$ with respect to i) x_n , $n \in \mathcal{E} \setminus (\mathcal{A}_e^- \cup \{e\} \cup \mathcal{B}_e^+)$, and ii) x_m , $m \in \mathcal{A}_{e, \text{inf}}^- \cup \{e\}$, in a similar way to that in Section III-C, and thus prove (43). $\square$

IV. RESILIENT CONTROL DESIGN

In this section, we study control design with guaranteed throughput. We will extend the classical notion of MCC to stochastic networks and present a series of results characterizing the resiliency under various classes of control laws.

Two extensions of the MCC, i.e. the MECC and the EMCC, are considered in the stochastic setting. The MECC is the minimum cut capacity evaluated with the expected capacity of each link, and the EMCC is the expected minimum cut capacity over various modes. Noting that $C(\mathcal{Q}; \mathcal{G})$ denotes the MCC of network $\mathcal{G}$ with a set of link capacities $\mathcal{Q}$, we have

$$\text{MECC} := C(\bar{\mathcal{Q}}; \mathcal{G}), \quad \text{EMCC} := \sum_{s \in \mathcal{S}} p_s C(\mathcal{Q}_s; \mathcal{G}),$$

where $\bar{\mathcal{Q}} := \{\sum_{s \in \mathcal{S}} p_s Q_{se} \mid e \in \mathcal{E}\}$ and $\mathcal{Q}_s := \{Q_{se} \mid e \in \mathcal{E}\}$. In general, the MECC and the EMCC are not equal. Noting that the MCC is concave with respect to link capacities [18], one can show that the MECC is never less than the EMCC by Jensen's inequality. For instance, consider the numerical example in Section III-B; it is shown that the EMCC equals 0.75 and MECC equals 1 in Fig. 6.

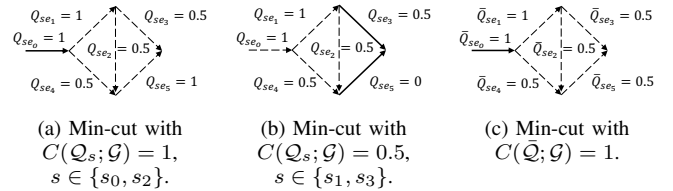


Fig. 6: EMCC and MECC; the link capacities are labeled, and the min-cuts are highlighted with solid links: a cut is a subset of links whose removal divides the node set into two disconnected subsets and min-cuts are those with the minimum sum of link capacities, i.e., the min-cut capacity.

A. Main results

The following investigates in which case there exist controls that attain the EMCC/MECC. We start from the simplest open-loop control. It is shown that if every link $e \in \mathcal{E}$ has infinite storage space without restricted receiving flows, there exists an open-loop control μ^{ol} that guarantees the MECC. The control design is inspired by the solution of the maximum flow problem (P_1). Before presenting it, we denote by $u^*(\bar{\mathcal{Q}}) := \{u_{ej}^*(\bar{\mathcal{Q}}) \mid (e, j) \in \mathcal{P}\}$ the solution of (P_1) given the set of link capacities $\bar{\mathcal{Q}}$ and define

$$\bar{\gamma}_{ej} := \frac{u_{ej}^*(\bar{\mathcal{Q}})}{\sum_{j' \in \mathcal{E}_e^+} u_{ej'}^*(\bar{\mathcal{Q}})}, \quad (e, j) \in \mathcal{P}. \quad (44)$$

Clearly, $\bar{\gamma}_{ej}$ denotes the proportion of the outflow of link e routed to link j per the solution $u^*(\bar{\mathcal{Q}})$.

Then the open-loop control is given by

$$\mu_{ej}^{\text{ol}} = \bar{\gamma}_{ej} Q_e^{\max}, \quad (e, j) \in \mathcal{P}, \quad (45)$$

where $Q_e^{\max} := \max_{s \in \mathcal{S}} Q_{se}$. The formal result of the open-loop control is stated below:

Theorem 3 (Max-flow min-expected-cut). Consider an acyclic dynamic flow network with $x_e^{\max} = \infty$ and $r_e(s, x_e) =$

∞ for any link $e \in \mathcal{E}$. There exists an open-loop control μ^{ol} given by (45) such that the network throughput equals the min-expected-cut capacity.

However, few links in practice have infinite storage, which indicates the MECC is generally unattainable. For networks possibly with finite storage space, we consider mode-dependent control given observable disruption mode $S(t)$. Suppose a network with the following physical disruptions:

$$f_e(s, x_e) = \min\{f_e(x_e), F_{se}\}, \quad (46a)$$

$$r_e(s, x_e) = \min\{r_e(x_e), R_{se}\}. \quad (46b)$$

These disruptions restrict maximal outflows or inflows, which satisfy Assumption 3.2. For the sake of simplicity of analysis, we also assume the critical density x_e^c given by (9) is finite for every $e \in \mathcal{E}$. If any disruption mode $s \in \mathcal{S}$ can be correctly observed, we design a mode-dependent control as follows:

$$\mu_{ej}^{\text{md}}(s) := u_{ej}^*(Q_s), \quad (e, j) \in \mathcal{P}, \quad (47)$$

where $u_{ej}^*(Q_s)$ is an optimal solution to the maximum flow problem (P_1) with the set of link capacities Q_s . Then we state the following theorem:

Theorem 4 (Max-flow expected-min-cut). *Consider an acyclic dynamic flow network with a set of observable disruptions (46a)-(46b). There exists a mode-dependent control μ^{md} given by (47) such that the network throughput equals the expected-min-cut capacity.*

Note that EMCC is less than the MECC in general. Hence the mode-dependent control above may appear conservative, provided a gap between the EMCC and the MECC. Besides, the mode-dependent control requires quick detection of disruptions and informs local controllers of disruption occurrences. If some of controllers are unaware of changes of disruption modes, Theorem 4 does not hold and the EMCC may not be guaranteed. In that case, we can apply Theorems 1 and 2 to the throughput analysis by modeling actuation faults.

By contrast, it is more practical to measure the network state $X(t)$. Thus we consider density-dependent control using the network state feedback. For a general network (with possibly incorrect sensing or with finite link storage spaces), one can apply the results of our resiliency analysis to design closed-loop controls. Suppose the control law is parameterized by θ . Then the parameter θ is designed by solving the following programming problem:

$$(P_3) \max_{\alpha \geq 0, \theta \in \Theta} \alpha \quad \text{s.t.} \quad (24).$$

Let θ^* and α^* denote optimal solutions of the programming problem P_3 . Then it follows from Theorem 1 that the throughput under the control law parameterized by θ^* is lower bounded by α^* :

Corollary 1 (Throughput-guaranteed control). *Consider an acyclic dynamic flow network with a density-dependent control μ^{dd} deriving from P_3 . Then, μ^{dd} can attain a throughput lower-bounded by the optimal α^* .*

To facilitate the control design, we can restrict the control structure $\theta \in \Theta$ so that $\mathcal{E}_{\text{inf}}$ and $\{\mathcal{B}_e^+ | e \in \mathcal{E}_{\text{inf}}\}$ are fixed.

In practice, the determination of these sets may depend on infrastructure capability or control preferences, e.g., whether link e has sufficient storage space, or whether link e is allowed to be a bottleneck.

The rest of this section first illustrates the above results with a numerical example (Section IV-B) and then proves them respectively (Sections IV-C-IV-D).

B. Numerical example

Again, consider the network in Fig. 3. We use Theorems 3, 4 and Corollary 1 to design open-loop control μ^{ol} , mode-dependent control μ^{md} and density-dependent control μ^{dd} , respectively. Table III lists the resiliency scores in various scenarios. Note the nominal MCC equals one. Our discussions are as follows:

- (i) It is demonstrated that the open-loop control can attain the MECC when the links have infinite storage space. Besides, the results also show that the MECC is not guaranteed when the link storage space becomes finite.
- (ii) It is verified that the mode-dependent control can attain the EMCC. Although this control seems conservative, it is still able to outperform the logit routing in case of links with infinite storage space; see Table II.
- (iii) Compared with the open-loop control, density-dependent control can achieve higher resiliency scores when dealing with physical disruptions. However, it is vulnerable to cyber disruptions.
- (iv) The results indicate that we can improve network resiliency by designing control laws with higher lower bounds of the resiliency scores.

Next, we present the design of three controls.

1) Open-loop control: Solving the maximum flow problem (P_1) with the expected link capacities gives $u_{e_0e_1}^* = u_{e_1e_3}^* = u_{e_0e_4}^* = u_{e_4e_5}^* = 0.5$ and $u_{e_1e_2}^* = 0$. According to (45)-(44), we have $\mu_{e_0e_1}^{\text{ol}} = \mu_{e_1e_3}^{\text{ol}} = \mu_{e_0e_4}^{\text{ol}} = \mu_{e_4e_5}^{\text{ol}} = 0.5$ and $\mu_{e_1e_2}^{\text{ol}} = 0$.

2) Mode-dependent control: The control μ^{md} is obtained from solving the maximum flow problem (P_1) for each disruption mode s . Then we have $\mu_{e_0e_1}^{\text{md}} = \mu_{e_1e_3}^{\text{md}} = 0.5$, $\mu_{e_1e_2}^{\text{md}} = 0$ and

$$\mu_{e_0e_4}^{\text{md}}(s) = \mu_{e_4e_5}^{\text{md}}(s) = \begin{cases} 0.5, & \text{if } s \in \{s_0, s_2\}, \\ 0, & \text{if } s \in \{s_1, s_3\}. \end{cases}$$

3) Density-dependent control: We consider the density-dependent routing at node v_o , namely $\mu_{e_0e_1}^{\text{dd}}(s, x) = w_c(u_1 - x_1)$ and $\mu_{e_0e_4}^{\text{dd}}(s, x) = w_c(u_4 - T_{e_4}(s, x))$, where the gains are set as the congestion wave speed w_c , the parameters u_1 and u_4 are to be designed, and $T_{e_4}(s, x)$ is the fault mapping given in Table I. We interpret u_1 (resp. u_4) as the maximum density allowed by the control for link e_1 (resp. e_4). We also let $\mu_{e_1e_3}^{\text{dd}}(s, x) = f_{e_1}(s, x_{e_1})$ and $\mu_{e_1e_2}^{\text{dd}}(s, x) = 0$.

When all links $e_0, e_1, \dots, e_5$ have infinite storage space, solving P_3 gives $u_1 = u_4 = \infty$. It means that the density-dependent control is reduced to an open-loop control that routes flows into links e_1 and e_4 with the same proportion. For links $e_1, \dots, e_5$ only with finite storage space, our method yields $u_1 = u_4 = 1.5$ for the network subject to either the cyber or physical disruption and $u_1 = 2.3, u_4 = 1.5$ in case

TABLE III: Comparison of resiliency scores of different control laws.

Infinite space	Cyber disruptions	Physical disruptions	Open-loop			Mode-dependent			Density-dependent		
			η_1^{ol}	η_2^{ol}	$\eta_{\text{sim}}^{\text{ol}}$	η_1^{md}	η_2^{md}	$\eta_{\text{sim}}^{\text{md}}$	η_1^{dd}	η_2^{dd}	$\eta_{\text{sim}}^{\text{dd}}$
yes	no	no	1	1	1	1	1	1	1	1	1
yes	yes	no	1	1	1	1	1	1	1	1	1
yes	no	yes	1	1	1	0.750	0.750	0.750	1	1	1
yes	yes	yes	1	1	1	0.750	0.750	0.750	1	1	1
no	no	no	1	1	1	1	1	1	1	1	1
no	yes	no	1	1	1	1	1	1	1	1	1
no	no	yes	0.747	0.850	0.859	0.750	0.750	0.750	0.746	0.867	0.873
no	yes	yes	0.747	0.850	0.859	0.750	0.750	0.750	0.739	0.851	0.863

of both the cyber and physical disruptions. More details can be found in the supplementary material.

C. Proof of Theorem 3

1) *Sufficiency*: The proof is similar to that of Theorem 1. Noting $r_e(s, x_e) = \infty$ for any $e \in \mathcal{E}$, we obtain $\mathcal{B}_e^+ = \emptyset$ for $e \in \mathcal{E}_{\text{inf}}$. Consider the following Lyapunov function:

$$V^{\text{ol}}(s, x) := \sum_{e \in \mathcal{E}_{\text{inf}}} a_e x_e \left[\frac{1}{2} x_e + \sum_{m \in \mathcal{A}_e^-} \bar{\gamma}_{me} x_m + b_{se} \right],$$

where $\bar{\gamma}_{me}$ is recursively defined by

$$\bar{\gamma}_{me} = \sum_{i \in \mathcal{A}_m^+ \cap \mathcal{E}_e^-} \bar{\gamma}_{mi} \bar{\gamma}_{ie} \quad (48)$$

with $\bar{\gamma}_{ie}$ given by (44). Noting the physical meaning of $\bar{\gamma}_{ie}$, we interpret $\bar{\gamma}_{me}$ as the proportion of the outflow of link m routed to link e per the solution $u^*(\bar{\mathcal{Q}})$. Specially, for link e_o with a demand $\alpha = C(\bar{\mathcal{Q}}; \mathcal{G})$, we have

$$\bar{\gamma}_{e_o e} C(\bar{\mathcal{Q}}; \mathcal{G}) = \sum_{i \in \mathcal{E}_j^-} u_{ie}^*(\bar{\mathcal{Q}}). \quad (49)$$

The above equation holds because, as specified by the solution $u^*(\bar{\mathcal{Q}})$, the optimal demand α^* equals $C(\bar{\mathcal{Q}}; \mathcal{G})$ and the flow routed from link e_o to link e equals $\sum_{i \in \mathcal{E}_j^-} u_{ie}^*(\bar{\mathcal{Q}})$.

The Lyapunov function indicates that we only need to show

$$\sum_{s \in \mathcal{S}} p_s \max_{x \in \mathcal{D}_e^{\text{ol}}} N_e(s, x) < 0, \quad \forall \alpha < C(\bar{\mathcal{Q}}; \mathcal{G}), \forall e \in \mathcal{E}_{\text{inf}}, \quad (50)$$

where $\mathcal{D}_e^{\text{ol}} = \{x \in \mathcal{X} | x_e \geq x_e^*\}$ and $N_e(s, x) = \sum_{m \in \mathcal{A}_e^-} \bar{\gamma}_{me} G_m(s, x) + G_e(s, x)$.

The open-loop control yields $q_{mj}^{\text{ol}}(s, x) = \bar{\gamma}_{mj} f_m(s, x_m)$. Thus, for any link $m \in \mathcal{A}_e^-$, we have

$$\begin{aligned} & - \sum_{j \in \mathcal{E}_m^+} \bar{\gamma}_{me} q_{mj}^{\text{ol}}(s, x) + \sum_{j' \in \mathcal{E}_m^+ \cap \mathcal{A}_e^-} \bar{\gamma}_{j'e} q_{mj'}^{\text{ol}}(s, x) \\ &= - \sum_{j \in \mathcal{E}_m^+} \bar{\gamma}_{me} \bar{\gamma}_{mj} f_m(s, x_m) + \sum_{j' \in \mathcal{E}_m^+ \cap \mathcal{A}_e^-} \bar{\gamma}_{j'e} \bar{\gamma}_{mj'} f_m(s, x_m) \\ &\stackrel{(44)}{=} - \bar{\gamma}_{me} f_m(s, x_m) + \bar{\gamma}_{me} f_m(s, x_m) = 0. \end{aligned}$$

So we obtain $N_e(s, x) = \bar{\gamma}_{e_o e} \alpha - f_e(s, x_e)$ and

$$\sum_{s \in \mathcal{S}} p_s \max_{x \in \mathcal{D}_e^{\text{ol}}} N_e(s, x) = \bar{\gamma}_{e_o e} \alpha - \sum_{s \in \mathcal{S}} p_s Q_{se}, \quad \forall e \in \mathcal{E}_{\text{inf}}. \quad (51)$$

Recalling (49), we arrive at

$$\bar{\gamma}_{e_o e} C(\bar{\mathcal{Q}}; \mathcal{G}) = \sum_{i \in \mathcal{E}_e^-} u_{ie}^*(\bar{\mathcal{Q}}) \leq \sum_{s \in \mathcal{S}} p_s Q_{se} \quad (52)$$

where the last inequality holds because of the constraint in the maximum flow problem. Combining (51) and (52), we complete the proof of (50). $\square$

2) *Necessity*: We prove the necessity by definition. Consider the cut-set $\mathcal{C}$ with the min-cut capacity $C(\bar{\mathcal{Q}}; \mathcal{G})$, where $\mathcal{C}$ is a set of links that divide the network into two disjoint parts containing the origin v_o and the destination v_d respectively. Let $\mathcal{E}_{\mathcal{C}}^-$ denote the set of links upstream of the cut-set. By the open-loop control, the outflow from $\mathcal{E}_{\mathcal{C}}^- \cup \mathcal{C}$ does not exceed $C(\bar{\mathcal{Q}}; \mathcal{G})$. If the inflow α is not less than $C(\bar{\mathcal{Q}}; \mathcal{G})$ and $Z < \infty$ is given, there must exist some initial condition such that (16) fails to hold. It concludes the proof. $\square$

D. Proof of Theorem 4

1) *Sufficiency*: The proof is based on Theorem 1. Noting Lemma 3 proved in Appendix E, we only need to show

$$\sum_{s \in \mathcal{S}} p_s \max_{x \in \mathcal{D}_{e_o}^{\text{md}}} (\alpha - \sum_{j \in \mathcal{E}_{e_o}^+} q_{e_o j}^{\text{md}}(s, x)) < 0, \quad \forall \alpha < \sum_{s \in \mathcal{S}} p_s C(\mathcal{Q}_s; \mathcal{G}). \quad (53)$$

Lemma 3. *The mode-dependent control μ^{md} admits an invariant set*

$$\mathcal{X}^{\text{md}} = [\underline{x}_{e_o}^{\text{md}}, \infty) \times \prod_{e \neq e_o} [\underline{x}_e^{\text{md}}, \bar{x}_e^{\text{md}}] = [\alpha, \infty) \times \prod_{e \neq e_o} [0, x_e^{c*}],$$

which leads to $\mathcal{E}_{\text{inf}} = \{e_o\}$ and $\mathcal{B}_{e_o}^+ = \emptyset$.

Then it follows

$$\begin{aligned} & \sum_{s \in \mathcal{S}} p_s \max_{x \in \mathcal{D}_{e_o}^{\text{md}}} (\alpha - \sum_{j \in \mathcal{E}_{e_o}^+} q_{e_o j}^{\text{md}}(s, x)) \\ &= \sum_{s \in \mathcal{S}} p_s (\alpha - \sum_{j \in \mathcal{E}_{e_o}^+} \mu_{e_o j}^{\text{md}}(s)), \end{aligned}$$

where the equality is due to i) $\mathcal{B}_{e_o}^+ = \emptyset$, ii) $f_{e_o}(s, x_{e_o}) \geq Q_{se_o}$ for $x_{e_o} \geq x_{e_o}^{c*}$ and iii) the constraint (19d). By the constraint (19a), we obtain $\sum_{j \in \mathcal{E}_{e_o}^+} \mu_{e_o j}^{\text{md}}(s) = \alpha^*(\mathcal{Q}_s; \mathcal{G}) = C(\mathcal{Q}_s; \mathcal{G})$, which completes the proof of (53). $\square$

2) *Necessity*: We consider the set $\{e_o\}$. Then the remaining proof is similar to that for the necessity of Theorem 3. $\square$

V. CONCLUDING REMARKS

This paper investigates resiliency and control design of dynamic flow networks suffering cyber-physical disruptions. First, we apply piecewise-deterministic Markov process to modeling disruptions and their impacts on networks. Then, a

set of stability criteria are proposed with the physical insights into flow networks. We also simplify these criteria with the monotone network dynamics. The stability conditions enable the measurement of resiliency in terms of throughput. They also contribute to two specific resiliency-by-design controls. The first one attains the min-expected-cut capacity if every link has infinite storage space; the second one attains the expected-min-cut capacity if the disruptions decrease maximum sending and/or receiving flows. In the general case, we propose a density-dependent control with a lower-bounded throughput. Our numerical examples show that the stability condition can yield tight lower bounds of resiliency scores and enhance resiliency by designing control laws that raise the lower bounds.

This work can serve as a basis for multiple future studies. First, our sufficient stability conditions only yield lower bounds of throughput. It is tempting to derive upper bounds by investigating the necessary conditions. Besides, this paper exploits the property of monotone flow networks to reduce the computation costs of verifying our stability conditions. However, monotone dynamics could be unavailable in multi-commodity networks [38]. It is worthwhile considering networks with weaker properties, such as mixed monotonicity [19]. Finally, this paper formulates the control design problem for the whole network. Solving it for large-scale networks could not be easy. Thus how to design control laws with guaranteed performances but only using a part of links/nodes is worth investigating.

APPENDIX

A. Open-loop control in the nominal case

We consider an open-loop control policy as follows:

$$\hat{\mu}_{ej}^{\text{ol}} = \hat{\gamma}_{ej} Q_{s_0 e}, \quad (54)$$

where

$$\hat{\gamma}_{ej} := \frac{u_{ej}^*(Q_{s_0})}{\sum_{j' \in \mathcal{E}_e^+} u_{ej'}^*(Q_{s_0})}, \quad (e, j) \in \mathcal{P}. \quad (55)$$

Clearly, it has the same structure as the open-loop control μ_{ej}^{ol} given by (45).

Before we state the lemma, we define

$$\hat{\gamma}_{me} = \sum_{i \in \mathcal{A}_m^+ \cap \mathcal{E}_e^-} \hat{\gamma}_{mi} \hat{\gamma}_{ie}. \quad (56)$$

Noting (48) and (52), for any $\alpha < C(Q_{s_0}; \mathcal{G})$, we obtain in a similar way

$$\hat{\gamma}_{e_0 e} \alpha < \hat{\gamma}_{e_0 e} C(Q_{s_0}; \mathcal{G}) = \sum_{i \in \mathcal{E}_j^-} u_{ie}^*(Q_{s_0}) \leq Q_{s_0 e}. \quad (57)$$

Lemma 4. *Consider an acyclic network satisfying Assumption 3 and only admitting the nominal mode s_0 . Given any demand $\alpha < C(Q_{s_0}; \mathcal{G})$ and the open-loop control given by (54), there exists a stable equilibrium such that $f_e(s_0, \hat{x}_e) = \hat{\gamma}_{e_0 e} \alpha$ for any $e \in \mathcal{E}$. Moreover, if there exists a neighborhood $U(\hat{x})$ of $\hat{x}$ such that $f_e(s_0, x_e)$ is strictly increasing in x_e for every $e \in \mathcal{E}$, the equilibrium $\hat{x}$ is also globally asymptotically stable.*

Obviously, the global asymptotic stability stated in Lemma 4 implies that the open-loop control (54) can enable the throughput to achieve the nominal min-cut capacity. Note that the required condition is really mild. As we will see in the proof, all of the link densities are fewer than their corresponding critical densities. It means that none of the links are congested. In that case, the sending flow $f_e(s_0, x_e)$ is typically sensitive to density changes, i.e., strictly increasing in x_e .

Below we give the proof of Lemma 4.

Proof. We first show the existence of $\hat{x}$. For any demand $\alpha < C(Q_{s_0}; \mathcal{G})$, we conclude $\hat{x} \in \prod_{e \in \mathcal{E}} [0, x_{s_0 e}^c]$ by using Assumptions 3.1-3.2 and $f_e(s_0, \hat{x}_e) = \hat{\gamma}_{e_0 e} \alpha < Q_{s_0 e}$, where $x_{s_0 e}^c$ is given by (8b).

Next, we prove that $\hat{x}$ is a stable equilibrium. Again noting (54) and (57), we conclude that there exists a neighbourhood $W(\hat{x})$ of $\hat{x}$ such that $q_{ej}(s_0, x) = \hat{\gamma}_{ej} f_e(s_0, x_e)$. Then $\hat{x}$ is an equilibrium by (56). The stability of $\hat{x}$ is directly implied by Theorem 6 (iii) in [19].

Note that $\hat{x}$ is globally asymptotically stable if and only if it is locally asymptotically stable; see Theorem 6 (iv) in [19]. Below we show that the network is locally asymptotically stable over the neighborhood $W(\hat{x}) \cap U(\hat{x})$. We first notice link x_{e_0} converges to $\hat{x}_{e_0}$ by using the monotonicity and Lipschitz continuity of $f_{e_0}(s_0, x_{e_0})$. Then we consider links from the upstream to the downstream. We can show x_e converges by using the convergence of its upstream links. $\square$

B. Extension for cyclic networks

We present stability analysis for cyclic networks. One major difference between cyclic and acyclic networks is $\mathcal{A}_e^- \cap \mathcal{A}_e^+ \neq \emptyset$ for some $e \in \mathcal{E}$ in a cyclic network. It induces that Lemma 2 may not hold because of $\mathcal{A}_{e, \text{inf}}^- \cap \mathcal{A}_{e, \text{inf}}^+ \neq \emptyset$. We address this problem by considering a partition of $\mathcal{E}_{\text{inf}}$.

We denote by $\{\mathcal{L}_z\}_{z=1}^Z$ a partition of $\mathcal{E}_{\text{inf}}$ such that given any $z_1, z_2 \in \{1, 2, \dots, Z\}$ with $z_1 \neq z_2$,

$$e \leftrightarrow j, \forall e, j \in \mathcal{L}_{z_1}, e \neq j, \quad (58a)$$

$$e \not\leftrightarrow j, \forall e \in \mathcal{L}_{z_1}, j \in \mathcal{L}_{z_2}, \quad (58b)$$

where $e \leftrightarrow j$ represents that the two links are reachable from each other, and $e \not\leftrightarrow j$ denotes that at least one link is unreachable from the other. For a cyclic network, a partition can be obtained by merging the sets above whose elements are accessible from each other, and we can prove by contradiction the uniqueness of the partition.

Then we show by the partition $\{\mathcal{L}_z\}_{z=1}^Z$ that Theorem 1 holds for acyclic networks. It implies that the other stability results in Section III also apply to acyclic networks.

Proposition 3. *Consider a cyclic network satisfying Assumptions 3.1-3.3. Suppose that the network admits a demand α and an invariant set $\mathcal{X}^\mu \subseteq \prod_{e \in \mathcal{E}} [\underline{x}_e^\mu, \bar{x}_e^\mu]$ under a control law $\mu: \mathcal{S} \times \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}^P$. The network is stable if (24) holds.*

Proof. Consider the following Lyapunov function:

$$\hat{V}(s, x) := \sum_{e \in \mathcal{E}_{\text{inf}}} a_{t_e} x_e \left[\frac{1}{2} \sum_{i \in \mathcal{L}_{t_e}} x_i + \sum_{m \in \mathcal{A}_e^- \setminus \mathcal{L}_{t_e}} x_m \right]$$

$$+ \sum_{\ell \in \mathcal{B}_e^+} \int_{x_\ell}^{x_\ell} \rho_\ell(\zeta) d\zeta + b_{se} \Big], \quad (59)$$

where ι_e denotes the set index such that $e \in \mathcal{L}_{\iota_e}$. The rest of the proof is similar to that of Theorem 1, except Lemma 2. We instead use the following lemma:

Lemma 5. *The system of equations*

$$\pi_e a_{\iota_e} + \bar{G} \sum_{n \in \mathcal{A}_{e,\text{inf}}^+ \setminus \mathcal{L}_{\iota_e}} a_{\iota_n} < 0, \quad \forall e \in \mathcal{E}_{\text{inf}} \quad (60)$$

has a positive solution for $\{a_{\iota_e} | e \in \mathcal{E}_{\text{inf}}\}$ if $\pi_e < 0$ for any $e \in \mathcal{E}_{\text{inf}}$.

Note that (58a)-(58b) reveals an acyclic structure of the partition. Thus we can prove Lemma 5 in a similar way to Lemma 2. $\square$

C. Proof of Lemma 1

To show the existence of a solution, note that (37) is equivalent to $\Lambda y = p^T z - z$ where $z = [z_{s_0}, \dots, z_{s_M}]^T$. Since the discrete state process is ergodic, the rank of the matrix Λ is M . Scaling row $i + 1$ with p_{s_i} for $i = 0, 1, \dots, M$ and adding the scaled rows $1, 2, \dots, M$ to row $M + 1$, we obtain

$$\begin{bmatrix} -p_{s_0} \sum_{i \neq 0} \lambda_{s_0 s_i} & \cdots & p_{s_0} \lambda_{s_0 s_M} \\ \vdots & \ddots & \vdots \\ \sum_{i \neq 0} p_{s_i} \lambda_{s_i s_0} - p_{s_0} \sum_{i \neq 0} \lambda_{s_0 s_i} & \cdots & \sum_{i \neq M} p_{s_i} \lambda_{s_i s_M} - p_{s_M} \sum_{i \neq M} \lambda_{s_M s_i} \end{bmatrix} y = \tilde{z} \quad (61)$$

where $\tilde{z} = [p_{s_0}(p^T z - z_{s_0}), \dots, (\sum_{i=0}^M p_{s_i})p^T z - p^T z]^T$. Noting $\Lambda^T p = \mathbf{0}$, we know that $\sum_{i \neq 0} p_{s_i} \lambda_{s_i s_j} - p_{s_j} \sum_{i \neq j} \lambda_{s_j s_i} = 0$ for any $j = 0, \dots, M$. Also note that $(\sum_{i=0}^M p_{s_i})p^T z - p^T z = 0$. Hence, the rank of the augmented coefficient matrix of the system of linear equations (61) is also M , equal to the rank of the coefficient matrix. Therefore, (61) must have a particular solution y^p . Noting that $\text{rank}(\Lambda) = M$ and that $y = \omega \mathbf{1}$ is a solution to $\Lambda y = \mathbf{0}$ for any $\omega \in \mathbb{R}$, we can conclude that the general solution of (37) is given by $y^g = y^p + \omega \mathbf{1}$. Clearly, a non-negative solution is available by letting ω be a sufficiently large number. $\square$

D. Proof of Lemma 2

If $\bar{G} \leq 0$, the lemma is trivial. Now we consider $\bar{G} > 0$ and construct $\{a_e | e \in \mathcal{E}_{\text{inf}}\}$ satisfying (39). Let $\bar{\pi} := \max_{e \in \mathcal{E}_{\text{inf}}} \pi_e < 0$ and $K := |\mathcal{E}_{\text{inf}}|$. Since this lemma focuses on acyclic networks, we can iteratively consider the links in $\mathcal{E}_{\text{inf}}$ backwards in some sense.

We first assume $\bar{G} \geq -\bar{\pi}$. The construction starts with initializing $\mathcal{N}^{(0)} := \mathcal{E}_{\text{inf}}$ and $i := 1$. In the i -th step, we consider $\mathcal{M}^{(i-1)} := \{e \in \mathcal{N}^{(i-1)} | \forall i \in \mathcal{N}^{(i-1)} \setminus \{e\}, i \not\rightarrow e\}$, where $i \not\rightarrow e$ implies that there are no directed paths from link i to link e . Intuitively, $\mathcal{M}^{(i-1)}$ denotes the collection of the far downstream links among $\mathcal{N}^{(i-1)}$. For any $e \in \mathcal{M}^{(i-1)}$, we let $a_e = (-\bar{G}K/\bar{\pi})^{i-1} > 0$ and then obtain

$$\pi_e a_e + \bar{G} \sum_{n \in \mathcal{A}_{e,\text{inf}}^+} a_n < \bar{\pi} \left(-\frac{\bar{G}K}{\bar{\pi}}\right)^{i-1} + \bar{G}K \left(-\frac{\bar{G}K}{\bar{\pi}}\right)^{i-2} = 0.$$

The i -th step ends up with $\mathcal{N}^{(i)} = \mathcal{N}^{(i-1)} \setminus \mathcal{M}^{(i-1)}$. The iteration stops when $\mathcal{N}^{(i)} = \emptyset$, which indicates that all the links of $\mathcal{E}_{\text{inf}}$ are considered.

If $\bar{G} < -\bar{\pi}$, the same iteration is conducted except letting $a_e = -(\bar{G}K^{i-1})/\bar{\pi} > 0$ in the i -th step. $\square$

E. Proof of Lemma 3

First, we note that the disruptions (46a)-(46b) imply

$$Q_{se} \leq r_e(s, x_e), \quad \forall s \in \mathcal{S}, \forall x_e \leq x_e^{c*}. \quad (62a)$$

$$Q_{se} \leq f_e(s, x_e), \quad \forall s \in \mathcal{S}, \forall x_e \geq x_e^{c*}, \quad (62b)$$

where x_e^{c*} is given by (9). Besides, for any $x \in \mathcal{X}$ with $x_j \leq x_j^{c*}$, we obtain

$$r_{ej}^{\text{md}}(s, x) = \frac{\mu_{ej}^{\text{md}}(s)}{\sum_{i \in \mathcal{E}_j^-} \mu_{ij}^{\text{md}}(s)} r_j(s, x_j) \stackrel{(19d), (62a)}{\geq} \mu_{ej}^{\text{md}}(s). \quad (63)$$

To show that $\mathcal{X}^{\text{md}}$ is an positively invariant set, we prove that on the boundary of $\mathcal{X}^{\text{md}}$, the vector field $G(s, x)$ points towards the interior regardless of $s \in \mathcal{S}$. That is, for any $e \neq e_o \in \mathcal{E}$ and $(s, x) \in \mathcal{S} \times \mathcal{X}^{\text{md}}$ with $x_e = x_e^{c*}$, we have

$$\begin{aligned} G_e(s, x) &\stackrel{(4), (63)}{\leq} \sum_{i \in \mathcal{E}_e^-} \mu_{ie}^{\text{md}}(s) - \sum_{j \in \mathcal{E}_e^+} \min\{\mu_{ej}^{\text{md}}(s), f_{ej}^{\text{md}}(s, x)\} \\ &\stackrel{(5a), (62b)}{\leq} \sum_{i \in \mathcal{E}_e^-} \mu_{ie}^{\text{md}}(s) - \min\left\{\sum_{j \in \mathcal{E}_e^+} \mu_{ej}^{\text{md}}(s), Q_{se}\right\} \\ &\stackrel{(19b), (19d)}{=} 0. \end{aligned}$$

To show that $\mathcal{X}^{\text{md}}$ is global attracting, we perform an induction on links. We first consider link e reaching the destination, i.e. $\tau_e = v_d$, since it does not have downstream bottlenecks. We see $G_e(s, x) \leq 0$ for any $(s, x) \in \{(s, x) \in \mathcal{S} \times \mathcal{X} | x_e > x_e^{c*}\}$. In addition, given $\alpha < \sum_{s \in \mathcal{S}} p_s C(Q_s; \mathcal{G})$, there exists some mode $s \in \mathcal{S}$ under which x_e converges to a value strictly less than x_e^{c*} . Since the process $\{S(t); t \geq 0\}$ is ergodic, it is concluded that x_e enters $[0, x_e^{c*}]$ almost surely. Then, as indicated by (63), link e is not a bottleneck for its upstream links under the mode-dependent control. The similar proof can be iteratively applied to links upstream of link e until all links are considered.

The invariant set $\mathcal{X}^{\text{md}}$ immediately gives $\mathcal{E}_{\text{inf}} = \{e_o\}$; by noting (63), we also obtain $\mathcal{B}_{e_o}^+ = \emptyset$. $\square$

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Yu Tang is a PhD student at Tandon School of Engineering in New York University. He received his B.Eng. degree and M.S. degree in Transportation Engineering from Tongji University in 2016 and 2019, respectively. He is interested in using ideas from stochastic processes and dynamic control to design resilient intelligent transportation systems.



Li Jin is John Wu & Jane Sun Assistant Professor of Electrical and Computer Engineering (2021–present) at the UM Joint Institute and the Department of Automation, Shanghai Jiao Tong University, China. He was Assistant Professor (2018–2020) in the Tandon School of Engineering, New York University, USA. He received his B.Eng. from Shanghai Jiao Tong University in 2011, M.S. from Purdue University in 2012, and Ph.D. from the Massachusetts Institute of Technology in 2018. His research uses ideas from stochastic processes, control, optimization, and machine learning to address key challenges in smart and connected system. Specific applications include connected and autonomous vehicles, and intelligent transportation systems.