Scrambling transition in free fermion systems induced by a single impurity

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In quantum many-body systems, interactions play a crucial role in the emergence of information scrambling. When particles interact throughout the system, the entanglement between them can lead to a rapid and chaotic spreading of quantum information, typically probed by the growth in operator size in the Heisenberg picture. In this study, we explore whether the operator undergoes scrambling when particles interact solely through a single impurity in generic spatial dimensions, focusing on fermion systems with spatial and temporal random hoppings. By connecting the dynamics of the operator to the symmetric exclusion process with a source term, we demonstrate the presence of an escape-to-scrambling transition when tuning the interaction strength for fermions in three dimensions. As a comparison, systems in lower dimensions are proven to scramble at arbitrarily weak interactions unless the hopping becomes sufficiently long-ranged. Our predictions are validated using both a Brownian circuit with a single Majorana fermion per site and a solvable Brownian SYK model with a large local Hilbert space dimension. This suggests the universality of the theoretical picture for free fermion systems with spatial and temporal randomness.

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I. INTRODUCTION

Generic interacting many-body systems can serve as their own bath, a pivotal element for the manifestation of quantum thermalization in isolated systems [1,2]. This involves the obscuring of all local initial conditions within the entire system after prolonged evolution, measured by the growth in operator size [3–9]. In this process, interactions play a crucial role. In the absence of interactions, excitations with infinite lifetimes can carry quantum information, remaining free from dissipation [10]. This occurs because quadratic Hamiltonians conserve the number of field operators, thereby preventing the growth of operator size. However, in the presence of interactions, a single excitation can undergo scattering, giving rise to multiple excitations. This iterative process leads to a rapid increase in complexity for simple initial operators subject to the Heisenberg evolution [11–32].

Recently, new insights into this problem have been gained from the study of fermionic systems that only interact through a single impurity [33]. The key question is whether a single impurity can effectively scramble the entire system. While one might typically anticipate that introducing a local impurity into a large many-body system does not lead to significant changes in bulk dynamics, it becomes apparent that even with just a single interaction, the Hamiltonian is no longer quadratic, thus permitting the growth of operator size. Indeed, previous study unveils the emergence of information scrambling in 1D systems with spatial and temporal random short-range hoppings for arbitrary weak interaction strength at the impurity [33]. (Many other papers have investigated similar setups—nonequilibrium dynamics in the presence of boundary perturbations. See Ref. [34] for a review on classical stochastic processes and see Ref. [35] for a review on quantum systems. Here, we study this from the perspective of quantum information. We also note a paper that studies information scrambling in an integrable Kondo model [36].) In this setup, both the operator and entanglement exhibit diffusive scaling, stemming from the random walk characteristics of fermion operators on the 1D lattice. However, it is acknowledged that the properties of random walks are greatly influenced by the connectivity among different sites across various geometries.

Building upon this insight, we delve deeper into investigating the growth of operator size within such systems across arbitrary dimensions. Utilizing a phenomenological percolation model on trees, we identify a notable distinction between systems in 3D and lower dimensions, summarized in Fig. 1. In 3D, interactions do not lead to a persistent growth of operator size unless their strength exceeds a critical value. Within this nonscrambling regime, the operator quickly escapes from the impurity, following the Pólya's theorem, and we term this as the escape phase. For stronger interactions, the operators near interacting sites can scramble into nonlocal operators, analogous to the scenario in lower dimensions with short-range hoppings. Additionally, when the hopping range is long enough, we also identify similar dynamical transitions in lower dimensions through the utilization of Lévy flight properties. Our predictions are demonstrated by both numerical

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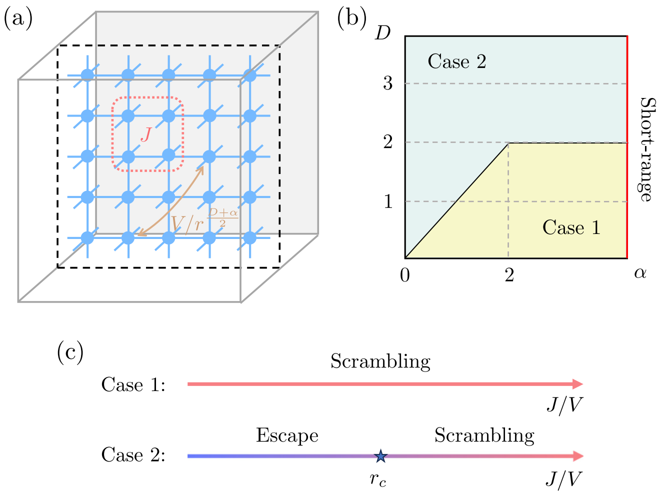

FIG. 1. (a) The schematics of our model in $D = 3$ with a single Majorana fermion per site. The model includes a solitary random interaction with strength J and Brownian hopping that decays as $1/r^{\frac{D+\alpha}{2}}$ for $\alpha > 0$. The case of short-range hopping corresponds to $\alpha \to \infty$. [(b) and (c)] The phase diagram of the model. Here, we extend the dimension D to an arbitrary real number. In the case 1 regime, arbitrarily weak interaction can scramble the entire system, whereas in the case 2 regime, a critical interaction strength is necessary to realize the scrambling of operators near the interacting sites.

simulations based on a small N Brownian circuits [28–32] and analytical calculations in a solvable large N Brownian Sachdev-Ye-Kitaev (SYK) chain [37–44]. This suggests universality regardless of the dimensionality of the local Hilbert space.

II. BROWNIAN CIRCUITS

To be concrete, we first examine Brownian circuits with nearest-neighbor hoppings in a generic dimension D, where operator dynamics can be mapped to a classical stochastic process. As we will demonstrate, the physical picture obtained in this model is also applicable to Brownian SYK models with large Hilbert space dimensions. Our focus is on Brownian circuits of Majorana fermions. Each site hosts a single Majorana mode χ_x with canonical anti-commutation relations $\{\chi_x, \chi_y\} = 2\delta_{xy}$. The Hamiltonian reads

$$dH(t) = i \sum_{\langle xy \rangle} dV_{x,y}\, \chi_x \chi_y + dJ \prod_{x \in \square} \chi_x, \tag{1}$$

where the first term denotes the hopping of the free fermion between neighboring sites and the second term is the interaction term. Here $\square$ labels four sites in a single plaquette near the origin, as illustrated in Fig. 1 with $D \geqslant 2$. For $D = 1$, we can simply pick four contiguous sites near the origin. Independent Brownian variables $dV_{x,y}$ and dJ satisfy the Wiener process, with

$$\overline{dV_{x,y} dV_{x',y'}} = V dt\, \delta_{xx'} \delta_{yy'}, \quad \overline{dJ^2} = J dt. \tag{2}$$

In a short time interval dt, the unitary evolution is given by $dU = e^{-idH}$. We are interested in the operator dynamics, governed by the Heisenberg evolution $O(t + dt) = dU^\dagger O(t) dU$. To study the growth of operator size, we introduce a complete orthonormal basis of Hermitian operators $\{B_\mu\} = \{i^{q(q-1)/2} \chi_{x_1} \chi_{x_2} \dots \chi_{x_q}\}$. Each Majorana string B_μ can be labeled by its height $\mathbf{h}_\mu$, defined as $h_{\mu,\boldsymbol{x}} = 1$ for $\boldsymbol{x} \in \{\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_q\}$ and otherwise $h_{\mu,\boldsymbol{x}} = 0$. The size of B_μ is further defined as $n_\mu = \sum_{\boldsymbol{x}} h_{\mu,\boldsymbol{x}}$

We expand $O(t)$ in this set of basis operators as $O(t) = \sum_\mu \alpha_\mu(t) B_\mu$, where $\alpha_\mu(t)$ represents the wave function for the operator evolution. In Brownian circuits, the phase of $\alpha_\mu(t)$ is averaged out due to the temporal randomness, and the evolution can be formulated in terms of a classical stochastic process described by the probability distribution $f_\mu(t) = |\alpha_\mu(t)|^2$, which is normalized $\sum_\mu f_\mu(t) = 1$ due to unitarity [7,9]. The size of $O(t)$ is defined as $N(t) \equiv \sum_\mu n_\mu f_\mu(t)$. By generalizing the analysis in Ref. [33], this dynamics describes a symmetric exclusion process (SEP) with a single source term at the origin [45,46]. The governing master equation for this dynamics, along with its derivation details, are provided in Appendix. Based on this master equation, we conduct numerical simulation using the following update rules.

(1) We implement an unbiased random walk for each particle independently.

(2) Each time when one particle returns to the origin, we branch it by adding n_i particles with a probability p_i.

In the original model (1), we expect $n_i = 2$ and $p_i \propto J/V$. The system is initialized by putting a single particle at the origin, and the operator size growth is studied by counting the number of particles.

We note that in 1D and 2D, an operator originating from the origin exhibits diffusive spreading, independent of the parameter p_i, eventually encompassing the entire space. Complexity arises in 3D, where for large p_i, the operator continues to grow over time. However, decreasing p can mitigate its growth. Notably, for $p < p_c$, during time evolution, the operator's size saturates to a finite constant, indicating the presence of a scrambling transition as p_i varies.

To understand the presence or absence of the transitions across various dimensions, we recognize that the dynamics can be modeled as percolation on a tree. As shown in Fig. 2(a), in this model, each vertex is branching into n vertex, with each edge subject to removal with probability $1 - p$. It is known that this model exhibits a percolation transition at $pn = 1$ [47]. When $pn > 1$, the root is connected to an infinite number of the vertices in the tree. Conversely, if $pn < 1$, the root connects to only a finite number of vertices in the tree. Our model maps to the percolation model as follows: each edge represents a particle from the origin, with each vertex's branching number estimated as $n = n_i p_i + 1$. Edges from a vertex are removed together with probability $1 - p_r$ if the particle doesn't return to the origin. Here p_r represents the return probability of a random walker to return to its starting point. The transition of this model is expected to occur at $p_r n = 1$.

Given $n = n_i p_i + 1 \geqslant 1$, observing a scrambling transition requires p_r to be less than 1. In 1D and 2D, where the random walker returns to its starting point with probability one, there is no scrambling transition, and the model remains in the scrambling phase as long as $p_i > 0$. On the other hand, according to Pólya's theorem [48,49], the 3D random walker

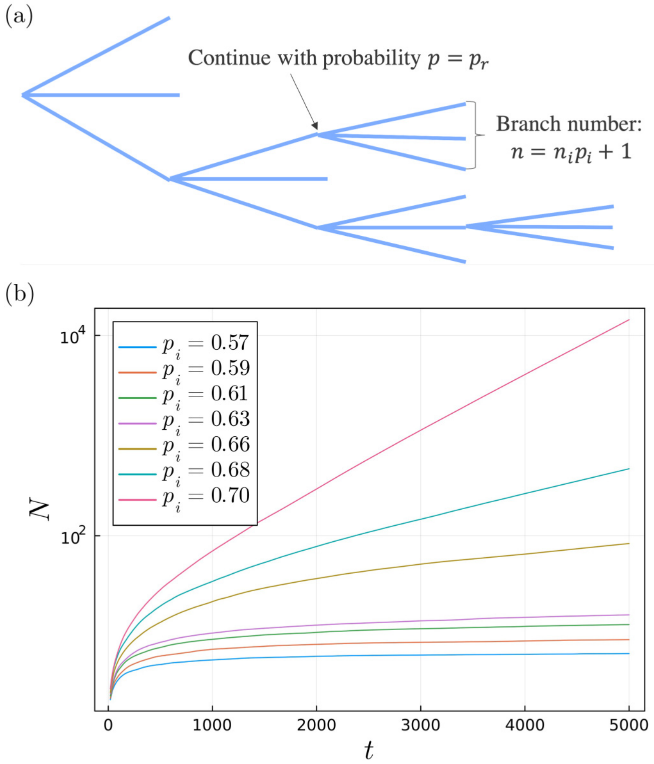

FIG. 2. (a) Schematics of the effective percolation model on trees: each edge represents a particle at the origin, which is removed with a probability of $1 - p_r$ if it escapes. Each remaining edge branches into $n = n_i p_i + 1$ edges. (b) Particle number N as a function of time t in log-lin scale in 3D. We average over 10 000 samples. We observe a transition of particle number between $p_i = 0.63$ and $p_i = 0.66$. The theoretical prediction of p_i^c is 0.646.

will return to its starting point with probability $p_r \approx 0.341$. A brief review of the derivation is presented in the supplementary material for clarity. The probability of returning being less than one allows us to induce a scrambling transition at some finite p_i^c, such that $(n_i p_i^c + 1)p_r = 1$. For $p_i < p_i^c$, particles escape from the origin, leading to saturation of the operator size. Specifically, we find $p_i^c \approx 0.646$ for $n_i = 3$, which aligns with the results from 3D classical particle model simulations illustrated in Fig. 2(b).

We can generalize the above discussion to fermion operators that are initially not located at the origin. As demonstrated in the supplementary material, the probability of reaching the origin is inversely proportional to the distance r from the origin. This suggests that these operators have a reduced probability of reaching the origin. Nevertheless, upon returning to the origin, they may undergo scrambling if $p_i > p_i^c$.

III. BROWNIAN SYK MODEL

We now develop an analytically solvable large-N model capable of demonstrating a scrambling transition. Specifically, we consider the Brownian SYK model [42,43] with a single interacting impurity. In this model, we have N Majorana fermions $\chi_{\boldsymbol{x},i}$ with $i \in \{1, 2, \dots, N\}$ on each site, which satisfies $\{\chi_{\boldsymbol{x},i}, \chi_{\boldsymbol{y},j}\} = 2\delta_{\boldsymbol{xy}}\delta_{ij}$. The Hamiltonian reads

$$dH(t) = i \sum_{\langle \boldsymbol{xy}\rangle, ij} dV^{ij}_{\boldsymbol{x},\boldsymbol{y}}\, \chi_{\boldsymbol{x},i}\chi_{\boldsymbol{y},j} + i^{q/2} \sum_{i_1 \leqslant i_2 \leqslant \cdots \leqslant i_q} dJ_{i_1 i_2 \dots i_q} \chi_{\boldsymbol{0},i_1}\chi_{\boldsymbol{0},i_2}\cdots\chi_{\boldsymbol{0},i_q}, \tag{3}$$

where the second term represents a q-body interaction on the impurity with $q \geqslant 4$. Brownian variables with different indices are independent and satisfy

$$\overline{\left(dV^{ij}_{\boldsymbol{x},\boldsymbol{y}}\right)^2} = V dt/4N, \qquad \overline{\left(dJ_{i_1 i_2 \dots i_q}\right)^2} = (q-1)! J dt/4N^{q-1}. \tag{4}$$

The model can be analyzed using the large-N expansion. Focusing on its real-time dynamics, we first introduce the retarded Green's functions $G^R_{\boldsymbol{x}}(t) \equiv -i\theta(t)\langle\{\chi_{\boldsymbol{x},i}(t), \chi_{\boldsymbol{x},i}(0)\}\rangle$, where $\theta(t)$ is the Heaviside step function. In SYK-like models, the self-energy is dominated by melon diagrams, which gives $\Sigma^R_{\boldsymbol{x}}(\omega) = i(zV + J\delta_{\boldsymbol{x},\boldsymbol{0}})/4 \equiv i\Gamma_{\boldsymbol{x}}/4$. Here, $z = 2D$ is the coordination number for the square lattice and $\Gamma_{\boldsymbol{x}}$ represents the quasi-particle lifetime on site $\boldsymbol{x}$. Transforming into the time domain, we find $G^R_{\boldsymbol{x}}(t) = -2ie^{-\Gamma t/2}$.

In large-N systems, due to the large on-site Hilbert space, the operator size usually experiences exponential growth in the early-time regime, characterized by a rate known as the quantum Lyapunov exponent $\varkappa$. Consequently, we are examining whether the system undergoes scrambling by assessing the existence of a nonvanishing $\varkappa$. It is known that the average size is related to out-of-time-order (OTO) commutator as [9,15]

$$N_{\boldsymbol{x}}(t) = \frac{1}{4}\sum_j \langle |\{\chi_{\boldsymbol{y}=\boldsymbol{0},i}(t), \chi_{\boldsymbol{x},j}(0)\}|^2\rangle, \tag{5}$$

where we introduce a subscript $\boldsymbol{x}$ to denote the location of the operator. In SYK-like models, the self-consistent equation for the OTO commutator arises from the ladder diagram [39,50]

$$N_{\boldsymbol{x}}(t_1) = -\int dt'\, G^R_{\boldsymbol{x}}(t_{12})^2 \times \left[\frac{V}{4}\sum_{\langle \boldsymbol{xy}\rangle} N_{\boldsymbol{y}}(t_2) + \frac{J(q-2)}{4}\delta_{\boldsymbol{x},\boldsymbol{0}} N_{\boldsymbol{x}}(t_2)\right]. \tag{6}$$

Here we neglect the inhomogeneous terms since it does not contribute to the asymptotic behavior. In Brownian models, this is equivalently expressed as a differential equation:

$$-\frac{dN_{\boldsymbol{x}}}{dt} = V\left(zN_{\boldsymbol{x}} - \sum_{\langle \boldsymbol{xy}\rangle} N_{\boldsymbol{y}}\right) - J(q-2)\delta_{\boldsymbol{x},\boldsymbol{0}} N_{\boldsymbol{x}}. \tag{7}$$

The first term describes the diffusive spreading, while the second term is a source term at $x = 0$, which can potentially result in an exponential growth of N_x. This equation takes the form of the imaginary-time Schrödinger equation on the lattice with an attractive delta potential with depth $\sim J$ at the origin. Assuming $N_{\boldsymbol{x}}(t) = \exp(\varkappa t)N_{\boldsymbol{x}}$ for sufficiently long time t, we recognize that a positive Lyapunov exponent $\varkappa$ corresponds to a bound state with energy $-\varkappa$. The solution

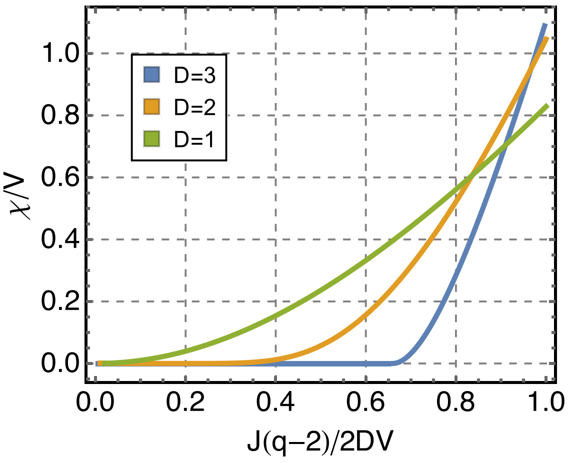

FIG. 3. Results for the quantum Lyapunov exponent $\varkappa$ as a function of interaction strength J/V in different dimensions $D \in \{1, 2, 3\}$, obtained by solving Eq. (8). The result is consistent with the theoretical prediction that a scrambling transition occurs at the critical strength given by $J(q-2)/2DV = (1-p_r) = 0.659$.

is given by the Lippmann-Schwinger equation

$$\frac{V}{J(q-2)} = \int \frac{d^D k}{(2\pi)^D} \frac{1}{\varkappa/V + 2D - \sum_\alpha 2\cos(k_\alpha)}. \quad (8)$$

Here, α labels different spatial directions $\alpha \in \{x, y, \dots\}$. We focus on the regime with a shallow bound state $\varkappa/V \ll 1$, where we can transition to the continuum limit by expanding $1-\cos k \approx k^2/2$. In this limit, Eq. (8) aligns with the standard Schrödinger equation with a quadratic dispersion. It is well-established that in 3D, a finite depth of the potential is required to sustain a bound state. In contrast, in 1D or 2D, a bound state emerges under infinitely weak attractions, leading to exponential growth in operator size. We can further make a direct connection between the bound state problem and the random walk picture in the last section. To determine the critical point J_*, we set $\varkappa = 0$ on the right-hand side (R.H.S.) of (8). Then, as reviewed in the supplementary material, it can be related to the returning probability p_r by R.H.S. $= [2(1-p_r)D]^{-1}$. In particular, $p_r = 1$ for $D \leqslant 2$ originates from the divergence of the integral $\sim \int k^{D-1} dk/k^2$. The critical point J_* is then given by $J_*/V = 2D(1-p_r)/(q-2)$, a close analog of results in Brownian circuits. For $D = 3$, this predicts $J(q-2)/2DV = (1-p_r) = 0.659$. We can further obtain the quantum Lyapunov exponent by solving Eq. (8) exactly, using the analytical results for lattice Green's functions [51]. The result is plotted in Fig. 3 for $D \in \{1, 2, 3\}$. In 3D, we can clearly observe a transition from escape phase with $\chi = 0$ to scrambling phase with $\chi > 0$.

IV. LONG-RANGE HOPPING

We now generalize our results to systems with long-range random hoppings [30,31,52–56]. Since we have demonstrated that both $N = 1$ Brownian circuits and the large N Brownian SYK model share the same phase diagram, here we take the Brownian SYK model as an example. The hopping term in the Hamiltonian now becomes

$$dH_0(t) = \sum_{x\neq y, ij} \frac{i}{2|\boldsymbol{x}-\boldsymbol{y}|^{\frac{\alpha+D}{2}}} dV^{ij}_{\boldsymbol{x},\boldsymbol{y}}\, \chi_{\boldsymbol{x},i}\chi_{\boldsymbol{y},j}, \quad (9)$$

which $dV^{ij}_{\boldsymbol{x},\boldsymbol{y}}$ still has variance (4). Carrying out the similar calculation as in the last section, we find firstly the decay rate becomes $\Gamma_{\boldsymbol{x}} = \sum_{\boldsymbol{y}\neq \boldsymbol{x}} \frac{V}{|\boldsymbol{x}-\boldsymbol{y}|^{\alpha+D}} + J\delta_{\boldsymbol{x},\boldsymbol{0}}$. Therefore, to ensure the convergence of the decay rate, we focus on $\alpha > 0$. Otherwise, the model is effectively all-to-all connected, whose scrambling dynamics in the presence of a singular interaction has been analyzed in Ref. [33]. Secondly, the random walk is replaced by a Lévy flight. This is reflected in the Lippmann-Schwinger equation, which now leads to

$$\frac{V}{J_*(q-2)} = \int \frac{d^D k}{(2\pi)^D} \frac{1}{\sum_{\boldsymbol{y}\neq \boldsymbol{0}} |\boldsymbol{y}|^{-\alpha-D}[1-\cos(\boldsymbol{k}\cdot\boldsymbol{y})]}.$$

To determine whether the R.H.S. diverges, we first perform the small k expansion of the denominator, which gives

$$\sum |\boldsymbol{y}|^{-\alpha-D}[1-\cos(\boldsymbol{k}\cdot\boldsymbol{y})] = \begin{cases} k^\alpha & \text{if } \alpha < 2, \\ k^2 & \text{if } \alpha \geqslant 2. \end{cases} \quad (10)$$

As a result, the returning probability of the Lévy flight is 1 for $D \leqslant \alpha$ when $\alpha \in (0, 2)$ and $D \leqslant 2$ when $\alpha \in [2, \infty)$. This defines the parameter regime referred to as case 1 in Fig. 1. Conversely, when the integral converges for higher dimensions, an escape-to-scrambling transition typically occurs, denoted as case 2 in Fig. 1. In particular, for $\alpha \geqslant 2$, the phase diagram is the same as the short-range hopping case.

V. DISCUSSIONS

In this study, we investigate the information dynamics in free fermion systems with spatial and temporal randomness that interacts through a single impurity. Our findings reveal that a solitary interaction can trigger a scrambling phase transition depending on the system's dimensionality and hopping range. We establish a universal phase diagram by employing both $N = 1$ Brownian circuits and large N Brownian SYK models. In high dimensions or with a long hopping range, the model undergoes an escape-to-scrambling transition upon tuning the interaction strength J. Within the scrambling phase, operators proximate to the interacting site can scramble into an extensive operator under time evolution. On the other hand, in low-dimensional systems with a short hopping range, even arbitrarily weak interactions lead to the scrambling of the entire system. We expect that these problems can also be characterized by other information quantities, such as the entanglement entropy and mutual information [33].

The dynamical transitions of information scrambling have been observed in various contexts. Specifically, Refs. [57,58] identify an environment-induced scrambling transition in systems embedded in environments. One can also treat Majorana fermions in bulk as an environment, achieving the transition by increasing the system-environment coupling V. In Refs. [57,58], the information never returns to the system once it enters the environment, corresponding to $p_r = 0$. Our analysis in this work proposes a refinement of the picture for environments with memory. (A more recent preprint [59] studied the scrambing transition where the information backflow can be tuned. Consistent with our results, they didn't find scrambling transition in low dimensions when the information backflow was nonzero, corresponding to $p_r = 1$.) The discussions in our paper are also pertinent to the intriguing question of whether a single thermal island can thermalize the entire system, a query crucial for understanding the existence of many-body localization phases. To address this question, it is imperative to extend the current discussions to models with static hopping strength, where localization is feasible. Nevertheless, the theoretical analysis in such cases becomes considerably more challenging, and we defer this task to future works.

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APPENDIX A: MASTER EQUATION IN BROWNIAN CIRCUITS

In this section, we give a detailed derivation of the master equation in Brownian circuits. The Hamiltonian reads

$$dH(t) = i\sum_{\langle xy\rangle} dV_{x,y}\chi_x\chi_y + dJ\prod_{x\in\square}\chi_x, \tag{A1}$$

where $dV_{x,y}$ and dJ satisfy the Wiener process, with

$$\overline{dV_{x,y}dV_{x',y'}} = V\,dt\,\delta_{xx'}\delta_{yy'}, \quad \overline{dJ^2} = J\,dt, \tag{A2}$$

and $\square$ labels four sites in a single plaquette near the origin as illustrated in the main text. We can expand the evolution of the operator $O(t)$ to second order:

$$\begin{aligned} dO(t) &= e^{idH(t)}O(t)e^{-idH(t)} - O(t)\\ &= [idH(t), O(t)] + \frac{1}{2}[idH(t), [idH(t), O(t)]]\\ &= i[dH(t), O(t)] - \frac{1}{2}\{dH(t)dH(t), O(t)\}\\ &\quad + dH(t)O(t)dH(t)\\ &= i[dH(t), (t)] - \sum_{\langle xy\rangle} O(t)V dt - O(t)J dt\\ &\quad - \sum_{\langle xy\rangle}\chi_x\chi_y O(t)\chi_x\chi_y V dt + \prod_{x\in\square}\chi_x O(t)\prod_{x'\in\square}\chi_{x'} J dt. \end{aligned} \tag{A3}$$

We introduce a complete orthonormal basis of Hermitian operators $\{B_\mu\} = \{i^{q(q-1)/2}\chi_{x_1}\chi_{x_2}\ldots\chi_{x_q}\}$ and the expansion coefficient $\alpha_\mu(t)$ is

$$\alpha_\mu(t) = \frac{1}{\mathrm{tr}(B_\mu B_\mu)}\mathrm{tr}(B_\mu O(t)). \tag{A4}$$

Its time evolution is given by

$$\begin{aligned} d\alpha_\mu(t) &= \frac{1}{\mathrm{tr}\left(B_\mu^2\right)}\mathrm{tr}(B_\mu dO(t))\\ &= \frac{i}{\mathrm{tr}\left(B_\mu^2\right)}\mathrm{tr}(B_\mu[dH(t), O(t)])\\ &\quad - \sum_{\langle xy\rangle}\alpha_\mu(t)V dt - \alpha_\mu(t)J dt\\ &\quad - \frac{1}{\mathrm{tr}\left(B_\mu^2\right)}\sum_{\langle xy\rangle}\mathrm{tr}(B_\mu\chi_x\chi_y O(t)\chi_x\chi_y)V dt\\ &\quad + \frac{1}{\mathrm{tr}\left(B_\mu^2\right)}\mathrm{tr}\left(B_\mu\prod_{x\in\square}\chi_x O(t)\prod_{x'\in\square}\chi_{x'}\right)J dt, \end{aligned} \tag{A5}$$

here,

$$\begin{aligned} &\frac{1}{\mathrm{tr}\left(B_\mu^2\right)}\sum_{\langle xy\rangle}\mathrm{tr}(B_\mu\chi_x\chi_y O(t)\chi_x\chi_y)V dt\\ &\quad = \frac{1}{\mathrm{tr}\left(B_\mu^2\right)}\sum_{\langle xy\rangle}\mathrm{tr}(\chi_x\chi_y B_\mu\chi_x\chi_y O(t))V dt\\ &\quad = -\sum_{\langle xy\rangle} q_{\mu,x,y}\alpha_\mu(t)V dt, \end{aligned} \tag{A6}$$

$$\begin{aligned} &\frac{1}{\mathrm{tr}\left(B_\mu^2\right)}\mathrm{tr}\left(B_\mu\prod_{x\in\square}\chi_x O(t)\prod_{x'\in\square}\chi_{x'}\right)J dt\\ &\quad = \frac{1}{\mathrm{tr}\left(B_\mu^2\right)}\mathrm{tr}\left(\prod_{x'\in\square}\chi_{x'}B_\mu\prod_{x\in\square}\chi_x O(t)\right)J dt\\ &\quad = q_{\mu,\prod_{x\in\square}\chi_x}\alpha_\mu(t)J dt, \end{aligned} \tag{A7}$$

where $q_{\mu,x,y} = 1$ if $\chi_x, \chi_y \in B_\mu$ or $\chi_x, \chi_y \notin B_\mu$; $q_{\mu,x,y} = -1$ if $\chi_x \in B_\mu, \chi_y \notin B_\mu$ or $\chi_y \in B_\mu, \chi_x \notin B_\mu$, and $q_{\mu,\prod_{x\in\square}\chi_x} = 1$ if $|\ \chi \in \prod_{x\in\square}\chi_x \mid \chi \in B_\mu\ | \equiv 0 \mod 2$; $q_{\mu,\prod_{x\in\square}\chi_x} = -1$ if $|\ \chi \in \prod_{x\in\square}\chi_x \mid \chi \in B_\mu\ | \equiv 1 \mod 2$; here $|C|$ is the cardinality of set C. We finally get

$$\begin{aligned} d\alpha_\mu(t) &= \frac{i}{\mathrm{tr}\left(B_\mu^2\right)}\mathrm{tr}(B_\mu[dH(t), O(t)])\\ &\quad - \sum_{\langle xy\rangle}\alpha_\mu(t)V dt - \alpha_\mu(t)J dt\\ &\quad + \sum_{\langle xy\rangle} q_{\mu,x,y}\alpha_\mu(t)V dt + q_{\mu,\prod_{x\in\square}\chi_x}\alpha_\mu(t)J dt\\ &= \frac{i}{\mathrm{tr}\left(B_\mu^2\right)}\mathrm{tr}(B_\mu[dH(t), O(t)]) - 2\\ &\quad \times \sum_{\{\langle xy\rangle | q_{\mu,x,y}=-1\}}\alpha_\mu(t)V dt - 2\delta_{q_{\mu,\prod_{x\in\square}\chi_x},-1}\alpha_\mu(t)J dt. \end{aligned} \tag{A8}$$

Define $f_\mu(t)$ to be the average probability at time t

$$f_\mu(t) = \overline{|\alpha_\mu(t)|^2} = \overline{\alpha_\mu^2(t)}, \tag{A9}$$

the evolution is given by

$$df_\mu(t) = 2\overline{\alpha_\mu(t)d\alpha_\mu(t)} + \overline{d\alpha_\mu(t)d\alpha_\mu(t)}. \tag{A10}$$

We have

$$\begin{aligned} df_\mu(t) =& -4\sum_{\{\langle \boldsymbol{xy}\rangle | q_{\mu,x,y}=-1\}} f_\mu(t)Vdt - 4\delta_{q_{\mu,\prod_{x\in\square}\chi_x},-1} f_\mu(t)Jdt \\ & - \frac{1}{\mathrm{tr}^2(B_\mu^2)}\mathrm{tr}^2(O(t)[B_\mu, dH(t)]) \\ =& -4\sum_{\{\langle \boldsymbol{xy}\rangle | q_{\mu,x,y}=-1\}} f_\mu(t)Vdt - 4\delta_{q_{\mu,\prod_{x\in\square}\chi_x},-1} f_\mu(t)Jdt \\ & + \sum_\nu\sum_{\langle \boldsymbol{xy}\rangle}\frac{1}{\mathrm{tr}^2(B_\mu^2)}\mathrm{tr}^2(B_\nu[B_\mu,\chi_x\chi_y])f(B_\nu,t)Vdt \\ & - \sum_\nu \frac{1}{\mathrm{tr}^2(B_\mu^2)}\mathrm{tr}^2\left(B_\nu\left[B_\mu,\prod_{\boldsymbol{x}\in\square}\chi_{\boldsymbol{x}}\right]\right)f(B_\nu,t)Jdt \\ =& -4\sum_{\{\langle \boldsymbol{xy}\rangle | q_{\mu,x,y}=-1\}} f_\mu(t)Vdt - 4\delta_{q_{\mu,\prod_{x\in\square}\chi_x},-1} f_\mu(t)Jdt \\ & + 4\sum_{\{\langle \boldsymbol{xy}\rangle | q_{\mu,x,y}=-1\},\{\nu || B_\mu\chi_x\chi_y| = |B_\nu|\}} f(B_\nu,t)Vdt \\ & + 4\sum_{\{\nu || B_\mu \prod_{x\in\square}\chi_x| = |B_\nu|\}} \delta_{q_{\mu,\prod_{x\in\square}\chi_x},-1} f(B_\nu,t)Jdt, \end{aligned} \tag{A11}$$

where the first equality uses the cyclicity of trace

$$\mathrm{tr}(B_\mu[dH(t),O(t)]) = \mathrm{tr}(O(t)[B_\mu,dH(t)]). \tag{A12}$$

Now we can consider the operator height distribution function

$$f(\mathbf{h},t) = |\alpha_\mu(t)|^2|_{\mathbf{h}_\mu=\mathbf{h}}. \tag{A13}$$

which satisfies the master equation

$$\begin{aligned} \frac{\partial f(\mathbf{h},t)}{\partial t} =& -4\sum_{\langle \boldsymbol{xy}\rangle}\delta_{h_x\oplus h_y,1}Vf(\mathbf{h},t) - 4\delta_{\sum^{\oplus}_{x\in\square}h_x,1}Jf(\mathbf{h},t) \\ & + 4\sum_{\langle \boldsymbol{xy}\rangle}\delta_{h_x\oplus h_y,1}Vf(\mathbf{h}\oplus\mathbf{e}_x\oplus\mathbf{e}_y,t) \\ & + 4\delta_{\sum^{\oplus}_{x\in\square}h_x,1}Jf\left(\mathbf{h}\oplus\sum^{\oplus}_{\boldsymbol{x}\in\square}\mathbf{e}_x,t\right) \\ =& 4V\sum_{\langle \boldsymbol{xy}\rangle}\delta_{h_x\oplus h_y,1}(f(\mathbf{h}\oplus\mathbf{e}_x\oplus\mathbf{e}_y,t) - f(\mathbf{h},t)) \\ & + 4J\delta_{\sum^{\oplus}_{x\in\square}h_x,1}\left(f\left(\mathbf{h}\oplus\sum^{\oplus}_{\boldsymbol{x}\in\square}\mathbf{e}_x,t\right) - f(\mathbf{h},t)\right), \end{aligned} \tag{A14}$$

where $\mathbf{e}_x$ represents a vector that takes the value 1 on site $\boldsymbol{x}$ and 0 at all other sites. The sum $\oplus$ is taken modulo 2.

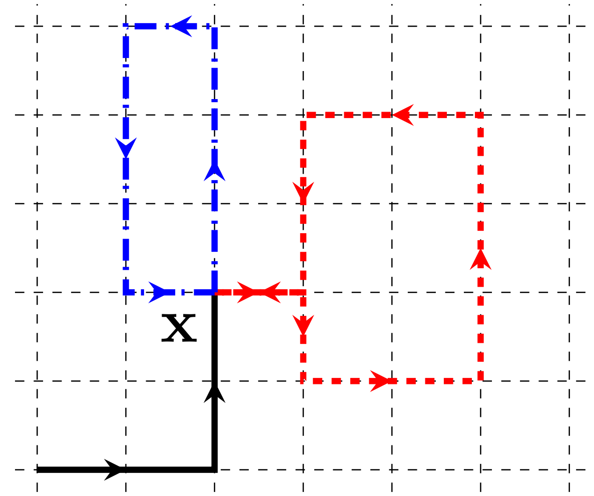

FIG. 4. A two-dimensional random walk starts from the origin at time 0 and reaches position $\boldsymbol{x}$ at $t = 24$. The path is reducible that can be decomposed in three irreducible ones: the black path which contributes to $q_4(\boldsymbol{x})$ with $\boldsymbol{x} = (2, 2)$, and the blue and red paths which contribute to $q_8(\mathbf{0})$ and $q_{12}(\mathbf{0})$, respectively.

APPENDIX B: THE RETURN PROBABILITY OF A RANDOM WALK

The return probability of a random walk is the cumulative probability up to time infinity that the walker comes back where it starts. In 1921, Pólya [48] proved that for a simple random walk, the return probability is 1 when the dimension $d \leqslant 2$, and strictly less than 1 when $d \geqslant 3$. This section reviews the computation of the return probability for a (potentially long range) random walk in a d-dimensional cubic lattice using Fourier transform.

To set the stage, let the walker start from the origin $\mathbf{0}$ of the lattice $\mathbb{Z}^d$ and perform a (Markovian) random walk. For each time step, there is a probability distribution $f(\mathbf{x})$ that determines the displacement $\boldsymbol{x}$. Define $q_n(\boldsymbol{x})$ to be the probability of the walker to reach position $\boldsymbol{x}$ *for the first time* at $t = n$. The probability the walker ever returns to the origin after $t = 0$ is

$$p_{\text{return}} = \sum_{n=0}^{\infty} q_n(\mathbf{0}). \tag{B1}$$

For consistency, we require $q_n(\mathbf{0}) = 0$. Slightly generalizing of the notion, the return probability of walker ever reaching position $\boldsymbol{x}$ is

$$p_{\text{return}}(\boldsymbol{x}) = \sum_{n=0}^{\infty} q_n(\boldsymbol{x}). \tag{B2}$$

1. The recursion relation

The probability $q_n(\boldsymbol{x})$ is given by an irreducible path with the constraint of arriving at $\boldsymbol{x}$ for the first time. It is much easier to figure out the probability $p_n(\boldsymbol{x})$ of unconstrained path that arrives at $\boldsymbol{x}$ at time t. Our initial condition is $p_0(\boldsymbol{x}) = \delta_{\boldsymbol{x},\mathbf{0}}$. The probability $p_n(\boldsymbol{x})$ corresponds to reducible paths which can be decomposed into irreducible paths for each visit of $\boldsymbol{x}$, see Fig. 4. This decomposition can be encapsulated into a recursion relation

$$p_n(\boldsymbol{x}) = \sum_{k=0}^{n} p_{n-k}(\boldsymbol{0}) q_k(\boldsymbol{x}) \begin{cases} n \geqslant 1 & \boldsymbol{x} = \boldsymbol{0} \\ n \geqslant 0 & \boldsymbol{x} \neq \boldsymbol{0} \end{cases}. \tag{B3}$$

In words, the probability of arriving at $\boldsymbol{x}$ at time t is the sum of probability of arriving at $\boldsymbol{x}$ for the first time at time k and the probability of not moving for the rest $n-k$ steps.

We can solve q from p through the generating functions $P(z, \boldsymbol{x}) = \sum_{n=0}^{\infty} p_n(\boldsymbol{x}) z^n$ and $Q(z, \boldsymbol{x}) = \sum_{n=0}^{\infty} q_n(\boldsymbol{x}) z^n$. The recursion relation in Eq. (B3) translates to

$$P(z, \boldsymbol{x}) - \delta_{\boldsymbol{x},\boldsymbol{0}} = P(z, \boldsymbol{0}) Q(z, \boldsymbol{x}). \tag{B4}$$

Therefore

$$Q(z, \boldsymbol{x}) = \begin{cases} 1 - \frac{1}{P(z,\boldsymbol{0})} & \boldsymbol{x} = \boldsymbol{0} \\ \frac{P(z,\boldsymbol{x})}{P(z,\boldsymbol{0})} & \boldsymbol{x} \neq \boldsymbol{0} \end{cases}. \tag{B5}$$

The return probability is

$$p_{\text{return}}(\boldsymbol{x}) = Q(1, \boldsymbol{x}) = \begin{cases} 1 - \frac{1}{\sum_{n=0}^{\infty} p_n(\boldsymbol{0})} & \boldsymbol{x} = \boldsymbol{0} \\ \frac{\sum_{n=0}^{\infty} p_n(\boldsymbol{x})}{\sum_{n=0}^{\infty} p_n(\boldsymbol{0})} & \boldsymbol{x} \neq \boldsymbol{0} \end{cases}. \tag{B6}$$

2. Solution of the reducible probability

For the random walk we consider, the probability satisfies a master equation:

$$p_n(\boldsymbol{x}) = \sum_{\boldsymbol{x}' \in \mathbb{Z}^d} f(\boldsymbol{x} - \boldsymbol{x}') p_{n-1}(\boldsymbol{x}'). \tag{B7}$$

The convolution becomes a product in Fourier space. Define the Fourier transform,

$$\tilde{p}_n(\boldsymbol{k}) = \sum_{\boldsymbol{x} \in \mathbb{Z}^d} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} p_n(\boldsymbol{x}),$$

$$p_n(\boldsymbol{x}) = \int_{[0,2\pi]^d} \prod_{j=1}^{d} \frac{dk_j}{2\pi} \tilde{p}_n(\boldsymbol{k}) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{B8}$$

Then with the initial condition $\tilde{p}_0(\boldsymbol{k}) = 1$, the master equation Eq. (B7) becomes

$$\tilde{p}_n(\boldsymbol{k}) = \tilde{f}(\boldsymbol{k})^n \tilde{p}_0(\boldsymbol{k}) = \tilde{f}(\boldsymbol{k})^n. \tag{B9}$$

Hence

$$p_n(\boldsymbol{x}) = \int_{[0,2\pi]^d} \prod_{j=1}^{d} \frac{dk_j}{2\pi} \tilde{f}(\boldsymbol{k})^n e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{B10}$$

and

$$\sum_{n=0}^{\infty} p_n(\boldsymbol{x}) = \int_{[0,2\pi]^d} \prod_{j=1}^{d} \frac{dk_j}{2\pi} \frac{1}{1 - \tilde{f}(\boldsymbol{k})} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{B11}$$

3. Simple random walks

Let us specialize to simple random walk on $\mathbb{Z}^d$, which means the probability is isotropic in each lattice direction. The transition probability in Fourier space is

$$f(\boldsymbol{k}) = \frac{1}{d} \sum_{j=1}^{d} \cos(k_j). \tag{B12}$$

For small k, the leading order expansion gives $f(\boldsymbol{k}) \sim 1 - \frac{1}{2d}|\boldsymbol{k}|^2$.

The return probability $p_{\text{return}}(\boldsymbol{0})$ is 1 when $\sum_{n=0}^{\infty} p_n(\boldsymbol{0})$ diverges. From the power counting of Eq. (B11) around $k = |\boldsymbol{k}| \sim 0$

$$\sim \int_{[0,2\pi]^d} \prod_{j=1}^{d} \frac{dk_j}{2\pi} \frac{1}{k^2} \sim \int k^{d-1} dk \frac{1}{k^2}, \tag{B13}$$

we can see that the integral diverges for $d \leqslant 2$ and is finite for $d > 2$ (meaning $d \geqslant 3$ for integer dimension), thus verifies Pólya's theorem.

For $d = 3$, the return probability is

$$p_{\text{return}}(\boldsymbol{0}) = 1 - \frac{1}{\int_{[0,2\pi]^3} \prod_{j=1}^{3} \frac{dk_j}{2\pi} \frac{1}{1 - \frac{1}{3}(\cos k_1 + \cos k_2 + \cos k_3)}} \approx 0.340537. \tag{B14}$$

We can also analyze the scalings of $x = \boldsymbol{x}$ for $p_{\text{return}}(\boldsymbol{x})$

$$\sim \int_{[0,2\pi]^d} \prod_{j=1}^{d} \frac{dk_j}{2\pi} \frac{1}{k^2} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \sim \int k^{d-1} dk \frac{1}{k^2} e^{-ikx} \sim \frac{1}{x^{d-2}}. \tag{B15}$$

4. Lévy flight

A d-dimensional Lévy flight has a transition probability with long range tail

$$f(\boldsymbol{x}) \sim \frac{1}{|\boldsymbol{x}|^{d+\alpha}} \quad \alpha \in (0, 2]. \tag{B16}$$

In Fourier space,

$$\tilde{f}(\boldsymbol{k}) \sim 1 - \#|\boldsymbol{k}|^{\alpha} \tag{B17}$$

for small $k = |\boldsymbol{k}|$.

The analysis of $p_{\text{return}}(\boldsymbol{0})$ and $p_{\text{return}}(\boldsymbol{x})$ can be carried over.

$$p_{\text{return}}(\boldsymbol{0}) \sim \int k^{d-1} dk \frac{1}{k^{\alpha}}, \tag{B18}$$

which is convergent when $d > \alpha$. In other words, the return probability is 1 when $\alpha > d$ and less than 1 when $\alpha < d$.

For the latter case,

$$p_{\text{return}}(\boldsymbol{x}) \sim \int k^{d-1} dk \frac{1}{k^{\alpha}} e^{-ikx} \sim \frac{1}{x^{d-\alpha}}. \tag{B19}$$

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