One-Tape Turing Machine and Branching Program Lower Bounds for MCSP

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Abstract

For a size parameter $s : \mathbb{N} \to \mathbb{N}$, the Minimum Circuit Size Problem (denoted by MCSP$[s(n)]$) is the problem of deciding whether the minimum circuit size of a given function $f : \{0,1\}^n \to \{0,1\}$ (represented by a string of length $N := 2^n$) is at most a threshold $s(n)$. A recent line of work exhibited "hardness magnification" phenomena for MCSP: A very weak lower bound for MCSP implies a breakthrough result in complexity theory. For example, McKay, Murray, and Williams (STOC 2019) implicitly showed that, for some constant $\mu_1 > 0$, if MCSP$[2^{\mu_1 \cdot n}]$ cannot be computed by a one-tape Turing machine (with an additional one-way read-only input tape) running in time $N^{1.01}$, then P $\neq$ NP.

In this paper, we present the following new lower bounds against one-tape Turing machines and branching programs:

1. A randomized two-sided error one-tape Turing machine (with an additional one-way read-only input tape) cannot compute MCSP$[2^{\mu_2 \cdot n}]$ in time $N^{1.99}$, for some constant $\mu_2 > \mu_1$.
2. A non-deterministic (or parity) branching program of size $o(N^{1.5}/\log N)$ cannot compute MKTP, which is a time-bounded Kolmogorov complexity analogue of MCSP. This is shown by directly applying the Nečiporuk method to MKTP, which previously appeared to be difficult.
3. The size of any non-deterministic, co-non-deterministic, or parity branching program computing MCSP is at least $N^{1.5-o(1)}$.

These results are the first non-trivial lower bounds for MCSP and MKTP against one-tape Turing machines and non-deterministic branching programs, and essentially match the best-known lower bounds for any explicit functions against these computational models.

The first result is based on recent constructions of pseudorandom generators for read-once oblivious branching programs (ROBPs) and combinatorial rectangles (Forbes and Kelley, FOCS 2018; Viola 2019). En route, we obtain several related results:

1. There exists a (local) hitting set generator with seed length $\widetilde{O}(\sqrt{N})$ secure against read-once polynomial-size non-deterministic branching programs on N-bit inputs.
2. Any read-once co-non-deterministic branching program computing MCSP must have size at least $2^{\widetilde{\Omega}(N)}$.

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1 Introduction

The Minimum Circuit Size Problem (MCSP) asks whether a given Boolean function $f : \{0,1\}^n \to \{0,1\}$ can be computed by some Boolean circuit of size at most a given threshold s. Here the function f is represented by the truth table of f, i.e., the string of length $N := 2^n$ that is obtained by concatenating all the outputs of f. For a size parameter $s : \mathbb{N} \to \mathbb{N}$, its parameterized version is denoted by MCSP[s]: That is, MCSP[s] asks if the minimum circuit size of a function $f : \{0,1\}^n \to \{0,1\}$ is at most $s(n)$.

MCSP is one of the most fundamental problems in complexity theory, because of its connection to various research areas, such as circuit complexity [38, 27, 24, 33, 23, 2], learning theory [9], and cryptography [38, 18, 20]. It is easy to see that MCSP $\in \mathsf{NP}$ because, given a circuit C of size s as an NP certificate, one can check whether C computes the given function f in time $N^{O(1)}$. On the other hand, its NP-completeness is a long-standing open question, which dates back to the introduction of the theory of NP-completeness (cf. [4]), and it has an application to the equivalence between the worst-case and average-case complexity of NP (cf. [20]).

Recently, a line of work exhibited surprising connections between very weak lower bounds of MCSP and important open questions of complexity theory, informally termed as "hardness magnification" phenomena. Oliveira and Santhanam [37] (later with Pich [36]) showed that, if an approximation version of MCSP cannot be computed by a circuit of size $N^{1.01}$, then $\mathsf{NP} \not\subseteq \mathsf{P/poly}$ (in particular, $\mathsf{P} \neq \mathsf{NP}$ follows). Similarly, McKay, Murray, and Williams [32] showed that, if MCSP[$s(n)$] cannot be computed by a 1-pass streaming algorithm of poly $(s(n))$ space and poly $(s(n))$ update time, then $\mathsf{P} \neq \mathsf{NP}$. Therefore, in order to obtain a breakthrough result, it is sufficient to obtain a very weak lower bound for MCSP.

Are hardness magnification phenomena plausible approaches for resolving the P versus NP question? We do not know the answer yet. However, it should be noted that, as argued in [3, 37], hardness magnification phenomena appear to bypass the *natural proof* barrier of Razborov and Rudich [38], which is one of the major barriers of complexity theory for resolving the P versus NP question. Most of lower bound proof techniques of complexity theory are "natural" in the following sense: Given a lower bound proof for a circuit class $\mathfrak{C}$, one can interpret it as an efficient average-case algorithm for solving $\mathfrak{C}$-MCSP (i.e., one can efficiently decide whether a given Boolean function f can be computed by a small $\mathfrak{C}$-circuit when the input f is chosen uniformly at random; cf. Hirahara and Santhanam [22]). Razborov and Rudich [38] showed that such a "natural proof" technique is unlikely to resolve $\mathsf{NP} \not\subseteq \mathsf{P/poly}$; thus we need to develop fundamentally new proof techniques. There seems to be no simple argument that naturalizes proof techniques of hardness magnification phenomena; hence, investigating hardness magnification phenomena could lead us to a new non-natural proof technique.

1.1 Our results

1.1.1 Lower bounds against one-tape Turing machines

Motivated by hardness magnification phenomena, we study the time required to compute MCSP by using a one-tape Turing machine. We first observe that the hardness magnification phenomena of [32] imply that a barely superlinear time lower bound for a one-tape Turing machine is sufficient for resolving the P versus NP question.

▶ **Theorem 1** (A corollary of McKay, Murray, and Williams [32]; see Appendix A)**.** *There exists a small constant* $\mu > 0$ *such that if* $\mathrm{MCSP}[2^{\mu \cdot n}] \notin \mathsf{DTIME}_1\left[N^{1.01}\right]$*, then* $\mathsf{P} \neq \mathsf{NP}$*.*

Here, we denote by $\mathsf{DTIME}_1[t(N)]$ the class of languages that can be computed by a Turing machine equipped with a one-way read-only input tape and a two-way read/write work tape running in time $O(t(N))$ on inputs of length N. We note that it is rather counter-intuitive that there is a *universal* constant $\mu > 0$; it is instructive to state Theorem 1 in the following logically equivalent way: If $\mathrm{MCSP}[2^{\mu \cdot n}] \notin \mathsf{DTIME}_1\left[N^{1.01}\right]$ for *all constants* $\mu > 0$, then $\mathsf{P} \neq \mathsf{NP}$.[1]

One of our main results is a nearly quadratic lower bound on the time complexity of a *randomized* one-tape Turing machine (with one additional read-only one-way input tape) computing MCSP.

▶ **Theorem 2.** *There exists some constant* $0 < \mu < 1$ *such that* $\mathrm{MCSP}[2^{\mu \cdot n}]$ *is not in* $\mathsf{BPTIME}_1\left[N^{1.99}\right]$*.*

Here, $\mathsf{BPTIME}_1[t(N)]$ denotes the class of languages that can be computed by a *two-sided-error randomized* Turing machine equipped with a one-way read-only input tape and a two-way read/write work tape running in time $t(N)$ on inputs of length N; we say that a two-sided-error randomized algorithm *computes* a problem if it outputs a correct answer with high probability (say, with probability at least $2/3$) over the internal randomness of the algorithm.

Previously, no non-trivial lower bound on the time complexity required for computing MCSP by a Turing machine was known. Moreover, Theorem 2 essentially matches the best-known lower bound for this computational model; namely, the lower bound due to Kalyanasundaram and Schnitger [28], who showed that Element Distinctness is not in $\mathsf{BPTIME}_1\left[o\left(N^2/\log N\right)\right]$.

Our lower bound against $\mathsf{BPTIME}_1\left[N^{1.99}\right]$ is much stronger than the required lower bound (i.e, $\mathsf{DTIME}_1\left[N^{1.01}\right]$) of the hardness magnification phenomenon of Theorem 1. However, Theorem 2 falls short of the hypothesis of the hardness magnification phenomenon of Theorem 1 because of the choice of the size parameter. In the hardness magnification phenomenon, we need to choose the size parameter to be $2^{\mu \cdot n}$ for some small constant $\mu > 0$, whereas, in our lower bound, we will choose μ to be some constant close to 1. That is, what is missing for proving $\mathsf{P} \neq \mathsf{NP}$ is to decrease the size parameter from $2^{(1-o(1)) \cdot n}$ to $2^{o(n)}$ in Theorem 2, or to increase the size parameter from $2^{o(n)}$ to $2^{(1-o(1)) \cdot n}$ in Theorem 1.

Next, we investigate the question of whether hardness magnification phenomena on $\mathrm{MCSP}[s(n)]$ such as Theorem 1 can be proved when the size parameter $s(n)$ is large, as posed by Chen, Jin, and Williams [11]. As observed in [10], most existing proof techniques on hardness magnification phenomena are shown by constructing an oracle algorithm which

[1] Observe that $\exists \mu, (P(\mu) \Rightarrow Q)$ is logically equivalent to $\exists \mu, (\neg P(\mu) \vee Q)$, which is equivalent to $\neg(\forall \mu, P(\mu)) \vee Q$.

makes short queries to some oracle. For example, behind the hardness magnification phenomena of Theorem 1 is a nearly-linear-time oracle algorithm that solves MCSP[$2^{o(n)}$] by making queries of length $2^{o(n)}$ to some PH oracle (see Corollary 22 for a formal statement). Chen, Hirahara, Oliveira, Pich, Rajgopal, and Santhanam [10] showed that most lower bound proof techniques can be generalized to such an oracle algorithm, thereby explaining the difficulty of combining hardness magnification phenomena with lower bound proof techniques. Following [10], we observe that our lower bound (Theorem 3) can be generalized to a lower bound against an oracle algorithm which makes short queries.

▶ **Theorem 3.** *Let $O \subseteq \{0,1\}^*$ be any oracle. Then, for every constant $1/2 < \mu < 1$, MCSP[$2^{\mu \cdot n}$] on truth tables of size $N := 2^n$ is not in $\mathsf{BPTIME}_1^O\left[N^{1+\mu'}\right]$ for some constant $\mu' > 0$, where all of the strings queried to O are of length $N^{o(1)}$.*

Theorem 3 can be seen as a partial answer to the question posed by [11]: It is impossible to extend the hardness magnification phenomena of Theorem 1 to MCSP[$2^{\mu n}$] for $\mu > 1/2$ by using similar techniques used in [32]. Recall that the proof techniques behind [32] are to construct a nearly-linear-time oracle algorithm that solves MCSP[$2^{\mu n}$] by making short queries to some oracle; the existence of such an oracle algorithm is ruled out by Theorem 3 when $\mu > 1/2$. Therefore, in order to obtain a hardness magnification phenomenon for MCSP[$2^{0.51n}$], one needs to develop a completely different proof technique that does not rely on constructing an oracle algorithm that makes short queries.

1.1.2 Lower bounds against branching programs

Another main result of this work is a lower bound against non-deterministic branching programs. We make use of *Nečiporuk's method*, which is a standard proof technique for proving a lower bound against branching programs. However, it appeared previously that Nečiporuk's method is not directly applicable to the problems such as MCSP [22]. In this paper, we develop a new proof technique for applying Nečiporuk's method to a variant of MCSP, called MKTP. MKTP is the problem of deciding whether $\mathrm{KT}(x) \leq s$ given (x, s) as input. Here $\mathrm{KT}(x)$ is defined as the minimum, over all programs M and integers t, of $|M| + t$ such that, for every i, M outputs the i-th bit of x in time t given an index i as input [1]. We prove lower bounds against general branching programs and non-deterministic branching programs by using Nečiporuk's method.

▶ **Theorem 4.** *The size of a branching program computing* MKTP *is at least $\Omega(N^2/\log^2 N)$. The size of a non-deterministic branching program or a parity branching program computing* MKTP *is at least $\Omega(N^{1.5}/\log N)$.*

Theorem 4 gives the first non-trivial lower bounds against non-deterministic and parity branching programs for MKTP and, in addition, these are the best lower bounds which can be obtained by using Nečiporuk's method (cf. [7]). Previously, by using a pseudorandom generator for branching programs constructed by [25], it was shown in [36, 12] that (deterministic) branching programs require $N^{2-o(1)}$ size to compute MCSP and MKTP.[2] Surprisingly, Theorem 4 is proved without using a pseudorandom generator nor a weaker object called a hitting set generator.

[2] It is worthy of note that Theorem 4 mildly improves the lower bounds of [36, 12] to $\Omega(N^2/\log^2 N)$ by directly applying Nečiporuk's method, which matches the state-of-the-art lower bound for any explicit function up to a constant factor.

At this point it is interesting to mention that the results of Theorem 4 hold for MKtP too — a resource-bounded Kolmogorov complexity measure that is similar to MKTP. We emphasize that it is surprising that a lower bound for MKtP can be obtained without using a hitting set generator; indeed, the complexity of MKtP is closely related to a hitting set generator, and in many settings (especially when the computational model is capable of computing XOR), a lower bound for MKtP and the existence of a hitting set generator are equivalent [20, 21].

The proof technique of Theorem 4 is applicable to problems of computing various resource-bounded Kolmogorov complexity measures, such as MKtP. However, we fail to apply Nečiporuk's method to MCSP, despite that circuit complexity can also be regarded as a version of resource-bounded Kolmogorov complexity. The KT-complexity of the truth table of a function f and the minimum circuit size of f are polynomially related to each other [1]; unfortunately, the relationship between circuit complexity and KT-complexity is not tight enough for our argument to work. Nevertheless, we were able to use a different approach to present the first non-trivial lower bound for MCSP against non-deterministic branching programs.

▶ **Theorem 5.** *The size of any non-deterministic, co-non-deterministic, or parity branching program computing* MCSP *is at least* $N^{1.5-o(1)}$.

The proof of Theorem 5 is based on a pseudorandom generator construction of Impagliazzo, Meka, and Zuckerman [25]. We show that their construction actually provides a pseudorandom generator of seed length $s^{2/3+o(1)}$ that fools non-deterministic, co-non-deterministic, and parity branching programs of size s.

Along the way, we obtain several new results regarding a lower bound for MCSP and a hitting set generator. A *hitting set generator* (HSG) $H : \{0,1\}^{\lambda(N)} \to \{0,1\}^N$ for a circuit class $\mathfrak{C}$ is a function such that, for any circuit C from $\mathfrak{C}$ that accepts at least $(1/2) \cdot 2^N$ strings of length N, there exists some seed $z \in \{0,1\}^{\lambda(N)}$ such that C accepts $H(z)$.

We present a hitting set generator secure against read-once non-deterministic branching programs, based on a pseudorandom generator constructed by Forbes and Kelley [14].

▶ **Theorem 6.** *There exists an explicit construction of a (local) hitting set generator* $H : \{0,1\}^{\widetilde{O}\left(\sqrt{N \cdot \log s}\right)} \to \{0,1\}^N$ *for read-once non-deterministic branching programs of size* s.

Previously, Andreev, Baskakov, Clementi, and Rolim [6] constructed a hitting set generator with non-trivial seed length for read-k-times non-deterministic branching programs, but their seed length is as large as $N - o(N)$. Theorem 6 improves the seed length to $\widetilde{O}(\sqrt{N \cdot \log s})$. As an immediate corollary, we obtain a lower bound for MCSP against read-once non-deterministic branching programs.

▶ **Corollary 7.** *Any read-once co-non-deterministic branching program that computes* MCSP *must have size at least* $2^{\widetilde{\Omega}(N)}$.

1.2 Our techniques

1.2.1 Local HSGs for MCSP lower bounds

For a circuit class $\mathfrak{C}$, a general approach for obtaining a $\mathfrak{C}$-lower bound for MCSP is by constructing a "local" hitting set generator (or a pseudorandom generator (PRG), which is a stronger notion) secure against $\mathfrak{C}$. Here, we say that a function $G : \{0,1\}^s \to \{0,1\}^N$ is *local* if, for every z, the ith bit of $G(z)$ is "easy to compute" from the index i; more precisely, for

every seed z, there exists some circuit C of size at most s such that C outputs the ith bit of $G(z)$ on input $i \in [N]$. Note here that $G(z)$ is a YES instance of MCSP$[s]$, whereas a string w chosen uniformly at random is a NO instance of MCSP$[s]$ with high probability. This means that any $\mathfrak{C}$-algorithm that computes MCSP$[s]$ distinguishes the pseudorandom distribution $G(z)$ from the uniform distribution w, and hence the existence of $\mathfrak{C}$-algorithm for MCSP$[s]$ implies that there exists no local hitting set generator secure against $\mathfrak{C}$. This approach has been used in several previous works, e.g., [38, 1, 22, 12]. In fact, it is worthy of note that, in some sense, this is *the only approach* — at least for a general polynomial-size circuit class $\mathfrak{C} = \mathsf{P/poly}$, because Hirahara [20] showed that a lower bound for an approximation version of MCSP is equivalent to the existence of a local HSG.

At the core of our results is the recent breakthrough result of Forbes and Kelley [14], who constructed the first pseudorandom generator with $\mathsf{polylog}(n)$ seed length that fools unknown-order read-once oblivious branching programs. Viola [42] used their construction to obtain a pseudorandom generator that fools *deterministic* Turing machines (DTMs). Herein, we generalize his result to the case of *randomized* Turing machine (RTMs), and the case of *two-sided-error* randomized Turing machine ($\mathsf{BPTIME}_1[t(N)]$).[3] At a high level, our crucial idea is that Viola's proof does not exploit the uniformity of Turing machines, and hence a good coin flip sequence of a randomized oracle algorithm and all of its (small enough) oracle queries and corresponding answers can be fixed as non-uniformity (Lemma 26). In addition, by a careful examination of the Forbes-Kelley PRG, we show that their PRG is local; this gives rise to a local PRG that fools $\mathsf{BPTIME}_1[t(N)]$, which will complete a proof of our main result (Theorem 3).

We note that the proof above implicitly shows an exponential-size lower bound for MCSP against read-once oblivious branching programs, which was previously not known. Corollary 7 generalizes this lower bound to the case of co-non-deterministic read-once (not necessarily oblivious) branching program. In order to prove this, we make use of PRGs that fool combinatorial rectangles (e.g., [14, 29]). We present a general transformation from a PRG for combinatorial rectangles into a HSG for non-deterministic read-once branching program, by using the proof technique of Borodin, Razborov, and Smolensky [8]; see Theorem 6.

1.2.2 Nečiporuk's method for MKTP lower bounds

In order to apply Nečiporuk's method to MKTP, we need to give a lower bound on the number of distinct subfunctions that can be obtained by fixing all but $O(\log n)$ bits.

The idea of counting distinct subfunctions of MKTP is to show that a random restriction which leaves $O(\log n)$ variables free induces different subfunctions with high probability. Specifically, partition the input variables $[n]$ into $m := n/O(\log n)$ blocks, pick $m-1$ strings $\rho := \rho_2 \cdots \rho_m \in (\{0,1\}^{O(\log n)})^{m-1}$ randomly, and consider the restricted function $f\restriction_\rho(\rho_1) := \mathrm{MKTP}(\rho_1\rho_2\cdots\rho_m, \theta)$ for some threshold function θ to be chosen later. Then, the string $\rho_i\rho_2\cdots\rho_m$ is compressible when $i \in \{2, \cdots, m\}$ whereas the string $\rho_1\rho_2\cdots\rho_m$ is not compressible when ρ_1 is chosen randomly. This holds as, in the former case, there exists a $k \in \{2, \ldots, m\}$ such that $\rho_i = \rho_k$ and this yields a description for the string $\rho_i\rho_2\cdots\rho_m$ that is shorter than most of its descriptions in the latter case. Let now θ be an upper bound on the KT complexity of $\rho_i\rho_2\cdots\rho_m$ in the case where $i \in \{2, \cdots, m\}$. Therefore, $f\restriction_\rho(\rho_i) = 1$ for any ρ and $i \in \{2, \cdots, m\}$, and $f\restriction_\rho(\rho_1) = 0$ with high probability over random ρ and ρ_1.

[3] We emphasize that the notion of PRGs secure against these three computational models is different. See Definition 14, Definition 17, and Lemma 19.

This implies that, with high probability over the random restrictions ρ and ρ', it is the case that $f\restriction_\rho \not\equiv f\restriction_{\rho'}$. This is so as, for every $i \in \{2, \ldots, m\}$, the probability over the random restrictions ρ and ρ' that the string ρ_i is such that $f\restriction_{\rho'}(\rho_i) = f\restriction_\rho(\rho_i)$ is small, by the fact that $f\restriction_\rho(\rho_i) = 1$ for any ρ and the fact that $f\restriction_{\rho'}(\rho_i) = 0$ with high probability over random ρ_i and ρ' (and therefore with high probability over random ρ and ρ' as well).

Unfortunately, the probability that $f\restriction_\rho \equiv f\restriction_{\rho'}$ holds may not be exponentially small. As a consequence, a lower bound on the number of distinct subfunctions that can be directly obtained from this fact may not be exponential. In contrast, we need to prove an exponential lower bound on the number of distinct subfunctions in order to obtain the state-of-the-art lower bound via Nečiporuk's method.

In order to make the argument work, we exploit symmetry of information for (resource-unbounded) Kolmogorov complexity and Kolmogorov-randomness. Instead of picking ρ and ρ' randomly, we keep a set P which contains restrictions ρ that induce distinct subfunctions. Starting from $P := \emptyset$, we add one Kolmogorov-random restriction ρ to P so that the property of P is preserved. By using symmetry of information for Kolmogorov complexity, we can argue that one can add a restriction to P until P becomes as large as $2^{\Omega(n)}$, which proves that the number of distinct subfunctions of MKTP is exponentially large. Details can be found in Section 6.

1.3 Related work

Chen, Jin, and Williams [11] generalized hardness magnification phenomena to arbitrary sparse languages in NP. Note that MCSP$[2^{\mu n}]$ is a *sparse* language in the sense that the number of YES instances of MCSP$[2^{\mu n}]$ is at most $2^{\widetilde{O}(2^{\mu n})}$, which is much smaller than the number 2^{2^n} of all the instances of length 2^n. Hirahara [21] proved that a super-linear-size lower bound on co-non-deterministic branching programs for computing an approximation and space-bounded variant of MKtP implies the existence of a hitting set generator secure against read-once branching programs (and, in particular, $\mathsf{RL} = \mathsf{L}$).

Regarding unconditional lower bounds for MCSP, Razborov and Rudich [38] showed that there exists no AC^0-natural property useful against $\mathsf{AC}^0[\oplus]$, which in particular implies that MCSP $\notin \mathsf{AC}^0$; otherwise, the complement of MCSP would yield an AC^0-natural property useful against $\mathsf{P/poly} \supseteq \mathsf{AC}^0[\oplus]$. Hirahara and Santhanam [22] proved that MCSP essentially requires quadratic-size de Morgan formulas. Cheraghchi, Kabanets, Lu, and Myrisiotis [12] proved that MCSP essentially requires cubic-size de Morgan formulas as well as quadratic-size (general, unconstrained) branching programs. Golovnev, Ilango, Impagliazzo, Kabanets, Kolokolova, and Tal [16] proved that, for any prime p, MCSP requires constant-depth circuits, that are augmented with MOD_p gates, of weakly-exponential size.

The state-of-the-art time lower bound against DTMs on inputs of size n is $\Omega\big(n^2\big)$, proved by Maass [30], for the Polydromes function (which is a generalization of Palindromes). Regarding the case when the considered DTMs have a two-way read-only input tape, Maass and Schorr [31] proved that there is some problem in $\Sigma_2\mathsf{TIME}[n]$ that requires $\Omega\big(n^{3/2}/\log^6 n\big)$ time to compute on such machines. As mentioned earlier, in Section 1.1, the state-of-the-art time lower bound against RTMs is due to Kalyanasundaram and Schnitger [28], who showed that Element Distinctness is not in $\mathsf{BPTIME}_1\big[o\big(N^2/\log N\big)\big]$.

Viola [42] gave a PRG that fools RTMs that run in time $n^{1+\Omega(1)}$; this also yields a $n^{1+\Omega(1)}$ time lower bound against such machines. To do this, Viola extended prior work [31, 41] on simulating any RTM by a sum of ROBPs (see Lemma 24) and then employed the PRG by

Haramaty, Lee, and Viola [17] that fools ROBPs;[4] it is a straightforward observation [42], then, that the Forbes-Kelley PRG [14] (which appeared afterwards and was inspired by the PRG by Haramaty, Lee, and Viola) yields a PRG of nearly quadratic stretch that fools RTMs and, therefore, a nearly quadratic lower bound against the same model as well. Moreover, Viola [42] showed that there exists some problem in $\Sigma_3\mathsf{TIME}[n]$ that requires $n^{1+\Omega(1)}$ time to compute on any RTM that has the extra feature of a two-way read-only input tape; one of the ingredients of this result, is again the PRG by Haramaty, Lee, and Viola [17].

For the case of one-tape TMs with no extra tapes, Hennie [19] proved in 1965 that the Palindromes function requires $\Omega\big(n^2\big)$ time to compute. Van Melkebeek and Raz [41] observed fixed-polynomial time lower bounds for SAT against non-deterministic TMs with a d-dimensional read/write two-way work tape and a random access read-only input tape; these lower bounds depend on d.

1.4 Organization

In Section 2, we give the necessary background. The main ideas of Theorem 3 and Theorem 6 are described in Section 3 and Section 4, respectively. In Section 5, we show that the pseudorandom generators we make use of are local, which will complete the proofs of Theorem 3 and Theorem 6. We prove Theorem 5 in Section 7 and Theorem 1 in Appendix A.

2 Preliminaries

All logarithms are considered to be taken to the base 2. Let $x \in \{0,1\}^n$ be a string; we denote by x_i the i-th bit of x. If $L \subseteq \{0,1\}^*$ is a language, then $\mathsf{co}L$ denotes $\{0,1\}^* \setminus L$. If $\mathfrak{C}$ is a class of circuits (say), then $\mathsf{co}\mathfrak{C}$ denotes a class of circuits C such that $C = \neg C'$ for some $C' \in \mathfrak{C}$.

2.1 Restrictions

Let $\rho \in \{0,1,*\}^n$ be a restriction. We define the set of *active (or unrestricted) variables of* ρ to be the set $\{i \in [n] \mid \rho(i) = *\}$.

Let $f : \{0,1\}^n \to \{0,1\}$ be a Boolean function and $\rho \in \{0,1,*\}^n$ a restriction. The *ρ-restricted version of* f is a function, denoted by $f\restriction_\rho$, such that for any $x \in \{0,1\}^n$ it is the case that $f\restriction_\rho(x) := f(y)$ where $y \in \{0,1\}^n$ and, for all $1 \leq i \leq n$, $y_i := \rho(i)$ if $\rho(i) \in \{0,1\}$, else $y_i := x_i$.

We say that a distribution of restrictions $\mathcal{D}$ on $\{0,1,*\}^n$ is *p-regular* if for every $i \in [n]$ it is the case that $\mathbf{Pr}_{\rho\sim\mathcal{D}}[\rho(i) = *] = p$.

We say that a distribution of restrictions $\mathcal{D}$ on $\{0,1,*\}^n$ is *k-wise independent* if any k coordinates of $\mathcal{D}$ are independent.

2.2 Circuit complexity

Let $f : \{0,1\}^n \to \{0,1\}$. We define the *circuit complexity of* f, denoted by $\mathrm{CC}(f)$, to be equal to the size (i.e., the number of gates) of the smallest bounded fan-in unbounded fan-out Boolean circuit, over the $\{\mathrm{AND}, \mathrm{OR}, \mathrm{NOT}\} = \{\wedge, \vee, \neg\}$ basis, that, on input x, outputs

[4] It should be noted that before Haramaty, Lee, and Viola [17] and Viola [42], the problem of designing PRGs of polynomial stretch that fool RTMs was wide open despite intense research efforts.

$f(x)$. For a string $y \in \{0,1\}^{2^n}$, we denote by $\mathrm{CC}(y)$ the circuit complexity of the function $f_y : \{0,1\}^n \to \{0,1\}$ encoded by y; i.e., $f_y(x) = y_x$, for any $x \in \{0,1\}^n$.

A standard counting argument shows that a random function attains nearly maximum circuit complexity with high probability.

▶ **Proposition 8** ([39]). *For any function $s : \mathbb{N} \to \mathbb{N}$ with $s(n) = o(2^n/n)$, it holds that*

$$\Pr_{x \sim \{0,1\}^{2^n}}[\mathrm{CC}(x) \leq s(n)] = o(1),$$

for all large $n \in \mathbb{N}$.

▶ **Definition 9** (Minimum Circuit Size Problem [27]). *We define* MCSP *as*

$$\mathrm{MCSP} := \left\{(x, \theta) \in \{0,1\}^{2^n} \times \{0,1\}^n \mid \mathrm{CC}(x) \leq \theta\right\}_{n \in \mathbb{N}},$$

and its parameterized version *as*

$$\mathrm{MCSP}[s(n)] := \left\{x \in \{0,1\}^{2^n} \mid \mathrm{CC}(x) \leq s(n)\right\}_{n \in \mathbb{N}},$$

for a size parameter $s : \mathbb{N} \to \mathbb{N}$.

Note that in Definition 9 the size θ denotes a number in $\{0, \ldots, 2^n - 1\}$.

2.3 Turing machines

Throughout this paper, we consider a Turing machine that has one work tape and a one-way input tape. In this context, "one-way" means that the tape-head may move only from left to right.

A *deterministic Turing machine (DTM)* is a Turing machine with two tapes: A two-way read/write work tape and a one-way read-only input tape. Let $x \in \{0,1\}^*$ and M be a DTM; we write $M(x)$ to denote the output of M when its input tape is initialized with x and its work tape is empty. Let $t : \mathbb{N} \to \mathbb{N}$ be time-constructible. The class of languages $L \subseteq \{0,1\}^*$ decided by some $O(1)$-state time-t DTM is denoted by $\mathsf{DTIME}_1[t]$.

We also consider a randomized variant of DTMs. A *randomized Turing machine (RTM)* is a Turing machine with three tapes: A two-way read/write work tape, a one-way read-only input tape, and a one-way read-only random tape. Let $x, r \in \{0,1\}^*$ and M be a RTM; we write $M(x, r)$ to denote the output of M when its input tape contains x, its work tape is empty, and its random tape contains r. Let $t : \mathbb{N} \to \mathbb{N}$ be time-constructible. For a language $L \subseteq \{0,1\}^*$ and a RTM M, we say that M *decides L with two-sided error* if $\Pr_r[M(x, r) = 1] \geq \frac{2}{3}$ for every input $x \in L$ and $\Pr_r[M(x, r) = 0] \geq \frac{2}{3}$ for every input $x \notin L$. The class of languages $L \subseteq \{0,1\}^*$ decided by some $O(1)$-state time-t RTM with two-sided error is denoted by $\mathsf{BPTIME}_1[t]$.

A *randomized oracle Turing machine (oracle RTM)* is a Turing machine with four tapes: A two-way read/write work tape, a one-way read-only input tape, a one-way read-only random tape, and an oracle tape. This model is identical to the randomized Turing machine model apart from the oracle tape, which is a standard oracle tape. The class of languages $L \subseteq \{0,1\}^*$ decided by some $O(1)$-state time-t oracle RTM, with access to some oracle $O \subseteq \{0,1\}^*$, with two-sided error is denoted by $\mathsf{BPTIME}_1^O[t]$.

2.4 Streaming algorithms

A *space-$s(n)$ streaming algorithm* with update time $u(n)$ on an input $x \in \{0,1\}^n$ has a working storage of $s(n)$ bits. At any point the algorithm can either choose to perform one operation on $O(1)$ bits in storage or it can choose to read the next bit from the input. The total time between two next-bit reads is at most $u(n)$ and the final outcome is reported in $O(u(n))$ time. A streaming algorithm A is said to be *one-pass* if A reads its input exactly once.

▶ **Lemma 10.** *Any one-pass streaming algorithm with $t(N)$ update time, on inputs of length N, can be simulated by a one-tape Turing machine with a one-way read-only input tape running in time $O(N \cdot \text{poly}(t(N)))$.*

Proof. Recall that a streaming algorithm reads one bit of its input from left to right, and each consecutive read operation occurs within $t(N)$ time steps. Thus, it takes $N \cdot \text{poly}(t(N))$ time-steps in total to finish the computation on inputs of length N in the standard multi-tape Turing machine model, as the size of the input is N and $\text{poly}(t(N))$ time-steps suffice for some multi-tape Turing machine to perform an update [13]. For any time constructible function $T : \mathbb{N} \to \mathbb{N}$, a one-tape Turing machine can simulate a $T(n)$-time multi-tape Turing machine within $O(T(n)^2)$ steps. Thus, a streaming algorithm can be simulated in time $N \cdot (\text{poly}(t(N)))^2 = N \cdot \text{poly}(t(N))$ by a one-tape Turing machine. ◀

2.5 Branching programs

A *branching program (BP)* is a directed acyclic graph with three special vertices: a start vertex s (the *source*) and two finish vertices, namely an accepting vertex h_1 and a rejecting vertex h_0 (the *sinks*).

On input $x \in \{0,1\}^n$, the computation starts at s and follows a directed path from s to some h_b, with $b \in \{0,1\}$. On this occasion, the output of the computation is b. In each step, the computation queries some input x_i, for $i \in [n]$, and then visits some other node depending on the value of the variable just queried namely 0 or 1, through an edge with label "$x_i = 0$" or "$x_i = 1$," respectively.

A branching program P *decides a language* $L \subseteq \{0,1\}^*$ in the natural way, i.e., $x \in L$ if and only if, on input x, the computation path that P follows starts at s and finishes at h_1. If the branching program is layered (whereby the nodes are partitioned into a number of layers, and edges go only from nodes in one layer to nodes in the next; the start node is in the first layer and the sink nodes in the last) and the variable queried within each layer is the same, then the branching program is called *oblivious*. If the branching program queries each variable at most once, then the branching program is called a *read-once branching program (ROBP)*. If the branching program is oblivious and always queries the variables in some known order, where it is known beforehand which variable is queried at each layer, then the branching program is called *known-order*, else it is called *unknown-order*.

A branching program is called *non-deterministic* if some of its vertices have an arbitrary number of outgoing edges (i.e., if this number is not 2) or if some of its vertices have edges that do not refer to the same input variable. Non-deterministic branching programs may also have *unlabelled* edges, as well. Due to the nature of a non-deterministic branching program, it is possible that a computation never reaches either h_0 or h_1 as there can be some node with edges that their labels are all false according to the input at hand; in this case, we assume that the computation halts in a rejecting state.

A *non-deterministic branching program computes a function* $f : \{0,1\}^n \to \{0,1\}$ if, for every $x \in \{0,1\}^n$ such that $f(x) = 1$, there is some s-h_1 path and for every $x \in \{0,1\}^n$ such

that $f(x) = 0$, all computations end in a rejecting state. (In this case, we may call h_1 a *target* state.)

A *co-non-deterministic branching program computes a function* $f : \{0,1\}^n \to \{0,1\}$ if, for every $x \in \{0,1\}^n$ such that $f(x) = 1$, all source-to-sink paths are s-h_1 paths and for every $x \in \{0,1\}^n$ such that $f(x) = 0$, there exists some rejecting computation.

A *parity branching program* is a branching program that has counting semantics. That is, a parity branching program computes a function $f : \{0,1\}^n \to \{0,1\}$ if, for every $x \in \{0,1\}^n$ such that $f(x) = 1$, there is an odd number of s-h_1 paths and for every $x \in \{0,1\}^n$ such that $f(x) = 0$, there is an even number of s-h_1 paths.

We define the *size* of a branching program to be the number of its labelled edges. The following two lemmas will come handy later.

▶ **Lemma 11.** *Let f be a Boolean function. Then, f can be computed by a size-s parity branching program if and only if the complement of f can be computed by a size-s parity branching program.*

Proof. It would suffice to prove only one of the directions, say the forward direction. Let f be a Boolean function that is computed by a non-deterministic parity branching program P of size s, and let $g := \neg f$ be the complement of f. Let σ be the start vertex of P and let τ be the target vertex of P, whereby an input x is accepted by P if and only if the number of σ-τ paths is odd.

Let P' be a non-deterministic branching program that is identical to P but has an additional unlabelled edge from σ to τ. (If P already contains such an edge, then we add another one to P' anyway.)

Let x be such that $g(x) = 1$; then, $f(x) = 0$ and so there are an even number of σ-τ paths in P; therefore there are an odd number of σ-τ paths in P'. The case of x such that $g(x) = 0$ is identical. This concludes the correctness of P' as a parity branching program for g.

As unlabelled edges are not counted towards the size of branching programs, we get that the size of P' is equal to that of P, which is s. This concludes the proof. ◀

Lemma 11 yields the following corollary.

▶ **Corollary 12.** *The computational model of parity branching programs is closed under taking complements.*

▶ **Lemma 13.** *A non-deterministic, co-non-deterministic, or parity branching program of size s can be described by $O\big(s^2 \log s\big)$ bits.*

Proof. This is true as a size-s non-deterministic, co-non-deterministic, or parity branching program P has s labelled edges and therefore at most $s + 1$ vertices, as the vertices that are incident only on unlabelled edges can be removed and the edges that they used to touch can be redirected or removed, without loss of generality. (Also, new unlabelled edges can be added during this step, so that the new non-deterministic, co-non-deterministic, or parity branching program operates as the original.) For that matter, P has $O\big(s^2\big)$ unlabelled edges. As $O(\log s)$ bits suffice to describe any edge, the result follows. ◀

2.6 Pseudorandom generators and hitting set generators

We recall the standard notions of pseudorandom generators and hitting set generators. In what follows, $\mathbf{Pr}[E]$ denotes the probability that the event E occurs and $\mathbf{Exp}[X]$ denotes the expected value of the random variable X.

▶ **Definition 14.** *Let $s : \mathbb{N} \to \mathbb{N}$ be a function, $\mathfrak{C}$ be a circuit class, and $0 < \varepsilon < 1$. A* pseudorandom generator (PRG) that ε-fools $\mathfrak{C}$ *is a function $G : \{0,1\}^{s(n)} \to \{0,1\}^n$ such that*

$$\left| \mathop{\mathbf{Exp}}_{x \sim \{0,1\}^n} [C(x)] - \mathop{\mathbf{Exp}}_{y \sim \{0,1\}^{s(n)}} [C(G(y))] \right| \leq \varepsilon,$$

for any circuit $C \in \mathfrak{C}$. The value $s(n)$ is referred to as the seed length *of G.*

▶ **Definition 15.** *Let $s : \mathbb{N} \to \mathbb{N}$ be a function, $\mathfrak{C}$ be a circuit class, and $0 < \varepsilon < 1$. A* hitting set generator (HSG) ε-secure against $\mathfrak{C}$ *is a function $G : \{0,1\}^{s(n)} \to \{0,1\}^n$ such that*

$$\mathop{\mathbf{Pr}}_{x \sim \{0,1\}^n} [C(x) = 1] \geq \varepsilon \implies C(H(y)) = 1 \text{ for some } y \in \{0,1\}^{s(n)},$$

for any circuit $C \in \mathfrak{C}$. By default, we choose $\varepsilon := 1/2$.

We observe that a PRG is a notion stronger than that of a HSG.

▶ **Lemma 16.** *Let $\mathfrak{C}$ be a circuit class. If $G : \{0,1\}^s \to \{0,1\}^n$ is a PRG that ε-fools $\mathfrak{C}$, then G is a HSG that is ε'-secure against $\mathfrak{C}$ for any $\varepsilon' > \varepsilon$.*

Proof. Let $C \in \mathfrak{C}$ be a circuit. Towards a contradiction, assume that G is not a HSG that is ε'-secure against $\mathfrak{C}$; that is, $\mathbf{Pr}_y[C(y) = 1] \geq \varepsilon'$ and for all $y \in \{0,1\}^n$ such that $C(y) = 1$ there is no $x \in \{0,1\}^s$ such that $G(x) = y$. However, by the definition of G,

$$\left| \mathop{\mathbf{Pr}}_{x}[C(G(x)) = 1] - \mathop{\mathbf{Pr}}_{y}[C(y) = 1] \right| \leq \varepsilon < \varepsilon',$$

or $\mathbf{Pr}_x[C(G(x)) = 1] > 0$; that is, there exists some $x \in \{0,1\}^s$ such that $C(G(x)) = 1$. This establishes the desired contradiction. ◀

For our purpose, it is useful to extend the notion of PRG to a pseudorandom generator that fools *randomized* algorithms.

▶ **Definition 17.** *For a function $s : \mathbb{N} \to \mathbb{N}$ and a parameter $0 < \varepsilon < 1$, a function $G : \{0,1\}^{s(n)} \to \{0,1\}^n$ is said to be a pseudorandom generator that* ε-fools q-state time-t RTMs *if*

$$\left| \mathop{\mathbf{Exp}}_{\substack{x \sim \{0,1\}^n, \\ r \sim \{0,1\}^t}} [M(x,r)] - \mathop{\mathbf{Exp}}_{\substack{y \sim \{0,1\}^{s(n)}, \\ r \sim \{0,1\}^t}} [M(G(y),r)] \right| \leq \varepsilon,$$

for any q-state time-t RTM M.

2.7 MCSP lower bounds from local HSGs

For a function $G : \{0,1\}^s \to \{0,1\}^n$, we say that G is *local* [12] if $\mathrm{CC}(G(z)) \leq s$ for every string $z \in \{0,1\}^s$. We make use of the following standard fact.

▶ **Lemma 18.** *Let $s : \mathbb{N} \to \mathbb{N}$ be a function such that $s(n) = o(2^n/n)$, and $N := 2^n$. Suppose that there exists a local hitting set generator $H : \{0,1\}^{s(n)} \to \{0,1\}^N$ that is secure against a circuit class $\mathfrak{C}$. Then,* $\mathrm{MCSP}[s(n)] \notin \mathsf{co}\mathfrak{C}$.

Proof. We prove the contrapositive. Let $C \in \mathsf{co}\mathfrak{C}$ be a circuit that computes $\mathrm{MCSP}[s(n)]$. Since $\mathrm{CC}(H(z)) \leq s(n)$, we have $H(z) \in \mathrm{MCSP}[s(n)]$; thus $C(H(z)) = 1$, for every $z \in \{0,1\}^{s(n)}$. For a random $w \sim \{0,1\}^N$, it follows from Proposition 8 that $w \notin \mathrm{MCSP}[s(n)]$ with probability $1 - o(1)$; hence $C(w) = 0$ for most w. Therefore, $\neg C \in \mathfrak{C}$ accepts at least a half of $\{0,1\}^N$ but rejects every string in the range of H, which contradicts the security of the hitting set generator H. ◀

We observe that a local pseudorandom generator for time-t RTMs also "fools" $\mathsf{BPTIME}_1[t(N)]$ in the following sense.

▶ **Lemma 19.** *Let $s, t : \mathbb{N} \to \mathbb{N}$ be functions, such that $s(n) = o(2^n/n)$, and $N := 2^n$. Suppose that there exists a family of local pseudorandom generators $G = \{G_n : \{0,1\}^{s(n)} \to \{0,1\}^N\}_{n\in\mathbb{N}}$ such that, for every $n \in \mathbb{N}$, G_n $(1/6)$-fools time-$t(N)$ RTMs. Then, $\mathrm{MCSP}[s(n)]$ is not in $\mathsf{BPTIME}_1[t(N)]$.*

Proof. We prove the contrapositive. Let M be a time-t RTM that decides $\mathrm{MCSP}[s(n)]$. Fix any $n \in \mathbb{N}$. For any seed $z \in \{0,1\}^{s(n)}$, we have $G_n(z) \in \mathrm{MCSP}[s(n)]$ since G_n is local. Thus, $\Pr_r[M(G_n(z), r) = 1] \geq 2/3$. On the other hand, pick a string $w \in \{0,1\}^N$ chosen uniformly at random. By the counting argument of Proposition 8, we get $\Pr_w[w \notin \mathrm{MCSP}[s(n)]] \geq 1 - o(1)$. Thus, we have $\Pr_{w,r}[M(w, r) = 1] \leq o(1) + 1/3 < 1/2$. Therefore,

$$\Pr_{z,r}[M(G_n(z), r) = 1] - \Pr_{w,r}[M(w, r) = 1] > \frac{2}{3} - \frac{1}{2} = \frac{1}{6},$$

which means that G_n does not fool RTMs. ◀

3 MCSP lower bounds against one-tape oracle RTMs

In this section, we present a proof of our main result.

▶ **Theorem 20** (Theorem 3, restated). *Let $O \subseteq \{0,1\}^*$ be any language. Then, for every constant $1/2 < \mu < 1$, $\mathrm{MCSP}[2^{\mu\cdot n}]$ on truth tables of size $N := 2^n$ is not in $\mathsf{BPTIME}_1^O\left[N^{2\cdot(\mu' - o(1))}\right]$ for all $1/2 < \mu' < \mu$, where all of the strings queried to O are of length $N^{o(1)}$.*

3.1 Connections to hardness magnification

As discussed in Section 1.1.1, Theorem 20 implies that establishing hardness magnification phenomena for MCSP, when the circuit size threshold parameter is $2^{0.51n}$, would require the development of new techniques; see Remark 23. To explain why this is true, we shall first require the following result by McKay, Murray, and Williams [32] that gives an oracle streaming algorithm for MCSP.

▶ **Lemma 21** ([32, Theorem 1.2]). *Let $s : \mathbb{N} \to \mathbb{N}$ be a size function, with $s(n) \geq n$ for all n, and $N := 2^n$. Then, there is a one-pass streaming algorithm for $\mathrm{MCSP}[s(n)]$ on N-bit inputs running in $N \cdot \widetilde{O}(s(n))$ time with $\widetilde{O}\left(s(n)^2\right)$ update time and $\widetilde{O}(s(n))$ space, using an oracle for $\Sigma_3\mathrm{SAT}$ with queries of length $\widetilde{O}(s(n))$.*

A corollary of Lemma 21 and Lemma 10 is the following.

▶ **Corollary 22** (Consequences of hardness magnification from currently known techniques). *Let $s : \mathbb{N} \to \mathbb{N}$ be a size function. Then, $\mathrm{MCSP}[s(n)]$ on truth tables of length $N := 2^n$ is in $\mathsf{DTIME}_1^O[N \cdot \mathrm{poly}(s(n))]$, for some $O \in \Sigma_3^{\mathsf{P}}$, where all of the strings queried to O are of length at most $\mathrm{poly}(s(n))$.*

The following remark summarizes the main idea of this subsection.

▶ Remark 23. By Corollary 22, we see that if $s(n) = 2^{\mu \cdot n}$, for $\mu = o(1)$, then MCSP$[s(n)]$ is in $\mathsf{DTIME}_1^O\left[N^{1+o(1)}\right]$, where all of the strings queried to O are of length $N^{o(1)}$. In light of this observation, Theorem 20 is important for the following reason. As $\mathsf{DTIME}_1^O\left[N^{1+o(1)}\right]$ is a subset of $\mathsf{BPTIME}_1^O\left[N^{2\cdot\left(\mu'-o(1)\right)}\right]$ for all $1/2 < \mu' < 1$ and all languages $O \subseteq \{0,1\}^*$, Theorem 20 shows that establishing hardness magnification phenomena for MCSP$[s(n)]$ like that of Theorem 1, when $s(n) = 2^{\mu \cdot n}$ for any constant $1/2 < \mu < 1$, would require the development of techniques that do not rely on designing oracle algorithms that make short oracle queries.

3.1.1 Comparison with the locality barrier

Chen, Hirahara, Oliveira, Pich, Rajgopal, and Santhanam [10] introduced the "locality barrier" to explain why it will be difficult to acquire a major complexity breakthrough through the lens of hardness magnification. Their reasoning goes as follows:

> Existing magnification theorems unconditionally show that problems, against which some circuit lower bound implies a complexity-theoretic breakthrough, admit highly efficient small fan-in oracle circuits, while lower bound techniques against weak circuit models quite often easily extend to circuits containing such oracles.

Our Remark 23, therefore, is close in spirit to the results of Chen et al. [10]: We make use of a lower bound (Theorem 20) to motivate the development of new techniques for proving hardness magnification phenomena while Chen et al. make use of hardness magnification phenomena to motivate the development of new techniques for acquiring lower bounds; a notable difference is that we consider one-tape Turing machines while they consider Boolean circuits.

3.2 Proof of Theorem 20

In order to prove Theorem 20, our goal is to construct a local pseudorandom generator that fools oracle RTMs and then apply Lemma 19. Viola [42] constructed a pseudorandom generator that fools the one-tape Turing machine model (DTM).[5] We will show that, in fact, the same construction fools oracle RTMs as well. In order to do so, we recall the idea of Viola [42]. The idea is that, in order to fool DTMs, it is sufficient to use a PRG that ε-fools ROBPs for an exponentially small ε. This is because time-t DTMs can be written as the sum of an exponential number of ROBPs.

▶ **Lemma 24** (Viola [42]). *Let $n \in \mathbb{N}$ and M be a q-state time-t DTM. Then, there is a family $\{P_\alpha\}_{\alpha \in A}$ of n-input ROBPs of width $\exp\left(O\left(\sqrt{t} \cdot \log(tq)\right)\right)$ such that, for any $x \in \{0,1\}^n$,*

$$M(x) = \sum_{\alpha \in A} P_\alpha(x),$$

where $|A| \leq (tq)^{O(\sqrt{t})}$.

[5] We note that our definition of PRG is different from that of [42] in that a random tape is not regarded as an input tape.

By a simple calculation (cf. Section 5), any pseudorandom generator that $\varepsilon/|A|$-fools ROBPs also ε-fools DTMs. Viola [42] then used the pseudorandom generator of Forbes and Kelley [14] that fools ROBPs. By a careful examination, we will show that the Forbes-Kelley pseudorandom generator is local; see Corollary 44 in Section 5.

▶ **Theorem 25** (Forbes-Kelley PRG is local)**.** *There exists a local pseudorandom generator with seed length* $\widetilde{O}\big((\sqrt{t}+\log(1/\varepsilon))\cdot\log q\big)$ *that* ε*-fools* q*-state time-*t n*-input DTMs for any* $t \geq n$*.*

Our main idea for obtaining an oracle randomized Turing machine lower bound is that Viola's reduction can be applied to *non-uniform computational models*, i.e., q-state Turing machines where q can become large as the input length becomes large. More specifically, it is possible to incorporate all possible oracle queries (along with their answers) and any good coin flip sequence r into the internal states of DTMs.

▶ **Lemma 26.** *For an input length* $n \in \mathbb{N}$*, for any* q*-state time-*t *oracle RTM* M*, that only queries strings of length at most* ℓ *to its oracle* O*, and a coin flip sequence* $r \in \{0,1\}^t$*, there exists some* $\big(q\cdot 2^\ell \cdot t\big)$*-state time-*$t$ *DTM* M' *such that* $M'(x) = M^O(x,r)$ *for every input* $x \in \{0,1\}^n$*.*

Proof. Let Q_M denote the set of the states of M. We define the set of the states of M' as

$$Q_{M'} := \Big\{(q,s,b,i) \in Q_M \times \{0,1\}^\ell \times \{0,1\} \times [t] \mid O(s) = b\Big\}.$$

The transition from the state $(q,s,b,i) \in Q_{M'}$ can be defined in a natural way, by using the i-th bit of r, namely r_i, the state q, and the fact that $O(s) = b$. ◀

▶ **Corollary 27.** *There exists a local pseudorandom generator with seed length* $\sigma(t,q,\varepsilon) = \widetilde{O}\big((\sqrt{t}+\log(1/\varepsilon))\cdot\log(q\cdot 2^\ell\cdot t)\big)$ *that* ε*-fools* q*-state time-*t n*-input oracle RTMs that may only query strings of length at most* ℓ *to their oracle, for any* $t \geq n$*.*

Proof. We hard-code the oracle queries and their answers in the internal states and, moreover, we use an averaging argument to fix one good coin flip sequence r. Let M be any q-state time-t oracle RTM that may query to its oracle O strings of length at most ℓ. Let G be a PRG from Theorem 25. We have that

$$\begin{aligned}
&\left|\mathop{\mathbf{Exp}}_{r\sim\{0,1\}^t}\left[\mathop{\mathbf{Exp}}_{x\sim\{0,1\}^n}\big[M^O(x,r)\big]\right] - \mathop{\mathbf{Exp}}_{r\sim\{0,1\}^t}\left[\mathop{\mathbf{Exp}}_{y\sim\{0,1\}^{\sigma(t,q,\varepsilon)}}\big[M^O(G(y),r)\big]\right]\right| \\
&\quad= \left|\mathop{\mathbf{Exp}}_{r}\left[\mathop{\mathbf{Exp}}_{x}\big[M^O(x,r)\big] - \mathop{\mathbf{Exp}}_{y}\big[M^O(G(y),r)\big]\right]\right| \\
&\quad\leq \mathop{\mathbf{Exp}}_{r}\left[\left|\mathop{\mathbf{Exp}}_{x}\big[M^O(x,r)\big] - \mathop{\mathbf{Exp}}_{y}\big[M^O(G(y),r)\big]\right|\right] \\
&\quad\leq \left|\mathop{\mathbf{Exp}}_{x}\big[M^O(x,r^*)\big] - \mathop{\mathbf{Exp}}_{y}\big[M^O(G(y),r^*)\big]\right|,
\end{aligned}$$

for some $r^* \in \{0,1\}^t$, by an averaging argument. By applying Lemma 26, for M^O, O, and r^*, we obtain an equivalent $\big(q\cdot 2^\ell\cdot t\big)$-state time-$t$ DTM M'. The result now follows from Theorem 25. Specifically,

$$\left|\mathop{\mathbf{Exp}}_{x}\big[M^O(x,r^*)\big] - \mathop{\mathbf{Exp}}_{y}\big[M^O(G(y),r^*)\big]\right| = \left|\mathop{\mathbf{Exp}}_{x}[M'(x)] - \mathop{\mathbf{Exp}}_{y}[M'(G(y))]\right| \leq \varepsilon.$$ ◀

Proof of Theorem 20. Take the local pseudorandom generator G of Corollary 27 with parameter $\varepsilon := 1/6$. Let $1/2 < \mu' < \mu < 1$ be arbitrary constants. Let $t, s, \ell : \mathbb{N} \to \mathbb{N}$ be functions such that $t(N) = N^{2\cdot(\mu'-o(1))}$, $s(n) = 2^{\mu\cdot n}$, and $\ell(n) = 2^{o(n)}$. Then, the seed length of G is at most

$$\widetilde{O}\left(\sqrt{t(N)}\cdot(\log q + \ell(n))\right) \leq \widetilde{O}(N^{\mu'-o(1)+o(1)}) \leq s(n),$$

where $N = 2^n$. Since $s(n) = o(2^n/n)$, by Lemma 19, we obtain that $\mathrm{MCSP}[s(n)] \notin \mathsf{BPTIME}^O_1[t(N)]$, where all of the strings queried to O are of length $N^{o(1)}$. ◀

4 HSGs against non-deterministic ROBPs

In this section, we present a construction of hitting set generator secure against non-deterministic ROBPs.

▶ **Theorem 28** (Theorem 6, restated). *There exists a local hitting set generator $H : \{0,1\}^{\widetilde{O}\left(\sqrt{n\cdot\log s}\right)} \to \{0,1\}^n$ for n-input size-s read-once non-deterministic branching programs.*

As a corollary, we obtain an exponential-size lower bound for co-non-deterministic read-once branching programs that compute MCSP.

▶ **Corollary 29** (Corollary 7, restated). *Any read-once co-non-deterministic branching program that computes* MCSP *must have size at least $2^{\widetilde{\Omega}(N)}$.*

Proof. Immediate from Lemma 18 and Theorem 28. ◀

Herein, we present a general connection from a pseudorandom generator for combinatorial rectangles to a hitting set generator for non-deterministic ROBPs. Below, for $x \in \{0,1\}^n$ and $S \subseteq [n]$, we denote by $x|_S$ the $|S|$-bit string obtained by concatenating x_i for each $i \in S$.

▶ **Definition 30** (Combinatorial rectangles). *Let $n \in \mathbb{N}$. A* combinatorial rectangle of k products and width m *is a function $\pi : \{0,1\}^n \to \{0,1\}$ of the form*

$$\pi(x) = \bigwedge_{j=1}^{k} f_j\left(x|_{S_j}\right),$$

for every $x \in \{0,1\}^n$, where $k \in \mathbb{N}$, $f_j : \{0,1\}^m \to \{0,1\}$ and $|S_j| \leq m$, for all $1 \leq j \leq k$, and the sets $\{S_j\}_{j=1}^k$ are disjoint subsets of $[n]$.

▶ **Theorem 31** (Local PRG for combinatorial rectangles). *There exists a local pseudorandom generator $G : \{0,1\}^r \to \{0,1\}^n$ that ε-fools the class of combinatorial rectangles of k products and width m, where the seed length r is $\widetilde{O}(m + \log(k/\varepsilon)) \cdot \mathsf{polylog}(n)$.*

We defer the proof of Theorem 31 to Section 5 (cf. Theorem 45). For the proof of Theorem 28, we shall invoke the following lemma first, by Borodin, Razborov, and Smolensky [8].

▶ **Lemma 32** (Borodin, Razborov, and Smolensky [8]). *Let $n, k, s \in \mathbb{N}$, $m := n/k \geq 1$, and $t := O\left((2s)^{2k}\right)$. Let $f : \{0,1\}^n \to \{0,1\}$ be a function that can be computed by a read-once non-deterministic branching program of size s. Then, there exist combinatorial rectangles $\pi_1, \ldots, \pi_t$ of k products and width m such that, for all $x \in \{0,1\}^n$,*

$$f(x) = \bigvee_{i=1}^{t} \pi_i(x).$$

Proof of Theorem 28. Let $G : \{0,1\}^r \to \{0,1\}^n$ be the local PRG of Theorem 31 that fools k-product combinatorial rectangles of width m (where k and m are parameters that we shall set later). We prove that G is a hitting set generator secure against read-once non-deterministic branching programs of size s. To this end, let f be any read-once non-deterministic branching program of size s such that $f(G(z)) = 0$ for every seed $z \in \{0,1\}^r$. We claim that $\mathbf{Pr}_w[f(w) = 1] < \frac{1}{2}$. By Lemma 32, f can be written as an OR of t combinatorial rectangles $\pi_1, \ldots, \pi_t$. By the assumption, for every seed z, we have $\bigvee_i \pi_i(G(z)) = f(G(z)) = 0$, and thus $\pi_i(G(z)) = 0$. By the fact that the pseudorandom generator G ε-fools combinatorial rectangles, it holds that, for all $1 \leq i \leq t$,

$$\left| \mathop{\mathbf{Pr}}_{x \sim \{0,1\}^n}[\pi_i(x) = 1] - \mathop{\mathbf{Pr}}_{z \sim \{0,1\}^r}[\pi_i(G(z)) = 1] \right| \leq \varepsilon$$

or

$$\mathop{\mathbf{Pr}}_{x \sim \{0,1\}^n}[\pi_i(x) = 1] \leq \varepsilon.$$

Hence,

$$\mathop{\mathbf{Pr}}_{x \sim \{0,1\}^n}[f(x) = 1] = \mathop{\mathbf{Pr}}_{x \sim \{0,1\}^n}\left[\bigvee_{i=1}^{t} \pi_i(x) = 1\right] \leq \sum_{i=1}^{t} \mathop{\mathbf{Pr}}_{x \sim \{0,1\}^n}[\pi_i(x) = 1] \leq t \cdot \varepsilon.$$

Choosing $\varepsilon := 1/(4t)$, this probability is bounded by $1/4$. It remains to choose the parameter k so that the seed length r is minimized. We have

$$r = \mathsf{polylog}(n) \cdot \widetilde{O}(m + \log(k/\varepsilon)) = \widetilde{O}(n/k + k \cdot \log s) = \widetilde{O}\left(\sqrt{n \cdot \log s}\right),$$

by setting $k := \sqrt{n / \log s}$. ◀

5 The Forbes-Kelley PRG is local

In this section, we show that the Forbes-Kelley PRG is local (up to some overhead in the seed length), which will complete the proofs of Section 3 and Section 4.

In the following proofs, for each pseudorandom generator $G : \{0,1\}^s \to \{0,1\}^n$ with seed length s, we will show that $\mathrm{CC}(G(z)) \leq \widetilde{O}(s) \cdot \mathsf{polylog}(n)$ for every seed $z \in \{0,1\}^s$. This naturally gives rise to a local pseudorandom generator of seed length $\widetilde{O}(s) \cdot \mathsf{polylog}(n)$, by simply ignoring a part of the seed.

We first recapitulate biased and almost k-wise independent distributions; these are primitives used in Forbes-Kelley PRG.

5.1 Biased and almost k-wise independent distributions

▶ **Definition 33.** *Let $n \in \mathbb{N}$ and $0 < \delta < 1$. A random variable $X = (x_1, \ldots, x_n)$ over $\{0,1\}^n$ is said to be* δ-biased with respect to the distribution D *if, for any $S \subseteq [n]$, it is the case that*

$$\left| \mathop{\mathbf{Pr}}_{X \sim D}\left[\prod_{i \in S} (-1)^{x_i} = 1\right] - \mathop{\mathbf{Pr}}_{X \sim \{0,1\}^n}\left[\prod_{i \in S} (-1)^{x_i} = -1\right] \right| \leq \delta.$$

On this occasion, we also say that D is a δ-biased distribution over $\{0,1\}^n$.

Below, we define almost k-wise independent distributions.

▶ **Definition 34.** *Let $n \in \mathbb{N}$, $0 < \gamma < 1$, and $k > 0$. A random variable $X = (x_1, \ldots, x_n)$ over $\{0,1\}^n$ is said to be* γ-almost k-wise independent with respect to the distribution D *if, for any k indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, it is the case that*

$$\sum_{\alpha \in \{0,1\}^k} \left| \mathop{\mathbf{Pr}}_{X \sim D}[x_{i_1} \cdots x_{i_k} = \alpha] - \mathop{\mathbf{Pr}}_{X \sim \{0,1\}^n}[x_{i_1} \cdots x_{i_k} = \alpha] \right| \leq \gamma.$$

On this occasion, we also say that D is a γ-almost k-wise independent distribution over $\{0,1\}^n$.

▶ **Definition 35.** *Let $n, k \in \mathbb{N}$, with $k \leq n$. A random variable $X = (x_1, \ldots, x_n)$ over $\{0,1\}^n$ is said to be* k-wise independent with respect to the distribution D *if X is 0-almost k-wise independent with respect to the distribution D. On this occasion, we also say that D is a* k-wise independent distribution.

It is known that a k-wise independent distribution over $\{0,1\}^n$ may be sampled by using $O(k \log n)$ random bits [40]. The following lemma upper-bounds the circuit complexity of some k-wise independent distribution.

▶ **Theorem 36** ([12, 15, 40])**.** *There exists a local k-wise independent generator $G : \{0,1\}^s \to \{0,1\}^n$ with seed length $s = k \cdot \widetilde{O}(\log n)$.*

We next show that there exists a local ε-biased generator.

▶ **Theorem 37** (The complexity of multiplication; cf. [15])**.** *For an integer $m > 0$, let the elements in $\mathbb{F}_{2^m}$ be represented by m-bit strings. Then, there exists a circuit of size $\widetilde{O}(m)$ that, on input $x, y \in \mathbb{F}_{2^m}$, outputs the m-bit representation of the product $x \cdot y$.*

▶ **Theorem 38.** *There exists a local ε-biased generator $G : \{0,1\}^s \to \{0,1\}^n$ with seed length $s = \widetilde{O}(\log(n/\varepsilon)) \cdot \log n$.*

Proof. We use the standard construction of an ε-biased generator G_0 of [5]. Let $m := \log(n/\varepsilon)$, and take a field of size 2^m. For random seeds $a, b \in \mathbb{F}_{2^m}$ of length $2m$, the ith bit of $G_0(a,b)$ is defined as $\langle a^i, b \rangle$, i.e., the inner product of the binary representations of a^i and b. It was shown in [5] that, for a, b chosen uniformly at random from $\mathbb{F}_{2^m}$, the distribution of $G_0(a,b)$ is ε-biased.

We claim that, for every $a, b \in \mathbb{F}_{2^m}$, there exists a circuit of size $O(\log(n/\varepsilon) \cdot \log n)$ that takes $i \in [n]$ as an input of length $\log n$ and computes $\langle a^i, b \rangle$. Indeed, we hardwire a^{2^j} for all $j \leq \log n$ into a circuit; given an input $i \in [n]$, one can compute a^i by multiplying a^{2^j} for all j such that the j-th bit of the binary representation of i is 1. This can be done with a circuit of size $\widetilde{O}(m) \cdot \log n$ by using Theorem 37. It remains to compute the inner product of a^i and b, which can be done with a linear-size circuit. Overall, we obtain a circuit of size $\widetilde{O}(m) \cdot \log n$.

The local ε-biased generator G is defined as a version of G_0 such that a part c of the seed is ignored: i.e., $G(a,b,c) := G_0(a,b)$, where $a, b \in \mathbb{F}_{2^m}$ and $|c| = \widetilde{O}(m) \cdot \log n$. ◀

Finally, we present a local δ-almost k-wise independent generator.

▶ **Lemma 39** ([34])**.** *Let $0 < \delta < 1$ and D be an δ-biased distribution over n-bit strings. Then, for any $k \in \mathbb{N}$, D is a $\left(2^{k/2} \cdot \delta\right)$-almost k-wise independent distribution over n-bit strings.*

▶ **Theorem 40.** *There exists a local δ-almost k-wise independent generator $G : \{0,1\}^s \to \{0,1\}^n$ with seed length $s = \widetilde{O}(k + \log(n/\delta)) \cdot \log n$.*

Proof. By setting $\varepsilon := 2^{-k/2} \cdot \delta$, the local ε-biased pseudorandom generator of Theorem 38 is a local δ-almost k-wise independent generator by using Lemma 39. ◀

5.2 The Forbes-Kelley PRG

▶ **Definition 41** (Forbes-Kelley PRG [14]). *Let n, δ, γ, r, k be some parameters chosen later. Let $D_1, \ldots, D_r$ be r independent δ-biased distributions and let $T_1, \ldots, T_r$ be r independent γ-almost k-wise independent distributions over $\{0,1\}^n$. We define G_r^{FK} inductively as follows. Let G_1^{FK} be some $320k$-wise independent distribution and let*

$$G_{i+1}^{\mathrm{FK}} := D_i + T_i \wedge G_i^{\mathrm{FK}},$$

for all $1 \leq i \leq r-1$, where $\wedge$ denotes bitwise AND and $+$ denotes bitwise XOR.

▶ **Theorem 42** (Correctness of Forbes-Kelley PRG [14]). *For parameters $n, w \in \mathbb{N}$ and $\varepsilon > 0$, choose the parameters as follows: $r := \lceil \log n \rceil$, $k := \lceil 3\log(nw/\varepsilon) \rceil$, $\gamma := (nw/\varepsilon)^{-9}$, and $\delta := (nw\mathcal{L}/\varepsilon)^{-3}$, where $\mathcal{L} := \binom{n}{k} w^{1/2}$. Then, $G_r^{\mathrm{FK}} : \{0,1\}^s \to \{0,1\}^n$ is a pseudorandom generator that ε-fools unknown-order ROBPs of width w, where the seed length s is $O\left(\log(nw/\varepsilon) \cdot \log^2 n\right)$.*

▶ **Theorem 43.** *There exists a local pseudorandom generator of seed length $\widetilde{O}\left(\log(nw/\varepsilon) \cdot \log^3 n\right)$ that ε-fools unknown-order n-input ROBPs of width w.*

Proof. We instantiate the Forbes-Kelley pseudorandom generator G_r^{FK} by using the parameters of Theorem 42 and using the construction of local generators of Theorem 38, Theorem 40, and Theorem 36 for D_i, T_i, and G_1^{FK}, respectively (this achieves the seed length given in the statement of Theorem 42).

For a distribution $\mathcal{D}$, we denote by $\mathrm{CC}(\mathcal{D})$ the maximum of $\mathrm{CC}(D)$ over all strings D in the range of $\mathcal{D}$. We claim that $\mathrm{CC}\big(G_r^{\mathrm{FK}}\big)$ is at most $\widetilde{O}\left(\log(nw/\varepsilon) \cdot \log^3 n\right)$. By Theorem 38 and Theorem 40, we have $\mathrm{CC}(D_i) \leq \widetilde{O}(\log(n/\delta)) \cdot \log n$ and $\mathrm{CC}(T_i) \leq \widetilde{O}(k + \log(n/\gamma)) \cdot \log n$ for all i. Therefore, since $G_{i+1}^{\mathrm{FK}} = D_i + T_i \wedge G_i^{\mathrm{FK}}$, we obtain

$$\mathrm{CC}\big(G_{i+1}^{\mathrm{FK}}\big) \leq \mathrm{CC}(D_i) + \mathrm{CC}(T_i) + \mathrm{CC}\big(G_i^{\mathrm{FK}}\big) + O(1)$$

for any $i \in [r-1]$. We also have $\mathrm{CC}\big(G_1^{\mathrm{FK}}\big) \leq k \cdot \widetilde{O}(\log n)$ from Theorem 36. Therefore,

$$\mathrm{CC}\big(G_r^{\mathrm{FK}}\big) \leq r \cdot \widetilde{O}(k + \log(n/\gamma\delta)) \cdot \log n + k \cdot \widetilde{O}(\log n) = \widetilde{O}(\log(nw/\varepsilon) \log^3(n)). \qquad \blacktriangleleft$$

▶ **Corollary 44** (A restatement of Theorem 25). *There exists a local pseudorandom generator with seed length $\widetilde{O}\big((\sqrt{t} + \log(1/\varepsilon)) \cdot \log q\big)$ that ε-fools q-state time-t n-input DTMs, for any $t \geq n$.*

Proof. By Lemma 24, any q-state time-t DTM M can be written as the sum of ROBPs $\{P_\alpha\}_{\alpha \in A}$ so that $M(x) = \sum_{\alpha \in A} P_\alpha(x)$, where $|A| \leq (tq)^{O(\sqrt{t})}$. We set $\varepsilon' := \varepsilon/|A|$ and use the local pseudorandom generator G_r^{FK} of Theorem 43 that ε'-fools ROBPs of width w, where $w = (tq)^{O(\sqrt{t})}$. Then, G_r^{FK} is a PRG that ε-fools DTMs because

$$\left| \mathop{\mathbf{Exp}}_{z,w} \left[M(G_r^{\mathrm{FK}}(z)) - M(w) \right] \right| \leq \sum_{\alpha \in A} \left| \mathop{\mathbf{Exp}}_{z,w} \left[P_\alpha(G_r^{\mathrm{FK}}(z)) - P_\alpha(w) \right] \right| \leq \varepsilon. \qquad \blacktriangleleft$$

5.3 Local PRG for combinatorial rectangles

▶ **Theorem 45** (Local PRG for combinatorial rectangles). *There exists a local pseudorandom generator $G : \{0,1\}^s \to \{0,1\}^n$ that ε-fools the class of combinatorial rectangles of k products and width m, where the seed length s is $\widetilde{O}\big((m + \log(k/\varepsilon)) \cdot \log^3 n\big)$.*

We mention that Lee [29] showed that in the case of product tests (which contain combinatorial rectangles as a special case), the analysis of the Forbes-Kelley PRG can be improved, and obtained a PRG with nearly optimal seed length. This optimization would improve our results at most a $\mathsf{polylog}(n)$ factor; for the sake of simplicity, we make use of the Forbes-Kelley PRG.

Proof of Theorem 45. Suppose that a function f is computed by a combinatorial rectangle of k products and width m. Namely, there exist some functions $\{f_i\}_i$ and disjoint subsets $S_1, \cdots, S_k \subseteq [n]$, where $|S_i| \leq m$, such that $f(x) = \bigwedge_{i\in[k]} f_i(x|_{S_i})$, for every x. Since any function f_i on m inputs can be written as a width-2^m ROBP, one can observe that f can be computed by a ROBP of width $2^m + 1$. Using the local PRG of Theorem 43 for width $w := 2^m + 1$, we obtain a local PRG for combinatorial rectangles. ◀

6 MKTP lower bounds against branching programs

In this section, we develop a proof technique for applying Nečiporuk's method to MKTP and prove Theorem 4. The KT-complexity is formally defined as follows.

▶ **Definition 46.** *Let U be an efficient universal Turing machine. For a string $x \in \{0,1\}^*$, the* KT-complexity of x *is defined as follows.*

$$\mathrm{KT}(x) := \min\{|d| + t \mid U^d(i) \text{ outputs } x_i \text{ in time } t \text{ for every } i \in [|x|+1]\}.$$

Here we define x_i as the ith bit of x if $i \leq |x|$ and $\perp$ otherwise.

For a threshold $\theta : \mathbb{N} \to \mathbb{N}$, we denote by $\mathrm{MKTP}[\theta]$ the problem of deciding whether $\mathrm{KT}(x) \leq \theta(|x|)$ given a string $x \in \{0,1\}^*$ as input.

For a function $f : \{0,1\}^n \to \{0,1\}$, we partition the input variables $[n]$ into disjoint blocks $V_1, \cdots, V_m$, where $|V_i| = v$ for each $i \in [m]$ and $n = vm$. ($v = O(\log n)$ will be chosen later.) The idea of the Nečiporuk's method is to lower-bound the number of subfunctions. For each $i \in [m]$, we define $c_i(f)$ to be the number of distinct functions $f\restriction_\rho$ such that $\rho : [n] \to \{0,1,*\}$ is a restriction with $\rho^{-1}(*) = V_i$.

The Nečiporuk method can be then summarized as follows.

▶ **Theorem 47** (Nečiporuk [35]; cf. [26, Theorem 15.1])**.** *The size of a branching program computing f is at least $\Omega\left(\sum_{i=1}^m \log c_i(f) / \log\log c_i(f)\right)$. The size of a non-deterministic branching program or a parity branching program computing f is at least $\Omega\left(\sum_{i=1}^m \sqrt{\log c_i(f)}\right)$. In both of these results the value of m is $n/O(\log n)$.*

Our main technical result of this section is the following.

▶ **Theorem 48.** *Let $f : \{0,1\}^n \to \{0,1\}$ be $\mathrm{MKTP}[\theta]$ on n-bit inputs for $\theta := n - 3c\log n - 4$, where $c > 0$ is a universal constant. Then, for every $i \in [m]$, it holds that $c_i(f) = 2^{\Omega(n)}$.*

The lower bounds for branching programs (Theorem 4) immediately follow from Theorem 48 and Theorem 47.

In our proof of Theorem 48, we only need the following two properties of KT-complexity.

1. The resource-unbounded Kolmogorov complexity[6] provides a lower bound on the KT-complexity. That is, $\mathrm{K}(x) \leq \mathrm{KT}(x)$ for any $x \in \{0,1\}^*$.

[6] Let U be an efficient universal Turing machine. For a string $x \in \{0,1\}^*$, the *resource-unbounded Kolmogorov complexity of x* is defined as $\mathrm{K}(x) := \min\{|d| \mid U^d(i) \text{ outputs } x_i \text{ for every } i \in [|x|+1]\}$.

2. For any strings $\rho_1, \cdots, \rho_m \in \{0,1\}^v$ such that there exist distinct indices $i \neq j \in [m]$ such that $\rho_i = \rho_j$, we have $\mathrm{KT}(\rho_1 \cdots \rho_m) \leq (m-1) \cdot v + O(\log n)$. This is because each bit of the string $\rho_1 \cdots \rho_m$ can be described by the strings $\{\rho_1, \cdots, \rho_m\} \setminus \{\rho_j\}$ and the index $j \in [m]$ in time $O(\log n)$.[7]

For simplicity, we focus on the case when $i = 1$; the other cases can be proved similarly. The idea of the proof is the following. Imagine that we pick $\rho \in \{*\}^{V_1} \times \{0,1\}^{V_2 \cup \cdots \cup V_m}$ uniformly at random. (Here we identify a restriction with a string in $\{0,1,*\}^{[n]}$.) We denote by $\rho_2 \in \{0,1\}^{V_2}, \cdots, \rho_m \in \{0,1\}^{V_m}$ the random bits such that $\rho = *^{V_1}\rho_2 \cdots \rho_m$. We will sometimes identify $\rho_2 \cdots \rho_m$ with ρ.

Let $f := \mathrm{MKTP}[\theta]$, and consider the function $f\restriction_\rho : \{0,1\}^{V_1} \to \{0,1\}$ obtained by restricting f by ρ. Then, we expect that $f\restriction_\rho(\rho_i) = 1$ for any $i \in \{2, \cdots, m\}$ since $\mathrm{KT}(\rho_i \rho_2 \cdots \rho_m)$ is small, whereas $f\restriction_\rho(U) = 0$ for a random $U \sim \{0,1\}^{V_1}$ with high probability. Thus, the function $f\restriction_\rho$ is likely to be distinct for a randomly chosen ρ.

In order to make the argument formal, we proceed as follows. Pick ρ randomly. Then we add it to a set P while keeping the promise that the map $\rho \in P \mapsto f\restriction_\rho$ is injective (meaning that each restriction ρ in P yields a *unique* restricted function $f\restriction_\rho$). We will show that one can keep adding ρ until the size of P becomes exponentially large. This will conclude the proof of Theorem 48.

We will make use of symmetry of information of (resource-unbounded) Kolmogorov complexity.

▶ **Lemma 49.** *There exists a constant $c > 0$ such that, for any strings $x, y \in \{0,1\}^*$,*

$$\mathrm{K}(xy) \geq \mathrm{K}(x) + \mathrm{K}(y \mid x) - c \log \mathrm{K}(xy).$$

We focus on restrictions ρ such that ρ is Kolmogorov-random. To this end, define

$$R := \{\rho \in \{0,1\}^{V_2 \cup \cdots \cup V_m} \mid \mathrm{K}(\rho) \geq |\rho| - 1\}$$

as the set of Kolmogorov-random restrictions ρ. By the standard counting argument, we have

$$\Pr_\rho[\rho \notin R] \leq \sum_{i=1}^{|\rho|-2} 2^i / 2^{|\rho|} \leq \frac{1}{2}.$$

The following lemma is the key for counting the number of distinct subfunctions.

▶ **Lemma 50.** *Let $\rho' \in R$ be an arbitrary restriction and define $\theta := n - v + c \log n$. If $f\restriction_\rho \equiv f\restriction_{\rho'}$, then $\mathrm{K}(\rho_i \mid \rho') \leq 2c \log n + 1$ for any $i \in \{2, \cdots, m\}$.*

Proof. For each $i \in [m] \setminus \{1\}$,

$$\mathrm{KT}(\rho_i \rho_2 \cdots \rho_m) \leq |\rho_2| + \cdots + |\rho_m| + O(\log n) \leq (m-1) \cdot v + c \log n \leq \theta.$$

This means that $\rho_i \rho_2 \cdots \rho_m$ is a YES instance of $\mathrm{MKTP}[\theta]$. Therefore, we have $1 = f\restriction_\rho(\rho_i) = f\restriction_{\rho'}(\rho_i)$, which implies that $\mathrm{KT}(\rho_i \rho'_2 \cdots \rho'_m) \leq \theta$. By the symmetry of information,

$$\theta \geq \mathrm{KT}(\rho_i \rho'_2 \cdots \rho'_m) \geq \mathrm{K}(\rho_i \rho'_2 \cdots \rho'_m) \geq \mathrm{K}(\rho'_2 \cdots \rho'_m) + \mathrm{K}(\rho_i \mid \rho'_2 \cdots \rho'_m) - c \log n.$$

Since $\rho' \in R$, we have $\mathrm{K}(\rho'_2 \cdots \rho'_m) \geq v(m-1) - 1 = n - v - 1$. Therefore,

$$\mathrm{K}(\rho_i \mid \rho'_2 \cdots \rho'_m) \leq \theta + c \log n - (n - v - 1) = 2c \log n + 1. \qquad \blacktriangleleft$$

[7] Here we assume that the universal Turing machine is efficient. If the universal Turing machine is slower and the time is $\mathrm{polylog}(n)$, we obtain a branching program size lower bound of $n^2/\mathrm{polylog}(n)$.

Now we set $v := 4c \log n + 4$. Then, for any $\rho' \in R$,

$$\begin{aligned}\Pr_{\rho}[f\restriction_{\rho} \equiv f\restriction_{\rho'}] &\leq \Pr[\forall i \in [m] \setminus \{1\},\ \mathrm{K}(\rho_i \mid \rho') \leq v/2 - 1] \\ &\leq (2^{v/2}/2^v)^{m-1} \\ &= 2^{-n/2+v/2} \\ &\leq 2^{-n/3}.\end{aligned}$$

In particular, for any $P \subseteq R$, by the union bound, we obtain

$$\Pr_{\rho}[\exists \rho' \in P,\ f\restriction_{\rho} \equiv f\restriction_{\rho'}] \leq |P| \cdot 2^{-n/3}.$$

Therefore,

$$\Pr_{\rho}[\rho \notin R \text{ or } \exists \rho' \in P,\ f\restriction_{\rho} \equiv f\restriction_{\rho'}] \leq 1/2 + |P| \cdot 2^{-n/3},$$

which is strictly less than 1 if $|P| < 2^{n/3-1}$. To summarize, we established the following property.

▶ **Corollary 51.** *For any $P \subseteq R$ such that $|P| < 2^{n/3-1}$, there exists a restriction ρ such that $\rho \in R$ and $f\restriction_{\rho} \not\equiv f\restriction_{\rho'}$ for any $\rho' \in P$.*

In light of this, we can construct a large set P such that the map $\rho \in P \mapsto f\restriction_{\rho}$ is injective as follows: Starting from $P := \emptyset$, add a restriction $\rho \in R$ such that $f\restriction_{\rho} \not\equiv f\restriction_{\rho'}$ for any $\rho' \in P$, whose existence is guaranteed by Corollary 51 if $|P| < 2^{n/3-1}$. In this way, we obtain a set P such that $|P| \geq 2^{n/3-1}$ and each $f\restriction_{\rho}$ is distinct for any $\rho \in P$. We conclude that $c_1(f) \geq |P| \geq 2^{n/3-1}$. This completes the proof of Theorem 48.

7 MCSP lower bounds against non-deterministic, co-non-deterministic, and parity branching programs

In this section, we prove our MCSP lower bound against non-deterministic, co-non-deterministic, and parity branching programs.

▶ **Theorem 52** (Theorem 5, restated)**.** *The size of any non-deterministic, co-non-deterministic, or parity branching program computing* MCSP *is at least $N^{1.5-o(1)}$.*

We will prove Theorem 52 by first showing that a PRG by Impagliazzo, Meka, and Zuckerman [25], henceforth IMZ PRG, fools non-deterministic, co-non-deterministic, and parity branching programs. As shown by Cheraghchi, Kabanets, Lu, and Myrisiotis [12], an IMZ PRG can be implemented as an *almost* local PRG. This makes it possible to apply Lemma 18 and obtain Theorem 52.

7.1 The IMZ PRG fools non-deterministic, co-non-deterministic, and parity branching programs

We first show that the IMZ PRG fools non-deterministic, co-non-deterministic, and parity branching programs (apart from formulas and deterministic branching programs).

▶ **Theorem 53** (Following Impagliazzo, Meka, and Zuckerman [25])**.** *For any constant $c > 0$, there is an explicit PRG using a seed of $s^{2/3+o(1)}$ random bits that s^{-c+1} fools non-deterministic, co-non-deterministic, or parity branching programs of size at most s.*

7.1.1 Proof of Theorem 53

We first recall the notion of shrinkage by random restrictions.

▶ **Definition 54** ([25]). *Let $\mathcal{F}$ be a class of functions with an associated size function $s : \mathcal{F} \to \mathbb{R}_{>0}$ and let $\mathcal{D}_p$ be a p-regular distribution on $\{0,1,*\}^n$. We say that $\mathcal{F}$* has ε-shrinkage exponent Γ with respect to $\mathcal{D}$ *if there exists a constant c such that for all $f \in \mathcal{F}$ it is the case that*

$$\mathop{\mathbf{Pr}}_{\rho \sim \mathcal{D}_p} \left[s\big(f\restriction_\rho\big) > c \cdot \left(p^\Gamma \cdot s(f) + 1\right) \cdot \log(1/\varepsilon) \right] \leq \varepsilon.$$

We will make use of the template of Impagliazzo, Meka, and Zuckerman [25] that gives a generic connection between shrinkage by pseudorandom restrictions and PRGs.

▶ **Theorem 55** (Following Impagliazzo, Meka, and Zuckerman [25]). *Fix $\varepsilon > 0$ and let $\mathcal{F}$ be a class of functions with an associated size function $s : \mathcal{F} \to \mathbb{N}$. Fix $s > 0$ and let $p := 1/s^{2/(2\Gamma+1)}$. Let $\mathcal{D}_p$ be a p-regular distribution on $\{0,1,*\}^n$ such that $\mathcal{F}$ has ε-shrinkage exponent Γ with respect to $\mathcal{D}_p$. Then, there exists an explicit PRG $G : \{0,1\}^r \to \{0,1\}^n$ that δ-fools all functions of size at most s in $\mathcal{F}$ for $\delta = O(\varepsilon \cdot r)$ and has seed length*

$$r = O\Big((R(s) + \log(s/\varepsilon)) \cdot \log(n/\varepsilon) \cdot s^{2/(2\Gamma+1)} \Big),$$

where $R(s)$ denotes the number of bits needed to efficiently sample from $\mathcal{D}_p$.

Proof sketch. The original version of Theorem 55, namely [25, Theorem 3.3], refers to Boolean computational models $\mathcal{F}$ that have the following property: *Size-s devices from $\mathcal{F}$ may be described by $O(s \log s)$ bits.* However, this is not known to be true for the model of non-deterministic branching programs; for that matter, we adjusted Theorem 55 to account for this case, whereby any size-s from $\mathcal{F}$, where $\mathcal{F}$ is the model of non-deterministic branching programs, may be described by $O\big(s^2 \log s\big)$ bits (Lemma 13).

So, the proof of Theorem 55 follows closely the proof of [25, Theorem 3.3] and the only difference lies, as illustrated above, at the description size of non-deterministic branching programs. For that matter, the last math display of [25, page 9], where an expression about the seed length r of the (instantiation of the) IMZ PRG that fools $\mathcal{F}$ is given, namely

$$r = O(R(s) + \log(s_0/\varepsilon)) \cdot \log(n/\varepsilon) \,/p + O(s_0 \log s_0)$$

where $s_0 := cp^\Gamma s \log(1/\varepsilon)$, for some $c > 0$, is the size of a size-s device P from $\mathcal{F}$ after P is being hit by a pseudorandom restriction ρ (with high probability over the choice of $\rho \sim \mathcal{D}_p$), in our case reads

$$r = O(R(s) + \log(s_0/\varepsilon)) \cdot \log(n/\varepsilon) \,/p + O\big(s_0^2 \log s_0\big)\,;$$

that is, $O(s_0 \log s_0)$ is replaced by $O\big(s_0^2 \log s_0\big)$. Then, the result follows by setting $p := 1/s^{2/(2\Gamma+1)}$, similarly to [25, Theorem 3.3], to balance out $1/p$ and $\left(p^\Gamma s\right)^2$. ◀

The following result, whose proof is identical to that of [25, Lemma 5.3], establishes the fact that non-deterministic, co-non-deterministic, and parity branching programs shrink in size after being hit by a k-wise independent restrictions, with high probability over the choice of the restriction. The reason that all these models behave the same after being hit by such a pseudorandom restriction, is that they are structurally alike and, moreover, they share a *common* notion of size (i.e., the number of labelled edges).

▶ **Lemma 56** (Following Impagliazzo, Meka, and Zuckerman [25]). *For any constant c and non-deterministic, co-non-deterministic, or parity branching program f a $\left(p := 1/s^{2/3}\right)$-regular $(c \log s)$-wise independent random restriction ρ yields*

$$\mathbf{Pr}\left[s\left(f\restriction_\rho\right) \geq 2^{3\sqrt{c \log s}} \cdot p \cdot s\right] \leq 2s^{-c}.$$

Lemma 56 yields the following corollary, about the shrinkage exponent of non-deterministic, co-non-deterministic, or parity branching programs, by referring to Definition 54.

▶ **Corollary 57.** *Let $\mathcal{F}$ be the class of non-deterministic, co-non-deterministic, or parity branching programs with an associated size function $s : \mathcal{F} \to \mathbb{N}$, $c > 0$, $\varepsilon := 2s^{-c}$, $p := 1/s^{2/3}$, and $\mathcal{D}_p$ be a p-regular distribution on $\{0, 1, *\}^n$. Then, $\mathcal{F}$ has ε-shrinkage exponent $\Gamma := 1$ with respect to $\mathcal{D}_p$.*

We now turn to the proof of Theorem 53. To this end, the following bound on the number of random bits that suffice to sample from a k-wise independent distribution will be useful.

▶ **Lemma 58** ([40]). *Let $0 < p < 1$ and $\mathcal{D}_p$ be a p-regular k-wise independent distribution on $\{0, 1, *\}^n$. Then, the number of bits needed to efficiently sample from $\mathcal{D}_p$ is at most $O(k \cdot \log n \cdot \log(1/p))$.*

Proof of Theorem 53. By Corollary 57, we get that non-deterministic, co-non-deterministic, or parity branching programs have ε-shrinkage exponent $\Gamma := 1$ with respect to $\mathcal{D}_p$, for $\varepsilon := 2s^{-c}$ and $p := 1/s^{2/3}$. Plugging Γ in the expression that gives r from Theorem 55, along with ε, p, and the upper bound on $R(s)$ from Lemma 58, we get $r = s^{2/3+o(1)}$. What is left, is to calculate $\delta := O(\varepsilon \cdot r) \leq s^{-c+1}$; the proof is now complete. ◀

7.2 Proof of Theorem 52

First, observe that Lemma 18 also holds for the case where $\mathfrak{C}$ is some class of non-deterministic, co-non-deterministic, or parity branching programs. We may now prove Theorem 52.

Proof of Theorem 52. Let $c > 0$ and $G : \{0,1\}^{s^{2/3+o(1)}} \to \{0,1\}^N$ be the instantiation of the IMZ PRG that s^{-c+1}-fools non-deterministic, co-non-deterministic, or parity branching programs of size s on N variables, as it is implied to exist by Theorem 53. By Cheraghchi, Kabanets, Lu, and Myrisiotis [12], for every $z \in \{0,1\}^{s^{2/3+o(1)}}$ it is the case that $\mathrm{CC}(G(z)) \leq \widetilde{O}(|z|) \cdot \mathsf{polylog}(N)$. By the discussion at the top of page 17, this fact yields a *local* PRG that fools non-deterministic, co-non-deterministic, or parity branching programs.

We will now apply Lemma 18. First, recall that a PRG is a notion stronger than that of a HSG (Lemma 16). Therefore, there is a local HSG $H : \{0,1\}^{s^{2/3+o(1)}} \to \{0,1\}^N$ that is secure against non-deterministic, co-non-deterministic, or parity branching programs of size s on N variables. Second, let $s := N^{1.5-\varepsilon}$ for an arbitrary small constant $0 < \varepsilon < 1$ and observe that $s^{2/3+o(1)} = o(N/\log N)$. In the notation of Lemma 18, let $\mathfrak{C}$ be the class of non-deterministic, co-non-deterministic, or parity branching programs of size s on N variables. Then, by Lemma 18, we get that $\mathrm{MCSP}\left[s^{2/3+o(1)}\right]$ cannot be computed by co-non-deterministic, co-co-non-deterministic, or co-parity branching programs of size s.

Two notes are in order. First, co-co-non-deterministic branching programs coincide with non-deterministic branching programs. Second, the class of parity branching programs is closed under taking complements (Corollary 12). Therefore, we get that $\mathrm{MCSP}\left[s^{2/3+o(1)}\right]$ cannot be computed by co-non-deterministic, non-deterministic, or parity branching programs of size s. Moreover, $\mathrm{MCSP}\left[s^{2/3+o(1)}\right]$, and for that matter MCSP as well, requires co-non-deterministic, non-deterministic, or parity branching programs of size $N^{1.5-o(1)}$. ◀

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A Hardness magnification for one-tape Turing machines

In this section, we obtain the following hardness magnification result for one-tape Turing machines.

▶ **Theorem 59** (A corollary of McKay, Murray, and Williams [32]; Theorem 1, restated)**.** *There exists a constant $\mu > 0$ such that, if* $\mathrm{MCSP}[2^{\mu n}]$ *is not in* $\mathsf{DTIME}_1\left[N^{1.01}\right]$*, then* $\mathsf{P} \neq \mathsf{NP}$*.*

Proof. McKay, Murray, and Williams [32, Theorem 1.3] showed that if $\mathsf{P} = \mathsf{NP}$, then there exists a polynomial p such that, for any time-constructible function $s(n)$, there exists a one-pass streaming algorithm with update time $p(s(n))$ that computes $\mathrm{MCSP}[s(n)]$. By Lemma 10, we obtain $\mathrm{MCSP}[s(n)] \in \mathsf{DTIME}_1[N \cdot p(s(n))]$, where $N = 2^n$. Depending on p, we choose a small constant $\mu > 0$ and set $s(n) := 2^{\mu n}$ so that $N \cdot p(s(n)) = N^{1+O(\mu)} \leq N^{1.01}$.

To summarize, we have proved that if $\mathsf{P} = \mathsf{NP}$, then for some constant $\mu > 0$, $\mathrm{MCSP}[2^{\mu n}] \in \mathsf{DTIME}_1[N^{1.01}]$. This statement is logically equivalent to the following: There exists a constant $\mu > 0$ such that $\mathsf{P} = \mathsf{NP}$ implies that $\mathrm{MCSP}[2^{\mu n}] \notin \mathsf{DTIME}_1[N^{1.01}]$ (because the statement that $\mathsf{P} = \mathsf{NP}$ is independent of μ). Taking its contrapositive, we obtain the desired result. ◀