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RESEARCH ARTICLE

Middle school students' development of an understanding of the concept of function using an applet with no algebraic representations

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Abstract

Middle school students ($n = 144$) worked with an applet specially designed to introduce the concept of function without using algebraic representations. The purpose of the study was to examine whether the applet would help students understand function as a relationship between a set of inputs and a set of outputs and to begin to develop a definition of function based on that relationship. Results indicate that, by focusing on consistency of the outputs, the students, at a rate of approximately 80%, are able to distinguish functions from nonfunctions. Also, students showed some promise in recognizing constant functions as functions, a known area of common misconceptions. Students' main conceptual difficulty, likely caused by the context, was accepting nonintuitive outputs even if those outputs were consistent.

KEYWORDS

middle school education, representations and visualization, technology

The concept of function is considered to be one of the most important underlying and unifying concepts of mathematics (e.g., Leinhardt et al., 1990; Thompson & Carlson, 2017). Students have experiences with functions, or function behavior, from the very earliest grades, usually through pattern exploration. The study of functions continues up to and through high school with a formal treatment of functions as arbitrary mappings between sets. Indeed, in the Common Core State Standards for Mathematics, the function is given its own domain in Grades 9–12 (Common Core State Standards Initiative, 2010).

Much of the lack of depth of knowledge of the concept can be attributed to the privileging of algebraic representations (function as algebraic rule) or graphical representations (function as graph that passes the vertical line test) and a consequent lack of focus on the general relationship, that is, a mapping between two sets (see, e.g., Best & Bikner-Ahsbahs, 2017; Breidenbach et al., 1992; Carlson, 1998; Thompson, 1994).

Novel representations of functions (e.g., dynagraphs, arrow diagrams, and a vending machine model) (see, e.g., Dubinsky & Wilson, 2013; McCulloch et al., 2022; Sinclair et al., 2009) have shown some success in refining conceptions of function for those with prior experience of function. In this paper, we examine what students with no prior experience of function might be able to learn about the general relationship by using a non-algebraic, nongeometric representation.

1 | RELATED LITERATURE

Before secondary school, opportunities for study of functions are limited in scope (Best & Bikner-Ahsbahs, 2017; Carlson & Oehrtman, 2005; Gueudet & Thomas, 2020; Vinner & Dreyfus, 1989; Zan & Di Martino, 2020) and focus mainly on pattern recognition and study of covarying quantities, most often related to an underlying linear structure (Blanton et al., 2015; Ellis, 2011; Stephens et al., 2017). For example, in Blanton et al. (2015), sixth-grade students are given the task "People and Ears: The relationship between the number of people and the total number of ears on the people (assuming each person has two ears)" (p. 520) to study the function type $y = x + x$ and "Age Difference: If Janice is 2 years younger than Keisha, the relationship between Keisha's age and Janice's age (Carraher et al., 2006)" (p. 521) to study the function type $y = x + 2$. In other words, the functional relationships typically encountered in elementary and middle school years are designed to prepare the (mathematical) ground for studying linear relationships ($y = mx$, $y = x + b$, $y = mx + b$), that is, the subsequent privileging of algebraic representations has its roots in the early in the study of functions. This existence of this phenomenon is also reported by Mesa (2004), in a study of 24 middle grades textbooks from 15 countries where the preponderance of algebraic representations in those textbooks was noted. Leinhardt et al. (1990), in a meta-study of research on function, note the difficulty for students in apprehending the modern, abstract definition of function depending, as it does, on the mapping of one set of elements to another, emphasizing the difference between function and relation (many-to-one acceptable, one-to-many not acceptable); whereas, the work on function in early grades builds on the intuitive notion of a 1–1 correspondence and furthermore, the historical development of function rested on the study of covarying quantities.

Even in secondary school, functions are typically introduced as very limited classes such as linear, quadratic, and exponential, with attendant graphs and tables, with the result that students regularly consider functions to be mathematics objects solely defined by an algebraic formula (e.g., Best & Bikner-Ahsbahs, 2017; Breidenbach et al., 1992; Carlson, 1998) and have difficulty identifying particular instantiations of functions (e.g., constant functions) as functions (Bakar & Tall, 1991; Carlson, 1998; Rasmussen, 2000). Instruction and curricular materials often emphasize procedures and algebraic manipulations when studying functions, and research shows that students then have difficulty in understanding different representations and different contexts for functions (Carlson & Oehrtman, 2005; Cooney et al., 2010). At the heart of many student difficulties is a shallow understanding of the definition (Ayalon et al., 2017; Panaoura et al., 2017). Students who have an algebraic view of function and who use procedural techniques to identify functions and nonfunctions (e.g., the vertical line test) struggle to comprehend the notion of function as a general mapping between sets (Carlson, 1998; Thompson, 1994).

Exposure to, and facility with, various representations of functions, that is, "flexible use of functions… within and between all kinds of representations and also between different functions" (Best & Bikner-Ahsbahs, 2017, p. 877), has been shown to be a critical component of a rich understanding of function (Best & Bikner-Ahsbahs, 2017; Dubinsky & Wilson, 2013; Martínez-Planell & Trigueros Gaisman, 2012). Researchers have found promising results when using novel contexts and nonstandard representations of functions such as dynagraphs, arrow diagrams, and directed graphs (Dubinsky & Wilson, 2013; Sinclair et al., 2009). Our own previous research (McCulloch et al., 2022), a study of preservice teachers working with a specially designed applet with no algebraic representation, showed that the applet was effective in initiating a series of dilemmas in preservice teachers' conception of function. The result was that the majority of the participants changed their conception of function in a positive direction. Specifically, in that study, the participants wrote a definition of function before working with the applet and then revised their definitions afterward. Using the applet helped improve their definitions. Building on that study, which involved preservice teachers who were already familiar with the concept of function, we designed a new study to introduce the concept of function to middle school students who do not know the concept. The goal was to examine the effect of a specially designed applet on middle school students' ability to develop an understanding of the concept of function.

Specifically, the research question is: what understanding of function can seventh-grade students, who have not encountered the term function, develop through using a specially designed applet (using a vending machine) with no algebraic representations?

2 | METHODS

2.1 | Participants

The *Introduction to Function* applet was used in 15 seventh-grade classrooms. These classrooms were across two different states (one Northeastern state and one Southeastern state) and five different teachers for a total of 144 students who engaged with the task. These students engaged with the applet and worksheet for a single class period toward the end of their seventh-grade year and had not yet learned about the definition of function or about function notation.

2.2 | Context

Previous research (McCulloch et al., 2022) has shown the promise of a vending machine representation as a "cognitive root" (Tall et al., 2000) for the study of functions with preservice teachers who were already familiar with the concept of function. Thus, we designed a version of the applet, *Introduction to Function* (https://tinyurl.com/y2dramsb), as a mechanism for learners who have never encountered the concept of a mathematical function and, therefore, do not associate the concept with any particular representation to learn the basic elements of function. The goal was for the students to learn that a function is a relationship between a set of inputs that are matched with a set of outputs in a consistent and, therefore, predictable manner, with each input matched to exactly one output.

The context we provide for the learners is a vending machine. There is a considerable amount of research on the use of contexts in mathematical learning. Boaler (1993) and Clarke and Helme (1998) argue that the context of a mathematical task is not fixed but, rather, forms the basis of an interactive, dynamic process as learners work on the task. In other words, as van Oers (1998) states, "what counts as context depends on how a situation is interpreted in terms of activity to be carried out" (p. 481). Our concern was to provide a model of function that was not based in algebra and, while mindful of the research on contexts, were encouraged by the research study with the preservice teachers that the model provided more affordances than limitations to student learning.

The *Introduction to Function* task is a GeoGebra book that consists of seven pages and has an accompanying worksheet. On the first two pages are two vending machines, each of which consists of four buttons (Red Cola, Diet Blue, Silver Mist, and Green Dew). When a button is clicked, it produces none, one, or more than one of the four different colored cans (red, blue, silver, and green), which may or may not correspond to the color of the button pressed (see Figure 1). The students are told that the first machine on each page is an example of something called a function, and the other is not a function, with their task being to identify what is the difference between the behavior of the machines that make one a function and the other not.

The machines on the first two pages work as follows (Figure 2).

Note that Machines B and D are not functions because one of the buttons when clicked, will produce a random can (i.e., not always the same result). Note also that in Machine C, the color of the output cannot correspond to the input button pressed, but the nonmatching can is consistently produced. After users work through the first two pages, there should be a whole group discussion led by an instructor in which users share their thoughts on what machine behavior makes a function, the goal being the consolidation of their ideas. The goal is that, at the end of the discussion, students will agree that what manifests in a machine that is a function is consistent behavior. This goal was achieved in each instantiation of the study, as evidenced by the video recording of each class.

The next four pages of the GeoGebra book consist of pairs of machines, with the students being told that one of each pair is a function. In each case, there is a random element in the nonfunction. The machines work as follows (Figure 3).

On the worksheet, students are asked to note whether each machine is a function or not a function and how they know. After they complete these pages, students are given the prompt: "Using the terms 'input' and 'output,' write a definition for function based on your exploration of the machines." This activity then served as the basis of a whole class discussion with the goal of agreeing on a class definition of mathematical function.

2.3 | Data collection and analysis

Students worked in pairs ($N = 72$) to engage with the applet on a laptop that screen captured their work. Data collected were their worksheets, which included their definitions, screen recordings while they worked on the task, and audio recordings of their discussion while they worked on the task. Owing to some technical issues, there were 69 complete pairs of data. All data was coded

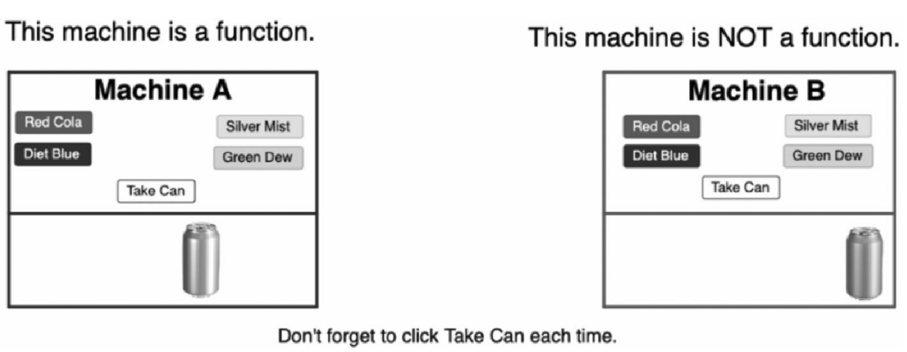

FIGURE 1 Screenshot of introduction to function.

FIGURE 2 Machines A–D.

This One is a Function		This One is Not a Function	
A	Red – Red Blue – Blue Silver – Silver Green – Green	B	Red – Red Blue – Blue Silver – Random Green – Green
This One is a Function		**This One is Not a Function**	
C	Red – Blue Blue – Red Silver – Silver Green – Green	D	Red – Red Blue – Random Silver – Silver Green – Green

FIGURE 3 Machines E–L.

	Which One is a Function?			
Page 3	E	Red Cola – red Diet Blue – blue Silver Mist – silver Green Dew – random color	F	Red Cola – silver Diet Blue – green Silver Mist – red Green Dew – blue
	Which One is a Function?			
Page 4	G	Red Cola – random color Diet Blue – random color Silver Mist – random color Green Dew – random color	H	Red Cola– blue Diet Blue – silver Silver Mist – green Green Dew – red
	Which One is a Function?			
Page 5	I	Red Cola – 2 silver cans Diet Blue – green Silver Mist – red Green Dew – blue	J	Red Cola – red Diet Blue – blue & random color Silver Mist – silver Green Dew – green
	Which One is a Function?			
Page 6	K	Red Cola – pair of random color Diet Blue – blue Silver Mist – silver Green Dew – green	L	Red Cola – green Diet Blue – green Silver Mist – green Green Dew – green

by two researchers. Any disagreements were discussed until any discrepancies were resolved.

In this study, we report our analysis in four stages:

i. Analysis of the worksheet responses to tally the correct or incorrect identification of functions/nonfunctions; in terms of the identification of functions/nonfunctions, for the pairs of machines E and F, G and H, I and J, and K and L, since students were told one was a function and one was not, it was possible to simply count the classification. Of course, the percentages should mirror each other, that is, the number of "corrects" for Machine E should match the number of "incorrects" for Machine F.

ii. Analysis of the students' justifications of their classifications as written on the worksheets; the students' written justifications for their machine classifications were open coded using a constant comparative method to look for themes (Creswell, 2014). In addition, some codes that had been developed in a previous project using a version of the applet with preservice teachers (McCulloch et al., 2022) were considered for their appropriateness to this data. The final codes for students' justifications are shown in Figure 4. Justification codes were not mutually exclusive, as a justification could have been coded based on inconsistency as well as using the context of the vending machines.

iii. Analysis of the video/audio recordings to examine why pairs of students were ever incorrect and, in particular, "critical events" (Powell et al., 2003) when they changed from correct to incorrect (or vice versa) as they worked through the four pages. After the tallying of correct/incorrect classifications, approximately 80% of the students were able to correctly identify functions, with their reasoning being consistency/predictability of outputs. The next stage of the analysis focused on the remaining 20% and consisted of going through each video to find "critical events" in order to understand why students were misidentifying machines and, in particular, why they changed from correct to incorrect identifications or vice versa.

Code	Description
Justification based on inconsistency of output	Students' justifications use phrases indicating an attention to the inconsistency of the outputs. Examples include: "different colors", "random", "it changes".
Justification based on consistency of output	Students' justifications use phrases indicating an attention to the consistency of the outputs. Examples include: "consistent", "constant", "pattern".
Justification uses the context of the machine	Students' justifications describe the relationships between inputs and outputs using the vending machine context. Examples include: "because it always gives the wrong drink", "it gives random colored cans"
Justification unclear	There is not enough detail in the students' written response to classify their justification.

FIGURE 4 Justification codes.

iv. Analysis of the definitions written by the students as a culmination of the activity. For the definitions we coded for use of the terms input/output, focus, that is, how students thought of the function, and attention to output (McCulloch et al., 2022). In terms of input/output, each definition was read for the use of those terms in the definition, for example, "M49_M62 [i.e., participant 49 working with participant 62]: No matter what input the output is the same" and "M117_M118: A function is when you get the same output."

In terms of focus, each definition was coded regarding whether the definition indicated a function was a relationship (or mapping), an object, or neither. Therefore, if the definition indicated that the function relates to the input and output, then the definition was coded as a relationship. For example, "M91_M96 The word function may mean when you input something, even though you may not get what you asked for, you will only get one type of it." The code "object" was used when the definition referred to a function as something, such as a button or a machine.

Finally, definitions were coded according to whether or not they attended to output. For this code, the definition needed to refer to an output having a pattern, being the same, or being consistent. For example, "M54_M59: Function is when you put in the input, and the output will never change/will always be the same."

3 | RESULTS

3.1 | Identification of the machines as functions/nonfunctions

The percentage of correctly identified functions for the first four pairs of machines was at least 80% and ranged from 80.7% to 95.8%. Students' classification of the machines is shown in Figure 5.

Machines	Non-function reason	% Correct
E & F	Machine E: Green Dew has random output	81.3
G & H	Machine G: all outputs are random	95.8
I & J	Machine J: Diet Blue output is Blue & random	86.1
K & L	Machine K: Red Cola output is 2 random cans	80.7

FIGURE 5 Participants' correct identification of functions.

3.2 | Students' justification of functions and nonfunctions

To better understand the ways in which students were making sense of the machines, we analyzed their justifications for whether or not each machine was a function or nonfunction. Some of the justifications were unclear, but the majority were clear and were based on the language of consistency. In particular, machines identified as functions were described as consistent (e.g., Pair M15 and M23 on Machine I, "Green Dew always gives Diet Blue"), and those identified as nonfunctions were described as inconsistent (e.g., Pair M48 and M65 on Machine K "Because Red Cola doesn't give out a consistent can color."). A tally of the justifications can be seen in Figure 6.

An interesting slight outlier here is Machine F, which was being compared to Machine E (see Figure 7) and which 11 pairs of students justified as not a function based on consistency.

As one student (M90) put it, Machine E is "more consistent" than Machine F, which "randomizes things." As will be discussed in more detail in the video analysis below, we see that it appears that these 11 pairs of

FIGURE 6 Characterizations of students' justifications for each machine.

Machine	Justification based on consistency	Justification based on inconsistency	Justification unclear
E	5	54	13
F	45	11	17
G	2	68	2
H	62	1	8
I	54	6	12
J	2	59	9
K	4	54	11
L	58	1	10

FIGURE 7 Machines E and F.

Which One is a Function?			
E	Red Cola – red Diet Blue – blue Silver Mist – silver Green Dew – random color	F	Red Cola – silver Diet Blue – green Silver Mist – red Green Dew – blue

students understood a machine as "inconsistent" if it gave them a different color output from the input button pressed, even if it did so consistently.

Examples of students' justifications based on inconsistency are shown in Figure 8 below.

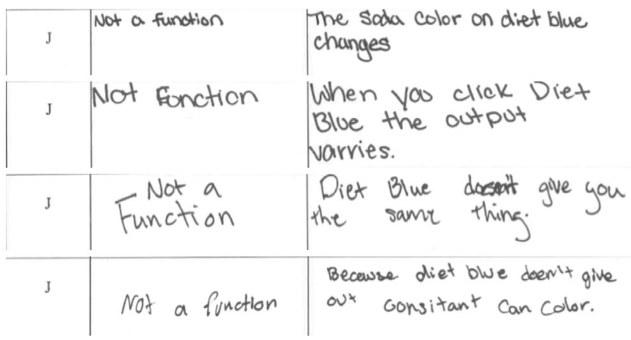

FIGURE 8 Examples of justifications based on attention to inconsistency of outputs.

3.2.1 | Video and audio recordings: "Critical events"

Recall that the percentage of correctly identified functions for the first four pairs of machines was at least 80%. Pairs of students who correctly identified all of the Machines E–L ($n = 49$) did so in a similar manner, being heard to say things like "E is not a function because it's always random when you click Green. [On F] The Red and Silver are inverted but this one's a function because it always gives you the same result." (Pair M18/M19). "G is not a function because it's inconsistent... it's always random. H is a function because it has a pattern and you can rely on it." (Pair M51/M52).

For those misidentifying machines, the principal reason was that the students considered the machine to be giving the "wrong" colors. The output can does not match the button pressed. For example, Pair M162/163 are heard to say, "F is not a function because it gives you the opposite colours."

There were occasional instances of students getting "better" as they worked through the four pages, that is, misidentifying machines in the early pages but correctly identifying them in later pages. The most common reason for improvement is instructor intervention ($n = 6$), that is, the instructor reminding students of the discussion of consistency at the end of the work on Machines A–D. The other notable reason for improvement ($n = 3$) was students essentially "working it out" and deciding that consistent "wrong" outputs were better than random outputs.

3.3 | Definitions

One of the 72 pairs of students did not complete a definition on their worksheet. The remaining 71 definitions were coded using the codebook. In terms of the use of input/output, 62 out of 71 (87.3%) definitions used the word input, and 65 out of 73 (89.0%) definitions used the term output.

In terms of focus, none of the participants described a relationship between inputs and outputs explicitly as a mapping between sets, and most definitions (43/71 [60.6%]) were coded as "neither object nor relationship." A large number of participants' definitions (27/71 [38.0%]) were coded as "object" since they made explicit reference to the vending machine or the buttons of the machine. For example, "Whenever you input into the vending machine, you know the output, which makes it reliable."

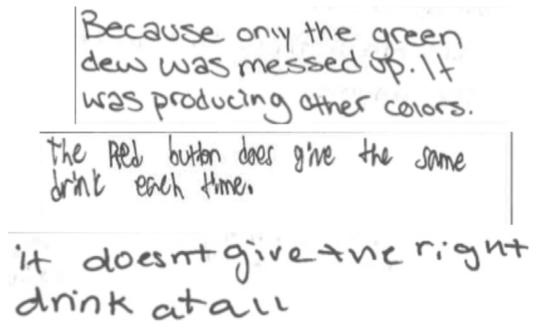

FIGURE 9 Examples of justifications that use the context of a vending machine.

4 | DISCUSSION

4.1 | Identification of the machines as functions/nonfunctions

The percentage of correctly identified functions for the first four pairs of machines was at least 80% and ranged from 80.7% to 95.8%. At the first level of analysis, this shows that broadly speaking, the pairs of students were able to correctly identify which machines were functions. Looking more closely at the incorrect answers for the first four pairs of machines we see that it is often the same pairs of students getting incorrect answers. A total of 10 of the 14 (71.4%) pairs of students who made a misidentification of the E and F pair misidentified at least one other machine, with 5 pairs misidentifying all of the first four sets of machines except the G and H pairing.

4.1.1 | Constant function

The result for Machine L, with 80.0% of participants identifying it as a function, is a potentially significant result since researchers have shown that students exhibit difficulties identifying constant functions as functions (e.g., Carlson, 1998; Rasmussen, 2000). The successful pairs, when working on this machine, note that K red is giving two random cans and, while some have some hesitation when they get to Machine L (e.g., Pair M105/M110 discuss if both can be nonfunctions), they settle on Machine K's randomness as "more important." It might be interesting to see if reversing the order of Machines K and L would give more students hesitation with regard to the constant function.

4.2 | Students' justification of functions and nonfunctions

As is evident in the Machine F example above, the students' justifications provide insight into their misidentification of both functions and nonfunctions. For example, looking at the 13 pairs of students that misidentified Machine K (R → random pair) as a function, it is evident that they either did not test the machine enough to see the random outputs that occurred when clicking Red Cola (e.g., "every color is functional, red produces two greens"), or they decided that since the rest of the buttons were consistent it was "close enough." For example, one pair wrote "mostly consistent" and another wrote "3 of the 4 functions correctly." Furthermore, the understanding that machines give a different output from the button pressed, even if it does so consistently, persisted for a number of pairs as they continued to work through the machines. For example, Pair M17 and M20 said of machine J (R → R, B → B and random, S → S, G → G "The Blue one gives two but the others work."

It is notable that 80% of the student pairs used the language of the machine context in their justifications (see, e.g., Figure 8). This suggests that having a realistic context in which to both think about and test their conjectures proved to be helpful in explaining their thinking for many students and answers the research question inasmuch as they developed a strong understanding based on the notion of consistency of output (Figure 9).

4.3 | Video and audio recordings: "Critical events"

As mentioned above, the students identifying the machines correctly did so in a similar manner. The analysis of the work of the students misidentifying machines focuses on two questions:

i. Why were students misidentifying particular machines as functions/nonfunctions? and
ii. Were there instances of students getting "better" or "worse" as they worked through the four pages, that is, misidentifying machines in the early pages but

correctly identifying them in later pages (or vice versa) and, if so, how and why were they able to adjust their understanding?

4.3.1 | Machines E and F

Of the 69 pairs of students who completed the worksheet, 12 misidentified Machines E and F, saying that E (G → random) is a function and that F (R → S, B → G, S → R, G → B) is not a function. While working on these machines, successful Pairs can be heard saying things like "Green Dew gave out random cans instead of one consistent can" (Pair M66/M67).

Of the 12 pairs that were incorrect, all state, in essence, that F is not a function because it is giving the "wrong" colors, that is, the output can does not match the button pressed. For example, possibly picking up on the use of the word "random" in the class discussion that came after the first two pages of the applet, but misinterpreting it, Pair M76/M92 are heard to say "When I click on Red it comes out Silver, when I click on Blue it comes out Green, so random... it's going to be random." Pair M162/163 are heard to say "F is not a function because it gives you the opposite colours."

Interestingly, five pairs of students who see that the Green button in Machine E gives random outputs still consider Machine E to be a function because, as, for example, Pair M145/M146 put it "In this one [E], it was only green that was getting messed up but in this one [F] every one is getting messed up." Another version of this interpretation came from Pair M80/M85 who say "Machine E is a function because it gives out all or mostly right colours. It got most all the colours right and Machine F is not a function because it got all the colours wrong."

Finally, it is noteworthy that for two pairs, they did not click on F's Green button enough to see the randomness.

Overall, we can see here that for 11 of the 69 pairs of students (c. 16%), after the class discussion about Machines A–D, they have developed an understanding that a machine with consistent outputs is not a function if those outputs do not color match the button pressed. In particular, for a group of students, consistent mismatching is "worse" behavior by the vending machine than having one with several matches but one button with completely unpredictable outcomes. This points to the limitations of the context for some students. The intention is for the vending machine to provide a non-algebraic cognitive root for students to focus on the concept of function. For some students, however, the context is not metaphorical but, rather, is an actual machine with which they are trying to interact, and they are frustrated that it does not work the way they expect.

As we will see below, this idea that a machine where the outputs do not match the buttons cannot be a function persists, for a number of pairs of students, throughout the activity.

4.3.2 | Machines G and H

Of the 69 pairs, just 2 misidentified Machine G (all random) as a function, with an additional pair saying that neither G nor H (R → B, B → S, S → R, G → B) is a function. All three pairs were pairs that had misidentified E and F. Therefore, of the 11 pairs that misidentified Machines E and F, 9 of them (c. 82%) correctly identified Machines G and H, and 1 more Pair correctly identified G.

The first reason for improvement is instructor intervention ($n = 6$). As mentioned above, instructors circulated as students worked on the task. Here, for example, is the exchange between Pair M31/M36 and an instructor:

Instructor (referring to G):	Why is it not a function?
Student (clicking on various cans but only once each):	Red equals Blue, Blue gives you Green, Green gives Blue and Silver will be Green. So it's not a function.
Instructor:	Ok, why not? Using those words ('random' 'consistent') [the teacher] said.
Student:	They're all random
Instructor:	They're all random. Ok, what about on H?
Student (clicking on various cans but again only once each):	Red equals Blue, Blue equals Silver, Green equals Red and Silver equals Green.
Instructor:	Ok, so it's not just that the colours are messed up it's that when you try Red *multiple* times that it's random. So have you tried each button more than once in a row? Student clicks on some buttons on Machine H.

Instructor (noting output):	So Red is *always* Blue. So let's go back to G.
Student (clicking on Red gets Red):	Red.
Instructor:	Ok what happens the next time you press Red?
	Student clicks and gets various outputs after clicking Red.
Instructor:	So, is that consistent?
Student:	No.
Instructor:	So G is not your function because it's not consistent. H is your function. It's Ok that the colours are messed up but so long as Red *always* gives Blue that's Ok.
Student:	Ok. That makes more sense now.

Following interactions like the above six of the Pairs who had misidentified E and F now correctly identified G and H.

The second most common reason ($n = 3$) for a change from E and F incorrect to G and H correct was students essentially "working it out" and deciding that consistent "wrong" outputs were better than random outputs. For example, when Pair M145/M146 who, as seen above, had said about E and F, "In this one [E], it was only green that was getting messed up but in this one [F] every one is getting messed up" arrive at Machines G and H they proceed as follows:

Starting on Machine G, they click once on each button and say that it is "messed up." They then click once on each button in H, and there is a hesitant pause. They go back to G, click on Silver three times and get three different results. They then go back to H, click on Silver three times, getting Green each time, and are heard to say, "Hey, this one's more functional." This student then, while clicking on Machine G and getting random outputs, says to his partner: "Look you get multiple sodas with each button." They then write on their worksheet: "G is not functional. Because when you click each color, each time you click it, you get something different" and "H is Functional. Because when you clicked a button, it might have given a diff. color, but it was the same each time."

The pairs of students ($n = 3$) who misidentified G and/or H after misidentifying E and F maintained the understanding they developed, that is, that if the outputs do not match the button pressed, the machine is not a function. For example, Pair M17/M20 are heard to say that for Machine G, "Well, it's sometimes right," but Machine F, "it's always wrong."

4.3.3 | Machines I and J

For Machines I (a function) (R → Silver pair, B → G, S → R, G → B) and J (not a function) (R → R, B → B + random, S → S, and G → G) the pairs ($n = 3$) who misidentified Machines E–H continued to misidentify the machines and for the same reason: the output colors do not match the button pressed. Interestingly, this misidentification persists even when it is clear from the video with audio recordings that the students are seeing the random outputs from machines they decide are functions, that is, a machine with three matches and one random button is seen as preferable to a machine with four "mismatches" even when those "mismatches" are consistent.

Two pairs of students who misidentified E and F, but correctly identified G and H, in each case without instructor intervention, then reverted to misidentifying I and J. In the case of Pair M145/M146, it became clear that only one of the pair had "worked it out" in the case of Machines G and H and his partner now convinced him that the output should actually match the button pressed and that this occurs more often for I. The student who had said earlier, "Look you get multiple sodas with each button" in correctly identifying G as a nonfunction, clicks on Machine I enough to see consistency and on Machine J enough to see the randomness in the second can be outputted by Blue, however, his partner claims that J is "more functional because only the Blue is messed up." After a lot of clicking on both machines, the argument that more buttons are "messed up" on Machine I than on Machine J holds sway. This example is interesting in terms of the power that correct matching, as opposed to consistent matching, holds for students. The other reverting pair only clicks on J Blue once, so they do not see the randomness, and it is not clear from their discussion why they decide J is a function.

Finally, for Machines I and J, three pairs of students are "wrong" for the first time. In the case of two of those pairs, they do not click on J Blue enough to really register the randomness and adopt the thinking that the machine where more outputs match inputs is "better." For the third pair, their thinking is not clear.

4.3.4 | Machines K and L

For Machines K (R → random pair, B → B, S → S, G → G) and L (all go to Green), seven pairs of students either continue to misidentify (having started back on E and F or become "wrong" later) and all for the same reason, to wit, to be a function the output color must match the button pressed and some occasional randomness is acceptable. It should be noted that none of these pairs experienced an instructor intervention.

It is interesting here that five pairs of students misidentify Machines K and L, having correctly identified all of Machines E–J, that is, they misidentify for the first time when they encounter the constant function. In one case (Pair M168/M169), they did not click on K enough to see the randomness. For the remaining four pairs, they did not like that Machine L only gave Green. "This gives only Green: if I come up to this machine and ask for Red Cola and I get Green Dew, I'm going to the county to complain." (Pair M70/M75). This pair and others argue that Machine K is "better" because it gives all the colors. Again, we see here that the real-world context seems to actually interfere with students' ability to develop the understanding we are looking for.

In terms of the research question, the understanding that approximately 80% of the students developed was a robust one based on consistency of behavior and outputs, while for approximately 20% of the students, their understanding was based on how they literally expected the vending machine behaved rather than the understanding of the machine as a cognitive root that we had hoped for.

4.4 | Definitions

Recall that in terms of the use of input/output, 62 out of 71 (87.3%) definitions used the word input, and 65 out of 73 (89.0%) definitions used the term output. Of course, the participants were asked to use these terms and, therefore, the result is not entirely surprising. Nevertheless, the result is promising in terms of establishing sets of inputs and outputs as a central aspect of the definition of function.

Perhaps the most interesting aspect of the activity was to examine the extent to which the participants would pay due attention to the outputs from the machines. Analysis of the definitions shows that 45 of 71 (61.6%) of the participants did pay attention to the output with definitions such as "When you input something, the output always will stay the same." However, 14 of 71 (19.7%) of participants, while paying attention to the output, made an incorrect statement such as "Your input is your output and does not change."

4.5 | The word "function"

One final result that emerged from the analysis of the "critical events" was the students' understanding of the word "function." The participants in this study were chosen precisely because they had not yet encountered the word function in a mathematical context. When the task was introduced, the students were told that we would be talking about something called a function and were reminded that they had not yet studied this mathematical concept lest they be concerned about trying to remember something from a previous mathematics class. As can be seen above, the analysis of the video with audio of the students working on the task showed that the most common reason for students misidentifying machines as function/nonfunctions was their understanding that the output color should match the button pressed, seemingly looking for the context to match their real-life experience. It was clear from the language they used that a mismatch was an indication to them that the machine was not working properly. On many, many occasions, they used the non-mathematical meaning of the word function and said things like "This machine is not functioning," "It's not functional," "This machine is more functional than the other." Therefore, relying on their existing meaning for the word "function" may have exacerbated, for some students, the challenge they had with accepting an output that was different from the input button, even if that output was consistent.

5 | CONCLUSION

The research question was: what understanding of function can seventh-grade students, who had not encountered the term function, develop through using a specially designed applet (using a vending machine) with no algebraic representations? Specifically, the applet was designed to help them develop an understanding of a function as a relationship between inputs and outputs with some restrictions on the outputs. The nonstandard representation of the *Introduction to Function* applet served to introduce the concept of function without algebraic representations. With the focus on the consistency, or otherwise, of the outputs, the participants were able to correctly distinguish between functions and nonfunctions at least 80% of the time. This suggests that the applet may be a good way to introduce functions to middle school students and may overcome some of the difficulties that arise from traditional approaches in middle school (Mesa, 2004). The main challenge to correct identification of functions is the real-world context of a vending machine, and the attendant challenge of accepting an output can color that does not match the can color of the input button perhaps exacerbated, somewhat ironically, for some students, by the use of the word function. Some limitations of the study may be that the results were overdetermined by the discussion after the first two pairs of machines, followed by some instructor intervention as students worked on the task, and that the participants might be seen to be simply playing a pattern recognition "game" with the rule "random bad, not random good." Therefore, more study would be needed to establish if the basic concept learned here transfers effectively to further study of function. However, even within this study, more than 60% used some appropriate language to describe the nature of the output in their definitions of function. In addition, contrary to a well-known misconception, participants may be able to recognize a constant function as a function. Finally, it should be noted that the purpose of this activity was to set the scene for a class discussion of their definitions with the goal of arriving at a shared definition. The results of the study suggest that there is a good foundation for that discussion.

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REFERENCES

Ayalon, M., Watson, A., & Lerman, S. (2017). Students' conceptualisations of function revealed through definitions and examples. *Research in Mathematics Education*, *19*(1), 1–19. https://doi.org/10.1080/14794802.2016.1249397

Bakar, M., & Tall, D. (1991). Students' mental prototypes for functions and graphs. In F. Furinghetti (Ed.), *Proceedings of the 15th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 104–111). International Group for the Psychology of Mathematics Education.

Best, M., & Bikner-Ahsbahs, A. (2017). The function concept at the transition to upper secondary school level: Tasks for a situation of change. *ZDM*, *48*, 865–880. https://doi.org/10.1007/s11858-017-0880-6

Blanton, M., Brizuela, B. M., Gardiner, A. M., Sawrey, K., & Newman-Owens, A. (2015). A learning trajectory in 6-year-olds' thinking about generalizing functional relationships. *Journal for Research in Mathematics Education*, *46*(5), 511–558.

Boaler, J. (1993). The role of contexts in the mathematics classroom: Do they make mathematics more "real"? *For the Learning of Mathematics*, *13*(2), 12–17.

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics*, *23*(3), 247–285.

Carlson, M., & Oehrtman, M. (2005). *Key aspects of knowing and learning the concept of function*. Mathematical Association of America.

Carlson, M. P. (1998). A cross-sectional investigation of the development of the function concept. In E. Dubinsky, A. H. Schoenfeld, & J. J. Kaput (Eds.), *Research in collegiate mathematics education, III* (pp. 115–162). American Mathematical Society.

Clarke, D. J., & Helme, S. (1998). Context as construction. In *Mathematics teaching from a constructivist point of view* (pp. 129–147).

Common Core State Standards Initiative. (2010). Common core state standards for mathematics. The National Governors Association center for Best Practices and the Council of Chief State School Officers.

Cooney, T. J., Beckman, S., & Lloyd, G. M. (2010). *Developing essential understanding of functions for teaching mathematics in grades 9–12*. National Council of Teachers of Mathematics.

Creswell, J. (2014). *Research design: Qualitative, quantitative, and mixed methods approaches* (4th ed.). SAGE Publications.

Dubinsky, E., & Wilson, R. T. (2013). High school students' understanding of the function concept. *The Journal of Mathematical Behavior*, *32*(1), 83–101.

Ellis, A. B. (2011). Algebra in the middle school: Developing functional relationships through quantitative reasoning. In J. Cai & E. Knuth (Eds.), *Early algebraization* (pp. 215–238). Springer.

Gueudet, G., & Thomas, M. O. (2020). Secondary-tertiary transition in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 762–766). Springer.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research*, *60*(1), 1–64.

Martínez-Planell, R., & Trigueros Gaisman, M. (2012). Students' understanding of the general notion of a function of two variables. *Educational Studies in Mathematics*, *81*, 365–384.

McCulloch, A. W., Lovett, J. N., Meagher, M. S., & Sherman, M. F. (2022). Challenging preservice secondary mathematics teachers' conceptions of function. *Mathematics Education Research Journal*, *34*, 343–368.

Mesa, V. (2004). Characterizing practices associated with functions in middle school textbooks: An empirical approach. *Educational Studies in Mathematics*, *56*(2–3), 255–286.

Panaoura, A., Michael-Chrysanthou, P., Gagatsis, A., Elia, I., & Philippou, A. (2017). A structural model related to the understanding of the concept of function: Definition and problem solving. *International Journal of Science and Mathematics Education*, *15*, 723–740. https://doi.org/10.1007/s10763-016-9714-1

Powell, A. B., Francisco, J. M., & Maher, C. A. (2003). An analytical model for studying the development of learners' mathematical ideas and reasoning using videotape data. *The Journal of Mathematical Behavior*, *22*(4), 405–435.

Rasmussen, C. L. (2000). New directions in differential equations: A framework for interpreting students' understandings and difficulties. *Journal of Mathematical Behavior*, *20*, 55–87.

Sinclair, N., Healy, L., & Sales, C. O. R. (2009). Time for telling stories: Narrative thinking with dynamic geometry. *ZDM*, *41*(4), 441–452.

Stephens, A. C., Fonger, N., Strachota, S., Isler, I., Blanton, M., Knuth, E., & Murphy Gardiner, A. (2017). A learning progression for elementary students' functional thinking. *Mathematical Thinking and Learning*, *19*(3), 143–166.

Tall, D., McGowen, M., & DeMarois, P. (2000). The function machine as a cognitive root for the function concept. In M. L. Fernandez (Ed.), *Proceedings of the annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 255–261). Tucson, AZ: PME-NA.

Thompson, P. W. (1994). Students, functions, and the undergraduate curriculum. In E. Dubinsky, A. H. Schoenfeld, & J. J. Kaput (Eds.), *Research in collegiate mathematics education, 1* (Vol. 4, pp. 21–44). American Mathematical Society.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421–456). National Council of Teachers of Mathematics.

van Oers, B. (1998). From context to contextualizing. *Learning and Instruction*, *8*(6), 473–488.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education*, *20*(4), 356–366.

Zan, R., & Di Martino, P. (2020). Students' attitude in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 813–817). Springer.

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