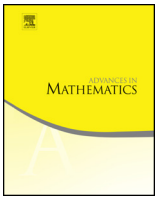
Advances in Mathematics 430 (2023) 109203

Contents lists available at ScienceDirect

Advances in Mathematics

journal homepage: www.elsevier.com/locate/aim



The graph minor theorem in topological combinatorics

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ARTICLE INFO

Article history:
Received 12 December 2020
Received in revised form 12 June 2023
Accepted 23 June 2023
Available online 14 July 2023
Communicated by Ezra Miller

Keywords:
Graph minors
Matching complexes
Topological combinatorics
Representation stability

ABSTRACT

We study a variety of natural constructions from topological combinatorics, including matching complexes as well as other graph complexes, from the perspective of the graph minor category of [13]. We prove that these complexes must have universally bounded torsion in their homology across all graphs of bounded genus. One may think of these results as arising from an algebraic version of the graph minor theorem of Robertson and Seymour [19,20].

1. Introduction

We study a category $\mathcal{G}_{\leq g}$ whose objects are finite connected graphs whose first Betti number, or (combinatorial) genus, is at most g and whose morphisms are built out of automorphisms, deletions, and contractions. For instance, a single vertex with g loops is an object of $\mathcal{G}_{\leq g}$, whereas trees form, by definition, the objects of $\mathcal{G}_{\leq 0}$. A precise definition

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https://doi.org/10.1016/j.aim.2023.109203
0001-8708/

of these categories will appear in Section 3.1. We then prove finite generations results about the representation theory of this category and apply these results in topological combinatorics. In particular we study spaces arising from graphs that behave well with respect to automorphisms, deletions, and contractions.

Throughout this introduction, a **graph** will refer to an at most one-dimensional CW complex that is both connected and finite. We say that a graph G is a **minor** of a graph G' if G can be obtained from G' by a sequence of edge deletions and contractions. In their seminal series of papers, Robertson and Seymour proved, among many other things, that the minor relation is actually a well-quasi-order [19,20]. That is to say, in *any* infinite collection of graphs, there must be a pair where one is a minor of the other. In this paper, we prove a weakened categorical version of the graph minor theorem which we outline below.

Fix a nonnegative integer g and write $\mathcal{G}_{\leq g}$ for the category whose objects are graphs of genus at most g, and whose morphisms are what are known as **minor morphisms** (see Section 3.1). While the precise definition of a minor morphism is a bit technical, for the purposes of this introduction you can think of them as maps built out of edge deletions, contractions, and graph automorphisms. In particular, one has a minor morphism $\varphi : G' \to G$ if and only if G is a minor of G'. A $\mathcal{G}^{\mathrm{op}}$-module over a Noetherian ring R is then a covariant functor $M : \mathcal{G}^{\mathrm{op}} \to R\text{-mod}$. Concretely, a $\mathcal{G}^{\mathrm{op}}$-module may be thought of as a collection of R-modules $\{M(G)\}_G$, one for each graph G, such that whenever G is a minor of G', one has an induced map $M(G) \to M(G')$. We say that a $\mathcal{G}^{\mathrm{op}}$-module M is finitely generated if there is some finite list of graphs $\{G_i\}_i$ such that each $M(G_i)$ is a finitely generated R-module and for any graph G, the R-module $M(G)$ is spanned by the images of the $M(G_i)$ under the aforementioned maps induced by the minor relation.

Our first result follows from results of [17] and shows that $\mathcal{G}^{\mathrm{op}}_{\leq g}$-modules satisfy a **Noetherian Property** and should be viewed as a weakened categorical version of the graph minor theorem.

Theorem 1.1. *Any submodule of a finitely generated $\mathcal{G}^{\mathrm{op}}_{\leq g}$-module is itself finitely generated.*

This weak categorical graph minor theorem translates the combinatorics of well-quasi-orders present in the original graph minor theorem to an algebraic statement about submodules of finitely generated modules. While it is the case that the categorical graph minor theorem is equivalent to the original, it is also presented in a language that is more amenable to application in topology and algebra. In particular, see [17][16] for applications of the categorical graph minor theorem to Kazhdan–Lusztig polynomials of graphical matroids, as well as configuration spaces of graphs. In the following work, we focus our attention to two topics that are important in topological combinatorics – simplicial complexes and hyperplane arrangements – in order to illustrate the power of the categorical graph minor theorem.

1.1. Homology of the matching complex

For a graph G, a **matching of** G is a collection of pairwise non-adjacent edges. The **matching complex** $\mathcal{M}(G)$ **of** G is the simplicial complex whose simplices are in bijection with matchings on G. Matching complexes of complete graphs and complete bipartite graphs have been studied extensively using a wide variety of techniques including discrete Morse theory. It is notable, however, that there is relatively little known about the matching complexes of general graphs. In this work, we aim to fill this gap in the literature by considering a new perspective on the problem: instead of focusing on $\mathcal{M}(G)$ for some particular graph G, or even for some particular family of graphs, we consider the matching complexes of all graphs at once.

The key observation that allows us to do this is that whenever G is a minor of G' there is a natural way to embed the edges of G into those of G'. Moreover, this embedding preserves the condition that the edges are disjoint, as any "undoing" of an edge deletion or contraction can only push things further apart. We therefore obtain simplicial complex maps $\mathcal{M}(G) \to \mathcal{M}(G')$ whenever G is a minor of G', which induce maps on homology $H_i(\mathcal{M}(G)) \to H_i(\mathcal{M}(G'))$. In particular, for any fixed $i \geq 0$, the assignment

$$G \mapsto H_i(\mathcal{M}(G))$$

is a well-defined $\mathcal{G}_{\leq g}^{\mathrm{op}}$-module over $\mathbb{Z}$. Using the categorical graph minor theorem, we will prove the following

Theorem 1.2. *The $\mathcal{G}_{\leq g}^{\mathrm{op}}$-module*

$$G \mapsto H_i(\mathcal{M}(G))$$

is finitely generated. In particular, there exists an integer $\epsilon_{i,g}$, depending only on i and g, such that the torsion part of $H_i(\mathcal{M}(G))$ is annihilated by $\epsilon_{i,g}$, for all graphs G.

Remark 1.3. Having seen a theorem such as Theorem 1.2, it is natural for one to ask whether one can obtain *effective* bounds on the items involved. For instance, bounds on the sizes of the graphs which generate $H_i(\mathcal{M}(G))$, or even bounds on the constant $\epsilon_{i,g}$ as a function of g. While there are certain limited cases in which this has been done – $\epsilon_{i,0} = 1$ [4] while $\epsilon_{1,g} = 2$ for $g \geq 4$ [11], for instance – it seems like it will be quite difficult to do in general. One particular observation that makes the problem somewhat tractible is that our proof of Theorem 1.1 is partially constructive. What we mean by this is that the proof proceeds in a way that is very similar to the classical **Gröbner basis** proof of the Hilbert Basis Theorem. While the tools of computational commutative algebra are not immediately applicable, it seems plausable that one might be able to adapt these tools to our setting.

The statement about torsion in the above theorem should be particularly interesting, as the torsion appearing the homology of the matching complex has been a subject of considerable intrigue in recent years [24][9][8].

We also note that, by the very general nature of the weak categorical graph minor theorem, the conclusions of Theorem 1.2 will remain true when the matching complex is replaced by a large variety of graph-based complexes, such as those discussed in [7]. See Section 4.2 for more on this.

1.2. Graphical linear subspace arrangements

Recall that, to any graph G, one may define a linear subspace arrangement induced by the edge relation of its vertices. We define the complement of this arrangement in $(\mathbb{C}^d)^{V(G)}$ by $\mathrm{Conf}(G, \mathbb{C}^d)$. More precisely, we have

$$\mathrm{Conf}(G, \mathbb{C}^d) = \{(x_v)_{v \in V(G)} \in (\mathbb{C}^d)^{V(G)} \mid x_v \neq x_w \text{ if } \{v, w\} \text{ is an edge of } G\}.$$

For instance, if $G = K_n$ is the complete graph, $\mathrm{Conf}(G, \mathbb{C}^d)$ is the classical configuration space of points in $\mathbb{C}^d$. If instead $G = K_{a,b}$ is the complete bipartite graph, then one recovers the colored configuration spaces $\widetilde{\mathcal{Z}}^D_{a+b}$ as studied by Farb, Wolfson, and Wood [6]. In this work we will specialize to the family of *line graph complements*.

Given a graph G, we write $\mathcal{L}^c(G)$ to denote the simple graph whose vertices are indexed by the edges of G, and whose edges indicate the corresponding edges in G are non-adjacent. These graphs have been called Kneser graphs by some authors [3], as the usual Kneser graph $K(n, 2)$ is seen to be $\mathcal{L}^c(K_n)$.

The same observation made above tells us that whenever G is a minor of G', one obtains a graph embedding

$$\mathcal{L}^c(G) \hookrightarrow \mathcal{L}^c(G').$$

This embedding induces a "forgetful" map

$$\mathrm{Conf}(\mathcal{L}^c(G'), \mathbb{C}^d) \to \mathrm{Conf}(\mathcal{L}^c(G), \mathbb{C}^d),$$

which when composed with cohomology yields

$$H^i(\mathrm{Conf}(\mathcal{L}^c(G), \mathbb{C}^d)) \to H^i(\mathrm{Conf}(\mathcal{L}^c(G'), \mathbb{C}^d)).$$

Our second theorem therefore can be stated as follows.

Theorem 1.4. *For any fixed i, d, g, the assignment*

$$G \mapsto H^i(\mathrm{Conf}(\mathcal{L}^c(G), \mathbb{C}^d))$$

defines a finitely generated $\mathcal{G}^{\mathrm{op}}_{\leq g}$-module over $\mathbb{Z}$.

One observes that the complete bipartite graph $K_{a,b}$ can (essentially) be realized as the line graph complement of some other graph. In particular, the finite generation result of Theorem 1.4 can be seen as a generalization of some results in [6] (see Theorem 3.21).

The paper is organized as follows. In Section 2 we outline and expand on the theory of Gröbner categories introduced by Sam and Snowden in [22]. In Section 3 we define the category $\mathcal{G}_{\leq g}$ and use the results of Section 2 to show that $\mathcal{G}_{\leq g}^{\mathrm{op}}$ has a Noetherian property. Finally in Section 4 we use the Noetherian property of $\mathcal{G}_{\leq g}^{\mathrm{op}}$ to study spaces in topological combinatorics arising from graphs.

Remark 1.5. In previous versions of this work, the above results were all stated without the dependency on the genus parameter g. These earlier versions of this paper were based on the Categorical Graph Minor Theorem of [13], and worked with the category of all graphs with minor morphisms. Unfortunately, a gap was found in the proof of the main theorem of [13], and so for now we can only state our results in the setting where genus is bounded. That being said, it is still very much the belief of the authors that the Categorical Graph Minor Theorem is true, and the techniques of this paper would then imply stronger versions of Theorems 1.2 and 1.4 where the parameter g does not appear.

Acknowledgments

The first author was supported by NSF grants DMS-1954050, DMS-2039316, and DMS-2053243. The second author was supported by NSF grants DMS-1704811 and DMS-2137628. Both authors would like to send their thanks to Nicholas Proudfoot for his various thoughts and advice related to this project.

2. Gröbner theory of categories

Let $\mathcal{C}$ be an essentially small category and k a ring. We define $\mathrm{Rep}_k(\mathcal{C})$ to be the category of functors from $\mathbb{C}$ to the category of k-modules. A module $\mathcal{M} \in \mathrm{Rep}_k(\mathbb{C})$ is called **finitely generated** if there exist finitely many objects $c_1, \ldots, c_r$ of $\mathbb{C}$ along with elements $v_i \in \mathcal{M}(c_i)$ such that, for any object c of $\mathbb{C}$, $\mathcal{M}(c)$ is spanned over k by the images of the elements v_i along the maps induced by all possible morphisms $\varphi_i : c_i \to c$. If every submodule of $\mathcal{M}$ is finitely generated, then $\mathcal{M}$ is said to be **Noetherian**. If every finitely generated module is Noetherian, the category $\mathrm{Rep}_k(\mathbb{C})$ is said to be **locally Noetherian**. Sam and Snowden have developed powerful machinery for proving that module categories are locally Noetherian which we summarize below.

Given an object x of $\mathbb{C}$, let $\mathbb{C}_x$ be the set of equivalence classes of morphisms out of x where $\varphi \in \mathrm{Hom}_{\mathcal{C}}(x, y)$ is equivalent to $\psi \in \mathrm{Hom}_{\mathcal{C}}(x, z)$ if there exists an isomorphism $\rho \in \mathrm{Hom}_{\mathcal{C}}(y, z)$ such that $\rho \circ \varphi = \psi$. The category $\mathcal{C}$ satisfies **property (G1)** for exevery object x of $\mathcal{C}$ there exists a linear order $\prec$ on $\mathcal{C}_x$ preserved under post composition. That is to say, if $\varphi, \psi \in \mathrm{Hom}_{\mathcal{C}}(x, y)$ and $\varphi \prec \psi$, then for any $\rho \in \mathrm{Hom}_{\mathcal{C}}(y, z)$, we have

$h \circ f \prec h \circ g$. There is a natural quasi-order on $\mathcal{C}_x$ where we say $\varphi \leq \psi$ if and only if there exists a morphism $\rho \in \mathrm{Hom}_{\mathcal{C}}(y, z)$ such that $\rho \circ \varphi = \psi$. Note, the orders $\leq$ and $\prec$ are independent. We say that $\mathcal{C}$ satisfies **property (G2)** if $\leq$ is a well quasi-order on $\mathcal{C}_x$. Namely, for any infinite sequence $\varphi_1, \varphi_2, \varphi_3, \dots$ of elements of $\mathcal{C}_x$, there exists a pair of indices $i < j$ such that $\varphi_i \leq \varphi_j$. The category $\mathcal{C}$ is **Gröbner** if $\mathcal{C}$ is a directed category and satisfies properties (G1) and (G2).

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $\Phi : \mathcal{D} \to \mathcal{C}$ a functor. The functor Φ satisfies **property (F)** if for any object x of $\mathcal{C}$ there exist finitely many objects $y_1, \dots, y_n$ of $\mathcal{D}$ and morphisms $\varphi_i \in \mathrm{Hom}_{\mathcal{C}}(x, \Phi(y_i))$ such that for any object y of $\mathcal{D}$ and any morphism $\varphi \in \mathrm{Hom}_{\mathcal{C}}(x, \Phi(y))$, there exists a morphism $\psi \in \mathrm{Hom}_{\mathcal{D}}(y_i, y)$ such that $\varphi = \Phi(\psi) \circ \varphi_i$. The category $\mathcal{C}$ is **quasi-Gröbner** if there exists a Gröbner category $\mathcal{D}$ and a functor $\Phi : \mathcal{D} \to \mathcal{C}$ satisfying property (F).

Theorem 2.1. *[22, Theorem 1.1.3] If $\mathcal{C}$ is a quasi-Gröbner category and k is a Noetherian commutative ring, then $\mathrm{Rep}_k(\mathcal{C})$ is locally Noetherian.*

2.1. Modules over algebras over categories

Let $\mathcal{C}$ be a category equipped with a functor $S : \mathcal{C} \to \mathrm{FI}$ and let k be a commutative ring. There is a natural functor from $\mathcal{C}$ to k-algebras taking an object c to the polynomial ring

$$\mathcal{A}_S(c) := k[x_e \mid e \in S(c)].$$

Equivalently, we can think of $\mathcal{A}_S \in \mathrm{Rep}_k(\mathcal{C})$ as a module equipped with a product $\mathcal{A}_S \otimes \mathcal{A}_S \to \mathcal{A}_S$ that is both associative and commutative. Let $\mathrm{Rep}_k(\mathcal{C}, S)$ be the category of modules over $\mathcal{A}_S$. Formally, an object of $\mathrm{Rep}_k(\mathcal{C}, S)$ is an object $\mathcal{M} \in \mathrm{Rep}_k(\mathcal{C})$ along with a multiplication $\mathcal{A}_S \otimes \mathcal{M} \to \mathcal{M}$ such that the two natural maps $\mathcal{A}_S \otimes \mathcal{A}_S \otimes \mathcal{M} \to \mathcal{M}$ coincide. More intuitively, an object $\mathcal{M}$ of $\mathrm{Rep}_k(\mathcal{C}, S)$ consists of an $\mathcal{A}_S(c)$-module $\mathcal{M}(c)$ for each object c of $\mathcal{C}$ and an $\mathcal{A}(c)$-module map $\mathcal{M}(c) \to \mathcal{M}(c')$ for each morphism $\varphi : c \to c'$, where $\mathcal{M}(c')$ is an $\mathcal{A}_S(c)$-module via the ring homomorphism $\mathcal{A}_S(c) \to \mathcal{A}_S(c')$ induced by φ.

A module $\mathcal{M} \in \mathrm{Rep}_k(\mathcal{C}, S)$ is called **finitely generated** if there exist finitely many objects $c_1, \dots, c_r$ of $\mathcal{C}$ along with elements $v_i \in \mathcal{M}(c_i)$ such that, for any object c of $\mathcal{C}$, $\mathcal{M}(c)$ is spanned over $\mathcal{A}_S(c)$ by the images of the elements v_i along the maps induced by all possible morphisms $\varphi_i : c_i \to c$. If every submodule of $\mathcal{M}$ is finitely generated, then $\mathcal{M}$ is said to be **Noetherian**. If every finitely generated module is Noetherian, the category $\mathrm{Rep}_k(\mathcal{C}, S)$ is said to be **locally Noetherian**. Now, we outline some basic facts about finitely generated modules and Noetherian modules, the proofs of which are completely standard. Let $\mathcal{C}$ be a category, $S : \mathcal{C} \to \mathrm{FI}$ a functor, and k a commutative ring. For any object c of $\mathcal{C}$, define the **principal projective** $\mathcal{P}_c \in \mathrm{Rep}_k(\mathcal{C}, S)$ to be the module that takes an object c' to the free $\mathcal{A}_S(c')$-module spanned by the set $\mathrm{Hom}_{\mathcal{C}}(c, c')$, with maps defined via composition.

Lemma 2.2. *A module $\mathcal{M} \in \mathrm{Rep}_k(\mathcal{C}, S)$ is finitely generated if and only if there exists a surjection*

$$\bigoplus_{i=1}^{r} \mathcal{P}_{c_i} \twoheadrightarrow \mathcal{M}$$

for some list of (not necessarily distinct) objects $c_1, \ldots, c_r$ of $\mathcal{C}$.

Proof. Suppose that $c_1, \ldots, c_r$ are objects of $\mathcal{C}$ and $v_i \in \mathcal{M}(c_i)$ for all i. These classes generate $\mathcal{M}$ if and only if the map

$$\bigoplus_{i=1}^{r} \mathcal{P}_{c_i} \to \mathcal{M}$$

taking $\mathrm{id}_{c_i} \in \mathcal{P}_{c_i}(c_i)$ to v_i is surjective. □

Recall from the introduction that a module $\mathcal{M} \in \mathrm{Rep}_k(\mathcal{C}, S)$ is **Noetherian** if every submodule of $\mathcal{M}$ is finitely generated.

Lemma 2.3. *A module $\mathcal{M}$ is Noetherian if and only if every ascending chain of submodules of $\mathcal{M}$ eventually stabilizes.*

Proof. Suppose that $\mathcal{M}$ is Noetherian and $(\mathcal{N}_i \mid i \in \mathbb{N})$ is an ascending chain of submodules of $\mathcal{M}$. Let $\mathcal{N} := \bigcup_{i \in \mathbb{N}} \mathcal{N}_i \subset \mathcal{M}$. Since $\mathcal{M}$ is Noetherian, $\mathcal{N}$ is finitely generated. If we choose i large enough so that $\mathcal{N}_i$ contains all of the finitely many generating classes, then we have $\mathcal{N}_i = \mathcal{N}$.

Conversely, suppose that $\mathcal{M}$ has a submodule $\mathcal{N} \subset \mathcal{M}$ that is not finitely generated. We will define an ascending chain of finitely generated submodules $(\mathcal{N}_i \mid i \in \mathbb{N})$ as follows. Let $\mathcal{N}_0 = 0$. Once we have defined $\mathcal{N}_i$, the fact that $\mathcal{N}_i$ is finitely generated means that $\mathcal{N}_i \subsetneq \mathcal{N}$, so we may choose an object c_i of $\mathcal{C}$ and an element $v_i \in \mathcal{N}(c) \setminus \mathcal{N}_i(c)$. Let $\mathcal{N}_i$ be the smallest submodule of $\mathcal{N}$ containing both $\mathcal{N}_i$ and v_i. This chain of submodules clearly does not stabilize. □

Lemma 2.4. *Suppose that $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ is short exact sequence in $\mathrm{Rep}_k(\mathcal{C}, S)$. Then $\mathcal{M}$ is Noetherian if and only if both $\mathcal{M}'$ and $\mathcal{M}''$ are Noetherian.*

Proof. If $\mathcal{M}$ is Noetherian, then $\mathcal{M}'$ is Noetherian by definition. If $\mathcal{N}'' \subset \mathcal{M}''$ is a submodule, let $\mathcal{N} \subset \mathcal{M}$ be the preimage of $\mathcal{N}''$ in $\mathcal{M}$. Since $\mathcal{M}$ is Noetherian, $\mathcal{N}$ is finitely generated, thus so is $\mathcal{N}''$ by Lemma 2.2.

Conversely, suppose that both $\mathcal{M}'$ and $\mathcal{M}''$ are Noetherian, and let $(\mathcal{N}_i \mid i \in \mathbb{N})$ be an ascending chain of submodules of $\mathcal{M}$. For each i, let $\mathcal{N}_i' := \mathcal{N}_i \cap \mathcal{M}'$ and let $\mathcal{N}_i''$ be the image of $\mathcal{N}_i$ in $\mathcal{M}''$. Since $\mathcal{M}'$ and $\mathcal{M}''$ are both Noetherian, Lemma 2.3 tells us that

there is an index n such that, for all $i > n$, $\mathcal{N}'_i = \mathcal{N}'_{i+1}$ and $\mathcal{N}''_i = \mathcal{N}''_{i+1}$. We can then conclude that $\mathcal{N}_i = \mathcal{N}_{i+1}$ by applying the Five Lemma to the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{N}'_i & \longrightarrow & \mathcal{N}_i & \longrightarrow & \mathcal{N}''_i & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle =} & & \Big\downarrow & & \Big\downarrow{\scriptstyle =} & & \\ 0 & \longrightarrow & \mathcal{N}'_{i+1} & \longrightarrow & \mathcal{N}_{i+1} & \longrightarrow & \mathcal{N}''_{i+1} & \longrightarrow & 0 \end{array}$$

Thus $\mathcal{M}$ satisfies the ascending chain condition, and is therefore Noetherian by Lemma 2.3. □

In the following sections, we build on the work of [22] and define what it means for the pair $(\mathcal{C}, S)$ to be **Gröbner** (resp. **quasi-Gröbner**) and prove the following generalization of Theorem 2.1.

Theorem 2.5. *Let $\mathcal{C}$ be a category and $S : \mathcal{C} \to \mathrm{FI}$ a functor. If the pair $(\mathcal{C}, S)$ is quasi-Gröbner, then $\mathrm{Rep}_k(\mathcal{C}, S)$ is locally Noetherian for any Noetherian commutative ring k.*

Remark 2.6. Theorem 2.5 is motivated by the work of Nagel and Römer [15]. Though they do not make these definitions in the same generality, they essentially prove that the pair $(\mathrm{FI}, \mathrm{id})$ is quasi-Gröbner, and they use this result to show that $\mathrm{Rep}_k(\mathrm{FI}, \mathrm{id})$ is locally Noetherian for any Noetherian commutative ring k. Moreover, they show that if $S_d : \mathrm{FI} \to \mathrm{FI}$ is the functor taking a set T to the set of unordered d-tuples of distinct elements of T, then the pair (FI, S_d) is quasi-Gröbner and the category $\mathrm{Rep}_k(\mathrm{FI}, \mathrm{id})$ is locally Noetherian if and only if $d \leq 1$ [15, Proposition 4.8].

Remark 2.7. Note that we have $\mathrm{Rep}_k(\mathcal{C}) = \mathrm{Rep}_k(\mathcal{C}, \emptyset)$, where $\emptyset : \mathcal{C} \to \mathrm{FI}$ is the constant functor that takes every object of $\mathcal{C}$ to the empty set. We will see that the category $\mathcal{C}$ is Gröbner (resp. quasi-Gröbner) in the sense of [22] if and only if the pair $(\mathcal{C}, \emptyset)$ is quasi-Gröbner.

2.2. Gröbner pairs

Let OI be the category whose objects are totally ordered finite sets and whose morphisms are ordered inclusions, and let $\Psi : \mathrm{OI} \to \mathrm{FI}$ be the functor that forgets the order on a finite set. Let D be an essentially small category and $T : \mathrm{D} \to \mathrm{OI}$ any functor. The purpose of this section is to define what it means for the pair (D, T) to be Gröbner.

A **quartet** for the pair (D, T) is a quadruple $\mu = (d, d', \varphi, m)$, where d and d' are objects of D, $\varphi : d \to d'$ is a morphism, and $m : T(d') \to \mathbb{N}$ is a map of sets. For

any morphism $\psi : d' \to d''$ in D, we will write $T(\psi) : T(d') \to T(d'')$ for the induced morphism in OI, and we will write

$$\psi(\mu) := (d, d'', \psi \circ \varphi, m_\psi),$$

where m_ψ is determined by the conditions that $m_\psi \circ T(\psi) = m$ and m_ψ is identically zero outside of the image of $T(\psi)$. For any map $n : T(d') \to \mathbb{N}$, we will write

$$\mu + n := (d, d', \varphi, m + n).$$

If $\mu_1 = (d, d'_1, \varphi_1, m_1)$ and $\mu_2 = (d, d'_2, \varphi_2, m_2)$, we say that $\mu_1 \leq \mu_2$ if there exists a morphism $\psi : d'_1 \to d'_2$ and a map $n : T(d'') \to N$ such that $\mu_2 = \psi(\mu_1) + n$.

Remark 2.8. The motivation for these definitions is that, once we choose a commutative ring k, the quartet μ determines a monomial

$$x^m := \prod_{a \in T(d')} x_a^{m(a)} \in \mathcal{R}(d')$$

along with an element

$$b_\mu := x^m \cdot \varphi \in \mathcal{P}_d(d') \in \mathrm{Rep}_k(\mathrm{D}, \Psi \circ T).$$

Then $\mu_1 \leq \mu_2$ if and only if φ_2 factors through φ_1 via a map ψ and we have

$$b_{\mu_2} = x^n \psi(b_{\mu_1})$$

for some monomial $x^n \in \mathcal{A}(d'_2)$.

We say that μ_1 and μ_2 are **equivalent** if $\mu_1 \leq \mu_2 \leq \mu_1$. For each object d of D, let $|\mathrm{D}_d^T|$ denote the poset of equivalence classes of quartets with first coordinate d. Given a quartet $\mu = (d, d', \varphi, m)$, we will write $[\mu]$ to denote its equivalence class in $|\mathrm{D}_d^T|$. A well-order $\prec$ of $|\mathrm{D}_d^T|$ is called **admissible** if, given two quartets $\mu_1 = (d, d', \varphi_1, m_1)$ and $\mu_2 = (d, d', \varphi_2, m_2)$ with the same source and target along with a morphism $\psi : d' \to d''$ and a map $n : T(d'') \to \mathbb{N}$, we have

$$[\mu_1] \prec [\mu_2] \implies [\psi(\mu_1) + n] \prec [\psi(\mu_2) + n].$$

We say that the pair (D, T) satisfies **property (G1)** if, for every object d of D, the poset $|\mathrm{D}_d^T|$ admits an admissible well-order. A poset P is said to be **Noetherian** if, for any sequence $(p_i \mid i \in \mathbb{N})$ in P, there exist natural numbers $i < j$ such that $p_i \leq p_j$. We say that the pair (D, T) satisfies **property (G2)** if, for every object d of D, the poset $|\mathrm{D}_d^T|$ is Noetherian. The category D is said to be **directed** if, for any object d of D, the only morphism from d to d is the identity. We call the pair (D, A) **Gröbner** if D is directed and (D, A) satisfies properties (G1) and (G2).

Remark 2.9. Property (G1) for the pair $(\mathrm{D}, \emptyset)$ is equivalent to property (G1) for D as defined in [22, Section 1.1], and similarly property (G2) for the pair $(\mathrm{D}, \emptyset)$ is equivalent to property (G2) for D. Thus a directed category D is Gröbner in the sense of [22] if and only if the pair $(\mathrm{D}, \emptyset)$ is Gröbner.

The following Proposition says that the functor T does not add anything interesting to property (G1). In other words, the distinction between a Gröbner category and a Gröbner pair lies entirely in the property (G2).

Proposition 2.10. *The pair (D, T) satisfies property (G1) if and only if the pair $(\mathrm{D}, \emptyset)$ satisfies property (G1).*

Proof. If $\prec$ is an admissible order of $|\mathrm{D}_d^T|$, then restriction to quartets with $m = 0$ gives an admissible order of $|\mathrm{D}_d^\emptyset|$. Conversely, if we have an admissible order of $|\mathrm{D}_d^\emptyset|$, we can compare the classes of two quartets $\mu_1 = (d, d_1', \varphi_1, m_1)$ and $\mu_2 = (d, d_2', \varphi_2, m_2)$ for (D, T) by first comparing the classes of the quartets $(d, d_1', \varphi_1, 0)$ and $(d, d_2', \varphi_2, 0)$ for $(\mathrm{D}, \emptyset)$ and then, if they are equal, breaking the tie by comparing m_1 and m_2 lexicographically. □

2.3. Gröbner bases

Let D be an essentially small category and $T : \mathrm{D} \to \mathrm{OI}$ a functor such that the pair (D, T) is Gröbner, and choose an admissible well-order $\prec$ of $|\mathrm{D}_d^T|$ for each object d of D as in the definition of property (G1). For any pair of objects d and d' in D, let $Q_{d,d'}$ be the set of quartets of the form $\mu = (d, d', \varphi, m)$. The fact that D is directed implies that the natural map from $Q_{d,d'}$ to $|\mathrm{D}_d^T|$ is injective, thus $Q_{d,d'}$ is well-ordered by $\prec$.

Fix a commutative ring k, so that we may define the representation category $\mathrm{Rep}_k(\mathrm{D}, \Psi \circ T)$. For any nonzero element

$$p \;=\; \sum_{\mu \in Q_{d,d'}} \lambda_\mu b_\mu \;\in\; \mathcal{P}_d(d'),$$

we define the **leading quartet** $\mathrm{LQ}(p)$ to be the maximal μ with respect to the well-order $\prec$ such that the coefficient $\lambda_\mu \in k$ is nonzero. If $\mu = \mathrm{LQ}(p)$, we define the **leading term** $\mathrm{LT}(p) := \lambda_\mu b_\mu \in \mathcal{P}_d(d')$ and the **leading coefficient** $\mathcal{L}^c(p) := \lambda_\mu \in k$.

Lemma 2.11. *Suppose we have a morphism $\psi : d' \to d''$, a map $n : T(d') \to \mathbb{N}$, and an element $0 \neq p \in \mathcal{P}_d(d')$. Then*

$$\mathrm{LT}(x^n \psi(p)) = x^n \psi(\mathrm{LT}(p)).$$

Proof. This is precisely the definition of admissibility of the well-order $\prec$. □

Given a submodule $\mathcal{N} \subset \mathcal{P}_d$, we define a subset

$$\mathrm{LQ}(\mathcal{N}) := \left\{ [\mathrm{LQ}(p)] \;\middle|\; 0 \neq p \in \mathcal{N}(d') \right\} \subset |\mathrm{D}_d^T|.$$

For each object d' of D, we define

$$\mathrm{LT}(\mathcal{N})(d') := \{0\} \cup \{\mathrm{LT}(p) \mid 0 \neq p \in \mathcal{N}(d') \subset \mathcal{P}_d(d')\}.$$

For each quartet $\mu = (d, d', \varphi, m)$, we define the ideal

$$\mathcal{L}^c(\mathcal{N}, \mu) := \{0\} \cup \{\mathcal{L}^c(p) \mid 0 \neq p \in \mathcal{N}(d') \text{ and } \mathrm{LQ}(p) = \mu\} \subset k.$$

Lemma 2.11 implies that $\mathrm{LT}(\mathcal{N}) \subset \mathcal{P}_d$ is a submodule and that we have an inclusion of ideals $\mathcal{L}^c(\mathcal{N}, \mu_2) \subset \mathcal{L}^c(\mathcal{N}, \mu_1)$ whenever $[\mu_1] \leq [\mu_2]$.

Suppose we are given a finite set $B = \{(d'_1, p_1), \ldots, (d'_r, p_r)\}$ of pairs with $0 \neq p_i \in \mathcal{N}(d'_i)$ for all i. We say that B is a **Gröbner basis** for $\mathcal{N}$ if the module $\mathrm{LT}(\mathcal{N})$ is generated by the classes $\mathrm{LT}(p_i)$ for $1 \leq i \leq r$.

Lemma 2.12. *If B is a Gröbner basis for $\mathcal{N}$, then B generates $\mathcal{N}$.*

Proof. If not, choose an element $p \in \mathcal{N}(d')$ that is not in the submodule generated by B, and choose it in such a way that the leading quartet $\mathrm{LQ}(p)$ is minimal with respect to the admissible well-order on $|\mathrm{D}_d^T|$. Since B is a Gröbner basis, we may choose an index i, a morphism $\psi : d'_i \to d'$, a function $n : T(d') \to \mathbb{N}$, and a scalar $\lambda \in k$ such that $\mathrm{LT}(p) = \lambda x^n \psi(\mathrm{LT}(p_i))$. By Lemma 2.11, this is equal to $\mathrm{LT}(\lambda x^n \psi(p_i))$. But then $p - \lambda x^n \psi(p_i)$ is not in the submodule generated by B and has a leading quartet strictly smaller than $\mathrm{LQ}(p)$, which gives a contradiction. □

Proposition 2.13. *Suppose that the ring k is Noetherian. For every object d of* D*, the principal projective $\mathcal{P}_d \in \mathrm{Rep}_k(\mathrm{D}, \Psi \circ T)$ is Noetherian.*

Proof. By Lemma 2.12, it is sufficient to show that every submodule $\mathcal{N} \subset \mathcal{P}_d$ has a Gröbner basis. By property (G2), the set $\mathrm{LQ}(\mathcal{N}) \subset |\mathrm{D}_d^T|$ has only finitely many minimal elements with respect to the partial order. Choose finitely many quartets $\mu_1, \ldots, \mu_r$ representing these minimal classes, and write $\mu_i = (d, d'_i, \varphi_i, m_i)$. For each i, the fact that k is Noetherian implies that the ideal $\mathcal{L}^c(\mathcal{N}, \mu_i)$ is generated by finitely many elements

$$\lambda_i^1, \ldots, \lambda_i^{s_i} \in k.$$

For each $j \leq s_i$, choose an element $0 \neq p_i^j \in \mathcal{N}(d'_i)$ with $\mathrm{LT}(p_i^j) = \lambda_i^j b_{\mu_i}$, and let

$$B := \{(d'_i, p_i^j) \mid 1 \leq i \leq r,\ 1 \leq j \leq s_i\}.$$

We claim that B is a Gröbner bases for $\mathcal{N}$.

Let $0 \neq p \in \mathcal{N}(d')$ be given; we will show that $\mathrm{LT}(p)$ is in the submodule of $\mathcal{P}_d$ generated by the classes $\mathrm{LT}(p_i^j)$. Let $\nu := \mathrm{LQ}(p)$. By definition of the quartets $\mu_1, \dots, \mu_r$, there exists an index i such that $[\mu_i] \leq [\nu]$. That means that we can choose a morphism $\psi : d_i' \to d'$ and a map $n : T(d') \to \mathbb{N}$ such that $\nu = \psi(\mu_i) + n$. Since $[\mu_i] \leq [\nu]$, we have $\mathcal{L}^c(\mathcal{N}, \nu) \subset \mathcal{L}^c(\mathcal{N}, \mu_i)$, and therefore there exist elements $\zeta_i^1, \dots, \zeta_i^{s_i} \in k$ such that

$$\mathcal{L}^c(p) = \zeta_i^1 \lambda_i^1 + \cdots + \zeta_i^{s_i} \lambda_i^{s_i}.$$

Then

$$\begin{aligned}
\mathrm{LT}(p) &= \mathcal{L}^c(p) b_\nu \\
&= (\zeta_i^1 \lambda_i^1 + \cdots + \zeta_i^{s_i} \lambda_i^{s_i}) b_{\psi(\mu_i)+n} \\
&= x^n \psi(\zeta_i^1 \lambda_i^j b_{\mu_i} + \cdots + \zeta_i^{s_i} \lambda_i^j b_{\mu_i}) \\
&= x^n \psi\left(\zeta_i^1 \,\mathrm{LT}(p_i^1) + \cdots + \zeta_i^{s_i} \,\mathrm{LT}(p_i^{s_i})\right)
\end{aligned}$$

is in the submodule of $\mathcal{P}_d$ generated by the classes $\mathrm{LT}(p_i^j)$. □

Corollary 2.14. *Let* D *be an essentially small category,* $T : \mathrm{D} \to \mathrm{OI}$ *a functor, and* k *a Noetherian commutative ring. If the pair* (D, T) *is Gröbner, then* $\mathrm{Rep}_k(\mathrm{D}, \Psi \circ T)$ *is locally Noetherian.*

Proof. Suppose that $\mathcal{M} \in \mathrm{Rep}_k(\mathrm{D}, \Psi \circ T)$ if finitely generated. By Lemma 2.2, $\mathcal{M}$ is a quotient of a direct sum of principal projectives. Proposition 2.13 tells us that each of these principal projectives is Noetherian, and Lemma 2.4 then tells us that the same is true of $\mathcal{M}$. □

2.4. Quasi-Gröbner pairs

Let $\Phi : \mathrm{D} \to \mathcal{C}$ be a functor. Following Sam and Snowden [22, Definition 3.2.1], we say that Φ has **property (F)** if, for any object c of $\mathcal{C}$, there exist finitely many objects $d_1, \dots, d_r$ of D along with morphisms $\varphi_i : c \to \Phi(d_i)$ such that, for any object d of D and any morphism $\psi : c \to \Phi(d)$, there exists an i and a morphism $\rho : d_i \to d$ such that $\psi = \Phi(\rho) \circ \varphi_i$. Given a functor $S : \mathcal{C} \to \mathrm{FI}$, we say that the pair $(\mathcal{C}, S)$ is **quasi-Gröbner** if there exists a Gröbner pair (D, T) and an essentially surjective functor $\Phi : \mathrm{D} \to \mathcal{C}$ with property (F) such that $S \circ \Phi$ is naturally isomorphic to $\Psi \circ T$.

Remark 2.15. The pair $(\mathcal{C}, \emptyset)$ is quasi-Gröbner if and only if the category $\mathcal{C}$ is quasi-Gröbner in the sense of [22].

Let $\Phi : \mathrm{D} \to \mathcal{C}$ and $S : \mathcal{C} \to \mathrm{FI}$ be any functors. For any commutative ring k, we have an exact functor $\Phi^* : \mathrm{Rep}_k(\mathcal{C}, S) \to \mathrm{Rep}_k(\mathrm{D}, S \circ \Phi)$ that takes a module $\mathcal{M} \in \mathrm{Rep}_k(\mathcal{C}, S)$ to

$$\Phi^*\mathcal{M} := \mathcal{M} \circ \Phi \in \mathrm{Rep}_k(\mathrm{D}, S \circ \Phi).$$

Proposition 2.16. *Let $\Phi : \mathrm{D} \to \mathcal{C}$ be a functor with property F, let $S : \mathcal{C} \to \mathrm{FI}$ be any functor, and let k be a commutative ring. If $\mathcal{M} \in \mathrm{Rep}_k(\mathcal{C}, S)$ is finitely generated, then $\Phi^*\mathcal{M} \in \mathrm{Rep}_k(\mathrm{D}, S \circ \Phi)$ is finitely generated.*

Proof. Since $\mathcal{M}$ is finitely generated, Lemma 2.2 tells us that $\mathcal{M}$ is a quotient of a direct sum of principal projectives. Since Φ^* is exact, $\Phi^*\mathcal{M}$ is a quotient of a direct sum of pullbacks of principal projectives. Thus, it is sufficient to show that, for any object c of $\mathcal{C}$, $\Phi^*\mathcal{P}_c$ is finitely generated. Choose finitely many objects $d_1, \ldots, d_r$ of D along with morphisms $\varphi_i : c \to \Phi(d_i)$ as in the definition of property (F). Consider the maps

$$\mathcal{P}_{d_i} \to \Phi^*\mathcal{P}_{\Phi(d_i)} \to \Phi^*\mathcal{P}_c,$$

where the first map is induced by Φ and the second is induced by φ_i. Property (F) says precisely that the direct sum map

$$\bigoplus_{i=1}^{r} \mathcal{P}_{d_i} \to \Phi^*\mathcal{P}_c$$

is surjective, which implies that $\Phi^*\mathcal{P}_c$ is finitely generated. □

Proof of Theorem 2.5. Let $(\mathcal{C}, S)$ be quasi-Gröbner pair. That means that there exists a Gröbner pair (D, T), an essentially surjective functor $\Phi : \mathrm{D} \to \mathcal{C}$ with property (F), and a natural isomorphism $\Psi \circ T \cong S \circ \Phi$. Fix a commutative ring k, a finitely generated module $\mathcal{M} \in \mathrm{Rep}_k(\mathcal{C}, S)$, and a submodule $\mathcal{N} \subset \mathcal{M}$. We need to prove that $\mathcal{N}$ is finitely generated, as well.

Proposition 2.16 tells us that $\Phi^*\mathcal{M} \in \mathrm{Rep}_k(\mathrm{D}, S \circ \Phi) \simeq \mathrm{Rep}_k(\mathrm{D}, \Psi \circ T)$ is finitely generated, and Corollary 2.14 then implies that $\Phi^*\mathcal{N} \subset \Phi^*\mathcal{M}$ is also finitely generated. Choose a generating set consisting of objects $d_1, \ldots, d_r$ of D and elements $v_i \in \Phi^*\mathcal{N}(d_i)$. This means that, for any object d of D, $\Phi^*\mathcal{N}(d)$ is spanned over $\mathcal{A}(d)$ by the images of the elements v_i along the maps induced by all possible morphisms $\varphi_i : d \to d_i$. This is equivalent to saying the $\mathcal{N}(\Phi(d))$ is spanned over $\mathcal{A}(\Phi(d))$ by the images of the elements $v_i \in \mathcal{N}(\Phi(d_i))$ along the maps induced by all morphisms $\Phi(\varphi_i) : \Phi(d) \to \Phi(d_i)$. Since Φ is essentially surjective, this means that $\mathcal{N}$ is finitely generated. □

3. Graphs

We define the category $\mathcal{G}_{\leq g}$ and use the results of Section 2 to prove Noetherianity results about the representation theory of this category.

3.1. Defining the graph categories

A **directed graph** is a quadruple (V, A, h, t), where V and A are finite sets (vertices and arrows), and h and t are each maps from A to V (head and tail). A **graph** is a quintuple (V, A, h, t, σ), where (V, A, h, t) is a directed graph and σ is a fixed-point-free involution of A with the property that $h = t \circ \sigma$. If (V, A, h, t, σ) is a graph, elements of the quotient A/σ are called **edges**. Given a directed graph $D = (V, A, h, t)$, we define the **underlying graph** $\bar{D} = (V, \bar{A}, h, t, \sigma)$, where $\bar{A} = A \times \{\pm 1\}$, $h(a, 1) = h(a) = t(a, -1)$, $t(a, 1) = t(a) = h(a, -1)$, and σ acts by toggling the second factor. We will usually suppress h and t from the notation and simply write (V, A, σ) for a graph.

Remark 3.1. This might seem to be an unnecessarily complicated definition of a graph. For example, one might try defining a graph to consist of a vertex set, and edge set, and a map from edges to unordered pairs of vertices. However, we want a graph with a loop to have a nontrivial automorphism that reverses the orientation of the loop. It is difficult to formalize this with the unordered pair definition.

If $G = (V, A, \sigma)$ is a graph and $v, v' \in V$, a **walk** in G from v to v' is a finite sequence $(a_1, \ldots, a_n)$ of arrows with $t(a_1) = v$, $h(a_n) = v'$, and $h(a_i) = t(a_{i+1})$ for all $1 \leq i < n$. A **path** in G from v to v' is a walk from v to v' of minimal length. We say that G is **connected** if there exists at least one path between any pair of vertices, and we say that G is a **forest** if there exists at most one path between any pair of vertices. A nonempty connected forest is called a **tree**.

Let $G = (V, A, \sigma)$ and $G' = (V', A', \sigma')$ be graphs. A **minor morphism** from G to G' is a map

$$\varphi : V \sqcup A \sqcup \{\star\} \to V' \sqcup A' \sqcup \{\star\}$$

satisfying the following properties:

- $\varphi(\star) = \star$.
- For every vertex $v \in V$, $\varphi(v) \in V'$.
- For every arrow $a \in A$, $\varphi \circ \sigma(a) = \sigma' \circ \varphi(a)$ where σ' acts trivially on $V' \sqcup \{\star\}$.
- For every arrow $a' \in A'$, there exists a unique arrow $a \in A$ with $\varphi(a) = a'$.
- If $\varphi(a) \in A$, then $\varphi \circ h(a) = h' \circ \varphi(a)$ and $\varphi \circ t(a) = t' \circ \varphi(a)$.
- If $\varphi(a) \in V$, then $\varphi \circ h(a) = \varphi(a) = \varphi \circ t(a)$.
- For every $v' \in V'$, $\varphi^{-1}(v')$ consists of the edges and vertices of a tree.

The edges that map to vertices are called **contracted edges** and the edges that map to $\star$ are called **deleted edges**. Note that the second condition and third conditions imply that the edges of G that are neither deleted nor contracted map bijectively to the edges of G'. In particular, φ induces an injection $\varphi^* : A'/\sigma' \to A/\sigma$.

Remark 3.2. If G and G' are connected graphs, then a minor morphism $\varphi : G \to G'$ is determined by its restriction to the set of arrows of G. (Connectedness is necessary because the graph with two vertices and no arrows has a nontrivial automorphism swapping the vertices.) However, it is convenient to define φ on the whole set $V \sqcup A \sqcup \{\star\}$ so that minor morphisms can be composed simply by composing functions. Note that φ is *not* determined by the map on edges φ^*. To see this, consider the example where G has two vertices and two parallel edges between them, and G' consists of a single vertex with no edges. There are two minor morphisms from G to G', corresponding to the choice of which edge is deleted and which edge is contracted.

Let $\mathcal{G}$ denote the category whose objects are nonempty connected graphs and whose morphisms are minor morphisms.

Conjecture 3.3. *The category $\mathcal{G}^{\mathrm{op}}$ is quasi-Gröbner.*

Remark 3.4. The Gröbner cover of $\mathcal{G}^{\mathrm{op}}$ should be the category $\mathcal{OD}^{\mathrm{op}}$ of directed graphs whose arrows are ordered with opposite minor morphisms that preserve the order of arrows. That is if $\varphi : (V, A, h, t) \to (V', A', h', t')$ is a minor morphism of directed graphs whose arrows are ordered, then we also require that whenever $a'_1 \leq a'_2$ in A', then $\varphi^*(a'_1) \leq \varphi^*(a'_2)$ in A. However, to prove that $\mathcal{OD}^{\mathrm{op}}$ satisfies property (G2) one would need a stronger version of Robertson and Seymour's labeled graph minor theorem [20, p. 1.7] where the order of the labels on edges is preserved in the above sense. If Conjecture 3.3 is proven, then all of the results in Section 4 can be upgraded from statements about graphs with genus at most g to statements about all graphs with no restriction on genus.

In light of Remark 3.4, we restrict our attention to a family of full subcategories of $\mathcal{G}^{\mathrm{op}}$. The **combinatorial genus** of a graph $G = (V, A, \sigma)$ is $|A/\sigma| - |V| + 1$. For any $g \geq 0$, let $\mathcal{G}_{\leq g}$ be the full subcategory of $\mathcal{G}$ whose objects are non-empty connected graphs of genus at most g.

Theorem 3.5. *For any positive integer g, $\mathcal{G}^{\mathrm{op}}_{\leq g}$ is quasi-Gröbner.*

The proof of Theorem 3.5 relies on the work in [17] which we summarize here.

A **rooted tree** is a pair (T, v) where T is a tree and v is a vertex of T called the **root**. There is a natural partial order on the vertex set of rooted tree where $w \leq u$ if and only if the unique path from w to the root passes through u. A **direct descendant** of a vertex w is a vertex covered by w with respect to this partial order. A **planar rooted tree** is a rooted tree equipped with a linear order on the set of direct descendants of each vertex. Note that this gives a depth first linear order on the set of vertices.

Let L be a finite set. An **L-labeled planar rooted tree** is a triple (T, v, ℓ) where (T, v) is a rooted tree and ℓ is a function from the set of vertices of T to L.

In a nonempty connected graph G, a **spanning tree** for G is a subgraph that is a tree and contains all vertices of G. Now, for each genus g, fix once and for all a graph R_g with one vertex and g loops. Define a **rigidified graph** of genus g to be a quadruple (G, T, v, τ) where G is a graph of genus g, (T, v) is a planar rooted spanning tree of G, and $\tau : G \to R_g$ is a minor morphism where the contracted edges are exactly the edges of T. Note that rigidified graphs come equipped with a linear order on the vertices coming from their planar rooted spanning trees. A **morphism of rigidified graphs** $(G, T, v, \tau) \to (G', T', v', \tau')$ between rigidified graphs of genus g is a minor morphism $\varphi : G \to G'$ that restricts to a minor morphism $T \to T'$ such that $\varphi(v) = v'$, and if $w' \leq u'$ under linear order on vertices of G', then the smallest vertex in the preimage of w' comes before the smallest vertex in the preimage on u' under the linear order on the vertices of G. Note that a minor morphism between graphs of the same genus necessarily has no deleted edges.

Let $\mathcal{RG}_g$ be the category whose objects are rigidified graphs of genus g and whose morphisms are minor morphisms of rigidified graphs of genus g. Let $\mathcal{RG}^{\mathrm{op}}_{\leq g}$ denote the category whose objects are rigidified graphs of genus at most g and whose morphisms are minor morphisms of rigidified graphs between graphs of the same genus. One can think of $\mathcal{RG}^{\mathrm{op}}_{\leq g}$ as a kind of disjoint union of the finitely many categories $\mathcal{RG}^{\mathrm{op}}_h$ for $h \leq g$.

Theorem 3.6 *([17]). For any $g \geq 0$, the category $\mathcal{RG}^{\mathrm{op}}_g$ is Gröbner.*

Corollary 3.7. *For any $g \geq 0$, the category $\mathcal{RG}^{\mathrm{op}}_{\leq g}$ is Gröbner.*

Proof. It is clear from the definitions that $\mathcal{RG}^{\mathrm{op}}_{\leq g}$ satisfies properties (G1) and (G2) since $\mathcal{RG}^{\mathrm{op}}_h$ satisfies both properties for each of the finitely many $h \leq g$. □

Note that there is a forgetful functor

$$\Phi_{\leq g} : \mathcal{RG}^{\mathrm{op}}_{\leq g} \to \mathcal{G}^{\mathrm{op}}_{\leq g}$$

which sends a rigidified graph to its underlying graph and a minor morphism of rigidified graphs to the underlying minor morphism of graphs.

Lemma 3.8. *The functor $\Phi_{\leq g}$ satisfies property (F).*

Proof. This proof is similar to the proof of [17, Lemma 3.11]. The fact that $\Phi_{\leq g}$ is essentially surjective is clear as every graph can be given a rigidified graph structure.

Let G be a graph of genus $h \leq g$ with exactly n edges. To show $\Phi_{\leq g}$ has property (F), we need to produce a finite collection (G_i, T_i, v_i, τ_i) of Rigidified graphs of genus at most g along with minor morphisms $\varphi_i : G_i \to G$ such that, given any rigidified graph (G', T', v', τ') and minor morphism $\varphi : G' \to G$, there exists an index i and a morphism of rigidified graphs $\rho : (G', T', v', \tau') \to (G_i, T_i, v_i, \tau_i)$ with $\varphi = \varphi_i \circ \Phi_{\leq g}(\rho)$.

Consider all possible rigidified graphs with at most $n + 2g - h$ edges and genus at least h and at most g. For each isomorphism class of such rigidified graph, choose a representative (G'', T'', v'', τ'') and add it to our collection (G_i, T_i, v_i, τ_i) as many times as there are minor morphisms $G'' \to G$. Then, take our collection of minor morphisms φ_i so that for each i, every possible minor morphism $G_i \to G$ appears in our collection. Note, there are only finitely many isomorphism classes of rigidified graphs with at most $n + 2k - h$ in $\mathcal{RG}_h^{\mathrm{op}}$ and for such a rigidified graph there are only finitely many minor morphisms from the underlying graph to G. Thus our collection of rigidified graphs and minor morphisms is indeed finite.

Now, fix a rigidified graph (G', T', v', τ') with genus k where $h \leq k \leq g$ and a minor morphism $\varphi : G' \to G$. Let E' be the set of edges that are contracted under φ. In a morphism of rigidified graphs, we are only allowed to contract edges in the designated spanning tree. Thus, let ρ be the morphism of rigidified graphs $\rho : (G', T', v', \tau') \to (G'/(E' \cap T'), T'/(E' \cap T'), v', \tau')$ corresponding to contracting the edges of $E' \cap T'$. Clearly φ factors through $\Phi_{\leq g}(\rho)$. All that we must verify is that $(G'/(E' \cap T'), T'/(E' \cap T'), v', \tau')$ is a member of our collection. In otherwords, we must show that $G'/(E' \cap T')$ has at most $n + 2g - h$ edges. Let n' be the number of edges of G'. Since deleting an edge lowers the genus of the graph by 1, we know that under φ, exactly $k - h$ edges are deleted. Thus, $n' = n + k - h + |E'|$ or equivalently,

$$|E'| = n' - (n + k - h).$$

Furthermore, we know that the number of edges in T' is equal to $n' - k$. Thus,

$$|E' \cap T'| \geq n' - (n + k - h) - k = n' - n - 2k + h.$$

This implies that the number of edges in $G'/(E' \cap T')$ is at most

$$n' - (n' - n - 2k + h) = n + 2k + h$$

which is no more than $n + 2g + h$ since $k \leq g$. □

3.2. Edge functors

Recall that given a minor morphism of graphs $\varphi : (V, A, \sigma) \to (V', A', \sigma')$, we get an inclusion of edge sets $\varphi^* : A'/\sigma' \to A/\sigma$. Thus, we have an **edge functor**

$$E : \mathcal{G}_{\leq g}^{\mathrm{op}} \to \mathrm{FI}$$

taking a graph G to its set of edges. Similarly, the edges of a rigidified graph are totally ordered and given minor morphism of rigidified graphs, the pullback inclusion on edge sets preserves this order. This gives an ordered edge functor

$$\Omega : \mathcal{RG}^{\mathrm{op}}_{\leq g} \to \mathrm{OI}$$

taking a rigidified graph to its ordered set of edges. Recall from Section 2.2 that $\Psi : \mathrm{OI} \to \mathrm{FI}$ is the forgetful functor that sends an ordered set to its underlying (unordered) set.

Lemma 3.9. *$\Psi \circ \Omega$ and $E \circ \Phi_{\leq g}$ are naturally isomorphic functors $\mathcal{RG}^{\mathrm{op}}_{\leq g} \to \mathrm{FI}$.*

Proof. This is clear from the definitions of the functors. □

Theorem 3.10. *The pair $(\mathcal{RG}^{\mathrm{op}}_{\leq g}, \Omega)$ is Gröbner.*

To prove Theorem 3.10 we will need a labeled version of a planar rooted tree. This is a slightly more general version of the notion of an S-labeled planar rooted tree in [17]. Let L be a set equipped with a well quasi order $\leq$. An L-labeled planar rooted tree is a triple (T, v, ℓ) where (T, v) is a planar rooted tree and ℓ is a function from the set of vertices of T to L. An **L-labeled minor morphism of planar rooted trees** or **L-labeled minor morphism** $(T, v, \ell) \to (T', v', \ell')$ is a minor morphism $\varphi : (T, v) \to (T', v')$ of rigidified graphs of genus 0 (note planar rooted trees are exactly the rigidified graphs of genus 0) such that $\ell'(w') \leq \ell'(\max \varphi^{-1}(w'))$ where $\max \varphi^{-1}(w')$ is the first vertex in the preimage of w' under φ with respect to the natural depth first order on the vertices of T. Let $\mathcal{PT}_L$ denote the category whose objects are L-labeled planar rooted trees and whose morphisms are L-labeled minor morphisms. For a fixed L-labeled planar rooted tree (T, v, ℓ), we may give a quasi ordered $\leq$ to the L-labeled minor morphisms into (T, v, ℓ). Namely, if $\varphi' : (T', v', \ell') \to (T, v, \ell)$ and $\varphi'' : (T'', v'', \ell'') \to (T, v, \ell)$ are L-labeled minor morphisms, then $\varphi \leq \varphi''$ if and only if there exists an L-labeled minor morphism $\varphi : (T'', v'', \ell'') \to (T', v', \ell')$ such that $\varphi'' = \varphi' \circ \psi$. Let $|(\mathcal{PT}^{\mathrm{op}}_L)_{(T,v,\ell)}|$ denote the poset of equivalence classes of L-labeled minor morphisms into (T, v, ℓ) under this quasi order.

Lemma 3.11. *Let L be a well quasi ordered set. We partially order the isomorphism classes of L-labeled planar rooted trees where $[(T', v', \ell')] \leq [(T, v, \ell)]$ if and only if there is a minor morphism of L-labeled planar rooted trees $(T, v, \ell) \to (T', v', \ell')$.*

Corollary 3.12. *Let L be a well quasi ordered set. Fix an L-labeled planar rooted tree (T, v, ℓ). The poset $|(\mathcal{PT}^{\mathrm{op}}_L)_{(T,v,\ell)}|$ is a well partial order.*

Remark 3.13. Note that the notion of L-labeled planar rooted tree and L-labeled minor morphism are slight generalizations of the notions of an S-labeled planar rooted tree and an S-labeled contraction defined in [17] where the set of labels was an (unordered) finite set. Nevertheless, Lemma 3.11 and Corollary 3.12 are analogues of and have nearly identical proofs to [17, Theorem 3.6] and [17, Corollary 3.7] respectively. Thus, we omit their proofs here.

Recall from Section 2.2 that a quartet $\mu = (R, R', \varphi, m)$ for the pair $(\bigsqcup_{h\leq g} \mathcal{RG}_h^{\mathrm{op}}, OE)$ consists of two rigidified graphs of genus at most g, $R = (G, T, v, \tau)$, $R' = (G', T', v', \tau')$, a minor morphism of rigidified graphs $\varphi : R' \to R$, and a map of sets m from the edges of R' to $\mathbb{N}$. We should think of m as assigning to each edge of R' a natural number. For any fixed rigidified graph R, We also have a quasi order on quartets whose first coordinate is R where $(R, R', \varphi', m') \leq (R, R'', \varphi'', m'')$ if there exists a minor morphism of rigidified graphs $\psi : R'' \to R'$ such that $\varphi'' = \varphi' \circ \psi$ and if e' is an edge in R', $m'(e') \leq m''(\varphi^*(e'))$ where φ^* is the natural inclusion of the edges of R' to the edges of R''. Denote the corresponding poset of equivalence classes under this quasi order by $|(\mathcal{RG}_{\leq g}^{\mathrm{op}})_R^{OE}|$.

Fix a rigidified graph $R = (G, T, v, \tau)$ with genus $h \leq g$. Now for a suitable choice of L, we wish to encode each quartet with first coordinate R as an L-labeled planar rooted tree so that the poset $|(\mathcal{RG}_{\leq g}^{\mathrm{op}})_R^{OE}|$ is equivalent to the poset $|(\mathcal{PT}_L^{\mathrm{op}})_{(T,v,\ell)}|$ where (T, v) is the planar rooted tree of R and ℓ is some suitable label. To this end, let $L = (\mathbb{N} \cup \{\star\})^{2h} \times \mathbb{N}$. The order on $(\mathbb{N} \cup \star)$ comes from the usual order on $\mathbb{N}$ along with setting $\star$ to be incomparable to all elements of $\mathbb{N}$. Then, the order on L is the usual order on the cartesian product of posets.

Given a quartet of the form (R, R', φ, m), where $R' = (G', T', v', \tau')$ the corresponding L-labeled planar rooted tree will be of the form (T', v', ℓ') for some labeling ℓ'. Note that R' must also have genus h since we only have morphisms of rigidified graphs between rigidified graphs of the same genus. Thus, R' has h extra edges not in T'. The planar rooted structure of (T', v') gives an orientation and ordering to these h extra edges (we orient them from smaller to larger vertex and then order them by the order on their terminating vertex). Call these extra edges $e'_1, \ldots, e'_h$. For each $1 \leq i \leq h$, let w'_{2i-1} be the vertex at which e_i originates and let w'_{2i} be the vertex at which e_i terminates. Then for each vertex w' of T', and each $1 \leq j \leq 2h$, define the jth coordinate of $\ell'(w')$ to be $m(e'_j)$ if $w \geq w_j$ and $\star$ otherwise. Then for each edge e' in T', if w' is the vertex of e' further from the root, set the last coordinate of w' to $m(e')$. Finally set the last coordinate of $\ell'(v')$ to 0. The intuition behind this labeling is that the first $2h$ coordinates encode the location and weights given by m for the h edges not in T'. The last coordinate encodes the weights given by m of the edges in T'. Finally, let ℓ be the fixed labeling of (T, v) corresponding to the quartet $(R, R, \mathrm{Id}_R, 0)$.

Lemma 3.14. *Let (R, R', φ, m) be a quartet and (T', v', ℓ') be the corresponding L-labeled planar rooted tree as defined above. Then φ induces an L-labeled morphism $\varphi_L : (T', v', \ell') \to (T, v, \ell)$.*

Proof. One simply uses a nearly identical argument to the one given in the proof of Lemma [17, Lemma 3.8]. □

Lemma 3.15. *Let $\mu' = (R, R', \varphi', m')$ and $\mu'' = (R, R'', \varphi'', m'')$ be quartets with corresponding L-labeled planar rooted trees (T', v', ℓ') and (T'', v'', ℓ'') and define φ'_L and φ''_L as in Lemma 3.14. Then $\mu' \leq \mu''$ if and only if $\varphi'_L \leq \varphi''_L$ in $|(\mathcal{PT}_L^{\mathrm{op}})_{(T,v,\ell)}|$*

Proof. Note that $\mu' \leq \mu''$ means there exists a minor morphism of rigidified graphs $\psi : R'' \to R'$ such that $\varphi'' = \varphi' \circ \psi$ and for any edge e' of R', $m'(e') \leq m''(\psi^*(e'))$. Using a nearly identical argument to that given in the proof of [17, Lemma 3.8] we see that such a minor morphism ψ is equivalent to an L-labeled minor morphism of L-labeled planar rooted trees $\psi_L : (T'', v'', \ell'') \to (T', v', \ell')$ such that $\varphi''_L = \varphi'_L \circ \psi_L$. This is equivalent to saying $\varphi'_L \leq \varphi''_L$. □

proof of Theorem 3.10. That the pair $(\mathcal{RG}^{\mathrm{op}}_{\leq g}, \Omega)$ satisfies property (G1) follows from Corollary 3.7 and Proposition 2.10. The fact that $(\mathcal{RG}^{\mathrm{op}}_{\leq g}, OE)$ satisfies property (G2) follows from Lemma 3.15 and Corollary 3.12. □

Theorem 3.16. *The pair $(\mathcal{G}^{\mathrm{op}}_{\leq g}, E)$ is quasi Gröbner*

Proof. This follows from Lemmas 3.8 and 3.9. □

3.3. Finite generation

Theorems 3.5 and 3.16 tell us that the categories of representations $\mathrm{Rep}_k(\mathcal{G}^{\mathrm{op}}_{\leq g})$ and $\mathrm{Rep}_k(\mathcal{G}^{\mathrm{op}}_{\leq g}, E)$ are locally Noetherian. That is, every submodule of any finitely generated module of either of these categories, is itself finitely generated.

The properties of finitely generated $\mathcal{G}^{\mathrm{op}}$-modules fall into two broad categories: global and local. Global properties are those which universally bound or otherwise restrict algebraic behaviors present in the constituent modules $M(G)$. Local properties, on the other hand, are those that can be observed when one limits their attention to the modules $M(G)$, where G ranges within certain natural families of graphs. We note that the study of the category $\mathcal{G}^{\mathrm{op}}$ and its representations is still fairly new, and thus the following list should by no means seen as exhaustive. We believe that future study into understanding the inner mechanisms of finitely generated $\mathcal{G}^{\mathrm{op}}$-modules is a very interesting direction for future research.

To begin, we have the following, which is essentially just a reformulation of the definition of finite generation.

Theorem 3.17. *Let M be a finitely generated $\mathcal{G}^{\mathrm{op}}_{\leq g}$ Then there exists a non-negative integer N such that for all graphs G one has*

$$M(G) = \mathrm{colim}_{G' < G} M(G'),$$

where the colimit ranges over all proper minors G' of G that have no more than N edges.

One may view this theorem as stating that given a finitely generated $\mathcal{G}^{\mathrm{op}}_{\leq g}$-module M, the presentation of $M(G)$ becomes standardized once G has sufficiently many edges. Our next result relates with how fast $\mathcal{G}^{\mathrm{op}}_{\leq g}$-modules can grow. For the following statement, we write $e(G)$ for the number of edges of G, and $v(G)$ for the number of vertices.

Theorem 3.18. *Let M be a finitely generated $\mathcal{G}_{\leq g}^{\mathrm{op}}$-module over a field k, and assume that the generators of M have no more than N edges, for some non-negative integers N. Then there exists a polynomial $P \in \mathbb{Q}[x, y]$ of degree at most N such that,*

$$\dim_k M(G) \leq \tau(G) P(e(G), v(G)),$$

where $\tau(G)$ is the number of spanning trees of G.

Proof. For simplicity during this proof, write $g(G) = e(G) - v(G) + 1$. It will suffice to prove the theorem in the case where M is the principal projective module $P_{G'}$, for some fixed graph G'. In this case, for any graph G, the dimension of $M(G)$ is precisely the number of minor morphisms from G to G'. Any such morphism, up to an automorphism of G', can be determined in the following way: First one picks a spanning tree of G in which all of the contractions will take place. One then chooses $g(G')$ edges not to delete outside of this spanning tree, and $e(G') - g(G')$ edges to not contract within the spanning tree. Therefore,

$$\dim_k M(G) \leq |\operatorname{Aut}(G')| \tau(G) \binom{g(G)}{g(G')} \binom{v(G) - 1}{e(G') - g(G')}.$$

This then implies,

$$\dim_k M(G) \leq |\operatorname{Aut}(G')| \tau(G) g(G)^{g(G')} (v(G) - 1)^{e(G') - g(G')}$$

as desired. □

Remark 3.19. The bound of Theorem 3.18 is an improvement of a bound found in [17], and can be seen to be sharp. Consider the principal projective module $P_\bullet$, over the graph with no edges. In this case, a minor morphism from a graph G to the graph $\bullet$ is precisely determined by a choice of spanning tree for G. In particular,

$$\dim_k P_\bullet(G) = \tau(G)$$

The example of the dimension growth of $P_\bullet$ is also notable, as it illustrates just how complicated computing the dimensions in finitely generated $\mathcal{G}_{\leq g}^{\mathrm{op}}$-modules can be. In other words, while one can in principal compute $\dim_k(P_\bullet(G))$ for any graph G using, for instance, the Matrix Tree Theorem, this is fairly non-trivial counting. Moreover, $P_\bullet$ is the simplest of the principal projective modules, and this is not even to mention the kinds of behaviors present in the dimensions of the submodules of principal projective modules. It is for this reason, as we shall see, that it is often times more mathematically fruitful to consider what we call the local properties of $\mathcal{G}_{\leq g}^{\mathrm{op}}$-modules.

Our final global property of finitely generated $\mathcal{G}_{\leq g}^{\mathrm{op}}$-modules relates with the kinds of torsion that can appear in the modules $M(G)$.

Theorem 3.20. *Let M be a finitely generated $\mathcal{G}_{\leq g}^{\mathrm{op}}$-module over $\mathbb{Z}$. Then there exists a non-negative integer ϵ such that for all graphs G of genus at most g, the torsion part of $M(G)$ has exponent at most ϵ.*

Proof. If $t \in M(G)$ is a torsion element, then for any minor morphism $\varphi : G' \to G$, the induced map $M(G) \to M(G')$ must send t to a torsion element. Thus, we have a $\mathcal{G}_{\leq g}^{\mathrm{op}}$-submodule T of M, where T sends a graph G to the torsion part of $M(G)$. By assumption $M(G)$ is finitely generated, so Theorem 1.1 tells us that T is also finitely generated, by say the graphs $G_1, \ldots, G_n$. We may take ϵ to be the least common multiple of the annihilators of $T(G_1), \ldots, T(G_n)$. □

Moving on, we next consider two local consequences of finite generation. Our first such result describes the growth of the modules associated to families of graphs obtained through sprouting and subdividing.

Theorem 3.21 *(Corollaries 4.5 and 4.7, [17]). Let M be a finitely generated $\mathcal{G}_{\leq g}^{\mathrm{op}}$-module over a field k, and let G be a fixed graph, with a distinguished collection of vertices $v_1, \ldots, v_r$, and edges $e_1, \ldots, e_s$. We write $G^{(n_1,\ldots,n_r)}$ for the graph obtained from G by attaching n_i leaves to the vertex v_i, and $G_{(m_1,\ldots,m_s)}$ for the graph obtained from G by subdividing the edge e_j, m_j times. Then there exist polynomials $P_1(n_1, \ldots, n_r)$ and $P_2(m_1, \ldots, m_s)$ such that*

$$\dim_k M(G^{(n_1,\ldots,n_r)}) = P_1(n_1, \ldots, n_r)$$
$$\dim_k M(G_{(m_1,\ldots,m_s)}) = P_2(m_1, \ldots, m_s),$$

for all vectors $(n_1, \ldots, n_r)$ and $(m_1, \ldots, m_s)$ whose each component is sufficiently large.

Through subdivision one can, for instance, study $M(G)$ as G ranges within the cycle graphs. This approach was particularly useful in the case of Kazhdan-Lusztig polynomials of graphical matroids (see [17]). Later, we will relate a space considered by Farb Wolfson and Wood to sprouting on the two vertices of a single edge.

Our final local property relates with restriction to the family of trees. Recall that a **Dyck-path** is a properly nested set of parentheses. Equivalently, it is a word in the symbols 1 and -1 such that all partial sums beginning from the start of the word are never negative. To each Dyck-path w, one may associate a (rooted and planar) tree as follows: reading w from left to right, draw a new edge going upward each time a left parenthesis is read, while you backtrack down the nearest edge whenever a right parenthesis is read. For instance, the Dyck-path (()()) is associated to the tree that looks like the letter Y. Importantly, this association is not one-to-one – For instance, ()()() is also associated to the graph that looks like the letter Y – though every tree

arises in this way.[1] We write $T(w)$ for the tree associated to the Dyck-path w. We also write $r(w)$ for the number of left parenthesis in the Dyck-path w – i.e. half of its length.

Theorem 3.22 *(Theorem 1, [18]). Let M be a finitely generated $\mathcal{G}_0^{\mathrm{op}}$-module over a field k. Then the generating function*

$$HD_M(t) := \sum_w \dim_k(M(T(w)))t^{r(w)},$$

where the right hand sum is over all Dyck-paths, is algebraic.

If M is the module that assigns the field k to every graph, then the above theorem specializes to say that the generating function for the number of Dyck-paths is algebraic. In other words, the generating function of the Catalan numbers is algebraic. This is a very well known fact about these numbers.

4. Applications to topological combinatorics

Throughout this section, let $g \geq 0$ be a fixed integer. In this section, we present a number of applications of the representation theory of $\mathcal{G}_{\leq g}^{\mathrm{op}}$ to topological combinatorics.

4.1. The matching complex

Now, we introduce matching complexes of graphs and show that for any i, the map assigning to each graph G with genus at most g the ith homology group of its matching complex forms a finitely generated $\mathcal{G}_{\leq g}^{\mathrm{op}}$-module.

For a graph G, a **matching on** G is a subset $S \subset E(G)$ of non-loop edges such that no two edges in S share a vertex. Note that any subset of a matching is also a matching and the empty set, $\varnothing$, is a matching. Thus, the collection of matchings on G forms a simplicial complex $\mathcal{M}(G)$ whose vertices are the edges of G, which we call the **matching complex on** G.

The topology of matching complexes on the complete graphs K_n and the complete bipartite graphs $K_{m,n}$ have been well studied. Much of what is known about the topology of these complexes is outlined in [25] and [7]. We outline a few notable results on these complexes below.

It was shown by Björner, Lovász, Vrećica, and Živaljević in [1] that, for $n \geq 2$, $\mathcal{M}(K_n)$ is $\nu_n - 1$ connected and, for $1 \leq m \leq n$, $\mathcal{M}(K_{m,n})$ is $\nu_{n,m} - 1$ connected where

$$\nu_n = \left\lfloor \frac{n+1}{3} \right\rfloor - 1 \text{ and } \nu_{m,n} = \min\left\{m, \left\lfloor \frac{m+n+1}{3} \right\rfloor\right\} - 1.$$

[1] Note however that the association is one-to-one if we instead consider rooted trees with a cyclic order on the edges moving away from the root for every vertex.

Later, this was result was strengthened by Shareshian and Wachs, who showed that the ν_n-skeleton of $\mathcal{M}(K_n)$ is shellable [24] and Ziegler, who showed that the $\nu_{n,m}$-skeleton of $\mathcal{M}(K_{m,n})$ is shellable.

Remark 4.1. In view of our main Theorem 4.3, the above results may seem a bit concerning. Indeed, finite generation would be meaningless if it was known that the module was eventually — i.e. perhaps for all graphs with sufficiently many edges — constantly zero! To see that this is not the case, consider the tree G_n, which has two vertices of degree $n+1$, connected to each other by a single edge. In this case the matching complex of G_n is seen to be one dimensional. Indeed, it is precisely the disjoint union of the complete bipartite graph $K_{n,n}$ and a single point. This particular topological space only has non-trivial homology in degrees 0 and 1, where it is isomorphic to a free group of ranks 2 and $n^2 - 2n + 1$, respectively. Similar examples can be constructed to show that, in any homological degree, the homology of the matching complex is not eventually constantly zero.

The rational homology of these complexes is known due to the work of Bouc [2] and Friedman and Hanlon [5]. However, much is still unknown regarding the torsion in the integral homology of these complexes. Shareshian and Wachs show that the homology of $\mathcal{M}(K_n)$ exhibits 3-torsion for sufficiently large n and the homology of $\mathcal{M}(K_{m,n})$ also contains 3-torsion for certain (but infinitely many) values of m and n [24]. Later, Jonsson showed that 5-torsion in present in the homology of $\mathcal{M}(K_n)$ for sufficiently large n and found that there are elements of order 5, order 7, order 11, and order 13 appearing in the homology of $\mathcal{M}(K_n)$ for varying values of n [8]. For graphs other than K_n and $K_{m,n}$, not much is known about the topology of their matching complex.

There is also a natural generalization of the matching complex. Note, that for a graph G, and any subset $F \subset E(G)$, we have the induced the subgraph G_F of G. Namely, G_F is the graph with $V(G_F) = V(G)$ and $E(G_F) = F$. Thus, F is a matching on G if and only if each vertex of G_F has degree at most 1. More generally, for any integer $d \geq 1$ we can consider subsets $F \subset E(G)$ such that each vertex of G_F has degree at most d. Call such a subset a **d-matching of G**. Note that the collection of all d-matchings on G forms a simplicial complex which we will denote $\mathcal{M}_d(G)$. In particular, $M_1(G) = \mathcal{M}(G)$, the matching complex.

In the case where G is a forest, Singh showed that $\mathcal{M}_d(G)$ is either contractible or homotopy equivalent to a wedge of spheres [21]. For $d \geq 2$, Jonsson showed that if n is sufficiently large, then there exist certain values of d depending on n, where the homology of $\mathcal{M}_d(K_n)$ contains 3-torsion [10].

In this paper, we approach the problem of studying the homology of $\mathcal{M}_d(G)$ for general graphs G in a completely new way, by realizing the map sending a graph G to the homology of $\mathcal{M}_d(G)$ as a finitely generated $\mathcal{G}^{\mathrm{op}}_{\leq g}$-module.

To do this, we will need a particular $\mathcal{G}^{\mathrm{op}}_{\leq g}$-module. Let M_E be the $\mathcal{G}^{\mathrm{op}}_{\leq g}$-module which assigns to each graph G the free R-module with basis indexed by the edges of G. For

each minor morphism $\varphi : G \to G'$, the natural inclusion $\varphi^* : E(G') \to E(G)$ gives us a natural inclusion $M_E(\varphi) : M_E(G') \to M_E(G)$. We will call M_E, the **edge module**. This $\mathcal{G}_{\leq g}^{\mathrm{op}}$-module satisfies the following very important property.

Lemma 4.2. *For any i, the ith tensor power of the edge module, $M_E^{\otimes i}$, is a finitely generated $\mathcal{G}_{\leq g}^{\mathrm{op}}$-module.*

Proof. For any graph G, $M_E^{\otimes i}(G)$ has basis given by i-tuples of edges of G. Let G be a graph with strictly more than i edges. Then for any tuple $(e_1, \ldots, e_i)$ of edges of G, one may find an edge e of G that is not among the e_j. By contracting e, or deleting it in the case where e is a loop, one obtains a minor morphism $\varphi : G \to G'$. It is clear that the tuple $(e_1, \ldots, e_i)$ will be in the image of the map induced by φ. In other words, $M_E^{\otimes i}$ is generated by graphs with at most i edges, of which there are only finitely many up to isomorphism in $\mathcal{G}_{\leq g}^{\mathrm{op}}$. □

Theorem 4.3. *For any $i \geq 0$ and any $d \geq 1$, $H_i(\mathcal{M}_d(-); \mathbb{Z})$ is a finitely generated $\mathcal{G}_{\leq g}^{\mathrm{op}}$-module.*

Proof. For this proof, we are working with $\mathbb{Z}$-coefficients, though we will suppress this from the notation.

First, we argue that $H_i(\mathcal{M}_d(-))$ is in fact a $\mathcal{G}_{\leq g}^{\mathrm{op}}$-module. Recall that given a minor morphism $\varphi : G \to G'$, we have an induced inclusion on edge sets $\varphi^* : E(G') \to E(G)$ sending an edge in G' to the unique edge in its preimage under φ. Now, let $F' \subset E(G')$. If some vertex $v \in V(G)$ were incident to more than d edges in $\varphi^*(F')$, then $\varphi(v) \in V(G')$ would be incident to more than d edges in F'. Thus, if F' is a d-matching on G', then $\varphi^*(F')$ is a d-matching on G. We may therefore, consider φ^* as a function

$$\varphi^* : \{d\text{-matchings on } G'\} \to \{d\text{-matchings on } G\}.$$

Furthermore, this map is compatible with taking boundaries of simplices and so we get a chain map

$$\varphi_\bullet : C_\bullet(\mathcal{M}_d(G')) \to C_\bullet(\mathcal{M}_d(G))$$

where $C_\bullet(\mathcal{M}_d(G))$ denotes the simplicial chain complex of $\mathcal{M}_d(G)$. This induces a map $H_i(\mathcal{M}_d(G')) \to H_i(\mathcal{M}_d(G))$. The above construction is compatible with taking compositions of minor morphisms and so indeed $H_i(\mathcal{M}_d(-))$ is a $\mathcal{G}_{\leq g}^{\mathrm{op}}$-module.

To see that it is finitely generated, note that following the above construction, for any i, $C_i(\mathcal{M}_d(-))$ also forms a $\mathcal{G}_{\leq g}^{\mathrm{op}}$-module and $H_i(M_d(-))$ is a subquotient of $C_i(M_d(-))$. Because $C_i(\mathcal{M}_d(-))$ is a submodule of $\bigwedge^i M_E$, the ith wedge power of the edge module, and $\bigwedge^i M_E$ is a quotient of $M_E^{\otimes i}$, we see that $H_i(M_d(-))$ is a subquotient of $M_E^{\otimes i}$. By Lemma 4.2, $M_E^{\otimes i}$ is a finitely generated $\mathcal{G}_{\leq g}^{\mathrm{op}}$-module and so Theorem 1.1 tells us $H_i(\mathcal{M}_d(-))$ is itself finitely generated. □

This theorem leads to an immediate corollary regarding the torsion that can appear in the homology which says that for fixed i and d, there is some uniform maximum torsion that can appear in $H_i(\mathcal{M}_d(G))$ as G ranges over all graphs of genus at most g.

Corollary 4.4. *For any $i \geq 0$ and any $d \geq 1$, there exists a positive integer $\epsilon_{i,d,g}$ such that, for any graph G with genus at most g, the torsion part of $H_i(\mathcal{M}_d(-))$ is annihilated by $\epsilon_{i,d}$.*

Proof. This follows immediately from the previous theorem as well as Theorem 3.20. □

4.2. Other graph complexes

The matching complex of a graph is just one example of a simplicial complex on the edges of a graph. More generally, one can consider what is called a **monotone graph property**. This is a collection $\mathcal{P}$ of graphs, closed under isomorphisms, such that if $G \in \mathcal{P}$ and G' is another graph with $V(G') = V(G)$ and $E(G') \subset E(G)$, then $G' \in \mathcal{P}$. In other words, $\mathcal{P}$ is closed under edge deletions. Note that, for any graph G, we obtain a simplicial complex $\Delta_{\mathcal{P}}(G)$ on the edges of G where the n-simplices of $\Delta_{\mathcal{P}}(G)$ correspond to graphs $G' \in \mathcal{P}$ with $V(G') = V(G)$, $E(G') \subset E(G)$, and $|E(G')| = n + 1$. Intuitively if we identify a graph in $\mathcal{P}$ with its set of edges, the n-simplices are just sets of $n+1$ edges of G in $\mathcal{P}$. See [7] for a comprehensive reference on graph complexes. In particular [7, Table 7.1] gives a list of monotone graph properties and what is known about the homotopy type of their corresponding simplicial complexes.

Recall that a minor morphism $\varphi : G \to G'$ induces an inclusion $\varphi^* : E(G') \to E(G)$. Suppose that $\mathcal{P}$ is a monotone graph property with the extra condition that, if $\varphi : G \to G'$ is a minor morphism and H' is a simplex in $\Delta_{\mathcal{P}}(G')$ (that is, $H' \in \mathcal{P}$, $V(H') = V(G')$, and $E(H') \subset E(G')$), then the subgraph of G induced by the image $\varphi^*(E(H'))$ is also in $\mathcal{P}$. Call such a $\mathcal{P}$ a **$\mathcal{G}^{\mathrm{op}}_{\leq g}$-monotone graph property**. In rough terms, a $\mathcal{G}^{\mathrm{op}}_{\leq g}$-monotone graph property is a monotone graph property that is also preserved under "uncontracting" edges. Note that by construction, if $\mathcal{P}$ is a $\mathcal{G}^{\mathrm{op}}_{\leq g}$-monotone graph property, then for any minor morphism $\varphi : G \to G'$, the n-simplices of $\Delta_{\mathcal{P}}(G')$ naturally include in the n-simplices of $\Delta_{\mathcal{P}}(G)$. This observation and the argument used in the proof of Theorem 4.3 yields the following result.

Proposition 4.5. *Let $\mathcal{P}$ be a $\mathcal{G}^{\mathrm{op}}_{\leq g}$-monotone graph property. For any i, the assignment of G to the ith simplicial homology group $H_i(\Delta_{\mathcal{P}}(G))$ forms a finitely generated $\mathcal{G}^{\mathrm{op}}_{\leq g}$-module.*

Thus, for any $\mathcal{G}^{\mathrm{op}}_{\leq g}$-monotone graph property, one would obtain an analogous result to Corollary 4.4 about the torsion that can appear in the ith homology group of $\Delta_{\mathcal{P}}(G)$ as G ranges over all graphs with genus at most g.

4.3. Commutative algebra of graph complexes

Famously, the study of simplicial complexes is, in a formal sense, dual to the commutative algebra of square-free monomial ideals through the Stanley-Reisner correspondence. We consider this perspective in this section. Fix a field K. Recall that our edge functor $E : \mathcal{G}_{\leq g}^{\mathrm{op}} \to \mathrm{FI}$ gives rise to another functor $\mathcal{A}_E$ from $\mathcal{G}_{\leq g}^{\mathrm{op}}$ to K-algebras where

$$\mathcal{A}_E(G) = K[x_e \,|\, e \in E(G)].$$

In [15], Nagel and Römer show there are uniform bounds on the degrees for nonvanishing graded Betti numbers for certain families of ideals that form modules over a particular FI-algebra. Using similar techniques, we start by describing some of the homological algebra for $\mathcal{A}_E$-modules and prove analogous results about the graded Betti numbers for families of ideals forming modules over the edge algebra $\mathcal{A}_E$.

Lemma 4.6. *For any finitely generated $\mathcal{A}_E$-module M there exists a projective resolution $F_\bullet$ of M by finitely generated $\mathcal{A}_E$-modules.*

Proof. By Lemma 2.2, since M is finitely generated, there exists graphs $G_1, \ldots, G_n$ and a surjection

$$\bigoplus_{i=1}^{n} P_{G_i}^{\mathcal{A}_E} \to M$$

where P_{G_i} is the principal projective module at G_i as defined in Section 2.1. Let $F_0 = \bigoplus_{i=1}^{n} P_{G_i}^{\mathcal{A}_E}$. The kernel K of this surjection must be finitely generated by Theorem 3.10 and so again, we may find finitely many graphs $H_1, \ldots, H_m$ such that there is a surjection

$$\bigoplus_{i=1}^{m} P_{H_i}^{\mathcal{A}_E} \to K.$$

Let $F_1 = \bigoplus_{i=1}^{m} P_{H_i}^{\mathcal{A}_E}$. Continuing in this fashion, we build the entire projective resolution $F_\bullet$ of M. □

Given two finitely generated $\mathcal{A}_E$-modules, M and N, we may take their tensor product over $\mathcal{A}_E$ by taking the pointwise tensor product. This gives an $\mathcal{A}_E$-module $M \otimes_{\mathcal{A}_E} N$. Explicitly, for any graph G,

$$(M \otimes_{\mathcal{A}_E} N)(G) = M(G) \otimes_{\mathcal{A}_E(G)} N(G),$$

where the morphisms are defined in the obvious way. Now, take a projective resolution $F_\bullet$ of M as in Lemma 4.6. Tensoring this resolution with N gives a chain complex $F_\bullet \otimes_{\mathcal{A}_E} N$ of $\mathcal{A}_E$ modules. We define $\mathrm{Tor}_i(M, N)$ to be the $\mathcal{A}_E$-module where

$$\mathrm{Tor}_i(M,N)(G) = H_i(F_\bullet(G) \otimes_{\mathcal{A}_E(G)} N(G)).$$

For each G, $\mathcal{A}_E(G)$ is a graded K-algebra, where x_e is in degree 1 for each $e \in E(G)$. A **graded $\mathcal{A}_E$-module** M is an $\mathcal{A}_E$-module such that each $M(G)$ is a graded $\mathcal{A}_E(G)$-module and for any minor morphism $\varphi : G \to G'$, the image under $M(\varphi)$ of the degree i part of $M(G')$, denoted $M(G')_i$, is contained in $M(G)_i$.

Remark 4.7. Given a finitely generated graded $\mathcal{A}_E$-module M, we may upgrade the surjection of Lemma 4.6 to a surjection of graded $\mathcal{A}_E$-modules in the following way. Suppose M is generated by the elements $m_1, \ldots, m_n$ where for each i, $m_i \in M(G_i)$. We may assume that each m_i is a homogeneous element of $M(G_i)$ of degree d_i. Then, for each i, we give a grading to the principal projective $\mathcal{A}_E$-module $P_{G_i}^{\mathcal{A}_E}$ the grading where $b_\varphi \in P_{G_i}^{\mathcal{A}_E}(G)$ is homogeneous of degree d_i for any graph G, and minor morphism $\varphi : G \to G_i$. Then, the morphism of $\mathcal{A}_E$-modules

$$\bigoplus_{i=1}^{n} P_{G_i}^{\mathcal{A}_E} \to M$$

that sends b_{id_i} to m_i is a surjection of graded $\mathcal{A}_E$-modules. With this fact, we see that the projective resolution of Lemma 4.6 may, in fact, be upgraded to a projective resulotion of graded $\mathcal{A}_E$-modules.

Let I_E be the $\mathcal{A}_E$-submodule of $\mathcal{A}_E$ itself where $I_E(G)$ is the ideal $\langle x_e | e \in E(G)\rangle \subset \mathcal{A}_E(G)$ for each graph G. Call I_E the **edge ideal of $\mathcal{A}_E$**. Then, $K_{\mathcal{A}_E} = \mathcal{A}_E/I_E$ is the graded $\mathcal{A}_E$-module taking each graph to the base field K in degree 0, where all morphisms are the identity and for each graph G, monomials of $\mathcal{A}_E(G)$ act by zero, and elements of $K \subset \mathcal{A}_E(G)$ act by multiplication.

Lemma 4.8. *If M is a finitely generated graded $\mathcal{A}_E$-module, then $\mathrm{Tor}_i(M, K_{\mathcal{A}_E})$ is also a finitely generated graded $\mathcal{A}_S$-module for all i.*

Proof. Let $F_\bullet$ be a projective resolution of M where each term is finitely generated as in Lemma 4.6. By Remark 4.7 we may assume $F_\bullet$ is a projective resolution by graded $\mathcal{A}_E$-modules. For each i, $F_i \otimes K_{\mathcal{A}_E}$ is isomorphic as an $\mathcal{A}_E$-module to the module $F_i/I_E F_i$ taking a graph G to the $\mathcal{A}_E(G)$-module $F_i(G)/I_E(G)F_i(G)$. Thus, $\mathrm{Tor}_i(M, K_{\mathcal{A}_E})$ is a subquotient of a finitely generated $\mathcal{A}_E$-module and is thus, finitely generated by Theorem 3.10. □

Given a finitely generated graded $\mathcal{A}_E$-module M, a graph G, and integers $i, a \geq 0$, the **ith graded Betti number of M in degree a with respect to G** is defined to be

$$\beta_{i,a}^G(M) := \dim_K(\mathrm{Tor}_i(M, K_{\mathcal{A}_E})(G))_a.$$

Theorem 4.9. *Let M be a finitely generated graded $\mathcal{A}_E$-module and fix $i \geq 0$. Then, there exists an integer n_i such that for any graph G,*

$$\beta_{i,a}^G(M) = 0 \quad \textit{for all} \quad a > n_i.$$

Proof. By Lemma 4.8, we know that $T_i = \mathrm{Tor}_i(M, K_{\mathcal{A}_E})$ is a finitely generated $\mathcal{A}_E$ module. Therefore, we can find a finite list $t_1, \ldots, t_n$ where $t_j \in M(G_j)$ for some (not necessarily distinct) graphs $G_1, \ldots, G_n$, such that for any graph G, $T_i(G)$ is generated by the images of the t_j under maps induced by minor morphisms $G_j \to G$. Moreover, we may assume that each t_j is homogeneous of degree d_j. Then, define $n_i := \max\{d_j\}_{j=1}^n$. We know that $\beta_{i,a}^G(M) = \dim_K(T_i(G))_a$ is the number of generators of degree a in a minimal homogeneous generating set of $T_i(G)$ (see for instance [14, Lemma 1.32]). Thus, since the images of the t_j under maps induced by minor morphisms $G_j \to G$ contain a minimal homogeneous generating set of $T_i(G)$, if $a > n_i$, we must have $\beta_{i,a}^G(M) = 0$. □

For any fixed graph G, we can consider $\mathcal{A}_E(G) = K[x_e | e \in E(G)]$ as an $\mathbb{N}^{E(G)}$-graded K-algebra where x_e is in degree $\mathbf{v}_e \in \mathbb{N}^{E(G)}$ which has a 1 in the e coordinate and 0's elsewhere. We call this grading the **edge-grading on** $\mathcal{A}_E(G)$. If M is a graded $\mathcal{A}_E$-module such that for each graph G, $M(G)$ is in fact an $\mathbb{N}^{E(G)}$-graded $\mathcal{A}_E(G)$-module, we call M an **edge-graded $\mathcal{A}_E$-module**. Note that $K_{\mathcal{A}_E}$ is an edge-graded $\mathcal{A}_E$-module where $K_{\mathcal{A}_E}(G)$ is concentrated in degree $0 \in \mathbb{N}^{E(G)}$. Thus, if M is an edge-graded $\mathcal{A}_E$-module, this induces an edge-graded $\mathcal{A}_E$-module structure on $\mathrm{Tor}_i(M, K_{\mathcal{A}_E})$ for any i.

Now, given an edge-graded $\mathcal{A}_E$-module M, a graph G, an integer $i \geq 0$, and $\mathbf{a} \in \mathbb{N}^{E(G)}$, we can consider the **$i^{\text{th}}$ edge-graded Betti number of $M(G)$ in degree $\mathbf{a}$** denoted $\beta_{i,\mathbf{a}}(M(G))$ where

$$\beta_{i,\mathbf{a}}(M(G)) = \dim_K(\mathrm{Tor}_i(M, K_{\mathcal{A}_E})(G))_{\mathbf{a}}.$$

Let $\mathrm{sum}(\mathbf{a})$ denote the sum of the entries of $\mathbf{a}$. We see that

$$\beta_{i,a}^G(M) = \sum_{\mathrm{sum}(\mathbf{a})=a} \beta_{i,\mathbf{a}}(M(G)).$$

By Theorem 4.9, we have the following immediate result.

Corollary 4.10. *Let M be a finitely generated edge-graded $\mathcal{A}_E$-module and fix $i \geq 0$. Then, there exists an integer n_i such that for any graph G,*

$$\beta_{i,\mathbf{a}}(M(G)) = 0 \quad \textit{whenever} \quad \mathrm{sum}(\mathbf{a}) > n_i.$$

Let us now consider the case where M is a finitely generated edge-graded $\mathcal{A}_E$-module such that for each graph G, $M(G)$ is a square free monomial ideal of $\mathcal{A}_E(G) = K[x_e | e \in E(G)]$. In this case, using the Stanely-Reisner construction, one can associate to $M(G)$ a

simplicial complex $\Delta_M(G)$ on the set $E(G)$. Namely, the simplices of $\Delta_M(G)$ are given by squarefree monomials of $\mathcal{A}_E(G)$ not contained in $M(G)$. Now, let us identify each subset $\sigma \subseteq E(G)$ with its indicator vector in $\mathbb{N}^{E(G)}$ so that each such subset of $E(G)$ corresponds to a squarefree degree in the edge-grading of $\mathcal{A}_E(G)$. The commutative algebra of $M(G)$ is intimately linked with the topology and combinatorics of $\Delta_M(G)$. See [23] for an in-depth treatment of this relationship. In particular, there is a nice relationship between the edge-graded Betti numbers of $M(G)$ and the simplicial complex $\Delta_M(G)$ due to Hochster (see [14, Corollary 5.12]).

Theorem 4.11 *(Hochster's formula). The nonzero Betti numbers of $M(G)$ lie only in square free degrees, namely degrees corresponding to subsets $\sigma \subset E(G)$. Furthermore,*

$$\beta_{i,\sigma}(M(G)) = \dim_K \tilde{H}^{|\sigma|-i-2}(\Delta_M(G)|_\sigma; K)$$

where $\Delta_M(G)|_\sigma = \{\tau \in \Delta_M(G) \,|\, \tau \subseteq \sigma\}$.

Hochster's formula together with Corollary 4.10 gives the following.

Corollary 4.12. *Fix $i \geq 0$. There exists an integer n_i such that for any graph G, if $\sigma \subseteq E(G)$ with $|\sigma| > n_i$, then*

$$\dim_K \tilde{H}^{|\sigma|-i-2}(\Delta_M(G)|_\sigma; K) = 0.$$

As an application of the above results, we first define a specific $\mathcal{A}_E$-module which will be denoted $I_{\mathcal{L}^c}$. For a graph G, the *edge ideal* of G is the ideal $I \subseteq K[x_v | v \in V(G)]$ generated by monomials $x_v x_w$ where v and w are connected by an edge in $E(G)$. Given a graph G, define the **line graph of** G denoted $\mathcal{L}(G)$ to be the simple graph whose vertices are the edges of G and two vertices of $\mathcal{L}(G)$ are adjacent if and only if they share a vertex in G. Then, define the **complement line graph of** G, denoted $\mathcal{L}^c(G)$, to be the complement of $\mathcal{L}(G)$. Namely, the vertices of $\mathcal{L}^c(G)$ are the edges of G and two vertices of $\mathcal{L}^c(G)$ are adjacent if and only if they do not share a vertex in G.

Example 4.13. If G is the star graph - i.e. the tree with one vertex of degree n and all other vertices of degree 1 - then $\mathcal{L}^c(G)$ is immediately seen to be a disjoint collection of points. On the other hand, if G is the complete graph K_n, then $\mathcal{L}^c(G)$ is the Kneser graph $K(n, 2)$. indeed, certain authors refer to line graph complements as being generalized Kneser graphs for this reason [3].

Now, let $I_{\mathcal{L}^c}$ be the $\mathcal{A}_E$-module taking a graph G to the edge ideal of $\mathcal{L}^c(G)$. We see that

$$I_{\mathcal{L}^c}(G) = \langle x_e x_f | e, f \in E(G) \text{ don't share a vertex} \rangle \subseteq \mathcal{A}_E(G).$$

Indeed, $I_{\mathcal{L}^c}(G)$ is a square free monomial ideal of $\mathcal{A}_E(G)$. Furthermore, if $\varphi : G \to G'$ is a minor morphism, and $e, f \in E(G)$ don't share a vertex, then $\varphi^*(e), \varphi^*(f) \in E(G')$ don't share a vertex so $I_{\mathcal{L}^c}$ does, in fact, give an $\mathcal{A}_E$-module of square free monomial ideals.

To see that $I_{\mathcal{L}^c}$ is a finitely generated $\mathcal{A}_E$-module, note that it is a submodule of the edge ideal I_E. We see I_E is finitely generated by the graphs G_1 and G_2, where G_1 consists of two vertices and a single edge connecting them and the graph G_2 consists of a single vertex and a loop at that vertex. This is because for any graph G and any edge $e \in E(G)$, we can find a minor morphism $G \to G_i$ for some $i \in \{1, 2\}$ such that e gets sent to the single edge in G_i. Thus, the Noetherian property in Theorem 3.10 tells us that $I_{\mathcal{L}^c}$ is a finitely generated $\mathcal{A}_E$-module.

Applying Corollary 4.10 for any fixed i, we immediately get the existence of a bound on the degrees of the nonzero edge-graded Betti numbers $\beta_{i,\mathbf{a}}(I_{\mathcal{L}^c}(G))$ that is uniform as G ranges over all graphs. Moreover, Corollary 4.12 tells us about the cohomology of some subcomplexes of the simplicial complexes $\Delta_{I_{\mathcal{L}^c}}(G)$ as G ranges over all graphs. We note that $\Delta_{I_{\mathcal{L}^c}}(G)$ is precisely the flag complex - or clique complex - of the line graph $\mathcal{L}(G)$.

4.4. Linear subspace arrangements of line graph complements

In this section we study the cohomology of a certain family of hyperplane arrangement complements. In particular, we prove a finite generation result that recovers and expands upon similar results present in [6].

Let $G = (V, E)$ be a graph, let d be a positive integer, and let K be either $\mathbb{C}$ or $\mathbb{R}$. For each $e \in E$, if $v, w \in V$ are the endpoints of e, let $W_e \subset (K^d)^V$ be the subspace

$$W_e = \{x \in (K^d)^V \mid x_v - x_w = 0\}.$$

The collection of all W_e as e ranges over the edges of G is a subspace arrangement called the **graphical arrangement of** G denoted $\mathscr{A}(G)$. In the case where $d = 1$, each W_e is a hyperplane of K^V. In general, W_e is a codimension d subspace of $(K^d)^V$. Now, let $\mathrm{Conf}(G, K^d) = (K^d)^V - \mathscr{A}(G)$ be the space obtained by removing the subspaces of $\mathscr{A}(G)$. In the case where G is the complete graph K_m, $\mathrm{Conf}(G, K^d)$ is the usual ordered configuration space of m points in K^d.

To each graph G, instead of assigning to it the space $\mathrm{Conf}(G, K^d)$, we will assign to it the space $\mathrm{Conf}(\mathcal{L}^c(G), K^d)$. We see that if $\varphi : G \to G'$ is a minor morphism, then the inclusion $\varphi^* : E(G') \to E(G)$ is an inclusion $V(\mathcal{L}^c(G')) \to V(\mathcal{L}^c(G))$. Thus, φ^* induces a natural projection

$$\overline{\varphi} : (K^d)^{V(\mathcal{L}^c(G))} \to (K^d)^{V(\mathcal{L}^c(G'))}$$

Furthermore, φ^* preserves pairs of edges that do not share a vertex. Namely, if $e'_1, e'_2 \in E(G')$ do not share a vertex, then $\varphi^*(e'_1)$ and $\varphi^*(e'_2)$ do not share a vertex. This obser-

vation yields a natural inclusion $E(\mathcal{L}^c(G')) \to E(\mathcal{L}^c(G))$. Thus, if $x \in \mathrm{Conf}(\mathcal{L}^c(G), K^d)$, then $\overline{\varphi}(x) \in \mathrm{Conf}(\mathcal{L}^c(G), K^d)$ so $\overline{\varphi}$ restricts to a map

$$\mathrm{Conf}(\mathcal{L}^c(G'), K^d) \to \mathrm{Conf}(\mathcal{L}^c(G), K^d).$$

The construction of the above map is functorial with respect to compositions of minor morphisms. Thus, we have a functor from $\mathcal{G}$ to the category of topological spaces sending a graph G to $\mathrm{Conf}(\mathcal{L}^c(G), K^d)$. Then, taking the cohomology ring with $\mathbb{Z}$ coefficients, gives a functor $\mathcal{G}_{\leq g}^{\mathrm{op}}$ to the category of abelian groups.

Theorem 4.14. *For any i, the functor $H^i(\mathrm{Conf}(\mathcal{L}^c(-), \mathbb{C}^d); \mathbb{Z})$ from $\mathcal{G}_{\leq g}^{\mathrm{op}}$ to the category of abelian groups taking a graph G to $H^i(\mathrm{Conf}(\mathcal{L}^c(G), \mathbb{C}^d); \mathbb{Z})$ is a finitely generated $\mathcal{G}_{\leq g}^{\mathrm{op}}$-module.*

Proof. For a graph G, we know that $\mathrm{Conf}(\mathcal{L}^c(G), K^d)$ is the complement of a subspace arrangement where each subspace has real codimension r where $r = d$ when $K = \mathbb{R}$ and $r = 2d - 1$ when $K = \mathbb{C}$. In [12, Corollary 5.6], de Longueville and Schultz give a presentation for the cohomology ring of the complement of a subspace arrangement over $\mathbb{R}$ where each subspace has the same codimension. This result tells says that the cohomology ring $H^*(\mathrm{Conf}(\mathcal{L}^c(G), K^d); \mathbb{Z})$ has the following presentation:

$$H^*(\mathrm{Conf}(\mathcal{L}^c(G), K^d); \mathbb{Z}) \cong \wedge^* \mathbb{Z}^{E(\mathcal{L}^c(G))}/I$$

when r is even and

$$H^*(\mathrm{Conf}(\mathcal{L}^c(G), K^d); \mathbb{Z}) \cong \mathrm{Sym}^* \mathbb{Z}^{E(\mathcal{L}^c(G))}/I$$

when r is odd. Moreover, if $e \in E(\mathcal{L}^c(G))$, then under this isomorphism $e \in H^{r-1}(\mathrm{Conf}(\mathcal{L}^c(G), K^d))$. If r is even, I is generated by

$$\sum_{i=0}^{k} (-1)^i \epsilon(e_{j_1}, \ldots, \hat{e}_{j_i}, \ldots, e_{k_i}) e_{j_0} \wedge \cdots \wedge \widehat{e}_{j_i} \wedge \cdots \wedge e_{j_k}$$

and if r is odd, I is generated by

$$e^2 \quad \text{and} \quad \sum_{i=0}^{k} (-1)^i \epsilon(e_{j_1}, \ldots, \hat{e}_{j_i}, \ldots, e_{k_i}) e_{j_0} \ldots \widehat{e}_{j_i} \ldots e_{j_k}$$

for all $e \in E(\mathcal{L}^c(G))$ and all sets $\{e_{a_0}, \ldots, e_{a_k}\} \subset E(\mathcal{L}^c(G))$ that form a cycle. Here, $\epsilon(e_{j_1}, \ldots, \hat{e}_{j_i}, \ldots, e_{k_i})$ is a sign coming from a choice of orientation on the ambient real vector space.

We know cycles of $\mathcal{L}^c(G)$ correspond to sets of edges $\{f_1, \ldots, f_k\} \subset E(G)$ such that f_i and f_{i+1} do not share a vertex and f_1 and f_k do not share a vertex. Given a minor

morphism $\varphi : G \to G'$, the inclusion $\varphi^* : E(G') \to E(G)$ preserves pairs of edges that don't share a vertex, and so the induced map $E(\mathcal{L}^c(G')) \to E(\mathcal{L}^c(G))$ sends cycles to cycles. Furthermore, the inclusion $\varphi^* : E(G') \to E(G)$ gives an inclusion of the ambient real vector space for $\mathrm{Conf}(E(G'), K^d)$ into the ambient real vector space for $\mathrm{Conf}(E(G), K^d)$ which in particular, allows us to pick compatible orientations on these ambient vector spaces. Thus, the presentation for $H^*(\mathrm{Conf}(\mathcal{L}^c(G)), \mathbb{C}^d); \mathbb{Z})$ is compatible minor morphisms. Thus, we see that for each i, $H^{ir}(\mathrm{Conf}(\mathcal{L}^c(-), \mathbb{C}^d); \mathbb{Z})$ as $\mathcal{G}^{\mathrm{op}}_{\leq g}$-module, is a quotient of the ith tensor power of $\mathbb{Z}^{E(\mathcal{L}^c(G))}$,

$$T^i \mathbb{Z}^{E(\mathcal{L}^c(-))}.$$

Finally, we notice that $\mathbb{Z}^{E(\mathcal{L}^c(-))}$ is the submodule of the second tensor power of the edge module $M_E^{\otimes 2}$, where $\mathbb{Z}^{E(\mathcal{L}^c(G))}$ generated by elements of the form $e_i \otimes e_j$ where $e_i, e_j \in E(G)$ do not share a vertex. Thus, by Theorem 1.1, we have the desired result. □

Remark 4.15. If $d = 1$, $\mathrm{Conf}(\mathcal{L}^c(G), K)$ is the complement of a hyperplane arrangement in K^V. When $K = \mathbb{C}$, $H^*(\mathrm{Conf}(\mathcal{L}^c(G), K))$ is just the Orlik-Solomon algebra of the complex hyperplane arrangement where the generators live in degree 1. If $d > 1$, the above presentation shows that $H^*(\mathrm{Conf}(\mathcal{L}^c(G), K^d))$ is still isomorphic to the aforementioned Orlik-Solomon algebra, only now the generators live in degree $2d - 1$. When $K = \mathbb{R}$, $H^*(\mathrm{Conf}(\mathcal{L}^c(G), K))$ is the Cordovil algebra of the real hyperplane arrangement. If d is odd, then $H^*(\mathrm{Conf}(\mathcal{L}^c(G), K^d))$ is isomorphic to $H^*(\mathrm{Conf}(\mathcal{L}^c(G), K))$, only now the generators live in degree $d - 1$.

Example 4.16. Let $a, b \geq 1$ and let G be the graph with two vertices of degrees $a + 1$ and $b + 1$, respectively, connected to one another by a single edge. Put another way, G is two copies of star graphs, of degrees a and b, glued together by an edge. Then $\mathcal{L}^c(G)$ is easily seen to be a complete bipartite graph $K_{a,b}$, disjoint union a point. Then we have

$$\mathrm{Conf}(\mathcal{L}^c(G), \mathbb{C}^d) = \widetilde{\mathcal{Z}}^D_{a+b}(\mathbb{C}^d) \times \mathbb{C}^d$$

where $\widetilde{\mathcal{Z}}^D_{a+b}(\mathbb{C}^d)$ are the colored configuration spaces considered by Farb Wolfson and Wood in [6], with D being the vertices of the complete bipartite graph colored in the obvious way. Observe moreover that the graph G can be seen as an edge with a and b leaves sprouted on its two vertices, respectively. Therefore, Theorem 3.21 implies that our result can be seen as a generalization of the stabilization phenomena observed by [6].

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