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NUMERICAL APPROXIMATION AND ANALYSIS FOR PARTIAL INTEGRODIFFERENTIAL EQUATIONS OF HYPERBOLIC TYPE

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We consider the numerical analysis for a partial integrodifferential equation of hyperbolic type. The central difference formula and the second-order convolution quadrature are applied to construct the numerical scheme, where the convolution quadrature method could accommodate the complicated case that the explicit form of the memory kernel is not available. We propose a novel analysis to prove the finite-time stability of the numerical solutions and specify the condition that ensures the long time stability. We also prove error estimates for the numerical scheme based on a newly developed approximate result of convolution quadrature for convolution of nonsmooth functions. Finally, we extend the developed methods to construct and analyze a numerical scheme for the corresponding nonlinear problems. Numerical experiments are performed to substantiate the theoretical findings.

1. Introduction

This work considers numerical approximation of the following partial integrodifferential equation of hyperbolic type proposed in, e.g., [1; 2; 6; 8; 20; 25]:

$$u''(t) + Au(t) - (\beta * Au)(t) = f(t, u(t)), \quad t > 0, \tag{1}$$

subject to the initial conditions

$$u(0) = u_0, \quad u'(0) = u_1. \tag{2}$$

Here A is a positive self-adjoint densely defined linear operator on the Hilbert space H, $u_0, u_1 \in H$ are given data, and $\beta(t) \in L_1(\mathbb{R}_+)$ is a scalar memory kernel. By [4, Theorems 9 and 11], there exists a unique positive self-adjoint operator $A^{1/2}$ such that $(A^{1/2})^2 = A$. The $*$ represents the convolution defined by

$$(\beta * \psi)(t) := \int_0^t \beta(t-s)\psi(s)\, ds, \quad t \geq 0. \tag{3}$$

Integrodifferential equations such as (1)–(2) with $f(t, u(t)) = f(t)$ arise in several fields such as the linear viscoelasticity or heat conduction with memory, and the operator A usually takes the form of the negative Laplacian, the Stokes operator, or the biharmonic operator, etc., equipped with appropriate

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boundary conditions [18; 19; 20]. The kernel β is assumed to satisfy the following conditions in the aforementioned works:

(4) $\beta(t) \geq 0$ is nonincreasing, locally absolutely continuous on $(0, \infty)$ with $\int_0^\infty \beta(t)\,dt < 1$.

Some typical examples of such kernels are the weak singular kernel [13; 21]

$$\beta(t) = \gamma_0 \frac{t^{\alpha-1}}{\Gamma(\alpha)} e^{-\rho t}, \quad t > 0,\ \rho > 0,\ 0 < \alpha < 1,\ \gamma_0 \in (0, \rho^\alpha), \tag{5}$$

and the smooth kernel [24]

$$\beta(t) = \frac{e^{-t}(1 - e^{-t})}{t}. \tag{6}$$

There exist several theoretical studies for the linear case of problem (1)–(2) on the existence and decay properties of the solutions [1; 2; 5; 6; 8; 16; 20], and some numerical studies have also been considered. For instance, Pani et al. [17] considered the interpolation quadrature to solve (1) with a smooth kernel. Then, Larsson et al. considered the continuous Galerkin method [11] and discontinuous Galerkin method [12] for the linear case of (1)–(2) with weak singular kernels. Karaa and Pani developed the mixed finite element method [9] and the discontinuous Galerkin method [10] for the linear case of (1)–(2) with smooth kernels.

For nonlinear problems, a pioneering work investigated the attenuated Westervelt equation [3], which, compared with the problem (1)–(2), contained an additional nonlinear term $k(u^2)''$ for some $k > 0$ and the two specific kernels, i.e., the tempered fractional kernel and the Mittag-Leffler type kernel. The existence and regularity of the solutions were rigorously proved via sophisticated analysis, which in turn supported the error estimates of the numerical discretization based on the trapezoidal rule and A-stable convolution quadrature (CQ).

Despite the aforementioned significant progress, these works rely on properties of the kernel, which are not always available. For instance, a class of variable-order fractional kernels was considered in, for instance, [7], to account for the varying nature of nonlocalities, which varied the order of the operators in the Laplace domain such that the explicit form of the resulting kernel and thus its properties were in general not available. For such complicated problems, Xu applied the Laplace transform for model (1)–(2) to split the solution into two parts, and the CQ in which the underlying multistep method was the trapezoidal rule was utilized for temporal discretization [25]. For nonlinear problems, it is difficult to apply the Laplace transform method, which motivates us to develop a direct and CQ-based computation method.

We consider a direct discretization scheme for model (1)–(2) where the second-order difference and CQ schemes are applied for approximating the second-order time derivative and the convolution term, respectively, which is feasible to treat nonlinear problems without explicit expressions of kernels. The main contributions are enumerated as follows:

- We prove the finite-time stability of the numerical solutions to the linear case of problem (1)–(2) based only on the imposed properties of the Laplace transform of the kernel, and specify the condition that ensures the long-time stability of the numerical solutions. In particular, a novel norm is introduced (see (25)), which captures the structure of the scheme and thus significantly simplifies the analysis procedure.

• We prove error estimates for the numerical scheme of the linear case of problem (1)–(2). A key ingredient lies in developing a new approximate result of CQ for the convolution of nonsmooth functions (see Lemma 4.2) by technical derivations, which accounts for the possible singularity of the solutions caused by that of the memory kernel.

• We extend the developed methods to construct and analyze a numerical scheme for the nonlinear problem (1)–(2), which circumvents the limitation of the Laplace transform method and generalizes the application of the proposed numerical discretization method for more complicated problems.

The rest of this paper is organized as follows: We propose a discrete-in-time scheme for the linear problem in Section 2. The stability analysis of numerical solutions is given in Section 3. Section 4 presents the error estimates for the proposed scheme. In Section 5, we extend the developed methods to construct a numerical scheme of the semilinear problem. Numerical experiments are carried out to substantiate the theoretical results in Section 6.

2. Discrete-in-time scheme for linear problem

Define the norms $\|w\| = \sqrt{\langle w, w\rangle}$ and $\|w\|_m = \|A^{m/2}w\|$ for $m = 1, 2, 3, 4$, and define

(7)
$$b(t) := \int_t^\infty \beta(q)\,dq \quad \text{and} \quad b_0 := b(0) = \int_0^\infty \beta(q)\,dq,$$

where $\beta(t)$ is given in (4). Based on the assumptions in (4), $b(t)$ is nonnegative, nonincreasing and convex on $(0, \infty)$. We then follow the ideas in [18; 25] to apply the integration by parts to obtain

$$Au(t) - (\beta * Au)(t) = (1 - b_0)Au(t) + b(t)Au_0 + (b * Au')(t).$$

This reformulation eliminates the negative sign in the convolution term in (1) that facilitates the analysis. Then for $f(t, u(t)) = 0$, we first consider the linear case of problem (1)–(2), and rewrite (1) as

(8)
$$u''(t) + (1 - b_0)Au(t) + b(t)Au_0 + (b * Au')(t) = 0.$$

To discretize (8), we set the time step size k and consider (8) at $t_n = nk$

(9)
$$u''(t_n) + (1 - b_0)Au(t_n) + b_n Au_0 + (b * Au')(t_n) = 0, \quad n \ge 1,$$

where $b_n = \int_{t_n}^\infty \beta(s)\,ds$. Then we discretize the terms in (9) one by one. Let $u^n = u(t_n)$ and

(10)
$$\delta_t u^n = \frac{u^n - u^{n-1}}{k}, \quad \delta_t^{(2)} u^n = \delta_t(\delta_t u^{n+1}), \quad \tilde{u}^n = \frac{u^{n+1} + 2u^n + u^{n-1}}{4}, \quad \bar{u}^n = \frac{u^{n+1} - u^{n-1}}{2k}.$$

We approximate $u''(t_n)$ by $\delta_t^{(2)} u^n$ with the remainder expressed by the Taylor expansion

(11)
$$\begin{aligned} u''(t_n) - \delta_t^{(2)} u^n &= \frac{-1}{6k^2}\left(\int_{t_n}^{t_{n+1}} (t_{n+1} - t)^3 u''''(t)\,dt + \int_{t_{n-1}}^{t_n} (t - t_{n-1})^3 u''''(t)\,dt\right) \\ &\equiv [R_{t,1}]^n, \quad n \ge 2, \\ u''(t_1) - \delta_t^{(2)} u^1 &= \frac{-1}{2k^2}\left(\int_{t_1}^{t_2} (t_2 - t)^2 u'''(t)\,dt + \int_0^{t_1} t^2 u'''(t)\,dt\right) \equiv [R_{t,1}]^1. \end{aligned}$$

Then we follow [17] to approximate $Au(t_n)$ by $A\tilde{u}^n$ with the remainder expressed by the Taylor expansion

$$Au(t_n) - A\tilde{u}^n = \frac{-1}{4}\left(\int_{t_n}^{t_{n+1}} (t_{n+1}-t)Au''(t)\,dt + \int_{t_{n-1}}^{t_n} (t-t_{n-1})Au''(t)\,dt\right) \equiv [R_{t,2}]^n, \quad n \ge 1. \tag{12}$$

To approximate the convolution term $b * Au'$, we adopt the following second-order CQ [14; 15] for the convolution $b*\varphi$:

$$\widetilde{\mathscr{Q}}_{t_n}(\varphi) = \mathscr{Q}_{t_n}(\varphi) + \chi_{n0}(k)\varphi(0), \quad \text{where } \mathscr{Q}_t(\varphi) = \sum_{0 \le t_p \le t} \omega_p(k)\varphi(t-t_p), \tag{13}$$

with the quadrature weights $\omega_n(k)$ being the coefficients of the power series

$$\hat{b}\left(\frac{\zeta(z)}{k}\right) = \sum_{n=0}^{\infty} \omega_n(k) z^n, \quad |z|<1; \qquad \zeta(z) = \frac{(3-z)(1-z)}{2}, \quad z \in \mathbb{C}, \tag{14}$$

where $\hat{b}(s)$ represents the Laplace transform of $b(t)$. The starting weight in (13) is given in order to maintain the second-order accuracy

$$\chi_{n0}(k) = (b*1)\,(t_n) - \sum_{p=0}^{n} \omega_p(k). \tag{15}$$

Based on [15], $\zeta(z)$ satisfies the following conditions:

(i) $\zeta(z)$ is analytic and without zeros in a neighborhood of the closed unit disc $|z| \le 1$, with the exception of a zero at $z=1$;

(ii) $|\arg \zeta(z)| \le \pi/2$ for $|z|<1$ and $\frac{1}{k}\zeta(e^{-k}) = 1 + O(k^2)$.

Then the convolution term $b*Au'$ is approximated by integrating the second-order CQ with the leapfrog scheme

$$(b*Au')(t_n) = \widetilde{\mathscr{Q}}_{t_n}(A\bar{u}) + [R_{t,3}]^n,$$

with $\bar{u}^0 := u_1$, the initial value of u', and the error

$$[R_{t,3}]^n = ((b*Au')(t_n) - \widetilde{\mathscr{Q}}_{t_n}(Au')) + \widetilde{\mathscr{Q}}_{t_n}(A(u'-\bar{u})) =: [R_{t,3,1}]^n + [R_{t,3,2}]^n. \tag{16}$$

Invoking (11)–(13) in (9) we have

$$\delta_t^{(2)} u^n + (1-b_0)A\tilde{u}^n + b_n A u_0 + \widetilde{\mathscr{Q}}_{t_n}(A\bar{u}) = [R_t]^n, \quad n \ge 1, \tag{17}$$

where $[R_t]^n = -\sum_{j=1}^{3}[R_{t,j}]^n$. The initial data (2) provides

$$\delta_t u^1 - \left(u_1 + \frac{k}{2}u_2\right) = \frac{1}{2k}\int_0^k (k-t)^2 u'''(t)\,dt \equiv [R_t]^0, \quad u^0 = u_0, \tag{18}$$

where $u_2 := u''(0) = -Au_0$ from (8). Let U^n be the approximated solution of u^n with $\bar{U}^0 = \bar{u}^0$. Then we drop the truncation errors in (17) and (18) to get the discrete-in-time scheme

$$\delta_t^{(2)} U^n + (1-b_0)\, A\widetilde{U}^n + b_n A u_0 + \widetilde{\mathscr{Q}}_{t_n}(A\bar{U}) = 0, \quad n \geq 1, \tag{19}$$

$$\delta_t U^1 = u_1 + \frac{k}{2} u_2, \quad U^0 = u_0. \tag{20}$$

Throughout this paper, C denotes a positive constant that is independent of the time step size but may assume different values at different occurrences.

3. Numerical stability

We first establish the finite-time stability of the numerical solution, and then specify the condition that ensures the long-time stability. We follow [15] to make conventional assumptions for the Laplace transform of b for the sake of numerical analysis:

Assumption A: $\hat{b}(s)$ is analytic in a sector $|\arg(s-c)| < \pi - \theta$ with $\theta < \pi/2$ and $c \in \mathbb{R}$, and satisfies $|\hat{b}(s)| \leq \mathscr{M}|s|^{-\mu}$ for some $\mathscr{M} < \infty$ and $\mu > 0$.

Theorem 3.1. *Suppose (4) and Assumption A hold. Then for $T < \infty$ with $N+1 = T/k$, the finite-time stability holds*

$$\|U^n\| + \|\delta_t U^n\| \leq C(T)(\|u_0\|_2 + k^2\|u_0\|_3 + \|u_1\| + k\|u_1\|_2), \quad 1 \leq n \leq N+1.$$

Proof. First, we take the inner product of (19) with $\bar{U}^n$ to get

$$\begin{aligned}\frac{1}{2k^3}&(\|U^{n+1}\|^2 - \|U^{n-1}\|^2 - 2\langle U^n, U^{n+1}\rangle + 2\langle U^{n-1}, U^n\rangle)\\ &+ b_n\langle Au_0, \bar{U}^n\rangle + \frac{1-b_0}{8k}(\|U^{n+1}\|_1^2 - \|U^{n-1}\|_1^2)\\ &+ \frac{1-b_0}{8k}(2\langle A^{1/2}U^n, A^{1/2}U^{n+1}\rangle - 2\langle A^{1/2}U^{n-1}, A^{1/2}U^n\rangle)\\ &+ \sum_{p=0}^{n}\omega_p(k)\langle A^{1/2}\bar{U}^{n-p}, A^{1/2}\bar{U}^n\rangle + \chi_{n0}(k)\langle Au_1, \bar{U}^n\rangle = 0,\end{aligned} \tag{21}$$

where we use the results that

$$\begin{aligned}\langle \delta_t^{(2)}U^n, \bar{U}^n\rangle &= \frac{1}{2k^3}(\|U^{n+1}\|^2 - \|U^{n-1}\|^2 - 2\langle U^n, U^{n+1}\rangle + 2\langle U^n, U^{n-1}\rangle)\\ &= \frac{1}{2k^3}(\|U^{n+1} - U^n\|^2 - \|U^n - U^{n-1}\|^2),\end{aligned} \tag{22}$$

and that

$$\begin{aligned}\langle A\widetilde{U}^n, \bar{U}^n\rangle &= \frac{1}{8k}(\|U^{n+1}\|_1^2 - \|U^{n-1}\|_1^2 - 2\langle A^{1/2}U^n, A^{1/2}U^{n+1}\rangle + 2\langle A^{1/2}U^n, A^{1/2}U^{n-1}\rangle)\\ &= \frac{1}{8k}(\|U^{n+1} + U^n\|_1^2 - \|U^n + U^{n-1}\|_1^2).\end{aligned} \tag{23}$$

Then summing (21) for n from 1 to M for some $M \le N$, we have

$$(24)\quad \|U^M\|_A^2 = \|U^0\|_A^2 - 2k^3 \sum_{n=1}^{M} b_n \langle Au_0, \bar{U}^n\rangle + 2k^3\omega_0(k)\langle A^{1/2}u_1, A^{1/2}u_1\rangle - 2k^3 \sum_{n=0}^{M}\sum_{p=0}^{n} \omega_p(k)\langle A^{1/2}\bar{U}^{n-p}, A^{1/2}\bar{U}^n\rangle - 2k^3\sum_{n=1}^{M}\chi_{n0}(k)\langle Au_1, \bar{U}^n\rangle,$$

in which $\bar{U}^0 = u_1$ and the norm $\|\cdot\|_A$ is defined as

$$(25)\quad \|V^n\|_A := \sqrt{\|V^{n+1}-V^n\|^2 + \frac{(1-b_0)k^2}{4}\|(V^{n+1}+V^n)\|_1^2}, \quad n \ge 0.$$

By using [23, Lemma 3.1], we have

$$(26)\quad \sum_{n=0}^{M}\sum_{p=0}^{n} \omega_p(k)\langle A^{1/2}\bar{U}^{n-p}, A^{1/2}\bar{U}^n\rangle \ge 0.$$

Using (26), (20) and Cauchy–Schwarz inequality, (24) implies

$$\|U^M\|_A^2 \le \|U^0\|_A^2 + 2k^3\sum_{n=1}^{M} b_n\|Au_0\|\,\|\bar{U}^n\| + 2k^2\omega_0(k)\,\|Au_1\|\,\|U^1\|_A + 2k^3\sum_{n=1}^{M}|\chi_{n0}(k)|\,\|Au_1\|\,\|\bar{U}^n\|,$$

which, together with

$$(27)\quad \|\bar{U}^n\| \le \frac{\|U^{n+1}-U^n\| + \|U^n - U^{n-1}\|}{2k} \le \frac{\|U^n\|_A + \|U^{n-1}\|_A}{2k},$$

leads to

$$(28)\quad \|U^M\|_A^2 \le \|U^0\|_A^2 + k\|u_0\|_2\left(k\sum_{n=1}^{M} b_n(\|U^n\|_A + \|U^{n-1}\|_A)\right) + k\|u_1\|_2\left(k\sum_{n=1}^{M}|\chi_{n0}(k)|(\|U^n\|_A+\|U^{n-1}\|_A) + 2k\omega_0(k)\|U^1\|_A\right).$$

Define $\|U^J\|_A := \max_{0\le n\le N}\|U^n\|_A$, and (28) leads to

$$\|U^J\|_A^2 \le \|U^0\|_A\|U^J\|_A + k\|u_0\|_2\left(k\sum_{n=1}^{J} b_n(\|U^n\|_A+\|U^{n-1}\|_A)\right) + k\|u_1\|_2\left(k\sum_{n=1}^{J}|\chi_{n0}(k)|(\|U^n\|_A+\|U^{n-1}\|_A) + 2k\omega_0(k)\|U^1\|_A\right)$$
$$\le \|U^0\|_A\|U^J\|_A + 2k\|u_0\|_2\left(k\sum_{n=1}^{J} b_n\right)\|U^J\|_A + 2k\|u_1\|_2\left(k\sum_{n=1}^{J}|\chi_{n0}(k)| + k\omega_0(k)\right)\|U^J\|_A,$$

which naturally implies

$$\|U^J\|_A \le \|U^0\|_A + 2k\,\|u_0\|_2\left(k\sum_{n=1}^{J} b_n\right) + 2k\,\|u_1\|_2\left(k\sum_{n=1}^{J}|\chi_{n0}(k)| + k\omega_0(k)\right)$$
$$\le \|U^0\|_A + 2k\,\|u_0\|_2\left(k\sum_{n=1}^{N} b_n\right) + 2k\,\|u_1\|_2\left(k\sum_{n=1}^{N}|\chi_{n0}(k)| + k\omega_0(k)\right). \tag{29}$$

By (7), the properties of $b(t)$ and (4), we have

$$k\sum_{n=1}^{N} b_n \le Tb_0, \tag{30}$$

and from (14) with $\frac{\zeta(0)}{k} = \frac{3}{2k}$, we also have

$$\omega_0(k) = \hat{b}(\frac{\zeta(0)}{k}) = \int_0^\infty e^{-(3/(2k))t} b(t)\,dt \le \frac{2b_0}{3}k. \tag{31}$$

Inserting (30)–(31) into (29) and utilizing (25), we have

$$\begin{aligned} \|U^M\|_A \le \|U^J\|_A &\le \|U^1 - U^0\| + \frac{\sqrt{1-b_0}}{2}k\,\|U^1 + U^0\|_1 \\ &\quad + 2Tb_0k\,\|u_0\|_2 + \frac{4b_0}{3}k^3\,\|u_1\|_2 + 2k^2\left(\sum_{n=1}^{N}|\chi_{n0}(k)|\right)\|u_1\|_2 \\ &\le k\,\|u_1\| + \frac{k^2}{2}\|u_0\|_2 + \frac{\sqrt{1-b_0}}{2}k\left(2\,\|u_0\|_1 + k\,\|u_1\|_1 + \frac{k^2}{2}\|u_0\|_3\right) \\ &\quad + 2Tb_0k\,\|u_0\|_2 + \frac{4b_0}{3}k^3\,\|u_1\|_2 + 2k^2\left(\sum_{n=1}^{N}|\chi_{n0}(k)|\right)\|u_1\|_2, \end{aligned} \tag{32}$$

where we used (20), i.e.,

$$U^1 = U^0 + ku_1 + \frac{k^2}{2}u_2 = u_0 + ku_1 - \frac{k^2}{2}Au_0. \tag{33}$$

By $|\chi_{n0}(k)| \le Ct_n^{\mu-1}k$ [15], we have $c_N^* := \sum_{n=1}^N |\chi_{n0}(k)| \le C(T)$ such that (32) leads to

$$\begin{aligned} \|U^M\|_A \le k\Bigg(&\sqrt{1-b_0}\,\|u_0\|_1 + \left(2Tb_0 + \frac{k}{2}\right)\|u_0\|_2 + \frac{\sqrt{1-b_0}}{4}k^2\,\|u_0\|_3 + \|u_1\| \\ &+ \frac{\sqrt{1-b_0}}{2}k\,\|u_1\|_1 + 2k\left(\frac{2b_0}{3}k + c_N^*\right)\|u_1\|_2\Bigg) := k\Phi(u_0, u_1). \end{aligned} \tag{34}$$

Then (25) and (34) give

$$\|U^{M+1} - U^M\| \le k\Phi(u_0, u_1), \qquad \|\delta_t U^{M+1}\| \le \Phi(u_0, u_1), \tag{35}$$

and we use the property of A to obtain

$$\frac{2}{\sqrt{1-b_0}}\Phi(u_0,u_1) \geq \|(U^{M+1}+U^M)\|_1 \geq c'\|U^{M+1}+U^M\|. \tag{36}$$

Combining (35) and (36), we further arrive at

$$\begin{aligned}\|U^{M+1}\| &\leq \frac{\|U^{M+1}+U^M\|+\|U^{M+1}-U^M\|}{2}\\ &\leq \left(\frac{k}{2}+\frac{1}{c'\sqrt{1-b_0}}\right)\Phi(u_0,u_1).\end{aligned} \tag{37}$$

Thus the proof is completed by combining (35) and (37). □

Based on the above proof, we shall extend to establish the long-time stability of numerical solutions for the model (1)–(2) with exponential decay kernels such as (5) and (6), an important class of kernels that satisfy (4) and Assumption A.

Theorem 3.2. *Let U^n be numerical solution of (19)-(20) for $n \geq 0$. Under (4), Assumption A and the exponential decay condition*

$$\beta(t) = e^{-\rho t}\beta_0(t) \text{ for some } 0<\rho<\infty \text{ such that } \beta_0(t)\geq 0 \text{ is nonincreasing},$$

then the following long-time stability holds if the derivative initial condition $u_1 = 0$

$$\|U^n\|+\|\delta_t U^n\| \leq C(\|u_0\|_2+k^2\|u_0\|_3), \quad n\geq 1.$$

Proof. Let $\|U^K\|_A := \max_{n\geq 0}\|U^n\|_A$ where K might be a finite number or infinity. Then similar to the analysis of (29), we apply $u_1=0$ to get

$$\|U^M\|_A \leq \|U^K\|_A \leq \|U^0\|_A + 2k\|u_0\|_2\left(k\sum_{n=1}^{K} b_n\right). \tag{38}$$

We use the exponential decay condition in this theorem to obtain

$$\begin{aligned}k\sum_{n=1}^{\infty} b_n &\leq \int_0^\infty b(t)\,dt = \int_0^\infty\left(\int_t^\infty e^{-\rho s}e^{\rho s}\beta(s)\,ds\right)dt\\ &\leq \int_0^\infty \beta_0(t)\int_t^\infty e^{-\rho s}\,ds\,dt = \frac{1}{\rho}\int_0^\infty \beta(t)\,dt < \frac{1}{\rho}.\end{aligned} \tag{39}$$

Furthermore, (33) with $u_1=0$ implies

$$\begin{aligned}\|U^0\|_A &\leq \|U^1-U^0\| + \frac{\sqrt{1-b_0}}{2}k\|U^1+U^0\|_1\\ &\leq \frac{k^2}{2}\|u_0\|_2 + \frac{\sqrt{1-b_0}}{2}k\left(2\|u_0\|_1+\frac{k^2}{2}\|u_0\|_3\right).\end{aligned} \tag{40}$$

Invoking (39) and (40) in (38), we obtain

$$\begin{aligned}\|U^M\|_A &\le k\left(\frac{2}{\rho}+\frac{k}{2}\right)\|u_0\|_2+\frac{\sqrt{1-b_0}}{2}k\left(2\|u_0\|_1+\frac{k^2}{2}\|u_0\|_3\right)\\ &:= k\Phi_0(u_0).\end{aligned} \tag{41}$$

The rest of the proof could be performed in analogous of (34)-(37), which completes the proof. □

4. Regularity assumption and error estimate

We shall give the regularity assumptions and error estimates for the linear case of the problem (1)–(2).

Regularity assumption. To establish the convergence, we give some necessary assumptions about the regularity of the solutions motivated from the ordinary differential equation analogue of (1) with $f=0$, that is,

$$\begin{aligned}u''(t)+\lambda u(t)-\lambda\int_0^t\beta(t-s)u(s)\,ds=0,\quad t\ge 0,\\ u(0)=u_0,\quad u'(0)=u_1.\end{aligned} \tag{42}$$

Here λ is some positive constant. And then, by defining $w(t)=u''(t)$, we yield for $t\ge 0$,

$$u(t)=u_0+tu_1+\int_0^t(t-s)w(s)\,ds. \tag{43}$$

By putting (43) into (42), we thus get

$$w(t)=-\lambda\left(u_0+tu_1+\int_0^t(t-s)w(s)\,ds\right)+\lambda\int_0^t\beta(t-s)\left(u_0+su_1+\int_0^s(s-\tau)w(\tau)\,d\tau\right)ds,\quad t\ge 0.$$

If $t\to 0^+$, we have

$$\begin{aligned}u_0+tu_1+\int_0^t(t-s)w(s)\,ds\to u_0,\quad \lambda\int_0^t\beta(t-s)u_0\,ds=\lambda u_0\int_0^t\beta(s)\,ds,\\ \lambda\int_0^t\beta(t-s)\left(su_1+\int_0^s(s-\tau)w(\tau)\,d\tau\right)ds\to\lambda\int_0^t\beta(t-s)o(1)\,ds.\end{aligned}$$

Hence, for the kernels (5) and (6), noting that $\beta(t)\in L^1(0,\infty)$, we get the asymptotic behavior of $u''(t)$ as follows:

$$u''(t)=w(t)\simeq-\lambda u_0+\lambda u_0\int_0^t\beta(s)\,ds+\lambda\int_0^t\beta(s)o(1)\,ds,\quad t\to 0^+. \tag{44}$$

Based on this asymptotic behavior, we assume that

$$\|u''(t)\|+\|Au''(t)\|\le C,\quad \|u'''(t)\|+\|Au'''(t)\|\le C|\beta(t)|,\quad \|u''''(t)\|\le C|\beta'(t)|. \tag{45}$$

Error estimate. We derive auxiliary estimates to support the error estimate of the time-discrete scheme (19)–(20). We first cite the following classical approximate result of the convolution quadrature $\tilde{Q}_t$ from [15].

Lemma 4.1. *Under (4) and Assumption A, for* $1 \le n \le N$,

$$|\widetilde{\mathscr{Q}}_{t_n}(\varphi) - (b*\varphi)(t_n)| \le C t_n^{\mu-1} k^2 \quad \text{for } \varphi \in C^2[0,T].$$

For the case that $\varphi''(t)$ is singular at the initial point $t=0$, we propose an alternative approximate result in the following lemma.

Lemma 4.2. *Under (4) and Assumption A, the following approximate result holds for* $1 \le n \le N$:

$$|\widetilde{\mathscr{Q}}_{t_n}(\varphi) - (b*\varphi)(t_n)| \le Ck^2|\varphi'(0)| + Ck^2 \int_0^{t_n} |\varphi''(\vartheta)|\, d\vartheta.$$

Proof. We introduce the notation $r(t) = (1*\varphi')(t)$, $E_k[\varphi](t) = \mathscr{Q}_t(\varphi) - (b*\varphi)(t)$ and $\widetilde{E}_k[\varphi](t) = \widetilde{\mathscr{Q}}_t(\varphi) - (b*\varphi)(t)$. Then we apply $\widetilde{E}_k[1](t) = 0$ to obtain

$$\widetilde{E}_k[\varphi](t_n) = \widetilde{E}_k[r](t_n) + \varphi(0)\widetilde{E}_k[1](t_n) = E_k[r](t_n) + r(0)\chi_{n0}(k) = E_k[r](t_n),$$

and we apply $r(t) = r'(0)t + (t*r'')(t)$ to find that

$$\begin{aligned} E_k[r](t_n) &= r'(0)E_k[t](t_n) + E_k[(t*r'')](t_n) \\ &= r'(0)E_k[t](t_n) + (E_k[t]*r'')(t_n), \end{aligned} \tag{46}$$

where μ is determined via the properties of $\hat{b}$ in Assumption A. We first estimate the first right-hand side term of (46). By [15, Theorem 5.2] and the assumptions of this lemma, we have

$$|E_k[t^{\beta-1}](x)| \le Cx^{\mu-1}k^{\beta} \quad \text{for } 0<\beta\le 2,\ k\le x\le T. \tag{47}$$

We apply (47) with $\beta = 2$ and $x = t_n$ to get

$$|E_k[t](t_n)| \le Ct_n^{\mu-1}k^2 \quad \text{for } n \ge 1. \tag{48}$$

To bound the second right-hand side term of (46), we apply (47) to obtain

$$|E_k[t](\tau)| \le C\tau^{\mu-1}k^2 \quad \text{for } k \le \tau \le t_n, \tag{49}$$

while for $0 \le \tau < k$, we follow the definition to obtain

$$|E_k[t](\tau)| = \left|\omega_0(k)\tau - \int_0^{\tau} b(\tau-\vartheta)\vartheta\, d\vartheta\right| \le \frac{2b_0}{3}k^2 + \frac{b_0}{2}k^2 = \frac{7b_0}{6}k^2. \tag{50}$$

Therefore, (49) and (50) provide

$$\begin{aligned} |(E_k[t]*r'')(t_n)| &\le \int_0^k |E_k[t](\tau)||r''(t_n-\tau)|\, d\tau + \int_k^{t_n} |E_k[t](\tau)||r''(t_n-\tau)|\, d\tau \\ &\le \frac{7b_0}{6}k^2 \int_{t_{n-1}}^{t_n} |r''(\vartheta)|\, d\vartheta + Ck^2 \int_0^{t_{n-1}} (t_n-\vartheta)^{\mu-1}|r''(\vartheta)|\, d\vartheta. \end{aligned} \tag{51}$$

To determine μ, we obtain from (7) that $\hat{b}(s) = \frac{b_0}{s} - \frac{1}{s}\int_0^\infty \beta(t)e^{-st}\, dt$, which implies $|\hat{b}(s)| \le C|s|^{-1}$ such that $\mu = 1$ in Assumption A. We thus invoke $\mu = 1$ in (48) and (51) to complete the proof. □

Next, we shall derive the error estimate of the scheme (19)–(20).

Lemma 4.3. *The following estimate holds for* $1 \le n \le N+1$:

$$\|u^n - U^n\| + \|\delta_t(u^n - U^n)\| \le C\bigg(\|[R_t]^0\| + k\,\|[R_t]^0\|_1 + k\sum_{n=1}^{N}\|[R_t]^n\|\bigg).$$

Proof. Define $\eta^n = u^n - U^n$ and we subtract (17)–(18) from (19)–(20) to get the error equations

$$\delta_t^{(2)}\eta^n + (1-b_0)A\tilde{\eta}^n + \sum_{p=0}^{n}\omega_p(k)A\bar{\eta}^{n-p} = [R_t]^n, \quad 1 \le n \le N, \tag{52}$$

$$\delta_t\eta^1 = [R_t]^0, \quad \eta^0 = 0. \tag{53}$$

We take the inner product of (52) with $\bar{\eta}^n$ and use (22)–(23) to get

$$\frac{1}{2k^3}(\|\eta^{n+1}\|^2 - \|\eta^{n-1}\|^2 - 2\langle\eta^n, \eta^{n+1}\rangle + 2\langle\eta^{n-1}, \eta^n\rangle) + \frac{1-b_0}{8k}(\|\eta^{n+1}\|_1^2 - \|\eta^{n-1}\|_1^2) + \frac{1-b_0}{8k}(2\langle A^{1/2}\eta^n, A^{1/2}\eta^{n+1}\rangle - 2\langle A^{1/2}\eta^{n-1}, A^{1/2}\eta^n\rangle) + \sum_{p=0}^{n}\omega_p(k)\langle A^{1/2}\bar{\eta}^{n-p}, A^{1/2}\bar{\eta}^n\rangle = \langle [R_t]^n, \bar{\eta}^n\rangle. \tag{54}$$

With the definition of $\|\cdot\|_A$, we sum (54) for n from 1 to M to get

$$\|\eta^M\|_A^2 = \|\eta^0\|_A^2 - 2k^3\sum_{n=0}^{M}\sum_{p=0}^{n}\omega_p(k)(A^{1/2}\bar{\eta}^{n-p}, A^{1/2}\bar{\eta}^n) + 2k^3\sum_{n=1}^{M}([R_t]^n, \bar{\eta}^n),$$

based on which we follow the analysis of (26) to find

$$\|\eta^M\|_A^2 \le \|\eta^0\|_A^2 + k^2\sum_{n=1}^{M}\|[R_t]^n\|\,(\|\eta^n\|_A + \|\eta^{n-1}\|_A). \tag{55}$$

Let L be such that $\|\eta^L\|_A := \max_{0\le n\le N}\|\eta^n\|_A$, and (55) provides

$$\|\eta^L\|_A \le \|\eta^0\|_A + 2k^2\sum_{n=1}^{L}\|[R_t]^n\| \le \|\eta^0\|_A + 2k^2\sum_{n=1}^{N}\|[R_t]^n\|. \tag{56}$$

Furthermore, we apply (25) and (53) to find

$$\|\eta^0\|_A \le \|\eta^1\| + \frac{\sqrt{1-b_0}}{2}k\,\|\eta^1\|_1 = k\,\|[R_t]^0\| + \frac{\sqrt{1-b_0}}{2}k^2\,\|[R_t]^0\|_1. \tag{57}$$

Combining (56) and (57) we obtain

$$\|\eta^M\|_A \le k\,\|[R_t]^0\| + \frac{\sqrt{1-b_0}}{2}k^2\,\|[R_t]^0\|_1 + 2k^2\sum_{n=1}^{N}\|[R_t]^n\|. \tag{58}$$

Analogous to the analysis of (35)–(37), we have

$$\|\eta^{M+1}\| + \|\delta_t \eta^{M+1}\| \le C\Big(\|[R_t]^0\| + \frac{\sqrt{1-b_0}}{2} k\, \|[R_t]^0\|_1 + 2k \sum_{n=1}^{N} \|[R_t]^n\| \Big), \tag{59}$$

which completes the proof. □

Based on Lemma 4.3, we intend to prove the convergence order based on reasonable regularity assumptions of the solutions to problem (1)–(2).

Theorem 4.4. *Under the regularity assumptions in (45), for $1 \le n \le N+1$, it holds that*

$$\|u^n - U^n\| + \|\delta_t (u^n - U^n)\| \le C\Big(k^2 \int_0^T \beta(t)\, dt\Big) + C\Big(\Big(k \int_0^{2k} \beta(t)\, dt\Big) + \Big(k^2 \int_k^T |\beta'(t)|\, dt\Big)\Big).$$

Remark 4.5. *It is worth mentioning that for smooth kernels β such as (6), this theorem implies the second-order accuracy*

$$\|u^n - U^n\| + \|\delta_t (u^n - U^n)\| \le Ck^2,$$

while for nonsmooth kernels, the accuracy may be deteriorated. For instance, for the weak singular kernel (5), the above theorem implies the accuracy of $1+\alpha$ order

$$\|u^n - U^n\| + \|\delta_t (u^n - U^n)\| \le Ck^2 + Ck \int_0^{2k} t^{\alpha-1}\, dt + Ck^2 \int_k^T t^{\alpha-2}\, dt \le Ck^{1+\alpha}.$$

Proof. At first, (45) and (18) lead to

$$\|[R_t]^0\| + \frac{\sqrt{1-b_0}}{2} k\, \|[R_t]^0\|_1 \le C\Big(k \int_0^k \beta(t)\, dt + k^2 \int_0^k \beta(t)\, dt\Big). \tag{60}$$

Then we apply (11) and (45) to find that

$$\begin{aligned} k \sum_{n=1}^{N} \|[R_{t,1}]^n\| &= k\, \|[R_{t,1}]^1\| + k \sum_{n=2}^{N} \|[R_{t,1}]^n\| \\ &\le C\Big(k \int_0^{2k} \beta(t)\, dt + k^2 \int_k^T |\beta'(t)|\, dt\Big), \end{aligned} \tag{61}$$

and we use (12) and (45) to obtain

$$k \sum_{n=1}^{N} \|[R_{t,2}]^n\| \le Ck^2.$$

It remains to estimate $[R_{t,3}]^n$ in (16). First, Lemma 4.2 and (45) imply

$$\begin{aligned} k \sum_{n=1}^{N} \|[R_{t,3,1}]^n\| &\le C(T) k^2\, \|Au''(0)\| + C(T) k^2 \int_0^T \|Au'''(t)\|\, dt \\ &\le C(T)\Big(1 + \int_0^T \beta(t)\, dt\Big) k^2. \end{aligned} \tag{62}$$

We then combine the Taylor expansion with the integral remainder

$$A(u'-\bar{u})(t_n)=\frac{-1}{4k}\left(\int_{t_n}^{t_{n+1}}(t_{n+1}-t)^2Au'''(t)\,dt+\int_{t_{n-1}}^{t_n}(t-t_{n-1})^2Au'''(t)\,dt\right)$$

with (45) to obtain

$$\begin{aligned}(63)\qquad k\sum_{n=1}^{N}\|[R_{t,3,2}]^n\| &\le k\sum_{n=1}^{N}\sum_{p=1}^{n}|\omega_{n-p}(k)|\,\|A(u'-\bar{u})(t_p)\|\\ &\le\left(k\sum_{n=1}^{N}\max_{0\le p\le n-1}|\omega_p(k)|\right)\frac{k}{2}\int_0^T\|Au'''(t)\|\,dt\\ &\le Ck^2\left(\sum_{n=1}^{N}\max_{0\le p\le n-1}|\omega_p(k)|\right)\int_0^T\beta(t)\,dt.\end{aligned}$$

Following from Assumption A and [15, (4.2) and Theorem 4.1] that

$$(64)\qquad |\omega_0(k)|\le Ck^{\mu},\qquad |\omega_n(k)|\le Ck(t_n)^{\mu-1},\quad 1\le n\le N,$$

which implies that the summation on the right-hand side of (63) is bounded. Then we invoke (60)–(64) in (59) to get the desired result. □

5. A nonlinear extension

Based on the discussion of the linear problem, we extend the developed methods and results to numerically study the nonlinear problem (1)

$$(65)\qquad u'(t)+Au(t)-(\beta*Au)(t)=f(t,u(t)),\quad t>0,$$

which satisfies the initial condition (2), and the semilinear source term is Lipschitz continuous with the Lipschitz constant $\mathscr{L}>0$

$$(66)\qquad \|f(t,u)-f(t,v)\|\le\mathscr{L}\|u-v\|.$$

Similar to the analysis of (8), we rewrite (65) as

$$(67)\qquad u''(t)+(1-b_0)Au(t)+b(t)Au_0+A(b*u')(t)=f(t,u(t)),\quad t\ge 0.$$

Then we discretize (67) at $t=t_n$ via (11)–(15) to obtain

$$(68)\qquad \delta_t^{(2)}u^n+(1-b_0)A\tilde{u}^n+b_nAu_0+\sum_{p=0}^{n}\omega_p(k)A\bar{u}^{n-p}+\chi_{n0}(k)Au_1=f(t_n,u^n)+[R_t]^n$$

with $1\le n\le N$, where $[R_t]^n$ is defined by (17). Moreover, (68) subjects to

$$(69)\qquad \delta_t u^1=\left(u_1+\frac{k}{2}u_2^*\right)+[R_t]^0,\quad u^0=u_0,$$

where $u_2^* = -Au_0 + f(0, u_0)$ is obtained by (67). We then drop the truncation errors to get the time-discrete scheme

$$\delta_t^{(2)} U^n + (1-b_0)A\widetilde{U}^n + b_n A u_0 + \sum_{p=0}^{n} \omega_p(k) A\overline{U}^{n-p} + \chi_{n0}(k) A u_1 = f(t_n, U^n), \quad 1 \le n \le N, \tag{70}$$

$$\delta_t U^1 = u_1 + \frac{k}{2} u_2^*, \quad U^0 = u_0. \tag{71}$$

We then prove the stability of the time-discrete scheme (70)–(71).

Theorem 5.1. *Let U^n be the numerical solution of the time-discrete scheme (70)–(71). Then we have*

$$\|U^n\| \le C\Big(\|u_0\|_2 + k^2\|u_0\|_3 + \|u_1\| + k\|u_1\|_2 + k\sum_{n=1}^{N}\|f(t_n, u_0)\|\Big), \quad 1 \le n \le N+1.$$

Proof. We apply the triangle inequality and (66) to arrive at

$$\begin{aligned} \|f(t_n, U^n)\| &\le \|f(t_n, U^n) - f(t_n, U^0)\| + \|f(t_n, U^0)\| \\ &\le \mathscr{L}\|U^n - U^0\| + \|f(t_n, U^0)\| \le \mathscr{L}(\|U^n\| + \|U^0\|) + \|f(t_n, U^0)\|. \end{aligned} \tag{72}$$

We incorporate this with a similar analysis as (27) to get

$$\begin{aligned} \|U^M\|_A^2 \le{}& \|U^0\|_A^2 + 2k^3\sum_{n=1}^{M} b_n\|Au_0\|\,\|\overline{U}^n\| + 2k^2\omega_0(k)\|Au_1\|\,\|U^1\|_A \\ &+ 2k^3\sum_{n=1}^{M}|\chi_{n0}(k)|\,\|Au_1\|\,\|\overline{U}^n\| + 2k^3\sum_{n=1}^{M}\|f(t_n, U^n)\|\,\|\overline{U}^n\| \\ \le{}& \|U^0\|_A^2 + k^2\sum_{n=1}^{M} b_n\|u_0\|_2(\|U^{n-1}\|_A + \|U^n\|_A) \\ &+ 2k^2\omega_0(k)\|u_1\|_2\|U^1\|_A + k^2c_N^*\|u_1\|_2(\|U^{n-1}\|_A + \|U^n\|_A) \\ &+ k^2\sum_{n=1}^{M}\big(\mathscr{L}(\|U^n\| + \|U^0\|) + \|f(t_n, U^0)\|\big)(\|U^{n-1}\|_A + \|U^n\|_A). \end{aligned}$$

Let $\|U^J\|_A := \max_{0\le n\le M}\|U^n\|_A$ such that

$$\begin{aligned} \|U^J\|_A \le{}& \|U^0\|_A + 2k^2\sum_{n=1}^{J} b_n\|u_0\|_2 + 2k^2\omega_0(k)\|u_1\|_2 + 2k^2c_N^*\|u_1\|_2 \\ &+ 2k^2\sum_{n=1}^{J}\big(\mathscr{L}(\|U^n\| + \|u_0\|) + \|f(t_n, u_0)\|\big) \\ \le{}& \|U^0\|_A + 2k^2\sum_{n=1}^{M} b_n\|u_0\|_2 + 2k^2\omega_0(k)\|u_1\|_2 + 2k^2c_N^*\|u_1\|_2 \\ &+ 2k^2\sum_{n=1}^{M}\big(\mathscr{L}(\|U^n\| + \|u_0\|) + \|f(t_n, u_0)\|\big), \end{aligned}$$

and we incorporate this with (30) and (31) to further get

$$\begin{aligned}(73)\qquad \|U^M\|_A \le k\|u_1\| + \frac{k^2}{2}\|u_0\|_2 + \frac{\sqrt{1-b_0}}{2}k\Big(2\|u_0\|_1 + k\|u_1\|_1 + \frac{k^2}{2}\|u_0\|_3\Big)\\ + 2kTb_0\|u_0\|_2 + 2k^2\frac{2b_0}{3}k\|u_1\|_2 + 2k^2c_N^*\|u_1\|_2\\ + 2k^2\sum_{n=1}^{N}(\mathscr{L}\|u_0\| + \|f(t_n,u_0)\|) + 2\mathscr{L}k^2\sum_{n=1}^{N}\|U^n\|\\ := k\Big(\Phi_1(u_0,u_1) + 2\mathscr{L}k\sum_{n=1}^{M}\|U^n\|\Big).\end{aligned}$$

By similar analysis as (35)–(37), (73) yields

$$(74)\qquad \|U^{M+1}\| \le \Big(\frac{k}{2} + \frac{1}{c'\sqrt{1-b_0}}\Big)\Big(\Phi_1(u_0,u_1) + 2\mathscr{L}k\sum_{n=1}^{M}\|U^n\|\Big).$$

Then we apply the discrete Grönwall inequality to complete the proof. □

We then derive the error estimate of the time-discrete scheme (70)–(71).

Lemma 5.2. *Let u^n and U^n satisfy (68)–(69) and the time-discrete scheme (70)–(71), respectively. Then for $1 \le n \le N+1$*

$$\|u^n - U^n\| + \|\delta_t(u^n - U^n)\| \le C(T)\Big(\|[R_t]^0\| + k\|[R_t]^0\|_1 + k\sum_{m=1}^{n}\|[R_t]^m\|\Big).$$

Proof. Based on (68)–(71), we obtain the error equations in terms of $\eta^n = u^n - U^n$

$$(75)\qquad \delta_t^{(2)}\eta^n + (1-b_0)A\tilde{\eta}^n + \sum_{p=0}^{n}\omega_p(k)A\bar{\eta}^{n-p} = f(t_n,u^n) - f(t_n,U^n) + [R_t]^n, \quad n \ge 1,$$

$$(76)\qquad \delta_t\eta^1 = [R_t]^0, \quad \eta^0 = 0.$$

Analogous to the proof of Lemma 4.3, we obtain

$$\|\eta^L\|_A \le \|\eta^0\|_A + 2k^2\sum_{n=1}^{N}\|[R_t]^n\| + 2k^2\sum_{n=1}^{M}\|f(t_n,u^n) - f(t_n,U^n)\|,$$

where $\|\eta^L\|_A := \max_{0\le n\le M}\|\eta^n\|_A$. Then, we use (57) and (66) to get

$$\begin{aligned}(77)\qquad \|\eta^M\|_A \le k\|[R_t]^0\| + \frac{\sqrt{1-b_0}}{2}k^2\|[R_t]^0\|_1 + 2k^2\sum_{n=1}^{N}\|[R_t]^n\|\\ + 2\mathscr{L}k^2\sum_{n=1}^{M}\|\eta^n\| := k\Big(\Phi_2(u_0,u_1) + 2\mathscr{L}k\sum_{n=1}^{M}\|\eta^n\|\Big),\end{aligned}$$

which in turn implies

$$\|\eta^{M+1}\| \le C\bigg(\Phi_2(u_0, u_1) + 2\mathscr{L}k\sum_{n=1}^{M}\|\eta^n\|\bigg).$$

We incorporate this with the discrete Grönwall inequality to obtain

$$\max_{1\le n\le N+1}\|\eta^n\| \le C(T)\Phi_2(u_0, u_1), \tag{78}$$

and we invoke (78) in the right-hand side of (77) and adopt similar analysis as (34)–(37) to get

$$\|\eta^{M+1}\| + \|\delta_t\eta^{M+1}\| \le C(T)\Phi_2(u_0, u_1).$$

The proof is thus completed. □

Finally, we combine the analysis in Theorem 4.4 and the conclusion of Lemma 5.2 to obtain the following convergence result.

Theorem 5.3. *Let u^n and U^n satisfy (68)–(69) and the time-discrete scheme (70)–(71), respectively. Then under the regularity assumptions in (45), it holds for $1 \le n \le N+1$*

$$\|u^n - U^n\| + \|\delta_t(u^n - U^n)\| \le C(T)\bigg(k^2\int_0^T \beta(t)\,dt\bigg) + C(T)\bigg(\bigg(k\int_0^{2k}\beta(t)\,dt\bigg) + \bigg(k^2\int_k^T|\beta'(t)|\,dt\bigg)\bigg).$$

6. Numerical experiments

We perform numerical examples to substantiate the analysis of the time-discrete schemes. We consider a concrete problem of the form (1) or (65) in one space dimension with the spatial domain $\Omega = (0, 1)$ and the operator $A = -d^2/dx^2$ with boundary conditions $u(t, 0) = u(t, 1) = 0$ for $t \in (0, T]$. We apply the second-order center difference for spatial discretization with a uniform mesh size $h = 1/M$ for some $M > 0$, and we define the discrete L^2 norm for the finite difference method as in [26]:

$$\|U^n\| = \sqrt{h\sum_{j=1}^{M-1}|U_j^n|^2}.$$

Let the time step size $k = T/(N+1)$ with $N \ge 1$ and $t_{N+1} = T$. To illustrate the convergence of proposed schemes, we define the spatial error in L_2 norm at $t_{N+1} = T$ and the corresponding temporal convergence order as

$$E_2(N+1) = \|U^{N+1} - U^{2(N+1)}\|, \quad \text{Rate} = \log_2\frac{E_2(N+1)}{E_2(2(N+1))}.$$

We shall consider smooth and nonsmooth kernels $\beta(t)$ in the following examples.

A weak singular kernel. We consider the weak singular kernel (5) with $\gamma_0 = \rho^\alpha/2$. By (7), we have

$$b(t) = \frac{\Gamma(\alpha, \rho t)}{2}, \quad \hat{b}(s) = \frac{b_0}{s} - \frac{(1+s/\rho)^{-\alpha}}{2s},$$

	$\alpha = 0.25$		$\alpha = 0.5$		$\alpha = 0.95$	
N	$E_2(N+1)$	rate	$E_2(N+1)$	rate	$E_2(N+1)$	rate
32	$1.1027 \cdot 10^{-3}$	–	$1.2056 \cdot 10^{-3}$	–	$1.0619 \cdot 10^{-3}$	–
64	$4.5125 \cdot 10^{-4}$	1.289	$3.9058 \cdot 10^{-4}$	1.626	$2.7041 \cdot 10^{-4}$	1.973
128	$1.9591 \cdot 10^{-4}$	1.204	$1.3180 \cdot 10^{-4}$	1.567	$6.8386 \cdot 10^{-5}$	1.983
256	$8.5399 \cdot 10^{-5}$	1.198	$4.5700 \cdot 10^{-5}$	1.528	$1.7244 \cdot 10^{-5}$	1.988
512	$3.6869 \cdot 10^{-5}$	1.212	$1.6069 \cdot 10^{-5}$	1.508	$4.3421 \cdot 10^{-6}$	1.990

Table 1. Example 6.1: L_2 errors and temporal convergence rates with different α.

where the upper incomplete gamma function

$$\Gamma(\alpha, y) := \frac{1}{\Gamma(\alpha)} \int_y^\infty t^{\alpha-1} e^{-t}\, dt$$

with $\Gamma(\alpha, 0) = 1$. Then we provide the approach to compute the quadrature weights. By (14), we have the representation [23]

$$\omega_n(k) = \frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} z^{-n-1} \hat{b}\left(\frac{\zeta(z)}{k}\right) dz = \Re\left(\frac{1}{\pi} \int_0^\pi G_n(y)\, dy\right), \tag{79}$$

in which $\mathrm{i}^2 = -1$, $\Re$ indicates the real part of a complex number and $G_n(y) = e^{\mathrm{i}ny}\, \hat{b}\big(\frac{\zeta(e^{-\mathrm{i}y})}{k}\big)$. In subsequent numerical implementations, we apply the composite rectangle formula to approximate the last integral of (79). Specifically, given $\mathcal{J} = N^2$, the quadrature weights $\omega_n(k)$ are generated by

$$\omega_n(k) \approx \Re\left(\frac{1}{\pi}\left(\sum_{j=0}^{\mathcal{J}-1} G_n\left(\frac{y_j + y_{j+1}}{2}\right)\right)\Delta y\right), \tag{80}$$

where $y_j = j\Delta y$ with $\Delta y = \pi/\mathcal{J}$ and $j = 0, 1, 2, \ldots, \mathcal{J}$.

Example 6.1 (the linear case). Let the initial conditions $u_0(x) = \sin(\pi x)$ and $u_1(x) = \sin(2\pi x)$ in (1), $h = \frac{1}{128}$, $T = 1$ and $\rho = 5$. We list the L_2 errors and temporal convergence rates in Table 1, from which we observe that the convergence rate is approximately $1 + \alpha$ that is consistent with the estimates in Theorem 4.4.

To demonstrate the finite-time and long-time stability proved in Section 3, we present $\max_{0 \le n \le N+1} \|U^n\|$ under $h = \frac{1}{128}$, $k = 0.1$, $u_0 = \sin(\pi x)$ and different u_1 and α in Table 2, which shows that the numerical solution may diverge with the increment of N (and thus $T = Nk$) if $u_1 \neq 0$. When $u_1 = 0$, the numerical solution is stable for large N, which indicates the long-time stability of the numerical solution and thus validates the theorems in Section 3.

Example 6.2 (the nonlinear case). Let the initial conditions $u_0(x) = \sin(\pi x)$ and $u_1(x) = \sin(2\pi x)$ with $f(t, u) = u - u^3$ in (65). We set $h = \frac{1}{128}$, $T = 1$ and $\rho = 5$, and list L_2 errors and temporal convergence rates in Table 3, which indicates the $1 + \alpha$ accuracy as predicted in Theorem 5.3.

	$\alpha=0.1$		$\alpha=0.5$		$\alpha=0.9$	
N	$u_1=\sin(2\pi x)$	$u_1=0$	$u_1=\sin(2\pi x)$	$u_1=0$	$u_1=\sin(2\pi x)$	$u_1=0$
2	$7.0711\cdot10^{-1}$	$7.0711\cdot10^{-1}$	$7.0711\cdot10^{-1}$	$7.0711\cdot10^{-1}$	$7.0711\cdot10^{-1}$	$7.0711\cdot10^{-1}$
4	$7.0711\cdot10^{-1}$	$7.0711\cdot10^{-1}$	$7.0711\cdot10^{-1}$	$7.0711\cdot10^{-1}$	$7.0711\cdot10^{-1}$	$7.0711\cdot10^{-1}$
8	$1.2710\cdot10^{0}$	$1.0965\cdot10^{0}$	$7.0711\cdot10^{-1}$	$7.0711\cdot10^{-1}$	$7.0711\cdot10^{-1}$	$7.0711\cdot10^{-1}$
16	$1.8653\cdot10^{0}$	$1.7085\cdot10^{0}$	$8.4259\cdot10^{-1}$	$8.2922\cdot10^{-1}$	$7.0711\cdot10^{-1}$	$7.0711\cdot10^{-1}$
32	$1.9507\cdot10^{0}$	$1.7085\cdot10^{0}$	$9.3824\cdot10^{-1}$	$8.2922\cdot10^{-1}$	$7.0711\cdot10^{-1}$	$7.0711\cdot10^{-1}$
64	$4.1844\cdot10^{0}$	$1.7085\cdot10^{0}$	$1.7257\cdot10^{0}$	$8.2922\cdot10^{-1}$	$7.0711\cdot10^{-1}$	$7.0711\cdot10^{-1}$
128	$8.0698\cdot10^{0}$	$1.7085\cdot10^{0}$	$3.6807\cdot10^{0}$	$8.2922\cdot10^{-1}$	$7.0711\cdot10^{-1}$	$7.0711\cdot10^{-1}$
256	$1.6150\cdot10^{1}$	$1.7085\cdot10^{0}$	$7.6179\cdot10^{0}$	$8.2922\cdot10^{-1}$	$1.0862\cdot10^{0}$	$7.0711\cdot10^{-1}$
512	$3.2341\cdot10^{1}$	$1.7085\cdot10^{0}$	$1.5504\cdot10^{1}$	$8.2922\cdot10^{-1}$	$2.2483\cdot10^{0}$	$7.0711\cdot10^{-1}$
1024	$6.4737\cdot10^{1}$	$1.7085\cdot10^{0}$	$3.1280\cdot10^{1}$	$8.2922\cdot10^{-1}$	$4.5731\cdot10^{0}$	$7.0711\cdot10^{-1}$

Table 2. Example 6.1: values of $\max_{0\le n\le N+1}\|U^n\|$ with different α and derivative initial conditions u_1.

A smooth kernel. We choose the smooth kernel (6) and shall give $\hat{\beta}(s)$ by means of Laplace and Stieltjes transforms. Define the piecewise continuous function

$$\alpha_0(x)=\begin{cases}0, & x=0,\\ x, & 0<x\le 1,\\ 1, & 1<x<\infty,\end{cases}$$

and let

$$\beta_0(t):=\int_0^\infty e^{-xt}\,d\alpha_0(x)$$

such that

$$\beta_0(t)=\int_0^1 e^{-xt}\,dx=\frac{1-e^{-t}}{t}.$$

From [22, Chapter 8], we have

$$\widehat{\beta_0}(s)=\int_0^\infty\frac{d\alpha_0(x)}{s+x}=\log\left(1+\frac{1}{s}\right).$$

	$\alpha=0.1$		$\alpha=0.5$		$\alpha=0.9$	
N	$E_2(N+1)$	rate	$E_2(N+1)$	rate	$E_2(N+1)$	rate
64	$3.4089\cdot10^{-4}$	–	$4.1244\cdot10^{-4}$	–	$2.7405\cdot10^{-4}$	–
128	$1.4278\cdot10^{-4}$	1.255	$1.4108\cdot10^{-4}$	1.548	$7.0160\cdot10^{-5}$	1.966
256	$6.6698\cdot10^{-5}$	1.098	$4.9327\cdot10^{-5}$	1.516	$1.7919\cdot10^{-5}$	1.969
512	$3.1762\cdot10^{-5}$	1.070	$1.7425\cdot10^{-5}$	1.501	$4.5745\cdot10^{-6}$	1.970
1024	$1.5061\cdot10^{-5}$	1.076	$6.1791\cdot10^{-6}$	1.496	$1.1686\cdot10^{-6}$	1.969

Table 3. Example 6.2: L_2 errors and temporal convergence rates with different α.

	$T=0.2$		$T=1$		$T=4$	
N	$E_2(N+1)$	rate	$E_2(N+1)$	rate	$E_2(N+1)$	rate
32	$3.9702\cdot 10^{-5}$	–	$5.8641\cdot 10^{-3}$	–	$6.2497\cdot 10^{-2}$	–
64	$1.0717\cdot 10^{-5}$	1.889	$1.6584\cdot 10^{-3}$	1.822	$8.8076\cdot 10^{-3}$	2.827
128	$2.7897\cdot 10^{-6}$	1.942	$4.3955\cdot 10^{-4}$	1.916	$1.1642\cdot 10^{-3}$	2.919
256	$7.1196\cdot 10^{-7}$	1.970	$1.1303\cdot 10^{-4}$	1.959	$4.8802\cdot 10^{-4}$	1.254
512	$1.7985\cdot 10^{-7}$	1.985	$2.8651\cdot 10^{-5}$	1.980	$1.3746\cdot 10^{-4}$	1.828

Table 4. Example 6.3: L_2 errors and temporal convergence rates under different T.

Thus, (6) gives $\beta(t)=e^{-t}\beta_0(t)$, which leads to $\hat\beta(s)=\widehat{\beta_0}(s+1)$. Note that

$$\int_0^\infty \beta(t)\,dt=\hat\beta(0)=\log 2<1,$$

which implies that (6) satisfies (4). Furthermore, we apply (7) to obtain

$$\begin{aligned}b(t)=\int_t^\infty \beta(s)\,ds&=\int_0^\infty \beta(s)\,ds-\int_0^t \beta(s)\,ds=\hat\beta(0)-\int_0^t \beta(s)\,ds\\&=\hat\beta(0)-(\beta*1)(t)=\log(2)-(\beta*1)(t),\end{aligned}$$

which yields

$$\hat b(s)=\frac{b_0}{s}-\frac{1}{s}\hat\beta(s)=\frac{\log 2}{s}-\frac{1}{s}\log\Big(1+\frac{1}{s+1}\Big)=\frac{1}{s}\log\frac{2(s+1)}{s+2}. \tag{81}$$

Inserting (81) into (79) we obtain the weights $\omega_n(k)$ by the approximate method (80).

Example 6.3 (the linear case). Let $u_0(x)=\sin(\pi x)$ and $u_1(x)=\sin(2\pi x)$ for model (1), and we set $h=\frac{1}{128}$. In Table 4, we test L_2 errors and temporal convergence rates, which indicate that the proposed

	$k=0.01$		$k=0.1$		$k=1$	
N	$u_1=\sin(2\pi x)$	$u_1=0$	$u_1=\sin(2\pi x)$	$u_1=0$	$u_1=\sin(2\pi x)$	$u_1=0$
2	$7.0711\cdot 10^{-1}$	$7.0711\cdot 10^{-1}$	$7.0711\cdot 10^{-1}$	$7.0711\cdot 10^{-1}$	$2.8706\cdot 10^{0}$	2.7822
4	$7.0711\cdot 10^{-1}$	$7.0711\cdot 10^{-1}$	$7.0711\cdot 10^{-1}$	$7.0711\cdot 10^{-1}$	$3.7904\cdot 10^{0}$	2.7822
8	$7.0711\cdot 10^{-1}$	$7.0711\cdot 10^{-1}$	$7.0711\cdot 10^{-1}$	$7.0711\cdot 10^{-1}$	$9.5557\cdot 10^{0}$	2.7822
16	$7.0711\cdot 10^{-1}$	$7.0711\cdot 10^{-1}$	$7.0711\cdot 10^{-1}$	$7.0711\cdot 10^{-1}$	$2.2250\cdot 10^{1}$	2.7822
32	$7.0711\cdot 10^{-1}$	$7.0711\cdot 10^{-1}$	$1.5768\cdot 10^{0}$	$7.0711\cdot 10^{-1}$	$4.7803\cdot 10^{1}$	2.7822
64	$7.0711\cdot 10^{-1}$	$7.0711\cdot 10^{-1}$	$5.7869\cdot 10^{0}$	$7.0711\cdot 10^{-1}$	$9.8916\cdot 10^{1}$	2.7822
128	$7.0711\cdot 10^{-1}$	$7.0711\cdot 10^{-1}$	$1.5719\cdot 10^{1}$	$7.0711\cdot 10^{-1}$	$2.0114\cdot 10^{2}$	2.7822
256	$8.2968\cdot 10^{-1}$	$7.0711\cdot 10^{-1}$	$3.6143\cdot 10^{1}$	$7.0711\cdot 10^{-1}$	$4.0559\cdot 10^{2}$	2.7822
512	$3.8432\cdot 10^{0}$	$7.0711\cdot 10^{-1}$	$7.7033\cdot 10^{1}$	$7.0711\cdot 10^{-1}$	$8.1450\cdot 10^{2}$	2.7822
1024	$1.1529\cdot 10^{1}$	$7.0711\cdot 10^{-1}$	$1.5881\cdot 10^{2}$	$7.0711\cdot 10^{-1}$	$1.6323\cdot 10^{3}$	2.7822

Table 5. Example 6.3: values of $\max_{0\le n\le N+1}\|U^n\|$ under different k and derivative initial conditions u_1.

	$f(t,u)=u-u^3$		$f(t,u)=\sin u$		$f(t,u)=e^{-u}\cos u$	
N	$E_2(N+1)$	rate	$E_2(N+1)$	rate	$E_2(N+1)$	rate
32	$5.6500\cdot 10^{-3}$	–	$5.2910\cdot 10^{-3}$	–	$5.8688\cdot 10^{-3}$	–
64	$1.5964\cdot 10^{-3}$	1.823	$1.4956\cdot 10^{-3}$	1.823	$1.6624\cdot 10^{-3}$	1.820
128	$4.2317\cdot 10^{-4}$	1.916	$3.9640\cdot 10^{-4}$	1.916	$4.4184\cdot 10^{-4}$	1.912
256	$1.0883\cdot 10^{-4}$	1.959	$1.0194\cdot 10^{-4}$	1.959	$1.1433\cdot 10^{-4}$	1.950
512	$2.7589\cdot 10^{-5}$	1.980	$2.5839\cdot 10^{-5}$	1.980	$2.9353\cdot 10^{-5}$	1.962

Table 6. Example 6.4: L_2 errors and temporal convergence rates under different nonlinear terms.

scheme could achieve the second-order temporal accuracy for T not large enough. For large T, the convergence order is not stable, which may be caused by the loss of long-time stability of numerical solutions as we will show in Table 5.

In Table 5 we compute $\max_{0\le n\le N+1}\|U^n\|$ under $h=\frac{1}{128}$, $u_0=\sin(\pi x)$ and different u_1 and k, from which we observe that when $u_1\neq 0$, the numerical solution exhibits instability with the increment of N. When the derivative initial condition $u_1=0$, the numerical solution is stable for large N, which is consistent with Theorem 3.2.

Example 6.4 (the nonlinear case). Let the initial conditions $u_0(x)=\sin(\pi x)$ and $u_1(x)=\sin(2\pi x)$ in (65) with $h=\frac{1}{128}$, $T=1$ and different source term $f(t,u)$. We present L_2 errors and temporal convergence rates of the scheme (70)–(71) in Table 6, which indicates its second-order temporal accuracy proved in Theorem 5.3.

7. Concluding remarks

We investigated the numerical approximation for a nonlinear hyperbolic-type partial integrodifferential equation. For the linear case of this equation, we discretized it by the central difference formula for space and the second-order convolution quadrature for time, where smooth and nonsmooth memory kernels were considered. The stability and convergence were deduced by means of the energy argument. Then we extended the theoretical results to the corresponding nonlinear problem. Numerical experiments support the theoretical findings.

There are several places in this work that could be improved. For instance, in numerical experiments the composite rectangle formula is used to calculate the weights $\omega_n(k)$ for simplicity. Indeed, the fast Fourier transform method is a more efficient technique to obtain weights, and we will adopt this to develop the fast solution method in the future work. The proof of the regularity of the solutions is not straightforward, and we will investigate this challenging issue in the near future.

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