TRACE FORMULAS REVISITED AND A NEW REPRESENTATION OF KDV SOLUTIONS WITH SHORT-RANGE INITIAL DATA

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Abstract. We put forward a new approach to Deift-Trubowitz type trace formulas for the 1D Schrodinger operator with potentials that are summable with the rst moment (short-range potentials). We prove that these formulas are preserved under the KdV ow whereas the class of short-range potentials is not. Finally, we show that our formulas are well-suited to study the dispersive smoothing e⁄ect.

We dedicate this paper to Vladimir Marchenko on the occasion of his centennial birthday. This paper is also dedicated to the memory of Vladimir Zakharov who has recently left us.

1. Introduction

We are concerned with the Cauchy problem for the Korteweg-de Vries (KdV) equation

$$\begin{cases} @_t q \; 6q @_x q + @_x^3 q = 0; \; x \; 2 \; \mathrm{R}; t \; 0 \\ q(x;0) = q(x): \end{cases} \tag{1.1}$$

As is well-known, (1.1) is the rst nonlinear evolution PDE solved in the seminal 1967 Gardner-Greene-Kruskal-Miura paper [7] by the method which is now referred to as the inverse scattering transform (IST). Conceptually, the IST is similar to the Fourier method but is based on the direct/inverse scattering (spectral) theory for the 1D Schr dinger operator $\mathrm{L}_q = @_x^2 + q(x)$. Explicit formulas, however, are in short supply and trace formulas are among a few available. Historically, for short-range potentials $q(x)$ (i.e. summable with the rst moment) such a formula (see (5.7)) was put forward by Deift-Trubowitz in [5] in the late 70s (we call it the Deift-Trubowitz trace formula). However, no adaptation of the trace formula (5.7) to the solution $q(x;t)$ to (1.1) is o⁄ered in [5] and, to the best of our knowledge, it has not been done in the literature. The main goal of our contribution is to address this problem.

To this end, we rst put forward an elementary approach to generate trace formulas for the Schr dinger operator L_q with a decaying (but not necessarily shortrange) potential q. More precisely, we start out with considering potentials $q \; 2 \; L^1(\mathrm{R})$ such that the right Jost solution $(x;k)$ of $\mathrm{L}_q = k^2$ satis es the condition:

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for all real x

$$2\mathrm{i}k\left(e^{-\mathrm{i}kx}\psi(x,k)-1\right)+\int_x^\infty q\left(s\right)\mathrm{d}s\in H^2,$$

where H^2 is the usual Hardy space consisting of analytic functions on the upper half plane with L^2 non-tangential boundary values on the real line. We show that for any > 0 and almost every x

$$q\,(x) = \partial_x \int_{R+\mathrm{i}} \mathrm{Re}\,\frac{1\ e^{\mathrm{i}kx}\,(x;k)\ 2k\mathrm{d}k}{k}, \qquad (1.2)$$

the integral being absolutely convergent. (see Theorem 5.1 for the complete statement.)

While the proof, based on Hardy space arguments, is totally elementary, the formula (1.2) is surprisingly convenient. First of all, if q is short range then (1.2) readily recovers the Deift-Trubowitz trace formula (5.7) (see the Appendix). For this reason we call (1.2) a Deift-Trubowitz-type trace formula.

The main advantage of (1.2) is that it is particularly convenient in the KdV context. We rst show that under additional assumptions on q it admits various derivations (5.4), (5.8), (5.12) that serve di⁄erent purposes. In particular, (5.12) remains valid for $q\,(x;t)$ (see section 6). The problem with the original DeiftTrubowitz trace formula (5.7) is that, as we show below (see Corollary 7.3), $q\,(x;t)$ need not remain short-range for $t > 0$ and therefore the approach of [5], where (5.7) is derived, breaks down in a serious way. We emphasize that we actually demonstrate as a corollary that (5.7) does hold for $q\,(x;t);t\ 0$. This also appears to be a new result.

It should be noticed that (5.8) is well-suited for subtle analysis of the gain of regularity (aka dispersive smoothing) phenomenon for the KdV equation (section 7). We study this phenomenon in [16], [17], [18] where we rely on the Dyson formula (aka the second log determinant formula) and the theory of Hankel operators for extension of the IST to initial data $q\,(x)$ that is essentially arbitrary at 1 (but still short-range at +1). Comparing with the Dyson formula considerations, our trace approach (which also crucially uses Hankel operators) is more robust for analysis of KdV solutions (see Remark 7.2). To the best of our knowledge Theorem 7.1 is new.

Note that for periodic potentials the trace formula was studied in great detail the 70s by McKean-Moerbeke [25], Trubowitz [32] and many others (see e.g. [9] for a nice historic review) before (5.7). It was generalized by Craig in [4] in the late 80s to arbitrary bounded continuous potentials (the so-called Craig s trace formula). In the 90s Gesztesy et al [10] - [14] developed a general approach to Craig type trace formulas based on the Krein trace formula (the "true" trace formula) under the only condition of essential boundedness from below. The general trace formulas studied in [10] - [14] yield previously known ones. In the 2000s we [28] introduced a new way of generating trace-type formulas that is not based upon Krein s trace formula but rests on the Titchmarsh-Weyl theory for second order di⁄erential equations and asymptotics of the Titchmarsh-Weyl m-function. The approach is quite elementary and essentially free of any conditions. Recently, in Binder et al [1] Craig s trace formulas was used in the KdV context to address some open problems related to almost periodic initial data.

The paper is organized as follows. In section 2 we introduce our notations Section 3 is devoted to basics of Hardy spaces and Hankel operator our approach is based upon. In section 4 we review the classical direct/inverse scattering theory for Schrodinger operators on the line using the language of Hankel operators. Section 5 is where our trace formulas are introduced. We do not claim their originality but believe that the approach is new. In Section 6 we derive a representation for the solution to the KdV equation with short-range initial data. To the best of our knowledge it is new. In the nal section 7 we demonstrate how our trace formula for the KdV is well-suited for the analysis of dispersive smoothing. The approach builds upon our recent [18] and suggests an e⁄ective way to understanding how the KdV ow trades the decay of initial data for gain of regularity. In Appendix we demonstrate that the Deift-Trubowitz trace formula is actually a "nonlinearization" of ours.

2. Notations

Our notations are quite standard:

Unless otherwise stated, all integrals are Lebesgue and, as is commonly done, we drop limits of integration if the integral (absolutely convergent) is over the whole line. For convergent integrals that are not absolutely convergent we always use the Cauchy principal value

$$(PV)\int = \lim_{a\to\infty}\int_{-a}^{a}.$$

S is the characteristic function of a (measurable) set S.

SAs usual,. If S = LR^{p}then we abbreviate(S); 0 <p 1, is the Lebesgue space on a (measurable) setL_p (R) = L_p. We include L_p in the family

of weighted L^p spaces de ned by

L^p = f j Z jf (x)j^p hxi dx< 1 ; > 0:

where hxi = p1 + x^2. The classL^1_1 is basic to scattering theory for 1D Schr dinger operators (short-range potentials).

k kX =$_X$ Lstands for a norm in a Banach space$_2$. We merely write k k in this case and also$X$. The most common space is

kfk^2 = hf;fi where hf;gi = Z f (x)g ($\overline{x}$)dx:

We write x ' y if x = Cy for some universal constant C; x.$_a$ y if x;y 0 and x C (a)y with a positive C dependent on a. We drop a if C is a universal constant.

We do not distinguish between classical and distributional derivatives. A statement A means two separate statements: A and A_+.

3. Hardy spaces and Hankel operators

To x our notation we review some basics of Hardy spaces and Hankel operators following [26].

A function f analytic in C = fz 2 C : Imz> 0g is in the Hardy space H^p for some 0 <p 1 if

$$\|f\|^p_{H^p_\pm} \stackrel{\text{def}}{=} \sup_{y>0} \|f(\cdot \pm iy)\|_p < \infty.$$

We set $H^p = H^p_+$: It is a fundamental fact of the theory of Hardy spaces that any $f(z)$ 2 H^p with 0 < p 1 has non-tangential boundary values $f(x$ i0) for almost every (a.e.) x 2 R and

kfk$_{H^p}$ = kf (i0)k$_{L^p}$ = kfk$_{L^p}$: (3.1)

Classes H^1 and H^2 will be particularly important. H^1 is the algebra of uniformly bounded in C functions and H^2 is the Hilbert space with the inner product induced from L^2.

It is well-known that $L^2 = H^2_+ \oplus H^2_-$, the orthogonal (Riesz) projection P onto H^2 being given by

$$(\mathbb{P}_\pm f)(x) = \pm\frac{1}{2\pi \mathrm{i}} \lim_{\varepsilon\to 0+} \int \frac{f(s)\mathrm{d}s}{s-(x\pm \mathrm{i}\varepsilon)} =: \pm\frac{1}{2\pi \mathrm{i}} \int \frac{f(s)\ \mathrm{d}s}{s-(x\pm \mathrm{i}0)}. \tag{3.2}$$

Observe that the Riesz projections can also be rewritten in the form

$$(\mathbb{P}_\pm f)(x) = \left(\widetilde{\mathbb{P}}_\pm f\right)(x) \mp \frac{1}{2\pi \mathrm{i}} \int \frac{f(s)}{s+\mathrm{i}} \mathrm{d}s, \tag{3.3}$$

where

$$\left(\widetilde{\mathbb{P}}_\pm f\right)(x) := (x+\mathrm{i}) \left(\mathbb{P}_\pm \frac{f}{\cdot + \mathrm{i}}\right)(x)$$

is well-de ned for any $f \in L^\infty$. This representation is very important in what follows.

If f 2 L^2 then P f is by de nition in H^2 but of course not in L^1. However under a stronger decay condition we have the following statement.

Lemma 3.1. *If* hxi$f(x)$ 2 L^2 *then*

$$(PV) \int \mathbb{P}\ f = \frac{1}{2} \int f. \tag{3.4}$$

Proof. Note rst that if hxi$f(x)$ 2 L^2 then f is of course integrable as one sees from

$$\int |f| = \int |\langle x\rangle f(x)| \frac{\mathrm{d}x}{\langle x\rangle} \le \|\langle\cdot\rangle f\| \left\|\langle\cdot\rangle^{-1}\right\| < \infty.$$

It follows then that for a $a > 0$ nite we have

$$\int_{-a} \mathbb{P}_- f = \left\langle \mathbb{P}_- f, \chi_{|\cdot|\le a} \right\rangle = \left\langle f, \mathbb{P}_- \chi_{|\cdot|\le a} \right\rangle \quad (\text{by } (3.3)) \qquad \mathbb{P} \ \frac{\chi_{|\cdot|\le}}{\cdot} \Big\rangle \quad \pi \quad \chi \quad - \quad \int$$

$$\Big\langle \quad \cdot \quad \frac{\chi \quad a}{\cdot} \Big\rangle + \frac{1}{2\pi \mathrm{i}} \int \frac{\chi_{|\cdot|\le a}}{s - \mathrm{i}} \mathrm{d}s \qquad f f, \mathbb{P}_- \frac{\chi_{|\cdot|\le a}}{\cdot + \mathrm{i}} \Big\rangle + \left(\frac{1}{2} - \frac{1}{\pi} \arctan \frac{1}{a} \right) \int f.$$

= f;(+ i) (3.5)

+ i

= (i) $\overline{\qquad}\Big\rangle \quad \Big(- - -$ $\quad -$

Here we have used

$$\frac{1}{2\pi \mathrm{i}} \int \frac{\chi_{|\cdot|\le a}}{s - \mathrm{i}} \mathrm{d}s = \frac{1}{2\pi \mathrm{i}} \int_{-a}^{a} \frac{\mathrm{d}s}{s - \mathrm{i}} = \frac{1}{2} - \frac{1}{\pi} \arctan \frac{1}{a}$$

$$\to \frac{1}{2}, \quad a \to +\infty. \tag{3.6}$$

Since j j a = 1 j j>a and 1=(x + i) 2 H^2 we have

$$\mathbb{P}_- \frac{\chi_{|\cdot|\le a}}{\cdot + \mathrm{i}} = \mathbb{P}_- \frac{1}{\cdot + \mathrm{i}} - \mathbb{P}_- \frac{\chi_{|\cdot|>a}}{\cdot + \mathrm{i}} = -\mathbb{P}_- \frac{\chi \quad \text{j j>}a}{\cdot \quad + \mathrm{i}}$$

and therefore

$$(\ \mathrm{i})^{f, \mathbb{P}_- \frac{\chi_{|\cdot|\le a}}{\cdot + \mathrm{i}}} \Big\rangle = \underline{\qquad} \quad (\cdot - \mathrm{i}) f, \mathbb{P}_- \frac{\chi_{|\cdot|>a}}{\cdot + \mathrm{i}} :$$

It follows that

$$(\cdot - \mathrm{i}) f, \mathbb{P}_- \frac{\chi_{|\cdot|\le a}}{\cdot + \mathrm{i}} \Big\rangle \Big| \le \|(\cdot - \mathrm{i}) f\| \Big\| \mathbb{P}_- \frac{\chi}{\cdot} \qquad \text{j j>}a \quad + \mathrm{i}$$

$$\le \|\langle \cdot \rangle f\| \left\| \frac{\chi_{|\cdot|>a}}{\cdot + \mathrm{i}} \right\| \to 0, \ a \to \infty :$$

This means that

$$\lim \quad (\ \mathrm{i}) \quad f, \mathbb{P}_- \frac{\chi_{|\cdot|\le a}}{\cdot + \mathrm{i}} \Big\rangle = 0$$

a!1

$a \to \infty$:

and we can pass in (3.5) as $\displaystyle \lim_{a \to \infty} \int_{-a}^{a} \mathbb{P} \ \ f = \frac{1}{2} \int f.$

We now de ne the Hankel operator on H^2. Let (Jf)(x) = f(x) be the operator of re ection. Given ’ 2 L^1 the operator H(’) : H^2 ! H^2 given by the formula

$$\mathbb{H}(\varphi) f = \mathbb{J} \mathbb{P}_- \varphi f, \quad f \in H^2_+, \tag{3.7}$$

is called the Hankel operator with symbol ’. Clearly kH(’)k k’k$_{L_1}$, H(’) is self-adjoint if (J’)(x) = $\overline{’(x)}$ (this is always our case), H(’) = 0 if ’ is a constant, and

$$\mathbb{H}(\varphi) = \mathbb{H}(\widetilde{\mathbb{P}}_-\varphi)$$
$$= \mathbb{H}(\mathbb{P} \ \ \varphi) \ (\text{if } \varphi \in L^2 \cap L^\infty). \quad (3.8)$$

The relevance of the Hankel operator in our setting is on the surface as the Marchenko operator, the cornerstone of the IST, is a Hankel operator. However, while in the literature on integrable systems it is rarely used in the form (3.7), we nd it particularly convenient due, among others, to the property (3.8), which is less transparent in the integral representation.

Finally we note that reliance on the theory of Hankel operator in the study of completely integrable systems has recently picked up momentum (see e.g. [2], [6], [8], [15], [23] and the references cited therein).

4. Overview of short-range scattering

Unless otherwise stated all facts are taken from [24]. Through this section we assume that q is short-range, i.e. $q \in L^1_1$. Associate with q the full line Schr dinger operator L_q = @$_x{}^2$ + $q(x)$. As is well-known, L_q is self-adjoint on L^2 and its spectrum consists of J simple negative eigenvalues $\{-\kappa_j^2 : 1 \le j \le J\}$, called bound states ($J$ = 0 if there are no bound states), and two fold absolutely continuous component lling $(0;1)$. There is no singular continuous spectrum. Two linearly independent (generalized) eigenfunctions of the a.c. spectrum $(x;k);k$ 2 R, can be chosen to satisfy

$$(x;k) = e^{ikx} + o(1);@_x (x;k) \qquad ik (x;k) = o(1); x ! \quad 1: \quad (4.1)$$

The function , referred to as right/left Jost solution of the Schr dinger equation

$$L_q = k^2 ; \quad (4.2)$$

is analytic for Imk> 0. It is convenient to introduce

$$y (k;x) := e^{ikx} (x;k) \qquad 1;$$

$(1 + y (k;x)$ is sometimes referred to as the Faddeev function), which is H^2 for each x. Since q is real, also solves (4.2) and one can easily see that the pairs

f +; +g and f ; g form fundamental sets for (4.2). Hence is a linear combination of f ; g. We write this fact as follows

$$T(k) \ (x;k) = \overline{(x;k) + R} \ (k) \ (x;k); \ k \ 2 \ \mathrm{R}, \quad (4.3)$$

where T and R ; are called transmission, right/left re ection coe¢ cients respectively. The function T (k) is meromorphic for Imk> 0 with simple poles at (i $_j$) and continuous for Imk = 0. Generically, T (0) = 0. The re ection coe¢ cient R (k) 2 L^2 but need not admit be analytic.

In the context of the IST Zakharov-Faddeev trace formulas [34] (conservation laws) play very important role. For Schwarz potentials q they are in nitely many.
Explicitly,

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$$\_\mathrm{Z} \log 1 \; \mathrm{j}R(k)\mathrm{j}2 \qquad \mathrm{d}k = \mathrm{Z}\; q + \mathrm{X}\; n \qquad \text{(rst trace formula)} \qquad (4.4)$$

$$\frac{8}{\pi}\int k^2 \log\left(1-|R_\pm(k)|^2\right)^{1} \mathrm{d}k = \int q^2 - \frac{16}{3}\sum \kappa_n^3 \qquad \text{(second trace formula) (4.5)}$$

It is shown in the recent [19] that (4.4) holds for any q 2 L^1, each term being nite. Since jR (k)j 1 and

$$\mathrm{j}R\,(k)\mathrm{j}^2; \qquad 1 \log 1 \; \mathrm{j}R\,(k)\mathrm{j}^2$$

one concludes that $R\,(k)$ 2 L^2 for q 2 L^1. The second one (4.5) holds for q 2 $L^1 \setminus L^2$ [21] and readily implies that $kR\,(k)$ 2 L^2. Note, that Zakharov-Faddeev trace formulas are not directly related to the trace formulas we discuss in Introduction but they are also related to the trace of some operators.

The identities (4.3) are totally elementary but serve as a basis for inverse scattering theory and for this reason they are commonly referred to as basic scattering relations. As is well-known (see, e.g. [24]), the triple fR ;($_j$;c $_{;j}$)g, where c $_{;j}$ = k (;i $_j$)k ^{1}, determines q uniquely and is called the scattering data for L$_q$. We emphasize that in order to come from a L^1_1 potential the scattering data fR ;($_n$;c $_{;n}$)g must satisfy some conditions known as Marchenko s characterization [24]. The actual process of solving the inverse scattering problem necessary for the IST is historically based on the Marchenko theory (also knows as FaddeevMarchenko or Gelfand-Levitan-Marchenko). In fact, this procedure is quite transparent from the Hankel operator point of view. Indeed, replacing in (4.3) with y and applying the operator JP , a straightforward computation [16] leads to

$$y + \mathrm{H}(')y = \mathrm{H}(')1; \quad \text{(Marchenko s equation)} \qquad (4.6)$$

where H(′) is the Hankel operator (3.7) with symbol

$$\varphi_\pm(k,x) = \sum_{n=1}^{N} \frac{-\mathrm{i}c_{\pm,n}^2 e^{\mp\,\kappa_n}}{k-\mathrm{i}\kappa} + R_\pm(k)\, e^{\pm}, \quad \frac{-\;\pm\quad \mp 2\kappa_n x}{} \;\; \pm \quad \pm 2\mathrm{i}kx \qquad (4.7)$$

and H(′)1 is understood as

$$\mathrm{H}(')1 = \mathrm{JP}\,' = \mathrm{P}_+\mathrm{J}' = \mathrm{P}_+\bar{'}:$$

We call (4.6) the Marchenko equation as its Fourier image is the Marchenko integral equation. It is proven in [16, Theorem 8.2] that I + H(′) is positive de nite and therefore

$$y = \quad [I + \mathrm{H}(')]^{\;1}\,\mathrm{H}(')1 \; 2 \; H^2: \qquad (4.8)$$

Thus, given data fR ;($_j$;c $_{;j}$)g we compute ′ by (4.7) and form the Hankel operator H(′). The function $y\,(k;x)$ is found by (4.8). The potential $q\,(x)$ can then be recovered in a few ways. Our method is, of course, to apply a suitable trace formula, which we derive in the next section.

Since many of our proofs below are based on limiting arguments we need to understand in what sense scattering data converges as we approximate q in the L^1_1. In particular the following statement plays an important role.

Proposition 4.1. *If $q_n(x)$ converges in L^1_1 to $q(x)$ then the sequence of re ection coe¢ cients $R_{;n}(k)$ corresponding to $q_n(x)$ converges in L^2 to $R(k)$:*

Proof. We consider the + case only and we suppress + sign. We use the following a priori estimates (see e.g. [5])

jy (x;k)j .q hxi=hki (4.9)

$$|y_-(x,k) - y_{-,n}(x,k)| \lesssim_q \langle x\rangle \|q-q_n\|_{L^1_1} \tag{4.10}$$

$$|T(k) - T_n(k)| \lesssim_q |k|^{\ \ 1} \|q-q_n\|_{L^1_1}. \tag{4.11}$$

Consider kR R_nk^2 and rewrite it as (" is any)

kR R_nk^2 = (R R_n) j j " ^{2} + (R R_n) j j>" ^{2} (4.12)

2

8" + (R R_n) j j>"*:*

It follows from the general formula [5]

$$R(k) = \frac{T(k)}{2\mathrm{i}k}\int e^{\ \ 2\mathrm{i}kx} q(x)(1+y_-(x,k))\,\mathrm{d}x \tag{4.13}$$

that

$$\begin{aligned} R(k) - R_n(k) &= \frac{T(k)-T_n(k)}{2\mathrm{i}k}\int e^{-2\mathrm{i}kx} q(x)(1+y\quad(x,k))\,\mathrm{d}x \\ &+ \frac{T_n(k)}{2\mathrm{i}k}\int e^{-2\mathrm{i}kx}(q(x)-q_n(x))\,\mathrm{d}x \\ &+ \frac{T_n(k)}{2\mathrm{i}k}\int e^{-2\mathrm{i}kx} q(x)(y_-(x,k)-y_{-,n}(x,k))\mathrm{d}x \\ &= I_1(k) + I_2(k) + I_3(k) \end{aligned}$$

and hence

(R R_n) j j>" I_1 j j>" + I_2 j j>" + I_3 j j>"*:*

Estimate each term separately. For I_1 j j>" we have

$$\begin{aligned} \left\|I_1\chi_{|\cdot|>\varepsilon}\right\|^2 &\lesssim_q \int |q|\,(1+|y_-(x,k)|)\,\mathrm{d}x \cdot \int_{|k|>\varepsilon}\left|\frac{T(k)-T_n(k)}{k}\right|^2 \mathrm{d}k \\ &\lesssim_q \|q-q_n\|^2_{L^1_1}\int_{|k|>\varepsilon} k^{-4}\mathrm{d}k \quad \text{(by (4.9),(4.11))} \\ &\lesssim \varepsilon^{-3}\|q-q_n\|^2_{L^1_1}. \end{aligned}$$

Thus

$$\left\|I_1\chi_{|\cdot|>\varepsilon}\right\| \lesssim_q \varepsilon^{-3/2}\left\|q-q_n\right\|_{L^1_1}.$$

For I_2 j j>" we have

$$\left\|I_2\chi_{|\cdot|>\varepsilon}\right\|^2 \le \left\|q-q_n\right\|_{L^1}^2 \int_{|k|>\varepsilon}\left|\frac{T_n(k)}{k}\right|^2 \mathrm{d}k$$
$$\lesssim \frac{1}{\varepsilon}\left\|q-q_n\right\|_{L^1_1}^2$$

$$\left\|I_2\chi_{|\cdot|>\varepsilon}\right\| \lesssim_q \varepsilon^{\ 1/2}\left\|q-q_n\right\|_{L^1}.$$

and hence ||

Finally for I_2 j j>" one has in a similar manner

I_3 j j>" $_q$ " 1=2 sup$_k$ Z e 2ikx$q$ $(x)(y$ $(x;k)$ y $_{;n}$ $(x;k))$dx

$_q$ " 1=2 kq q_nk$_{L11}$ (by (4.10))

and hence

$$\left\|I_3\chi_{|\cdot|>\varepsilon}\right\| \lesssim_q \varepsilon^{-1/2}\left\|q-q_n\right\|_{L^1_1}$$

|| || :

One can now sees that each I_j j j>", j = 1;2;3; vanishes as $\left\|q-q_n\right\|_{L^1_1}$ does and hence, since " is arbitrary, it follows from (4.12) that

kR R_nk ! 0;n ! 1:

Note that the question in what sense the re ection coe¢ cient converges when we approximate the potential in a certain way is a subtle one [27].

Finally we observe that ;y ;T;R as functions of k (momentum) satisfy

(Jf)(k) = f (k) = f $(\overline{k})$ (symmetry property). (4.14)

5. Trace formulas

In this section we put forward a new approach to generate Deift-Trubowitz type trace formulas. It is based on Hardy spaces and Hankel operators.

Theorem 5.1. *Suppose that* q 2 L^1 *and*

$$Q_+(x) := \int_x^\infty q(s)\,\mathrm{d}s, Q\quad (x) := \int_{-\infty}^x q(s)\,\mathrm{d}s.$$

Let $(x;k)$ *be right/left Jost solution and*

y $(k;x)$ = e ikx $(x;k)$ 1:

If for all real x

$$2iky\ (k;x) + Q\ then(x)\ 2\ H^2; \qquad (5.1)$$

for any > 0 and a.e. x

$$q(x) = \mp\frac{2}{\pi}\partial_x \int \operatorname{Re}\frac{y_\pm(k,x)}{k+\mathrm{i}\alpha}k\mathrm{d}k \quad \text{(trace formula)}. \qquad (5.2)$$

If for every real x

$$y\ (\ ;x)\ 2\ H^2 \qquad (5.3)$$

then (5.2) simpli es to read

$$q\ (x) = \frac{2}{\ } @_x \mathrm{Z}\ \mathrm{Re} y\ (k;x)\mathrm{d}k: \qquad (5.4)$$

Proof. Note rst that both Jost solutions exist for $q\ 2\ L^1$ (not only for L^1_1). Multiplying (5.1) by $\mathrm{i}=(k + \mathrm{i}\ )\ 2\ H^2$ (> 0) and recalling that a product of two H^2 functions is in H^1, we have

$$(5.5) \quad \frac{2k}{k+\mathrm{i}\alpha} y\ (k;x) \ \frac{\mathrm{i}}{k+\mathrm{i}} \ Q\ (x)\ H^1_2:$$

But it is well-known that

$$f\ 2\ H^1 =)\ \mathrm{Z} f(k + \mathrm{i}0)\mathrm{d}k = 0 \qquad (5.6)$$

and therefore

$$\int\left[\frac{2k}{k+\mathrm{i}\alpha}y_\pm(k,x) - \frac{\mathrm{i}}{k+\mathrm{i}\alpha}Q_\pm(x)\right]\mathrm{d}k = 0 :$$

For its real part we have Z Re

$$\left[\frac{2k}{k+\mathrm{i}\alpha}y_\pm(k,x)\right] - \frac{\alpha}{k^2+\alpha^2}Q_\pm(x)\Big\}\,\mathrm{d}k = 0,$$

which can be rearranged to read

$$Q\ (x) = 2\mathrm{Z}\ \mathrm{Re}\ \frac{y\ (k;x)}{k + \mathrm{i}}\ k\mathrm{d}k$$

and (5.2) follows upon di⁄erentiating in x.

We show now (5.4). To this end, we just split (5.2) as

$$\int \operatorname{Re}\frac{ky_\pm(k,x)}{k+\mathrm{i}\alpha}\mathrm{d}k = \int \operatorname{Re} y_\pm(k,x)\mathrm{d}k + \mathrm{Im}\int\frac{y_\pm(k,x)}{k+\mathrm{i}\alpha}\mathrm{d}k$$

and observe that by (5.6) the second integral on the right hand side is zero (both y and $1=(k + \mathrm{i}\ )$ are in H^2).

Remark 5.2. *Under the condition* $q \in L_1^1$ *the following formula is proven in* [5] *(only + sign is considered):*

$$q\,(x) = -4\sum_{n=1}^{N}\kappa_n c_{+,n}^2 \psi_+\,(x, \mathrm{i}\kappa_n)^2 \tag{5.7}$$

$$+ \_\_2\mathrm{i}\,(PV)\int R_+(k)\;{}_+(x;k)^2\,k\mathrm{d}k:$$ *(Deift-Trubowitz trace formula)*

Visually it is very di⁄erent from (5.4) (${}_+(x;k)$ *appears in (5.7) squared whereas in (5.4) it does not). One can however show that (5.4) implies (5.7). We demonstrate this fact in the Appendix. Theorem 5.1 is an extension of (5.7) as it accepts certain singularities of* $(k;x)$ *at* $k = 0$*. The latter may occur if* $q \notin L_1^1$*. Thus following the terminology of* [5] *we may refer to our (5.2) and (5.4) as trace formulas.*

The next statement o⁄ers a version of (5.7) that is linear with respect to the Jost solution .

Corollary 5.3. *Suppose* $q \in L_1^1 \cap L^2$ *and let* $\{R;\ _n;c\ _{;n}\}$ *be its scattering data. Then*

$$q\,(x) = \pm\partial_x\left\{2\sum_{n=1}^{N}c_{\pm,n}^2 e^{\mp\kappa_n x}\psi_\pm(x,\mathrm{i}\kappa_n) + \frac{1}{\pi}\int e^{\pm\mathrm{i}kx}R_\pm\,(k)\,\psi_\pm(x,k)\mathrm{d}k\right\}. \tag{5.8}$$

Proof. As is well-known (see e.g. [24]), for $q \in L^1$

$$y\,(k;x) = \frac{\mathrm{i}}{2k}Q\,(x) + O\,k^{2};\ k \to \infty; \tag{5.9}$$

and furthermore $y\,(k;x)$ is bounded at $k = 0$ for $q \in L_1^1$. It immediately follows that the condition (5.1) is satis ed. Also, the condition (5.3) holds due to (4.8). Therefore (5.4) holds for short-range q. To show (5.8) we turn to the Marchenko equation (4.6). Applying the operator of re ection J to this equation and recalling the symmetry property (4.14) we have

$$\overline{y} + \mathbb{P}\,(\varphi y) = \mathbb{P}\,\varphi;$$

which together with (4.6) yield

$$2\mathrm{Re}\,y = \mathbb{J}\mathbb{P}\,(\varphi y) \quad \mathbb{P}\,(\varphi y) \quad \mathbb{J}\mathbb{P}\,\varphi\,\mathbb{P}\,\varphi:$$

Since obviously

$$\int_{-a}^{a}\mathbb{J}f = \int_{-a}^{a}f$$

we have

$$\int_{a}^{a}\mathrm{Re}\,y_\pm = -\int_{a}^{a}\mathbb{P}_-\varphi_\pm - \int_{a}^{a}\mathbb{P}_-\left(\varphi_\pm y_\pm\right): \tag{5.10}$$

Consider each term on the right hand side of (5.10). Observing that $(k\ \mathrm{i}\ _n)^{1} \in H^2$ and hence $\mathbb{P}\,(k\ \mathrm{i}\ _n)^{1} = (k\ \mathrm{i}\ _n)^{1}$, we have

$$\mathbb{P}_-\varphi_\pm = \mathbb{P}_-\left[\sum_{n=1}^N \frac{-\mathrm{i}c_{\pm,n}^2 e^{\mp 2\kappa_n x}}{k-\mathrm{i}\kappa_n}\right] + \mathbb{P}_-\left[R_\pm(k)\, e^{\pm 2\mathrm{i}kx}\right]$$
$$= \sum_{n=1}^N \frac{-\mathrm{i}c_{\pm,n}^2 e^{\mp 2\kappa_n x}}{k-\mathrm{i}\kappa_n} + \mathbb{P}_-\left[R_\pm(k)\, e^{\pm 2\mathrm{i}kx}\right]$$

and thus

$$\int_{-a}^{a} \mathbb{P}_-\varphi_\pm = \int_{-a}^{a} \sum_{n=1}^{N} \frac{-\mathrm{i}c_{\pm,n}^2 e^{\mp 2\kappa_n x}}{k-\mathrm{i}\kappa_n}\,\mathrm{d}k + \int_{-a}^{a} \mathbb{P}_-\left[R_\pm(k)\, e^{\pm 2\mathrm{i}kx}\right]\mathrm{d}k$$

$$= -\mathrm{i}\sum_{n=1}^N c_{\pm,n}^2 e^{\mp 2\kappa_n x} \int_{a}^{a} \frac{\mathrm{d}k}{k-\mathrm{i}\kappa_n} + \int_{a}^{a} \mathbb{P}_-\left[R_\pm(k)\, e^{\pm 2\mathrm{i}kx}\right]\mathrm{d}k. \quad (5.11)$$

$$\int_{-} \mathbb{P}_-\varphi_\pm = \int_{-} \sum \frac{-\mathrm{i}\ \ \pm,n \ \ \mp\ \kappa}{k-\mathrm{i}\kappa_n}\,\mathrm{d}k + \int_{-} \mathbb{P}_-\left[R_\pm(k)\, e^{\pm 2\mathrm{i}kx}\right] k$$

Pass in (5.11) now to the limit as a ! 1. By (3.6)

$$\lim_{a\to\infty}\int_{a}^{a} \frac{\mathrm{d}k}{k-\mathrm{i}\kappa_n} = \lim_{a\to\infty}\int_{a}^{a} \frac{\mathrm{d}k}{k-\mathrm{i}} = \quad \mathrm{i}$$

and hence substituting this into (5.11) one has

$$(PV)\int_{-a}^{a} \mathbb{P}_-\varphi_\pm = \pi\sum_{n=1}^N c_{\pm,n}^2 e^{\mp 2\kappa_n x}$$
$$+ (PV)\int_{-a}^{a} \mathbb{P}\quad \left[R_\pm(k)\, e^{\pm 2\mathrm{i}kx}\right]\mathrm{d}k.$$

It remains to evaluate the integral on the right hand side. As we have shown in Section 4, hkiR (k) 2 L^2. By Lemma 3.1 then

$$(PV)\int_{a}^{a} \mathbb{P}_-\left[R_\pm(k)\, e^{\pm 2\mathrm{i}kx}\right]\mathrm{d}k = \frac{1}{2}\int R_\pm(k)\, e^{\pm 2\mathrm{i}kx}\mathrm{d}k$$

and nally

$$(PV)\int \mathbb{P}_-\varphi_\pm = \pi\sum_{n=1}^N c_{\pm,n}^2 e^{\mp 2\kappa_n x}$$
$$+ \frac{1}{2}\int R_\pm(k)\, e^{\pm 2\mathrm{i}kx}\mathrm{d}k.$$

Similarly,

$$(PV)\int \mathbb{P}_-(\varphi_\pm y_\pm) = \frac{1}{2}\int \varphi_\pm y$$

and from (5.10) we obtain

$$\begin{aligned}\int \operatorname{Re} y_\pm &= -\int \mathbb{P} \quad \varphi_\pm - \int \mathbb{P} \quad (\varphi_\pm y_\pm) \\ &= -\pi \sum_{n=1}^{N} c_{\pm,n}^2 e^{\mp 2\kappa_n x} - \frac{1}{2}\int R_\pm(k)\, e^{\pm 2\mathrm{i}kx} \mathrm{d}k \\ &- \frac{1}{2}\int \varphi_\pm y_\pm .\end{aligned}$$

Inserting this into (5.4) one has

$$\begin{aligned}q(x) &= \mp\frac{2}{\pi}\partial_x \int \operatorname{Re} y_\pm(k,x)\,\mathrm{d}k \\ &= \pm\partial_x\left\{2\sum_{n=1}^{N} c_{\pm,n}^2 e^{\mp 2\kappa_n x} + \int R_\pm(k)\, e^{\pm 2\mathrm{i}kx}\frac{\mathrm{d}k}{\pi} + \int \varphi_\pm(k,x)\, y_\pm(k,x)\frac{\mathrm{d}k}{\pi}\right\}:\end{aligned}$$

It remains to evaluate the last integral on the right hand side. By the Cauchy formula

$$\begin{aligned}\int \varphi_\pm y_\pm &= \int \sum_{n=1}^{N} \frac{-\mathrm{i}c_{\pm,n}^2 e^{\mp 2\kappa_n x}}{k - \mathrm{i}\kappa_n} y_\pm(k,x) \\ &+ \int R_\pm(k)\, e^{\pm 2\mathrm{i}kx} y_\pm(k,x)\,\mathrm{d}k \\ &= 2\pi\sum_{n=1}^{N} c_{\pm,n}^2 e^{\mp 2\kappa_n x} y_\pm(\mathrm{i}\kappa_n, x) + \int R_\pm(k)\, e^{\pm 2\mathrm{i}kx} y_\pm(k,x)\,\mathrm{d}k\end{aligned}$$

and hence

$$q(x) = \pm\partial_x\left\{2\sum_{n=1}^{N} c_{\pm,n}^2 e^{\mp 2\kappa_n x}\left[1 + y_\pm(\mathrm{i}\kappa_n, x)\right] + \int R_\pm(k)\, e^{\pm 2\mathrm{i}kx}\left[1 + y_\pm(k,x)\right]\frac{\mathrm{d}k}{\pi}\right\}:$$

Recalling that *y* (*k;x*) = *e* *ikx* (*x;k*) 1 we nally obtain

$$q(x) = \pm\partial_x\left\{2\sum_{n=1}^{N} c_{\pm,n}^2 e^{\mp\kappa_n x}\psi_\pm(x, \mathrm{i}\kappa_n) + \int e^{\pm\mathrm{i}kx} R_\pm(k)\,\psi_\pm(x,k)\frac{\mathrm{d}k}{\pi}\right\},$$

which is (5.8).

Remark 5.4. *It follows from (5.9) that y* (*k;x*) 2=L^1 *but* Re*y* (*k;x*) 2 L^1.

An important corollary of Theorem 5.1 is the following

Theorem 5.5. *Suppose that* $q \in L_1^1$*. Let* f*R;* *n;cn*g *be its right scattering data and* 0(*x;k*) *be the right Jost solution corresponding to the data* f*R;*?g*. Denote*

0 (*x*) = (0 (*x;*i *n*))*; C* = diag c^2_n*:*

Then

q (*x*) = q_0 (*x*) (5.12)

$$+ 2\partial_x \quad {}_0(x)\ C^{\ 1} + \int_x^{\infty} \mathbf{\Psi}_0(s)^T\, \mathbf{\Psi}_0(s)\,\mathrm{d}s\Big)^{\ 1} \mathbf{\Psi}_0(x)^T ,$$

where $q_0(x)$ *admits the following representations*

$$q_0(x) = 2\partial_x \mathrm{Z}\ \mathrm{Re}\ 1\ e^{\ ikx}\ {}_0(x;k)\ \_\_\mathrm{d}k$$

$$= \partial_x (PV) \int e^{\mathrm{i}kx} R(k)\, \psi_0(x,k) \frac{\mathrm{d}k}{\pi}$$

$$= \partial_x^2 \int \frac{e^{2\mathrm{i}kx} - 1}{2\mathrm{i}k} R(k) \frac{\mathrm{d}k}{\pi} + \partial_x \int e^{2\mathrm{i}kx} R(k)\, y_0(k,x)$$

$$\mathrm{d}k$$

$$\_\_:$$

Proof. We merely combine the formula (5.4) from Theorem 5.1 and the version of the binary Darboux transformation from our [30] $q\ (x) = q_0\ (x)\ 2\partial_x^2$ logdet C $^1 + \int_x^{\infty} \mathbf{\Psi}_0^T(s)\, \mathbf{\Psi}_0(s)\Big)$,

where $q_0(x)$ is the potential corresponding to f$R;$?g. Indeed, by the Jacobi formula on di⁄erentiation of determinants one has

$$\partial_x \quad \mathrm{logdet} \quad C^{\ 1} + \int_x^{\infty} \mathbf{\Psi}_0^T(s)\, \mathbf{\Psi}_0(s)$$

$$= \quad {}_0(x) \quad \mathrm{Z}\ C^{\ 1T} +{}_0(s) \quad {}_0(s)\mathrm{d}s \ ^{1} \quad {}_0(x)^{\ 1T} :$$

We will demonstrate below that the trace formula (5.12) is convenient for limiting arguments. Of course, a similar formula holds for the left scattering data.

6. Trace formula and KdV solutions

In this section we show that our trace formulas yield new representations for solutions to the KdV equation with short-range initial data. Note that the condition $q(x) \in L_1^1$ alone does not guaranty that $q(x,t) \in L_1^1$ for $t > 0$ (see Corollary 7.3) and therefore (5.4) does not apply. We cannot even be sure that (5.1) holds for $q\ (x;t)$*:* To overcome the problems we employ some limiting arguments. Through the rest of the paper we use the following convenient notation

$${}_{x;t}(k) := \mathrm{expi}\ 8k^3t + 2kx :$$

While highly oscillatory on the real line, this function has a rapid decay along R+ia for any $a > 0$.

Theorem 6.1. *If* $q(x) \in L_1^1$ *and* fR; $_j$;c$_j$g *are the associated right scattering data then the solution*[1] $q(x;t)$ *to the Cauchy problem for the KdV equation (1.1) with initial data* $q(x)$ *can be represented by*

$$q(x;t) = q_0(x;t) \tag{6.1}$$

$$+\, 2@_x \quad {}_0(x;t)\, C^{(t)^{-1}} + \int_x^\infty \Psi_0(s,t)^T\, \Psi_0(s,t)\, \mathrm{d}s\Big)^{-1} \Psi_0(x,t)^T,$$

where

$${}_0(x;t) = (\,{}_0(x;t;\mathrm{i}\,_j));\ C^{(t)} = \left(c_j \exp 8\kappa_j^3 t\right),$$

$${}_0(x;t;k) = e^{ikx}\,[1 + y_0(k;x;t)];$$

$y_0(\ ;x;t)$ *is the* H^2 *solution of the singular integral equation*

$$y + \mathbb{H}(R\xi_{x,t})y = -\mathbb{H}(R\xi_{x,t})1,$$

$$q_0(x,t) = \partial_x \left\{ (PV) \int R(k)\, \xi_{x,t}(k)\, \frac{\mathrm{d}k}{\pi} + \int R(k)\, \xi_{x,t}(k)\, y_0(k,x,t)\, \frac{\mathrm{d}k}{\pi} \right\}. \tag{6.2}$$

Proof. For Schwarz $q(x)$ there is nothing to prove as $q(x;t)$ is also a Schwarz function. Since KdV is well-posed in any Sobolev space $H^{"}$ with 0 <" 1 (see e.g. [22]) and $L_1^1 \subset H^{-\varepsilon}$, for any sequence of (real) Schwarz functions $q_n(x)$ approximating $q(x)$ in L_1^1 the sequence of $q_n(x;t)$ converges in $H^{"}$ to $q(x;t)$, the solution to (1.1) with the initial pro le $q(x)$. Thus, we only need to compute $\lim_{n!1} q_n(x;t)$. Note that convergence of norming constants is somewhat inconvenient to deal with but results of our recent [30] o⁄ers a simple detour of this circumstance. Take the scattering data fR;?g (i.e. no bound states) and construct by (5.4) the corresponding potential

$$q_0(x) = \quad 2@_x \mathrm{Z}\ \mathrm{Re}\, y_0(k;x)\mathrm{d}k:$$

Since by construction L_{q_0} is positive, q_0 is the Miura transformation

$$q_0(x) = @_x r(x) + r(x)^2$$

of some real r 2 L^2_{loc} [20]. Choose a sequence (r_n) of Schwarz function such that the sequence $q_{0;n} = r_n^2 + @_x r_n$ approximates q_0 in L_1^1. As is well-known, each $R_n(k)$ is also Schwarz and so is $q_{0;n}(x;t)$ for t 0. Therefore, by (5.8) and recalling that $_n = e^{ikx}(1 + y_n)$ we have

$$\begin{aligned} q_{0,n}(x,t) &= \partial_x \int R_n(k)\, \xi_{x,t}(k)\,[1 + y_n(k,x,t)]\, \frac{\mathrm{d}k}{\pi} \\ &= \partial_x^2 \int \frac{\xi_{x,t}(k) - 1}{2\mathrm{i}k} R_n(k)\, \frac{\mathrm{d}k}{\pi} + \partial_x \int R_n(k)\, \xi_{x,t}(k)\, y_n(k,x,t)\, \frac{\mathrm{d}k}{\pi} \\ &=: q_{0,n}^{(1)}(x,t) + q_{0,n}^{(2)}(x,t) \end{aligned}$$

[1] The general theory guaranties well-posedness at least in the L^2- base Sobolev space $H^{"}$ with any index 0<" 1 (see, e.g. [22]).

Here we have used a well-known regularization of the Fourier integral. This representation is convenient for passing to the limit as n ! 1. By Proposition 4.1 the sequence of re ection coe¢ cients R_n converges in L^2 to R. Paring this sequence, if needed, we may assume that R_n ! R a.e. Clearly $R_n\,_{x;t}$! $R\,_{x;t}$ a.e. too. But then, as is well-known (can also be easily shown), the corresponding sequence of Hankel operators H($R_n\,_{x;t}$) converges to H($R\,_{x;t}$) in the strong operator topology. Since (all) I + H($R_n\,_{x;t}$) and I + H(R $_{x;t}$) are positive de nite [16, Theorem 8.2] for all $x;t$ we conclude that in H^2 y_n = (I + H(R_n $_{x;t}$)) 1 H($R_n\,_{x;t}$)1

! (I + H($R\,_{x;t}$)) ^{1} H($R\,_{x;t}$)1 =: y_0 n ! 1;

where y_0 ($\in H^2$;$x;t$). Therefore, for all $x;t$

$\xi_{x,t}\left(k\right)-1$

Z 2ik R_n ($k;t$) dk ! Z $_{x;t}$ (2kik) 1R($k;t$) d k

and

Z R(k) $_{x;t}$ (k)y_n ($k;x;t$) ! Z R(k) $_{x;t}$ (k)y_0 ($k;x;t$) : dk

dk

Thus we conclude that for each t 0

$$w^{*}-\lim_{n\to\infty}q_{0,n}^{(1)}\left(x,t\right)=\partial_{x}^{2}\int\frac{\xi_{x,t}\left(k\right)-1}{2\mathrm{i}k}R\left(k\right)$$

dk

$$w^{*}-\lim_{n\to\infty}q_{0,n}^{(2)}\left(x,t\right)=\partial_{x}\int R_{n}\left(k\right)\xi_{x,t}\left(k\right)y_{n}\left(k,x,t\right)\frac{\mathrm{d}k}{\pi}$$

and (6.2) follows. Performing the binary Darboux transformation [30] we arrive at (6.1).

Remark 6.2. *Performing in (6.1) the inverse binary Darboux transformation* [30]*, we can conclude that we also have*

2

q ($x;t$) = @$_x$ Z Rey ($k;x;t$)dk (6.3)

but cannot claim that this integral is absolutely convergent as it was in (5.4). This of course would be true if the asymptotic (5.9) held for q ($x;t$)*. The problem with (5.9) is that the error in (5.9) depends on* kq (;t)k$_{L_1}$*, which need not be nite. We however conjecture that the integral in (6.3) is indeed absolutely convergent but we can no longer use tools and estimates from the short-range scattering theory.*

Corollary 6.3. *The trace Deift-Trubowitz trace formula (5.7) holds for t*> 0 *(not only for t* = 0*).*

Proof. Indeed, the approximating sequence q_n ($x;t$) that corresponds to the sequence fR_n; $_j$;c_jg where R_n is the same as constructed in the proof of Theorem 6.1 will do the job. The only question is why the rst term in (5.7) holds for $t > 0$. This easily follows from our arguments. Indeed, since $y_n(\ ;x;t) ! y(\ ;x;t)$ in H^2 we also have uniform convergence for $\mathrm{Im}k > 0$ on compacts. Therefore, $_n(\mathrm{i}\ _j;x;t) ! (\mathrm{i}\ _j;x;t)$ for all $x;t$.

Remark 6.4. *Extension of the Deift-Trubowitz trace formula (5.7) to KdV solution would be a hard problem back in the 70s as the breakthrough in the understanding of wellposedness in the L^2 based Sobolev spaces with negative indexes only occurred after the seminal 1993 Bourgain paper* [3]*, where wellposedness was proven in L^2.*
With no well-posedness at hand we cannot use limiting arguments even if $q(x)$ 2 $L_1^1 \cap L^2$.

7. How KdV trades decay for smoothness

The goal of this section is to show how the results of the previous section could be useful in understanding the phenomenon of dispersive smoothing (aka gain of regularity).

Theorem 7.1. *If* $q(x) \in L_1^1 \cap L^2$ *then* $q(x,t) \in L_{\mathrm{loc}}^\infty \cap L^2$ *for* $t > 0$.

Proof. We rst note that we may assume that the negative spectrum is absent. I.e. L_q is positive. Split our initial pro le as

$$q = q + q_+;\ q := q\mathrm{j}_R:$$

We may assume that L_{q_+} is positive as possible appearance of a negative eigenvalue could only lead to minor technical complications. We use the following representation from [18]:

$$R(k) = \quad _1(k) + {}_2(k) + A(k); \tag{7.1}$$

where

$$\phi_1(k) = \frac{T_0(k)}{2\mathrm{i}k}\widehat{q}(k),$$
$$\phi_2(k) = \frac{T_0(k)}{(2\mathrm{i}k)^2}\widehat{p}(k),$$
$$f(k) := \int^{\infty} e^{-2\mathrm{i}ks} p(s)\,\mathrm{d}s \qquad \mathrm{b} \qquad 0 \qquad ;$$

T_0 2 H^1 is the transmission coe¢ cient for q_+; p is the derivative of an absolutely continuous function and

$$|p(x)| \lesssim_{\|q_+\|_{L^1}} |q(x)| + C\int_x^\infty |q|, \quad x \geq 0; \tag{7.2}$$

and A 2 H^1 (which form is not important). Note that $\frac{T_0(k)}{2\mathrm{i}k}$ remains bounded at $k = 0$ as well as $\frac{T_0(k)}{(2\mathrm{i}k)^2}p(k)$ b .

It follows from (6.2) that

$$\begin{aligned} q_0(x,t) &= \partial_x \left\{ (PV) \int R(k)\, \xi_{x,t}(k) \frac{\mathrm{d}k}{\pi} + \int R(k)\, \xi_{x,t}(k)\, y_0(k,x,t) \frac{\mathrm{d}k}{\pi} \right. \\ &= \partial_x (PV) \int R(k)\, \xi_{x,t}(k) \frac{\mathrm{d}k}{\pi} \\ &+ \int R(k) \left[\partial_x \xi_{x,t}(k)\right] y_0(k,x,t) \frac{\mathrm{d}k}{\pi} \\ &+ \int R(k)\, \xi_{x,t}(k) \left[\partial_x y_0(k,x,t)\right] \frac{\mathrm{d}k}{\pi} \\ &=: q_1(x,t) + q_2(x,t) + q_3(x,t). \end{aligned} \tag{7.3}$$

Consider each term separately. By (7.1) for $q_1(x;t)$ we have

$$q_1(x;t) = q_{11}(x;t) + q_{12}(x;t) + q_{13}(x;t)$$

where

$$q_{1n}(x,t) := \partial_x (PV) \int \xi_{x,t}(k)\, \phi_n(k) \frac{\mathrm{d}k}{\pi}, \quad n = 1,2$$

and

$$q_{13}(x;t) = \partial_x (PV) \int \xi_{x;t}(k) A(k) \frac{\mathrm{d}k}{\pi}.$$

The simplest term is q_{13}. Since $\xi_{x;t}(k + \mathrm{i}a)$ (and all its x-derivatives) rapidly decays along $\mathbb{R}+\mathrm{i}a$ for any $a > 0$ we deform the contour of integration to $\mathbb{R}+\mathrm{i}a$ that provides a rapid convergence of the integral (the original integral need not be absolutely convergent).

The term q_{12} is also easy. Indeed, since ϕ_2 is clearly in L^1 we have

$$\begin{aligned} q_{12}(x,t) &= \partial_x \int \phi_2(k)\, \xi_{x,t}(k) \frac{\mathrm{d}k}{\pi} \\ &= \int \frac{T_0(k)}{2\mathrm{i}k} \widehat{p}(k)\, \xi_{x,t}(k) \frac{\mathrm{d}k}{\pi}. \end{aligned}$$

It remains to show that this integral is absolutely convergent. It follows from (7.2) that

1 1 2 1=2

kpk .kq+kL1 kqk + C Z0 Zx jqj dx!

Z 1 Z 1 1=2 1 kqk + C 0 xjq (x)j

x jqj dx q 2 L1

kqk + C kqk1L=12 kqk1L=112 < 1

and hence $p \in L^2$. Therefore, $q_{12}(x;t)$ is locally bounded for $t \geq 0$. (In fact, continuous).b

Consider the remaining term $q_{11}(x;t)$. In order to proceed we need rst to regularize the improper integral. It cannot be done by merely deforming R to R+ia as is done for q_3 since $q(k)$ need not admit analytic continuation into the upper

half plane. To detour this circumstance we de neb qb(k) by

$$\left(\overline{\ }\right) = \int_0^\infty e^{\ \ 2i\overline{k}s} q(s)\,\mathrm{d}s, \qquad \geq$$

and apply the Cauchy-Green formula for the strip $\leq$ $\leq$. We have

$$\begin{aligned} q_{11}(x,t) &= \partial_x (PV) \int \xi_{x,t}(k) \frac{T_0(k)}{2\mathrm{i}k} \widehat{q}\left(\overline{k}\right) \frac{\mathrm{d}k}{\pi} \\ &= \partial_x \int_{0 \leq \operatorname{Im} k \leq 1} \xi_{x,t}(k) \frac{T_0(k)}{2\mathrm{i}k} \partial_{\overline{k}} \widehat{q}\left(\overline{k}\right) \frac{\mathrm{d}u\mathrm{d}v}{\pi^2} \quad (k = u + \mathrm{i}v) \\ &+ \partial_x \qquad \xi_{x,t}(k) \frac{T_0(k)}{\qquad} q\left(\overline{k}\right) \frac{\mathrm{d}k}{\pi} \\ &=: q_{111}(x,t) + q_{112}(x,t). \end{aligned}$$

qb k Imk 0;

0 Imk 1

Z$_{R+\mathrm{i}}$ 2ik b

The second term $q_{112}(x;t)$ is treated the same way as $q_{13}(x;t)$ and one immediately concludes that $q_{112}(x;t)$ are bounded (in fact, smooth) for $t > 0$. Turn to q_{111}. We rearrange it by observing that the double integral is absolutely convergent and the order of integration may be interchanged:

$$\begin{aligned} q_{111}(x,t) &= \partial_x \int_{0 \leq \operatorname{Im} k \leq 1} \xi_{x,t}(k) \frac{T_0(k)}{2\mathrm{i}k} \left[\int_0^\infty (-2\mathrm{i}s)\, e^{\ \ 2\mathrm{i}\overline{k}s} q(s)\,\mathrm{d}s \right] \frac{\mathrm{d}u\mathrm{d}v}{\pi^2} \\ &= -\partial_x \int_0^\infty sq(s) \left[\int_{0 \leq \operatorname{Im} k \leq 1} \xi_{x,t}(k) \frac{T_0(k)}{k} e^{-2\mathrm{i}\overline{k}s} \frac{\mathrm{d}u\mathrm{d}v}{\pi^2} \right] \mathrm{d}s \\ &= \mathrm{i} \int_0^\infty \left[\int_{0 \leq \operatorname{Im} k \leq 1} e^{-2vs} \xi_{x\ \ s,t}(k)\, T_0(k) \frac{\mathrm{d}u\mathrm{d}v}{\pi^2} \right] sq(s)\,\mathrm{d}s, \quad \left(\overline{k} = k - 2\mathrm{i}v\right) \\ &\simeq \int_0^\infty \left\{ \int_0^1 e^{-2vs} I(s - x, t)\,\mathrm{d}v \right\} sq(s)\,\mathrm{d}s, \end{aligned}$$

where

$I(s\ x;t) :=$ Z $_{x\ s;t}(k) T_0(k) \mathrm{d}k$
Z
$R+\mathrm{i}v$

$= _{x\ s;t}(k) T_0(k) \mathrm{d}k$
$R+\mathrm{i}$

is independent of v 0. Thus

$$q_{111}(x,t) \simeq \partial_x \int_0^\infty \left\{ \int_0^1 e^{-2vs} \mathrm{d}v \right\} I(s-x,t)\, sq(s)\, \mathrm{d}s$$
$$= \int_0^\infty \frac{1-e^{-2s}}{2} I(s-x,t)\, q(s)\, \mathrm{d}s$$
$$= \int_0^\infty I(s-x,t) \frac{1-e \quad 2s}{2} q(s)\, \mathrm{d}s.$$

It remains to study the behavior of[1] as k ! 1 and x is xed we only need to worry about $I(s\ x;t)$ as s ! +1. Since $T_0(k) = 1 +$

$O\ k$

$I_0(s;t) =$ Z $_{s;t}(k) \mathrm{d}k;$

$R+\mathrm{i}$

which is closely related to the Airy function. For the reader convenience we o⁄er a direct treatment. Rewrite

$_{s;t}(k) = \exp \mathrm{i}[\ S(\ )];$

where $S(\ ) =$ ^{3}=3 and

$! = 2(s=3t)^{1=2}; = 3t(s=3t)^{3=2}; = k=!$

Noticing that we need not adjust the contour of integration, we then have

$I_0(s;t) := !$Z $e^{\mathrm{i}\ S(\ )} \mathrm{d}:$ (7.4)

$R+\mathrm{i}$

Apparently, the phase $S(\ ) =$ ^{3}=3 has stationary points at = 1 and we need to deform the contour in (7.4) to pass through points = 1. We denote such a contour . To apply the steepest descent we need to make sure that expi[$!S(\ )$] decay on away from 1. To this end must be in the lower half plane between points 1 and 1. Noticing that = $(3t=8)!^3$, $! =$ $O\ s^{1=2}$ by the steepest descent method (see e.g. [33]) one has

$$I_0(s,t) = \omega \int_\Gamma e^{\mathrm{i}\Omega S(\lambda)} \mathrm{d}\lambda = \omega O\left(\Omega \quad ^{1=2}\right.$$
$$= O\left(s^{-1/4}\right), \quad s \to +\infty \quad ; \quad ! +1;$$

$:$

Thus $q_{111}(x;t)$ is bounded for $t > 0$ (even if $q(x)$ decays slower than L^1). All four pieces $q_1(x;t)$ is made of are bounded and so is $q_1(x;t)$.

There is now only one term q_3 left in (7.3) to analyze. We are done if we show that $@_x y_0 \; 2 \; H^2$. Di⁄erentiating

$$y_0 + \mathrm{H} y_0 = \mathrm{H}1; \quad \mathrm{H} := \mathrm{H}(R_{x;t});$$

in x one has

$$@_x y_0 + \mathrm{H}(@_x y_0) = \quad @_x \mathrm{H}1 \quad (@_x \mathrm{H}) y_0:$$

Thus

$$@_x y_0 = \quad (I + \mathrm{H})^{\ 1} [(@_x \mathrm{H})1 + (@_x \mathrm{H}) y_0]:$$

It follows that we only need to show that $(@_x \mathrm{H})1 \; 2 \; H^2$ and $@_x \mathrm{H}$ is a bounded operator.

Note rst that

$$(@_x \mathrm{H}) = \mathrm{H}(2\mathrm{i}kR_{x;t}):$$

Since $kR(k) \; 2 \; L^2$ (from the second Zakharov-Faddeev trace formula),

$$(@_x \mathrm{H})1 = \mathrm{JP}\,(2\mathrm{i}kR(k)_{x;t}(k)) \; 2 \; H^2$$

as desired. The proof of boundedness of $(@_x \mathrm{H})$ is a bit more complicated. By (7.1) we have

$$\mathrm{H} = \mathrm{H}_1 + \mathrm{H}_2 + \mathrm{H}_3$$

where

$$\mathrm{H}_n := \mathrm{H}(\ _n\,_{x;t}); n = 1;2; \; \mathrm{H}_3 := \mathrm{H}(A_{x;t}):$$

For $n = 1;2$ both H_n admit a direct di⁄erentiation in x. Indeed, one can easily see that

$$@_x \mathrm{H}_n = \mathrm{H}(\ _n @_x\,_{x;t}) = \mathrm{H}(2\mathrm{i}k\ _n\,_{x;t}); n = 1;2:$$

Since $q;p \; 2 \; L^1$

$$2\mathrm{i}k\ _1(k) = T_0(k) q\mathrm{b}(k) \; 2 \; L^1 \tag{7.5}$$

and

$$\frac{T(k)}{\quad} \quad 2\mathrm{i}k\phi_2(k) = \ {}^{0}_{2\mathrm{i}k} \ \widehat{\ } \quad \partial_x \mathbb{H}(\phi_n), n = 1, 2, \qquad p(k) \; 2 \; L^1$$

and hence the operatorsare bounded. To di⁄erentiate H_3 we need rst to use (3.8). One has

$$\mathrm{H} = \mathrm{H}\,\mathrm{P}\,(R_{x;t}):$$

But

$$\begin{aligned}\mathbb{P}_-[A(k)\xi_{x,t}(k)] &= -\frac{1}{2\mathrm{i}\pi}\int \frac{A(\lambda)\xi}{\lambda-(k} \\ &= -\frac{1}{2\mathrm{i}\pi}\int_{\mathbb{R}+\mathrm{i}} \frac{A(\lambda)\xi_{x,t}(\lambda)}{\lambda-k}\mathrm{d}\lambda,\end{aligned}$$

$x;t\,(\,)$

___ __________d

i0)

where the integral is absolutely convergent, and therefore we may di⁄erentiate under the integral sign

$$\partial_x \mathbb{P}_- \left[A(k)\,\xi_{x,t}(k)\right] = -\frac{1}{2\mathrm{i}\pi}\int_{\mathbb{R}+\mathrm{i}} \frac{2\mathrm{i}\lambda A(\lambda)\,\xi_{x,t}(\lambda)}{\lambda - k}\mathrm{d}\lambda,$$

which is well-de ned and bounded. Consequently, $@_x$H3 is a bounded operator and so is $@_x$H. Thus, indeed $@_x y_0$ 2 H^2.

Remark 7.2. *Theorem 7.1 of* [17]*, which proof is based on the Dyson formula, relates smoothness of* q $(x;t)$ *with the decay of* q (x)*. In particular, it follows from that result that if* $q(x) \in L^1_{3/2} \cap L^2$ *then* q $(x;t)$ 2 $L^1_{\mathrm{loc}} \setminus L^2$ *for* $t>0$*: Stronger decay is due to the fact the Dyson formula involves* det$(I$ + H)*, which use requires to analyze di⁄erentiability of* H *in trace norm. (The latter is also technically much more involved. It was our attempt to dispose of trace norm considerations that led us to our trace formulas, which require uniform norms only.*

The following important consequence directly follows from Theorem 7.1 and invariance of the KdV with respect to $(x;t)$! (x; t).

Corollary 7.3. *The class* L^1_1 *is not preserved under the KdV ow.*

Proof. Suppose to the contrary that L^1_1 is preserved under the KdV ow. I.e. if $q(x) \in L^1_1$ then $q(x,t) \in L^1_1$ for any t. Take $q(x) \in L^1_1 \cap L^2$ but q (x) 2=L^1 and x $t_0>0$ By Theorem 7.1, $q_0(x) := q(x,t_0) \in L^\infty_{\mathrm{loc}} \cap L^2$. Take q_0 (x) as new initial data. By our assumption it is also in L^1_1. Thus $q_0(x) \in L^\infty_{\mathrm{loc}} \cap L^1_1 \cap L^2$. But this leads us to a contradiction as q_0 $(x;t_0)$ = q (x) was not assumed locally bounded.

In the conclusion we mention that much more general and precise statements can be made regarding how the KdV solutions gain regularity (smoothness) in exchange for loss of decay. We plan on showing elsewhere how the results of [17], [18], [29], and [31] may be improved to optimal statements.

8. Appendix

We demonstrate that the Deift-Trubowitz trace formula is actually a "nonlinearization" of our trace formulas. Assume for simplicity that there are no bound states (non-empty negative spectrum merely complicates the computations) and do our computation for the + sign only. The reader who has been able to get to this point should be able to follow the calculations below. Denoting H = H($R_{x;t}$), h := H1, 1_a := j j a, we have

$$\begin{aligned}
\pi q &= -2\partial_x \int \mathrm{Re}\, y = \frac{1}{\pi}\partial_x \int \mathrm{Re}\,(I+\mathbb{H})\ \ ^1 h \\
&= 2\partial_x \int \mathrm{Re}\,(I+\mathbb{H})^{-1} h \\
&= -2\int (I+\mathbb{H})^{-1}(\partial_x \mathbb{H})(\mathbb{I}+\mathbb{H})^{-1} h + 2\int (I+\mathbb{H})^{-1}\partial_x h \\
&=: q_1 + q_2.
\end{aligned}$$

For q_1 we have

$$q_1 = 2\lim_{a\to\infty} \left\langle (I+\mathcal{H})^{-1}(\partial_x \mathcal{H})(I+\mathcal{H})^{-1}h, 1_a \right\rangle.$$

For the inner product one has

$$\begin{aligned}
&\left\langle (I+\mathcal{H})^{-1}(\partial_x \mathcal{H})(I+\mathcal{H})^{-1}h, P_+1_a \right\rangle \\
&= \left\langle (\partial_x \mathcal{H})(I+\mathcal{H})^{-1}h, (I+\mathcal{H})^{-1}P_+1_a \right\rangle \\
&= \left\langle (\partial_x \mathcal{H})y, (I+\mathcal{H})^{-1}P_+1_a \right\rangle \\
&= \left\langle (\partial_x \mathcal{H})y, P_+1_a + (I+\mathcal{H})^{-1}\mathcal{H}P_+1_a \right\rangle \\
&= \langle (\partial_x \mathcal{H})y, P_+1_a \rangle + \left\langle (\partial_x \mathcal{H}), (I+\mathcal{H})^{-1}\mathcal{H}P_+1_a \right\rangle.
\end{aligned}$$

Passing to the limit yields

$$q_1 = 2\int (\partial_x \mathcal{H})y + \langle (\partial_x \mathcal{H})y, y\rangle.$$

One may now see how "nonlinear" dependence on y in (5.7) comes about. Indeed, the second term $\langle (\partial_x \mathcal{H})y, y\rangle$ is a quadratic form. For q_2 we similarly have

$$\begin{aligned}
q_2 &= 2\int (I+\mathcal{H})^{-1}\partial_x h = 2\lim_{a\to\infty} \left\langle (I+\mathcal{H})^{-1}(\partial_x h), P_+1_a \right\rangle = 2\lim_{a\to\infty} \left\langle \partial_x h, P_+1_a (I+\mathcal{H})^{-1}\mathcal{H}P_+1_a \right\rangle \\
&= \int \partial_x h + \langle \partial_x h, y\rangle.
\end{aligned}$$

Since

$$@_x \mathrm{H} f(k) = 2\mathrm{iJP}\; kR(k) e^{2ikx} f(k)$$

we have

$$q = q_1 + q_2$$

$$= \mathrm{h}(@_x \mathrm{H}) y; y\mathrm{i} + 2\mathrm{Z}\; (@_x \mathrm{H}) y + \mathrm{h} @_x h; y\mathrm{i} + 2\mathrm{Z}\; @_x h$$

$$= 2\mathrm{iZ}\; kR(k) e^{2ikx} y\; (k;x)^2\, \mathrm{d}k$$

$$+ 4\mathrm{iZ}\; kR(k) e^{2ikx} y\; (k;x) \mathrm{d}k + 2\mathrm{iZ}\; kR(k) e^{2ikx} \mathrm{d}k$$

$$= 2\mathrm{iZ}\; kR(k) e^{2ikx} [1 + y\; (k;x)]^2\, \mathrm{d}k$$

$$= 2\mathrm{iZ}\; kR(k)\; (x;k)^2\, \mathrm{d}k$$

and (5.7) with $c_n = 0$ follows.

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