Effect of Essential Skills Practice on Student Understanding of Probabilities in an Upper-Division Quantum Mechanics Course

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An important component of learning physics is being able to apply concepts and reasoning when solving problems. This can be especially challenging for students in quantum mechanics courses, in which the mathematical nature of the theory requires students adjust to new and unfamiliar ways of attaining "understanding". This paper describes the application of the Essential Skills Framework in an upper-division quantum mechanics course. A preliminary set of "essential skills" were proposed that underlie the solutions of common problems related to probabilities in quantum mechanics. Homework assignments were then developed that provided students with practice in applying these skills. The effect was assessed both by examining the accuracy and speed of students in using these skills over repeated homework assignments as well as through the impact on a standard course exam. Significant improvements were observed, although to a different extent on different types of questions. The results suggest that essential skills practice can be productively incorporated into courses on quantum mechanics, but certain skills are more difficult and may need special attention.

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I. INTRODUCTION

Quantum mechanics is recognized as a hard subject; not only are the concepts difficult and unintuitive, but it is expressed in abstract mathematical formalism, much of which is new to students. There has been increasing research examining the problems that students encounter in learning this topic, which ranges from probing their understanding of the many novel physical aspects of the theory [1, 2] to examining their ability to reason with and manipulate the mathematical objects in which the theory is encoded [3].

Ideally, when taking a course on quantum mechanics, students learn both the mathematics and the theory and become adept at translating between them. However, the unfamiliar language and the new ways in which concepts are expressed present a barrier [4, 5]. Even at the start of graduate instruction on quantum mechanics, many students have difficulty in relating mathematical quantities to the relevant concepts (*e.g.*, expectation values, probabilities, eigenvalues, *etc.*) [6].

One way of interpreting these results is that the combination of new tools and concepts induces considerable cognitive overhead for students [7]. When solving a problem or following an example presented in lecture, students may need to dedicate significant mental energy to simply recall which quantities are represented by different symbols and to distinguish between them. This can impede the ability of students to focus on and recognize the important takeaways of a lesson. It suggests that students need to develop skill in using the novel mathematics (*e.g.*, linear algebra) and learn how to quickly and coherently relate various mathematical representations to real-world phenomena.

In our courses we are using tutorials that are designed to address conceptual difficulties related to quantum mechanics [8]. However, we observe that students continue to struggle with applying the mathematical formalism. We decided to try to give students practice that could improve their fluency in applying the formalism and reduce this cognitive load.

To this end, we designed homework exercises using the online STEM Fluency system based on the Essential Skills Framework (ESF) by Andrew Heckler and Brendon Mikula at Ohio State University (OSU) [9]. It is intended to promote "fluency" by helping students develop both accuracy and speed in applying key concepts and skills. An example in the use of ESF is given by Heckler *et al.* in the context of vector algebra, which is ubiquitous in introductory sequences and presents difficulty for a wide variety of students [10].

This paper begins by discussing the STEM Fluency system used to provide students with focused practice and gives an overview of some of the skills we have targeted and the questions we have designed. It then presents preliminary results from student responses to a subset of these skills administered over the period of a quarter, as well as a comparison of results from an exam question given in courses with and without essential skills (ES) practice.

II. IMPLEMENTATION AND FORMS OF ASSESSMENT

A. Implementation

Essential skills practice was administered in weekly, required homework assignments via the STEM Fluency system at Ohio State University. This mastery-based, online system allows instructors to create assignments that consist of sets of questions organized into "categories" and "subcategories". Each assignment includes a set of categories and subcategories chosen by the instructor. The relevant questions administered are then given randomly within the assignment, so sequential questions come from different subcategories. Each category has a "mastery" level, which indicates the number of questions in that category that students must answer correctly in a row before they stop seeing questions from that category in that assignment (thus, having acquired "mastery" in that category). The assignment is complete, and students receive credit, once they acquire mastery in all the categories assigned. Individual students thus saw different numbers of questions based on their performance on prior questions. The individual questions in a given category are all very similar and differ only in the details of the context.

Nine assignments were administered in a junior-level quantum mechanics course ($N = 93$) in Fall 2021. The course used a positions-first textbook (as opposed to a spins-first textbook). On average, each assignment had four categories, each with a mastery level of three. About half of the skills were repeated at least once in the quarter. The assignments were expected to take less than 30 minutes on average. To compensate for the extra work, some homework was removed. All assignments were completed by the majority of the class.

Discussed in this paper is the category we developed for "Probabilities" (offered in Weeks 6 and 7). It had two subcategories *Math to Idea* and *Idea to Math*. These were developed by taking into account known student difficulties in quantum mechanics as well as our personal experience in interacting with students [11–13]. It seemed to us that students have difficulty in expressing probabilistic ideas in a mathematical form (*Idea to Math*) and, conversely, interpreting the mathematical expressions in terms of the relevant concepts (*Math to Idea*). Note that these questions were designed, to the extent possible, to test single ideas and skills and to be possible to complete quickly.

Figure 1 contains an example question from each subcategory. A full list of the categories and subcategories that were developed, as well as how we decided on the particular question hierarchy, will be discussed in a future article.

B. Forms of Assessment

We have used two methods to assess the impact of the STEM Fluency system on student understanding. The first is by examining the progression of student performance on a

Consider a spin$-\frac{1}{2}$ particle in the state $|\chi\rangle$. The eigenstates for spin in the z-direction are given by $|s_i\rangle$, where $s_i = \frac{1}{2}$ or $-\frac{1}{2}$. What do you call/how do you interpret the quantity given by $|\langle\frac{1}{2}|\chi\rangle|^2$?

(A) The probability of measuring the z-component of spin for our state to be $\frac{1}{2}$.
(B) The expectation value of the z-component of spin in the $|\chi\rangle$ state.
(C) The result of measuring the z-component of spin in our state $|\chi\rangle$.
(D) The wavefunction of $|\chi\rangle$ in the z-spin basis.
(E) The probability density for $|\chi\rangle$ in the z-spin basis.

(a)

Let H be a Hamiltonian with energy eigenstates $|n\rangle$ for $n = 1, 2, ...$ with associated energy eigenvalues E_n. In addition, consider a particle in the state $|\psi\rangle$. What is the probability of measuring E_2 for our particle?

(A) $|\langle 2|\psi\rangle|^2$
(B) $|\langle 2|H|\psi\rangle|^2$
(C) $\langle 2|\psi\rangle$
(D) $\langle 2|H|\psi\rangle$

(b)

FIG. 1. An example question from a) the Math to Idea subcategory and b) the Idea to Math subcategory of the Probabilities category. Questions in each subcategory had similar structures, involving many different observables and many different representations when available.

given category/subcategory after repeated exposures within STEM Fluency. The second is comparing student performance on exam questions that rely on those skills and that have also been given in prior classes that did not use STEM Fluency.

1. Assessment within STEM Fluency

STEM Fluency records each student response and the time taken for each question. We have been using subcategory score (the fraction of correctly answered questions in a given subcategory) as a proxy for "accuracy". A student who gets 100% accuracy in a given subcategory has not answered any of those questions incorrectly, whereas an accuracy of 50% indicates that the student had multiple incorrect answers. We use time per question (the total time spent on questions in a given subcategory divided by the number of attempted questions in that subcategory) as a proxy for "speed". Fluency is then a combination of both measures.

For each STEM Fluency assignment, we have found that a few students took a very long time on the assignment. The assignments are designed to be done in less than 30 minutes. Most students take less time, but some take up to 2 hours.

Consider a particle in the quantum mechanical infinite square well of width a. At $t = 0$, the normalized wavefunction for the particle is given by

$$\Psi(x, t=0) = \begin{cases} \frac{\sqrt{15}}{a^{5/2}}(x^2 - ax) & 0 < x < a \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

You may receive full credit on this problem without evaluating any integrals; however, writing down an integral without explaining where it comes from will not result in credit.

A. Suppose you were to measure the energy of this particle at $t = 0$. Determine the probability that the energy is equal to E_1, the energy of the ground state. Explain your reasoning.

C. Suppose you were to measure the position of this particle at $t = 0$ (no other measurements have been made). Determine the probability that the particle is measured to be between $x = 0$ and $x = a/3$. Explain your reasoning.

FIG. 2. Portions of an exam question administered in classes before and after STEM Fluency was incorporated. The questions assess student ability to determine probabilities for various observables.

We have dropped these students from the analysis, since they likely took time away from the assignment before finishing and their time is not reflective of the time they spent on individual questions.

2. Assessment through Exam Questions

To assess the effect of the ES practice, we analyzed results from two exam questions that were given before (Fall 2018) and after (Fall 2021) ES practice was incorporated into the course. Figure 2 shows the two relevant questions. Both questions can be regarded as testing the *Idea to Math* subcategory. The content and structure of the course between the two quarters was identical except for the ES practice. Although the instructors differed, our past experience in this and other courses suggest that typically the effect of the instructor is small. Thus, we believe that differences in performance on the question can likely be attributed to the ES practice.

III. RESULTS AND DISCUSSION

A. Assessment through STEM Fluency

The impact of the essential skills practice was assessed, in part, by analyzing changes in student accuracy and time per question (TpQ) from the first homework assignment (week 6) to the second (week 7). The results for the two probability subcategories *Math to Idea* and *Idea to Math* are shown

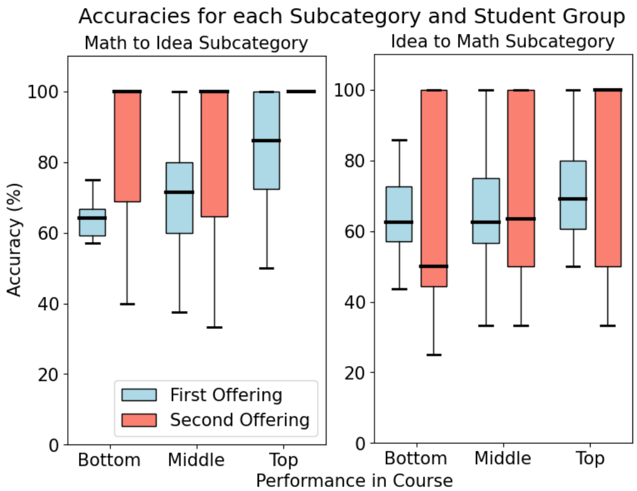

FIG. 3. Box plots for both the *Math to Idea* and *Idea to Math* subcategories of the Probability category (N = 93). Data are shown for the Top quartile, Middle two quartiles, and Bottom quartile based on final course performance. (Note: the Top and Bottom quartiles are divided into four parts each with about 6 students, and the Middle two quartiles have four parts with about 12 students each.)

through the box-plots in Figure 3. We were interested in determining the impact on all students, thus in each figure the data is divided according to class performance. The terms "Top", "Middle", and "Bottom" indicate whether the student's final course grade was in the top quartile, middle two quartiles, or in the bottom relative to final course grade. The large number of students with perfect accuracy on the second offering motivated the choice of using these three groups.

The "ideal" outcome from using STEM Fluency would be 1) accuracies would increase with a corresponding decrease in TpQ for each group, and 2) lower-performing students would have their final performance near that of the higher-performing students. Here, we focus on two aspects of the box-plots: the medians for each data set (represented by the solid horizontal black lines), and an overview of the distribution of student scores within the Top, Middle, and Bottom groups.

1. *Math to Idea Subcategory Results*

For the *Math to Idea* subcategory, the median accuracy increased substantially from the first to the second assignment. All three student groups (Top, Middle, and Bottom) started with a median accuracy between 65% and 85% on the first assignment, which increased to 100% on the second assignment. Thus, half or more of each group, and the class as a whole, performed perfectly with no errors before achieving mastery on the assignment. Interestingly, however, the median TpQ remained roughly the same across both assignments for all three groups. The medians started at 22, 22.4, and 22.9 seconds per question for the Top, Middle, and Bottom groups, respectively, and ended up at 20.8, 21, and 22.1 seconds per question. Thus, after a single repetition, students were completing questions on the *Math to Idea* subcategory at roughly the same rate, but with much higher accuracy.

Note that in the Top group, essentially every student had 100% accuracy. The bottom halves of the Middle and Bottom groups, despite the large number of students who achieved 100% accuracy, ended with a distribution of accuracies that was broader than it was on the first assignment.

2. *Idea to Math Subcategory Results*

The *Idea to Math* subcategory had a different outcome than the *Math to Idea* subcategory. On the first assignment, performance for the Top, Middle, and Bottom groups was more compressed, ranging from 65% to 70%, so these questions appeared to be harder for the Top group. On the second assignment, only the median accuracy of the top group increased, reaching 100%. The Middle and Bottom groups had mixed improvement. Each had a quarter of their students achieve 100% accuracy and the next highest performing quarter had a higher and broader range than was true for the first administration. However, the median accuracy of the Middle group was unchanged and that of the Bottom group may have had a decrease in median accuracy.

In addition, we saw that the distributions for the lower half of each group broadened considerably. Although many students achieved 100% accuracy on the second assignment, those who did not obtained a wider range of scores compared to those who did not obtain 100% on the first assignment.

Meanwhile, the TpQ of all three groups significantly decreased from the first iteration to the second. The medians started at 20.6, 20.6, and 23.4 seconds per question for the Top, Middle, and Bottom groups, respectively. They ended up at 14, 14.3, and 19.4 seconds per question. Thus, it seems that while the Top group had improvements in both their accuracy and TpQ, the bottom three quartiles of students answered the questions more quickly but with no positive shifts in their median accuracy.

3. *Summary and Interpretation*

Student performance on the *Math to Idea* and *Idea to Math* subcategories demonstrated somewhat different behaviors. The increases in the *Math to Idea* median accuracies for all groups after only one repetition might suggest that not much practice is required for students to be able to identify the concepts associated with mathematical expressions. The lesser improvement in the *Idea to Math* subcategory may suggest that translating between probabilistic concepts and mathematics is an asymmetric process, with one direction being more difficult than the other. It suggests that the Middle and Bottom group of students (and perhaps all) may need more practice in the *Idea to Math* subcategory for them to reach the improvements observed in *Math to Idea* subcategory.

On the other hand, the *Math to Idea* subcategory had no significant shifts in the TpQs from the first to the second assignment. This was true for all groups despite significant improvements in accuracy. The converse was seen in the *Idea to Math* subcategory in which all of the groups had a decrease in TpQ despite the bottom two groups having little or no change in their median accuracies.

One might expect that ES practice would initially result in a simultaneous increase in accuracy and decrease in TpQ and that these would level off with repeated practice. These results suggest that these two aspects do not necessarily improve together; extra practice may be necessary to achieve the desired improvements for both measures.

B. Assessment Through Exam Question

The essential skills practice was also assessed by using two exam questions given in a class ($N = 98$) without ES practice and in a class with it ($N = 77$)[14]. The questions are shown in Figure 2. Table 1 shows the percentage of students from each class who answered each question correctly. An answer was regarded as "correct" if students had a correct procedure. Questions of this form (short-answer and compound in nature) are different from those used in the ES practice, which had questions that were more atomistic (for example, requiring students to select which bra to use in a probability calculation, or to associate a physics or math concept with a particular mathematical object). The exam question thus requires that students be able to use the skills developed in STEM Fluency when they are not explicitly prompted.

TABLE I. Percentage of students who answered correctly the exam questions in Figure 2. These questions are of the type *Idea to Math* described previously.

Observable	**No SF** ($N = 98$)	**SF** ($N = 77$)
Energy	43%	66%
Position	80%	78%

On the energy probability question, about 43% of the students without STEM Fluency answered correctly. This increased to 66% for the class that used STEM Fluency. This 23% increase was statistically significant (with $\chi^2 = 8.547, df = 1$). On the position probability question, both classes did well (80% vs 78% correct), but there was no significant shift ($\chi^2 = .062, df = 1$). Given that the only difference was the introduction of STEM Fluency, it is possible that the ES practice may have played a role in improving student performance on the energy question. We believe that this question requires more steps in the solution than that for position and thus STEM Fluency may have played a larger role. However, additional research would be needed to draw conclusions.

IV. CONCLUSION

Weekly essential skills practice was administered in an upper-division quantum mechanics course. At the start of this project, we identified some skills that we believed research had demonstrated are difficult for students and are essential for students to be able to apply the concepts they were learning. In this paper, we focus on skills related to the concept of probability.

As part of our assessment, we looked at changes in student accuracy and the time required to answer questions within the STEM Fluency system (*e.g.*, performance on repeated assignments). Preliminary results have been encouraging. The scores of students on repeated assignments suggest that students at all levels (Top, Middle, and Bottom, as defined relative to their final course grade) can benefit. There were significant increases in the median accuracies for questions involving translating mathematical expressions in terms of a relevant probability (*Math to Idea*) for each of these three student groups. The results were not as strong for the converse, identifying the mathematical expression corresponding to a given probability (*Idea to Math*). However, even for this type of question we saw considerable improvement, especially among students in the Top group. The disparity in improvement for these two lines of reasoning may not be surprising, since coming up with a mathematical expression requires identifying the component parts (*e.g.*, bras and kets or elements of an integral) and then assembling them. This asymmetry does not appear to be documented in the literature on quantum mechanics and we plan to examine the implications as part of our future work.

We also found that increases in accuracy were not necessarily correlated with the time taken per question by students. This finding suggests that accuracy alone is not sufficient as a measure of fluency. Indeed, learning is a complex process where different aspects may improve at different rates.

Finally, we examined results from exam questions that required students to produce the mathematical expression corresponding to a particular probability (*Idea to Math*). Although not originally designed with STEM Fluency in mind, we saw large improvements on student performance on some of these questions as well.

The preliminary results presented here suggest that incorporating ES practice into a junior level course on quantum mechanics can be productive. This type of practice is not intended to replace other course components that help students develop a conceptual framework. However, the mathematics and formalism in quantum mechanics is sufficiently difficult and new for students that we believe targeted practice in interpreting and generating the mathematical expressions can be useful. In developing our practice questions, we drew on existing research into student difficulties associated with quantum mechanics. However, additional work is needed to determine whether the skills we identified and the questions we designed effectively address the needs of students.

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