SIAM J. MATH. ANAL.
Vol. 56, No. 4, pp. 5583–5603

APPROXIMATION OF CALOGERO–MOSER LATTICES BY BENJAMIN–ONO EQUATIONS*

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Abstract. We provide a rigorous validation that the infinite Calogero–Moser lattice can be well-approximated by solutions of the Benjamin–Ono equation in a long-wave limit.

Key words. Calogero–Moser, Benjamin–Ono, fractional KdV, FPUT lattices, modulation equations, approximation theory

MSC codes. 70F45, 35Q53, 35Q70, 34A33

DOI. 10.1137/24M1629869

1. Introduction.

The (generalized[1]) Calogero–Moser system is

$$\ddot{x}_j = -\alpha \sum_{m\geq 1} \left[\frac{1}{(x_{j+m}-x_j)^{\alpha+1}} - \frac{1}{(x_j - x_{j-m})^{\alpha+1}} \right]. \tag{1}$$

In the above, $j \in \mathbf{Z}$, $x_j \in \mathbf{R}$, $t \in \mathbf{R}$. The system can be interpreted as the governing equations for the positions $(x_j(t))$ of infinitely many particles arranged on a line and interacting pairwise through a power-law force.

Ingimarson and Pego in [7] state that for $\alpha \in (1,3)$ and in a certain scaling regime (the so-called *long-wave limit*) the system is formally approximated by a Benjamin–Ono-type equation. Here is a quick summary of their findings. Suppose that $u = u(X,\tau)$ solves the (generalized[2]) Benjamin–Ono equation

$$\kappa_1 \partial_\tau u + \kappa_2 u \partial_X u + \kappa_3 H|D|^\alpha u = 0. \tag{2}$$

In (2), H is the Hilbert transform on $\mathbf{R}$ and $|D| = H\partial_X$. We define these as Fourier[3] multiplier operators:

$$\widehat{Hf}(k) := -i\,\mathrm{sgn}(k)\widehat{f}(k) \quad \text{and} \quad \widehat{|D|^\alpha f}(k) := |k|^\alpha \widehat{f}(k).$$

The constants κ_1, κ_2, and κ_3 are determined from α by

$$\begin{aligned} &c_\alpha := \sqrt{\alpha(\alpha+1)\zeta_\alpha}, \quad \kappa_1 := 2c_\alpha, \quad \kappa_2 := \alpha(\alpha+1)(\alpha+2)\zeta_\alpha \\ &\quad \text{and} \quad \kappa_3 := \alpha(\alpha+1)\int_0^\infty \frac{1-\mathrm{sinc}^2(s/2)}{s^\alpha}\,ds. \end{aligned} \tag{3}$$

*Received by the editors January 8, 2024; accepted for publication (in revised form) April 29, 2024; published electronically August 6, 2024.
https://doi.org/10.1137/24M1629869
Funding: This work was supported by the National Science Foundation under grant DMS-2006172.
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[1]It is *the* Calogero–Moser system when $\alpha = 2$.
[2]It is *the* Benjamin–Ono equation when $\alpha = 2$.
[3]We use the following form of the Fourier transform: $\mathfrak{F}[f](k) := \widehat{f}(k) := (2\pi)^{-1}\int_{\mathbf{R}} f(X)e^{-ikX}dX$ and $\mathfrak{F}^{-1}[g](X) := \check{g}(X) := \int_{\mathbf{R}} g(k)e^{ikX}dk$. We use the Fourier transform to define Sobolev norms in the usual way: $\|f\|_{H^s} := \sqrt{\int_{\mathbf{R}}(1+k^2)^s|\widehat{f}(k)|^2dk}$.

Here $\zeta_s := \sum_{m\geq 1} 1/m^s$ is the ballyhooed zeta-function.

In [7], the authors show that if $u = -\partial_X v$ and

$$\tilde{x}_j(t) := j + \tilde{y}_j(t) \quad \text{and} \quad \tilde{y}_j(t) := \epsilon^{\alpha-2} v(\epsilon(j - c_\alpha t), \epsilon^\alpha t), \tag{4}$$

then

$$R_\epsilon(j,t) := \ddot{\tilde{x}}_j + \alpha \sum_{m\geq 1} \left[\frac{1}{(\tilde{x}_{j+m} - \tilde{x}_j)^{\alpha+1}} - \frac{1}{(\tilde{x}_j - \tilde{x}_{j-m})^{\alpha+1}} \right] \tag{5}$$

is formally $o(\epsilon^{2\alpha-1})$ as $\epsilon \to 0^+$. We call R_ϵ *the residual*, and it indicates the amount by which the approximation fails to satisfy (1). The scaling in (4) is what is referred to as the *long-wave scaling.*

Given the result from [7], one expects that if $x_j(t)$ solves (1) and $\tilde{x}_j(t)$ is given as in (4), then $x_j - \tilde{x}_j$ will be small in some appropriate sense. Indeed, our goal here is to provide a quantitative and rigorous error estimate on the difference. However, it turns out to be more natural to validate the approximation in terms of relative displacements and velocities instead of the position coordinates $x_j(t)$, that is, in terms of

$$r_j(t) := x_{j+1}(t) - x_j(t) - 1 \quad \text{and} \quad p_j(t) := \dot{x}_j(t).$$

The reason for this is that the total mechanical energy of (1) is expressed in terms of these variables and the validation process makes use of that energy (the interested reader can jump ahead to section 3 to see the details). This is the approach taken in many previous long-wave approximation results for Hamiltonian lattices (especially FPUT-type systems; see [11, 2, 3]).

And so if one suspects $x_j(t) \sim \tilde{x}_j(t)$, then some quick formal calculations show that one expects $r_j(t) \sim -\epsilon^{\alpha-1} u(\epsilon(j - c_\alpha t), \epsilon^\alpha t)$ and $p_j(t) \sim c_\alpha \epsilon^{\alpha-1} u(\epsilon(j - c_\alpha t), \epsilon^\alpha t)$. These arguments are made precise in our main result, which is the following.

THEOREM 1. *There exists $\alpha_* \in (1.45, 1.5)$ such that the following holds for $\alpha \in (\alpha_*, 3)$. Let*

$$\gamma_\alpha := \begin{cases} 2\alpha - 5/2, & \alpha \in (1, 2], \\ 3/2, & \alpha \in (2, 3), \end{cases}$$

and determine c_α, κ_1, κ_2, and κ_3 as in (3)*. Suppose that, for some $\tau_0 > 0$, $u(X,\tau)$ solves* (2) *for $|\tau| \leq \tau_0$ and $\sup_{|\tau|\leq\tau_0} \|u(\cdot,\tau)\|_{H^6} < \infty$. Then there exist $C_1, C_2, \epsilon_* > 0$, so the following holds for $\epsilon \in (0, \epsilon_*]$.*

If the initial data for (1) *satisfies*

$$r_j(0) = -\epsilon^{\alpha-1} u(\epsilon j, 0) + \bar{\mu}_j \quad \textit{and} \quad p_j(0) = c_\alpha \epsilon^{\alpha-1} u(\epsilon j, 0) + \bar{\nu}_j,$$

where

$$\|\bar{\mu}\|_{\ell^2} \leq C_1 \epsilon^{\gamma_\alpha} \quad \textit{and} \quad \|\bar{\nu}\|_{\ell^2} \leq C_1 \epsilon^{\gamma_\alpha},$$

then the solution of (1) *satisfies*

$$r_j(t) = -\epsilon^{\alpha-1} u(\epsilon(j - c_\alpha t), \epsilon^\alpha t) + \mu_j(t) \ \textit{and}\ p_j(t) = c_\alpha \epsilon^{\alpha-1} u(\epsilon(j - c_\alpha t), \epsilon^\alpha t) + \nu_j(t),$$

where

$$\sup_{|t|\leq\tau/\epsilon^\alpha} \|\mu(t)\|_{\ell^2} \leq C_2 \epsilon^{\gamma_\alpha} \quad \textit{and} \quad \sup_{|t|\leq\tau/\epsilon^\alpha} \|\nu(t)\|_{\ell^2} \leq C_2 \epsilon^{\gamma_\alpha}.$$

Remark 1. The theorem presents the absolute error made in the approximation. To compute the relative error we note that the long-wave scaling $X = \epsilon j$ implies $\|\epsilon^{\alpha-1}u(\epsilon(\cdot - c_\alpha t), \epsilon^\alpha t)\|_{\ell^2} \le C\epsilon^{\alpha-1/2}$ (see estimate (4.8) from Lemma 4.3 in [3]). This leads to a relative error like $C\epsilon^{\gamma_\alpha-\alpha+1/2} = C\epsilon^{1-|\alpha-2|}$. We do think the error estimates we compute here are sharp, though we do not have a proof of that.

Remark 2. In our proof, it comes out that we need $2\zeta_{\alpha+1} - \zeta_\alpha > 0$, and it is here that the restriction $\alpha > \alpha_*$ comes from. See Figure 1 below. Precisely, α_* is the positive solution of $2\zeta_{\alpha+1} - \zeta_\alpha = 0$. By deploying the tried and true method of using MAPLE to zoom in on the intersection of the graph of $2\zeta_{\alpha+1} - \zeta_\alpha$ with the α-axis, one finds that $\alpha_* \sim 1.478750785$. We do not claim the condition $2\zeta_{\alpha+1} - \zeta_\alpha > 0$ is necessary, but it does arise in a somewhat natural way.

Remark 3. The use of H^6 in the theorem is a worst-case scenario. It works for all $\alpha \in (\alpha_*, 3)$. If one wanted, one could determine a lower regularity condition on u which would depend on α. There is no pressing need for that in this article. One may wonder if H^6 solutions of (2) exist. The short answer is yes. To get more information, the introduction of [5] gives a terrific overview.

Remark 4. For $\alpha = 2$, there are known connections between special solutions of (1) and (2), which rely in part on the fact that both systems are integrable. In particular, in [8] it is shown that the poles of the multisoliton solutions of the Benjamin–Ono equation satisfy a (finite dimensional) Calogero–Moser system. This remarkable connection between the two systems is seemingly quite different from the long-wave limit uncovered in [7] and studied further here. Exploring the similarities and differences between the two reductions is a very interesting path for future research.

Remark 5. The Benjamin–Ono equation has served as long-wave limit in a variety of hydrodynamic problems; see [1] for an overview. The article [6] and recent preprint [9] contain rigorous validations of two different such limits, similar in some ways to what we have here.

Here is the plan of attack. First we make the formal estimates on R_ϵ from [7] rigorous in section 2. Then we prove a general approximation theorem in section 3. Last, in section 4 we put things together in the proof of Theorem 1.

2. Rigorous residual estimates. The first task is to make the formal estimate of the residual R_ϵ from [7] rigorous. Here is the result.

PROPOSITION 2. *If $u(X,\tau)$ is a solution of* (2) *with* $\sup_{|\tau|\le\tau_0} \|u(\cdot,\tau)\|_{H^6} < \infty$, *then there exist $C > 0$ and $\epsilon_0 > 0$ for which $\epsilon \in (0, \epsilon_0]$ implies*

$$\sup_{|t|\le\tau_0/\epsilon^\alpha} \|R_\epsilon(\cdot,t)\|_{\ell^2} \le C\epsilon^{\beta_\alpha},$$

where

$$\beta_\alpha := \begin{cases} 3\alpha - 5/2, & \alpha \in (1,2], \\ \alpha + 3/2, & \alpha \in (2,3). \end{cases}$$

Proof. The proof is technical, and we break it up into several parts: an analysis of the acceleration term, another for the force terms, and then a final section where we put everything together.

Part 1: the acceleration term. The chain rule and (4) give $\ddot{\tilde{x}}_j(t) = a_\epsilon(\epsilon(j - c_\alpha t), \epsilon^\alpha t)$, where

$$a_\epsilon := \epsilon^\alpha c_\alpha^2 \partial_X^2 v - 2c_\alpha \epsilon^{2\alpha-1}\partial_\tau\partial_X v + \epsilon^{3\alpha-2}\partial_\tau^2 v.$$

Using the relation $u = -\partial_X v$ along with the formula for κ_1 from above, we can convert this to

$$a_\epsilon = -\epsilon^\alpha c_\alpha^2 \partial_X u + \epsilon^{2\alpha-1}\kappa_1 \partial_\tau u + \epsilon^{3\alpha-2}\partial_\tau^2 v.$$

The first two terms on the right-hand side constitute the leading order part of a_ϵ and will ultimately combine and cancel with terms from the force down below. Thus the contribution from a_ϵ to R_ϵ will stem from $\epsilon^{3\alpha-2}\partial_\tau^2 v$.

With this in mind, we have

$$\sup_{|\tau|\le\tau_0} \|a_\epsilon + \epsilon^\alpha c_\alpha^2 \partial_X u - \epsilon^{2\alpha-1}\kappa_1\partial_\tau u\|_{H^1} = \sup_{|\tau|\le\tau_0} \epsilon^{3\alpha-2}\|\partial_\tau^2 v\|_{H^1}.$$

We need to estimate $\partial_\tau^2 v$ in terms of u, but this is somewhat ambiguous since all we have specified is that $u = -\partial_X v$. Here is what to do.[4] First we put

$$v(X,\tau) := -\int_0^X u(b,\tau)db + q(\tau) \tag{6}$$

for an as yet undetermined scalar function $q(\tau)$. This ensures that $u = -\partial_X v$ and $q(\tau)$ is in place to make sure that $\partial_\tau^2 v$ is in H^1. Differentiation of v with respect to τ followed by using (2) gives us

$$\partial_\tau v(X,\tau) = \int_0^X \left(\frac{\kappa_2}{2\kappa_1}(\partial_b u^2)(b,\tau) - \frac{\kappa_3}{\kappa_1}\partial_b |D|^{\alpha-1}u(b,\tau)\right) db + \dot q(\tau).$$

Note that in this computation we have used the fact that $H|D|^\alpha = -|D|^{\alpha-1}\partial_X$.

The fundamental theorem of calculus yields

$$\partial_\tau v(X,\tau) = \frac{\kappa_2}{2\kappa_1}u^2(X,\tau) - \frac{\kappa_3}{\kappa_1}|D|^{\alpha-1}u(X,\tau) - \frac{\kappa_2}{2\kappa_1}u^2(0,\tau) + \frac{\kappa_3}{\kappa_1}|D|^{\alpha-1}u(0,\tau) + \dot q(\tau).$$

We select

$$\dot q = \frac{\kappa_2}{2\kappa_1}u^2(0,\tau) - \frac{\kappa_3}{\kappa_1}|D|^{\alpha-1}u(0,\tau) \quad \text{and} \quad q(0) = 0 \tag{7}$$

so that

$$\partial_\tau v = \frac{\kappa_2}{2\kappa_1}u^2 - \frac{\kappa_3}{\kappa_1}|D|^{\alpha-1}u. \tag{8}$$

One more τ-differentiation gives us

$$\partial_\tau^2 v = \frac{\kappa_2}{\kappa_1}u\partial_\tau u - \frac{\kappa_3}{\kappa_1}|D|^{\alpha-1}\partial_\tau u.$$

Using the Sobolev inequality and counting derivatives provides us the estimate

$$\|\partial_\tau^2 v\|_{H^1} \le C\|u\|_{H^1}\|\partial_\tau u\|_{H^1} + C\|\partial_\tau u\|_{H^\alpha}.$$

Taking the H^s norm of both sides of (2) tells us that $\|\partial_\tau u\|_{H^s} \le C(\|u\|_{H^{s+1}}^2 + \|u\|_{H^{s+\alpha}})$. In turn this gives

$$\|\partial_\tau^2 v\|_{H^1} \le C\|u\|_{H^1}\left(\|u\|_{H^2}^2 + \|u\|_{H^{1+\alpha}}\right) + C\left(\|u\|_{H^{\alpha+1}}^2 + \|u\|_{H^{2\alpha}}\right).$$

[4]Special thanks go to one of the referees for pointing out an error in the first draft related to this part of the paper and suggesting this remedy.

Since $\alpha < 3$ and we have assumed a uniform bound on $u \in H^6$ for $|\tau| \le \tau_0$, we can conclude that

$$\sup_{|\tau|\le\tau_0} \|a_\epsilon + \epsilon^\alpha c_\alpha^2 \partial_X u - \epsilon^{2\alpha-1}\kappa_1 \partial_\tau u\|_{H^1} \le C\epsilon^{3\alpha-2}. \tag{9}$$

Part 2: the force term. The authors of [7] show that

$$\widetilde{x}_{j+m} - \widetilde{x}_j = m - m\epsilon^{\alpha-1} A_{\epsilon m} u(X,\tau) \quad \text{and} \quad \widetilde{x}_j - \widetilde{x}_{j-m} = m - m\epsilon^{\alpha-1} A_{-\epsilon m} u(X,\tau),$$

where

$$A_h u(X,\tau) := \frac{1}{h}\int_0^h u(X+z,\tau)dz.$$

If we let

$$V_m(g) := \frac{1}{(m+g)^\alpha} - \frac{1}{m^\alpha} + \frac{\alpha}{m^{\alpha+1}} g \tag{10}$$

so that

$$V_m'(g) = -\frac{\alpha}{(m+g)^{\alpha+1}} + \frac{\alpha}{m^{\alpha+1}},$$

then the force terms in R_ϵ can be rewritten as

$$\alpha \sum_{m\ge1} \left[\frac{1}{(\widetilde{x}_{j+m} - \widetilde{x}_j)^{\alpha+1}} - \frac{1}{(\widetilde{x}_j - \widetilde{x}_{j-m})^{\alpha+1}} \right] = F_\epsilon(\epsilon(j - c_\alpha t), \epsilon^\alpha t),$$

where

$$F_\epsilon(X,\tau) := -\sum_{m\ge1} \left[V_m'(-m\epsilon^{\alpha-1} A_{\epsilon m} u(X,\tau)) - V_m'(-m\epsilon^{\alpha-1} A_{-\epsilon m} u(X,\tau)) \right].$$

A combination of the fundamental theorem of calculus and Taylor's theorem implies

$$V_m'(g_+) - V_m'(g_-) = V_m''(0)(g_+ - g_-) + \frac{1}{2} V_m'''(0)(g_+^2 - g_-^2) + \int_{g_-}^{g_+} E_m(\sigma) d\sigma,$$

where

$$E_m(\sigma) := \int_0^\sigma V_m''''(\phi)(\sigma - \phi) d\phi.$$

This leads to the expansion

$$F_\epsilon = \alpha(\alpha+1) L_\epsilon + \frac{\alpha(\alpha+1)(\alpha+2)}{2} N_\epsilon + M_\epsilon, \tag{11}$$

where

$$L_\epsilon := \epsilon^{\alpha-1} \sum_{m\ge1} \frac{1}{m^{\alpha+1}} \left(A_{\epsilon m} - A_{-\epsilon m} \right) u,$$

$$N_\epsilon := \epsilon^{2\alpha-2} \sum_{m\ge1} \frac{1}{m^{\alpha+1}} \left((A_{\epsilon m} u)^2 - (A_{-\epsilon m} u)^2 \right),$$

$$M_\epsilon := -\sum_{m\ge1} \int_{-m\epsilon^{\alpha-1} A_{-\epsilon m} u}^{-m\epsilon^{\alpha-1} A_{\epsilon m} u} E_m(\sigma) d\sigma.$$

The terms L_ϵ and N_ϵ coincide with their forms in [7]. The remaining term, M_ϵ, is lumped into a generic $o(\epsilon^{2\alpha+1})$ term there.

Part 2a: *Estimates for* L_ϵ. We follow the blueprint provided by [7]. Using the fact that $\widehat{A_h u}(k) = \widehat{u}(k)(e^{ikh}-1)/ikh$, they show that

$$\widehat{L}_\epsilon(k) = \epsilon^\alpha \zeta_\alpha ik\widehat{u}(k) + \epsilon^{2\alpha-1} ik|k|^{\alpha-1}\widehat{u}(k)\left(|k|\epsilon \sum_{m\geq 1} \frac{\operatorname{sinc}^2(|k|\epsilon m/2)-1}{(|k|\epsilon m)^\alpha}\right).$$

Let

$$\eta_\alpha(h) := h\sum_{m\geq 1}\frac{1-\operatorname{sinc}^2(hm/2)}{(hm)^\alpha}.$$

A key observation from [7] is that $\eta_\alpha(h)$ is the approximation of

$$\eta_\alpha := \int_0^\infty \frac{1-\operatorname{sinc}^2(s/2)}{s^\alpha}ds$$

using the rectangular rule with right-hand endpoints. As such, $\lim_{h\to 0^+}\eta_\alpha(h) = \eta_\alpha$. Note that η_α is finite so long as $\alpha \in (1,3)$.

Then we have

$$\begin{aligned}\widehat{L}_\epsilon(k) &= \epsilon^\alpha\zeta_\alpha ik\widehat{u}(k) - \epsilon^{2\alpha-1}\eta_\alpha ik|k|^{\alpha-1}\widehat{u}(k)\\ &\quad + \epsilon^{2\alpha-1}ik|k|^{\alpha-1}\widehat{u}(k)\left(\eta_\alpha - \eta_\alpha(\epsilon|k|)\right).\end{aligned} \tag{12}$$

$\square$

What is the error made by approximating $\eta_\alpha(\epsilon|k|)$ by η_α? To determine this, we need the following.

LEMMA 3. *For* $\alpha\in(1,2]$, *there exists* $C>0$ *for which* $|\eta_\alpha(h)-\eta_\alpha|\leq Ch$ *for all* $h>0$. *If* $\alpha\in(2,3)$, *there exists* $C>0$ *for which* $|\eta_\alpha(h)-\eta_\alpha|\leq Ch^{3-\alpha}$ *for all* $h>0$.

Proof. If the integral were not improper, this would be an elementary estimate. But it is. In fact when $\alpha\in(2,3)$ it is improper at $s=0$ and that is why the estimate is worse in that setting. Also, when $\alpha\in(1,2)$ the derivative of the integrand

$$f_\alpha(s) := \frac{1-\operatorname{sinc}^2(s/2)}{s^\alpha}$$

diverges as $s\to 0^+$, which complicates things.

First we deal with $h\geq 1$. We have

$$|\eta_\alpha(h)-\eta_\alpha| \leq \eta_\alpha + h\sum_{m\geq 1}\frac{1-\operatorname{sinc}^2(hm/2)}{(hm)^\alpha}.$$

Since $\operatorname{sinc}^2(s)\in[0,1]$ for all $s\in\mathbf{R}$, we make an easy estimate:

$$|\eta_\alpha(h)-\eta_\alpha| \leq \eta_\alpha + h\sum_{m\geq 1}\frac{1}{(hm)^\alpha} = \eta_\alpha + h^{1-\alpha}\zeta_\alpha \leq (\eta_\alpha+\zeta_\alpha)h \leq Ch.$$

So $h\geq 1$ is taken care of for all $\alpha\in(1,3)$.

Now fix $h\in(0,1)$. We break things up:

$$\eta_\alpha(h)-\eta_\alpha = \underbrace{h\sum_{m=1}^{\lceil 1/h\rceil} f_\alpha(mh) - \int_0^{h\lceil 1/h\rceil} f_\alpha(s)ds}_{IN} + \underbrace{h\sum_{m\geq\lceil 1/h\rceil+1} f_\alpha(mh) - \int_{h\lceil 1/h\rceil}^\infty f_\alpha(s)ds}_{OUT}.$$

For OUT, by standard integral identities and the integral version of the mean value theorem we have

$$OUT = \sum_{m\geq\lceil 1/h\rceil+1}\left(hf_\alpha(mh) - \int_{(m-1)h}^{mh} f_\alpha(s)ds\right) = h\sum_{m\geq\lceil 1/h\rceil+1}\left(f_\alpha(mh) - f_\alpha(s_m)\right).$$

Here $s_m \in [(m-1)h, mh]$. Then we use the derivative version of the mean value theorem to get

$$OUT = h\sum_{m\geq\lceil 1/h\rceil+1} f'_\alpha(\sigma_m)(mh - s_m),$$

where $\sigma_m \in [s_m, mh]$. Note that $|mh - s_m| \leq h$.

Routine calculations show that there is a constant $C>0$ such that $|f'_\alpha(s)| \leq Cs^{-\alpha-1}$ for $s\geq 1$. Since $s^{-\alpha-1}$ is a decreasing function, these considerations lead to

$$|OUT| \leq Ch^2\sum_{m\geq\lceil 1/h\rceil+1}[(m-1)h]^{-\alpha-1} = Ch^2\sum_{m\geq\lceil 1/h\rceil}[mh]^{-\alpha-1}.$$

Next, $h\sum_{m\geq\lceil 1/h\rceil}[mh]^{-\alpha-1}$ is the approximation of $\int_{h\lceil 1/h\rceil-h}^\infty s^{-\alpha-1}ds$ using the rectangular rule with right-hand endpoints. Since $s^{-\alpha-1}$ is decreasing, we know that $h\sum_{m\geq\lceil 1/h\rceil}[mh]^{-\alpha-1} \leq \int_{h\lceil 1/h\rceil-h}^\infty s^{-\alpha-1}ds$. Also, since $h\in(0,1)$, we have $h\lceil 1/h\rceil - h \geq 1/2$, and so $\int_{h\lceil 1/h\rceil-h}^\infty s^{-\alpha-1} \leq \int_{1/2}^\infty s^{-\alpha-1} = 2^\alpha/\alpha$. Putting these together implies $|OUT| \leq Ch$.

For IN, we need to desingularize the integrand at $s=0$. Putting

$$f_\alpha(s) = \underbrace{\frac{1-(s^2/12)-\mathrm{sinc}^2(s/2)}{s^\alpha}}_{g_\alpha(s)} + \frac{1}{12}s^{2-\alpha}$$

gives

$$IN = \left(h\sum_{m=1}^{\lceil 1/h\rceil} g_\alpha(mh) - \int_0^{h\lceil 1/h\rceil} g_\alpha(s)ds\right) + \frac{1}{12}\left(h\sum_{m=1}^{\lceil 1/h\rceil}(mh)^{2-\alpha} - \int_0^{h\lceil 1/h\rceil} s^{2-\alpha}ds\right).$$

Taylor's theorem tells us that $|1-(s^2/12)-\mathrm{sinc}^2(s/2)|/s^4$ is bounded as $s\to 0$, and as a byproduct we see that $g_\alpha(s)$ is C^1 on the interval $[0,2]$. Routine error estimates for approximating integrals with rectangles tells us $|h\sum_{m=1}^{\lceil 1/h\rceil} g_\alpha(mh) - \int_0^{h\lceil 1/h\rceil} g_\alpha(s)ds| \leq Ch$.

So what remains is to estimate the singular piece

$$SING_\alpha = \left|h\sum_{m=1}^{\lceil 1/h\rceil}(mh)^{2-\alpha} - \int_0^{h\lceil 1/h\rceil} s^{2-\alpha}ds\right|.$$

Note that if $\alpha = 2$, then $SING_\alpha = 0$, so that case is pretty easy. But the cases $\alpha\in(1,2)$ and $\alpha\in(2,3)$ require some care.

We know that $h\sum_{m=1}^{\lceil 1/h\rceil}(mh)^{2-\alpha}$ is the rectangular approximation of $\int_0^{h\lceil 1/h\rceil} s^{2-\alpha}ds$ using right-hand endpoints, but it is also the rectangular approximation of $\int_0^{h\lceil 1/h\rceil}(s+h)^{2-\alpha}ds$ using left-hand endpoints. If $\alpha\in(2,3)$, then $s^{2-\alpha}$ is a decreasing function and we get the following chain of inequalities:

$$\int_0^{h\lceil 1/h\rceil} (s+h)^{2-\alpha} ds \le h \sum_{m=1}^{\lceil 1/h\rceil} (mh)^{2-\alpha} \le \int_0^{h\lceil 1/h\rceil} s^{2-\alpha} ds.$$

On the other hand, if $\alpha \in (1,2)$, then $s^{2-\alpha}$ is increasing and we have

$$\int_0^{h\lceil 1/h\rceil} (s+h)^{2-\alpha} ds \ge h \sum_{m=1}^{\lceil 1/h\rceil} (mh)^{2-\alpha} \ge \int_0^{h\lceil 1/h\rceil} s^{2-\alpha} ds.$$

Either of the chains tells us

$$\begin{aligned} SING_\alpha &\le \left| \int_0^{h\lceil 1/h\rceil} \left(s^{2-\alpha} - (s+h)^{2-\alpha}\right) ds \right| \\ &= \frac{1}{3-\alpha} \left| (h\lceil 1/h\rceil)^{3-\alpha} - (h\lceil 1/h\rceil + h)^{3-\alpha} + h^{3-\alpha} \right| \\ &\le \frac{1}{3-\alpha} \left| (h\lceil 1/h\rceil)^{3-\alpha} - (h\lceil 1/h\rceil + h)^{3-\alpha} \right| + \frac{1}{3-\alpha} h^{3-\alpha}. \end{aligned}$$

The mean value theorem gives $(3-\alpha)^{-1}|(h\lceil 1/h\rceil)^{3-\alpha} - (h\lceil 1/h\rceil + h)^{3-\alpha}| = hh_*^{2-\alpha}$, where h_* is in between $h\lceil 1/h\rceil$ and $h\lceil 1/h\rceil + h$. These numbers are in the interval $[1,3]$, and so we have $(3-\alpha)^{-1}|(h\lceil 1/h\rceil)^{3-\alpha} - (h\lceil 1/h\rceil + h)^{3-\alpha}| \le Ch$. Therefore, $|SING_\alpha| \le Ch + Ch^{3-\alpha}$.

Everything all together tells us that $h \in (0,1)$ and $\alpha \in (1,3)$ imply $|\eta_\alpha(h) - \eta_\alpha| \le Ch + Ch^{3-\alpha}$. If $\alpha \in (1,2]$, then $h \le h^{3-\alpha}$ and the inequality flips for $\alpha \in (2,3)$. That finishes the proof. ∎

With Lemma 3, (12) implies

$$\left| \widehat{L}_\epsilon(k) - \epsilon^\alpha \zeta_\alpha ik\widehat{u}(k) + \epsilon^{2\alpha-1}\eta_\alpha ik|k|^{\alpha-1}\widehat{u}(k) \right| \le C\epsilon^{2\alpha-1+r_\alpha}|k|^{\alpha+r_\alpha}|\widehat{u}(k)|,$$

where

$$r_\alpha := \begin{cases} 1, & \alpha \in (1,2] \\ 3-\alpha, & \alpha \in (2,3). \end{cases}$$

This in turn implies (along with the assumed uniform estimate for u) that

$$\sup_{|\tau|\le\tau_0} \|L_\epsilon - \epsilon^\alpha \zeta_\alpha \partial_X u - \epsilon^{2\alpha-1}\eta_\alpha H|D|^\alpha u\|_{H^1} \le C\epsilon^{2\alpha-1+r_\alpha}. \tag{13}$$

Part 2b: *Estimates for* N_ϵ. Some easy algebra leads to

$$\begin{aligned} N_\epsilon = {} & 2\epsilon^{2\alpha-2} u \sum_{m\ge1} \frac{1}{m^{\alpha+1}} (A_{\epsilon m}u - A_{-\epsilon m}u) \\ & + \epsilon^{2\alpha-2} \sum_{m\ge1} \frac{1}{m^{\alpha+1}} (A_{\epsilon m}u + A_{-\epsilon m}u - 2u)(A_{\epsilon m}u - A_{-\epsilon m}u). \end{aligned}$$

We recognize that L_ϵ is lurking in the first term. That and a little subtraction action get us to

$$\begin{aligned} N_\epsilon - 2\epsilon^{2\alpha-1}\zeta_\alpha u\partial_X u = {} & 2\epsilon^{\alpha-1} u\left(L_\epsilon - \epsilon^\alpha \zeta_\alpha \partial_X u\right) \\ & + \epsilon^{2\alpha-2} \sum_{m\ge1} \frac{1}{m^{\alpha+1}} (A_{\epsilon m}u + A_{-\epsilon m}u - 2u)(A_{\epsilon m}u - A_{-\epsilon m}u). \end{aligned}$$

We take the H^1 norm and use triangle and Sobolev:

$$\begin{aligned}&\|N_\epsilon - 2\epsilon^{2\alpha-1}\zeta_\alpha u\partial_X u\|_{H^1}\\ &\quad\le 2\epsilon^{\alpha-1}\|u\|_{H^1}\|L_\epsilon - \epsilon^\alpha\zeta_\alpha\partial_X u\|_{H^1}\\ &\qquad + \epsilon^{2\alpha-2}\sum_{m\ge1}\frac{1}{m^{\alpha+1}}\|A_{\epsilon m}u + A_{-\epsilon m}u - 2u\|_{H^1}\|A_{\epsilon m}u - A_{-\epsilon m}u\|_{H^1}.\end{aligned} \tag{14}$$

The estimate (13) tells us that $2\epsilon^{\alpha-1}\|u\|_{H^1}\|L_\epsilon - \epsilon^\alpha\zeta_\alpha\partial_X u\|_{H^1} \le C\epsilon^{3\alpha-2}\|u\|^2_{1+\alpha+r_\alpha}$.

To control the remaining term in (14), we will use the following estimates (see [4] for the proof).

LEMMA 4. *There is $C > 0$ such that for all $h > 0$ and $s \in \mathbf{R}$,*

$$\begin{aligned}\|A_h u\|_{H^s} &\le C\|u\|_{H^s},\\ \|A_h u + A_{-h}u - 2u\|_{H^s} &\le Ch^2\|u\|_{H^{s+2}},\\ \|A_h u - A_{-h}u\|_{H^s} &\le Ch\|u\|_{H^{s+1}},\\ \|A_h u - u\|_{H^s} &\le Ch\|u\|_{H^{s+1}}.\end{aligned}$$

We have to deal with terms like $A_{\epsilon m}$, and so the above result will be helpful when ϵm is "small" but not very useful otherwise. So we break things up:

$$\begin{aligned}&\epsilon^{2\alpha-2}\sum_{m\ge1}\frac{1}{m^{\alpha+1}}\|A_{\epsilon m}u + A_{-\epsilon m}u - 2u\|_{H^1}\|A_{\epsilon m}u - A_{-\epsilon m}u\|_{H^1}\\ &= \epsilon^{2\alpha-2}\sum_{m=1}^{\lfloor 1/\epsilon\rfloor}\frac{1}{m^{\alpha+1}}\|A_{\epsilon m}u + A_{-\epsilon m}u - 2u\|_{H^1}\|A_{\epsilon m}u - A_{-\epsilon m}u\|_{H^1}\\ &\quad + \epsilon^{2\alpha-2}\sum_{m\ge\lfloor 1/\epsilon\rfloor}\frac{1}{m^{\alpha+1}}\|A_{\epsilon m}u + A_{-\epsilon m}u - 2u\|_{H^1}\|A_{\epsilon m}u - A_{-\epsilon m}u\|_{H^1}\\ &= I + II.\end{aligned}$$

The second and third estimates from Lemma 4 give $I \le C\epsilon^{2\alpha+1}\|u\|^2_{H^3}\sum_{m=1}^{\lfloor 1/\epsilon\rfloor} m^{2-\alpha}$. A classic "integral comparison" tells us that $\sum_{m=1}^{\lfloor 1/\epsilon\rfloor} m^{2-\alpha} \le C\epsilon^{\alpha-3}$. So then $I \le C\epsilon^{3\alpha-2}\|u\|^2_{H^3}$.

For II, we use the first estimate in Lemma 4 to get $II \le C\epsilon^{2\alpha-2}\|u\|^2_{H^1}\sum_{m>\lfloor 1/\epsilon\rfloor}\frac{1}{m^{\alpha+1}}$. Then another integral-type estimate tells us $\sum_{m>\lfloor 1/\epsilon\rfloor}\frac{1}{m^{\alpha+1}} \le C\epsilon^\alpha$. So then $II \le C\epsilon^{3\alpha-2}\|u\|^2_{H^1}$. Therefore, we have our final estimate for N_ϵ:

$$\sup_{|\tau|\le\tau_0}\|N_\epsilon - 2\epsilon^{2\alpha-1}\zeta_\alpha u\partial_X u\|_{H^1} \le C\epsilon^{3\alpha-2}. \tag{15}$$

Part 2c: *Estimates for M_ϵ.* We need to treat $\|M_\epsilon\|_{L^2}$ and $\|\partial_X M_\epsilon\|_{L^2}$ separately, and we start with the former. A standard estimate shows

$$|M_\epsilon| \le \sum_{m\ge1} m\epsilon^{\alpha-1}|(A_{\epsilon m} - A_{-\epsilon m})u|\sup_{\sigma\in I_m}|E_m(\sigma)|,$$

where I_m is the interval between $-m\epsilon^{\alpha-1}A_{\epsilon m}u$ and $-m\epsilon^{\alpha-1}A_{-\epsilon m}u$.

If we assume $\sigma > 0$, then

$$|E_m(\sigma)| \le \int_0^\sigma |V_m''''(\phi)||\sigma-\phi|d\phi \le \sup_{0\le\phi\le\sigma}|V_m''''(\phi)|\int_0^\sigma|\sigma-\phi|d\phi = \frac{1}{2}\sup_{0\le\phi\le\sigma}|V_m''''(\phi)|\sigma^2.$$

$|V_m''''(\phi)|$ is decreasing, and so $\sup_{0\le\phi\le\sigma}|V_m''''(\phi)| = |V_m''''(0)| = C/m^{\alpha+4}$. So in this case

$$|E_m(\sigma)| \le \frac{C\sigma^2}{m^{\alpha+4}}.$$

Similarly, if $\sigma \le 0$, then

$$|E_m(\sigma)| \le \int_\sigma^0 |V_m''''(\phi)||\sigma-\phi|d\phi \le \frac{1}{2}\sup_{0\le\phi\le\sigma}|V_m''''(\phi)|\sigma^2 \le \frac{C\sigma^2}{(m+\sigma)^{\alpha+4}}.$$

In either case we had

$$|E_m(\sigma)| \le \frac{C\sigma^2}{(m-|\sigma|)^{\alpha+4}}. \tag{16}$$

Note that the constant C here is independent of m.

We have, using Lemma 4 and Sobolev,

$$\sigma \in I_m \implies |\sigma| \le Cm\epsilon^{\alpha-1}\|u\|_{H^1}. \tag{17}$$

In particular, by ensuring that ϵ is not so large we have

$$\sigma \in I_m \implies |\sigma| \le m/2. \tag{18}$$

So (16), (17), and (18) give

$$\sup_{\sigma\in I_m}|E_m(\sigma)| \le \left(\frac{Cm^2\epsilon^{2\alpha-2}\|u\|_{H^1}^2}{(m-m/2)^{\alpha+4}}\right) \le \frac{C\epsilon^{2\alpha-2}}{m^{\alpha+2}}\|u\|_{H^1}^2. \tag{19}$$

In turn, this gives $|M_\epsilon| \le C\epsilon^{3\alpha-3}\|u\|_{H^1}^2\sum_{m\ge1}\frac{1}{m^{\alpha+1}}|(A_{\epsilon m}-A_{-\epsilon m})u|$. Then Lemma 4 leads us to

$$\|M_\epsilon\|_{L^2} \le C\epsilon^{3\alpha-2}\|u\|_{H^1}^3\sum_{m\ge1}\frac{1}{m^\alpha} \le C\zeta_\alpha\epsilon^{3\alpha-2}\|u\|_{H^1}^3.$$

Next, we compute using the fundamental theorem and some algebra

$$\partial_X M_\epsilon = \epsilon^{\alpha-1}\sum_{m\ge1} m\left[E_m(-m\epsilon^{\alpha-1}A_{\epsilon m}u)A_{\epsilon m}\partial_X u - E_m(-m\epsilon^{\alpha-1}A_{-\epsilon m}u)A_{-\epsilon m}\partial_X u\right].$$

Adding zero takes us to

$$\begin{aligned}\partial_X M_\epsilon &= \epsilon^{\alpha-1}\sum_{m\ge1} mE_m(-m\epsilon^{\alpha-1}A_{\epsilon m}u)\left(A_{\epsilon m}-A_{-\epsilon m}\right)\partial_X u\\ &\quad + \epsilon^{\alpha-1}\sum_{m\ge1} m\left(E_m(-m\epsilon^{\alpha-1}A_{\epsilon m}u) - E_m(-m\epsilon^{\alpha-1}A_{-\epsilon m}u)\right)A_{-\epsilon m}\partial_X u\\ &= III + IV.\end{aligned}$$

Using (16) and the same reasoning that led to (18) yields

$$\begin{aligned}|III| &\le C\epsilon^{\alpha-1}\sum_{m\ge1} m\frac{\left(m\epsilon^{\alpha-1}A_{\epsilon m}u\right)^2}{m^{\alpha+4}}\left|(A_{\epsilon m}-A_{-\epsilon m})\partial_X u\right|\\ &\le C\epsilon^{3\alpha-3}\sum_{m\ge1}\frac{1}{m^{\alpha+1}}|A_{\epsilon m}u|^2\left|(A_{\epsilon m}-A_{-\epsilon m})\partial_X u\right|.\end{aligned}$$

Sobolev and the first estimate in Lemma 4 imply $|A_{\epsilon m} u(X)| \leq C\|u\|_{H^1}$, and so

$$|III| \leq C\epsilon^{3\alpha-3}\|u\|_{H^1}^2 \sum_{m\geq 1} \frac{1}{m^{\alpha+1}} \left|(A_{\epsilon m} - A_{-\epsilon m})\partial_X u\right|.$$

We take the L^2 norm of the above, use the third estimate in Lemma 4, and do the resulting sum to obtain

$$\|III\|_{L^2} \leq C\epsilon^{3\alpha-3}\|u\|_{H^1}^2 \sum_{m\geq 1} \frac{1}{m^{\alpha+1}} \epsilon m \|\partial_X u\|_{H^1} \leq C\epsilon^{3\alpha-2}\|u\|_{H^2}^3.$$

For IV, routine estimates and the mean value theorem give

$$|IV| \leq \epsilon^{2\alpha-2} \sum_{m\geq 1} m^2 |A_{-\epsilon m}\partial_X u||(A_{\epsilon m} - A_{-\epsilon m})u| \sup_{\sigma\in I_m} |E_m'(\sigma)|,$$

where I_m is consistent with its definition above. Reasoning analogous to that which led to (16) can be used to show that $|E_m'(\sigma)| \leq C|\sigma|/(m-|\sigma|)^{\alpha+4}$. And then (17) and (18) imply $\sup_{\sigma\in I_m} |E_m'(\sigma)| \leq \frac{C\epsilon^{\alpha-1}}{m^{\alpha+3}}\|u\|_{H^1}$. So

$$|IV| \leq C\epsilon^{3\alpha-3}\|u\|_{H^1} \sum_{m\geq 1} \frac{1}{m^{\alpha+1}} |A_{-\epsilon m}\partial_X u||(A_{\epsilon m} - A_{-\epsilon m})u|.$$

Using Sobolev and the estimates in Lemma 4 in ways we have done above give $\|IV\|_{L^2} \leq C\epsilon^{3\alpha-2}\|u\|_{H^1}^3$. This, in conjunction with the above estimates for $\|III\|_{L^2}$ and $\|M_\epsilon\|_{L^2}$, results in

$$\sup_{|\tau|\leq\tau_0} \|M_\epsilon\|_{H^1} \leq C\epsilon^{3\alpha-2}. \tag{20}$$

Part 3: finishing touches. Using the decomposition of F_ϵ from (11) and adding and subtracting a lot of terms, we get

$$\begin{aligned}
a_\epsilon + F_\epsilon &= a_\epsilon + \alpha(\alpha+1)L_\epsilon + \frac{\alpha(\alpha+1)(\alpha+2)}{2}N_\epsilon + M_\epsilon \\
&= a_\epsilon + \epsilon^\alpha c_\alpha^2 \partial_X u - \epsilon^{2\alpha-1}\kappa_1\partial_\tau u \\
&\quad + \alpha(\alpha+1)\left(L_\epsilon - \epsilon^\alpha\zeta_\alpha\partial_X u - \epsilon^{2\alpha-1}\eta_\alpha H|D|^\alpha u\right) \\
&\quad + \frac{\alpha(\alpha+1)(\alpha+2)}{2}\left(N_\epsilon - 2\epsilon^{2\alpha-1}\zeta_\alpha u\partial_X u\right) \\
&\quad + M_\epsilon \\
&\quad + \epsilon^\alpha\left(-c_\alpha^2\partial_X u + \alpha(\alpha+1)\zeta_\alpha\partial_X u\right) \\
&\quad + \epsilon^{2\alpha-1}\left(\kappa_1\partial_\tau u + \alpha(\alpha+1)\eta_\alpha H|D|^\alpha u + \alpha(\alpha+1)(\alpha+2)\zeta_\alpha u\partial_X u\right).
\end{aligned}$$

The second to last line is zero because of the definition of c_α in (3). The final line is zero because u satisfies (2) with the coefficients as given in (3). And the first four lines we have estimates for, namely (9), (13), (15), and (20). Thus, we have

$$\sup_{|\tau|\leq\tau_0} \|a_\epsilon + F_\epsilon\|_{H^1} \leq C\left(\epsilon^{2\alpha-1+r_\alpha} + \epsilon^{3\alpha-2}\right).$$

For $\alpha \in (1,2]$, we have $\epsilon^{2\alpha-1+r_\alpha} = \epsilon^{2\alpha} \leq \epsilon^{3\alpha-2}$, and so

$$\alpha \in (1,2] \implies \sup_{|\tau|\leq\tau_0} \|a_\epsilon + F_\epsilon\|_{H^1} \leq C\epsilon^{3\alpha-2}.$$

For $\alpha \in (2,3)$, we have $\epsilon^{2\alpha-1+r_\alpha} = \epsilon^{\alpha+2} \geq \epsilon^{3\alpha-2}$, and so

$$\alpha \in (2,3) \implies \sup_{|\tau|\leq \tau_0} \|a_\epsilon + F_\epsilon\|_{H^1} \leq C\epsilon^{\alpha+2}.$$

By design, $R_\epsilon(j,t) = a_\epsilon(\epsilon(j - c_\alpha t), \epsilon^\alpha t) + F_\epsilon(\epsilon(j - c_\alpha t), \epsilon^\alpha t)$. Estimate (4.8) from Lemma 4.3 in [3] states that

$$G(X) \in H^1 \implies \|G(\epsilon\cdot)\|_{\ell^2} \leq C\epsilon^{-1/2}\|G\|_{H^1}. \tag{21}$$

Thus, we have $\sup_{|t|\leq\tau_0/\epsilon^\alpha} \|R_\epsilon(\epsilon(\cdot - t), \epsilon^\alpha t)\|_{\ell^2} \leq C\epsilon^{\beta_\alpha}$, with β_α as in the statement of the proposition. That does it. ∎

3. A general approximation theorem.

The previous section provides a rigorous bound on the size of the residual R_ϵ, but this is only part of the approximation theory. We need to demonstrate that a true solution $x_j(t)$ is shadowed by the approximate solution $\widetilde{x}_j(t)$ in an appropriate sense. The argument is based on "energy estimates" and is a direct descendent of the validation of KdV as the long-wave limit for FPUT lattices in [11] (see also [2, 10, 3]). The estimates are much more transparent after a change of coordinates. After the recoordinatization, we prove a conservation law which will imply global in time existence of solutions. Then we prove a general approximation result.

3.1. Relative displacement/velocity coordinates.

As stated in the introduction, we work in terms of relative displacements ($r_j := x_{j+1} - x_j - 1$) and velocities ($p_j := \dot{x}_j$). Note that if $r_j = p_j = 0$, then the system is in the equilibrium configuration $x_j = j$. A calculation shows that

$$x_{j+m} - x_j = m + \mathcal{G}_m r_j \quad \text{and} \quad x_j - x_{j-m} = m + \mathcal{G}_{-m} r_j,$$

where

$$\mathcal{G}_m r_j := \sum_{l=0}^{m-1} r_{j+l} \quad \text{and} \quad \mathcal{G}_{-m} r_j := \sum_{l=0}^{m-1} r_{j-m-l}.$$

Also define operators S^k, $\delta_m^\pm$ via

$$S^k f_j := f_{j+k}, \quad \delta_m^+ f_j := f_{j+m} - f_j, \quad \text{and} \quad \delta_m^- f_j := f_j - f_{j-m}.$$

Here are a few useful formulas that are not too hard to confirm:

$$\mathcal{G}_m \delta_1^+ = \delta_m^+ \quad \text{and} \quad S^{-m}\mathcal{G}_m r_j = \mathcal{G}_m r_{j-m} = \mathcal{G}_{-m} r_j.$$

The above considerations allow us to reformulate (1) as a first order system in terms of r_j and p_j:

$$\dot{r}_j = \delta_1^+ p_j \quad \text{and} \quad \dot{p}_j = \sum_{m\geq 1} \delta_m^- V_m'(\mathcal{G}_m r)_j, \tag{22}$$

where V_m coincides with its definition in the previous section at (10).

3.2. Energy conservation.

Let

$$\mathcal{E} := K + P,$$

where

$$K := \frac{1}{2}\sum_{j\in\mathbf{Z}} p_j^2 \quad \text{and} \quad P := \sum_{j\in\mathbf{Z}}\sum_{m\geq 1} V_m(\mathcal{G}_m r)_j.$$

This quantity corresponds to the total mechanical energy of the system. It is constant and here is the extremely classical argument.

Differentiate $\mathcal{E}$ with respect to t:

$$\dot{\mathcal{E}} = \sum_{j\in\mathbf{Z}}\left(p_j\dot{p}_j + \sum_{m\geq 1} V_m'(\mathcal{G}_m r)_j \mathcal{G}_m \dot{r}_j\right).$$

Eliminate $\dot{p}$ and $\dot{r}$ on the right using (22):

$$\dot{\mathcal{E}} = \sum_{j\in\mathbf{Z}}\left(p_j \sum_{m\geq 1}\delta_m^- V_m'(\mathcal{G}_m r)_j + \sum_{m\geq 1} V_m'(\mathcal{G}_m r)_j \mathcal{G}_m \delta_1^+ p_j\right).$$

Rearrange the sums:

$$\dot{\mathcal{E}} = \sum_{m\geq 1}\sum_{j\in\mathbf{Z}}\left(p_j\delta_m^- V_m'(\mathcal{G}_m r)_j + V_m'(\mathcal{G}_m r)_j \mathcal{G}_m \delta_1^+ p_j\right).$$

Use $\mathcal{G}_m\delta_1^+ = \delta_m^+$ in the second term and the summation by parts formula in the first:

$$\dot{\mathcal{E}} = \sum_{m\geq 1}\sum_{j\in\mathbf{Z}}\left(-\delta_m^+ p_j V_m'(\mathcal{G}_m r)_j + V_m'(\mathcal{G}_m r)_j \delta_m^+ p_j\right) = 0.$$

So $\mathcal{E}$ is constant.

3.3. Norm equivalence and global existence. In certain circumstances, the (square root of the) energy $\mathcal{E}$ from the previous section is equivalent to the $\ell^2\times\ell^2$ norm. It is trivial that K is equivalent to $\|p\|_{\ell^2}^2$, but the part involving P is not obvious. Here is the result.

LEMMA 5. *Suppose that $\alpha > 1$ such that $2\zeta_{\alpha+1} - \zeta_\alpha > 0$. Then there exist $\rho > 0$ and a constant $C > 1$ such that*

$$\|r\|_{\ell^2} \leq \rho \implies C^{-1}\|r\|_{\ell^2} \leq \sqrt{P} \leq C\|r\|_{\ell^2}.$$

The proof of this will be a consequence of Proposition 8, below. For now, note that it implies that small initial data for (22) implies global existence of solutions. Specifically, we have the following.

COROLLARY 6. *Fix $\alpha > 1$ such that $2\zeta_{\alpha+1} - \zeta_\alpha > 0$. Then there exist $\rho, C > 0$ such that if $\|\bar{r},\bar{p}\|_{\ell^2\times\ell^2} \leq \rho$, then there exists $(r_j(t), p_j(t)) \in C^1(\mathbf{R};\ell^2\times\ell^2)$ which solves* (22) *and for which $(r_j(0), p_j(0)) = (\bar{r}_j, \bar{p}_j)$. Moreover,* $\sup_{t\in\mathbf{R}}\|r(t),p(t)\|_{\ell^2\times\ell^2} \leq C\|\bar{r},\bar{p}\|_{\ell^2\times\ell^2}$.

We omit the proof, as it is classical. In any case, it is nearly identical to the proof of Theorem 5.2 of [3].

Remark 6. It is nonobvious that the condition $2\zeta_{\alpha+1} - \zeta_\alpha > 0$ is met. In Figure 1, we plot $2\zeta_{\alpha+1} - \zeta_\alpha$ vs. α. One sees that there exists a root of $2\zeta_{\alpha+1} - \zeta_\alpha$, denoted by α_*, in the interval $(1.4, 1.5)$, such that $2\zeta_{\alpha+1} - \zeta_\alpha > 0$ for $\alpha > \alpha_*$ and is nonpositive otherwise. Thus, the condition is nonvacuous.

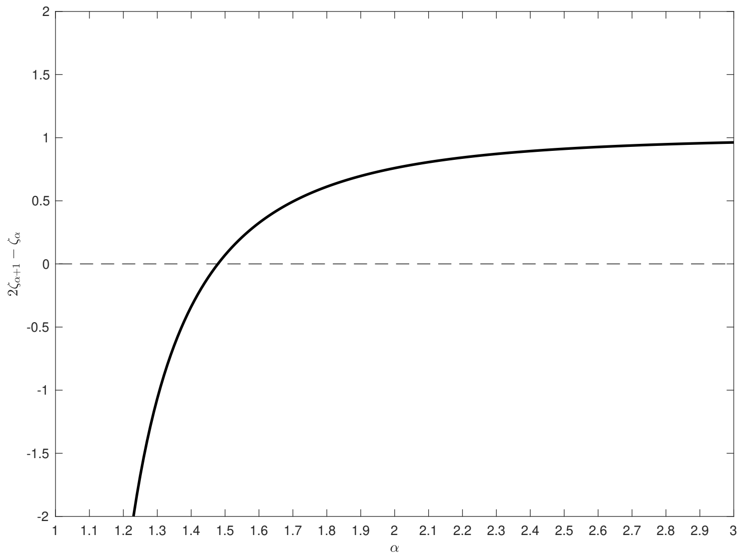

FIG. 1. $2\zeta_{\alpha+1}-\zeta_\alpha$ *vs.* α.

3.4. Approximation in general. Let

$$r_j(t)=\widetilde{r}_j(t)+\eta_j(t) \quad \text{and} \quad p_j(t)=\widetilde{p}_j(t)+\xi_j(t),$$

where $\widetilde{r}_j(t)$ and $\widetilde{p}_j(t)$ are some given functions which we expect to be good approximators to true solutions $r_j(t)$ and $p_j(t)$ of (22). Then the "errors" $\eta_j(t)$ and $\xi_j(t)$ solve

$$\dot{\eta}_j=\delta_1^+\xi_j+\operatorname{Res}_1 \quad \text{and} \quad \dot{\xi}_j=\sum_{m\geq 1}\delta_m^-\left[V_m'\left(\mathcal{G}_m\left(\widetilde{r}+\eta\right)\right)-V_m'\left(\mathcal{G}_m\widetilde{r}\right)\right]_j+\operatorname{Res}_2, \tag{23}$$

where

$$\operatorname{Res}_1=\delta_1^+\widetilde{p}_j-\dot{\widetilde{r}}_j \quad \text{and} \quad \operatorname{Res}_2=\sum_{m\geq 1}\delta_m^-V_m'\left(\mathcal{G}_m\widetilde{r}\right)_j-\dot{\widetilde{p}}_j. \tag{24}$$

The functions Res_1 and Res_2 are, like R_ϵ, residuals and quantify the amount by which the approximators $\widetilde{r}_j(t)$ and $\widetilde{p}_j(t)$ fail to satisfy (22). Ultimately, these will be expressed in terms of R_ϵ, but for now we leave things general.

Our goal in this section is to show that $\eta_j(t)$ and $\xi_j(t)$ remain small (in ℓ^2) over long-time periods, provided they are initially small. In particular, we prove the following.

THEOREM 7. *Suppose that* $\alpha>1$ *with* $2\zeta_{\alpha+1}-\zeta_\alpha>0$. *Assume further that for some* $\tau_0,C_1,\epsilon_1>0$ *and* $\beta>\alpha$, $\epsilon\in(0,\epsilon_1]$, *imply*

$$\sup_{|t|\leq\tau_0/\epsilon^\alpha}\left(\|\operatorname{Res}_1\|_{\ell^2}+\|\operatorname{Res}_2\|_{\ell^2}\right)\leq C_1\epsilon^\beta, \qquad \sup_{|t|\leq\tau_0/\epsilon^\alpha}\|\dot{\widetilde{r}}\|_{\ell^\infty}\leq C_1\epsilon^\alpha \tag{25}$$

and

$$\|\bar{\eta},\bar{\xi}\|_{\ell^2\times\ell^2}\leq C_1\epsilon^{\beta-\alpha}.$$

Then there exist constants $C_*,\epsilon_*>0$ *so that the following holds for* $\epsilon\in(0,\epsilon_*]$. *If* $\eta_j(t)$, $\xi_j(t)$ *solve* (23) *with initial data* $\bar{\eta}_j$, $\bar{\xi}_j$, *we have*

$$\sup_{|t|\leq\tau_0/\epsilon^\alpha}\|\eta(t),\xi(t)\|_{\ell^2\times\ell^2}\leq C_*\epsilon^{\beta-\alpha}.$$

Proof. We begin by rewriting (23) in a helpful way. For $a,b\in\mathbf{R}$, put

$$W_m(a,b):=V_m(b+a)-V_m(b)-V_m'(b)a$$

and let

$$W_m'(a,b):=\partial_a W_m(a,b)=V_m'(b+a)-V_m'(b).$$

With this (23) becomes

$$\dot{\eta}_j=\delta_1^+\xi_j+\mathrm{Res}_1\quad\text{and}\quad\dot{\xi}_j=\sum_{m\geq1}\delta_m^- W_m'(\mathcal{G}_m\eta,\mathcal{G}_m\widetilde{r})_j+\mathrm{Res}_2\,. \tag{26}$$

The point of introducing W_m here is that now (26) is structurally similar to (22) with V_m replaced by W_m. Then we hope we can recapture some of the glory of conservation of $\mathcal{E}$ from above but for the error equations.

So for a solution of (26) put

$$\mathcal{H}:=\frac{1}{2}\sum_{j\in\mathbf{Z}}\xi_j^2+\sum_{j\in\mathbf{Z}}\sum_{m\geq1}W_m(\mathcal{G}_m\eta,\mathcal{G}_m\widetilde{r})_j.$$

This is our replacement for $\mathcal{E}$. The following proposition contains the key properties of the second term in $\mathcal{H}$, chief of which that under some conditions it is equivalent to $\|\eta\|_{\ell^2}^2$.

PROPOSITION 8. *Fix $\alpha>1$ with $2\zeta_{\alpha+1}-\zeta_\alpha>0$. Then there exists $C>1$ such that the following hold when $\|\widetilde{r}\|_{\ell^2}\leq1/4$ and $\|\eta\|_{\ell^2}\leq1/4$:*

$$C^{-1}\|\eta\|_{\ell^2}\leq\sqrt{\sum_{j\in\mathbf{Z}}\sum_{m\geq1}W_m(\mathcal{G}_m\eta,\mathcal{G}_m\widetilde{r})_j}\leq C\|\eta\|_{\ell^2}, \tag{27}$$

$$\sum_{m\geq1}m\|W_m'(\mathcal{G}_m\eta,\mathcal{G}_m\widetilde{r})\|_{\ell^2}\leq C\|\eta\|_{\ell^2}, \tag{28}$$

$$\sum_{m\geq1}m\|\partial_b W_m(\mathcal{G}_m\eta,\mathcal{G}_m\widetilde{r})\|_{\ell^1}\leq C\|\eta\|_{\ell^2}^2. \tag{29}$$

∎

Remark 7. Note that $W_m(a,0)=V_m(a)$, and so if $\widetilde{r}_j(t)$ is identically zero, then (27) coincides exactly with the estimate in Lemma 5, with η swapped with r.

Proof. We begin with (27). Taylor's theorem tells us that $W_m(a,b)=\alpha(\alpha+1)a^2/2(m+b_*)^{\alpha+2}$ with b_* in between b and $b+a$. This leads to

$$\frac{\alpha(\alpha+1)}{2(m+|a|+|b|)^{\alpha+2}}a^2\leq W_m(a,b)\leq\frac{\alpha(\alpha+1)}{2(m-|a|-|b|)^{\alpha+2}}a^2. \tag{30}$$

We have assumed $\|\eta\|_{\ell^2}\leq1/4$. Thus, the classical estimate $\|f\|_{\ell^\infty}\leq\|f\|_{\ell^2}$ along with the triangle inequality tell us

$$|\mathcal{G}_m\eta_j|\leq\|\mathcal{G}_m\eta\|_{\ell^\infty}\leq\sum_{l=0}^{m-1}\|\eta_{\cdot+l}\|_{\ell^\infty}=m\|\eta\|_{\ell^\infty}\leq m/4.$$

Similarly, $\|\widetilde{r}\|_{\ell^2} \leq 1/4$ implies $|\mathcal{G}_m \widetilde{r}_j| \leq m/4$. So we have $m - |\mathcal{G}_m \eta_j| - |\mathcal{G}_m \widetilde{r}_j| \geq m/2$ and $m + |\mathcal{G}_m \eta_j| + |\mathcal{G}_m \widetilde{r}_j| \geq 3m/2$, and thus (30) gives

$$\frac{2^{\alpha+1}\alpha(\alpha+1)}{3^{\alpha+2}m^{\alpha+2}}(\mathcal{G}_m\eta_j)^2 \leq W_m(\mathcal{G}_m\eta, \mathcal{G}_m\widetilde{r})_j \leq \frac{2^{\alpha+1}\alpha(\alpha+1)}{m^{\alpha+2}}(\mathcal{G}_m\eta_j)^2.$$

Summing this gets us to

$$\frac{2^{\alpha+1}\alpha(\alpha+1)}{3^{\alpha+2}}P_2 \leq \sum_{j\in\mathbf{Z}}\sum_{m\geq 1} W_m(\mathcal{G}_m\eta, \mathcal{G}_m\widetilde{r})_j \leq 2^{\alpha+1}\alpha(\alpha+1)P_2,$$

where

$$P_2 := \sum_{j\in\mathbf{Z}}\sum_{m\geq 1}\frac{1}{m^{\alpha+2}}(\mathcal{G}_m\eta)_j^2.$$

Using the definition of $\mathcal{G}_m$ and multiplying out the square gives

$$P_2 = \sum_{j\in\mathbf{Z}}\sum_{m\geq 1}\frac{1}{m^{\alpha+2}}\left(\sum_{l=0}^{m-1}\eta_{j+l}^2 + 2\sum_{0\leq l<k\leq m-1}\eta_{j+l}\eta_{j+k}\right).$$

Rearranging sums and doing some computations on the "diagonal part" of the above gets

$$\sum_{j\in\mathbf{Z}}\sum_{m\geq 1}\frac{1}{m^{\alpha+2}}\sum_{l=0}^{m-1}\eta_{j+l}^2 = \sum_{m\geq 1}\frac{1}{m^{\alpha+2}}\sum_{l=0}^{m-1}\sum_{j\in\mathbf{Z}}\eta_{j+l}^2 = \sum_{m\geq 1}\frac{1}{m^{\alpha+2}}\sum_{l=0}^{m-1}\|\eta\|_{\ell^2}^2 = \zeta_{\alpha+1}\|\eta\|_{\ell^2}^2.$$

Therefore,

$$P_2 - \zeta_{\alpha+1}\|\eta\|_{\ell^2}^2 = 2\sum_{j\in\mathbf{Z}}\sum_{m\geq 1}\frac{1}{m^{\alpha+2}}\sum_{0\leq l<k\leq m-1}\eta_{j+l}\eta_{j+k} =: P_{22}.$$

Rearranging sums gives

$$P_{22} = 2\sum_{m\geq 1}\frac{1}{m^{\alpha+2}}\sum_{0\leq l<k\leq m-1}\sum_{j\in\mathbf{Z}}\eta_{j+l}\eta_{j+k}.$$

We use Cauchy–Schwarz to get

$$|P_{22}| \leq 2\sum_{m\geq 1}\frac{1}{m^{\alpha+2}}\sum_{0\leq l<k\leq m-1}\|\eta\|_{\ell^2}^2.$$

Since $\sum_{0\leq l<k\leq m-1} 1 = m(m-1)/2$, we obtain

$$|P_{22}| \leq \|\eta\|_{\ell^2}^2\sum_{m\geq 1}\frac{m(m-1)}{m^{\alpha+2}} = (\zeta_\alpha - \zeta_{\alpha+1})\|\eta\|_{\ell^2}^2.$$

Thus,

$$\left|P_2 - \zeta_{\alpha+1}\|\eta\|_{\ell^2}^2\right| \leq (\zeta_\alpha - \zeta_{\alpha+1})\|\eta\|_{\ell^2}^2$$

or rather

$$(2\zeta_{\alpha+1} - \zeta_\alpha)\|\eta\|_{\ell^2}^2 \leq P_2 \leq \zeta_\alpha\|\eta\|_{\ell^2}^2.$$

Therefore, $2\zeta_{\alpha+1} - \zeta_\alpha > 0$ implies that $\sqrt{P_2}$ is equivalent to $\|\eta\|_{\ell^2}$.

This in combination with (27) gives

$$\frac{2^{\alpha+1}\alpha(\alpha+1)}{3^{\alpha+2}}\left(2\zeta_{\alpha+1}-\zeta_\alpha\right)\|\eta\|_{\ell^2}^2 \le \sum_{j\in\mathbf{Z}}\sum_{m\ge 1} W_m(\mathcal{G}_m\eta,\mathcal{G}_m\widetilde{r})_j \le 2^{\alpha+1}\alpha(\alpha+1)\zeta_\alpha\|\eta\|_{\ell^2}^2.$$

That is, we have (27).

Next up is (28). The mean value theorem tell us that $W_m'(a,b)=\alpha(\alpha+1)a/(m+b_*)^{\alpha+2}$, where b_* lies between b and $b+a$. As above, we have $\|\mathcal{G}_m\widetilde{r}\|_{\ell^\infty}\le m/4$ and $\|\mathcal{G}_m\eta\|_{\ell^\infty}\le m/4$. So b_* would be controlled above by $m/2$ for all j. Thus,

$$|W_m'(\mathcal{G}_m\eta,\mathcal{G}_m\widetilde{r})_j| \le \frac{2^{\alpha+2}\alpha(\alpha+1)}{m^{\alpha+2}}|\mathcal{G}_m\eta_j|.$$

We take the ℓ^2 norm and use the triangle inequality

$$\|W_m'(\mathcal{G}_m\eta,\mathcal{G}_m\widetilde{r})\|_{\ell^2} \le \frac{2^{\alpha+2}\alpha(\alpha+1)}{m^{\alpha+2}}\|\mathcal{G}_m\eta\|_{\ell^2} \le \frac{2^{\alpha+2}\alpha(\alpha+1)}{m^{\alpha+1}}\|\eta\|_{\ell^2}.$$

Thus,

$$\sum_{m\ge 1} m\,\|W_m'(\mathcal{G}_m\eta,\mathcal{G}_m\widetilde{r})\|_{\ell^2} \le 2^{\alpha+2}\zeta_\alpha\alpha(\alpha+1)\|\eta\|_{\ell^2}.$$

This is (28).

To get (29) is more of the same. We have $\partial_b W_m(a,b) = -\alpha(\alpha+1)(\alpha+2)a^2/2(m+b_*)^{\alpha+3}$ with b_* in between b and $b+a$. Much as we did above, we get the estimate

$$|\partial_b W_m(\mathcal{G}_m\eta,\mathcal{G}_m\widetilde{r})_j| \le \frac{2^{\alpha+2}\alpha(\alpha+1)(\alpha+2)}{m^{\alpha+3}}|\mathcal{G}_m\eta_j|^2.$$

Summing over j and the triangle inequality leads to

$$\|\partial_b W_m(\mathcal{G}_m\eta,\mathcal{G}_m\widetilde{r})\|_{\ell^2} \le \frac{2^{\alpha+2}\alpha(\alpha+1)(\alpha+2)}{m^{\alpha+3}}\|\mathcal{G}_m\eta\|_{\ell^2}^2 \le \frac{2^{\alpha+2}\alpha(\alpha+1)(\alpha+2)}{m^{\alpha+1}}\|\eta\|_{\ell^2}^2.$$

Then

$$\sum_{m\ge 1} m\|\partial_b W_m(\mathcal{G}_m\eta,\mathcal{G}_m\widetilde{r})\|_{\ell^2} \le 2^{\alpha+2}\alpha(\alpha+1)(\alpha+2)\zeta_\alpha\|\eta\|_{\ell^2}^2$$

and we are done. ∎

With Proposition 8 taken care of, we can now get into the energy argument at the heart of the proof of Theorem 7. We begin with differentiation of $\mathcal{H}$ to get

$$\dot{\mathcal{H}} = \sum_{j\in\mathbf{Z}}\left(\xi_j\dot{\xi}_j + \sum_{m\ge 1} W_m'(\mathcal{G}_m\eta,\mathcal{G}_m\widetilde{r})_j\mathcal{G}_m\dot{\eta}_j\right) + \sum_{j\in\mathbf{Z}}\sum_{m\ge 1}\partial_b W_m(\mathcal{G}_m\eta,\mathcal{G}_m\widetilde{r})_j\mathcal{G}_m\dot{\widetilde{r}}_j.$$

Call the terms on the right I and II in the obvious way. Using (26) in I gets

$$I = \sum_{j\in\mathbf{Z}}\left(\xi_j\left(\sum_{m\ge 1}\delta_m^- W_m'(\mathcal{G}_m\eta,\mathcal{G}_m\widetilde{r})_j + \mathrm{Res}_2\right) + \sum_{m\ge 1} W_m'(\mathcal{G}_m\eta,\mathcal{G}_m\widetilde{r})_j\mathcal{G}_m(\delta_1^+\xi_j + \mathrm{Res}_1)\right).$$

Summing by parts and using $\mathcal{G}_m\delta_1^+ = \delta_m^+$ gives some cancellations:

$$I = \sum_{j\in\mathbf{Z}} \left(\xi_j \operatorname{Res}_2 + \sum_{m\geq 1} W'_m(\mathcal{G}_m\eta, \mathcal{G}_m\widetilde{r})_j \mathcal{G}_m \operatorname{Res}_1 \right).$$

Cauchy–Schwarz implies $|I| \leq \|\xi\|_{\ell^2}\|\operatorname{Res}_2\|_{\ell_2} + |I_2|$, where

$$I_2 := \sum_{m\geq 1}\sum_{j\in\mathbf{Z}} W'_m(\mathcal{G}_m\eta, \mathcal{G}_m\widetilde{r})_j \mathcal{G}_m \operatorname{Res}_1.$$

The dual of the operator $\mathcal{G}_m$ with respect to the ℓ^2-inner product is $\mathcal{G}_{-m}$, and so

$$I_2 = \sum_{m\geq 1}\sum_{j\in\mathbf{Z}} (\mathcal{G}_{-m} W'_m(\mathcal{G}_m\eta, \mathcal{G}_m\widetilde{r})_j) \operatorname{Res}_1.$$

Reorder the sum again:

$$I_2 = \sum_{j\in\mathbf{Z}} \operatorname{Res}_1 \sum_{m\geq 1} (\mathcal{G}_{-m} W'_m(\mathcal{G}_m\eta, \mathcal{G}_m\widetilde{r})_j).$$

Then Cauchy–Schwarz and the triangle inequality lead to

$$|I_2| \leq \|\operatorname{Res}_1\|_{\ell_2} \sum_{m\geq 1} \|\mathcal{G}_{-m} W'_m(\mathcal{G}_m\eta, \mathcal{G}_m\widetilde{r})\|_{\ell^2} \leq \|\operatorname{Res}_1\|_{\ell^2} \sum_{m\geq 1} m\, \|W'_m(\mathcal{G}_m\eta, \mathcal{G}_m\widetilde{r})\|_{\ell^2}.$$

The estimate (28) from Proposition 8 gives

$$|I_2| \leq C\|\operatorname{Res}_1\|_{\ell^2}\|\eta\|_{\ell^2}.$$

Thus, we have

$$|I| \leq C\left(\|\operatorname{Res}_1\|_{\ell^2}\|\eta\|_{\ell^2} + \|\operatorname{Res}_2\|_{\ell^2}\|\xi\|_{\ell^2}\right).$$

Now look at II. By using naive estimates we get

$$|II| \leq \sum_{j\in\mathbf{Z}}\sum_{m\geq 1} |\partial_b W_m(\mathcal{G}_m\eta, \mathcal{G}_m\widetilde{r})_j| |\mathcal{G}_m\dot{\widetilde{r}}\|_{\ell^\infty} \leq \|\dot{\widetilde{r}}\|_{\ell^\infty} \sum_{m\geq 1} m\|\partial_b W_m(\mathcal{G}_m\eta, \mathcal{G}_m\widetilde{r})\|_{\ell^1}.$$

Then (29) from Proposition 8 yields

$$|II| \leq C\|\dot{\widetilde{r}}\|_{\ell^\infty}\|\eta\|_{\ell^2}^2.$$

So all together

$$\dot{\mathcal{H}} \leq C\left(\|\operatorname{Res}_1\|_{\ell^2}\|\eta\|_{\ell^2} + \|\operatorname{Res}_2\|_{\ell^2}\|\xi\|_{\ell^2}\right) + C\|\dot{\widetilde{r}}\|_{\ell^\infty}\|\eta\|_{\ell^2}^2.$$

Using (27) we have

$$\dot{\mathcal{H}} \leq C\left(\|\operatorname{Res}_1\|_{\ell^2} + \|\operatorname{Res}_2\|_{\ell^2}\right)\sqrt{\mathcal{H}} + C\|\dot{\widetilde{r}}\|_{\ell^\infty}\mathcal{H}.$$

The assumptions made on Res_1, Res_2, and $\dot{\widetilde{r}}$ lead to

$$\dot{\mathcal{H}} \leq C\epsilon^\beta\sqrt{\mathcal{H}} + C\epsilon^\alpha\mathcal{H}.$$

Applying Grönwall's inequality yields

$$\sqrt{\mathcal{H}(t)} \leq e^{C\epsilon^\alpha t}\sqrt{\mathcal{H}(0)} + C\epsilon^{\beta-\alpha}\left(e^{C\epsilon^\alpha t} - 1\right).$$

Then we use (27) one last time to get

$$\|\eta(t), \xi(t)\|_{\ell^2\times\ell^2} \leq e^{C\epsilon^\alpha t}\|\bar{\eta}, \bar{\xi}\|_{\ell^2\times\ell^2} + C\epsilon^{\beta-\alpha}\left(e^{C\epsilon^\alpha t} - 1\right).$$

Taking the supremum over $|t| \leq \tau_0/\epsilon^\alpha$ and using the assumption on the size of the initial data gives the final estimate in Theorem 7. ∎

4. Proof of Theorem 1. The proof of Theorem 1 is more or less a direct application of the general approximation theorem, Theorem 7. There are a few small details to attend to, and that is what we do now.

Proof. (Theorem 1). Fix $u(X,\tau)$ a solution of (2) subject to the bound described in the statement of the theorem. Let $v(X,\tau)$ be given as in (6) and (7). Form $\widetilde{x}_j$ as in (4), namely $\widetilde{x}_j(t) := j + \epsilon^{\alpha-2} v(\epsilon(j - c_\alpha t), \epsilon^\alpha t)$. Then put

$$\widetilde{r}_j(t) := -1 + \delta_1^+ \widetilde{x}_j(t) \quad \text{and} \quad \widetilde{p}_j(t) := \dot{\widetilde{x}}_j(t).$$

We first compute Res_1 and Res_2 as in (24). We have

$$\text{Res}_1 = \delta_1^+ \widetilde{p} - \dot{\widetilde{r}} = \delta_1^+ (\dot{\widetilde{x}}) - \partial_t \left(-1 + \delta_1^+ \widetilde{x}\right) = 0.$$

For Res_2, we compute

$$\begin{aligned}
\text{Res}_2 &= \sum_{m \geq 1} \delta_m^- V_m' \left(\mathcal{G}_m \widetilde{r}\right)_j - \dot{\widetilde{p}}_j \\
&= \sum_{m \geq 1} \delta_m^- V_m' \left(-m + \delta_m^+ \widetilde{x}\right)\big)_j - \ddot{\widetilde{x}}_j \\
&= -\alpha \sum_{m \geq 1} \left(\frac{1}{(\widetilde{x}_{j+m} - \widetilde{x}_j)^{\alpha+1}} - \frac{1}{(\widetilde{x}_j - \widetilde{x}_{j-m})^{\alpha+1}} \right) - \ddot{\widetilde{x}}_j \\
&= -R_\epsilon,
\end{aligned}$$

with R_ϵ as above in (5). Thus, Proposition 2 tells us that the hypothesis on the residuals, (25), in Theorem 7 is met with $\beta = \beta_\alpha$. Note that in the statement of Theorem 1 the order of the error is ϵ^{γ_α}, where $\gamma_\alpha = \beta_\alpha - \alpha$.

The fundamental theorem of calculus gives

$$\delta_1^+ (f(\epsilon \cdot))_j = f(\epsilon(j+1)) - f(\epsilon j) = \int_{\epsilon j}^{\epsilon j + \epsilon} f_X(X) dX = \epsilon (A_\epsilon f_X)(\epsilon j). \tag{31}$$

If we use this and the relation $u = -\partial_X v$, we have

$$\begin{aligned}
\widetilde{r}_j(t) &= -\epsilon^{\alpha-1} (A_\epsilon u)(\epsilon(j - c_\alpha t), \epsilon^\alpha t) \\
&= -\epsilon^{\alpha-1} u(\epsilon(j - c_\alpha t), \epsilon^\alpha t) - \epsilon^{\alpha-1} ((A_\epsilon - 1)u)(\epsilon(j - c_\alpha t), \epsilon^\alpha t).
\end{aligned}$$

So (21) and the final estimate in Lemma 4 give us

$$\sup_{|t| \leq \tau_0/\epsilon^\alpha} \|\widetilde{r}(t) + \epsilon^{\alpha-1} u(\epsilon(\cdot - c_\alpha t), \epsilon^\alpha t)\|_{\ell^2} \leq C \epsilon^{\alpha - 1/2}. \tag{32}$$

The assumption on the initial conditions in Theorem 1 implies

$$\|r(0) + \epsilon^{\alpha-1} u(\epsilon \cdot, 0)\|_{\ell^2} = \|\bar{\mu}\|_{\ell^2} \leq C \epsilon^{\beta_\alpha - \alpha}.$$

It is easy enough to check that $\epsilon^{\alpha-1/2} \leq \epsilon^{\beta_\alpha - \alpha}$, and therefore (32) and the triangle inequality cough up

$$\|r(0) - \widetilde{r}(0)\|_{\ell^2} \leq C \epsilon^{\beta_\alpha - \alpha},$$

which is one of the hypotheses on the initial data in Theorem 7.

Similarly, we have

$$\widetilde{p}_j(t) = c_\alpha \epsilon^{\alpha-1} u(\epsilon(j - c_\alpha t), \epsilon^\alpha t) + \epsilon^{2\alpha-2} v_\tau(\epsilon(j - c_\alpha t), \epsilon^\alpha t).$$

It is straightforward to use (8) and (21) to show $\|v_\tau(\epsilon(\cdot - c_\alpha t), \epsilon^\alpha t)\|_{\ell^2} \leq C\epsilon^{-1/2}$, and so

$$\sup_{|t|\leq \tau_0/\epsilon^\alpha} \|\widetilde{p}(t) - c_\alpha \epsilon^{\alpha-1} u(\epsilon(\cdot - c_\alpha t), \epsilon^\alpha t)\|_{\ell^2} \leq C\epsilon^{2\alpha-3/2} \leq C\epsilon^{\beta_\alpha - \alpha}. \tag{33}$$

The assumption on the initial conditions in Theorem 1 tell us

$$\|p(0) - c_\alpha \epsilon^{\alpha-1} u(\epsilon\cdot, 0)\|_{\ell^2} = \|\bar{\nu}\|_{\ell^2} \leq C\epsilon^{\beta_\alpha - \alpha}.$$

Therefore,

$$\|p(0) - \widetilde{p}(0)\|_{\ell^2} \leq C\epsilon^{\beta_\alpha - \alpha},$$

which is the other hypothesis on the initial data in Theorem 7.

Next, since $\dot{\widetilde{r}}_j(t) = \delta_1^+ \widetilde{p}_j(t)$, (31) and $u = -\partial_X v$ give us

$$\dot{\widetilde{r}}_j(t) = c_\alpha \epsilon^\alpha (A_\epsilon u_X)(\epsilon(j - c_\alpha t), \epsilon^\alpha t) - \epsilon^{2\alpha-1}(A_\epsilon u_\tau)(\epsilon(j - c_\alpha t), \epsilon^\alpha t).$$

An easy estimate provides

$$\|\dot{\widetilde{r}}(t)\|_{\ell^\infty} \leq c_\alpha \epsilon^\alpha \|A_\epsilon u_X(\cdot, \epsilon^\alpha t)\|_{L^\infty} + \epsilon^{2\alpha-1}\|A_\epsilon u_\tau(\cdot, \epsilon^\alpha t)\|_{L^\infty}.$$

Using Sobolev, followed by the first estimate in Lemma 4,

$$\|\dot{\widetilde{r}}(t)\|_{\ell^\infty} \leq C\epsilon^\alpha \|u(\cdot, \epsilon^\alpha t)\|_{H^2} + C\epsilon^{2\alpha-1}\|u_\tau(\cdot, \epsilon^\alpha t)\|_{H^1}.$$

Then we use (2) and the uniform bound on u to get

$$\sup_{|t|\leq \tau_0/\epsilon^\alpha} \|\dot{\widetilde{r}}(t)\|_{\ell^\infty} \leq C\epsilon^\alpha + C\epsilon^{2\alpha-1} \leq C\epsilon^\alpha.$$

This gives the estimate on $\dot{\widetilde{r}}$ in (25).

We have now checked off all the hypotheses of Theorem 7, and thus its conclusions hold. And so we find that

$$\sup_{|t|\leq \tau/\epsilon^\alpha} \|r(t) - \widetilde{r}(t)\|_{\ell^2} \leq C\epsilon^{\beta_\alpha - \alpha}.$$

This, together with (32) and the triangle inequality, gives

$$\sup_{|t|\leq \tau/\epsilon^\alpha} \|r(t) + \epsilon^{\alpha-1} u(\epsilon(\cdot - t), \epsilon^\alpha t)\|_{\ell^2} \leq C\epsilon^{\beta_\alpha - \alpha},$$

which is the estimate on $\mu(t)$ in Theorem 1. The estimate on $\nu(t)$ follows from

$$\sup_{|t|\leq \tau/\epsilon^\alpha} \|p(t) - \widetilde{p}(t)\|_{\ell^2} \leq C\epsilon^{\beta_\alpha - \alpha}$$

and (33) in the same way. ∎

Acknowledgment. Thanks go to Bob Pego, who sent a preprint of [7] to the author and spurred this work into existence.

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