Landau levels for charged particles with anisotropic mass

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The problem of the two-dimensional motion of a charged particle with constant mass in presence of a uniform constant perpendicular magnetic field features in several undergraduate and graduate quantum physics textbooks. This problem is very important to studies of two-dimensional materials that manifest quantum Hall behavior as evidenced by several major discoveries over the last few years. Many real experimental samples are more complicated due to the anisotropic mass of the electrons. In this work, we provide the exact solution to this problem by means of a clever scaling of coordinates. Calculations are done for a symmetric gauge of the magnetic field. This study allows a broad audience of students and teachers to understand the mathematical techniques that lead to the solution of this quantum problem.

Keywords: Charged particle, Magnetic field, Anisotropic mass, Landau states.

I. INTRODUCTION

The quantum problem of the two-dimensional (2D) motion of a charged particle subject to a uniform constant perpendicular magnetic field was first solved in 1930 by Lev Landau[1]. He showed that when a charged particle is in a uniform constant magnetic field, the energy levels are degenerate with degeneracy directly proportional to the strength of the applied magnetic field. The quantization of the levels leads to oscillations of thermodynamic quantities (de Haas-van Alphen effect)[2] and transport coefficients (Shubnikov-de Haas effect)[3] upon variation of the magnetic field. The quantization of energies has especially dramatic consequences in 2D systems of electrons subject to a very strong perpendicular magnetic field. In this regime, novel unexpected phenomena known as the integer quantum Hall effect and the fractional quantum Hall effect were discovered opening a whole new set of research areas[4–7]. With improvement in the quality of samples and experimental designs, more and more quantum Hall states have been found in a variety of systems that include novel materials such as graphene[8]. The endeavor to grasp the intricate quantum properties of systems of electrons continues to be a topic of great ongoing research interest as evidenced by several Nobel Prizes in Physics awarded over the last few years for discoveries that have contributed to the understanding of the world of electrons under a variety of quantum conditions[9].

The solution to this problem is presented in several quantum physics textbooks including Gasiorowicz's Quantum Physics (Third Edition)[10] and Liboff's Introductory Quantum Mechanics (Fourth Edition)[11] (Chapter 10, pgs. 430-435) as well as Bransden's Quantum Mechanics (Second Edition) book[12] (Chapter 12, pgs. 571-574), though the latter seems to be more suitable for a lower-level graduate course. In most instances where this problem is considered, the charged particle has an isotropic mass. However, the situation is more nuanced for electrons confined to 2D semiconductors including quantum wells in $AlAs$ heterostructures[13,14], Si surfaces[15], $PbTe$ quantum wells[16] as well as a number of other types of semiconductors[17]. In these systems, the electrons move in the periodic potential structure of atomic potentials in the crystal. In such conditions the role of mass is played by the so-called effective mass of the electrons which might be anisotropic. Therefore, the solution of the Landau problem for a model that accommodates a charged particle with anisotropic mass is more than just an exercise in quantum mechanics since it is also important to understanding quantum Hall systems[18]. The effective mass of electrons in a semiconductor might be a tensor quantity. The simplest choice is to assume an anisotropic mass, m_x along the x direction and m_y along the y direction.

In this work, we show that this seemingly challenging problem can be solved through a transformation of coordinates that reduces the unknown problem to the case of a charged particle with isotropic mass, albeit in terms of "new" scaled coordinates. The solution is not only of research interest, but also of pedagogical value since the mathematical level of such transformations is not difficult to grasp by typical upper-level undergraduate or low-level graduate students. We organize the paper as follows: In Section II we explain briefly the model and the theory for the familiar case of a charged particle with constant isotropic mass in a uniform constant perpendicular magnetic field. In Section III we provide an easy to follow implementation of the solution method for the problem under consideration. In Section IV we discuss the results from a scientific and pedagogical point of view and provide some concluding remarks.

II. CHARGED PARTICLE WITH CONSTANT ISOTROPIC MASS IN A UNIFORM CONSTANT PERPENDICULAR MAGNETIC FIELD

We first briefly review the quantum problem of a particle with constant isotropic mass, m, and charge, q, constrained to move in the xy-plane in the presence of a uniform constant perpendicular magnetic field, $\vec{B} = (0, 0, B_z)$. Its solution is presented in much greater detail in Ref.[19].

The Hamiltonian is:

$$\hat{H} = \frac{1}{2\,m}\left[\hat{\vec{p}} - q\,\vec{A}(x,y)\right]^2 \ , \tag{1}$$

where $\hat{\vec{p}} = (\hat{p}_x, \hat{p}_y) = \left(-i\,\hbar\,\frac{\partial}{\partial x}, -i\,\hbar\,\frac{\partial}{\partial y}\right)$ is the 2D linear momentum operator and $\vec{A}(x,y)$ is the vector potential for the magnetic field. Here, $i = \sqrt{-1}$ is the imaginary unit and $\hbar$ is the reduced Planck's constant. The interaction of the particle's quantum spin with the magnetic field (the Zeeman effect) is not considered.

In this work, we adopt the symmetric gauge:

$$\vec{A}(x,y) = \frac{B_z}{2}\Big(-y, x, 0\Big) \ , \tag{2}$$

so that the Hamiltonian becomes

$$\hat{H} = \frac{1}{2\,m}\left(-i\,\hbar\,\frac{\partial}{\partial x} + q\,\frac{B_z}{2}\,y\right)^2 + \frac{1}{2\,m}\left(-i\,\hbar\,\frac{\partial}{\partial y} - q\,\frac{B_z}{2}\,x\right)^2 . \tag{3}$$

Because the operators within each bracket of Eq.(3) commute with each other, the Hamiltonian simplifies to:

$$\hat{H} = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2\,m} - \frac{q\,B_z}{2\,m}\hat{L}_z + \frac{m}{2}\left(\frac{q\,B_z}{2\,m}\right)^2 (x^2+y^2) \ , \tag{4}$$

where

$$\hat{L}_z = x\,\hat{p}_y - y\,\hat{p}_x \ , \tag{5}$$

is the z-component of the angular momentum operator.

To describe electrons ($q = -|q| < 0$), it is particularly convenient to choose $B_z = -|\vec{B}| < 0$, since this will enable us to describe the wave function in terms of a complex variable $z = x + i\,y$ instead of its complex conjugate. The above notation for the 2D complex position is standard in the quantum Hall literature[20] and readers can easily rely on the context not to confuse it with the z-direction.

For electrons, the Hamiltonian in Eq.(4) can be rewritten as:

$$\hat{H} = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2\,m} - \frac{\omega_c}{2}\,\hat{L}_z + \frac{m}{2}\left(\frac{\omega_c}{2}\right)^2 (x^2+y^2) \ , \tag{6}$$

where

$$\omega_c = \frac{|q|\,|\vec{B}|}{m} > 0 \ , \tag{7}$$

is the cyclotron angular frequency.

The stationary Schrödinger's equation is then solved in 2D polar coordinates. The energy eigenvalues (found in Ref.[19]) are:

$$E_{n_r m_l} = \hbar\,\omega_c\left(n_r + \frac{1}{2} + \frac{|m_l| - m_l}{2}\right) \ , \tag{8}$$

where $n_r = 0, 1, \ldots$ represents a radial quantum number and $m_l = 0, \pm 1, \ldots$ is the z-component angular momentum quantum number. Note that there are no restrictions on m_l for a given n_r. The normalized eigenfunctions are written as a product of a radial function with an angular-dependent function (the eigenstate of $\hat{L}_z$) as:

$$\Psi_{n_r m_l}(r,\varphi) = R_{n_r m_l}(r)\,\Phi_{m_l}(\varphi) \ , \tag{9}$$

where $r = \sqrt{x^2+y^2} \geq 0$ denotes the 2D radial distance and $0 \leq \varphi < 2\,\pi$ is the polar angle for a 2D polar system of coordinates. The normalized radial function is:

$$R_{n_r m_l}(r) = N_{n_r m_l}\left(\frac{r}{l_0}\right)^{|m_l|} \exp\left(-\frac{r^2}{4\,l_0^2}\right) L_{n_r}^{|m_l|}\left(\frac{r^2}{2\,l_0^2}\right) \ , \tag{10}$$

where $N_{n_r m_l}$ is a normalization constant, $L_n^k(x)$ are associated Laguerre polynomials and

$$l_0 = \sqrt{\frac{\hbar}{|q|\,|\vec{B}|}} \ , \tag{11}$$

is known as the magnetic length. The magnetic length may be viewed as representing the smallest radius of a circular orbit in a magnetic field that is allowed by the rules of quantum mechanics.

Since the radial wave function is purely real, its orthonormalization condition is:

$$\int_0^\infty dr\ r\ R_{n_r m_l}(r)\ R_{n_r' m_l}(r) = \delta_{n_r n_r'} \ , \tag{12}$$

where δ_{ij} is the Kronecker delta symbol. The condition in Eq.(12) leads to the value of the normalization constant:

$$N_{n_r m_l} = \sqrt{\frac{n_r!}{l_0^2\ 2^{|m_l|}\ (n_r + |m_l|)!}} \ . \tag{13}$$

The normalized angular function, $\Phi_{m_l}(\varphi)$ is the eigenstate of the operator $\hat{L}_z$ from Eq.(5) which, in 2D polar coordinates, can be written as:

$$\hat{L}_z = -i\,\hbar\,\frac{\partial}{\partial\varphi} \ . \tag{14}$$

One can verify that:

$$\Phi_{m_l}(\varphi) = \frac{e^{i\,m_l\,\varphi}}{\sqrt{2\pi}} \ , \tag{15}$$

is orthonormal and also satisfies the condition:

$$\hat{L}_z\ \Phi_{m_l}(\varphi) = \hbar\,m_l\ \Phi_{m_l}(\varphi) \quad ; \quad m_l = 0, \pm 1, \pm 2, \ldots \tag{16}$$

Note that the allowed energy eigenvalues in Eq.(8) are determined by the quantum number:

$$n = n_r + \frac{|m_l| - m_l}{2} \ . \tag{17}$$

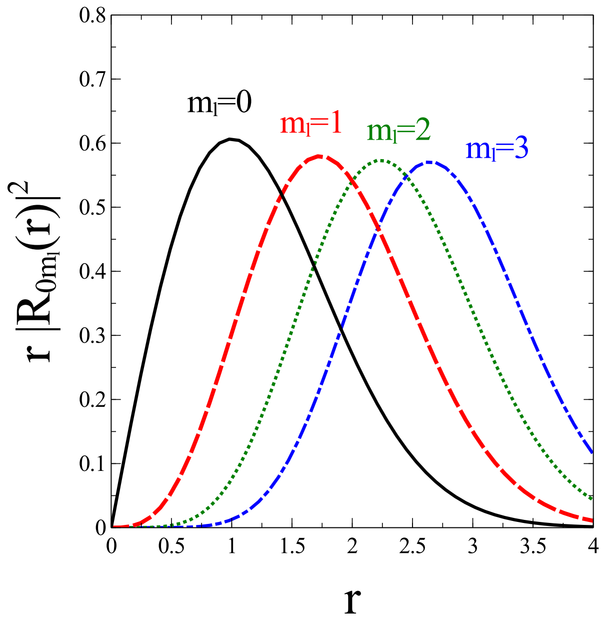

FIG. 1: The quantity, $r\,|R_{0\,m_l}(r)|^2$ is plotted as a function of r for values of $m_l = 0$ (solid line), $m_l = 1$ (dashed line), $m_l = 2$ (dotted line) and $m_l = 3$ (dash-dotted line). In our notation, r is a 2D radial distance. It is assumed that $l_0 = 1$.

The level with lowest possible energy (the ground state energy) is known as the lowest Landau level (LLL). The LLL energy has the value of $\hbar\,\omega_c/2$ and corresponds to $n = 0$; thus, it is associated with the quantum numbers:

$$n_r = 0 \quad ; \quad m_l = 0, 1, 2, \dots \; . \tag{18}$$

The other Landau levels (LLs) with higher energies correspond to $n = 1, 2, \dots$ values. All LLs are highly degenerate (through the quantum number m_l). The number of states, N_s with the same energy (degeneracy) for each LL in a macroscopic (disk) sample with area, A is calculated to be:

$$N_s = \frac{A}{2\,\pi\,l_0^2} \propto |\vec{B}| \; . \tag{19}$$

To explain briefly this result one observes that states with higher m_l extend to a larger radial distance. As a matter of fact, one can calculate exactly the expectation value, $\langle r^2\rangle$ for any given state. If a state with $n = 0$ is taken as an example, $\langle r^2\rangle = 2\,l_0^2\,(m_l + 1)$; $m_l = 0, 1, 2, \dots$. Since the particle should be within the area of the given sample, the maximum value of $\pi\langle r^2\rangle$ will be A. Thus, the maximum value of m_l denoted as m_l^{max} is such as to satisfy the condition, $2\,\pi\,l_0^2\,(m_l^{max} + 1) = A$. This means that the resulting degeneracy is $N_s = (m_l^{max} + 1) = A/(2\,\pi\,l_0^2)$ as seen from Eq.(19).

As a final observation, one can check that the LLL single-particle wave functions ($n_r = 0$; $m_l = 0, 1, \dots$) in Eq.(9) can be conveniently written using a complex notation as:

$$\Psi_{0\,m_l}(z) = \frac{1}{\sqrt{2\,\pi\,l_0^2\,2^{m_l}\,m_l!}} \left(\frac{z}{l_0}\right)^{m_l} \exp\left(-\frac{|z|^2}{4\,l_0^2}\right) \; , \tag{20}$$

where $z = x + i\,y = r\,e^{i\,\varphi}$ represents a complex variable for the 2D position vector and $m_l = 0, 1, \dots, N_s - 1$. A telling display is the plot of $r\,|R_{0\,m_l}(r)|^2$ for values $m_l = 0, 1, 2$ and 3 as shown in Fig. 1. For simplicity, we assume a unit length, $l_0 = 1$. One can visually get the idea from Fig. 1 that all the curves have the same integral consistent with the normalization condition in Eq.(12).

III. CHARGED PARTICLE WITH ANISOTROPIC MASS IN A UNIFORM CONSTANT PERPENDICULAR MAGNETIC FIELD

Now we consider the same setup for a particle with anisotropic mass of the form $m_x > 0$ and $m_y > 0$, so that the Hamiltonian becomes:

$$\hat{H} = \frac{1}{2\,m_x}\left(-i\,\hbar\,\frac{\partial}{\partial x} + q\,\frac{B_z}{2}\,y\right)^2 + \frac{1}{2\,m_y}\left(-i\,\hbar\,\frac{\partial}{\partial y} - q\,\frac{B_z}{2}\,x\right)^2 . \tag{21}$$

The Hamiltonian in Eq.(21) is the anisotropic mass counterpart to that for a constant isotropic mass in Eq.(3). Unfortunately, the counterpart to Eq.(4) is not useful, since the anisotropic mass makes it impossible to express the Hamiltonian in terms of $\hat{L}_z$. Instead, we check whether it is useful to rescale the variables, rewriting Eq.(21) as:

$$\hat{H} = \frac{\gamma^2}{2\,m_x}\left[-i\,\hbar\,\frac{\partial}{\partial(\gamma\,x)} + q\,\frac{B_z}{2}\left(\frac{y}{\gamma}\right)\right]^2 + \frac{1}{2\,m_y\,\gamma^2}\left[-i\,\hbar\,\frac{\partial}{\partial(\frac{y}{\gamma})} - q\,\frac{B_z}{2}\left(\gamma\,x\right)\right]^2 , \tag{22}$$

where γ is a real (positive) parameter to be determined.

In Eq.(3), the two coeffiecients in front of each bracket were equal, which we can achieve in Eq.(22) by choosing

$$\frac{\gamma^2}{2\,m_x} = \frac{1}{2\,m_y\,\gamma^2} \quad or \quad \gamma^2 = \sqrt{\frac{m_x}{m_y}} \; . \tag{23}$$

Then the Hamiltonian becomes

$$\hat{H} = \frac{1}{2\,m_c}\left(-i\,\hbar\frac{\partial}{\partial x'} + q\frac{B_z}{2}y'\right)^2 + \frac{1}{2\,m_c}\left(-i\,\hbar\frac{\partial}{\partial y'} - q\frac{B_z}{2}x'\right)^2 , \tag{24}$$

where

$$x' = \gamma\,x \quad ; \quad y' = \frac{y}{\gamma} \; , \tag{25}$$

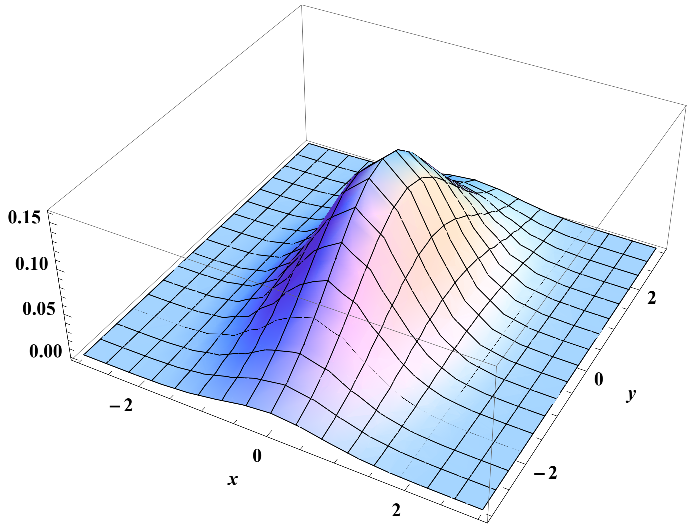

FIG. 2: Magnitude squared of the LLL wave function, $|\Psi_{0\,0}(z')|^2$ for a particle with anisotropic mass, $m_x = 2\,m_e$, $m_y = m_e/2$ as a function of coordinates x and y. The above values of the anisotropic mass give rise to parameter, $\gamma = \sqrt{2}$. As a result, the complex 2D position variable, z' reads $z' = x' + i\,y' = \gamma\,x + i\,y/\gamma = \sqrt{2}\,x + i\,y/\sqrt{2}$. For convenience, it is assumed that $l_0 = 1$.

are two scaled coordinate variables and

$$m_c = \sqrt{m_x\,m_y}\ , \tag{26}$$

denotes the effective cyclotron mass of a particle with anisotropic mass[21]. The Hamiltonian in Eq.(24) has exactly the same form as that in Eq.(3). Also, note that $dx\,dy = dx'\,dy'$. Therefore, the solutions presented in Section II are exactly the solutions to this problem, with the substitutions $x \to x'$, $y \to y'$, and $m \to m_c = \sqrt{m_x\,m_y}$.

In the following, we write only the result for the LLL states with a LLL energy of:

$$E_{LLL} = \frac{\hbar\,\omega_c'}{2}\ , \tag{27}$$

where $\omega_c' = |q||\vec{B}|/m_c$ is the cyclotron frequency. The normalized single-particle LLL states conveniently written using complex notation and the primed coordinates are:

$$\Psi_{0\,m_l}(z') = \frac{1}{\sqrt{2\,\pi\,l_0^2\,2^{m_l}\,m_l!}}\left(\frac{z'}{l_0}\right)^{m_l}\exp\left(-\frac{|z'|^2}{4\,l_0^2}\right)\ , \tag{28}$$

where $n_r = 0$ is the value of the radial quantum number for the LLL, $m_l = 0, 1, \ldots$ for the LLL, $z' = x' + i\,y'$ represents a complex variable in terms of the primed coordinates and l_0 is the magnetic length that does not depend on the mass of the particle.

To see the effects of mass anisotropy on the form of the Landau states, let us consider the simplest LLL state with $n_r = 0$ and $m_l = 0$, choosing $m_x = 2\,m_e$ and $m_y = m_e/2$ where m_e is the electron's bare mass. The effective cyclotron mass in this case with be $m_c = \sqrt{m_x\,m_y} = m_e$. This means that the LLL energy of this particle with anisotropic mass is exactly the same as that for an electron with constant isotropic mass, $m = m_e$. Since $\gamma^2 = 2$, the magnitude squared of this quantum state is:

$$\begin{aligned}
|\Psi_{0\,0}(z')|^2 &= \frac{1}{2\,\pi\,l_0^2}\exp\left(-\frac{|z'|^2}{2\,l_0^2}\right)\\
&= \frac{1}{2\,\pi\,l_0^2}\exp\left(-\frac{1}{2}\,\frac{\gamma^2\,x^2 + \frac{y^2}{\gamma^2}}{l_0^2}\right)\\
&= \frac{1}{2\,\pi\,l_0^2}\exp\left(-\frac{1}{2}\,\frac{2\,x^2 + \frac{y^2}{2}}{l_0^2}\right)\ ,
\end{aligned} \tag{29}$$

after we revert back to the coordinates x and y. A three-dimensional plot of $|\Psi_{0\,0}(z')|^2$ is shown in Fig. 2. For convenience, we assume $l_0 = 1$. As can be seen from the above plot, the magnitude squared of the wave function manifests the expected lack of circular symmetry induced by the anisotropic mass of the particle.

IV. CONCLUSIONS

There are not many problems in quantum mechanics that have analytical solutions and are straightforward enough to teach to undergraduate students. The list of such problems typically involves a free particle, various potential wells, the harmonic oscillator, the hydrogen atom and a particle in a magnetic field, with the latter featuring either in upper-level undergraduate courses or lower-level graduate courses. The problem of the quantum states of a charged particle moving in 2D space subject to a uniform constant perpendicular magnetic field is intimately connected to the understanding of two very important phenomena in condensed matter physics, namely, the integer quantum Hall effect and the fractional quantum Hall effect. To understand realistic experimental results, it's necessary to use a model with effective band mass anisotropy of electrons[13–17]. Therefore, we hope that this work will encourage future authors of quantum mechanics textbooks to incorporate the problem of a particle with anisotropic mass (with or without the presence of a magnetic field) more often as a very useful case study that illustrates the behavior of a system with inherent anisotropy.

Competing Interests

The authors have no conflicts to disclose.

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