Electrostatic potential of a uniformly charged annulus

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The calculation of the electrostatic potential and/or electrostatic field due to a continuous distribution of charge is a well-covered topic in all calculus-based undergraduate physics courses. The most common approach is to consider bodies with uniform charge distribution and obtain the quantity of interest by integrating over the contributions from all the differential charges. The examples of a uniformly charged disk and ring are prominent in many physics textbooks since they illustrate well this technique at least for special points or directions of symmetry where the calculations are relatively simple. Surprisingly, the case of a uniformly charged annulus, namely, an annular disk, is largely absent from the literature. One might speculate that a uniformly charged annulus is not extremely interesting since after all, it is a uniformly charged disk with a central circular hole. However, we show in this work that the electrostatic potential created by a uniformly charged annulus has features that are much more interesting than one might have expected. A uniformly charged annulus interpolates between a uniformly charged disk and ring. However, the results of this work suggest that a uniformly charged annulus has such electrostatic features that may be essentially viewed as ring-like. The ring-like characteristics of the electrostatic potential of a uniformly charged annulus are evident as soon as a hole is present no matter how small the hole might be. The solution of this problem allows us to draw attention to the pedagogical aspects of this overlooked, but very interesting case study in electrostatics. In our opinion, the problem of a uniformly charged annulus and its electrostatic properties deserves to be treated at more depth in all calculus-based undergraduate physics courses covering electricity and magnetism.

Keywords: Electrostatic potential, Uniform charge distribution, Annulus, Surface charge density, Linear charge density.

I. INTRODUCTION

The calculation of the electrostatic potential or electrostatic field due to a continuous charge distribution is a very important topic in electrostatics. The typical scenario considered is that of a uniform charge distribution over the length, surface or volume of a given body for the respective one-dimensional (1D), two-dimensional (2D) or three-dimensional (3D) cases. The typical objects studied in physics textbooks are regular charged bodies that have some form of symmetry. Prominent examples that feature in almost all calculus-based undergraduate physics textbooks that deal with the topics of electricity and magnetism [1–7] are objects like a uniformly charged finite line, ring, disk, spherical surface and solid sphere. The 1D case of a finite line with constant linear charge density is easy to handle by direct integration using a variety of methods [8]. The 3D case studies of a conducting spherical surface with constant surface charge density and a non-conducting solid sphere with constant volume charge density are perfect examples that illustrate the application of Gauss's law to derive the electric field. The 2D case of a disk with constant surface charge density represents a scenario in which the calculation of electrostatic potential/field is relatively straightforward for special points or directions of symmetry, for instance, at center of disk or along the axis of symmetry (line going through center of disk perpendicular to its plane). The same can be said for a uniformly charged ring. An expression for the electrostatic potential/field due to a uniformly charged disk or ring at an arbitrary point in space is not provided in most university physics textbooks because such a calculation is mathematically very demanding [9, 10]. Calculations for the electrostatic potential of a uniformly charged square/rectangular plate are mostly available from published peer-reviewed specialized papers [11, 12].

The problem of a uniformly charged annulus is absent in most of the common calculus-based undergraduate physics textbooks that cover electricity and magnetism [1–7]. A uniformly charged annulus, namely, an annular disk, can be seen as either a disk with a small circular hole when the inner radius is small or as a ring-like object when the inner radius of the annulus is about the same size as the outer radius. If one recalls, there are striking differences between the electrostatic potential due to a uniformly charged disk and that due to a uniformly charged ring even for points/directions of symmetry. For example, the electrostatic potential on the plane of a uniformly charged ring is singular (divergent) at points in the ring, but this is not the case for the potential at the edge of a uniformly charged disk. On the contrary, the electrostatic potential on the plane of a uniformly charged disk is smooth everywhere. It has its largest value at the center of the disk and monotonically decreases with distance (without having any peaks).

Therefore, a simple question that arises is whether the electrostatic potential of a uniformly charged annulus has disk-like or ring-like traits given that an annu-

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lus interpolates between these two objects. The naive expectation is that the electrostatic potential of a uniformly charged annulus should resemble that of a uniformly charged disk when the inner radius of the annulus is relatively small (the circular hole is small) and that of a uniformly charged ring, otherwise. It turns out that the correct answer to this question is, in our view, subtler than expected and a little bit surprising. Our results show that, for all instances in which the electrostatic potentials of a uniformly charged disk and ring differ from each other (like points on the respective planes of each object), the electrostatic potential of a uniformly charged annulus shows ring-like characteristics. In simple words, as soon as the annulus has a hole, its electrostatic potential on its plane does not look like the potential of a uniformly charged disk.

In this work, we calculate the electrostatic potential of a uniformly charged annulus at an arbitrary point in space. We introduce a special calculation method and explain the key mathematical steps in a clear pedagogical manner. The mathematical method that we use allows us to obtain a very useful compact exact 1D integral expression for the electrostatic potential of the annulus at any point in space as a function of its inner/outer radii, the two parameters that determine the shape of the object. Since the mathematical treatment quickly becomes a little bit too challenging for the audience that we have in mind, we move on to focus our attention on some special cases that are easy to understand. The results obtained illustrate that, indeed a uniformly charged annulus has fascinating properties that, unfortunately, have been largely overlooked in the wider literature.

The article is organized as follows. In Section II we explain the model and introduce the theoretical formalism. In Section III we discuss the main results. In Section IV we deliberate on the nature of the equilibrium state at the center of the uniformly charged annulus. In Section V we draw some conclusions and give a sense of the broader pedagogical and technical aspects of the work.

II. MODEL AND THEORY

A schematic view of an annulus is shown in Fig. 1. The annulus has an inner radius, R_1 and outer radius, R_2. We assume that:

$$0 \leq R_1 < R_2 \ . \tag{1}$$

The annulus becomes a disk with radius, R_2 when $R_1 = 0$. The annulus becomes a ring with radius, R_2 when $R_1 = R_2^-$. However, one must be very careful for this case and interpret the ring limit of the annulus as:

$$R_1 \rightarrow R_2^- \ , \tag{2}$$

where the "$^-$" sign means that the limit is reached from the side of R_1 that is smaller than R_2. The two parameters, R_1 and R_2 determine the shape and, thus, the geometry of the object. The annulus contains a net charge, Q

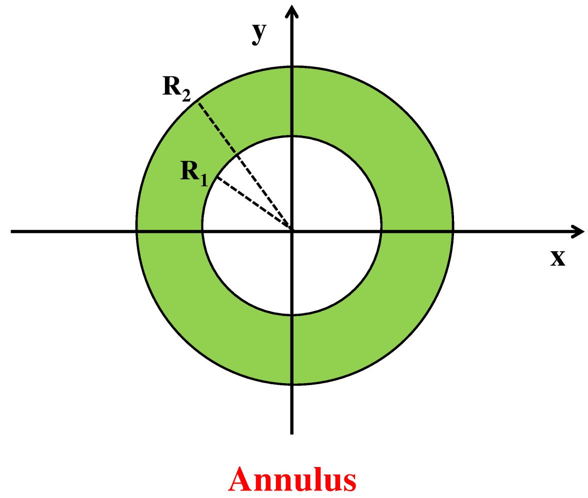

FIG. 1. Schematic view of a uniformly charged annulus lying on the $x-y$ plane. The origin of the Cartesian system of coordinates corresponds to the center of the annulus. The annulus has an inner radius, R_1 and an outer radius, R_2. The annnulus contains a total charge, Q that is spread uniformly on its surface.

that is uniformly distributed on its surface. As a result, the annulus has a constant surface charge density:

$$\sigma = \frac{Q}{\pi\, R_2^2 - \pi\, R_1^2} \ . \tag{3}$$

The system of coordinates is chosen so that the annulus lies on the $x-y$ plane with its center at the origin. For clarity, the z-axis is not drawn in Fig. 1 where we show only a 2D view of the domain occupied by the charge contained in the annulus. The annulus has circular symmetry. This means that a 3D cylindrical system of coordinates is the most suitable one to study this object. We denote by $\vec{r}\,' = (x', y', z' = 0)$ the position of an arbitrary point on the annulus. In 3D cylindrical coordinates, the 2D vector, $\vec{\rho}\,' = (x', y')$ has its components written as: $x' = \rho' \cos(\varphi')$ and $y' = \rho' \sin(\varphi')$ where $\rho' = |\vec{\rho}\,'| \geq 0$ is the 2D radial distance and $0 \leq \varphi' < 2\pi$ is the polar angle. On the other hand, the 3D vector, $\vec{r} = (x, y, z)$ represents an arbitrary point in space where the electrostatic potential is to be calculated. The corresponding 2D vector, $\vec{\rho} = (x, y)$ has its components expressed as its "primed" counterpart except that it is not "primed". The annulus surface domain reads:

$$D \ : \ \Big\{ R_1 \leq \rho' \leq R_2 \ ; \ 0 \leq \varphi' < 2\pi \Big\} \ . \tag{4}$$

We consider a differential element of charge, $dQ' = \sigma\, d^2\rho'$ where $d^2\rho' = dx'\, dy' = d\rho'\, \rho'\, d\varphi'$ is an elementary surface area on the annulus at location, $\vec{\rho}\,' = (x', y')$. The resulting electrostatic potential that this differential charge element creates at point, $\vec{r}$ is written as: $dV(\vec{r}) =$

$k_e\, dQ'/|\vec{r}-\vec{r}\,'|$, where k_e is Coulomb's electric constant. The electrostatic potential due to the whole uniformly charged annulus is then written as:

$$V_{R_1 R_2}(\vec{r}) = k_e\,\sigma \iint_D d^2\rho'\, \frac{1}{|\vec{r}-\vec{r}\,'|} = k_e\,\sigma \int_{R_1}^{R_2} d\rho'\,\rho' \int_0^{2\,\pi} d\varphi' \frac{1}{|\vec{r}-\vec{r}\,'|}\ , \tag{5}$$

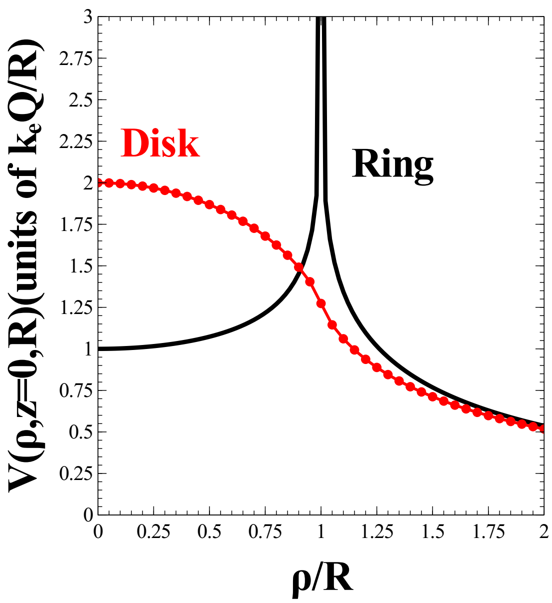

FIG. 2. Electrostatic potential on the plane of a uniformly charged disk, $V_{Disk}(\rho, z=0, R)$ (filled circles) and a uniformly charged ring, $V_{Ring}(\rho, z=0, R)$ (solid line) as a function of distance ρ/R. Both disk and ring have radius, R and contain a net charge, Q. The disk has a constant surface charge density, $\sigma = Q/(\pi\,R^2)$ while the ring has a constant linear charge density, $\lambda = Q/(2\,\pi\,R)$. Note the singularity (divergence) of the electrostatic potential due to a uniformly charged ring at $\rho = R$. The electrostatic potential is measured in units of $k_e\,Q/R$.

where the 2D integral is carried out over the annulus domain, D in Eq.(4) and $|\vec{r}-\vec{r}\,'| = \sqrt{|\vec{\rho}-\vec{\rho}\,'|^2+|z|^2}$. The integral in Eq.(5) cannot be calculated by direct integration and, thus, a special mathematical treatment is required. At this juncture, we take the opportunity to remind the reader about some already known results for a uniformly charged disk and ring that will be useful to us, later on.

The electrostatic potential due to a uniformly charged disk with radius, R and constant surface charge density, $\sigma = Q/(\pi\,R^2)$ can be expressed in compact form [see Eq.(11) of Ref. [13]] as:

$$V_{Disk}(\rho, z, R) = (2\,\pi)\,k_e\,\sigma\,R \int_0^\infty \frac{dk}{k}\, J_0(k\,\rho)\, J_1(k\,R)\, e^{-k\,|z|}\ , \tag{6}$$

where $J_m(x)$ are Bessel functions of the first kind of order, m. As we proceed with this work, we note that our preference is to use the 1D integral representation in Eq.(6) for the electrostatic potential of a uniformly charged disk instead of more explicitly written analytic expressions. Available explicit analytic expressions from the literature are very complicated even for the value of the electrostatic potential on the plane of the disk [13] and let alone its value everywhere in space [9, 10].

The electrostatic potential of a uniformly charged ring with radius, R and constant linear charge density, $\lambda = Q/(2\,\pi\,R)$ may be written [see Eq.(10) of Ref. [14]] as:

$$V_{Ring}(\rho, z, R) = (2\,\pi)\,k_e\,\lambda\,R \int_0^\infty dk\, J_0(k\,\rho)\, J_0(k\,R)\, e^{-k\,|z|}\ . \tag{7}$$

An explicit analytic result may be obtained [see Eq.(3) of Ref. [14]] in the form:

$$V_{Ring}(\rho, z, R) = \frac{k_e\,\lambda\,4\,R}{\sqrt{(\rho+R)^2+z^2}}\, K\left[\frac{4\,\rho\,R}{(\rho+R)^2+z^2}\right]\ , \tag{8}$$

where

$$K(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-m\,\sin^2(\theta)}}\ , \tag{9}$$

is the complete elliptic integral of the first kind defined for $0 \le m < 1$ as in Arfken and Weber [15] (see page 355). However, as already discussed for the case of a uniformly charged disk, we prefer to use the 1D integral representation in Eq.(7) even for the electrostatic potential of a uniformly charged ring because it is compact.

At this point, we note that one can easily rewrite the expressions in Eq.(6) and Eq.(7) in terms of charge, Q if one so desires. In fact, let us rewrite these two quantities for points that are on the plane of the respective objects ($z = 0$) by using charge, Q as a parameter instead of σ or λ. Although explicit analytical results are available [13, 14], the 1D integral form is preferred:

$$V_{Disk}(\rho, z=0, R) = 2\,\frac{k_e\,Q}{R} \int_0^\infty \frac{dq}{q}\, J_0\left(\frac{\rho}{R}\,q\right)\, J_1(q)\ , \tag{10}$$

and

$$V_{Ring}(\rho, z=0, R) = \frac{k_e\,Q}{R} \int_0^\infty dq\, J_0\left(\frac{\rho}{R}\,q\right)\, J_0(q)\ , \tag{11}$$

where $q = k\,R$ is a dummy variable (for both disk and ring). We note that the integrand in Eq.(10) is alternating in sign for both $\rho \neq R$ and $\rho = R$. However, the integrand in Eq.(11) is alternating in sign for $\rho \neq R$ but becomes non-negative for $\rho = R$. The outcome of this behavior is that the integral $\int_0^\infty dq\;[J_0(q)]^2$ diverges. Useful expressions for the value of the electrostatic potential at some special points of a uniformly charged disk are:

$$V_{Disk}(\rho = 0, z = 0, R) = 2\,\frac{k_e\,Q}{R} \tag{12}$$

and

$$V_{Disk}(\rho = R, z = 0, R) = \frac{4}{\pi}\,\frac{k_e\,Q}{R}\ . \tag{13}$$

It is also easy to obtain from Eq.(11) the expected result for the electrostatic potential of a uniformly charged ring at its center:

$$V_{Ring}(\rho = 0, z = 0, R) = \frac{k_e\,Q}{R}\ . \tag{14}$$

The results above can be checked with help from the following formulas: $J_0(x = 0) = 1$, $\int_0^\infty \frac{dx}{x} J_1(x) = 1$, $\int_0^\infty \frac{dx}{x} J_0(x)\,J_1(x) = 2/\pi$ and $\int_0^\infty dx\,J_0(x) = 1$.

As clearly shown in Fig. 2, the dependence of $V_{Disk}(\rho, z = 0, R)$ as a function of radial distance, ρ is strikingly different from that of $V_{Ring}(\rho, z = 0, R)$. Note that $V_{Disk}(\rho, z = 0, R)$ decreases monotonically from its largest value at the center of the disk as ρ increases. On the other hand, $V_{Ring}(\rho, z = 0, R)$ increases monotonically starting from its value at the center of the ring, it is divergent at $\rho = R$ and then decreases monotonically as ρ increases. Therefore, the question whether the electrostatic potential of a uniformly charged annulus resembles that of a uniformly charged disk (for a small inner radius) or that of a uniformly charged ring (for a much larger inner radius) and how much it resembles one or another object is not only interesting, but also demands a very careful analysis.

III. RESULTS AND DISCUSSIONS

The success of the mathematical method that is used to solve this problem hinges on the following expansion from Jackson's book [16] (see pg. 96):

$$\frac{1}{|\vec{r} - \vec{r}\,'|} = \sum_{m=-\infty}^{+\infty} \int_0^\infty dk\, e^{i\,m\,(\varphi - \varphi\,')}\, J_m(k\,\rho)\, J_m(k\,\rho\,')\, e^{-k\,|z - z\,'|}\ , \tag{15}$$

where $i = \sqrt{-1}$ is the imaginary unit and $J_m(x)$ was previously defined in the context of Eq.(6). The above transformation applies to any pair of 3D vectors expressed in cylindrical coordinates.

In many instances, integrals over angular variables of two-particle functions are very difficult [17–20]. However, one can see that axial symmetry in conjunction with Eq.(15) helps a lot in this case to obtain:

$$\int_0^{2\,\pi} d\varphi\,'\,\frac{1}{|\vec{r} - \vec{r}\,'|} = (2\,\pi)\int_0^\infty dk\,J_0(k\,\rho)\,J_0(k\,\rho\,')\,e^{-k\,|z|}\ , \tag{16}$$

since $z\,' = 0$. By substituting the result from Eq.(16) into Eq.(5) one has:

$$V_{R_1 R_2}(\rho, z) = (2\,\pi)\,k_e\,\sigma \int_0^\infty dk\,J_0(k\,\rho)\,e^{-k\,|z|} \int_{R_1}^{R_2} d\rho'\,\rho'\,J_0(k\,\rho')\ . \tag{17}$$

Note that the electrostatic potential of a uniformly charged annulus is written as $V_{R_1 R_2}(\rho, z)$ where the variables ρ and z are explicitly shown as arguments. This choice of notation reflects the circular symmetry of the problem under consideration. At this juncture, one uses the following indefinite integral formula involving Bessel functions:

$$\int dx\,x\,J_0(x) = x\,J_1(x)\ , \tag{18}$$

to obtain:

$$\int_{R_1}^{R_2} d\rho'\,\rho'\,J_0(k\,\rho') = \frac{1}{k}\Big[R_2\,J_1(k\,R_2) - R_1\,J_1(k\,R_1)\Big]\ . \tag{19}$$

As a result, we can write the electrostatic potential due to a uniformly charged annulus in a simple 1D integral form as:

$$V_{R_1 R_2}(\rho, z) = (2\,\pi)\,k_e\,\sigma \int_0^\infty \frac{dk}{k}\,J_0(k\,\rho)\Big[R_2\,J_1(k\,R_2) - R_1\,J_1(k\,R_1)\Big]\,e^{-k\,|z|}\ . \tag{20}$$

At this juncture, one can make an important observation with regard to the expression in Eq.(20) and notice that one can write it as:

$$V_{R_1 R_2}(\rho, z) = V_{Disk}(\rho, z, R = R_2) - V_{Disk}(\rho, z, R = R_1)\ , \tag{21}$$

where $V_{Disk}(\rho, z, R)$ is given from Eq.(6). This result can be interpreted from the standpoint of viewing the electrostatic potential of a uniformly charged annulus with inner radius R_1, outer radius R_2 and surface charge density σ as the electrostatic potential of a uniformly charged disk with radius R_2 minus the electrostatic potential of a uniformly charged disk with radius R_1 where both disks are concentric, coplanar and have the same surface charge density σ. Another equivalent way to reach the same conclusion is by decomposing the uniformly charged annulus with inner radius R_1, outer radius R_2 and surface charge density σ into two concentric coplanar uniformly charged disks: one with a radius of R_2 and a surface charge density, σ and the other one with a radius R_1 and a surface charge density, $-\sigma$. This decomposition method, allows one to capture even the nature of the electrostatic potential of a disk with an off-centered circular hole (let's say with a radius R_1) in it (with the assumption that the object has constant surface charge density) since the electrostatic potential of a uniformly charged disk is known. However, when using the decomposition approach, one must be careful while writing the electrostatic potential of the uniformly charged disk (that fills the off-centered circular hole) since the center of this uniformly charged disk is shifted away from the origin. If $\vec{\rho}_c = (x_c, y_c)$ denotes such a 2D center then one can conclude that the electrostatic potential at some point $\vec{r} = (x, y, z)$ can be formally written as a superposition: $V_{Disk}(\rho, z, R = R_2) - V_{Disk}(|\vec{\rho} - \vec{\rho}_c|, z, R = R_1)$ where the expression in Eq.(6) is used for $V_{Disk}(\rho, z, R)$. Obviously, the total electrostatic potential in this scenario does not have a circular symmetry.

Since it is preferable to express the electrostatic potential in terms of the net charge rather than the surface charge density, one writes:

$$(2\,\pi)\,k_e\,\sigma = \frac{2\,k_e\,Q}{R_2^2 - R_1^2}\;. \tag{22}$$

For more convenience, one can make the expression in Eq.(20) even more compact by introducing ratios of parameters. To this effect, we assume that:

$$R_2 \neq 0\;. \tag{23}$$

Now, we express R_1 in terms of R_2 by introducing a parameter, α defined as:

$$0 \leq \alpha = \frac{R_1}{R_2} < 1\;. \tag{24}$$

The range of α in Eq.(24) is such because of the condition, $0 \leq R_1 < R_2$ from Eq.(1).

To simplify the calculations, one introduces a dummy variable, $q = k\,R_2$ and after some careful manipulation of the terms in Eq.(20), the final result reads:

$$V_{\alpha R_2}(\rho, z) = \frac{k_e\,Q}{R_2}\,\frac{2}{1-\alpha^2}\int_0^\infty \frac{dq}{q}\,J_0\left(\frac{\rho}{R_2}\,q\right)\,\Big[J_1(q) - \alpha\,J_1(\alpha\,q)\Big]\,e^{-\frac{|z|}{R_2}\,q}\;. \tag{25}$$

Note that we explicitly put α and R_2 as parameters in the expression for the electrostatic potential function, $V_{\alpha R_2}(\rho, z)$ appearing in Eq.(25). The results in Eq.(20) and Eq.(25) are very general and allow one to treat various special cases by carefully taking the proper limits. Overall, the simplicity of the 1D integral presentation in Eq.(20) and Eq.(25) is very appealing. The 1D integrals can be easily calculated numerically with very high precision by using standard integration packages [21]. At this stage, one can use either Eq.(20) or Eq.(25) to verify whether the above quantities reduce to known results for special cases. To be more specific, we will look at the disk and ring limit of the electrostatic potential of the uniformly charged annulus, value of electrostatic potential at center of annulus, value of electrostatic potential on the plane of the annulus and value of electrostatic potential along the axis of symmetry of the annulus. A particular focus will be placed on the analysis of the two latter scenarios since such an analysis represents an excellent case study to check whether the electrostatic potential of a uniformly charged annulus manifests disk-like or ring-like traits.

A. Disk limit

A uniformly charged annulus becomes a uniformly charged disk with radius, R_2 when $R_1 = 0$. Therefore, one sets $R_1 = 0$ in Eq.(20), to obtain:

$$V_{R_1=0 R_2}(\rho, z) = (2\,\pi)\,k_e\,\sigma\,R_2\int_0^\infty \frac{dk}{k}\,J_0(k\,\rho)\,J_1(k\,R_2)\,e^{-k\,|z|}\;. \tag{26}$$

At this juncture, it is easy to see that:

$$V_{R_1=0 R_2}(\rho, z) = V_{Disk}(\rho, z, R = R_2)\;, \tag{27}$$

where the expression for $V_{Disk}(\rho, z, R)$ is given from Eq.(6).

B. Ring limit

A uniformly charged annulus becomes a uniformly charged ring with radius, R_2 when $R_1 = R_2^-$. However, this condition requires a very careful consideration of the

$R_1 \to R_2^-$ limit:

$$\lim_{R_1 \to R_2^-} (R_2^2 - R_1^2) = 2\, R_2 \lim_{R_1 \to R_2^-} (R_2 - R_1) \ . \qquad (28)$$

Based on the result from Eq.(28), one can write:

$$\lim_{R_1 \to R_2^-} \sigma = \lambda \lim_{R_1 \to R_2^-} \frac{1}{(R_2 - R_1)} \ , \qquad (29)$$

where

$$\lambda = \frac{Q}{2\,\pi\, R_2} \ , \qquad (30)$$

represents the constant linear charge density of a ring with radius, R_2 and net charge, Q. Imposition of the condition from Eq.(29) into Eq.(20) leads to:

$$V_{R_1 = R_2^-\, R_2}(\rho, z) = (2\,\pi)\, k_e\, \lambda \int_0^\infty \frac{dk}{k}\, J_0(k\,\rho) \lim_{R_1 \to R_2^-} \left[\frac{R_2\, J_1(k\, R_2) - R_1\, J_1(k\, R_1)}{R_2 - R_1} \right] e^{-k\,|z|} \ . \qquad (31)$$

The correct value of the limit under the integral sign is obtained with help from the following derivative formula:

$$\frac{d}{dx} \Big[x\, J_1(k\, x) \Big] = (k\, x)\, J_0(k\, x) \ . \qquad (32)$$

As a consequence, one has:

$$V_{R_1 = R_2^-\, R_2}(\rho, z) = (2\,\pi)\, k_e\, \lambda\, R_2 \int_0^\infty dk\, J_0(k\,\rho)\, J_0(k\, R_2)\, e^{-k\,|z|} \ . \qquad (33)$$

The final result in Eq.(35) allows one to verify that:

$$V_{R_1 = R_2^-\, R_2}(\rho, z) = V_{Ring}(\rho, z, R = R_2) \ , \qquad (34)$$

where the expression for $V_{Ring}(\rho, z, R)$ is provided in Eq.(7).

C. Center of annulus

The value of the electrostatic potential at the center of a uniformly charged annulus reads:

$$V_{R_1\, R_2}(\rho = 0, z = 0) = (2\,\pi)\, k_e\, \sigma \int_0^\infty \frac{dk}{k} \Big[R_2\, J_1(k\, R_2) - R_1\, J_1(k\, R_1) \Big] \ , \qquad (35)$$

where we used the fact that $J_0(x = 0) = 1$. The next step is to implement the following integral formula: $\int_0^\infty dx\, \frac{J_1(x)}{x} = 1$. At this point, it is straightforward to obtain:

$$V_{R_1\, R_2}(\rho = 0, z = 0) = (2\,\pi)\, k_e\, \sigma\, (R_2 - R_1) \ . \qquad (36)$$

In terms of charge, after writing $\sigma = Q/(\pi\, R_2^2 - \pi\, R_1^2)$, one derives:

$$V_{R_1\, R_2}(\rho = 0, z = 0) = \frac{2\, k_e\, Q}{R_2 + R_1} \ . \qquad (37)$$

D. Plane of annulus

Now, we turn our fullest attention to the expression for the electrostatic potential created by a uniformly charged annulus on its own plane. We prefer to rely on the more compact 1D expression in Eq.(25) for such an analysis:

$$V_{\alpha\, R_2}(\rho, z = 0) = \frac{k_e\, Q}{R_2} \frac{2}{1 - \alpha^2} \int_0^\infty \frac{dq}{q}\, J_0\left(\frac{\rho}{R_2}\, q \right) \Big[J_1(q) - \alpha\, J_1(\alpha\, q) \Big] \ . \qquad (38)$$

We know that the electrostatic potential of a uniformly charged disk is smooth and monotonically decreasing

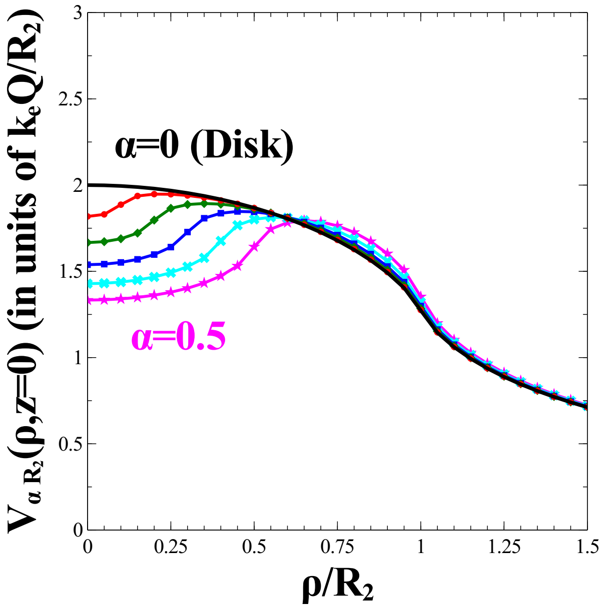

FIG. 3. Electrostatic potential, $V_{\alpha R_2}(\rho, z = 0)$ due to a uniformly charged annulus for points on its plane as a function of radial distance from its center, ρ/R_2. The total charge on the surface of the annulus is denoted by Q. The annulus has an inner radius, R_1 and an outer radius, R_2 (which is kept fixed). The inner radius may vary from $R_1 = 0$ to $R_1 = R_2^-$. Plots are given for a series of values of parameter, $\alpha = R_1/R_2 = 0$ (solid line), 0.1 (filled circles), 0.2 (filled diamonds), 0.3 (filled squares), 0.4 (crosses) and 0.5 (stars). The case $\alpha = 0$ represents a uniformly charged disk with radius, R_2 (without a central hole). The electrostatic potential is measured in units of $k_e\,Q/R_2$.

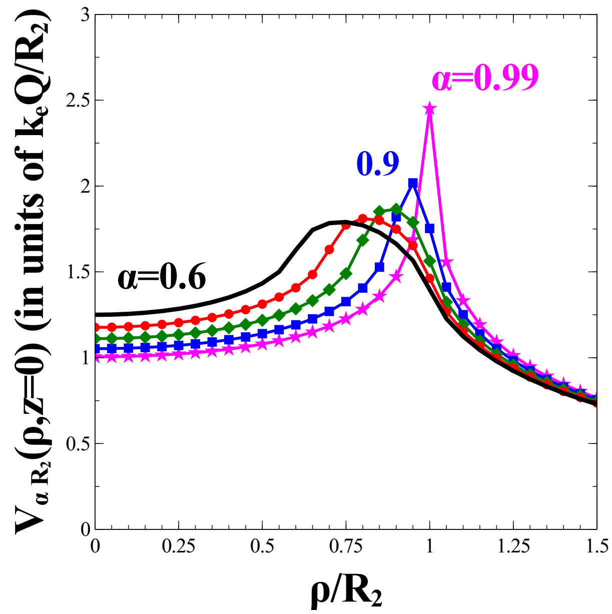

FIG. 4. The same as in Fig.3 but for different values of the parameter, $\alpha = 0.6$ (solid line), 0.7 (filled circles), 0.8 (filled diamonds), 0.9 (filled squares) and 0.99 (stars). The case $\alpha = 0.99$ is very close to representing a uniformly charged ring with radius, R_2 which has a diverging potential at $\rho = R_2$.

starting from its largest value at disk's center. On the other hand, the electrostatic potential of a uniformly charged ring increases at points away from the center, becames singular (infinite) on the ring and then gradually decreases. In crude terms, the electrostatic potential of a uniformly charged disk is smooth and falls monotonically as a function of distance without having any peaks, while the electrostatic potential of a uniformly charged ring has an (infinite) peak. At first sight, we were tempted to believe that, for a small inner radius, the electrostatic potential of a uniformly charged annulus may resemble that of a uniformly charged disk. To check this point, we considered six values of parameter, $\alpha = R_1/R_2$ starting from $\alpha = 0$ (disk case) and $\alpha = 0.1, 0.2, 0.3, 0.4$ and 0.5. In Fig.3 we show the electrostatic potential, $V_{\alpha R_2}(\rho, z = 0)$ as a function of radial distance from its center, ρ/R_2 for $0 \leq \alpha \leq 0.5$ (in steps of 0.1). The resulting electrostatic potential does not show any characteristics that resemble the electrostatic potential of a uniformly charged disk even for the smallest $\alpha \neq 0$ value considered ($\alpha = 0.1$). One can see from Fig.3 that the electrostatic potential of a uniformly charged annulus is not monotonic for values $\alpha \neq 0$ and always has a peak at some $\rho \neq 0$.

For all values of α considered in Fig.3, one notices that the peak of the electrostatic potential is broad and more or less develops at some radial distance $\rho \neq 0$ within the $R_1 \leq \rho \leq R_2$ region. One sees from Fig.3 that the peak grows progressively sharper as R_1 approaches R_2, namely, when parameter alpha grows from 0.1 to 0.5. We verified that similar patterns occur for values of α that are as small as $\alpha = 0.01$. Therefore, it is reasonable to conclude that a peak in the electrostatic potential of a uniformly charged annulus on its plane appears at some $\rho \neq 0$ for any arbitrary value $R_1 \neq 0$, no matter how small such a value might be. In a nutshell, once a central circular hole appears in the annulus ($R_1 \neq 0$), the electrostatic potential of a uniformly charged annulus on its plane develops a peak/maximum at a given point $\rho \neq 0$ between R_1 and R_2. Such traits are unlike those observed for a uniformly charged disk which produces an electrostatic potential that is smooth and has a peak at the center of the disk ($\rho = 0$). From this perspective, one might see the above features of the electrostatic potential of the annulus as manifesting a ring-like behavior. This is better seen in Fig.4 where we show the electrostatic potential of a uniformly charged annulus for points on its plane, $V_{\alpha R_2}(\rho, z = 0)$ as a function of the radial distance from its center, ρ/R_2 for larger values of the parameter, $\alpha = 0.6,\ 0.7,\ 0.8,\ 0.9$ and 0.99. The value $\alpha = 0.99$ is very close to representing a uniformly charged ring with radius, R_2 which has a diverging potential at $\rho = R_2$. The peak eventually becomes an infinity at $\rho = R_2$ for $R_1 = R_2^-$. However, the maximum appears, morphs, and blows up smoothly with the increase of the inner radius of the annulus (as α increases). Therefore, the above argument is not a water-tight one. As a matter of fact, it will be shown that an analysis of the electrostatic potential and the resulting electrostatic field along the z-axis of the uniformly charged annulus represents a more convincing evidence suggesting that, indeed, the physics of a uniformly charged annulus is closer to the physics of a uniformly charged ring than that of a disk.

E. Axis of annulus

The expression for the electrostatic potential along the z-axis of symmetry ($\rho = 0$) of a uniformly charged annulus is:

$$V_{R_1 R_2}(\rho = 0, z) = (2\,\pi)\, k_e\, \sigma \left[\sqrt{R_2^2 + |z|^2} - \sqrt{R_1^2 + |z|^2}\right] . \tag{39}$$

The result above can be easily verified if one starts from the expression in Eq.(20) and applies the following integral formula:

$$\int_0^\infty \frac{dx}{x}\, J_1(x)\, e^{-a\,x} = \sqrt{1 + a^2} - a \quad ; \quad a \geq 0 \; , \tag{40}$$

where parameter, $a \geq 0$ is considered to be real. If desirable, one may rewrite the expression in Eq.(39) as:

$$V_{\alpha R_2}(\rho = 0, z) = (2\,\pi)\, k_e\, \sigma\, R_2 \left[\sqrt{1 + \left(\frac{z}{R_2}\right)^2} - \sqrt{\alpha^2 + \left(\frac{z}{R_2}\right)^2}\right] , \tag{41}$$

where parameter α is defined in Eq.(24) and σ is defined in Eq.(3). In Fig.5 we show the electrostatic potential, $V_{\alpha R_2}(\rho = 0, z)$ in units of $(2\,\pi)\, k_e\, \sigma\, R_2$ as a function of z/R_2 for $\alpha = 0$ (disk) and $\alpha = 0.5$. The ring limit ($R_1 \to R_2^-$; $\alpha \to 1^-$) can be obtained from Eq.(39) [or from Eq.(41)] by carefully following the procedure already explained in Subsection.(III B).

For comparison, the electrostatic potential along the z-axis of symmetry for a uniformly charged disk with radius, R and constant surface charge density, $\sigma = Q/(\pi\, R^2)$ reads:

$$V_{Disk}(\rho = 0, z, R) = (2\,\pi)\, k_e\, \sigma \left[\sqrt{R^2 + |z|^2} - |z|\right] . \tag{42}$$

For a uniformly charged ring with radius, R and constant linear charge density, $\lambda = Q/(2\,\pi\, R)$ one has:

$$V_{Ring}(\rho = 0, z, R) = \frac{(2\,\pi)\, k_e\, \lambda\, R}{\sqrt{R^2 + |z|^2}} = \frac{k_e\, Q}{\sqrt{R^2 + |z|^2}} . \tag{43}$$

Knowledge of the electrostatic potential allows one to obtain the electrostatic field along the z-axis. For instance, when the annulus case is considered, one has:

$$E_{R_1 R_2\, z}(\rho = 0, z) = -\frac{\partial}{\partial z} V_{R_1 R_2}(\rho = 0, z) . \tag{44}$$

The electrostatic field along the z-axis can be calculated this way for each of the three objects under consideration (annulus, disk and ring). The result for a uniformly charged annulus is:

$$E_{R_1 R_2\, z}(\rho = 0, z) = (2\,\pi)\, k_e\, \sigma \left[\frac{z}{\sqrt{R_1^2 + |z|^2}} - \frac{z}{\sqrt{R_2^2 + |z|^2}}\right] . \tag{45}$$

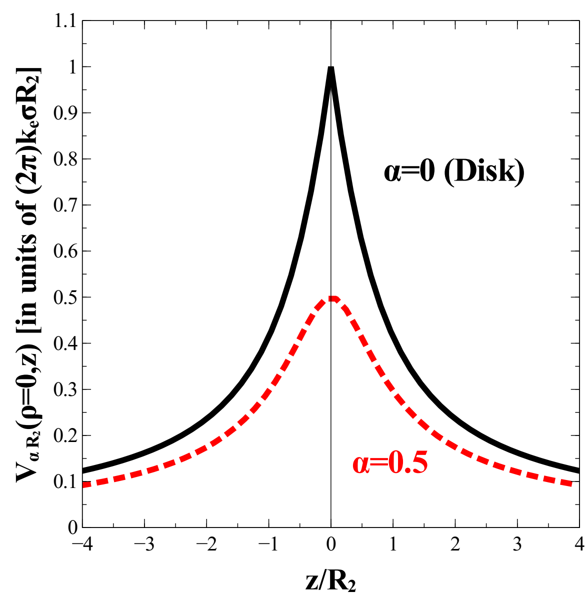

FIG. 5. Electrostatic potential, $V_{\alpha R_2}(\rho = 0, z)$ due to a uniformly charged annulus as a function of z/R_2 for $\alpha = R_1/R_2 = 0$ (solid line) and $\alpha = 0.5$ (dotted line). The case $\alpha = 0$ represents a uniformly charged disk with radius, R_2 (without a central circular hole). The electrostatic potential is measured in units of $(2\,\pi)\, k_e\, \sigma\, R_2$.

Note that as soon as $R_1 \neq 0$ (which means that there is circular hole at the center), one has:

$$E_{R_1 R_2\, z}(\rho = 0, z = 0) = 0 . \tag{46}$$

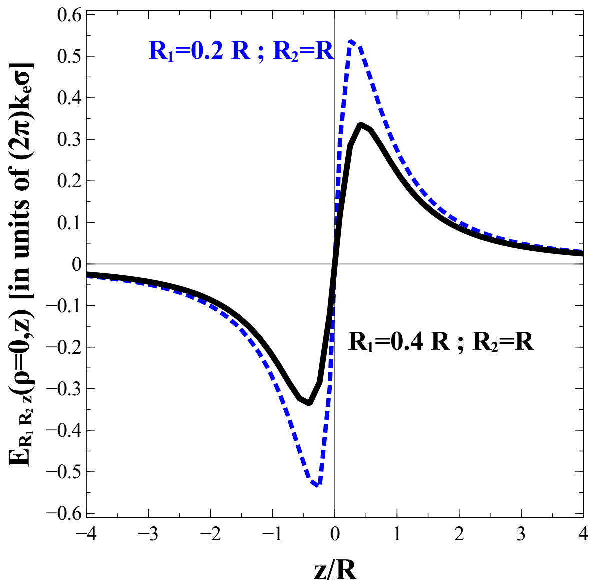

FIG. 6. Electrostatic field along the z-axis, for a uniformly charged annulus, $E_{R_1 R_2\, z}(\rho = 0, z)$ for values $R_1 = 0.4\, R$ and $R_2 = R$ (solid line) and $R_1 = 0.2\, R$ and $R_2 = R$ (dottd line) as a function of z/R. The field is measured in units of $(2\,\pi)\, k_e\, \sigma$.

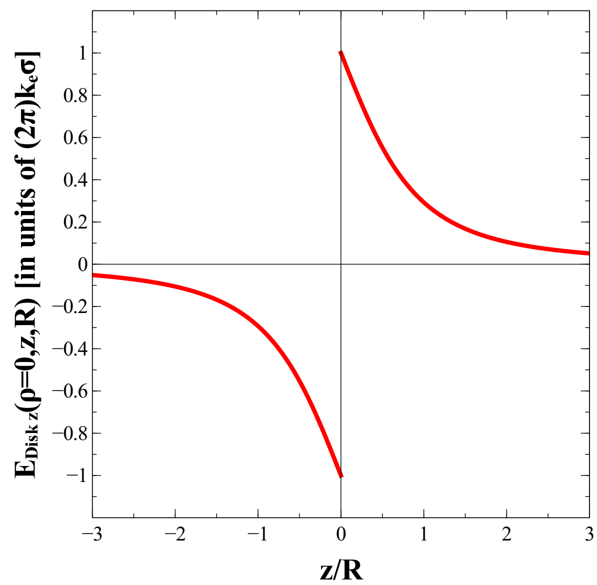

FIG. 7. Electrostatic field along the z-axis, $E_{Disk\, z}(\rho = 0, z, R)$ for a uniformly charged disk with radius, R and constant surface charge density, σ as a function of z/R. The field is measured in units of $(2\,\pi)\, k_e\, \sigma$.

The electrostatic field along the z-axis of a uniformly charged annulus is continous for all values of z. A simple plot of $E_{R_1 R_2\, z}(\rho = 0, z)$ measured in units of $(2\,\pi)\, k_e\, \sigma$ for values $R_1 = 0.4\, R$ and $R_2 = R$ (solid line) and $R_1 = 0.2\, R$ and $R_2 = R$ (dottd line) is shown in Fig.6.

The result for the electrostatic field of a uniformly charged disk along its z-axis of symmetry is:

$$E_{Disk\, z}(\rho = 0, z, R) = (2\,\pi)\, k_e\, \sigma \left[\frac{z}{|z|} - \frac{z}{\sqrt{R^2 + |z|^2}} \right] . \tag{47}$$

Note the discontinuity of the z-component of the electrostatic field at $z = 0$. A calculation of the electrostatic feld slightly above ($z = 0^+$) and slightly below ($z = 0^-$) the $z = 0$ plane of the disk at its center gives:

$$E_{Disk\, z}(\rho = 0, z = 0^+, R) = (2\,\pi)\, k_e\, \sigma = \frac{\sigma}{2\,\epsilon_0} , \tag{48}$$

and

$$E_{Disk\, z}(\rho = 0, z = 0^-, R) = -(2\,\pi)\, k_e\, \sigma = -\frac{\sigma}{2\,\epsilon_0} , \tag{49}$$

where $k_e = 1/(4\,\pi\,\epsilon_0)$. A plot of $E_{Disk\, z}(\rho = 0, z, R)$ measured in units of $(2\,\pi)\, k_e\, \sigma$ as a function of z/R is shown in Fig.7.

Finally, the result for the electrostatic field on the z-axis of a uniformly charged ring is:

$$E_{Ring\, z}(\rho = 0, z, R) = \frac{k_e\, Q\, z}{(R^2 + |z|^2)^{3/2}} . \tag{50}$$

It is easy to note that:

$$E_{Ring\, z}(\rho = 0, z = 0, R) = 0 . \tag{51}$$

Unlike the case of a uniformly charged disk, there is no discontinuity at $z = 0$ for the case of a uniformly charged ring (obviously, we consider $R \neq 0$). A plot of $E_{Ring\, z}(\rho = 0, z, R)$ measured in units of $k_e\, Q/R^2$ as a function of z/R is shown in Fig.8.

A quick comparison of Fig.6 and Fig.8 serves to illustrate what do we mean when we state that the physics of a uniformly charged annulus is closer to the physics of a uniformly ring than that of a uniformly charged disk. For a uniformly charged disk, the electric field along the z-axis goes through a jump (discontinuity) at the $z = 0$ plane because of the surface charge density present on the disk. On the other hand, this discontinuity does not appear for a uniformly charged annulus regardless of the size of the inner radius (as long as $R_1 \neq 0$). This is also the case for a uniformly charged ring. The similarity in shape when comparing the electrostatic field along the z-axis of a uniformly charged annulus with the ring counterpart is self-explanatory.

IV. EQUILIBRIUM AT THE CENTER OF ANNULUS

We have tacitly assumed throughout this work that the total charge that is uniformly spread over the area of the annulus is positive, $Q > 0$ (and the same assumption applies to the other bodies like the cases of disk or

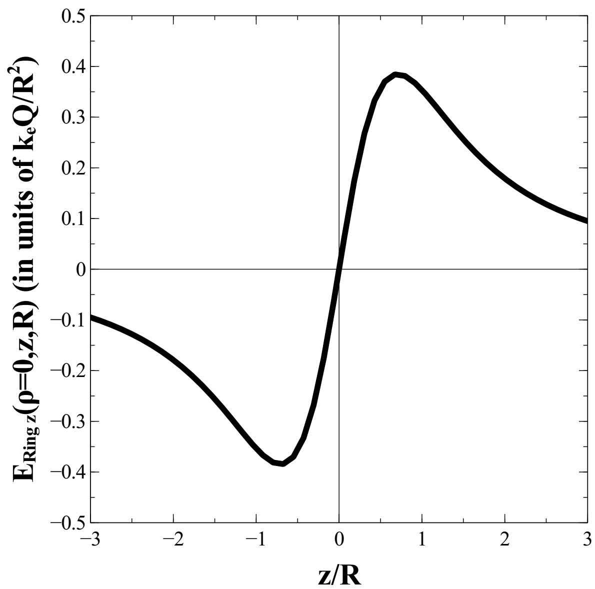

FIG. 8. Electrostatic field along the z-axis, $E_{Ring\,z}(\rho = 0, z, R)$ for a uniformly charged ring with radius, R and constant linear charge density, λ as a function of z/R. The field is measured in units of $k_e\, Q/R^2$.

ring). We also remark that the following discussion on the nature of the equilibrium at the center of a uniformly charged annulus applies to a situation in which the inner radius of the annulus is $R_1 \neq 0$. First, for a test charge to be in equilibrium at any particular point (in this case the point in question is the center of the annulus, $\rho = 0, z = 0$), the electrostatic field (thus, force) at that point must be zero. Second, if the equilibrium is to be a stable equilibrium, one must require that if we move the test charge away from that point by a little bit there should be a restoring force directed opposite to the displacement.

Let us consider a positive test charge $q_+ > 0$ initially located at ($\rho = 0, z = 0$). At that point the electrostatic field (and force) is zero. If we move the positive test charge away along the $z > 0$ (or $z < 0$) direction, the electrostatic force will not return it to the original location. Therefore, the test charge must be negative, $q_- < 0$ in order to experience a restoring force that causes it return toward the center. This means that a negative test charge, $q_- < 0$ will have a stable equilibrium along the z-axis at the center of the uniformly charged annulus.

Let us now review the situation along the radial direction, $\vec{\rho}$ on the $z = 0$ plane. As already explained, a negative test charge, $q_- < 0$ has a stable equilibrium at the center when displaced solely along the z-axis. Therefore, we consider this negative test charge, $q_- < 0$ initially at ($\rho = 0, z = 0$) and move it along the radial direction to a point ($\rho \neq 0, z = 0$) where ρ is small. The electric field due to the uniformly charged annulus along the radial direction at that point can be calculated from:

$$\vec{E}_{R_1 R_2\, \rho}(\rho, z = 0) = -\frac{dV_{R_1 R_2}(\rho, z = 0)}{d\rho}\, \frac{\vec{\rho}}{\rho}\ , \tag{52}$$

where $V_{R_1 R_2}(\rho, z = 0)$ imay be obtained from Eq.(20). One can easily note from Fig. 3 and Fig. 4 that $\frac{dV_{R_1 R_2}(\rho, z=0)}{d\rho} > 0$ for $\rho \neq 0$ (close to the center). This means that the electrostatic field for a small radial distance ρ is:

$$\vec{E}_{R_1 R_2\, \rho}(\rho, z = 0) \propto -\vec{\rho}\ . \tag{53}$$

As a result the electrostatic force felt by the negative test charge, $q_- < 0$ is:

$$\vec{F}(\rho, z = 0) = q_-\, \vec{E}_{R_1 R_2\, \rho}(\rho, z = 0) \propto +\vec{\rho}\ . \tag{54}$$

This radially outward force will move the negative test charge away from the center of the uniformly charged annulus. This means that there is an unstable equilibrium along the radial $\vec{\rho}$-axis. As a result, there is an overall unstable equilibrium at the center of a uniformly charged annulus. Mathematically speaking the point ($\rho = 0, z = 0$) represents a saddle point for the electrostatic potential due to a uniformly charged annulus. The electrostatic potential increases as we move away from the center of the annulus along the radial direction, $\vec{\rho}$ for small values of ρ, but decreases when moving away from the center along the z-axis direction.

This behavior is consistent with a venerable result in electrostatics known as Earnshaw's theorem [22] which states that point charges in empty space cannot be maintained in a stable stationary equilibrium solely by means of the electrostatic interaction of the charges. With few words, a stable equilibrium of a charged particle cannot exist in empty space and there must be an instability in some direction. The impossibility of having a point of stable equilibrium in any electrostatic field (with a few additional subleties) is neatly discussed on the beginning of Chapter 5 of "The Feynman Lectures on Physics" Vol. 2 using Gauss's law [23]. An important conclusion reached is that a test charge cannot be in stable equilibrium in empty space at a point where there is no some opposite charge [23]. The situation above precisely applies to the center of the uniformly charged annulus (which is a point surrounded by empty space in its immedate vicinity).

On the other hand, a test charge can be in stable equilibrium if it is in the middle of a distributed opposite charge with the understanding that the opposite charge distribution woud have to be held in place by other then electrical forces [23]. This would be the case of a negative test charge, $q_- < 0$ located at the center of a uniformly charged disk (it is assumed that the disk contains a total positive charge, thus, opposite charge, $Q > 0$ that is spread uniformly on its surface). This means that the negative test charge located at the center of the positively and uniformly charged disk (namely, a point surrounded by a distributed opposite positive charge in its immediate vicinity) can be in stable equilibrium.

V. CONCLUSIONS

In this work, we studied in detail the features of the electrostatic potential created by a uniformly charged annulus at an arbitraty point in space. We solved this problem with help from a special mathematical method that is valid in systems with circular symmetry. The final result for the electrostatic potential of a uniformly charged annulus is given in a compact 1D integral form that is easy to handle numerically. Naively, one might think that a uniformly charged annulus has properties that might resemble those of a uniformly charged disk when the inner radius is small. However, this work shows that the electrostatic properties of a uniformly charged annulus (for inner radius, $R_1 \neq 0$) resemble more those of a uniformly charged ring than disk. This means that the electrostatic properties of a uniformly charged annulus are much more interesting than one might have initially envisioned. It turns out that, while interpolating between a uniformly charged disk and ring, the electrostatic potential of a uniformly charged annulus is ring-like.

The most interesting scenario that we investigated concerned the attributes of the electrostatic potential of a uniformly charged annulus on its plane. For this situation, the electrostatic potential of a uniformly charged disk is fundamentally different from its ring counterpart. The electrostatic potential of a uniformly charged disk is finite at its center and then monotonically drops to a smaller finite value at its edge. On the other hand, the electrostatic potential of a uniformly charged ring is finite at its center and becomes infinite at points in the circumference of the ring. The evolution of the shape of the electrostatic potential of a uniformly charged annulus on its plane is seen clearly in Fig.3 and Fig.4. One can observe a singularity developing out of the initial broad peak by considering sequences of annuli with an inner radius that starts from zero (disk limit) and then progressively grows to come closer to the outer radius. An analysis of the electrostatic field along the z-axis of symmetry of the uniformly charged annulus represents a more convincing evidence that, indeed, the electrostatic properties of a uniformly charged annulus are closer to a uniformly charged ring than disk counterpart.

The electrostatic properties of a uniformly charged annulus can be used to illustrate fundamental principles of electrostatics from a pedagogical point of view. However, this system is also very important for many other applications, for instance, in the world of sensors. As a matter of fact, a charged annulus is a key component for building coplanar capacitive sensors which are used for a non-destructive evaluation of materials [24–27]. The most common design of a coplanar capacitive sensor has two coplanar electrodes, namely, plates with any given shape (square/rectangular, circular, etc.). Designs with circular symmetry are very common and typically they consist of two concentric coplanar electrodes [28, 29] where a disk acts as the inner electrode while a concentric annulus that surrounds the disk and is coplanar to it represents the outer electrode. In a nutshell, a charged annulus is relevant for both electrostatic and sensor technology applications.

By solving this problem that is largely overlooked in calculus-based undergraduate university physics textbooks, we want to draw particular attention to the very attractive pedagogical aspects of this case study. In our opinion, a uniformly charged annulus represents an object with such rich electrostatic properties that it deserves to be treated in more depth in physics courses dealing with electromagnetism. As a result, we believe that the insights gained from this work may appeal to a broad audience of undergraduate/graduate students, teachers and researchers working in various scientific disciplines.

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