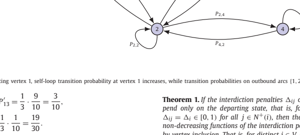


**Fig. 1.** By interdicting vertex 1, self-loop transition probability at vertex 1 increases, while transition probabilities on outbound arcs $(1,2)$ and $(1,3)$ decrease.

$$P'_{12}=\frac{1}{3}\cdot\frac{1}{5}=\frac{1}{15},\ P'_{13}=\frac{1}{3}\cdot\frac{9}{10}=\frac{3}{10},$$
$$P'_{11}=\frac{1}{3}+\frac{1}{3}\cdot\frac{4}{5}+\frac{1}{3}\cdot\frac{1}{10}=\frac{19}{30}.$$

Replacing these probabilities in the equations results in:

$$t_{13}(x)=1+\frac{19}{30}t_{13}(x)+\frac{1}{15}t_{23}(x)$$
$$t_{23}(x)=1+\frac{1}{3}t_{23}(x)+\frac{1}{3}t_{13}(x)+\frac{1}{3}t_{43}(x)$$
$$t_{43}(x)=1+\frac{1}{3}t_{43}(x)+\frac{1}{3}t_{23}(x),$$

which gives that $t_{13}(x)=3.72$, $t_{23}(x)=5.48$, and $t_{43}(x)=4.24$. Therefore, $t_{13}(x)<t_{13}$, and the FPT from between $S$ and $T$ in this case has decreased after interdiction according to $x$.

Intuitively, in the example in Fig. 2, interdiction of vertex 1 makes it more likely for the DTMC to remain in vertex 1 (increasing the probability from 1/3 to 19/30) while at the same time decreasing the probability of transitioning from 1 to 3 only by small amount (from 1/3 to 3/10). This imbalance is because $\Delta_{13}$ is much smaller compared to $\Delta_{12}$. Thus, one can expect interdiction to be successful, if for any $i\in V$, all the $\Delta_{ij}$ have similar values across all the outbound arcs. This observation is made precise in Theorem 1. Subsequently, in Theorem 2, we show that FPT interdiction is NP-hard even when restricted to such instances.

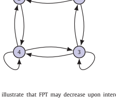

**Fig. 2.** An example to illustrate that FPT may decrease upon interdiction in some cases.

**Theorem 1.** *If the interdiction penalties $\Delta_{ij}$ on the outbound arcs depend only on the departing state, that is, for every $i\in V$, we have $\Delta_{ij}=\Delta_i\in[0,1)$ for all $j\in N^+(i)$, then the FPTs are monotonically non-decreasing functions of the interdiction policies (partially) ordered by vertex inclusion. That is, for distinct $i\in V$ and $j\in T$, $t_{ij}(\bar{x})\geq t_{ij}(\tilde{x})$ for $\bar{x},\tilde{x}\in\{0,1\}^{|V|}$ such that $\bar{x}\geq\tilde{x}$.*

**Proof.** For an arbitrary fixed vertex $j\in T$, after enforcing $t_{jj}(x)=0$ and $P_{ii}+\sum_{k\in N^+(i)}P_{ik}=1$, we can rewrite equations (3) as follows:

$$t_{ij}(x)-\frac{1}{(1-P_{ii})}\sum_{k\in N^+(i)\setminus\{j\}}P_{ik}t_{kj}(x)=\frac{1}{(1-\Delta_i x_i)(1-P_{ii})}\quad\forall i\in V\setminus\{j\}.$$

For given $x$ and $j$, we express the foregoing system of $|V|-1$ equations in as many unknown FPTs in matrix notation to facilitate our arguments. Let the unknown FPTs be denoted by the $(|V|-1)$-dimensional column vector:

$$T(x):=[t_{1j}(x),t_{2j}(x),\cdots,t_{j-1,j}(x),t_{j+1,j}(x),\cdots,t_{|V|,j}(x)]^\top,$$

and let $Q$ denote the $(|V|-1)\times(|V|-1)$ matrix whose entries are defined below (with rows and columns indexed by $V\setminus\{j\}$ for convenience):

$$Q_{ik}:=\begin{cases}\frac{P_{ik}}{1-P_{ii}}, & \text{for } k\in N^+(i)\setminus\{j\},\\ 0, & \text{otherwise,}\end{cases}\qquad\forall i\in V\setminus\{j\},$$

and let $I$ denote the $(|V|-1)\times(|V|-1)$ identity matrix. Note that $Q$ is well-defined because, from our assumptions, $P_{ii}=1$ if and only if $i=j$ and $T=\{i\}$. Furthermore, let $R(x)$ denote the right-hand side column vector with its $i$-th element defined as:

$$R(x)_i:=\frac{1}{(1-\Delta_i x_i)(1-P_{ii})}\qquad\forall i\in V\setminus\{j\}.$$

Then, the equations of FPTs in the new notations are as follows:

$$(I-Q)T(x)=R(x).$$

Since $V$ is finite, the rows of $Q$ sum to at most one, and at least one row of $Q$ sums to strictly less than one (otherwise, the probability of reaching state $j$ will be zero), then the absolute value of all eigenvalues of $Q$ is strictly less than one (see e.g., Theorem 2.8 of [11]). Thus, $(I-Q)^{-1}$ exists and is given by the following geometric series expansion:





$$(I-Q)^{-1}=\sum_{\ell=0}^{\infty}Q^\ell.$$

Consequently,

$$T(x)=\sum_{\ell=0}^{\infty}Q^\ell R(x).$$

For $i$ such that $\bar{x}_i>\tilde{x}_i$, we see that $R(\bar{x})_i>R(\tilde{x})_i$; with the components being equal otherwise. As each element in $Q^\ell$ is nonnegative for all $\ell\geq 0$ and $R(\bar{x})\geq R(\tilde{x})$, we conclude that $T(\bar{x})\geq T(\tilde{x})$, establishing our claim. □

**Corollary 1.** *If the assumptions of Theorem 1 hold, then for any interdiction decision $x$, we can compute lower and upper bounds on FPTs based on the inequality: $t_{ij}(\vec{0})\leq t_{ij}(x)\leq t_{ij}(\vec{1})$, where $\vec{0}$ and $\vec{1}$ are the $|V|$-dimensional all-zero and all-one vectors, respectively.*

### *3.2. FPT interdiction problem is NP-hard*

In this section, we show that problem (2) is NP-hard using a polynomial-time reduction from vertex cover [4]. Consider the following decision problems as given below. We assume that all data are rational.

VERTEX COVER: Given a simple, undirected graph $G=(V,E)$ and a positive integer $b$, does $G$ contain a vertex cover of size at most $b$? In other words, is there a subset $C$ with at most $b$ vertices that contains at least one end point of every edge in $G$?

FPT INTERDICTION: Given a digraph $D=(V,A)$, a DTMC $\{X_n:n\geq 0\}$ with state space $V$ and one-step transition probability matrix $P$, interdiction penalties $\Delta_{ij}\in[0,1)$ for $i\in V$, $j\in N^+(i)$, disjoint nonempty vertex subsets $S$, $T$, a positive integer $b$, and a positive rational number $\ell$, does there exist a feasible interdiction policy $x\in B$ such that $t_{ij}(x)\geq\ell$ for all $i\in S$ and $j\in T$?

**Theorem 2.** FPT INTERDICTION *is NP-complete.*

**Proof.** Given an interdiction policy $x$, in polynomial time we can verify its feasibility, compute the minimum FPT from $S$ to $T$ using the system of linear equations (3), and verify whether or not it is at least $\ell$. Therefore, FPT INTERDICTION belongs to class NP.

We demonstrate a polynomial time reduction from VERTEX COVER to establish NP-hardness. Let $\langle G=(V,E),b\rangle$ denote the VERTEX COVER instance. Without loss of generality, we assume that $G$ contains no isolated vertices. We create the digraph by adding self-loops at every vertex of $G$ and replacing every edge of $G$ with anti-parallel arcs. We also create a new auxiliary vertex $w\notin V$ to which every vertex in $G$ has outbound arcs (see Fig. 3). Denote this digraph by $D=(V\cup\{w\},A)$, where,

$$A:=\bigcup_{i\in V\cup\{w\}}\{(i,w),(i,i)\}\cup\bigcup_{\{i,j\}\in E}\{(i,j),(j,i)\}.$$

We set $S=V$, $T=\{w\}$, the interdiction budget is $b$ (same as the upper-bound of the vertex cover instance), and $\ell=2$. We also use a parameter $p\in[0,1)$ in defining the transition probabilities and interdiction penalties below, and eventually demonstrate the claim for a sufficiently large $p$. The one-step transition probability matrix of the DTMC is defined as follows, for each $(i,j)\in A$,

$$P_{ij}=\begin{cases}0, & \text{if } i=j\neq w,\\ 1, & \text{if } i=j=w,\\ p, & \text{if } j=w\neq i,\\ (1-p)/d_i, & \text{otherwise,}\end{cases}$$

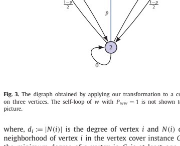


**Fig. 3.** The digraph obtained by applying our transformation to a complete graph on three vertices. The self-loop of $w$ with $P_{ww}=1$ is not shown to simplify the picture.

where, $d_i:=|N(i)|$ is the degree of vertex $i$ and $N(i)$ denotes the neighborhood of vertex $i$ in the vertex cover instance $G$. Note that the minimum degree of a vertex in $G$ is at least one. Finally, assume that the interdiction penalties are $\Delta_{ij}=p$ for every $i\in V$ and $j\in N^+(i)$. This completely specifies the FPT INTERDICTION instance which can be constructed in polynomial time. Note that the constructed interdiction instance also satisfies the conditions of Theorem 1.

The FPTs for the interdiction instance is given by the following equations for any interdiction policy $x\in B$ and $i\in V$ using equation (3) and substituting $\Delta_i$ as $p$ or zero as applicable:

$$t_{iw}(x)=\frac{1}{(1-px_i)}+\frac{(1-p)}{d_i}\sum_{k\in N(i)}t_{kw}(x).\qquad(4)$$

Next we show that the FPT INTERDICTION instance is a 'yes' instance if and only if $G$ contains a vertex cover of size at most $b$. Suppose $C\subset V$ is a vertex cover of $G$ containing at most $b$ vertices. Consider the interdiction policy $x$ in which we interdict vertices in $C$. We claim that the FPT from an arbitrary vertex $i\in S$ to $w$ is at least 2. We consider two cases: vertex $i\in C$ and vertex $i\notin C$.

*Case (i)* Vertex $i\in C$: In this case, $x_i=1$ and the value of the first passage time is:

$$t_{iw}(x)=\frac{1}{(1-p)}+\frac{(1-p)}{d_i}\sum_{k\in N(i)}t_{kw}(x)$$

In this case, if we choose $p\geq 1/2$, then $t_{iw}(x)\geq 2$.

*Case (ii)* Vertex $i\notin C$: In this case, $x_i=0$ and $x_k=1$ for all $k\in N(i)$ as $N(i)\subset C$ to cover all the edges incident at vertex $i$. The FPT in this case is:

$$\begin{aligned}t_{iw}(x)&=1+\frac{(1-p)}{d_i}\sum_{k\in N(i)}t_{kw}(x)\\&=1+\frac{(1-p)}{d_i}\sum_{k\in N(i)}\left[\frac{1}{(1-p)}+\frac{(1-p)}{d_k}\sum_{j\in N(k)}t_{jw}(x)\right]\\&=2+\frac{(1-p)^2}{d_i}\sum_{k\in N(i)}\frac{1}{d_k}\sum_{j\in N(k)}t_{jw}(x).\end{aligned}$$

by the solver. Although there are several Club instances that utilized general purpose cutting planes to arrive at an integral optimum (either at the root node or after enumeration), it is notable that for 9 Club instances and all DIMACS-10 instances the optimal solution of formulation (8) is obtained at the root node without the addition of any cutting planes. Our preliminary analysis did not uncover any simple explanation for this behavior. It would be interesting to see if graph properties like edge sparsity, low graph degeneracy, or small treewidth of the underlying graph make the problem easier to solve.

## 6. Conclusions

We introduced an interdiction model to maximize the minimum first passage time between two given sets of states in a DTMC. We demonstrate that the interdiction strategy could be counter-productive in some situations and provide a sufficient condition that guarantees that interdiction will not decrease the first passage times. We then established the NP-hardness of the problem. We present a MILP formulation that can be used to solve small to medium sized instances of the problem using a commercial solver within minutes. We observed from our preliminary experiments that the interdiction framework we propose is capable of producing significant increases in the minimum FPT, demonstrating its potential applicability. It would be interesting in light of some of our computational results to identify graph classes for which this problem may be polynomially solvable. More generalized interdiction budget constraints of the form $B := \left\{x \in \{0,1\}^{|V|} : \sum_{(i,j)\in A} c_{ij}\Delta_{ij}x_i \le b\right\}$ could also be explored in specific applications. In order to ensure that the methodology can scale well in practice, future work needs to focus on decomposition techniques to solve our formulation and also explore alternate formulation ideas.

## CRediT authorship contribution statement

**Niloufar Daemi:** Data curation, Formal analysis, Investigation, Methodology, Software, Writing – original draft, Writing – review & editing. **Juan S. Borrero:** Conceptualization, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Supervision, Writing – original draft, Writing – review & editing. **Balabhaskar Balasundaram:** Formal analysis, Investigation, Methodology, Supervision, Writing – original draft, Writing – review & editing.

## Acknowledgements

This research is partially funded by the National Science Foundation (NSF) (Awards ENG/CMMI # 2145553) and by the Air Force Office of Scientific Research (AFOSR) (Award # FA9550-22-1-0236).

## References

[1] S. Abraham, S. Nair, Cyber security analytics: a stochastic model for security quantification using absorbing Markov chains, J. Commun. 9 (12) (2014) 899–907.
[2] J. Bang-Jensen, G.Z. Gutin, Digraphs: Theory, Algorithms and Applications, Springer-Verlag, London, 2009.
[3] J. Berkhout, B.F. Heidergott, Analysis of Markov influence graphs, Oper. Res. 67 (3) (2019) 892–904.
[4] M.R. Garey, D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, W.H. Freeman and Company, New York, 1979.
[5] Gurobi Optimization, LLC, Gurobi Optimizer Reference Manual, http://www.gurobi.com, 2021.
[6] A. Gutfraind, A. Hagberg, F. Pan, Optimal interdiction of unreactive Markovian evaders, in: International Conference on Integration of Constraint Programming, Artificial Intelligence, and Operations Research, Springer, 2009, pp. 102–116.
[7] C. Haseltine, S. Wang, L.A. Albert, Dynamic cyber-physical system security planning using Markov chains and attack graphs, Online at https://guaguakai.github.io/aasg2022/assets/AASG2022_paper_1.pdf. (Accessed August 2023), 2022.
[8] M.P. Johnson, A. Gutfraind, K. Ahmadizadeh, Evader interdiction: algorithms, complexity and collateral damage, Ann. Oper. Res. 222 (1) (2014) 341–359.
[9] A. Kasenov, E. Kustov, A. Magazev, V. Tsyrulnik, A Markov model for optimization of information security remedies, J. Phys. Conf. Ser. (2020) 012043.
[10] M. Khouzani, Z. Liu, P. Malacaria, Scalable min-max multi-objective cyber-security optimisation over probabilistic attack graphs, Eur. J. Oper. Res. 278 (3) (2019) 894–903.
[11] V.G. Kulkarni, Modeling and Analysis of Stochastic Systems, CRC Press, 2016.
[12] A. Magazev, V. Tsyrulnik, Optimizing the selection of information security remedies in terms of a Markov security model, J. Phys. Conf. Ser. 1096 (2018) 012160.
[13] M.I. Meltzer, I. Damon, J.W. LeDuc, J.D. Millar, Modeling potential responses to smallpox as a bioterrorist weapon, Emerg. Infect. Dis. 7 (6) (2001) 959.
[14] F.M. Pajouh, B. Balasundaram, I.V. Hicks, On the 2-club polytope of graphs, Oper. Res. 64 (6) (2016) 1466–1481, https://doi.org/10.1287/opre.2016.1500.
[15] S.I. Resnick, Adventures in Stochastic Processes, Springer Science & Business Media, 1992.
[16] J.A. Sefair, J.C. Smith, M.A. Acevedo, R.J. Fletcher Jr, A defender-attacker model and algorithm for maximizing weighted expected hitting time with application to conservation planning, IISE Trans. 49 (12) (2017) 1112–1128.
[17] K. Zheng, L.A. Albert, Interdiction models for delaying adversarial attacks against critical information technology infrastructure, Nav. Res. Logist. 66 (5) (2019) 411–429.