

Inelastic effects in bulge formation of inflated polymer tubes

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Abstract

When a soft tube is inflated, it may sometimes show a bulge instability wherein a portion of the tube inflates much more than the rest. The bulge instability is well-understood for hyperelastic materials. We examine inflation of polyurethane tubes whose material behavior is not strictly hyperelastic. Upon inflating at constant rate, the tubes deform into a variety of shapes including irregular axisymmetric shapes with multiple localized bulges, a single axially-propagating bulge, or homogeneous cylindrical shapes. In all cases regardless of the inflation mode, the pressure first rises to a maximum, and then gradually reduces towards a plateau. We document numerous differences as compared to hyperelastic tubes. Most notably a pressure maximum can appear even without bulging, whereas for hyperelastic tubes, a pressure maximum is necessarily accompanied by bulging. Further, the decrease in pressure beyond the maximum occurs gradually over timescales as long as an hour, whereas bulging of hyperelastic tubes induces an instantaneous drop in pressure. We also observe permanent deformation upon deflation, a decrease in the pressure maximum during a subsequent second inflation, and more severe bulge localization at low inflation rates. Existing theory of hyperelastic tube inflation cannot capture the observed behaviors, even qualitatively. Finite element simulations suggest that many of the observations can be explained by viscoelasticity, specifically that a slow material response allows the pressure to remain high for long durations, which in turn allows growth of multiple bulges.

Keywords: Tube inflation, bulging instability, propagating instability, plasticity, viscoelasticity, aneurysm

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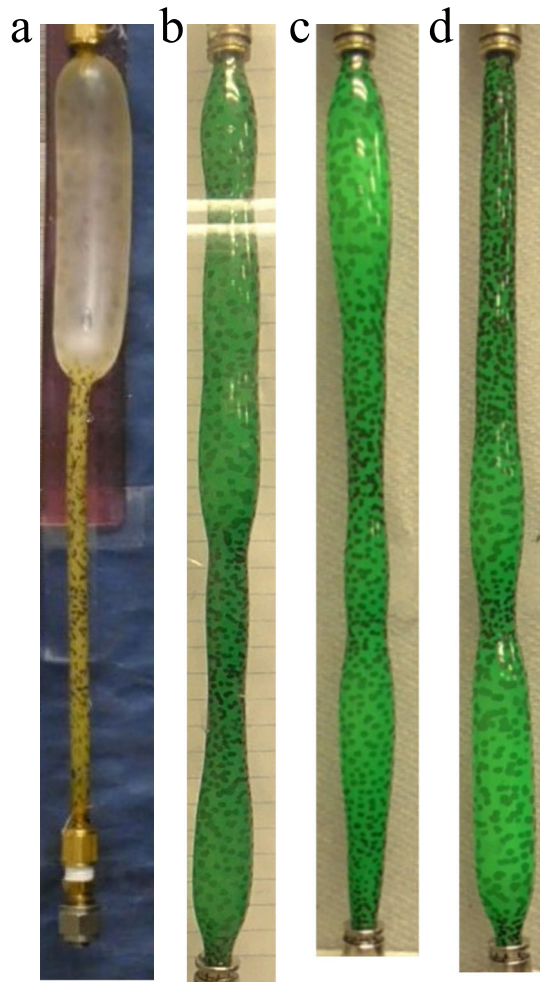
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1. Introduction

When a long elastomeric tube such as a rubber hose is inflated without constraining its length, it can exhibit two limiting behaviors: homogeneous expansion maintaining a cylindrical shape (not shown), or coexistence between two cylindrical regions one of which is much more inflated than the other (Figure 1a). The latter behavior is often called a *propagating instability* because as the tube is inflated, the more-inflated region propagates axially with no change in pressure or diameter of either region¹⁻¹¹. A third behavior – a localized bulge that expands to bursting^{7,12} – may be regarded as a special case of Figure 1a where the more-inflated region of the tube ruptures before coexistence between the two states is achieved. These behaviors can be captured by hyperelastic models of material behavior where the degree of strain hardening and the tube geometry (i.e. ratio of inner to outer diameter) determine which of these behaviors appear. If the ends of the tube are constrained, e.g. by providing a fixed axial stretch, the tubes may also buckle upon inflation¹³⁻¹⁶. Previous articles by Fu and coauthors have provided a succinct summary of the sequence of research starting from the 1960s regarding bulging of elastic tubes^{10,17}.

However, these limiting cases are overly-simplistic and more complex behavior may appear, as illustrated in Figure 1b-d. Even though all three specimens were cut from the same spool of polyurethane tubing and inflated identically, they show distinct inflations. These behaviors do not cleanly resemble either homogeneous inflation or the bulge propagation instability of Figure 1a; instead, the tubes inflated in an irregular fashion, sometimes with multiple bulges separated by less-inflated regions. More strikingly than the irregular inflation, the bulges in the polyurethane tubes grow over timescales of minutes to hours, whereas in hyperelastic tubes bulge growth is almost instantaneous. These tubes also show rate dependent inflation behavior, permanent deformation upon unloading, and loading-reloading hysteresis, all of which will be discussed below. None of these complexities can be captured by existing theories based on hyperelastic material models because hyperelastic models necessarily require the mechanical behavior to be captured by the instantaneous strain state with no dependence on deformation rate or deformation history.

This paper is an experimental study of tube inflation that includes mechanical behaviors beyond hyperelasticity. We examine the role of inelastic deformation, viscoelasticity, and strain-induced damage (i.e. permanent changes in properties upon first inflation) on the inflation behavior of polyurethane elastomer tubes. We document distinct differences as compared to hyperelastic tubes, yet, the bulges are axisymmetric, which is qualitatively similar to hyperelastic tubes. In later research to be published separately but available in a thesis¹⁸, we will also examine large deformation inflation of polyethylene “plastic” tubes which inflate in a non-axisymmetric manner which differ from hyperelastic tubes even qualitatively^{19,20}.



3 Figure 1: a. Bulge propagation in a natural rubber tube in which a more-bulged region coexists with
 4 a less-inflated region. As fluid is pumped in, the bulged region propagates axially with no change in
 5 diameter. The pressure-volume curve for this inflation is shown in Figure 8a. b-d. Irregular
 6 expansion of three polyurethane tubes inflated under the same conditions. All three were cut from
 7 the same spool of tubing had similar initial length. Both tubes had uninflated diameter of $0.25''$
 8 ≈ 6.35 mm. The dark splotches on each tube are ink marks to help visualize the local area changes.

10 Incidentally we note that even ordinary rubber balloons – which are often cited in the literature on inflation
 11 instabilities of hyperelastic tubes – can sometimes show some of these complexities. For example, inflating
 12 a rubber balloon into the bulge propagation regime such as Figure 1a and then deflating it induces a
 13 permanent increase in diameter. A second inflation can then yield three coexisting diameters, the smallest
 14 of which corresponds to the portion of the balloon that has never experienced large inflation (Figure S 1).
 15 Such complexities are rarely discussed in the literature. Indeed, experimental papers on inflation
 16 instabilities sometimes mention that they “preconditioned” their samples by stretching them repeatedly

prior to inflation^{6,21–25}, and it is only this preconditioning that allows them to be modeled as hyperelastic.

2. Experimental details

Tensile testing: Polyurethane tubes, with an outer diameter of 1/4 inch (6.35 mm) and an inner diameter of 3 5/32 inch (3.97 mm) with the product number of 5648K25, were purchased from McMaster-Carr Supply Co. The polyurethane has a durometer rating of 95A. The vendor states that the tubing was manufactured by Freelin Wade, and is a polyether-based polyurethane. The tubes were available in the form of rolled spools and hence have an intrinsic radius of curvature of 120-130 mm in their stress-free configuration. This intrinsic curvature had no noticeable effect on the inflation behavior. The rubber tube of Figure 1a (outer diameter 1/4 inch, inner diameter 1/8 inch, product number 5546K42) was also purchased from the same vendor.

Uniaxial tests were conducted using an Instron model 34TM-30 tensile testing machine equipped with a 30 kN load cell. The clamp-to-clamp length of the samples was 50 mm, and they were stretched at 25 mm/min (i.e. a nominal rate of 50% per minute) to various strains. The corresponding tensile data (Figure 2a) give a tensile modulus of 56 MPa. As with many polymeric materials, the material is strongly strain hardening at strains exceeding 200%.

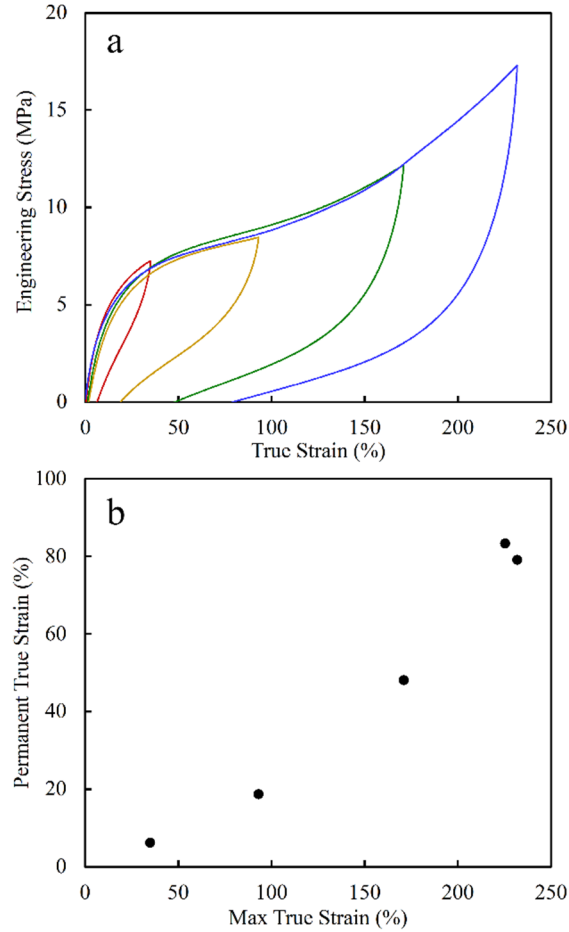


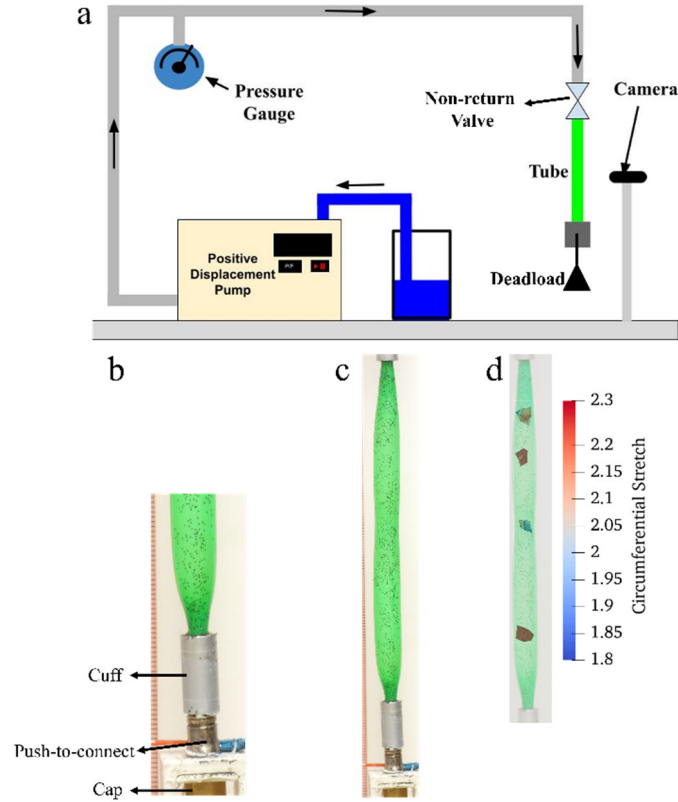
Figure 2: a. Uniaxial tensile testing data for polyurethane tubes during loading and unloading at a nominal strain rate of 50% per minute up to various values of maximum strain. b. Inelastic strain obtained from the data in a.

As mentioned in the Introduction, the tubes had readily-visible permanent deformation upon deflation. To test for inelastic behavior, the same specimens were also unloaded at the same speed. Pronounced loading-unloading hysteresis was noted, Figure 2a. A signature of inelastic deformation is that the force during unloading reduces to zero when the nominal strain is still non-zero. By this criterion, significant inelastic behavior (e.g. permanent strain exceeding 10%) appeared when the true strain exceeded about 50%.

The effect of elongation rate, ranging from nominal rate of 10%/min to 250%/min is shown in Supplementary Figure S 2. The tensile behavior remains qualitatively similar to that in Figure 2, with a modest decrease in stress at lower rate.

Inflation testing: The experimental setup is shown in Figure 3a. Tubes were inflated using a constant-flow rate piston pump (Isco Reaxys LS). A pressure gauge (Ralston LC10-GR2M) was used to continuously monitor pressure at 1 Hz frequency. The entire inflation process was imaged, either using a video camcorder (Panasonic HC-V180 operating at 60 frames/s) or a camera (Panasonic DC-FZ80 operating with a 1 - 10 s

1 duration between successive photos).



2

3 Figure 3: a. The inflation setup which includes water reservoir positive displacement pump,
 4 pressure gauge, and camera. b. Tube connections shown at higher magnification. c& d. Example of
 5 the snapshot and circumferential stretch of the tube at $\Delta V = 20$ mL calculated from tracking markers
 6 on the tube surface.

7

8 All tubes were cut to a length of 200 mm, and the water was supplied from the top of the tube, whereas the
 9 bottom was capped. The fittings on both ends were of the push-to-connect type. To reduce the chance of
 10 leakage at the connections, the end section immediately adjacent to the connectors was restrained by snugly
 11 fitting aluminum “cuffs” (Figure 3b). The section within the cuffs could only inflate axially, not radially.
 12 Thus, accounting for the cuff lengths and the length at each end that is inserted into the push-to-connect
 13 fittings, the section of the undeformed tube that could inflate freely was 115 mm long, corresponding to an
 14 undeformed aspect ratio L/R of 52.7, and an initial internal volume of 1.42 mL. Inflation rates ranged from
 15 0.1 to 20 mL/min, with a majority of the experiments being conducted at 2 mL/min. Incidentally, if the
 16 tubes expanded circumferentially uniformly, the rate of 2 mL/s corresponds to an initial expansion rate of
 17 70% per minute on the inner surface of the tube, which is comparable to the tensile testing rate in Figure 2.

18 All experiments were conducted with a load of 2.44 N suspended from the bottom of the tube, which served

to keep the tube approximately taut even before inflation was started. While previous experiments and theory show that inflation behavior can change with axial load^{3,5,6,15,26}, in fact the axial load used here is negligible. This may be judged by two criteria. First, the stress corresponding to this load is 0.13 MPa, which is too small to induce significant axial strain as judged from the uniaxial tensile data (Figure 2a). Second, typical pressures during inflation are on the order of 3 MPa, which (even using the cross-sectional area of the uninflated tube diameter) corresponds to an “internal” axial force of 37.7 N, which far exceeds the 2.44 N weight suspended from the ends. Incidentally we note that past literature has considered inflation with fixed axial load as well as fixed axial stretch^{14,15,23,27}. We only consider the former situation in this paper. In fact, we will show that the tubes undergo significant elongation during inflation, and therefore are expected to buckle if length was fixed. The axial stretch needed to avoid such inflation-induced buckling would be sufficiently large to induce inelastic effects even without inflation.

Prior to conducting the test, the tube was sprayed with droplets of paint, or with flakes of black “glitter” to serve as markers for motion-tracking. Subsequently, the displacement of these markers was monitored using digital correlation software (Blender). The radial and axial stretch of selected regions could then be calculated from these displacement fields as shown in Figure 3d.

As a measure of sample-to-sample variability, we note that at an inflation rate of 2 mL/min (which was used for most of the experiments in this paper), a total of 24 experiments were conducted. These showed a peak pressure of 2.93 MPa with a standard deviation of 0.16 MPa, which corresponds to a 5% variation in peak pressure. Tests conducted in succession, i.e. from adjacent sections of tubing, tended to show closer agreement suggesting that at least a part of the variation may be due to small variations or imperfections in the geometry or in the mechanical properties of the tube over long lengths.

3. Results

Figure 4a illustrates an exemplary inflation of a polyurethane tube at a rate of 2 mL/min, with ΔV denoting the increase in volume of the tube. Figure 4b shows a rise in pressure at low inflation volumes, followed by a peak at a volume of roughly $\Delta V=3$ mL and a pressure of roughly 2.8 MPa. Subsequently, the pressure decreases gradually towards a plateau value of about 1.8 MPa. This decrease in pressure is denoted *pressure-unloading* in the rest of this paper. During this inflation, the macroscopic length of the tube increased monotonically. At the final volume of $\Delta V=40$ mL, the increase in length corresponds to an average axial strain of roughly 80% based on the length of the uncuffed tube at the beginning of the experiment.

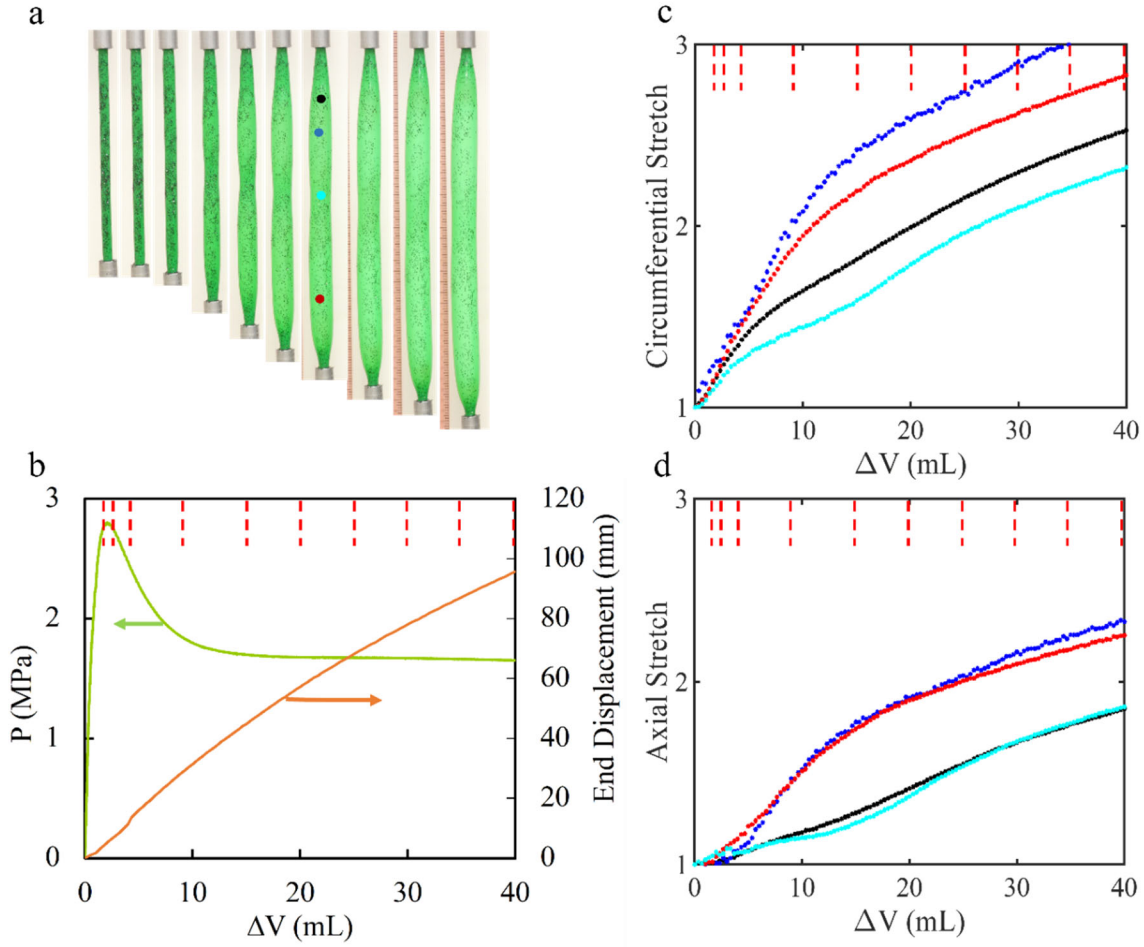


Figure 4: a. Appearance of the tube at various stages during inflation. b. Pressure vs volume during the same experiment. c&d. Quantification of the two stretches at the four locations marked in one of the images in a. The vertical red dashed lines in b-d are drawn at the volumes corresponding to the sequence of images in a.

The evolution of pressure and tube length during inflation is qualitatively similar to that of hyperelastic tubes undergoing stable propagation of an inflated region, and similar to hyperelastic tubes, the tube develops a distinct bulge after the pressure maximum. Yet, there are some key differences. First, the pressure-unloading is gradual, occurring over several mL of volume inflation, i.e. a period of 3-5 minutes. This is in sharp contrast to the behavior of rubber tubes where the pressure-unloading is nearly instantaneous^{6,22,23} (also Figure 8a discussed later). The second is that the inflation is irregular as was already mentioned in the Introduction. In the case of Figure 4a, the mid-section of the tube inflated less than the ends, and this was seen most frequently. In some cases, the inflation approximately resembled a bulge propagation instability where a more-bulged region coexists with a less-inflated region (two examples shown in Figure 5). Furthermore, although repeated trials on identical specimens had significant variations

1 in shape, they all had very similar pressure evolutions suggesting that the pressure is primarily related to
2 material behavior rather than the geometric details of bulging.

3 To quantify the irregular deformation, Figure 4c and d shows the circumferential and the axial stretches at
4 four distinct locations along the tube. Up to ΔV of roughly 5 mL, all marker positions show similar
5 deformation, indicating uniform expansion. Beyond ΔV of 5 mL, deformations at the various locations
6 steeply deviate from each other. Later during the inflation, the least-deformed region does “catch up” with
7 the more deformed regions, but at the final volume, there is still a difference of almost 0.8 units in the
8 circumferential stretch of the most vs least inflated region.

9 Next we examine the role of inflation rate. Tube specimens were inflated at rates ranging from 0.1 to 20
10 mL/min. Figure 5a shows that the peak pressure and the plateau pressure both increase modestly with
11 increasing flow rate. At the lowest rates examined, the pressure does not show a true plateau, but instead
12 continues to reduce gradually with continued inflation. The inflation rate strongly affects the tube shapes,
13 with low rates promoting a greater degree of localized bulging, and high rates appearing relatively more
14 homogeneous. It is noteworthy that the volume required for pressure-unloading remains roughly 5 mL
15 regardless of rate, and by implication, the time required for pressure-unloading increases sharply as the
16 flow rate reduces. For the lowest flow rate examined, the duration from the pressure peak to the end of the
17 experiment (when the pressure had still not plateaued) exceeded 1 hour. This long timescale for unloading
18 can be seen clearly in Supplementary Figure S 4a in which the same data are shown with time on the x-
19 axis. A remarkable observation from Figure 5 is that despite the diverse inflation behaviors, all cases share
20 similar pressure-volume curves. Even the cases of 6 mL/min and 20 mL/min, which happen to show nearly
21 homogeneous inflation, still show a clear maximum in the PV curve followed by pressure-unloading. In
22 contrast, homogeneous inflation of hyperelastic tubes gives monotonically increasing PV data. This will be
23 discussed further in Section 4.

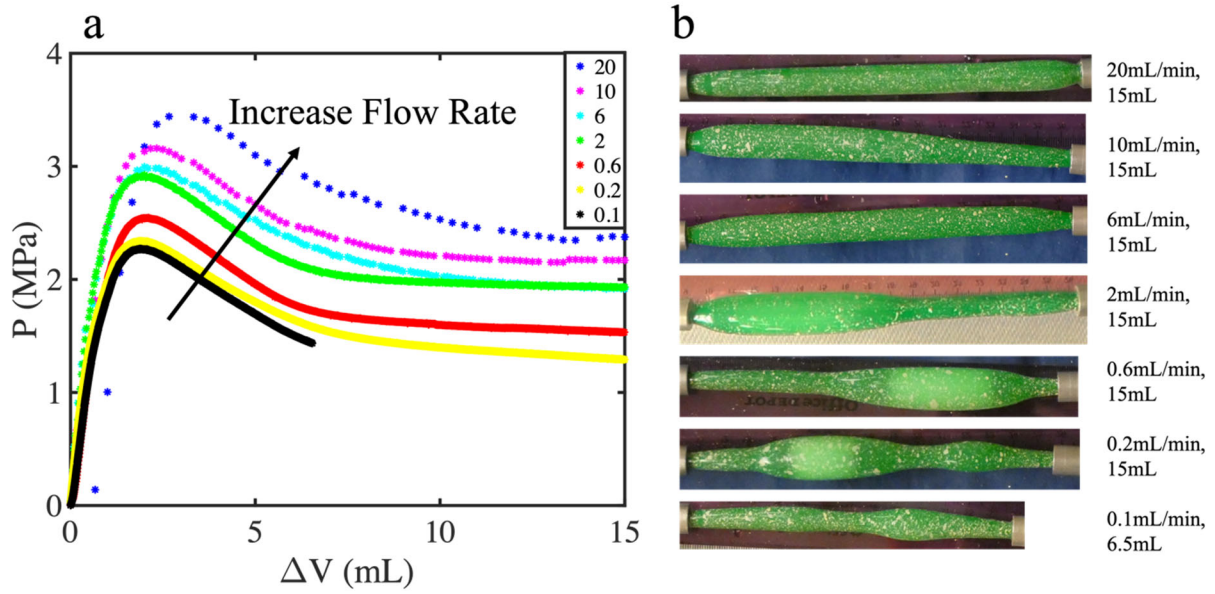


Figure 5: a. Pressure vs volume curve for the various rates listed in the legend (all in mL/min). b. Images of tubes during deformation. All the images correspond to $\Delta V = 15$ mL except the lowest for which $\Delta V = 6.5$ mL. Although the tubes were vertical during experiments, images have been rotated for convenience, with the sequence of images following the sequence of pressure-volume data.

We also examined the extent to which inflation induces irreversible softening (henceforth called *strain-induced damage*) of the material such that a second inflation is different from the first. Loading-unloading-reloading experiments were conducted where tubes were inflated by various volumes ΔV_I (ranging from 1.5 mL to 15 mL), then deflated to atmospheric pressure, and then reinflated. Figure 6a shows the images of the samples at three stages: at the volume ΔV_I during the first inflation, at zero pressure after deflation, and at volume ΔV_I during the second inflation. Figure 6b shows the pressures during the second inflation. Increasing ΔV_I first reduces, and then eliminates, the pressure peak during inflation.

Turning to the tube shapes, at first glance, the first and the second inflation appear similar, i.e. images in the third row of Figure 6a are almost identical to those in the first row. However closer examination reveals that for volumes exceeding 6 mL, the regions that were more (or less) inflated after the first inflation become even more inflated (or even less inflated) during the second. This is quantified Figure 6c which shows the difference in the profile between the first and second inflation for the 6 mL case.

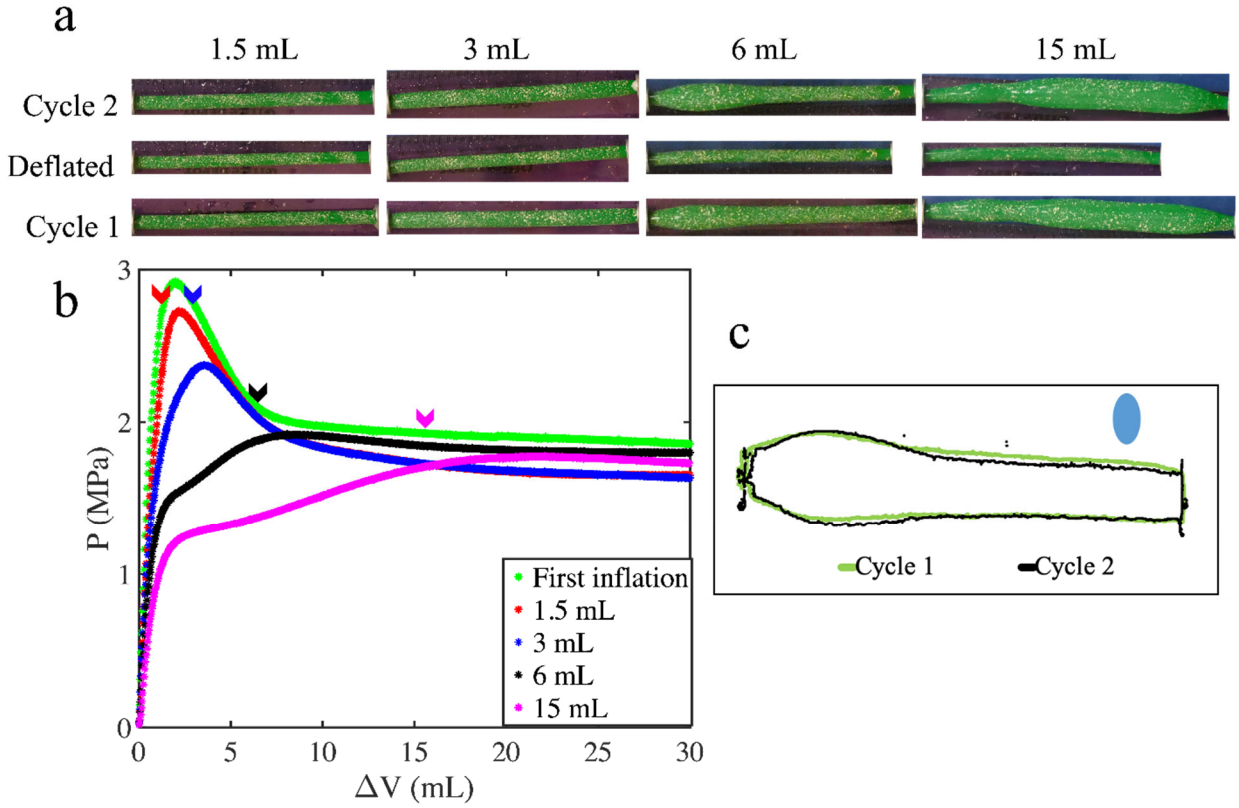


Figure 6: a. Images of samples during inflation-deflation-reinflation experiments. Although the tubes were vertical during experiments, images have been rotated for convenience. b. Pressure vs volume curves where green is a sample during its first inflation, and the other colors are samples being reinflated after deflating from various values of ΔV_1 listed in the legend. Arrows indicate the volumes at which each of the other four samples were deflated. c. A superposition of the profiles of the tube during the first and second cycle, both at $\Delta V = 6$ mL. Note that the axes of c are distorted so that the tube dimensions in the radial direction are magnified 2-fold, and the blue ellipse represents the initial, undeformed, circular cross section of the tube.

This issue of loading-unloading-reloading behavior was tested more thoroughly elsewhere¹⁸. Briefly, tubes were repeatedly cycled between two states: inflation by $\Delta V = 6$ mL, and zero pressure. Over 10 cycles, the bulge grew steadily suggesting a slow but incremental damage that caused an increasing degree of bulge localization.

Finally, we also observed that tubes deflated from a large volume did not return back to their original diameter, but were distinctly distorted. Accordingly, we examined (see Supplementary Figure S 3) inelastic deformation in more detail. Briefly, ΔV values exceeding 10 mL induced significant permanent deformation. Notably, a ΔV of 6 mL that completely eliminated the pressure peak (Figure 6b) did not induce significant permanent deformation, i.e. strain-induced damage precedes inelastic deformation.

4. Discussion

4.1. Interpretation based on theory of hyperelastic tubes

Figure 4-Figure 6, and Figure S 3 show that the inflation behavior of these polyurethane tubes have several features that deviate from those of hyperelastic tubes. However, before discussing these deviations, it is useful to consider whether some of the observations can be predicted by existing theory of hyperelastic tubes. Accordingly, this section proceeds on the assumption that the tubes are hyperelastic, with properties corresponding to the loading curve measured experimentally (Figure 2). The corresponding predictions for the pressure and for the deformation can then be compared against experimental measurements.

We adopt the incompressible Ogden model²⁸ for the constitutive behavior,

$$\widehat{W} = \sum_{n=1}^M \mu_n (\lambda_1^{\alpha_n} + \lambda_2^{\alpha_n} + \lambda_3^{\alpha_n} - 3) / \alpha_n \quad \text{Eq. 1}$$

where $\lambda_3 = (\lambda_1 \lambda_2)^{-1}$ is prescribed, and the value of M (i.e. the number of terms included in the sum) is sufficient to capture the measured mechanical behavior. The hat in Eq. 1 denotes that incompressibility is already incorporated into the constitutive equation. The corresponding true stress under uniaxial tension can then be calculated to be

$$\sigma_{11} = \sum_{n=1}^M \mu_n (\lambda_1^{\alpha_n} - \lambda_1^{-\frac{\alpha_n}{2}}) \quad \text{Eq. 2}$$

The parameters μ_n and α_n can now be obtained by fitting experimental data. Figure 7a shows the same data as the loading portion of the tensile data in Figure 2, but converted to true stress by multiplying the engineering stress by the stretch $(I + \epsilon)$. As a first attempt, we fitted these data to Eq. 1 with no constraints on μ_n and α_n . Adequate fits were obtained with $M=3$ (dashed black line in Figure 7a), and the second column of **Table 1** lists the corresponding fitting parameters. These parameters, and the corresponding predictions, are referred to as *unconstrained*.

The thin-wall approximation model developed by Kyriakides and Chang⁶ (see Supplementary Section S 2 for a derivation) was then applied wherein the pressure-volume behavior of homogenous inflation regime is obtained from three equations

$$P = \frac{1}{\lambda_\theta \lambda_z} \left(\frac{H}{R} \right) \left(\frac{\partial \widehat{W}}{\partial \lambda_\theta} \right) \quad \text{Eq. 3}$$

$$F = 2\pi RH \left[\frac{\partial \widehat{W}}{\partial \lambda_z} - \frac{\lambda_\theta}{2\lambda_z} \frac{\partial \widehat{W}}{\partial \lambda_\theta} \right] \quad \text{Eq. 4}$$

$$\frac{V}{V_0} = \frac{V}{\pi R^2 L} = \lambda_z \lambda_\theta^2 \quad \text{Eq. 5}$$

where R is taken as the mean value $R_m = (R_i + R_o)/2$, $H = R_o - R_i$, while H and L are the initial thickness and length of the tube, respectively. λ_θ and λ_z are circumferential and axial stretch, respectively. Since our experiments are conducted at negligible axial force, F can be set to zero in Eq. 4. The model predictions were then obtained by the following computational sequence. For each value of λ_θ , Eq. 4 was first solved (with $F=0$) to obtain λ_z . Then the pressure and volume were obtained from Eq. 3 and Eq. 5, respectively. In summary, these calculations predict how P , λ_θ and λ_z depend on $\Delta V/V_0$ if inflation is homogeneous.

Table 1: Parameters in the Ogden model from fitting experimental data.

	Unconstrained	Constraint of $\frac{d\lambda_z}{d\lambda_\theta} \geq 0$
μ_1	-10.378 MPa	77.6 MPa
α_1	2.8	0.019
μ_2	1.41 MPa	0.0165 MPa
α_2	3.65	6.24
μ_3	-12.39 MPa	-343.1 MPa
α_3	-4.98	-0.044

The prediction for pressure and λ_z can be compared against experiments such as Figure 4 directly. The predictions for the circumferential deformation cannot be compared against experiments directly because experiments measure the stretch on the outer surface, whereas λ_θ in the model corresponds to the mean radial location R_m . (Note that λ_z is independent of radius, and hence there is no need to distinguish between the axial stretch at the outer vs the mean radial position.) The circumferential stretch on the outer surface is related to the stretch at the mean radius by

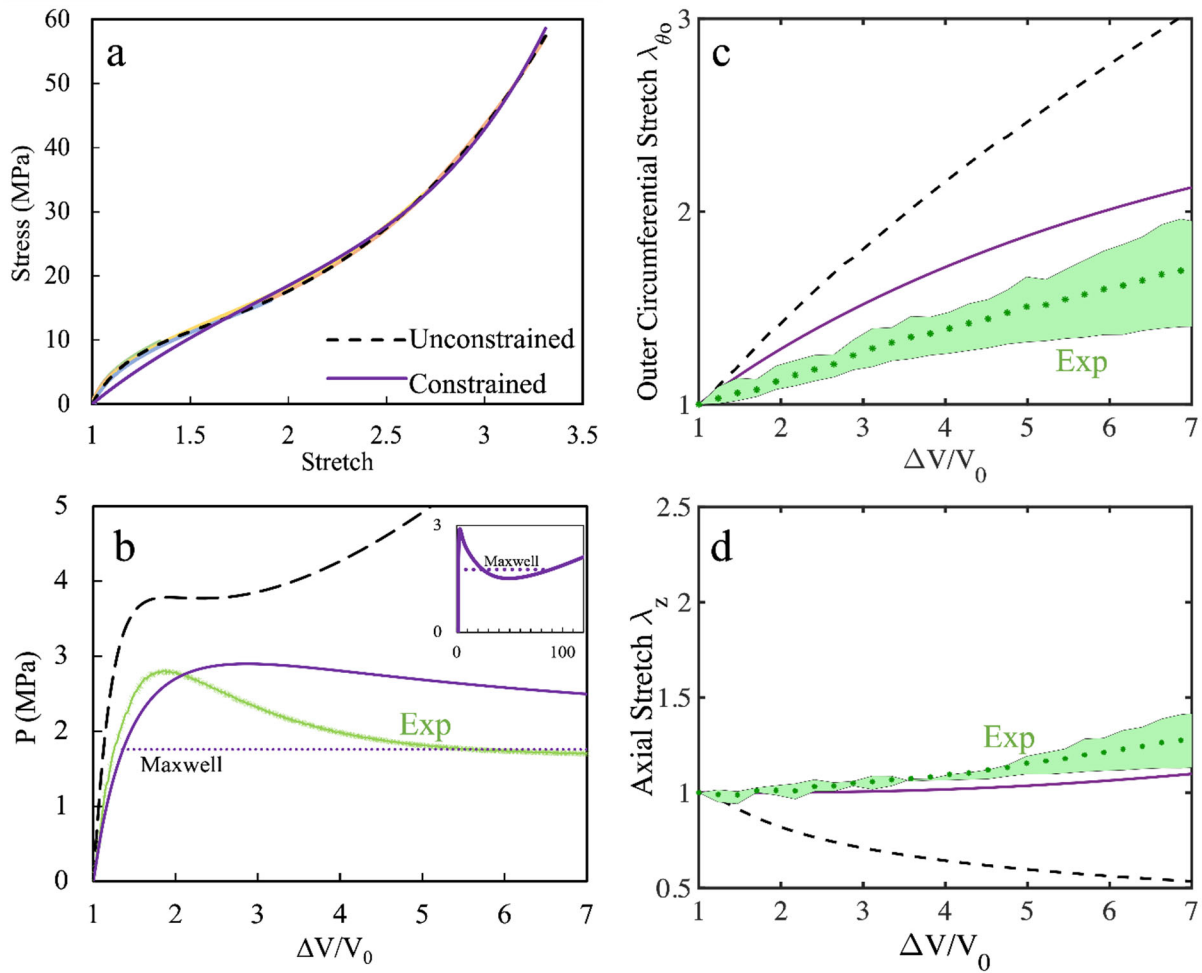
$$(\lambda_{\theta o}^2 \lambda_z - 1)(\pi R_o^2 L) = (\lambda_\theta^2 \lambda_z - 1)(\pi R_m^2 L) \quad \text{Eq. 6}$$

where the subscript o indicates outer. Physically, this equation states that the change in volume, ΔV , due to inflation must be the same regardless of which radius is used for normalization. Thus

$$\lambda_{\theta o}^2 = \frac{1}{\lambda_z} \left[(\lambda_\theta^2 \lambda_z - 1) \left(\frac{R_m^2}{R_o^2} \right) + 1 \right] \quad \text{Eq. 7}$$

This model prediction for the circumferential stretch on the outer surface can be compared against experiment.

Figures 7b-d now compare the predictions for pressure and the two stretches against experiments. The model significantly overpredicts the initial rise in pressure. Beyond this quantitative error however, a more significant qualitative discrepancy is that the predicted PV curve has a very weak maximum, indeed the curve becomes almost flat before rising again, whereas the measured pressure shows a clear maximum followed by pressure-unloading. The predictions for $\lambda_{\theta 0}$ are also much higher than experiments. Perhaps the greatest discrepancy is the model predicts $\lambda_z < 1$, whereas experimentally the tube lengthens upon inflation.



10

Figure 7: a. Fits of the loading portion of the tensile testing data to the Ogden model. All four datasets corresponding to Figure 2a are shown in light colors, whereas the two fitted models are shown as the dashed black and solid purple lines. b-d. Comparison of experimental data and theoretical predictions. In c and d, the green data are the average of the four markers in Figure 4c and d, whereas the green shaded region encompasses the range spanned by the markers. The inset in b shows the same pressure-volume graph, but up to higher volumes to show the dashed line Maxwell construction.

We therefore sought to fit the tensile data with a different set of μ_n, α_n parameters that guarantee that the tube lengthens at the beginning of the inflation. The Supplementary Section S 3 shows that the requirement that the initial inflation be accompanied by tube lengthening (not shortening) leads to the inequality

$$\sum_{n=1}^M \mu_n \alpha_n^2 \geq 0 \quad \text{Eq. 8}$$

Applying this inequality constraint when fitting the tensile data gives the parameters listed in the third column of **Table 1**, and the corresponding fit is shown as the solid purple curve in Figure 7. This model is referred to as *constrained*. It is noteworthy that the fit to the constrained model cannot capture the small-strain behavior of the uniaxial tensile tests accurately, i.e. no combination of μ_n, α_n that satisfies Eq. 8 can reproduce the initial curvature of the stress-strain data. Further, a careful inspection of Figure 7a shows that the purple curve overpredicts the stress between a stretch of 2 and 2.5. The corresponding prediction for pressure is shown as the solid purple curve in Figure 7b. While there is still a discrepancy at the earliest stages of inflation, the value of the maximum pressure is in reasonable agreement with experiments. Figure 7c shows a better prediction of the experimental $\lambda_{\theta 0}$, although the predicted λ_z in Figure 7d still remains lower than experimental values. It must be emphasized that strictly, comparisons of the model are only reasonable up to the pressure maximum. Beyond that, the model predicts non-homogeneous inflation and should not be compared against experiments.

The purple PV curve in Figure 7b is shown only to small values of volume. The inset to Figure 7b shows that at larger volumes, the pressure predicted by the constrained model rises to values above the peak pressure indicating that the bulge that initiates at the pressure maximum is limited by strain hardening. Thus, the model predicts that eventually the tube must show coexistence between a bulged and an unbulged state. The coexistence pressure calculated using the Maxwell construction (inset to Figure 7b) was found to be 1.76 MPa. This latter value is in remarkably good agreement with the plateau value of the pressure, even though the experiments do not actually show coexistence between a bulged and unbulged state! The corresponding stretches are found to be $\lambda_{\theta}=3.45$ (corresponding to $\lambda_{\theta 0}=2.8$ and $\lambda_z=1.97$), values that are larger than the stretches in Figure 4c and d. Thus, as per this interpretation, the stretches in Figure 4 have

not yet reached the limiting values needed for stable coexistence and bulge propagation. Yet the qualitative behavior of Figure 4 does not suggest stable bulge propagation will be approached at all; instead the tube already approaches homogeneity by the end of the inflation.

To summarize, the existing theory of thin-walled hyperelastic tubes, simply using the tensile loading curve to inform the constitutive behavior, is in qualitative and quantitative disagreement with experiments. If an additional constraint that the tubes must lengthen upon inflation is imposed, the small-strain tensile data are not well-captured by the constitutive equation. Yet, this ill-fitting model gives reasonable predictions for the value of the maximum pressure and the stable coexistence pressure.

Finally, note that the above judgements about the accuracy of the hyperelastic model depend on the tube radius R selected for the analysis. Eq. 3-Eq. 5 show that for $F=0$, the $\lambda_\theta - \lambda_z$ relationship is independent of the geometry, the magnitude of the predicted pressure scales with H/R , whereas the volume V_0 used for normalizing the experimental data scales with R^2 . Our calculations all used the value of the mean radius R_m . If on the other hand the value of R_i was used for the calculations, the predicted pressure curve would increase by a factor of $R_m/R_i=1.3$ whereas the value of V_0 would reduce by a factor of $(R_i/R_m)^2=0.59$. Accordingly, the constrained model (purple curves) would overestimate the peak pressure, but give good predictions for the value of the $\Delta V/V_0$ at the pressure maximum.

4.2. Inelastic effects and one further experiment

We now turn to the two noteworthy aspects of the inflation behavior that strongly differ from past experiments on rubber tubes^{3,6,22,23}. The first is that multiple specimens cut from the same spool of tubing show distinct behaviors even under identical inflation conditions. For the inflation rate of 2 mL/min, in more than 40 experiments, we have observed several examples of tubes with irregular shapes such as in Figure 1, several examples that approximately resemble bulge propagation, and occasional examples of nearly-homogeneous inflation. Even tubes that resemble bulge propagation show up to 20% variation in the diameter of the more-bulged and less-inflated regions. In contrast, hyperelastic tubes can only show coexistence of two well-defined strain states. Despite this variability in deformation however, all these specimens show very similar pressure-volume curves. Especially remarkable are two cases in Figure 5 where the inflation proceeds almost homogeneously even though the PV curve shows a prominent maximum. In contrast, for hyperelastic tubes, a pressure-volume curve with a negative slope is necessarily unstable and incompatible with homogeneous inflation. This suggests that pressure-unloading is attributable not just to bulging, but also to material behavior. This notion, that material behavior is a strong contributor to pressure-unloading is also supported by the observation (Figure 5) that despite a wide difference in the degree of bulging and a 100-fold difference in experimental timescales, the pressure-

unloading remains similar, and always requires roughly 5 mL of inflation volume.

A second noteworthy aspect is the simultaneous inflation of two or more bulges along a single tube. Intuition suggests that once a bulge starts growing, three factors encourage further localization. First, a bulge has a larger diameter and a thinner wall, both of which increase the local wall stress as per Laplace equation. This effect is present even in hyperelastic tubes. Second, if the material has a yield point (or undergoes strain-induced damage), the bulged region becomes more compliant. Third, as a consequence of the prior two factors, the pressure within the tube reduces, thus making it impossible to initiate new bulges. Accordingly, we expect that when a single bulge starts growing, the formation or growth of other bulges must be suppressed. Despite this however, we have noted numerous examples where more than one bulge grows simultaneously (although one always grows more than the others).

We propose that both these unusual aspects, sample-to-sample variability in bulging behavior and multiple bulges, are related to the remarkably slow pressure-unloading behavior of these tubes as compared to hyperelastic tubes. To illustrate this, Figure 8a compares the pressure-volume response of the polyurethane tubing with natural rubber tubing of similar diameter and wall thickness. The natural rubber tubing (same sample as shown in Figure 1a) was found to bulge “instantaneously”, i.e. the pressure reduces more steeply than can be resolved by the 1 Hz data acquisition rate. Such rapid bulging was also reported in previous studies^{6,22,23} and also appears in our own rubber balloon experiments from ESI Figure S 1. In contrast, the decrease in pressure of the polyurethane tubing occurs over several tens of seconds in Figure 8a, but might take over an hour at low flow rates.

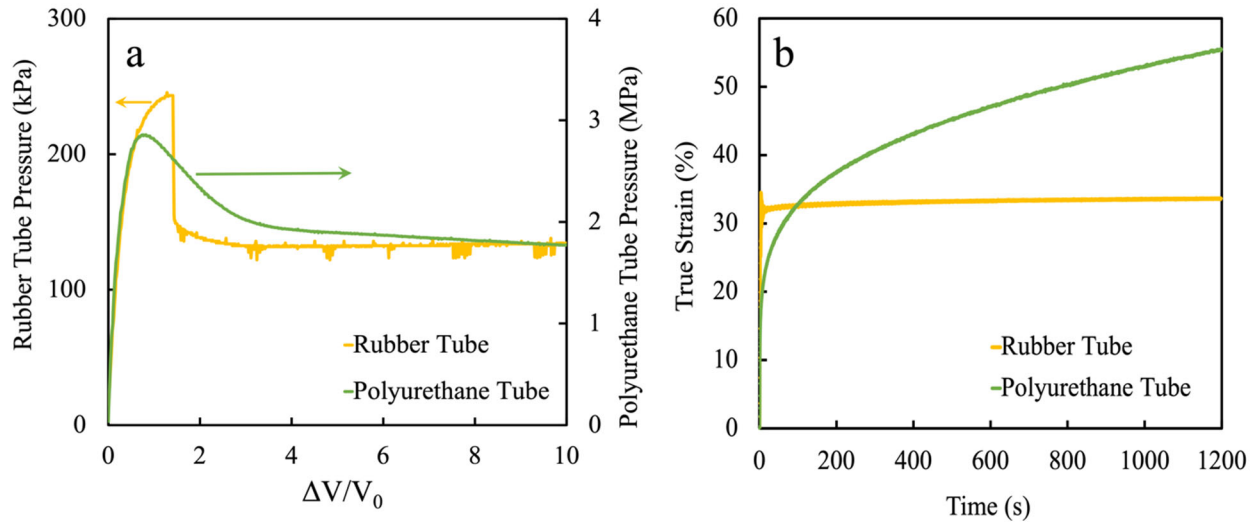


Figure 8: Delayed strain response of polyurethane tubes. a. Comparison of pressure-volume curves of polyurethane tubes vs natural rubber tubes. Note that the right and left y-axes have different scales because the natural rubber tube is much softer. The axes were scaled to approximately superpose the data at small inflation. The green data for the polyurethane tube in a is identical to Figure 4b. The rubber tube experiment corresponds to Figure 1a. b. The axial strain measured when a tube is subjected to a rapid increase in axial load from 40 N to 140 N.

Such slow unloading likely results from an intrinsically slow strain response of the polyurethane material to applied stress. To quantify this, we conducted an experiment where a polyurethane tube was first loaded with a weight of 40 N for several minutes, and then abruptly (within less than 2 s) the weight was raised to 140 N. This is conceptually similar to a creep experiment except that the force (rather than stress) is held constant. The strain evolution was obtained from quantitative analysis of images of ink marks on the tube taken during the experiment. Figure 8b shows that strain increases gradually over tens of minutes before approaching a stable value.

The consequence of the slow response to load is that in the inflation experiment, the strain in the tube wall lags behind the instantaneous pressure. Once the pressure reaches the maximum value expected from homogeneous expansion, the entire tube becomes susceptible to bulging, and at some location (presumably a small defect), a bulge initiates. However, the crucial point is that because this bulge grows slowly, the tube stays at high pressure for a long duration. Thus, locations sufficiently remote from the first bulge can also initiate bulging independently. Only when one or more bulges grows sufficiently does the pressure reduce gradually, and new bulges are no longer viable; beyond this point a single bulge grows more than the others. Nevertheless, some of the other bulges can still continue growing gradually. We hypothesize that this is because strain-induced damage has already rendered these regions softer, and local creep allows a steady increase in strain with time.

4.3. Simulations of viscoelastic tubes

To flesh out the physical picture from the previous paragraph, finite element simulations were conducted. The goal was not to reproduce all the physical phenomena quantitatively, but to test whether viscoelasticity can reproduce the qualitative differences between the inflation of the polyurethane tubes vs the theoretical expectations of hyperelastic tubes. Accordingly, we adopted the two simplest constitutive models that can highlight viscoelastic effects without confounding effects. In the first, denoted “hyperelastic”, the material was taken to be neo-Hookean with a small-strain modulus of μ_∞ . The second model, denoted “viscoelastic”, used the same strain-dependence as the neo-Hookean model, but the shear modulus was now taken to be time-dependent:

$$\mu(t) = \mu_{\infty} \left[\left(\frac{g}{1-g} \right) \exp \left(-\frac{t}{\tau} \right) + 1 \right] \quad \text{Eq. 9}$$

where $g < 1$. Eq. 9 corresponds to a single-term Prony model and Figure 9a illustrates a spring-dashpot analog to Eq. 9. In a step strain experiment, this model yields an initial modulus of $\mu_{\infty}/(1-g)$, a final modulus of μ_{∞} , and a single relaxation time τ . In a step stress (i.e. creep) experiment, this model yields an initial compliance of $(1-g)/\mu_{\infty}$, a final compliance of $1/\mu_{\infty}$, and a retardation time of $\tau/(1-g)$.

The tube was represented by a shell of radius R with hemispherical ends (Figure 9b). The straight section (i.e. excluding the hemispherical ends) had length $L \gg R$. The inner volume of the tube was increased at a fixed rate dV/dt . This inflation rate can be represented in non-dimensional terms as follows. Ignoring the hemispherical ends, the volume of the uninflated tube is $V_0 = \pi R^2 L$. If the tube expands homogeneously, $V = \lambda_{\theta}^2 \lambda_z V_0$. Since the initial expansion is expected to occur without axial stretching, at the early stages of expansion, λ_z remains 1, and hence

$$\frac{dV}{dt} \approx 2\lambda_{\theta} \frac{d\lambda_{\theta}}{dt} V_0 \quad \text{Eq. 10}$$

In the limiting case when inflation just starts, λ_{θ} is also 1, and hence the circumferential strain rate is

$$\frac{d\lambda_{\theta}}{dt} = \frac{1}{2V_0} \frac{dV}{dt} \quad \text{Eq. 11}$$

This initial circumferential expansion rate can now be rendered non-dimensional as

$$Wi = \tau \frac{d\lambda_{\theta}}{dt} = \tau \frac{1}{2V_0} \frac{dV}{dt} \quad \text{Eq. 12}$$

Here Wi stands for Weissenberg number which is based on the initial deformation rate of the tube. Wi is a measure of how rapidly the tube is inflated as compared to the relaxation time of the material.

Having defined the constitutive models, we can now pin down the parameters that affect inflation behavior. Assuming material incompressibility, and $L \gg R$, dimensional analysis requires that for the hyperelastic tube, the non-dimensional pressure P/μ_{∞} must depend only on $\Delta V/V_0$ with no further parameters. In contrast, for viscoelastic tubes, P/μ_{∞} must depend on $\Delta V/V_0$, g , and Wi . Thus the specific choice of μ_{∞} , τ , L and R are not expected to affect the results when represented in non-dimensional terms.

Simulations were conducted using the Abaqus software with a 3D model of the geometry of Figure 9b, using 4-node shell elements. The geometric parameters were set to $R=2.58$ mm (same as the mean value R_m for the experimental tube) and $L=150$ mm. μ_{∞} was set to be 18 MPa for both the hyperelastic and the viscoelastic model; this value is close to the measured modulus of the experimental tube. τ was set to 1 s. However, we reiterate that, as per the previous paragraph, these parameter values are not expected to affect

the non-dimensional pressure P/μ_∞ presented below. The Poisson's ratio was set to 0.5 to approximate incompressible behavior. The material parameter g was set to 0.9, and more comments on this are made below. Volume-controlled inflation was simulated by thermal expansion of a virtual fluid cavity inside the closed-end tube. The virtual fluid was assigned a thermal expansion coefficient, and a temperature rise was prescribed with a constant ramp rate. Accordingly, the cavity volume increased linearly with time, and the resulting pressure within the tube was reported.

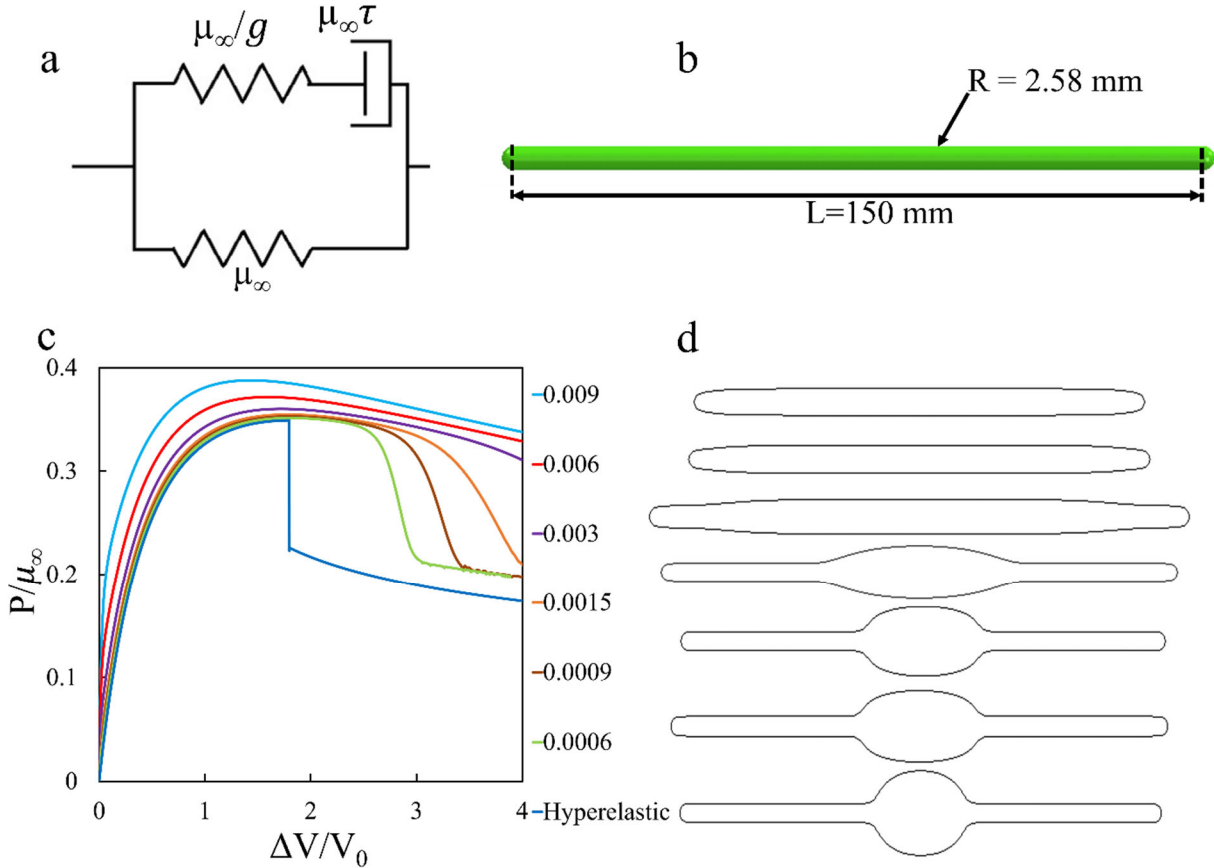


Figure 9: a. Spring dashpot model corresponding to the viscoelastic constitutive behavior of Eq. 9. b. Geometry of the tube used for simulations. c. Pressure evolution during inflation of tubes at the Wi values increasing from bottom to top as per the legend on the right of the graph. The lowest curve corresponds to the hyperelastic model. d. Configurations of each tube at a volume $\Delta V/V_0$ of 3.9 with Wi values increasing from bottom to top.

Figure 9c and d show the evolution of the non-dimensional pressure P/μ_∞ and the tube configurations during inflation for the hyperelastic (i.e. neo-Hookean) model, and for the viscoelastic model at various Wi values. As expected, the neo-Hookean tube inflates homogeneously up to a non-dimensional volume of $\Delta V/V_0$ of 1.8, upon which a bulged stage appears abruptly, i.e. the pressure-volume curve unloads almost

discontinuously. The viscoelastic model shows bulge formation only at low Wi , but the bulge initiates at a larger volume than the hyperelastic tube, and the bulge grows gradually (rather than abruptly) as reflected in the gradual decrease in pressure with increasing volume. Increasing Wi raises the pressure early during inflation, and also raises the volume at which the bulge grows; this was also observed in experiments and simulations of spherical membranes^{29,30}. These changes are qualitatively consistent with Figure 5. Most notably, since the bulge initiates at larger volume with higher Wi , at any selected volume, the tube inflation appears to become more homogeneous as rate increases, in agreement with experiments. Finally, at higher Wi values, a pressure maximum appears without bulge initiation, i.e. purely due to constitutive behavior of the material. This is in agreement with previous research on inflation of shells or tubes which also confirm that viscoelastic effects can give a pressure maximum even while the inflation remains homogeneous^{29,30}.

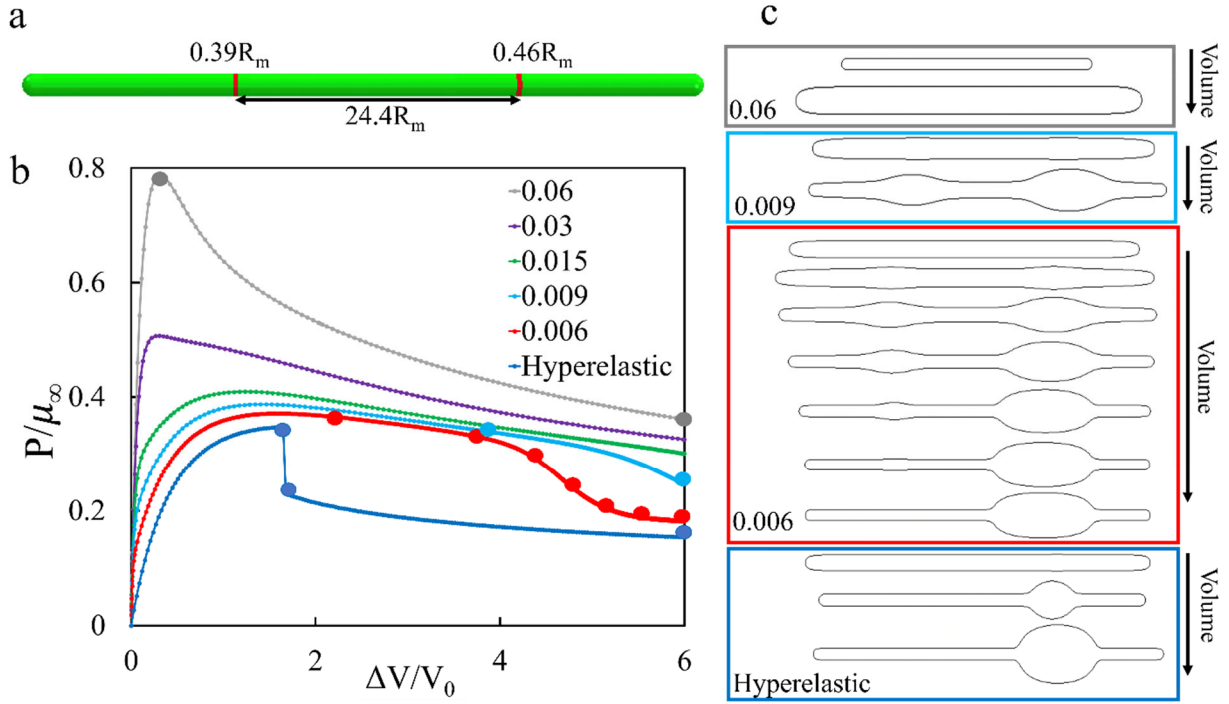


Figure 10: a. Tube showing defects (in red) where local modulus is lower than the rest of the tube. Overall tube dimensions are identical to that in Figure 9. b. Pressure evolution during inflation of tubes with defects. c. Configurations of each tube at selected volumes indicated by the filled points in b.

The previous section hypothesized that multiple bulges appear because the pressure remains high for long periods, and hence can initiate other bulges. To test this hypothesis, separate simulations were conducted where the tube was endowed with two axisymmetric defects lengths of $0.46R_m$ and $0.39R_m$ separated by a tube length of $24.4R_m$ (Figure 10a). Both defects had a local modulus that was 5.5% smaller than μ_∞ and

hence can serve as bulge initiation sites. In the hyperelastic case, the longer of the two defects initiated a bulge at a $\Delta V/V_0$ value of 1.7, which is close to that of the defect-free tube in Figure 9. This single bulge abruptly reduced the pressure (Figure 10b), thus suppressing growth of a second bulge, and only one bulge grew (Figure 10c, lowest frame). The simulation was repeated with the viscoelastic model, inflated at a Wi value of 0.006. In this case, the pressure remained high for an extended period after bulge initiation and – even though the longer defect initiated a larger bulge – both bulges started growing. Only later during the inflation did the larger bulge grow faster, causing the smaller bulge to disappear (Figure 10c, second frame from bottom). Raising the Wi value caused the second bulge to grow less and disappear sooner, until at $Wi = 0.06$, the tube expanded almost homogeneously ignoring the defects altogether disappear (Figure 10c, topmost frame). A video comparing all three simulations is available as Supplementary Information. In summary, simulations supports the physical picture that multiple bulges can grow because viscoelasticity causes the pressure to remain high for relatively long durations.

Limited simulations with shorter tubes show that, consistent with previous research⁷, the pressure unloading becomes more gradual as the tube length reduces, but qualitatively, the effects of increasing Wi remain the same. Simulations were also conducted with the same tube geometry at three other g values: 0.75, 0.8 and 0.95. As g approaches 1, the initial modulus increased as $\mu_\infty/(1 - g)$ and hence at any given Wi value, the pressure rose to a larger extent during the early, homogeneous phase of the expansion. The bulging behavior (ESI Figure S 5) remained qualitatively similar, except that when compared at fixed Wi , low g values caused the bulge to appear at a lower $\Delta V/V_0$ value.

Although viscoelastic simulations are in qualitative agreement with the results, viscoelasticity alone cannot fully describe the experiments. In the simulations, the bulge develops over the same timescale independent of the inflation rate. This is seen more clearly when the data of Figure 9c are plotted in the form of pressure vs time (Supplementary Figure S 4b). As a consequence, when plotted as a pressure-volume curve, the pressure-unloading appears increasingly sharp as rates reduces, e.g. compare the Wi values of 0.0006 and 0.0015 in Figure 9c. In stark contrast, Figure 5 shows that the pressure unloading occurs over a similar range of volumes, which corresponds to timescales as short as few ten seconds and as long as tens of minutes (Supplementary Figure S 4a). A possible reason for the discrepancy may be that strain-induced damage plays a significant role so that significant pressure unloading happens only once the material has reached a certain minimum strain, an effect that is not captured by the linear viscoelastic model. Finally we note that inelastic deformation is not incorporated into the model, and hence the permanent diameter change noted experimentally (Figure S 3) is not captured.

5. Summary and conclusions

To summarize, this paper explores the inflation behavior of tubes whose material behavior deviates from strict hyperelasticity. Hyperelastic tubes inflated under volume-controlled conditions are known to inflate along three possible pathways: tubes that are sufficiently strain-hardening inflate uniformly, maintaining their cylindrical shape. Tubes that have insufficient strain-hardening develop a bulge, which may either inflate to failure with a monotonic decrease in pressure, or propagate axially at constant pressure. This paper explores how the inflation behavior changes when the material comprising the tube wall has inelastic behaviors such as rate-dependent mechanical properties, strain-induced damage, and plastic deformation.

Experiments with polyurethane elastomer tubes inflated at fixed flow rate show that the pressure within the tubes first rises to a maximum, and then reduces towards a plateau. While this behavior is qualitatively similar to that of hyperelastic tubes undergoing axial bulge propagation, in fact there are major differences. First, the decrease in pressure from its peak value towards a plateau value can take from tens of seconds to tens of minutes, depending on the inflation rate. In contrast, rubber tubes inflate almost instantaneously they bulge. Second, the polyurethane tubes deform into a variety of shapes: uniform inflation maintaining cylindrical shape, irregular axisymmetric shapes with multiple bulges, or axial propagation of a bulge. In all these cases – even when tubes inflate uniformly – the pressure shows a maximum value followed by unloading. In contrast, for hyperelastic tubes at fixed axial force, a peak in pressure is a necessary and sufficient condition for bulge formation. Third, if tubes are reinflated a second time, the pressure-volume curve does not show a maximum, indicating that the first inflation induced damage, i.e. a permanent change in mechanical properties due to strain. Finally, after sufficient inflation, tubes are left permanently deformed indicating plastic deformation. One observation that underpins some of these behaviors is that the polyurethane material responds to an applied load relatively slowly, on the order of many minutes. We propose that it is this slow response that causes the pressure to remain high for a long duration during inflation, and hence allow multiple bulges to grow. In contrast, since a single bulge instantaneously reduces the pressure in hyperelastic tubes, growth of multiple bulges is suppressed. This physical picture is supported by finite element simulations using a linearly viscoelastic constitutive model with a single-relaxation time.

We also tested the extent to which existing theory of hyperelastic models of tube inflation can predict the first-inflation behavior of the tubes. The first-loading uniaxial tensile testing data were fitted to the Ogden model, and the corresponding predictions were tested against the observed behavior. The theory was found to underpredict the pressure and the deformations. More significant than the quantitative discrepancy, the theory does not predict a maximum in pressure, and predicts that the tubes shorten when inflated, contrary to experiments. We show that the Ogden model can make more reasonable predictions, but only at the cost

1 of poorly-fitting the initial portion of the uniaxial tensile testing data.

2 Broadly, we conclude that the tube-inflation characteristics noted in this paper appear from a complex
3 coupling of all four phenomena: bulge formation, strain hardening, time-dependent relaxation of the
4 material, and strain-induced damage. The first of these is a purely geometric effect, whereas the other three
5 are material behaviors. The latter two phenomena are (by definition) absent in hyperelastic models. An
6 especially interesting qualitative result from this paper is that a pressure peak followed by a pressure plateau
7 – a signature of bulge initiation and propagation in hyperelastic tubes – may appear without bulge
8 formation, and due to mechanical properties alone. While this article is fundamental in nature, it may guide
9 the design of actuators, e.g. used in soft robotics. In those applications, fully-reversible behavior of
10 inflatable is critical to consistently-repeatable actuation and any inelastic effects may build up over time to
11 reduce reproducibility. In addition, a large deformation even for a brief period (e.g. exposure to high
12 pressure, local kinking) may permanently change their behavior.

13 The polyurethane tubes studied here showed relatively modest deviations from hyperelastic behavior. They
14 retained one key feature of hyperelastic tube inflation, viz. the inflated tubes remain axisymmetric. In
15 contrast, if the tubes have a relatively large yield stress, non-axisymmetric deformations appear. These will
16 be explored in a separate publication.

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31

Electronic supplementary information

Inelastic effects in bulge formation of inflated polymer tubes

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Section S 1: Additional supplementary information

a No previous inflation



b Inflation cycle 1 (propagation)



c Deflation after cycle 1



d Inflation cycle 2



e Deflation after cycle 2



Figure S 1: A long rubber balloon during two cycles of inflation-deflation with air. After the first deflation (image c), the portion of the balloon that had bulged is permanently deformed. The second inflation (image d) shows three coexisting diameters, the smallest of which corresponds to the portion of the balloon that has never experienced large inflation, and the largest corresponds to the portion of the balloon that has bulged in both cycles. Black marks are ink marks to visualize the deformation.

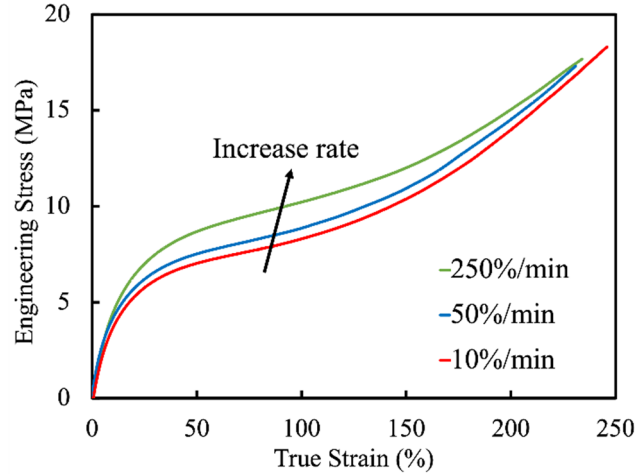


Figure S 2: Uniaxial tensile testing data for polyurethane tubes during loading and unloading at nominal strain rates of 10% per minute to 250% per minute.

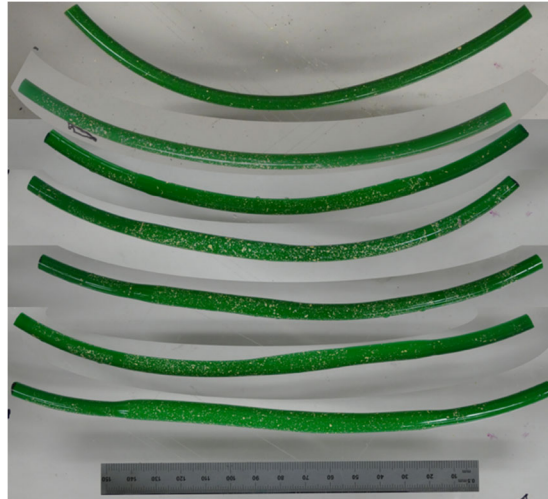


Figure S 3: Inelastic deformation of tubes after being inflated at 2 mL/min to the volumes listed alongside each figure. The ruler at the bottom is 150 mm long. The undeformed specimen (uppermost image) appears bent due to the intrinsic curvature of the tube (see Section 2 in the main text). Significant permanent deformation appears for $\Delta V \geq 10$ mL, both as a gradual straightening of the tube, as well as a permanent change in diameter. Deflation after $\Delta V = 6$ mL (far above $\Delta V = 2.5$ mL at the pressure maximum) induces only slight permanent deformation. This is consistent with Figure 2 and Figure 4 in the main text: at $\Delta V = 6$ mL, the axial stretch is less than 1.25 (Figure 4d), and the circumferential stretch is less than 1.6 (Figure 4c), values at which permanent deformation is modest (Figure 2b).

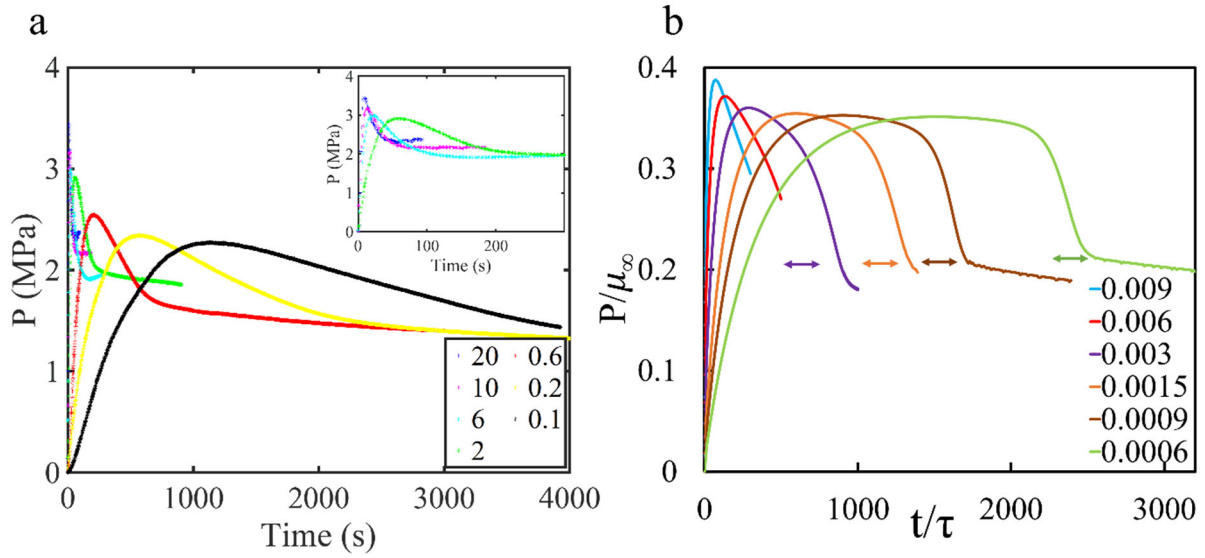


Figure S 4: a. Same data as Figure 4 in the main paper, but with time on the x-axis. Legend lists the various flow rate experiments listed in the legend (all in mL/min). b. Same data as Figure 9 in the main paper, but with nondimensional time in the x-axis. The double arrows (all 50 units wide) indicate that the timescale for pressure-unloading is approximately equal at all Wi values.

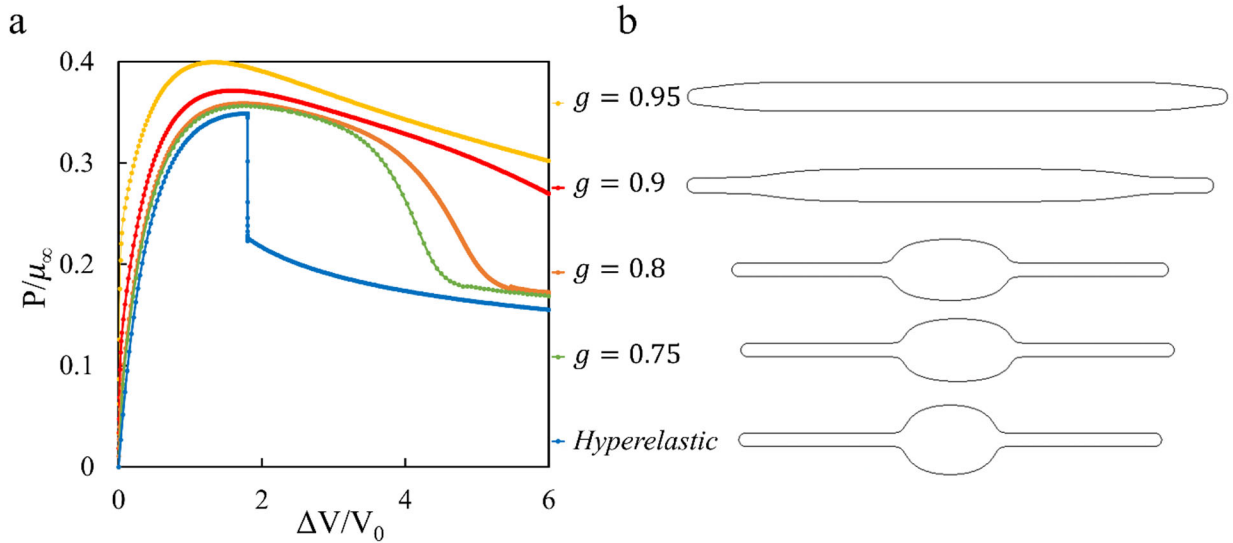


Figure S 5: Effect of the parameter g on tube inflation at one single Wi value of 0.006. b. Corresponding profiles of tubes at the same g values (increasing from bottom to top), at a single $\Delta V/V_0$ value of 6.

Section S 2: Hyperelastic tube inflation model

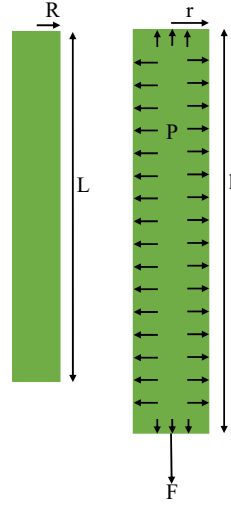


Figure S 6: External force and internal pressure induces axial and circumferential stress as the following equations.

Chang and Kyriakides^{7,12} examined the homogeneous inflation of thick-walled tubes, and then derived Eq. 3 and Eq. 4 in the main text in the limiting case of a thin-walled tube. Here we rederive the same equations, but more directly for a thin-walled tube. Consider the inflation of a thin-walled tube of undeformed radius R , undeformed wall thickness H , and undeformed length L , with an external axial force, F (which might be zero). The internal pressure causes both axial and circumferential stresses, whereas the external force causes axial stress only. Assuming homogeneous inflation, a force balance then leads to:

$$\sigma_z = \frac{F}{2\pi r h} + \frac{Pr}{2h} \quad \text{Eq. S. 1}$$

$$\sigma_\theta = \frac{Pr}{h} \quad \text{Eq. S. 2}$$

Here r and h are deformed radius and deformed wall thickness of the tube, respectively.

The deformed and undeformed tube dimensions can be related by incompressibility of the tube walls, i.e. by setting $\pi RHL = \pi rhl$, thus giving:

$$\lambda_z r h = RH \quad \text{Eq. S. 3}$$

where λ_z is axial stretch of the tube. It is useful to rewrite this in the form

$$\frac{h}{r} = \frac{1}{\lambda_z} \frac{RH}{r^2} = \frac{1}{\lambda_z} \frac{H R^2}{R r^2} = \frac{1}{\lambda_z} \frac{H}{R} \frac{1}{\lambda_\theta^2} \quad \text{Eq. S. 4}$$

Stress components based on derivatives of strain energy function, i. e. $\widehat{W}$, can be written as:

$$\sigma_i = \lambda_i \frac{\partial \widehat{W}}{\partial \lambda_i} \quad \text{Eq. S. 5}$$

Accordingly, Eq. S. 1 and Eq. S. 2 can be combined with Eq. S. 4 and Eq. S. 5 to give:

$$\lambda_\theta \frac{\partial \widehat{W}}{\partial \lambda_\theta} = \frac{Pr}{h} \quad \Rightarrow \quad P = \lambda_\theta \frac{\partial \widehat{W}}{\partial \lambda_\theta} \frac{h}{r} = \frac{\partial \widehat{W}}{\partial \lambda_\theta} \frac{H}{R} \frac{1}{\lambda_\theta \lambda_z} \quad \text{Eq. S. 6}$$

$$\frac{F}{2\pi r h} + \frac{Pr}{2h} = \lambda_z \frac{\partial \widehat{W}}{\partial \lambda_z} \quad \Rightarrow \quad F = 2\pi r h \left[\lambda_z \frac{\partial \widehat{W}}{\partial \lambda_z} - \frac{Pr}{2h} \right] = 2\pi R H \left[\frac{\partial \widehat{W}}{\partial \lambda_z} - \frac{\lambda_\theta}{2\lambda_z} \frac{\partial \widehat{W}}{\partial \lambda_\theta} \right] \quad \text{Eq. S. 7}$$

1 These correspond to Eq. 3 and Eq. 4 in the main text, and also to the equations derived by Chang and
2 Kyriakides^{7,12}. We have used these equations (setting $F=0$) to calculate the homogeneous inflation response
3 in the main paper.

4

5

Section S 3: Condition to guarantee lengthening when a thin-walled Ogden tube inflates homogeneously

As mentioned in the main text, the experiments show that the tubes stretch when they are inflated, whereas the unconstrained fit to the Ogden model predicts $\lambda_z < 1$. Here we will derive the condition that guarantees that the initial portion of the inflation (i.e. when the inflation just starts) induces lengthening of the tube.

The Ogden model (Eq. 1 in the main text) is substituted into the force equation (Eq. 4 in the main text). The resulting expression is then expanded to first order in $\lambda_z - 1$ to obtain an expression of the form :

$$F = f(\lambda_\theta) + g(\lambda_\theta)(\lambda_z - 1) \quad \text{Eq. S. 8}$$

Setting $F=0$, the axial stretch can be written as

$$\lambda_z = 1 - \frac{f(\lambda_\theta)}{g(\lambda_\theta)} \quad \text{Eq. S. 9}$$

For circumferential stretch close to one (i.e. at the early stages of inflation), it can be shown that the first derivative $\frac{\partial \lambda_z}{\partial \lambda_\theta}$ is always zero. I.e. regardless of the values of the Ogden equation parameters, for small inflations with zero axial force, the tube neither lengthens nor shortens. We must therefore examine the second derivative, which can be shown to be

$$\left. \frac{\partial^2 \lambda_z}{\partial \lambda_\theta^2} \right|_{\lambda_\theta=1} = \frac{2 \sum_{n=1}^M \mu_n \alpha_n^2}{3 \sum_{n=1}^M \mu_n \alpha_n} = \frac{\sum_{n=1}^M \mu_n \alpha_n^2}{3\mu} \quad \text{Eq. S. 10}$$

where the denominator μ is the shear modulus.

Lengthening is guaranteed by

$$\frac{\partial^2 \lambda_z}{\partial \lambda_\theta^2} \geq 0 \quad \text{Eq. S. 11}$$

Since the shear modulus is necessarily positive, Eq. S. 11, gives the final criterion

$$\sum_{n=1}^M \mu_n \alpha_n^2 \geq 0 \quad \text{Eq. S. 12}$$

This last equation is identical to Eq. 8 in the main text.