Teaching Geometry for Secondary Teachers: What are the Tensions Instructors Need to Manage?

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Abstract
This paper contributes to understanding the work of teaching the university geometry courses that are taken by prospective secondary teachers. We ask what are the tensions that instructors need to manage as they plan and teach these courses. And we use these tensions to argue that mathematics instruction in geometry courses for secondary teachers includes complexities that go beyond those of other undergraduate mathematics courses--an argument that possibly applies to other mathematics courses for teachers. Building on the notion that the work of teaching involves managing tensions, and relying on interviews of 32 instructors, we characterize 5 tensions (content, experiences, students, instructor, and institutions) that instructors of geometry for teachers manage in their work. We interpret these tensions as emerging from a dialectic between two normative understandings of instruction in these courses, using the instructional triangle to represent these.

Keywords: geometry, tensions, undergraduate, teacher preparation, secondary, instruction

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This paper inspects the work of teaching the geometry courses taken by prospective secondary teachers (hereafter, GeT[1] courses). Our research contributes to better understand how characteristics of the work of teaching mathematics relate to the institutional situatedness of this work. While geometry content is taught to prospective secondary teachers (hereafter, PTs) across the world (Tatto & Senk, 2011), there is considerable international variability in how the mathematical preparation of PTs is organized institutionally (Schmidt et al., 2011). This suggests that focusing on instructors working under similar institutional characteristics might offer the opportunity to ground some conjectures for later international comparison. We thus focus on GeT courses in the U.S., where it has been customary for secondary teacher education programs to require a geometry course usually taught in mathematics departments (Grover & Connor, 2000). As U.S. PTs reportedly do less well than those in other countries on items that measure their knowledge for teaching geometry and avowedly have less opportunity to learn geometry (Tatto & Senk, 2011), this suggests that room for improvement exists in how geometry is taught to them. But the design of such improvement likely requires understanding the conditions in which PTs study geometry. As instructors[2] are key agents in the construction of students' opportunity to learn (Masingila et al., 2012, 2022), this paper explores the perspective of instructors of geometry courses for prospective secondary teachers.

Mathematics courses for teachers are important for the development of PTs knowledge. Scholarship that informs the design and improvement of these courses has been growing internationally (e.g., Adler et al., 2014; Goulding et al., 2003; Leikin et al, 2018) as well as in the U.S. (Appova & Taylor, 2019; Wasserman et al., 2022). In particular, research that conceptualizes the mathematical knowledge teachers need to teach mathematics in school has started to influence the design of mathematics courses for teachers (Chapman, 2007; Lai, 2019; Ronau et al., 2020). While it is desirable that the mathematics curriculum of teacher preparation be informed by research on the mathematical demands of the work of teaching (Ball et al., 2008), such improvement should be informed by an understanding of the instructional practice in institutional context that might be influenced by such results of research (Reinholz et al., 2020).

[1] The acronym GeT stands for "geometry for teachers" but the courses being considered have a wide variety of names and are not solely taken by prospective teachers.
[2] Across the manuscript, *instructor* designates a role in the classroom rather than an institutional position. *Instructor* refers to any individual responsible for the teaching and assessment of students in a course of study, regardless of their university position (e.g., whether they are tenure-track faculty, graduate students, or part time lecturers). Instructor does not refer to graders or teaching assistants who perform a supporting role to instructors. *Instructor* is an alternate designation to teacher in a higher education context.

Speer, et al. (2010) noted that little research accounts for the work of college mathematics teaching (e.g., Nardi, 2008). While that research has been growing in the intervening decade (e.g., Petropolou et al., 2020; Pinto & Karsenty, 2018), our work addresses the specific demands of teaching mathematics courses for teachers from the perspective of the instructors who teach GeT courses. Cohen et al.'s (2003) definition of instruction as interactions among teacher, students, and content in environments has supported much mathematics education research on the work that teachers do in instruction, including at the university level (e.g., Mesa, et al., 2020). We use that framework to inspect the work that GeT instructors report doing.

GeT courses in the U.S. present an interesting opportunity for this examination as they are required by most secondary mathematics teacher preparation programs (Winsor et al., 2018). And this requirement owes to the longstanding presence of a high school course dedicated to geometry (González & Herbst, 2006), as well as the identification of geometry among the knowledge secondary school teachers need to know, as further asserted in guidelines for the certification of teacher education programs (AMTE, 2017; NCTM, 2020). In most of the U.S., secondary mathematics teachers are certified by teacher education programs, usually accredited by national organizations (e.g., CAEP; see http://caepnet.org/) which entitle programs to provide initial certification on behalf of each of the states (which establish their own teacher certification standards[3]). Teacher education programs provide such certification upon students' completion of courses and practicums in education and mathematics coursework taken in mathematics departments and equivalent to a major in the discipline (Winsor et al., 2018). The U.S.'s higher education system tends to allocate the teaching of disciplinary courses to discipline-specific departments (e.g., mathematics, physics) regardless of the professional orientation of those students. Hence, unlike in some other countries, U.S. professional schools (e.g., teacher education) tend not to offer mathematics courses for their students but send their students to take courses in mathematics departments (see Clark, 1983; cf. Adler et al., 2014). This institutional setup makes the case of mathematics courses for teachers in the U.S. particularly interesting to examine as two distinct[4] institutions–the mathematics department and the teacher education program–serve as environments for instruction. Thus, an inspection of the teaching of GeT courses may serve us to learn how environments support or constrain the relationships among instructor, students, and content, hence contributing to understanding college mathematics instruction. We undertake such inspection through an analysis of interviews with GeT instructors.

Literature Review

From developing curriculum to studying instruction in teacher preparation

The expectation that secondary mathematics teachers know the subject they teach has been a part of teacher preparation in the U.S. for more than a century (Smith, 1907) and continually

[3] For example, California Teacher Certification Requirements are approved by State Law and identified in https://www.ctc.ca.gov/credentials/leaflets/subject-matter-authorizations-(cl-852)
[4] We say *distinct* and not *disjoint* because occasionally individual faculty members have joint appointments in mathematics and in education.

reinforced by legislation and policy documents (e.g., No Child Left Behind [NCLB]). Mathematicians have traditionally played an important role in the preparation of secondary mathematics teachers because PTs have usually taken mathematics coursework comparable to mathematics majors and offered by instructors with degrees in mathematics. This involvement has been an important source for scholarship on the curriculum of mathematics courses for teachers, starting with Klein's (1924) books. Throughout the intervening century, mathematicians have shown a special interest in envisioning and discussing the mathematical preparation of teachers (Bass, 2005; CBMS, 2001, 2012; Wu, 2011).

Perhaps because of such investment, course offerings for PTs showcase diverse curricular choices. The courses required for PTs have included conventional offerings at the undergraduate level (e.g., linear algebra), as well as two types of courses for teachers described by Murray and Star (2013, p. 1298) as "secondary mathematics from an advanced standpoint and … tertiary mathematics with connections." In terms of geometry courses for teachers Moise's (1974) text written during the New Math movement and the more recent text by Clark (2012) illustrate courses in which the Euclidean geometry material for secondary schools is taught from an advanced standpoint, while Greenberg (1993) or Henderson and Taimina (2019) illustrate courses that offer advanced geometry with connections. In her examination of what types of courses for secondary teachers are valued by mathematicians, Lai (2019) noted that some of these courses also include development of pedagogical content knowledge (PCK; Shulman, 1986) though this is not frequent.

The plausibility that mathematical courses for teachers could cover PCK has been fueled by empirical research on mathematical knowledge for teaching (MKT; Ball et al., 2008). Rather than speculating on what knowledge of mathematics teachers should have, researchers have examined the actual mathematical demands of the work teachers do (Bass, 2005; Hoover et al., 2014). A compelling argument has been made that the work of teaching makes specific mathematical demands that are not always addressed in the courses that are usually included in a mathematics major (Ball & Bass, 2000). More recent policy documents on teacher preparation (AMTE, 2017; CMBS, 2012) have included MKT as a desirable source for the mathematics curriculum of teacher preparation.

More recent investigations in the development of curriculum for mathematics courses for secondary teachers have demonstrated what such curricula could look like: For instance, Wasserman et al.'s (2022) curriculum materials include cases of secondary instruction in which the teacher draws on mathematics from college courses. GeT courses could include PCK, as illustrated, for example by Chazan et al. (2018) or Lischka et al. (2020). The possibility of extended use of such materials might benefit from an understanding of the problem space in which regular instructors of geometry courses work.

Compared with the study of the curriculum for mathematics courses for teachers and of what PTs learn from their mathematical preparation, little is known about the instructional practices in GeT courses. Speer et al. (2010) noted how little research had been done on the practice of teaching college mathematics in general. In the intervening years, much has been done in the

RUME community to understand teaching practice at the college level (Johnson, 2013; Johnson et al., 2019; Mesa et al, 2014, 2020; Reinholz et al., 2019). Some of that work has been done to understand the mathematics instruction of PTs (e.g., Hart et al., 2013; Heyd-Metzuyanim et al., 2016; Kobiela et al., 2022; Rogers & Steele, 2016; Stevens et al., 2020). Blanton (2002) studied her own teaching of geometry instruction for PTs with a focus on enabling discourse. Other than that, Grover and Connor (2000) has been the only study focused on instruction across GeT courses in the U.S. They used a traditional survey to elicit instructional practices, curricular choices, and instructors' perceptions of students from GeT courses. Our work contributes to deepening our community's understanding of instruction in GeT courses by relying on in depth interviews of instructors.

Theoretical Framework: College Mathematics Instruction and its Possible Tensions

The instructional triangle represents instruction as the system of interactions among teacher, students, and content in environments (Cohen et al., 2003). The notion that teacher and students interact with content suggests that the content is not merely a finished object being passed around in instruction but rather something being shaped in and through mathematical work (see Kuzniak, 2018). In our studies of instruction, we have focused on the role of content inside instruction and noted that, from the teacher's perspective, the content of studies is both what is at stake in instruction (i.e., what teachers can recognize students have acquired as a course of study progresses) and the mathematical work in which teachers engage students in order to acquire the knowledge at stake (see Figure 1). Indeed, the teacher needs to attend to the possible exchange between those meanings present in the work experiences students have and those meanings at stake in the content designated to be learnt (Brousseau, 1997; Herbst, 2006). The themes of *experiences* and *content*, which label two of the tensions our interviews allowed us to characterize, arise from each of the terms in that exchange of meanings. The instructional triangle also asks us to think of the environments that make room for those interactions. As regards GeT courses, these environments include mathematics departments and teacher preparation programs within universities. This theoretical consideration suggests the *institution* as an important theme for understanding the context of instruction. Finally, the *student* and the *teacher* as two key agents being related within instruction suggest two more themes.

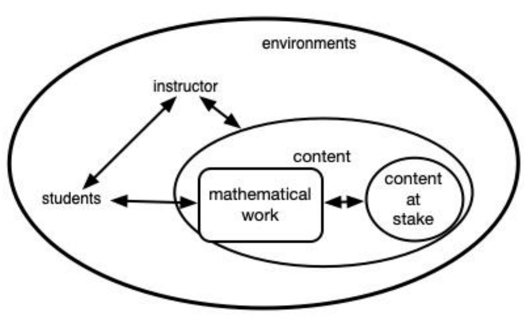

Figure 1. An elaboration of the instructional triangle

The instructional triangle does not by itself suggest that these five themes would name tensions in the work of instructors. But our present contribution argues that they do so for the case of

GeT instructors and suggests these tensions might also apply to the case of instructors of other mathematics courses for teachers. Indeed, when we started the interview study we report here, we expected to learn from the perspective of instructors what they saw as the demands in their work. The analysis of the data brought with it the overarching notion of tensions as applying to each of those themes identified by the instructional triangle. We thus elaborate briefly on the notion of tensions and its presence in prior research.

Research on teaching has often described teaching as a complex practice, fraught with dilemmas and tensions (Berlak & Berlak, 1981/2011; Cohen, 1990; Lampert, 1985). This literature posits that the decisions teachers have to make cannot be made once and for all but require constant considerations of context that create tensions for teachers. While the meaning of *dilemma* in philosophy is fairly standardized–a dilemma is the explicit choice an individual confronts between two distinct possible courses of action (the horns of the dilemma), neither of which is preferable (or both of which are problematic)--its use in education research, for example by Berlak and Berlak (1981/2011), has brought up the possibility that dilemmas be construals an observer proposes to describe apparent inconsistencies in teacher behavior. Adler (1999) and Brodie (2010) have used dilemmas as observer constructs to describe the apparent choice between two problematic possibilities. Yet, researchers who studied teaching by examining their own practice (e.g., Ball, 1993; Ball & Wilson, 1996; Chazan & Ball, 1999; Lampert, 1985) have used dilemmas with a meaning closer to its canonical interpretation, that is, as a conscious choice of the agent. At the same time, in none of those cases the two horns of the dilemma are necessarily unacceptable by themselves; rather, each is problematic inasmuch as it ignores some desirable aspects of the other. Peter Elbow (1983, 1986) has used the word *tension* to describe the teacher's experience of being pulled in two contrary directions and the possibility that a teacher may embrace the contradiction itself as characteristic of the work (hence, not necessarily see the circumstance as requiring a choice as in the classical notion of dilemma). Cooren et al. (2013) provide an in-depth elaboration of this set of concepts in the more general field of communication studies. Our choice to use the word *tensions* in this paper attempts to capture the sense that an actor (in this case the instructor) is confronted with two competing and contrary priorities, both of which make legitimate appeals to the actor and which the actor may or may not be conscious of. We found this notion of tension, as well as Elbow's (1983) proposition that tensions between contraries might be embraced, to be useful for describing the position in which the various components of the instructional triangle put GeT instructors.[5]

This manuscript attempts to answer the following questions: (1) How can we characterize the tensions specific to teaching GeT courses that can be detected beneath instructors' descriptions of the courses they teach? (2) How does the instructional triangle framework help organize and understand those tensions?

[5] We acknowledge that activity theory also speaks about tensions in activity systems (e.g., Skipper et al., 2021). As our research seeks to describe the work of instruction from the perspective of the instructor, we have found more leverage in the first person accounts of tensions and dilemmas to eventually integrate these notions to the theory which we are contributing to (Herbst & Chazan, 2012; in press).

Method

Study Background and Data Collection

The data was gathered as part of a multi-year project aimed at developing an inter-institutional network for instructors of GeT courses. In the first two years of that project, we interviewed GeT instructors to gain a better understanding of the problem space that individual instructors might identify as worthwhile for such a community to address. The interviews were conducted using online video-conferencing software that allowed us to capture audio and video records of the interaction, which were later transcribed for analysis.

Interviews followed a semi-structured protocol that had three sections. In the first section, we asked instructors 16 questions to help us understand the nature of the course (e.g., “Who are the students who take your geometry course?”, “How do faculty in the mathematics department come to teach the geometry for teachers course?”) as well as the role instructors saw the course playing in improving capacity for high school geometry teaching (e.g., “Could the mathematical experience of students in secondary schools be influenced by your geometry for teachers course? In what ways?”). In the second section, we asked instructors four questions to understand the particular content, experiences, and approaches they aimed for students to have in their course. In the final section, we asked instructors three questions about themselves and how they came to teach the course (e.g., “Can you tell us a little bit about some of the major stepping stones in your professional journey that led you to this point in your career?).

Participants

We interviewed 32 instructors (21 men and 11 women) from 30 universities across the U.S. Our recruitment focused on individuals drawn from mathematics departments situated in higher education institutions which have a secondary teacher preparation program, though one instructor was employed by a school of education. All of the institutions were public[6], but varied in terms of size and focus (some were doctoral granting, while others primarily served undergraduates). All interviewees had recently taught a geometry course taken by prospective secondary teachers. Most of the interviewees were faculty members (n=30), at various ranks, and two were graduate students in mathematics or mathematics education programs located in mathematics departments. In terms of research area, 13 participants identified it in mathematics while 19 did so in mathematics education.

Analysis

The institutional situatedness of GeT courses made us suspect that instructors could be facing a tension. But as the instructional triangle identifies not only the environments of instruction but

[6] This was not an outcome deliberately sought after, though there are many more public than private institutions that have secondary mathematics certification which may explain why none of our participants taught in a private university.

also the instructor, the students, the work students do to learn the content, and the content at stake, these components of instruction were also available to us as initial sensitizing themes with which to code the data. The analysis of the interview data was a multi-step process. In the first phase, we began by taking field notes during the interview. After the interviews, the video and audio records were converted into transcripts using natural language processing software. Those initial transcripts were then reviewed and cleaned up by interviewers using field notes along with one or two reviews of the video and audio records. After the transcripts were constructed, multiple researchers reviewed the interview artifacts (transcripts, video records) to look for how instructors described the various components of their work. Researchers drafted reflective memos regarding emerging themes and examples (Glaser, 1998; Strauss & Corbin, 1998). These reflective memos were shared and reviewed by the entire team. During the discussion of these memos, the team kept track of the themes that were emerging across the data. In some cases, the memos were revised to capture important details related to how emerging themes in the data evinced elements in tension connected to the sensitizing themes. These revisions were guided by a review of the interview artifacts. While the names for the five different tensions that we share about in this paper pre-existed the analysis as sensitizing themes issued from the theoretical framework, the poles that subtend tensions identified after those themes emerged from this first phase of analysis.

During that first phase, we noticed that even though the evidence for these five tensions were prevalent across the data, the way that instructors talked about them varied. In the second phase of analysis, we sought to understand more about the different ways these tensions could emerge in the data. After the construction, review, and refinement of reflective memos for all 32 interviews was complete, we returned to the primary data to explore the various ways that the tensions showed up across our interviews with instructors. In some cases, instructors were explicitly aware of tensions in their work. These more explicit expressions regarding the challenges instructors face in their daily work were useful in helping us to understand how the poles of the tensions contradicted each other. But individual instructors were not always explicit about a given tension, sometimes providing descriptions and sometimes justifications related to their personal alignment or misalignment with one or the other poles in tension. As the goal was to identify these tensions for the group of instructors, rather than describe the variability in how individuals experience these tensions, we used these testimonies to enrich the description of each of the poles in tension. In a separate report Brown et al. (in review) describe how individuals varied in their handling of the tensions.

Once we had a good understanding regarding the two poles of each of these tensions and how they contradicted each other, we used that understanding to code the entire corpus—treating each response instructors provided to the various interview questions as our unit of analysis. Three individuals coded participant responses for evidence of the tension, meeting together to compare, discuss, and reconcile our understanding of each coding category. All responses were then reconciled in a final codebook for total agreement across all three coders in synchronous meetings.

Results

In this section, we characterize the five tensions that emerged as such from our analysis of the interviews with instructors as we coded the data using the sensitizing themes of content, experiences, students, instructors, and institutions. In the subsections below, we define how each of these themes names a tension and illustrate each tension with data from the instructors' interviews. We propose these tensions, as instantiated in the set of interviews, as an underlying characteristic of the instructor's experience teaching GeT courses. We take each instructor primarily as an informant on the practice of GeT instruction more generally though we recognize they also provide glimpses of their own individual practice and context. Along those lines, the *tension* construct is useful to explain the practice in general and our characterization of the five tensions comes from looking at the aggregate dataset and the possibly opposing elements that influence GeT instruction that could be inferred from interview data. That said, the tensions were not always experienced in the same way: Some individuals experienced the tension as a source of stress, others have made commitments to one or another of the poles of each tension, yet others have found ways to do their job by satisficing the demands from both poles of the tension (see Brown et al., in review). Our present report attempts to inform our understanding of the professional work of teaching GeT and not to describe or differentiate the practices of individual instructors or to dwell on how individual instructors relate to each tension.

The content tension

The *content tension* was located in instructors' consideration of the kind of knowledge their students expect or are expected to learn. On the one hand, GeT courses exist in order to provide PTs with the knowledge that they will need to teach high school geometry (i.e., the content required by high school geometry standards; e.g., the properties of particular quadrilaterals, or how to do geometric proofs). On the other hand, these courses could be argued to exist in order to provide mathematics majors with the knowledge and tools developed over centuries of geometric research, including advanced geometric ideas and ways in which the field of mathematics has evolved its way of posing and addressing geometric questions.

Instructors communicated the challenges they find in achieving a balance between those poles. When asked about the expectations that shape the course Molly Vaughn[7] said,

> *From the department, I think there's an expectation that it's a serious math course with serious mathematical content and expectations. So you know, I had two students come up to me at the beginning of the course and say, 'The title of this course is geometry for future teachers. We thought this was going to be a course about pedagogy. We're very disappointed. Would you be willing to change the course?' And I said, 'No, I can't change the course because the expectation is this is a course taught in the math department, the content of the course is supposed to be mathematics.'*

Molly's report on students' missed expectations is framed in the context of recognizing the course as first and foremost a mathematics course. In contrast, Royce Andrews described how they were confronted by their department head for organizing the course around the knowledge

[7] The names used are pseudonyms. We use the pronouns they/them/their for all interviewees.

needed to teach high school geometry. In this case, the department head was coming to Royce on account of complaints they had received from mathematics majors enrolled in the course but not pursuing teaching certification who did not see the value of activities that focused on content that would support the teaching of secondary geometry. These accounts not only suggest that the content of the course attests of a tension, but also help us to understand that whether a GeT instructor elects to organize the content of the course around more advanced mathematical concepts or elects to focus on the concepts needed for teaching secondary geometry, they are liable to experience some tensions with their decision.

Distinct from these episodes of confrontation described by some instructors, other instructors described ways they have handled this tension. James Kerry explained how they help undergraduates to set realistic expectations about the focus of the course by saying:

> *I tell them on the first day, this stuff that I taught in this course content-wise is new—projective geometry—and it's not something that they're ever going to necessarily use directly in teaching anything in K-12.* [8]

Here, we see James describing ways they anticipate and address differing expectations students hold regarding content at the beginning of the semester. James made sure the interviewers knew that he does indeed prioritize “mak[ing] connections between projective geometry and the geometry that they learned in middle school and high school”—providing evidence for the awareness of the tension. Yet, in spite of this professed priority to making connections to the K-12 geometry content, it is notable that he found it necessary to start the semester by justifying the course’s content—and in this way helping students to have reasonable expectations about the content of the course.

Instructors’ ways of balancing attention to the content needed for high school teaching and advanced mathematical content varied. Larry Roland did this by ensuring that students knew the formal mathematical justification of geometric formulas, saying they

> *[make] sure that the students are familiar with the calculus proof of absolutely every formula that comes up.*

Likewise, some instructors make connections to group theory while other instructors use the course to help illustrate the underlying axiomatic structure of mathematics. In these courses, students are often engaged in the geometric content—sometimes Euclidean, sometimes non-Euclidean geometry—mostly as a context to learn about how mathematics is developed from a given set of axioms and definitions. In many cases, the treatment of Euclidean geometry is arguably distinct from high school content—with even the theorems, properties, and objects of study being distinct from those in high school (e.g., Ceva’s and Menelaus' theorems). They all show possible ways to reconcile the two poles in tension attending to both the mathematical knowledge needed for teaching high school geometry as well as treating the knowledge with similar mathematical rigor to that of other upper division mathematics courses.

The experiences tension

The *experiences tension* is concerned with the kinds of practices students are apprenticed into. On the one hand, as these are undergraduate mathematics courses, students

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[8] K-12 refers to Kindergarten to 12th grade, the range of compulsory education in the U.S.

are expected to be doing mathematics, thinking about mathematics, and engaging in mathematical practice. This could include opportunities to engage in the many activities that constitute mathematical practice (e.g., from conjecturing and proving to writing in LaTeX). On the other hand, as service courses for PTs, and considering the "apprenticeship of observation" (Lortie, 1975) according to which teachers learn to teach the way they were taught, students enrolled in the course are, at least by default, apprenticing into the work of teaching. This suggests that students apprentice into the practice of teaching by observing their instructor address the whole class, pose questions to students, respond to student contributions and questions, and so forth. As with the content tension, the data we share here suggests that it may be somewhat challenging for instructors to attend to these two practices. Consider what Ian Lawson said when asked about the expectations that shape the course:

> *I want them to have the experience of working on a problem that at first is too hard for them … I remember a student coming to me once [saying], 'This is challenging, but I can do it.' I think for her that was kind of a new experience. So that's the primary experience I want them to have. I also have them do presentations of homework problems, and for many of them this is the first time that they've actually explained a mathematical solution to a group of people. You know, stood in front of them and explained. Even though they're mostly secondary education majors, they don't do any [presentations] … So I want them to have the experience of standing in front of a group and explaining a problem.*

Ian simultaneously acknowledges the need for students to have both kinds of experiences—that of a mathematician and that of a teacher. Ian also helps illustrate an important challenge in apprenticing undergraduate students into both practices: For many students, the course provides them with their very first opportunity to engage in a crucial mathematical activity (i.e., persevering through a difficult problem) as well as a critical teaching activity (i.e., explaining a mathematical solution to a group of people). While the course may be able to provide students the opportunity to occasionally engage in activities drawn from both practices, time limitations may make it hard for an instructor to ensure students have meaningfully apprenticed into either professional practice.

The tension can be detected in the different kinds of experiences individual instructors elect to prioritize. In some cases, instructors seemed to prioritize giving undergraduates the opportunity to apprentice into mathematicians' practice. As Stephon Gage said,

> *So the most important thing for me is for the students to get an authentic experience of mathematical work ... it's very important to me that students get a sense of the mathematical cycle of an idea… I'm trying to get students in my particular geometry course to work in the style that mathematicians do just on really ancient material.*

In other cases, instructors seemed to value prioritizing experiences that provide opportunities for undergraduates to apprentice into the practice of a secondary mathematics teacher. James Kerry described a recent experience with an undergraduate student who had managed to create an application using dynamic geometry software, showing value for that experience by saying:

> *That [kind of experience] is the goal because eventually two years down the road, these students are going to be in their own classroom somewhere and they're going to have*

> *nobody to turn to…. So for me, providing them the opportunity to use a digital tool and to think about how the effective use of a digital tool can help them in both their learning and then forward their thinking in their teaching.*

This strategy of prioritizing students' experiences in ways that might help them gain fluency with the kinds of technological tools they would eventually have access to as teachers was common. Such a choice was often justified explicitly in terms of what PTs would be expected to implement in their future classrooms.

That said, the selection of technological tools was not the only way we saw instructors organizing the course to help apprentice students into the work of a secondary mathematics teacher. For example, when describing the kinds of experiences they aim for their students to have, Molly Vaughn said,

> *We spend an immense amount of time analyzing incorrect work ... It's usually framed in the context of their peers, so you know it's usually [their] work, not [high school geometry] student work, but I think that it's a related skill set... I think that's one way this course differs from a course I would teach to non-teaching majors.*

Thus, Molly named ways they elected to organize the course in ways different from other undergraduate courses to prioritize students' experiences gaining fluency with the instructional activity of analyzing student work. The choice to design the course to apprentice undergraduates into teaching practice was fairly common amongst the instructors we spoke with, though the particular teaching activities varied—including explaining solutions to problems to the class, teaching new content to others, providing feedback on a peer's work, examining classroom artifacts drawn from actual secondary geometry classrooms, designing lessons or activities for use in secondary geometry, designing problems that could be used in a secondary geometry course, or identifying standards aligned with their course's content.

The evidence suggests that organizing students' experiences around teaching activities is often deemed by students and instructors as non normative.[9] For example, in the context of the previous quote, Molly Vaughn noted, "I had one student complain about [their having students analyze incorrect work]. [That student was] like, 'We spent so much time looking at work that's incorrect. I'm not getting anything out of this.'" Indeed, instructors do not all agree that such work is an appropriate use of time.

The students tension

The *students tension* arises from instructors' need to consider the various kinds of individuals who populate the course and how instruction can be responsive to their needs. While PTs are included, GeT courses also contain students pursuing other majors[10]—including pure mathematics, physics, or engineering. As a group, PTs come with needs and expectations for this course. For example, it has been argued that PTs need more explicit conversation about how the mathematical ideas and practices learned in the course are linked with mathematical

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[9] We use *non normative* in the sense that neither students nor instructors perceive these activities as expected in mathematics classes.

[10] Two of the GeT courses taught by instructors in this data set were populated exclusively by prospective teachers. Presumably, such instructors may not experience the students tension. For these reasons, we have restricted our analysis of this tension to the other 30 instructors.

concepts and practices from the high school geometry course (Kilpatrick, 2019). It has also been argued that PTs can benefit from more explicit conversations about the pedagogical practices used for supporting mathematics learning (Wasserman et al., 2022). However, other students also come with competing expectations for the course, such as the expectation that the course would provide opportunities for them to learn new geometric content useful beyond the specific needs of a particular profession.

The tension between serving the two groups of students was apparent in the words of Laura Bachelor:

> *I feel that expectation of serving not only the education majors but the other majors in that class, I feel like I'm not able to spend time going back and digging deeply into the how's and why's of that high school geometry content.*

Thus, Laura names the multiple kinds of students the course needs to serve and the limited time they have available. Crucially, Laura perceived time as constraining them from digging more deeply into the connections between the course content and high school geometry in ways that might truly support the needs of PTs. This perspective was common amongst instructors, with some instructors stating that while they desired to make such connections for PTs, the needs of the other students in the class presented an important barrier that prevented them from fully engaging in that work with students.

Distinct from those who felt torn between the needs of two different groups of students, some instructors handled this tension by making considerations that added importance to meeting the specific needs of some students. They noted that future teachers whose high school experiences in geometry had been negative might need special attention to be willing to teach, and effective at teaching, high school geometry in the future. When describing experiences teaching the course, Qiana Roshan shared:

> *I had one student who actually wrote, "Geometry is boring [and] hard". I mean, that's its own can of worms about whether we want intending-high-school-teachers feeling that way. I do think they really need a second time around with geometry.*

These kinds of sentiments help illustrate why some instructors handle this tension by focusing centrally on the needs of PTs. The kind of negative disposition instructors identify in PTs can not only impact the individual students sitting in the course but can also have a compounding effect as those individuals take up their professional roles in schools.

Other instructors emphasized the needs of students as mathematics majors, independent of their role as future teachers. This sometimes came in the form of the instructor reminding the interviewers that PTs are, in fact, mathematics majors or mathematicians, and emphasizing the need that all mathematicians have of knowing Euclidean geometry. Ray Bucholz said

> *Geometry is … probably the oldest and most rigorous subject in mathematics. I think the history of Euclid and the Greek geometers is one that's very influential on mathematical traditions and ways of organizing information and establishing truth. So it's representative of an iconic subject that we pass along to the next generation … I think that every mathematician needs to learn the story of how geometry was developed and get a taste of it.*

This positioning of PTs as mathematicians while focusing on content that would be important for future teachers shows a way of handling the tension that looks for an argument that might address the needs of both groups of people.

The instructor tension

The *instructor tension* arises from instructors' considerations of the kinds of training and experiences that they feel they need to have to teach their course. On the one hand, the experiences of designing and teaching GeT could make an instructor acutely aware that there is much to know about the work of high school geometry teaching and much of that knowledge would be helpful in the design and implementation of a course for preparing PTs. To that end, some experience with high school geometry—having taught it or written textbooks for it, having supervised student teachers in it or done research on it—might qualify an instructor to teach a GeT course. On the other hand, the fact that the course prepares PTs who will have to teach a class in which students are introduced to proof and other aspects of theoretical mathematics might heighten the value of an instructor's experience doing mathematical research. Instructors' experiences doing research could be key in supporting a way of knowing geometry that highlights its instrumental role in developing a mathematical disposition in high school students. Many instructors provided evidence that they recognized these bodies of knowledge as crucial for the design and implementation of the course, and yet few identified themselves as experts in both kinds of knowledge. Some even described challenges that emerge in their work because they lack expertise with one of these bodies of knowledge. For example, when asked about the role of the mathematics department in supporting GeT students to gain the knowledge specific to the work of teaching secondary geometry, Ross Richardson said,

> *A lot more of what [math educators] talk about is how are [K-12] students thinking about these problems and what are some ways to help them. We actually have a tension in our department trying to figure that all out. … So we do have a responsibility there … I was trained as a mathematician—I was not given any formal training on student teaching learning ... So coming at it from the standpoint of a mathematician you think to yourself, 'Okay I'll make sure the students get this content, and then they'll be ready to teach it' ... I'm on the mathematician side—but I slide myself over to the math educator side and try to think about [K-12] students.*

Thus, this instructor demonstrated an acute awareness of how their own training, as a mathematician, fails to provide them with preparation on issues related to teaching and learning that other instructors might have had.

We note that recognition of limitations in their preparation was not unique to mathematicians. Furthermore, this kind of acknowledgement of their own limitations was not the only way this tension emerged. In some cases, instructors identified bodies of expertise held by other instructors, as something they appreciated and saw value for. For example, when asked about the role of mathematicians in teaching GeT courses, Laura Bachelor, who identifies as a mathematics educator, said,

> *I think mathematicians can always bring a contemporary perspective of what's going on. For example, my colleague who taught the course before me—his research areas*

> *related to this stuff so he can bring a lot of depth. I think that's pretty amazing and significant for the course to have someone that can bring that depth of understanding.*

Thus, some instructors perceive their own training as well as that of others—as privileging different kinds of assets useful for teaching GeT courses. But our data also contains evidence that both mathematicians and mathematics educators can hold similar deficit perspectives about the ability of mathematics educators or mathematicians respectively to play the role of instructor for GeT courses. For example, some of the instructors disqualified individuals with academic credentials different from their own with statements like "Well, you know mathematicians [as opposed to mathematics education specialists] know what proof is" (Ian Lawson) and "[Mathematicians] think they know what happens in schools … I would beg to differ" (James Kerry).

We take such underestimation of others', or overestimation of one's own, qualifications as evidence of recognition of the instructor tension. Of course, the tension could be handled in different ways, such as by seeking opportunities to grow professionally through professional development or collaboration. Indeed, most instructors found value in the prospect of working collaboratively with other instructors whose knowledge, training, and experiences differ from their own. For example, when asked about the role of mathematicians in teaching GeT courses, Louis Vaughan, who identifies as a mathematics educator said,

> *I'm a big fan of working collaboratively with mathematicians… math educators and mathematicians working together. They are the content experts. In my career, I have always sought out my mathematician colleagues to help me understand the mathematics at a deeper level. They always have great insight.*

The possibilities for such collaboration inside their institutions, however, were recognized as small, especially because Euclidean geometry has not been a prominent area of mathematical research for the last half-century (Willmore, 1970; see also Atiyah, 2002).

The institutions tension

The *institutions tension* arises when instructors consider the various institutions that are responsible for supporting and making demands on GeT courses. On the one hand, instructors look to the mathematics department for cues about how to shape GeT courses. On the other hand, instructors also take cues from teacher education programs. Our interviews with GeT instructors suggested that some see attending to both the mathematics department and the teacher education program as desirable. Yet, in practice, this is challenging. When asked about the expectations that shape GeT courses, Oscar Clark said:

> *The department was responding to a state mandate for a math course. But the math course was written by mathematicians with the mindset of what would be a viable proof-based course that was, in their eyes, worthy of being a 300- or 400-level math course … it wasn't done strategically with teachers in mind even though it was known that the course was going to be taken by teachers.*

This kind of narrative was quite common in our data. Instructors described GeT as a service course that emerged because: (1) some institution other than the mathematics department demanded it as a requirement in the preparation of secondary mathematics teachers and (2) the mathematics department complied with that demand by providing the requisite service (i.e., designing and implementing a course) at a level that seems acceptable by its own internal standards for what it means to be a service course. Such narratives help to illustrate ways that GeT courses can be understood as part of a “loosely coupled system” (Weick, 1976, p. 3)—in which one system (e.g., the state, the teacher education program) calls for an activity (i.e., designing and implementing a mathematics course for teachers) to be done by another system (i.e., university mathematics departments) which responds to that expectation, but does so in ways that preserve its own identity, separateness, and autonomy (Weick, 1976).

This loose coupling seems to be understood by instructors as lack of concern on the part of their departments. Quincey Dorne said that “there's only two of us [in the mathematics department] that like really give a sh*t about geometry” and Donovan Zanger said that he did not “think the math department cares enough about this course [because it] is really not a crucial part of the mission of the math department.” Teacher education programs and schools of education were also commonly described as uninvolved with the course. For example, Royce Andrews noted that while there may have been some expectations to meet the needs of PTs, “nobody [from the school of education is] looking over my shoulder.” Ojaswi Ogle added that “we have a teacher certification office that administers their certification exams. And really that's the only expectation that I've ever heard from them” and Quincey Dorne noted that “there's a state requirement about what we need to be able to say about our pre-service teachers who are going to be secondary math teachers. But it's not a very strong requirement. It's actually gotten weaker over the years. Essentially if we say this person gets our stamp of approval as a certified secondary teacher, the state will be happy with it no matter what.” On that point, however, the expectation that PTs should pass the course was a salient way in which the teacher education program’s influence was perceived. When asked about the expectations that shape GeT courses, Donovan Zanger said:

> *as far as the school of education goes the most important thing to them is that the students make it through the course. Since our math education majors sort of tend to be a little weaker than our math majors that poses some restrictions on the course because you can’t make it too demanding.*

The recent low recruitments of PTs have added to that pressure, as Royce Andrews said,

> *I feel pressure from the school of ed. They're trying to admit applicants into their program and we have required courses [their students] have to take. … there are other programs that we compete with—other universities that don't require any coursework; they just require you have a bachelor's degree and you pass the test.*

If the difficulties with recruitment of PTs made it important to attend to passing rates from the education side, from the side of the mathematics departments, instructors experienced a different pull. Again, Royce noted the very real pressure he experienced from his department head who told him that the course “wasn't a class for people wanting to become geometry teachers” and that he needed to attend to the diversity of majors among the students who take the course. This kind of explicit redirection from the mathematics department about the course was often noted by instructors as stemming from the economic needs of the departments to keep the courses viable.[11]

[11] NB. In the U.S., higher education economic models usually allocate financial resources to departments based on students’ registration in courses offered by each department.

Thus, while instructors noted lack of concern with curriculum and instruction and were uncertain of how their students' academic and practical experiences were being coordinated in ways that guaranteed the development of the requisite knowledge and skills, institutional pressures were perceived. Economic issues were the sources of pressure exerted by both teacher education programs and mathematics departments, bearing down on enrollment and assessment and because of the declining enrollments of PTs across the country. The declining enrollment of PTs (Sutcher et al., 2019; see also Holton et al., 2009) places pressure on (1) mathematics department to expand the audience of the course in order to keep the course viable and (2) teacher education programs to be concerned that if PTs do not succeed in GeT courses they might get their certification elsewhere.

The five preceding sections attest to five tensions experienced by instructors of GeT courses. Individual variability in how instructors experienced and handled the tensions is the focus of Brown et al. (in review). Here we are interested in relating these tensions to the nature of instruction in mathematics courses for teachers. We take this on in the next section.

Discussion

The analysis of these interviews sought to characterize tensions that pertain to the work of teaching GeT courses, taking the perspective that tensions are inherent to the work of teaching—that is, these tensions cannot be resolved but must be managed (Lampert, 1985), and perhaps even embraced (Elbow, 1983). In what follows we tie these tensions back to interpretations of the instructional triangle to contribute to the theorization of the work of teaching mathematics courses for teachers.

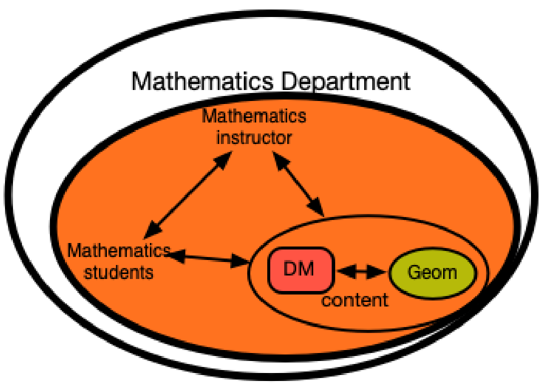
Cohen et al.'s (2003) *instructional triangle* describes instruction as the relationships entertained among teacher, students, and content in environments. Figure 1 shows an elaboration of the instructional triangle as used in the study of the practical rationality of mathematics teaching (Chazan et al., 2016). In this representation, the content vertex is elaborated to include two ways in which the content is present in those interactions: (1) as a set of ideas at stake, or curricular learning goals of a course of studies, and (2) as the mathematical work students engage in to lay claim on those ideas (Brousseau, 1997, p. 22-23). The following discussion provides an interpretive frame on the tensions identified in the interviews. This interpretation proposes that the five tensions can be understood as emergent from a more basic underlying tension between two instances of the instructional triangle (Figure 2).

Two interpretations of the instructional triangle

The tensions reported can be understood by considering that GeT instruction instantiates the instructional triangle in two different ways. First, GeT instruction can be seen as a case of undergraduate mathematics instruction (Figure 2a). In this interpretation of the instructional triangle, the teacher is a university mathematics instructor and the students are undergraduate mathematics students. The content at stake derives from the discipline of mathematics; in particular, it includes synthetic geometric knowledge developed or curated by mathematicians.

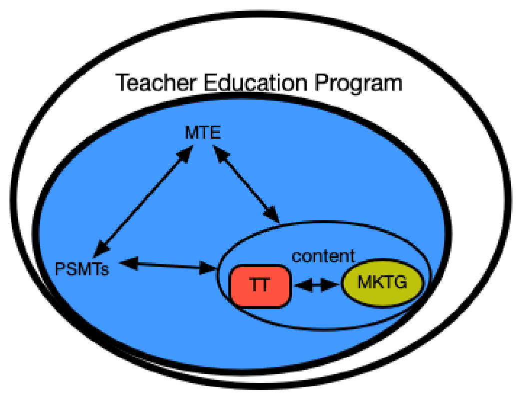
As in other undergraduate mathematics classes, the content is also present in the form of students' mathematical work, solving problems where those ideas are at stake. This instance of GeT instruction, like any undergraduate mathematics class, relies on the environment a university mathematics department provides.

Second, GeT courses can be seen as cases of secondary mathematics teacher preparation (Figure 2b). In this interpretation of the instructional triangle, the teacher is a mathematics teacher educator and the students are prospective secondary mathematics teachers. The content at stake is the knowledge needed for teaching high school geometry and the work students do involves encountering and practicing the use of that knowledge in the context of "mathematical tasks of teaching" (Ball et al., 2008, p. 400). This instance of GeT instruction relies on the environment that a university teacher preparation program provides. Carroll and Mumme (2007) and Nipper and Sztajn (2008) have used variations of this interpretation of the instructional triangle to examine the case of teacher educators and facilitators of professional development.



DM: Doing mathematics
Geom: Geometry

Figure 2a. GeT as an undergraduate mathematics course (UG)



TT: Mathematical Tasks of Teaching
MKTG: Mathematical Knowledge for Teaching Geometry
PSMTs: Preservice Secondary Mathematics Teachers
MTE: Mathematics Teacher Educator

Figure 2b. GeT as a secondary teacher education course (GTE)

These two interpretations of the instructional triangle are offered as ideal types that serve as reference for each instance of GeT instruction: The tensions the instructors experienced can thus be seen as emergent from a complex situation in which instructors are subject to two different sets of expectations sourced in the two triangles. Figure 3 represents how the two instances of the instructional triangle relate to each of the tensions.

The poles of each tension can be associated with each interpretation of the instructional triangle. While neither the generic instructional triangle nor each of these interpretations necessitates a specific didactical contract for a GeT course, we contend that each of these interpretations provides the grounds for legitimating some of the norms that might characterize different didactical contracts. And as they do so, they create opportunities for those tensions to come to the surface–as tensions between different normative practices. We elaborate this below.

We rely on Brousseau's notion of the didactical contract, which is defined by Brousseau et al. (2018) as:

> an interpretation of the commitments, the expectations, the beliefs, the means, the results, and the penalties envisaged by one of the protagonists of a didactical situation (student, teacher, parents, society) for him- or herself and for each of the others, à propos of the mathematical knowledge being taught (p. 1).

In this study, we have been particularly interested in the interpretations that GeT instructors have of the roles and responsibilities they have vis-à-vis students and the content of studies. We use the word *norm* to refer to each of the statements an observer might make to model the didactical contract being considered–a norm is therefore not a rule explicit for an actor but a rule the observer produces and that means that everything happens as if the actor was following a rule (Bourdieu, 1990, p. 61; see also Herbst & Chazan, 2012). Brousseau et al. (2018) posit that one aspect of the contract is devolution–or how "the teacher organizes the mathematical activity" (p. 1) in the class. The modeling of a didactical contract for devolution thus means to make descriptive statements of what the teacher takes (or the students take) to be expected in how the teacher organizes the mathematical activity in class. For example, in most mathematics classes teachers consider themselves entitled to assign problems for students to solve and to expect students to occupy themselves in those problems (see Dixon et al., 2009). We model those beliefs by stating two norms that purportedly describe what the teacher believes, namely (1) The teacher is entitled to assign problems for students to solve and (2) The teacher is entitled to expect students to work on the problems assigned. With that understanding of the word *norm*, we can now explicate the claim that two different interpretations of the instructional triangle provide grounds for legitimating some of the conflicting norms of the didactical contract in GeT courses.

As instructors describe their experience teaching GeT courses, they provide empirical material that can be used to model the didactical contract in their classes in terms of norms and those norms can be justified by recourse to one of those interpretations of the instructional triangle. For one such norm to be justified we mean that the norm can be judged as consistent with the interpretation of instruction offered. Clearly, many didactical contracts may be consistent with each of the two interpretations of the instructional triangle. For example, the norm (1) that entitles the instructor to assign problems for students to solve might be further specified as (1a) the instructor is entitled to assign geometry problems that students recognize as solvable with the geometric knowledge they are supposed to have at the moment or as (1b) the instructor is entitled to assign geometry problems for students to solve that they may not immediately recognize as ones they know how to solve. The first example (1a) describes classes where there is a contract of using worked out examples, while the second example describes classes where the contract might be described as inquiry-based. Both of those norms would be justified by an interpretation of the instructional triangle as the one provided in Figure 2a, as they are consistent with that interpretation of geometry instruction.

But our interviews also permit other norms to be stated that might be more consistent with an interpretation of the instructional triangle such as provided in Figure 2b. Specifically, instructors' descriptions of their experiences also suggested that (1c) the instructor is entitled to assign students to explain how to solve a problem to the whole class and (1d) the instructor is entitled to assign students to make comments or grade the responses other individuals gave to geometric problems they were assigned to solve. These norms might or might not have currency in the same class but both might need the assumption that an explanation of a geometric idea to a group and understanding how others solve geometric problems are aspects of the mathematical work GeT students need to do to show they know the geometry at stake in the course. These assumptions about the nature of the work that evinces knowledge are more consistent with the interpretation of the instructional triangle provided in Figure 2b. In what follows we use UG to refer to the first interpretation of the instructional triangle (Figure 2a) and GTE to refer to the second (Figure 2b).

Having explained how we are articulating the constructs *interpretation of the instructional triangle*, *didactical contract*, and *(contractual) norm*, we can now return to the five tensions. We state each tension in terms of conflicting contractual norms, each of which is warranted by one interpretation of the didactical contract.

The experiences tension. The examples provided above give us a head start on the Experiences tension, which can be defined as a tension between the following two norms

- E_{UG}: The instructor is entitled to assign geometric problems to individual students, whose mathematical work on those problems attests to their knowledge of geometry
- E_{GTE}: The instructor is entitled to assign mathematical tasks of teaching to individual students, whose work on such tasks attests to their knowledge of geometry for teaching

Those two norms are in conflict but are not necessarily contradictory. It is possible to envision a class in which students' experiences include solving problems as in any mathematics class as well as engaging in mathematical tasks of teaching as context to do other mathematics (Appova & Taylor, 2019; Leikin et al., 2018). But as resources, such as course credits and time, are limited, the conflict can be seen in terms of how resources are allocated (e.g., the grade points assigned to performing a task of teaching are not available to credit solutions to problems, the time spent listening to students' explanations is not available for students to work problems in class, etc.). All of which suggests also as possible that some instructors might only recognize norm E_{UG}; likewise, it is possible, though less likely given our data and what Lai (2019) reports, that some instructors might only recognize norm E_{GTE} and feel entitled to assign problems only if they can be posed in the context of a mathematical task of teaching (Ball et al., 2008; Blanton, 2002).

The content tension. Another aspect of the didactical contract is that "the teacher vouches for the part of [the students'] results that conforms to reference knowledge," which Brousseau et al. (2018, p. 1) call institutionalization: Recognizing the knowledge at stake in the students' completed work. This enables us to address the Content tension, which can be defined as a tension between the following two norms

- C_{UG}: The geometric content the instructor may recognize as acquired by students consists of geometric ideas (concepts, propositions, proofs, etc.) curated by the mathematical community under the names of synthetic, metric, Euclidean, or Non Euclidean geometry
- C_{GTE}: The geometric content the instructor may recognize as acquired by students consists of geometry knowledge for teaching (e.g., the geometric ideas school students learn, the ways in which school students understand or misunderstand geometric ideas; see Ball et al., 2008)

Again, while it is possible for a GeT course to include content that responds to both norms, it is also apparent that the content organizations available for study in different GeT courses might respond to one norm but not the other (see Appova & Taylor, 2019; Goulding et al., 2003; Leikin et al. 2018; Wasserman et al., 2022).

The students tension and the instructor tension. The students and instructor tensions can also be described in terms of conflicting norms. Namely

- S_{UG}: The student is (for the instructor) an undergraduate mathematics major
- S_{GTE}: The student is (for the instructor) a prospective secondary mathematics teacher

And

- In_{UG}: The instructor is (i.e., sees him or herself as) a mathematician who teaches college students
- In_{GTE}: The instructor is (i.e., sees him or herself as) a teacher educator who prepares mathematics teachers

As norms of the didactical contract, these describe how the two roles of student and instructor could be seen by the instructor, regardless of their objective truth. That is, an instructor could see themselves only as a mathematician when teaching even if they have had experiences teaching school mathematics that might otherwise entitle them to mentor future teachers (see Escudero-Avila et al., 2021). Reciprocally, an instructor could see themselves as a teacher educator regardless of whether they had professional preparation to teach teachers (see Goulding et al., 2003; Grover & Connor, 2000; Jackson et al., 2020; Masingila et al., 2012, 2022). Similarly, the instructor could see all students as undergraduate mathematics majors even if some of them are also prospective teachers. Or the instructor could see all students as (potential) PTs (see Goos & Bennison, 2019), considering that one job that many mathematics majors eventually do is to teach (e.g., if they seek graduate certification to teach school or tertiary levels later).

The institutions tension. Finally, the institutions tension can also be represented in terms of a conflict between two norms that purport to describe how the instructor sees the environment surrounding GeT instruction:

- It_{UG}: The GeT course is offered and overseen by the mathematics department
- It_{GTE}: The GeT course is offered as a service to the teacher education program

From an objective perspective, it is likely that both of those statements are true in most of the institutions where our interviewees work (Leikin et al., 2018; Masingila et al., 2012, 2022; cf. Adler et al., 2014). The conflict between those two statements as norms of the didactical contract alludes to the possibility that instructors might more readily see one than another of them when they consider how their course is situated in a larger context.

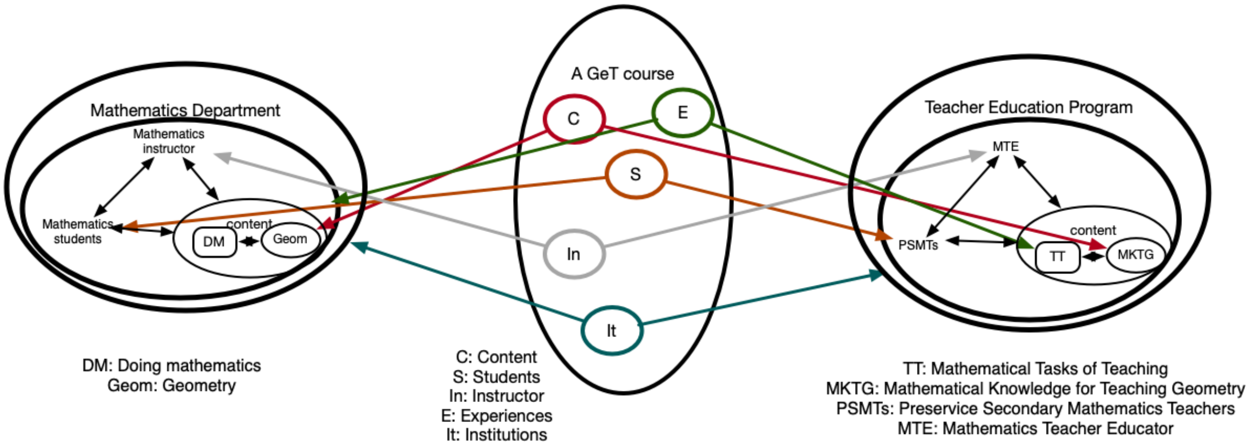


Figure 3. The five tensions in relation to two interpretations of the instructional triangle

Thus, we propose that experiences, content, students, instructor, and institutions are five arenas in which a more fundamental tension between the two interpretations of instruction (undergraduate mathematics instruction and mathematics teacher preparation) compete, generating a more fundamental tension. This competition is not necessarily one that needs to (or can) be resolved by one of the interpretations of instruction winning over the other one, though some instructors described their courses as closer to one than to the other. Rather, the conflict between the two interpretations of the instructional triangle provides a hypothesis for explaining why these five tensions exist and suggests that while the tensions are different, they spring from similar sources. Inasmuch as each instructional triangle pulls the instructor in its direction, the actions that an instructor might take to respond to the pull from any one of the poles in each of the tensions might be seen from the perspective of one triangle as a movement toward satisfying those instructional expectations but from the perspective of the other triangle as a movement away from satisfying those expectations. These tensions experienced by instructors attest to a complex practice.

Conclusion

We have documented five tensions experienced by instructors of geometry courses taken by prospective secondary mathematics teachers: content, experiences, students, instructors, and institutions. Each of those tensions requires instructors to cope with demands that pull them in different directions. Elbow (1983) observed that competing demands are inherent to the work of teaching and suggested that teachers need to embrace the tensions that issue from that complexity. In explaining the five tensions, we have proposed that GeT courses (and possibly other mathematics courses for teachers) can be seen from the perspective of two different applications of the instructional triangle that are at the basis of the tensions experienced.

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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