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Global Harvesting and Stocking Dynamics in a Modified Rosenzweig–MacArthur Model

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Abstract

Effective management of predator–prey systems is crucial for sustaining ecological balance and preserving biodiversity, which requires full understanding the dynamics of such systems with harvesting and stocking. This paper aims to investigate the global dynamics of a Rosenzweig–MacArthur model considering the interplay of these intervention practices. We reveal that this model undergoes a sequence of bifurcations, including cusp of codimensions 2 and 3, saddle-node bifurcation, Bogdanov–Takens (BT) bifurcation of codimensions 2 and 3, and degenerate Hopf bifurcation of codimension 2. In particular, a codimension-2 cusp of limit cycles is found, which indicates the coexistence of three limit cycles. An interesting and novel scenario is discovered: two distinct homoclinic cycle curves connect their respective BT bifurcation points. This differs from most models where a single homoclinic cycle curve may connect both BT bifurcation points. Moreover, we find that two families of limit cycles converge toward a heteroclinic cycle, signaling the risk of overexploitation. From a biological perspective, the prey population may undergo extinction for all initial states under large constant harvesting rate. Further, the simultaneous stocking of both populations is not conducive to the coexistence of both species; the stocking of one population and the harvesting of the other will promote the coexistence of two populations; while the simultaneous harvesting of two populations may result in multiple limit cycles, which effectively underscore the positive effect of harvesting and stocking. Identifying the optimal timing to harvest or stock predators and prey is crucial to prevent

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system collapse. This work promotes to a deeper understanding of the dynamics of ecosystems when harvesting and stocking occurs simultaneously. Further, it reveals the important roles of harvesting and stocking, contributing to the effective management of predator–prey systems.

Keywords Constant rate harvesting and stocking · Bogdanov–Takens bifurcation of codimensions 2 and 3 · Degenerate Hopf bifurcation of codimension 2 · Codimension-2 cusp of limit cycles · Overexploitation

Mathematics Subject Classification 34C07 · 34C23 · 34C25 · 34C37

1 Introduction

It is well recognized that the predator–prey interaction has been an important research issue in ecological systems. This concept originated from the innovative works of Lotka [1] and Volterra [2], and there have been abundant investigations [3–12] on the dynamics of predator–prey models in the last few decades. Particularly, harvesting has received extensive concerns [9, 13–19] because of its wide applications. The basic harvesting model is usually presented in the form of the following differential equations

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{K}\right) - m\phi(x)y - h_1, \\ \frac{dy}{dt} &= y(-d+em\phi(x)) - h_2, \end{aligned} \tag{1}$$

where $x(t)$ and $y(t)$ represent the population densities of prey and predators, respectively. Parameters $r, K > 0$ are the intrinsic growth rate and the carrying capacity of prey population, respectively. $e > 0$ represents the conversion rate from prey to predators, while $d > 0$ is the death rate of predators. Additionally, $h_1, h_2 \geq 0$ are the harvesting rates of two populations, respectively.

The functional response $m\phi(x)$, where $m > 0$, represents the density of prey captured per unit time per predator as the density of prey varies. It has different forms because it is influenced by diverse factors, such as the prey density, the physical state of predators, and environmental conditions. System (1) with the Holling type-II function

$$\phi(x) = \frac{x}{a+x} \tag{2}$$

and constant rate harvesting and stocking ($h_1, h_2 < 0$) has garnered considerable attention [11, 14, 20–27], where $a > 0$ is the half-saturation constant. In the scenario of $h_1 = h_2 = 0$, systme (1) becomes the classical Rosenzweig–MacArthur model as Lin et al. [28]. Hsu [29] established two criteria for the global stability of the locally stable equilibrium. In the case of $h_1 = 0$, Brauer and Soudack [22] investigated stability regions and transition phenomena for harvested predator–prey systems, but did not give global bifurcation analysis. Later, Xiao and Ruan [21] demonstrated that system (1) with constant rate predator harvesting exists a cusp of codimension 2 and a Bogdanov–Takens bifurcation, but they did not analyze bifurcation with higher codimension. Newly, Ruan and Xiao [27] showed that system (1) under human interventions (constant harvesting and stocking of predators) undergoes imperfect bifurcation and Bogdanov–Takens bifurcation, which induces much richer dynamical behaviors such as the existence of limit cycles or homoclinic loops. However, the constant harvesting and stocking of prey was not involved. In the context of $h_2 = 0$, Brauer and Soudack [20] conducted a similar analysis to their previous work [22] under constant-rate prey harvesting. They delineated the theoretically possible structures and transitions, and constructed examples to verify which of these transitions can be realized in a biologically plausible model through numerical simulation. Soon after, in the case where constant rate harvesting and stocking of both species exist, they provided similar results [14].

Moreover, for system (1) with Hollong II functional response and constant-rate prey harvesting, Peng et al. [26] analyzed the existence of equilibria and showed that this model can exhibit various bifurcation phenomena, containing the saddle-node bifurcation, degenerate Bogdanov–Takens bifurcation of codimension 3, supercritical and subcritical Hopf bifurcations, and generalized Hopf bifurcation, while the role of predator harvesting and stocking was not elucidated; Recently, Sarif and Sarwardi [11] studied the dynamics of system (1) in which both prey and predator are harvested with constant rate. They pointed out the existence of Hopf bifurcation and Bogdanov–Takens bifurcation of codimension 2, and showed the effect of harvesting on equilibria, stability, and bifurcations. However, higher codimension bifurcation analysis (such as Hopf bifurcation of codimension 2 and Bogdanov–Takens bifurcation of codimension 3) and the function of harvest are not displayed; When the constant rate harvesting and stocking of both species coexist, Myerscough et al. [24] investigated the qualitative properties of steady-state solutions, but they didn not analyzed the effects of these two factors on more complex dynamics. In general, there has been relatively little discussion of stocking, especially in terms of high-codimension bifurcations.

In order to uncover the effect of harvesting and stocking on the more complex dynamical behaviors, in this paper, we investigate the harvested system with Holling II functional response. Specifically, we allow either h_1, or h_2, or both, to be negative, representing stocking rather than harvesting of the corresponding species. Our model is as follows

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1 - \frac{x}{K}\right) - \frac{mxy}{a+x} - h_1, \\ \frac{dy}{dt} &= y\left(-d + \frac{emx}{a+x}\right) - h_2, \end{aligned} \tag{3}$$

where parameters r, K, m, a, d, e have the same meaning as those in model (1), parameters h_1 and h_2 are permitted to be arbitrary nonzero constant representing stocking (less than 0, which may be viewed as negative harvesting) or harvesting (greater than 0) of both species. For simplicity, we apply the coordinate and time transformations $X = \frac{x}{K}$, $Y = \frac{my}{rK}$, $\tau = rt$ to normalize system (3) as follows (we still denote X, Y, τ by x, y, t, respectively)

$$
\begin{aligned}
\frac{dx}{dt} &= x(1-x) - \frac{xy}{\alpha + x} - h, \\
\frac{dy}{dt} &= y\left(-\delta + \frac{\beta x}{\alpha + x}\right) - H,
\end{aligned} \tag{4}
$$

where $\alpha = \frac{a}{K}$, $\beta = \frac{em}{r}$, $\delta = \frac{d}{r}$ are always positive, parameters $h = \frac{h_1}{rK}$, $H = \frac{mh_2}{r^2 K}$ are permitted to be any nonzero constants.

In the remaining sections of this paper, we investigate the complex dynamics of system (4). In Sect. 2, we present the existence and type of positive equilibria. Section 3 provides a comprehensive bifurcation analysis, encompassing saddle-node bifurcation, cusp of codimensions 2 and 3, Bogdanov–Takens bifurcation of codimensions 2 and 3 for the double positive equilibria, and degenerate Hopf bifurcation of codimension 2 for the simple positive equilibrium. In Sect. 4, we perform numerical simulations, including bifurcation diagrams and phase portraits, to verify the theoretical results. Section 5 offers biological interpretations to elucidate the effect of harvesting and stocking on both species. Finally, we end this paper with a conclusion.

2 Existence and Type of Positive Equilibria

It is evident that system (4) has no boundary equilibrium, thus, we just focus on the existence and type of positive equilibria. Following, we will derive specific conditions for the existence of positive equilibria. Here, we restrict our analysis to the case of $\delta \neq \beta$ to save space; the other case, $\delta = \beta$, follows a similar method.

Assume that $E(x, y)$ is any positive equilibrium of system (4). From the first equation of system (4), we can get that $y = \frac{(\alpha+x)(-x^2+x-h)}{x}$, where $\max\{0, \frac{1-\sqrt{1-4h}}{2}\} < x < \frac{1+\sqrt{1-4h}}{2}$, $h < \frac{1}{4}$ to ensure that $y > 0$. Substituting the expression of y into the second equation shows that x is a positive root of the following cubic equation

$$
f(x) \doteq x^3 + a_2 x^2 + a_1 x + a_0 = 0, \tag{5}
$$

in which $a_2 = -\frac{\alpha\delta+\beta-\delta}{\beta-\delta}$, $a_1 = \frac{\alpha\delta+h(\beta-\delta)+H}{\beta-\delta}$, and $a_0 = \frac{\alpha\delta h}{\delta-\beta}$. Hence, the existence of positive equilibria for system (4) is equivalent to the existence of positive roots of Eq. (5) in the interval $(\bar{x}_1, \bar{x}_2)$, where $\bar{x}_1 = \max\{0, \frac{1-\sqrt{1-4h}}{2}\}$ and $\bar{x}_2 = \frac{1+\sqrt{1-4h}}{2}$. Since a_0 can be positive, based on Vieta's Theorem, Eq. (5) has at most three positive roots, which are expressed by size as $x_1 < x_2 < x_3$. Now, we begin to derive the specific conditions for the existence of 0, 1, 2 and 3 positive roots.

Differentiating $f(x)$ yields that $f'(x) = 3x^2 + 2a_2 x + a_1$ and the discriminant of $f'(x)$ is $\Delta = 4a_2^2 - 12a_1$. When $\Delta \leq 0$, we have that $f'(x) \geq 0$ and $f(x)$ is an increasing function. Thus, the equation $f(x) = 0$ has a unique positive root in the interval $(\bar{x}_1, \bar{x}_2)$ if $f(\bar{x}_1) < 0$, $f(\bar{x}_2) > 0$; has no positive root if $f(\bar{x}_1) \geq 0$ or $f(\bar{x}_2) \leq 0$. When $\Delta > 0$, we can concluede that $f'(x) = 0$ has two real roots, denoted by $\hat{x}_1 = \frac{-a_2-\sqrt{a_2^2-3a_1}}{3}$ and $\hat{x}_2 = \frac{-a_2+\sqrt{a_2^2-3a_1}}{3}$. Based on the positions of $\hat{x}_1$ and $\hat{x}_2$, we perform our analysis in the following five scenarios.

Table 1 The distribution of positive roots of $f(x) = 0$ in the interval $(\bar{x}_1, \bar{x}_2)$ when $\bar{x}_1 < \hat{x}_1 < \bar{x}_2 \leq \hat{x}_2$

Signs of $f(\bar{x}_1)$, $f(\bar{x}_2)$ and $f(\hat{x}_1)$			Existence of positive roots of $f(x) = 0$ in the interval $(\bar{x}_1, \bar{x}_2)$
$f(\bar{x}_1) \geq 0$	$f(\bar{x}_2) < 0$		A unique positive root, denoted by x_2
	$f(\bar{x}_2) \geq 0$		No positive root
$f(\bar{x}_1) < 0$	$f(\hat{x}_1) < 0$		No positive root
	$f(\hat{x}_1) = 0$		One positive root of multiplicity 2, denoted by $\hat{x}_1 = x_{1,2}$
	$f(\hat{x}_1) > 0$	$f(\bar{x}_2) < 0$	Two positive roots, denoted by $x_1 < x_2$
		$f(\bar{x}_2) \geq 0$	A unique positive root, denoted by x_1

(I) For $\bar{x}_2 \leq \hat{x}_1$ or $\bar{x}_1 \geq \hat{x}_2$, we can see that $f'(x) > 0, x \in (\bar{x}_1, \bar{x}_2)$, which shows that $f(x)$ is monotonically increasing in the interval $(\bar{x}_1, \bar{x}_2)$. Hence, we can get the same result as $\Delta \leq 0$.

(II) For $\bar{x}_1 < \hat{x}_1 < \bar{x}_2 \leq \hat{x}_2$, $f(x)$ first increases monotonously in the interval $(\bar{x}_1, \hat{x}_1)$, and then decreases monotonously in the interval $(\hat{x}_1, \bar{x}_2)$. At this point, our results are summarized in Table 1.

(III) For $\bar{x}_1 < \hat{x}_1 < \hat{x}_2 < \bar{x}_2$, $f(x)$ is first increases monotonously in the interval $(\bar{x}_1, \hat{x}_1)$, then decreases monotonously in the interval $(\hat{x}_1, \hat{x}_2)$, and finally increases monotonously in the interval $(\hat{x}_2, \bar{x}_2)$. Under this circumstance, the distribution of positive roots for the equation $f(x) = 0$ in the interval $(\bar{x}_1, \bar{x}_2)$ is summarized as follows.

(i) when $f(\bar{x}_1) \geq 0$, $f(x)$ has two positive roots if $f(\hat{x}_2) < 0$ and $f(\bar{x}_2) > 0$, denoted by $x_2 < x_3$; has a unique positive root if $f(\hat{x}_2) < 0$ and $f(\bar{x}_2) \leq 0$, denoted by x_2; has one positive root of multiplicity 2 if $f(\hat{x}_2) = 0$, denoted by $\hat{x}_2 = x_{2,3}$; and has no positive root if $f(\hat{x}_2) > 0$.

(ii) when $f(\bar{x}_1) < 0$, see Table 2.

(IV) For $\hat{x}_1 \leq \bar{x}_1 < \bar{x}_2 \leq \hat{x}_2$, we can conclude that $f'(x) < 0, x \in (\bar{x}_1, \bar{x}_2)$, which suggests that $f(x)$ is monotonously decreasing. Then the equation $f(x) = 0$ has a unique positive root if $f(\bar{x}_1) > 0$, $f(\bar{x}_2) < 0$; has no positive root if $f(\bar{x}_1) \leq 0$ or $f(\bar{x}_2) \geq 0$.

(V) For $\hat{x}_1 < \bar{x}_1 < \hat{x}_2 < \bar{x}_2$, the function $f(x)$ is monotonically decreasing in the interval $(\bar{x}_1, \hat{x}_2)$, and then monotonically increasing in the interval $(\hat{x}_2, \bar{x}_2)$. According to the signs of $f(\bar{x}_1)$, $f(\hat{x}_2)$ and $f(\bar{x}_2)$, we can conclude that the distribution of positive roots of $f(x) = 0$ in the interval $(\bar{x}_1, \bar{x}_2)$ is as follows.

(i) When $f(\bar{x}_1) \leq 0$, the equation $f(x) = 0$ has a unique positive root if $f(\bar{x}_2) > 0$, denoted by x_3; has no positive root if $f(\bar{x}_2) \leq 0$.

(ii) When $f(\bar{x}_1) > 0$, the equation $f(x) = 0$ has two positive roots if $f(\hat{x}_2) < 0$, $f(\bar{x}_2) > 0$, denoted by $x_2 < x_3$; has a unique positive root if $f(\hat{x}_2) < 0$, $f(\bar{x}_2) \leq 0$,

Table 2 The distribution of positive roots of $f(x) = 0$ in the interval $(\bar{x}_1, \bar{x}_2)$ when $\bar{x}_1 < \hat{x}_1 < \hat{x}_2 < \bar{x}_2$ and $f(\bar{x}_1) < 0$

Signs of $f(\hat{x}_1)$, $f(\hat{x}_2)$ and $f(\bar{x}_2)$			Existence of positive roots of $f(x) = 0$ in the interval $(\bar{x}_1, \bar{x}_2)$
$f(\hat{x}_1) > 0$	$f(\hat{x}_2) < 0$	$f(\bar{x}_2) > 0$	Three positive roots, denoted by $x_1 < x_2 < x_3$
		$f(\bar{x}_2) \leq 0$	Two positive roots, denoted by $x_1 < x_2$
	$f(\hat{x}_2) = 0$		Two positive roots, one of them is a positive root
			Of multiplicity 2, denoted by $x_1 < \hat{x}_2 = x_{2,3}$
	$f(\hat{x}_2) > 0$		A unique positive root, denoted by x_1
$f(\hat{x}_1) = 0$	$f(\bar{x}_2) > 0$		Two positive roots, one of them is a positive root
			Of multiplicity 2, denoted by $\hat{x}_1 = x_{1,2} < x_3$
	$f(\bar{x}_2) \leq 0$		One positive root of multiplicity 2, denoted by $\hat{x}_1 = x_{1,2}$
$f(\hat{x}_1) < 0$	$f(\bar{x}_2) > 0$		A unique positive root, denoted by x_3
	$f(\bar{x}_2) \leq 0$		No positive root

denoted by x_2; has one positive root of multiplicity 2 if $f(\hat{x}_2) = 0$, denoted by $\hat{x}_2 = x_{2,3}$; and has no positive root if $f(\hat{x}_2) > 0$.

For the positive roots $x_i, i = 1, 2, 3$ (if they exist) of equation $f(x) = 0$ in the interval $(\bar{x}_1, \bar{x}_2)$, system (4) has corresponding positive equilibria $E_i(x_i, y_i)$, in which $y_i = \frac{(\alpha+x_i)(-x_i^2+x_i-h)}{x_i}$. Similarly, for the double roots $x_{i,i+1}, i = 1, 2$ of equation $f(x) = 0$, system (4) has corresponding degenerate positive equilibria $E_{i,i+1}(x_{i,i+1}, y_{i,i+1})$, in which $y_{i,i+1} = \frac{(\alpha+x_{i,i+1})(-x_{i,i+1}^2+x_{i,i+1}-h)}{x_i}$. Next, we will discuss the type of these positive equilibria.

Primarily, the linear approximation of system (4) at any positive equilibrium $E(x, y)$ can be reduced to

$$J(E) = \begin{pmatrix} \frac{\alpha h - x^2(\alpha+2x-1)}{x(\alpha+x)} & -\frac{x}{\alpha+x} \\ \frac{\alpha\beta(-x^2+x-h)}{x(\alpha+x)} & \frac{x(\beta-\delta)-\alpha\delta}{\alpha+x} \end{pmatrix},$$

from which we can derive the determinant and trace of $J(E)$ as follows

$$\begin{aligned}\det(J(E)) &= \frac{-2(\beta-\delta)x^3+(\delta(\alpha-1)+\beta)x^2-\alpha\delta h}{x(\alpha+x)},\\ &= \frac{(\beta-\delta)(f(x)-xf'(x))}{x(\alpha+x)},\\ &= \frac{(\delta-\beta)f'(x)}{\alpha+x} \doteq m(x),\\ \mathrm{tr}(J(E)) &= \frac{-2x^3+(\beta+1-\alpha-\delta)x^2-\alpha\delta x+\alpha h}{x(\alpha+x)} \doteq n(x). \end{aligned} \tag{6}$$

Since $E(x, y)$ is any positive equilibrium of system (4) and the parameters α, δ and β are positive, we see that the sign of $\det(J(E))$ is the same as the sign of $(\delta-\beta)f'(x)$, which leads to the following conclusions.

Proposition 1 *For the simple positive equilibrium $E_*(x_*, y_*)$ of system (4), we have*
(i) *if $(\delta-\beta)f'(x_*) < 0$, then E_* is a saddle;*
(ii) *if $(\delta-\beta)f'(x_*) > 0$ and $tr(J(E_*)) < 0$, then E_* is a stable node or focus;*
(iii) *if $(\delta-\beta)f'(x_*) > 0$ and $tr(J(E_*)) > 0$, then E_* is an unstable node or focus.*

Remark 1 For the simple positive equilibria $E_i(x_i, y_i)$, $i = 1, 2, 3$ of system (4), we have that $f'(x_1)$, $f'(x_3) > 0$ and $f'(x_2) < 0$ from the analysis of the existence of positive roots for equation $f(x) = 0$ in the interval $(\bar{x}_1, \bar{x}_2)$. As a result, if $\delta > \beta$, then the equilibrium E_2 is a saddle; if $\delta < \beta$, then equilibria E_1 and E_3 are saddles.

3 Bifurcation Analysis

3.1 Saddle-Node Bifurcation, Cusp of Codimensions 2 and 3

Based on the existence of positive equilibria for system (4) and the relationship (6), we know that for the double roots $x_{i,i+1}$, $i = 1, 2$ of equation $f(x) = 0$, system (4) has corresponding degenerate positive equilibria $E_{i,i+1}(x_{i,i+1}, y_{i,i+1})$, which are marked as $E^*(x^*, y^*)$ for uniformity, where $y^* = \frac{(\alpha+x^*)(-x^{*2}+x^*-h)}{x^*}$. In this subsection, we discuss the type of $E^*(x^*, y^*)$. Actually, we have the following Theorem.

Theorem 1 *Assuming that $f(x^*) = f'(x^*) = 0$ and $f''(x^*) \neq 0$, where $\max\{0, \frac{1-\sqrt{1-4h}}{2}\} < x^* < \frac{1+\sqrt{1-4h}}{2}$, $h < \frac{1}{4}$, the following statements hold*
(I) *if $n(x^*) \neq 0$, then $E^*(x^*, y^*)$ is a saddle-node bifurcation point;*
(II) *if $n(x^*) = 0$ and $n'(x^*) \neq 0$, then $E^*(x^*, y^*)$ is a cusp of codimension 2;*
(III) *if $n(x^*) = n'(x^*) = 0$ and $\Phi \neq 0$, then $E^*(x^*, y^*)$ is a cusp of codimension 3.*
Here, $f(x)$ and $n(x)$ are defined in (5) and (6), respectively, and

$$\begin{aligned}\Phi = {} & (\delta-1)\delta^3+384x^{*8}+64(13\delta-12)x^{*7}-2(\delta-1)\delta^2(7\delta-2)x^*\\ & -16(\delta(3\delta+122)-36)x^{*6}-2\delta((\delta-1)\delta((\delta-33)\delta+23)+6)x^{*2}\\ & +16(\delta(\delta(17\delta-11)+111)-12)x^{*5}+4\delta(\delta(\delta(2(\delta-15)\delta+77)-48)+40)\\ & \times x^{*3}+4(\delta(\delta(\delta(13\delta-108)+83)-194)+6)x^{*4}. \end{aligned} \tag{7}$$

Proof (I) It is easy to derive $H=\frac{(2x^*-1)(\beta x^*-\delta(\alpha+x^*))^2}{\alpha\delta}$ and $h=\frac{x^{*2}(\beta-\delta+\delta(\alpha+2x^*)-2\beta x^*)}{\alpha\delta}$ from $f(x^*)=f'(x^*)=0$. Substituting them into system (4) and applying the change of variables $X=x-x^*$, $Y=y-y^*$ to transfer $E^*(x^*, y^*)$ to the origin, one obtains the Taylor expansion of system (4) around the origin as follows (for convenience, we still denote X, Y by x, y, respectively)

$$\begin{aligned}\frac{dx}{dt}&=\hat{a}_{10}x+\hat{a}_{01}y+\hat{a}_{20}x^2+\hat{a}_{11}xy+\hat{a}_{30}x^3+\hat{a}_{21}x^2y+o(|x, y|^3),\\ \frac{dy}{dt}&=\hat{b}_{10}x+\hat{b}_{01}y+\hat{b}_{20}x^2+\hat{b}_{11}xy+\hat{b}_{30}x^3+\hat{b}_{21}x^2y+o(|x, y|^3),\end{aligned}$$

where $\hat{a}_{ij}$ and $\hat{b}_{ij}$ are given in Appendix A.

Noting that $\hat{a}_{10}+\hat{b}_{01}=\frac{-2\beta x^{*2}+(\beta\delta+\beta-\delta^2)x^*-\alpha\delta^2}{\delta(\alpha+x^*)}=n(x^*)\neq 0$, let $X=\frac{\hat{b}_{01}x-\hat{a}_{01}y}{\hat{a}_{10}+\hat{b}_{01}}$, $Y=\frac{\hat{a}_{10}x+\hat{a}_{01}y}{\hat{a}_{10}+\hat{b}_{01}}$, then it follows that (for convenience, we still denote X, Y by x, y, respectively)

$$\begin{aligned}\frac{dx}{dt}&=\hat{c}_{20}x^2+\hat{c}_{11}xy+\hat{c}_{02}y^2+\hat{c}_{30}x^3+\hat{c}_{21}x^2y+\hat{c}_{12}xy^2+\hat{c}_{03}y^3+o(|x, y|^3),\\ \frac{dy}{dt}&=\hat{d}_{01}y+\hat{d}_{20}x^2+\hat{d}_{11}xy+\hat{d}_{02}y^2+\hat{d}_{30}x^3+\hat{d}_{21}x^2y+\hat{d}_{12}xy^2+\hat{d}_{03}y^3\\ &\quad+o(|x, y|^3),\end{aligned}\tag{8}$$

where $\hat{c}_{ij}$ and $\hat{d}_{ij}$ are given in Appendix A. Particularly, we have

$$\begin{aligned}\hat{c}_{20}&=\frac{\hat{a}_{10}(\hat{a}_{01}\hat{b}_{11}-\hat{a}_{11}\hat{b}_{01})+\hat{a}_{01}(\hat{a}_{20}\hat{b}_{01}-\hat{a}_{01}\hat{b}_{20})}{\hat{a}_{01}(\hat{a}_{10}+\hat{b}_{01})}=\frac{(\delta-\beta)f''(x^*)}{2\hat{d}_{01}(\alpha+x^*)}\neq 0,\\ \hat{d}_{01}&=\frac{\hat{a}_{10}(\hat{a}_{10}+\hat{b}_{01})+\hat{a}_{01}\hat{b}_{10}+\hat{b}_{01}^2}{\hat{a}_{10}+\hat{b}_{01}}=n(x^*)\neq 0,\end{aligned}$$

which indicates that there exists a center manifold

$$y=-\frac{\hat{d}_{20}x^2}{\hat{d}_{01}}x^2+o(x^2)$$

in a small neighborhood of the origin. Naturally, a simplified system on this center manifold can be obtained as follows

$$\frac{dx}{dt}=\hat{c}_{20}x^2+o(x^2).$$

According to Shan and Zhu [30], we can conclude that system (4) undergoes a saddle-node bifurcation around $E^*(x^*, y^*)$, which completes the proof of Theorem 1 (I);

(II) Next we verify the statement (II). By $f(x)=f'(x)=n(x)=0$, we have that $\beta=\frac{\delta^2(\alpha+x^*)}{x^*(\delta-2x^*+1)}$, $h=-\frac{x^*(\alpha\delta(x^*-1)+x^*(2x^*-1)(\alpha+2x^*-1))}{\alpha(\delta-2x^*+1)}$ and $H=\frac{\delta(\alpha+x^*)^2(2x^*-1)^3}{\alpha(\delta-2x^*+1)^2}$.

Substituting them into system (4) and implementing the following transformations successively

$$X = x - x^*, Y = y - y^*;$$
$$u = X, v = \frac{dX}{dt}, dt = (1 - a_{02}X)d\tau;$$
$$x = u, y = (1 - a_{02}u)v;$$

in which $a_{02} = \frac{\alpha}{x^*(\alpha+x^*)}$, system (4) can be written as follows (we still denote τ by t)

$$\frac{dx}{dt} = y,$$
$$\frac{dy}{dt} = b_{20}x^2 + b_{11}xy + o(|x, y|^2),$$

where

$$b_{20} = \frac{\delta(2x^* - 1)(\alpha\delta + x^*(\alpha + 3x^* - 1))}{x^*(\alpha + x^*)(\delta - 2x^* + 1)},$$
$$b_{11} = \frac{\alpha\delta^2 - \delta(\alpha + 2x^{*2}) + 2x^*(2x^* - 1)(\alpha + 3x^* - 1)}{x^*(\alpha + x^*)(\delta - 2x^* + 1)}.$$

Conditions $f''(x^*) \neq 0$ and $n'(x^*) \neq 0$ indicate that $b_{20}, b_{11} \neq 0$, which shows that $E^*(x^*, y^*)$ is a cusp of codimension 2.

(III) Finally, we show that $E^*(x^*, y^*)$ is a cusp of codimension 3. Conditions $f(x^*) = f'(x^*) = n(x^*) = n'(x^*) = 0$ lead to $\alpha = \frac{2x^*(x^*(\delta-6x^*+5)-1)}{(\delta-1)\delta+2x^*(2x^*-1)}$, $\beta = \frac{\delta^2(\delta+4x^*-2)}{(\delta-1)\delta+2x^*(2x^*-1)}$, $h = \frac{x^*(\delta+2x^*(x^*(3\delta+2x^*-1)-3\delta))}{2x^*(-\delta+6x^*-5)+2}$, $H = -\frac{\delta x^*(2x^*-1)^3(\delta+4x^*-2)^2}{2(x^*(-\delta+6x^*-5)+1)((\delta-1)\delta+2x^*(2x^*-1))}$. Similar to (I), using $X = x - x^*, Y = y - y^*$ to shift E^* to the origin and expanding the resulting system in a power series around the origin, we have (for convenience, in subsequent steps, we still denote X, Y and τ by x, y and t, respectively)

$$\frac{dx}{dt} = a^*_{10}x + a^*_{01}y + a^*_{20}x^2 + a^*_{11}xy + a^*_{30}x^3 + a^*_{21}x^2y + a^*_{40}x^4 + a^*_{31}x^3y$$
$$+o(|x, y|^4),$$
$$\frac{dy}{dt} = b^*_{10}x + b^*_{01}y + b^*_{20}x^2 + b^*_{11}xy + b^*_{30}x^3 + b^*_{21}x^2y + b^*_{40}x^4 + b^*_{31}x^3y$$
$$+o(|x, y|^4),$$

in which a^*_{ij} and b^*_{ij} are displayed in Appendix A.

By $X = x, Y = \frac{dx}{dt}$, the system above can be rewritten as

$$\begin{aligned}\frac{dx}{dt} &= y,\\ \frac{dy}{dt} &= c_{20}^{*}x^2 + c_{02}^{*}y^2 + c_{30}^{*}x^3 + c_{21}^{*}x^2y + c_{12}^{*}xy^2 + c_{40}^{*}x^4 + c_{31}^{*}x^3y + c_{22}^{*}x^2y^2\\ &\quad + o(|x, y|^4),\end{aligned}$$

where c_{ij}^{*} is displayed in Appendix A.

Letting $dt = (1 - c_{02}^{*}x)d\tau$, one obtains

$$\begin{aligned}\frac{dx}{dt} &= (1 - c_{02}^{*}x)y,\\ \frac{dy}{dt} &= (1 - c_{02}^{*}x)(c_{20}^{*}x^2 + c_{02}^{*}y^2 + c_{30}^{*}x^3 + c_{21}^{*}x^2y + c_{12}^{*}xy^2 + c_{40}^{*}x^4 + c_{31}^{*}x^3y\\ &\quad + c_{22}^{*}x^2y^2 + o(|x, y|^4)).\end{aligned} \tag{9}$$

Introducing $X = x$, $Y = (1 - c_{02}^{*}x)y$, system (9) becomes

$$\begin{aligned}\frac{dx}{dt} &= y,\\ \frac{dy}{dt} &= c_{20}^{*}x^2 + (c_{30}^{*} - 2c_{02}^{*}c_{20}^{*})x^3 + c_{21}^{*}x^2y + (c_{12}^{*} - c_{02}^{*2})xy^2\\ &\quad + (c_{40}^{*} - 2c_{02}^{*}c_{30}^{*} + c_{02}^{*2}c_{20}^{*})x^4`x + (c_{31}^{*} - c_{02}^{*}c_{21}^{*})x^3y\\ &\quad + (c_{22}^{*} - c_{02}^{*3})x^2y^2 + o(|x, y|^4).\end{aligned} \tag{10}$$

Condition $f''(x^*) \neq 0$ tells us $c_{20}^{*} \neq 0$. By carrying out the transformation $X = \pm x$, $Y = \pm\frac{y}{\sqrt{\pm c_{20}^{*}}}$, $\tau = \sqrt{\pm c_{20}^{*}}t$, one gets

$$\begin{aligned}\frac{dx}{dt} &= y,\\ \frac{dy}{dt} &= x^2 \pm \frac{c_{30}^{*} - 2c_{02}^{*}c_{20}^{*}}{c_{20}^{*}}x^3 + \frac{c_{21}^{*}}{\sqrt{\pm c_{20}^{*}}}x^2y + (c_{12}^{*} - c_{02}^{*2})xy^2\\ &\quad + \frac{c_{40}^{*} - 2c_{02}^{*}c_{30}^{*} + c_{02}^{*2}c_{20}^{*}}{c_{20}^{*}}x^4 \pm \left(\frac{c_{31}^{*} - c_{02}^{*}c_{21}^{*}}{\sqrt{\pm c_{20}^{*}}}\right)x^3y\\ &\quad \pm (c_{22}^{*} - c_{02}^{*3})x^2y^2 + o(|x, y|^4).\end{aligned} \tag{11}$$

According to the Proposition in Lemontagne et al. [31], we know that there exists a system equivalent to model (11), expressed as

$$\begin{aligned}\frac{dx}{dt} &= y,\\ \frac{dy}{dt} &= x^2 + Fx^3y + o(|x, y|^4),\end{aligned}$$

where

$$F = \frac{c_{21}^*(c_{02}^*c_{20}^* - c_{30}^*) + c_{20}^*c_{31}^*}{(\pm c_{20}^*)^{\frac{3}{2}}} = \frac{\delta\Phi}{(\pm c_{20}^*)^{\frac{3}{2}} x^{*4}(\delta - 2x^* + 1)^3(\delta + 4x^* - 2)^3},$$

with Φ is defined in (7). Condition $\Phi \neq 0$ shows that $E^*(x^*, y^*)$ is a cusp of codimension 3. The proof is completed. □

3.2 Bogdanov–Takens Bifurcation of Codimension 2

According to the formula (6), we know that the Jacobian matrix of system (4) at the positive equilibrium $E^*(x^*, y^*)$ has property $\det(J(E^*)) = \mathrm{tr}(J(E^*)) = 0$ if $f(x^*) = f'(x^*) = n(x^*)$=0, which means that Bogdanov–Takens bifurcation may occur around this equilibrium. Actually, we have the following Theorem.

Theorem 2 *Assume that* $f(x^*) = f'(x^*) = n(x^*) = 0$ *and* $f''(x^*), n'(x^*) \neq 0$, *where* $\max\{0, \frac{1-\sqrt{1-4h}}{2}\} < x^* < \frac{1+\sqrt{1-4h}}{2}$, $h < \frac{1}{4}$, $f(x)$ *and* $n(x)$ *are defined in* (5) *and* (6), *respectively. Further, selecting* h *and* H *as bifurcation parameters, system* (4) *undergoes a Bogdanov–Takens bifurcation of codimension 2 around* $E^*(x^*, y^*)$.

Proof Suppose the disturbed system is represented by

$$\begin{aligned}\frac{dx}{dt} &= x(1-x) - \frac{xy}{\alpha + x} - (h + \lambda_1),\\ \frac{dy}{dt} &= \frac{\beta xy}{\alpha + x} - \delta y - (H + \lambda_2),\end{aligned} \tag{12}$$

in which $\lambda = (\lambda_1, \lambda_2) \sim (0, 0)$, $\alpha, \beta, \delta > 0$, and h, H are any nonzero constants.

Same as Theorem 1 (II), one can easily have that $\beta = \frac{\delta^2(\alpha+x^*)}{x^*(\delta-2x^*+1)}$, $h = -\frac{x^*(\alpha\delta(x^*-1)+x^*(2x^*-1)(\alpha+2x^*-1))}{\alpha(\delta-2x^*+1)}$, $H = \frac{\delta(\alpha+x^*)^2(2x^*-1)^3}{\alpha(\delta-2x^*+1)^2}$ from $f(x) = f'(x) = n(x) = 0$. Using $X = x - x^*, Y = y - y^*$ to transform the positive equilibrium $E^*(x^*, y^*)$ of system (12) when $\lambda = 0$ to the origin and expanding the resulting system around the origin, we obtain (for briefness, in all of the following transformations, we still denote X, Y, τ as x, y, t, respectively)

$$\begin{aligned}\frac{dx}{dt} &= \bar{a}_{00} + \bar{a}_{10}x + \bar{a}_{01}y + \bar{a}_{20}x^2 + \bar{a}_{11}xy + o(|x, y, \lambda_1, \lambda_2|^2),\\ \frac{dy}{dt} &= \bar{b}_{00} + \bar{b}_{10}x + \bar{b}_{01}y + \bar{b}_{20}x^2 + \bar{b}_{11}xy + o(|x, y, \lambda_1, \lambda_2|^2),\end{aligned} \tag{13}$$

where $\bar{a}_{ij}, \bar{b}_{ij}$ and all coefficients in the transformations below are exhibited in Appendix B. It should be noted that $\bar{a}_{00}^* = \bar{b}_{00}^* = 0$ when $\lambda = 0$.

By nonsingular transformation $X = x$, $Y = \frac{dx}{dt}$, system (13) is changed into

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= \bar{c}_{00} + \bar{c}_{10}x + \bar{c}_{01}y + \bar{c}_{20}x^2 + \bar{c}_{11}xy + \bar{c}_{02}y^2 + o(|x, y, \lambda_1, \lambda_2|^2). \end{aligned} \tag{14}$$

Note that $\bar{c}_{00} = \bar{c}_{10} = \bar{c}_{01} = 0$ when $\lambda = 0$.

Further, by the transformation $dt = (1 - \bar{c}_{02}x)d\tau$, $X = x$, $Y = (1 - \bar{c}_{02}x)y$, we get an equivalent system

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= \bar{d}_{00} + \bar{d}_{10}x + \bar{d}_{01}y + \bar{d}_{20}x^2 + \bar{d}_{11}xy + o(|x, y, \lambda_1, \lambda_2|^2). \end{aligned} \tag{15}$$

It worth noting that $\bar{d}_{00} = \bar{d}_{10} = \bar{d}_{01} = 0$ when $\lambda = 0$.

With the change of variables $X = x + \frac{\bar{d}_{10}}{2\bar{d}_{20}}$, $Y = y$, system (15) becomes

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= \bar{e}_{00} + \bar{e}_{01}y + \bar{e}_{20}x^2 + \bar{e}_{11}xy + o(|x, y, \lambda_1, \lambda_2|^2). \end{aligned} \tag{16}$$

Noting that $\bar{e}_{00} = \bar{e}_{01} = 0$ when $\lambda = 0$.

Under the change of variables and time $X = \frac{\bar{e}_{11}^2}{\bar{e}_{20}}x$, $Y = \frac{\bar{e}_{11}^3}{\bar{e}_{20}^2}y$, $\tau = \frac{\bar{e}_{20}}{\bar{e}_{11}}t$, we obtain the universal unfolding of system (16) as follows

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= \mu_1 + \mu_2 y + x^2 + xy + o(|x, y, \lambda_1, \lambda_2|^2), \end{aligned} \tag{17}$$

where $\mu_1 = \frac{\bar{e}_{00}\bar{e}_{11}^4}{\bar{e}_{20}^3}$, $\mu_2 = \frac{\bar{e}_{01}\bar{e}_{11}}{\bar{e}_{20}}$ and we have $\mu_1 = \mu_2 = 0$ when $\lambda = 0$. Moreover, we obtain that

$$\left|\frac{\partial(\mu_1, \mu_2)}{\partial(\lambda_1, \lambda_2)}\right|_{\lambda=0} = -\frac{\alpha(-\alpha\delta - \alpha + 2x^*(2\alpha + x^*))B^5}{2\delta^3 x^*(\alpha + x^*)^3(2x^* - 1)^5(\delta - 2x^* + 1)A^5},$$

with $A = \alpha\delta + x^*(\alpha + 3x^* - 1)$, $B = \alpha\delta^2 - \delta(\alpha + 2x^{*2}) + 2x^*(2x^* - 1)(\alpha + 3x^* - 1)$. Taking $f''(x^*), n'(x^*) \neq 0$ into account, one has $A, B \neq 0$ and thus $\left|\frac{\partial(\mu_1,\mu_2)}{\partial(\lambda_1,\lambda_2)}\right|_{\lambda=0} \neq 0$. By Bogdanov [32, 33] and Takens [34], system (4) undergoes a Bogdanov–Takens bifurcation of codimension 2 around E^*. □

3.3 Degenerate Bogdanov–Takens Bifurcation of Codimension 3

In this subsection, we will further explore the degree of degradation of Bogdanov–Takens bifurcation. Before presenting the main result, we first recall the relevant definition [35] and property [36] to facilitate the understanding of derivation process.

Definition 1 The bifurcation that results from unfolding the following normal form of a cusp of codimension 3,

$$\frac{dx}{dt} = y, \\ \frac{dy}{dt} = x^2 \pm x^3 y, \tag{18}$$

is called a cusp type degenerate Bogdanov–Takens bifurcation of codimension 3.

Proposition 2 *A universal unfolding of the normal form* (18) *is expressed by*

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = \nu_1 + \nu_2 y + \nu_3 xy + x^2 \pm x^3 y + T(x, y, \varepsilon), \end{cases} \tag{19}$$

where $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \sim (0, 0, 0)$, $\frac{D(\nu_1,\nu_2,\nu_3)}{D(\varepsilon_1,\varepsilon_2,\varepsilon_3)} \neq 0$ *for small* ε *and*

$$T(x, y, \varepsilon) = y^2 O(|x, y|^2) + O(|x, y|^5) + O(\varepsilon)(O(y^2) + O(|x, y|^3)) \\ +O(\varepsilon^2)O(|x, y|). \tag{20}$$

The following Theorem is a crucial result of this paper.

Theorem 3 *Assume that* $f(x^*) = f'(x^*) = n(x^*) = n'(x^*) = 0$ *and* $f''(x^*), \Phi \neq 0$, *where* $\max\{0, \frac{1-\sqrt{1-4h}}{2}\} < x^* < \frac{1+\sqrt{1-4h}}{2}$, $h < \frac{1}{4}$, $f(x), n(x)$ *and* Φ *are given in* (5), (6) *and* (7), *respectively. Further, choosing* β, h *and* H *as bifurcation parameters and supposing that* $\Psi \doteq 24x^{*4} - 24x^{*3} + 2(\delta(9\delta - 8) + 3)x^{*2} - 2(\delta - 1)\delta(\delta + 5)x^* + (\delta - 1)\delta \neq 0$, *system* (4) *undergoes a degenerate Bogdanov–Takens bifurcation of codimension 3 around* $E^*(x^*, y^*)$.

Proof Perturbing β, h and H obtains the disturbed system as follows

$$\frac{dx}{dt} = x(1 - x) - \frac{xy}{\alpha + x} - (h + \epsilon_1), \\ \frac{dy}{dt} = \frac{(\beta + \epsilon_2)xy}{\alpha + x} - \delta y - (H + \epsilon_3), \tag{21}$$

in which $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3) \sim (0, 0, 0), \alpha, \beta, \delta > 0$ and h, H are arbitrary nonzero constant.

Same as Theorem 1 (III), conditions $f(x^*) = f'(x^*) = n(x^*) = n'(x^*) = 0$ result in $\alpha = \frac{2x^*(x^*(\delta-6x^*+5)-1)}{(\delta-1)\delta+2x^*(2x^*-1)}$, $\beta = \frac{\delta^2(\delta+4x^*-2)}{(\delta-1)\delta+2x^*(2x^*-1)}$, $h = \frac{x^*}{2x^*(-\delta+6x^*-5)+2}(\delta +$

$2x^*(x^*(3\delta+2x^*-1)-3\delta)), H = -\frac{\delta x^*(2x^*-1)^3(\delta+4x^*-2)^2}{2(x^*(-\delta+6x^*-5)+1)((\delta-1)\delta+2x^*(2x^*-1))}$. Substituting them into system (21) and by a linear coordinate change $X = x - x^*$, $Y = y - y^*$, we can get the Taylor expansion of system (21) around the origin as follows (for simplicity, in all of the following transformations, we still replace X, Y and τ with x, y and t, respectively)

$$\begin{aligned}\frac{dx}{dt} &= \bar{a}_{00}^* + \bar{a}_{10}^* x + \bar{a}_{01}^* y + \bar{a}_{20}^* x^2 + \bar{a}_{11}^* xy + \bar{a}_{30}^* x^3 + \bar{a}_{21}^* x^2 y + \bar{a}_{40}^* x^4 + \bar{a}_{31}^* x^3 y \\ &\quad + o(|x, y|^4), \\ \frac{dy}{dt} &= \bar{b}_{00}^* + \bar{b}_{10}^* x + \bar{b}_{01}^* y + \bar{b}_{20}^* x^2 + \bar{b}_{11}^* xy + \bar{b}_{30}^* x^3 + \bar{b}_{21}^* x^2 y + \bar{b}_{40}^* x^4 + \bar{b}_{31}^* x^3 y \\ &\quad + o(|x, y|^4),\end{aligned}$$

in which $\bar{a}_{ij}^*, \bar{b}_{ij}^*$ and all coefficients in the transformations below are shown in Appendix B. Note that $\bar{a}_{00}^* = \bar{b}_{00}^* = 0$ when $\epsilon = 0$.

Changing coordinates with

$$X = x, \ Y = \frac{dx}{dt},$$

we can rewrite the above system as follows

$$\begin{aligned}\frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= \bar{c}_{00}^* + \bar{c}_{10}^* x + \bar{c}_{01}^* y + \bar{c}_{20}^* x^2 + \bar{c}_{11}^* xy + \bar{c}_{02}^* y^2 + \bar{c}_{30}^* x^3 + \bar{c}_{21}^* x^2 y + \bar{c}_{12}^* xy^2 \\ &\quad + \bar{c}_{40}^* x^4 + \bar{c}_{31}^* x^3 y + \bar{c}_{22}^* x^2 y^2 + o(|x, y|^4).\end{aligned} \tag{22}$$

Notice that $\bar{c}_{00}^* = \bar{c}_{10}^* = \bar{c}_{01}^* = \bar{c}_{11}^* = 0$ when $\epsilon = 0$.

Next we will execute seven steps as in Li et al. [36] to transform system (22) into the universal unfolding form (19).

(I) Taking away the y^2-term from system (22). By making the scaling $x = X + \frac{\bar{c}_{02}^* X^2}{2}$, $y = Y + \bar{c}_{02}^* XY$, system (22) is transformed into

$$\begin{aligned}\frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= \bar{d}_{00}^* + \bar{d}_{10}^* x + \bar{d}_{01}^* y + \bar{d}_{20}^* x^2 + \bar{d}_{11}^* xy + \bar{d}_{30}^* x^3 + \bar{d}_{21}^* x^2 y + \bar{d}_{12}^* xy^2 + \bar{d}_{40}^* x^4 \\ &\quad + \bar{d}_{31}^* x^3 y + \bar{d}_{22}^* x^2 y^2 + o(|x, y|^4).\end{aligned} \tag{23}$$

Notice that $\bar{d}_{00}^* = \bar{d}_{10}^* = \bar{d}_{01}^* = \bar{d}_{11}^* = 0$ when $\epsilon = 0$.

(II) Taking away the xy^2-term from system (23). Making variable transformation $x = X + \frac{\bar{d}_{12}^*}{6} X^3$, $y = Y + \frac{\bar{d}_{12}^*}{2} X^2 Y$, we can obtain that

$$\begin{aligned}\frac{dx}{dt} &= y,\\ \frac{dy}{dt} &= \bar{e}^*_{00} + \bar{e}^*_{10}x + \bar{e}^*_{01}y + \bar{e}^*_{20}x^2 + \bar{e}^*_{11}xy + \bar{e}^*_{30}x^3 + \bar{e}^*_{21}x^2y + \bar{e}^*_{40}x^4 + \bar{e}^*_{31}x^3y\\ &\quad + \bar{e}^*_{22}x^2y^2 + o(|x, y|^4).\end{aligned} \tag{24}$$

Notice that $\bar{e}^*_{00} = \bar{e}^*_{10} = \bar{e}^*_{01} = \bar{e}^*_{11} = 0$ when $\epsilon = 0$.

(III) Taking away the x^2y^2-term from system (24). After a smooth coordinate change $x = X + \frac{\bar{e}^*_{22}}{12}X^4$, $y = Y + \frac{\bar{e}^*_{22}}{3}X^3Y$, system (24) can be rewritten as

$$\begin{aligned}\frac{dx}{dt} &= y,\\ \frac{dy}{dt} &= \bar{f}^*_{00} + \bar{f}^*_{10}x + \bar{f}^*_{01}y + \bar{f}^*_{20}x^2 + \bar{f}^*_{11}xy + \bar{f}^*_{30}x^3 + \bar{f}^*_{21}x^2y + \bar{f}^*_{40}x^4 + \bar{f}^*_{31}x^3y\\ &\quad + o(|x, y|^4).\end{aligned} \tag{25}$$

Notice that $\bar{f}^*_{00} = \bar{f}^*_{10} = \bar{f}^*_{01} = \bar{f}^*_{11} = 0$ when $\epsilon = 0$.

(IV) Taking away the x^3 and x^4-terms from system (25). We can easily get that $\overline{f}^*_{20} = \frac{\delta^2(x^*(10x^*-7)+1)}{x^*(\delta-2x^*+1)(\delta+4x^*-2)} + O(\epsilon)$, $\overline{f}^*_{20} \neq 0$ for small ϵ since $f''(x^*) \neq 0$. With

$$\begin{aligned}x &= X - \frac{\bar{f}^*_{30}}{4\bar{f}^*_{20}}X^2 + \frac{15\bar{f}^{*2}_{30} - 16\bar{f}^*_{20}\bar{f}^*_{40}}{80\bar{f}^{*2}_{20}}X^3,\\ y &= Y, \quad t = \left(1 - \frac{\bar{f}^*_{30}}{2\bar{f}^*_{20}}X + \frac{45\bar{f}^{*2}_{30} - 48\bar{f}^*_{20}\bar{f}^*_{40}}{80\bar{f}^{*2}_{20}}X^2\right)\tau,\end{aligned}$$

we have that

$$\begin{aligned}\frac{dx}{dt} &= y,\\ \frac{dy}{dt} &= \bar{g}^*_{00} + \bar{g}^*_{10}x + \bar{g}^*_{01}y + \bar{g}^*_{20}x^2 + \bar{g}^*_{11}xy + \bar{g}^*_{30}x^3 + \bar{g}^*_{21}x^2y + \bar{g}^*_{40}x^4 + \bar{g}^*_{31}x^3y\\ &\quad + o(|x, y|^4).\end{aligned} \tag{26}$$

Notice that $\bar{g}^*_{00} = \bar{g}^*_{10} = \bar{g}^*_{01} = \bar{g}^*_{11} = \bar{g}^*_{30} = \bar{g}^*_{40} = 0$ when $\epsilon = 0$.

(V) Taking away the x^2y-term from system (26). It's easy to know that $\bar{g}^*_{20} = \frac{\delta^2(x^*(10x^*-7)+1)}{x^*(\delta-2x^*+1)(\delta+4x^*-2)} + O(\epsilon)$, $\bar{g}^*_{20} \neq 0$ for small ϵ since $f''(x^*) \neq 0$. Introducing the new coordinates and time by

$$x = X, y = Y + \frac{\bar{g}^*_{21}}{3\bar{g}^*_{20}}Y^2 + \frac{\bar{g}^{*2}_{21}}{36\bar{g}^{*2}_{20}}Y^3, \tau = \left(1 + \frac{\bar{g}^*_{21}}{3\bar{g}^*_{20}}Y + \frac{\bar{g}^{*2}_{21}}{36\bar{g}^{*2}_{20}}Y^2\right)t,$$

we obtain

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= \bar{h}^*_{00} + \bar{h}^*_{10}x + \bar{h}^*_{01}y + \bar{h}^*_{20}x^2 + \bar{h}^*_{11}xy + \bar{h}^*_{31}x^3y + T_1(x, y, \epsilon). \end{aligned} \tag{27}$$

Notice that $\bar{h}^*_{00} = \bar{h}^*_{10} = \bar{h}^*_{01} = \bar{h}^*_{11} = 0$ when $\epsilon = 0$ and $T_1(x, y, \epsilon)$ possesses the property of (20).

(VI) Normalizing $\bar{h}^*_{20}$ and $\bar{h}^*_{31}$ to 1 in system (27). A simple calculation shows that $\bar{h}^*_{20}, \bar{h}^*_{31} \neq 0$ for small ϵ due to $f''(x^*), \Phi \neq 0$. Using the following rescaling transformation

$$x = \bar{h}^{*\frac{1}{5}}_{20}\bar{h}^{*-\frac{2}{5}}_{31}X, \quad y = \bar{h}^{*\frac{4}{5}}_{20}\bar{h}^{*-\frac{3}{5}}_{31}Y, \quad t = \bar{h}^{*-\frac{3}{5}}_{20}\bar{h}^{*\frac{1}{5}}_{31}\tau,$$

we have

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= \bar{j}^*_{00} + \bar{j}^*_{10}x + \bar{j}^*_{01}y + \bar{j}^*_{11}xy + x^2 + x^3y + T_2(x, y, \epsilon). \end{aligned} \tag{28}$$

Notice that $\bar{j}^*_{00} = \bar{j}^*_{10} = \bar{j}^*_{01} = \bar{j}^*_{11} = 0$ when $\epsilon = 0$ and $T_2(x, y, \epsilon)$ possesses the property of (20).

(VII) Taking away the h^*_{10}-term from system (28). Using

$$x = X - \frac{\bar{j}^*_{10}}{2}, \quad y = Y,$$

we eventually get

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= \bar{\nu}_1 + \bar{\nu}_2 y + \bar{\nu}_3 xy + x^2 + x^3y + T_3(x, y, \epsilon), \end{aligned} \tag{29}$$

in which $\bar{\nu}_1 = \bar{j}^*_{00} - \frac{\bar{j}^{*2}_{10}}{4}$, $\bar{\nu}_2 = \bar{j}^*_{01} - \frac{\bar{j}^*_{10}(\bar{j}^{*2}_{10} + 4\bar{j}^*_{11})}{8}$, $\bar{\nu}_3 = \bar{j}^*_{11} + \frac{3\bar{j}^{*2}_{10}}{4}$. Noticing that $\bar{\nu}_1 = \bar{\nu}_2 = \bar{\nu}_3 = 0$ when $\epsilon = 0$, system (29) is exactly the form of system (19) and the item $T_3(x, y, \epsilon)$ possesses the property of (20). Moreover, we can calculate that

$$\left|\frac{\partial(\bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3)}{\partial(\epsilon_1, \epsilon_2, \epsilon_3)}\right|_{\epsilon=0} = -\bar{h}^{*\frac{4}{5}}_{31}\bar{h}^{*-\frac{12}{5}}_{20}\Big|_{\epsilon=0}\frac{\Psi}{\delta^2 x^{*2} C} \neq 0,$$

where $\Psi = 24x^{*4} - 24x^{*3} + 2(\delta(9\delta - 8) + 3)x^{*2} - 2(\delta - 1)\delta(\delta + 5)x^* + (\delta - 1)\delta$ and $C = (2(\gamma + 1)x^* - 1)(5(\gamma + 1)x^* - 1)(\delta - 2(\gamma + 1)x^* + 1)^3(\delta + 4(\gamma + 1)x^* - 2)^3$. By the result of Li et al. [36], we know that system (29) is the versal unfolding of the Bogdanov–Takens singularity (cusp case) of codimension three. Hence, system (4) undergoes a degenerate Bogdanov–Takens bifurcation of codimension 3 around $E^*(x^*, y^*)$, which completes the proof. □

3.4 Degenerate Hopf Bifurcation of Codimension 2

In accordance with the existence analysis of positive equilibria, we can conclude that system (4) may undergo a Hopf bifurcation around the positive equilibria $E_i(x_i, y_i)$ $(i = 1, 3)$ in the case of $\delta > \beta$, or around the equilibrium $E_2(x_2, y_2)$ in the case of $\delta < \beta$. These equilibria are marked as $\tilde{E}(\tilde{x}, \tilde{y})$ for consistency in signs. In this subsection, we devote ourselves to exploring the degree of degradation of Hopf bifurcation around $\tilde{E}$. Firstly, we state the following Theorem.

Theorem 4 *Assuming that $f(\tilde{x}) = n(\tilde{x}) = 0$ and $f'(\tilde{x}) \neq 0$, where $\max\{0, \frac{1-\sqrt{1-4h}}{2}\} < \tilde{x} < \frac{1+\sqrt{1-4h}}{2}$, $h < \frac{1}{4}$, $f(x)$ and $n(x)$ are given in (5) and (6), respectively, we have*
(I) *if $\sigma_{11} < 0$, then $\tilde{E}(\tilde{x}, \tilde{y})$ is a stable weak focus with multiplicity one and one stable limit cycle bifurcates from $\bar{E}$ by the supercritical Hopf bifurcation;*
(II) *if $\sigma_{11} > 0$, then $\tilde{E}(\tilde{x}, \tilde{y})$ is an unstable weak focus with multiplicity one and one unstable limit cycle bifurcates from $\tilde{E}$ by the subcritical Hopf bifurcation;*
(III) *if $\sigma_{11} = 0$, then $\tilde{E}(\tilde{x}, \tilde{y})$ is a weak focus with multiplicity at least two and system (4) may exhibit a degenerate Hopf bifurcation of codimension at least 2, where*

$$\sigma_{11} = \sigma_{11}^0 + \sigma_{11}^1\beta + \sigma_{11}^2\beta^2, \tag{30}$$

with

$$\begin{aligned}\sigma_{11}^0 &= (\alpha h - \tilde{x}^2(\alpha + 2\tilde{x} - 1))(-\alpha^2 h(h - 2(\alpha - 1)\alpha) + \alpha\tilde{x}^3(-\alpha + 10h + 1) \\ &\quad + 3\alpha(6\alpha - 1)h\tilde{x}^2 + 3\alpha^2(4\alpha - 1)h\tilde{x} + 2\tilde{x}^6 + 6\alpha\tilde{x}^5 + 3(\alpha - 1)\alpha\tilde{x}^4), \\ \sigma_{11}^1 &= \alpha(-\alpha^3 h^2 + \tilde{x}^5(-2\alpha^2 - \alpha + 2h + 1) - \tilde{x}^4(\alpha^3 + \alpha + 2\alpha h) + 4\alpha h\tilde{x}^3 \\ &\quad + \alpha h\tilde{x}^2(\alpha - h) + \alpha^3 h\tilde{x} + 4\tilde{x}^7 + (\alpha - 5)\tilde{x}^6), \\ \sigma_{11}^2 &= \alpha^2\tilde{x}^2(\tilde{x}^2(\alpha + 2\tilde{x} - 1) - \alpha h).\end{aligned} \tag{31}$$

Proof Based on the choice of $\tilde{E}(\tilde{x}, \tilde{y})$, it is easy to see that $\det(J(\tilde{E})) > 0$. Additionally, we have $\mathrm{tr}(J(\tilde{E})) = n(\tilde{x}) = 0$. To investigate the degree of degradation of Hopf bifurcation in system (4), we first define a new time scale $dt = (\alpha + x)d\tau$, which leads to the polynomial system below (we still refer to τ as t)

$$\begin{aligned}\frac{dx}{dt} &= x(1 - x)(\alpha + x) - xy - h(\alpha + x), \\ \frac{dy}{dt} &= -\delta y(\alpha + x) + \beta xy - H(\alpha + x),\end{aligned} \tag{32}$$

in which $\delta = \frac{\alpha h+\tilde{x}^2(-\alpha+\beta-2\tilde{x}+1)}{\tilde{x}(\alpha+\tilde{x})}$ and $H = \frac{(h+(\tilde{x}-1)\tilde{x})(\alpha h-\tilde{x}^2(\alpha+2\tilde{x}-1))}{\tilde{x}^2}$ are obtained by the conditions $f(\tilde{x}) = n(\tilde{x}) = 0$. By performing the linear change $X = x - \tilde{x}$, $Y = y - \tilde{y}$, we shift $\tilde{E}$ to the origin and expand the generated system in power series, then system (32) becomes (we still write X, Y as x, y, respectively)

$$\begin{aligned}\frac{dx}{dt} &= \tilde{a}_{10}x + \tilde{a}_{01}y + \tilde{a}_{20}x^2 + \tilde{a}_{11}xy + \tilde{a}_{30}x^3,\\ \frac{dy}{dt} &= \tilde{b}_{10}x - \tilde{a}_{10}y + \tilde{b}_{11}xy,\end{aligned} \tag{33}$$

where

$$\begin{aligned}\tilde{a}_{10} &= \frac{\alpha h - \tilde{x}^2(\alpha + 2\tilde{x} - 1)}{\bar{x}},\ \tilde{a}_{01} = -\tilde{x},\ \tilde{a}_{20} = -\alpha - 3\tilde{x} + 1,\ \tilde{a}_{11} = \tilde{a}_{30} = -1,\\ \tilde{b}_{10} &= -\frac{\alpha\beta(h + (\tilde{x} - 1)\tilde{x})}{\tilde{x}},\ \tilde{b}_{11} = \frac{\tilde{x}(\alpha\beta + \tilde{x}(\alpha + 2\tilde{x} - 1)) - \alpha h}{\tilde{x}(\alpha + \tilde{x})}.\end{aligned}$$

We can deduce that $-\tilde{a}_{01}\tilde{b}_{10} - \tilde{a}_{10}^2 > 0$ since $\det(J(\tilde{E})) > 0$. Letting $\omega = \sqrt{-\tilde{a}_{01}\tilde{b}_{10} - \tilde{a}_{10}^2}$ and introducing the transformation $x = -\tilde{a}_{01}X$, $y = \tilde{a}_{10}X - \omega Y$, $dt = \frac{1}{\omega}d\tau$, system (32) can be transformed into (we rename X, Y and τ as x, y and t, respectively)

$$\begin{aligned}\frac{dx}{dt} &= y + \tilde{c}_{20}x^2 + \tilde{c}_{11}xy + \tilde{c}_{30}x^3,\\ \frac{dy}{dt} &= -x + \tilde{d}_{20}x^2 + \tilde{d}_{11}xy + \tilde{d}_{30}x^3,\end{aligned}$$

where

$$\begin{aligned}\tilde{c}_{20} &= \frac{\tilde{a}_{10}\tilde{a}_{11} - \tilde{a}_{01}\tilde{a}_{20}}{\omega},\ \tilde{c}_{11} = -\tilde{a}_{11},\ \tilde{c}_{30} = \frac{\tilde{a}_{01}^2\tilde{a}_{30}}{\omega},\\ \tilde{d}_{20} &= \frac{\tilde{a}_{10}(\tilde{a}_{01}(\tilde{b}_{11} - \tilde{a}_{20}) + \tilde{a}_{10}\tilde{a}_{11})}{\omega^2},\ \tilde{d}_{11} = -\frac{\tilde{a}_{01}\tilde{b}_{11} + \tilde{a}_{10}\tilde{a}_{11}}{\omega},\ \tilde{d}_{30} = \frac{\tilde{a}_{01}^2\tilde{a}_{10}\tilde{a}_{30}}{\omega^2}.\end{aligned}$$

Utilizing the formula in Perko [35] yields the first Lyapunov coefficient as follows

$$\sigma_1 = \frac{\sigma_{11}}{8\omega^3\tilde{x}(\alpha + \tilde{x})^2},$$

where σ_{11} is defined in (30) and the sign of σ_1 is the same as that of σ_{11}. This completes the proof. □

From the third scenario of Theorem 4, we know that system (4) may undergo a degenerate Hopf bifurcation around $\tilde{E}(\tilde{x}, \tilde{y})$ when $\sigma_{11} = 0$, i.e., $\beta = \beta_{\pm} \doteq$

$\frac{-\sigma_{11}^1 \pm \sqrt{(\sigma_{11}^1)^2 - 4\sigma_{11}^2\sigma_{11}^0}}{2\sigma_{11}^2}$, with σ_{11}^j, $j = 0, 1, 2$ being defined in (31). Further, the second Lyapunov coefficient can be calculated as follows

$$\sigma_2 = \frac{\sigma_{22}}{288\omega^7\tilde{x}^7(\alpha + \tilde{x})^4},$$

where σ_{22} is displayed in Appendix C. We can immediately receive the following results.

Theorem 5 *Assuming that* $f(\tilde{x}) = n(\tilde{x}) = 0$, $f'(\bar{x}) \neq 0$ *and* $\beta = \beta_\pm \doteq \frac{-\sigma_{11}^1 \pm \sqrt{(\sigma_{11}^1)^2 - 4\sigma_{11}^2\sigma_{11}^0}}{2\sigma_{11}^2}$, *where* $\max\{0, \frac{1-\sqrt{1-4h}}{2}\} < \tilde{x} < \frac{1+\sqrt{1-4h}}{2}$, $h < \frac{1}{4}$, $f(x), n(x)$ *and* σ_{11}^j, $j = 0, 1, 2$ *are given in* (5), (6) *and* (31), *respectively, we have*
(I) *if* $\sigma_{22} < 0$, *then* $\bar{E}(\tilde{x}, \tilde{y})$ *is a stable weak focus with multiplicity 2. System* (4) *undergoes a degenerate Hopf bifurcation of codimension 2 and there can be up to two limit cycles bifurcating from* $\tilde{E}$, *the outer one being stable;*
(II) *if* $\sigma_{22} > 0$, *then* $\bar{E}(\tilde{x}, \tilde{y})$ *is an unstable weak focus with multiplicity 2. System* (4) *undergoes a degenerate Hopf bifurcation of codimension 2 and there can be up to two limit cycles bifurcating from* $\tilde{E}$, *the outer one being unstable;*
(III) *if* $\sigma_{22} = 0$, *then* $\bar{E}(\tilde{x}, \tilde{y})$ *is a weak focus with multiplicity at least 3 and system* (4) *may undergo a degenerate Hopf bifurcation of codimension at least 3.*

4 Numerical Simulations

In this section, we will carry out numerical simulations to verify theoretical results and unveil the detailed transitions among different parameter regions by using the ODE software in Doedel et al. [37].

4.1 β as the Primary Bifurcation Parameter

Firstly, choose the initial parameter values as follows

$$\alpha = 1.1, \ \beta = 2.62, \ \delta = 0.43, \ h = -0.0004, \ H = 0.034, \tag{34}$$

and select parameter β as the primary bifurcation parameter, while the remaining parameter values are fixed. We observe two supercritical Hopf bifurcation points: $HB_1(1.07459 \times 10^{-1}, 1.08220)$ at $\beta = 5.18471$ and $HB_2(1.38684 \times 10^{-2}, 1.13055)$ at $\beta = 3.69519 \times 10^1$. Additionally, there are one saddle-node bifurcation point $SN(7.94669 \times 10^{-1}, 3.89989 \times 10^{-1})$ of equilibrium at $\beta = 1.23308$, and two saddle-node bifurcation points of limit cycles: $SNC_1(5.13032 \times 10^{-1}, 3.47922)$ at $\beta = 5.71963$ with a period of 1.30984×10^1, and $SNC_2(7.99950 \times 10^{-1}, 6.50564)$ at $\beta = 8.11771$ with a period of 1.58827×10^1 on their own limit cycle bifurcation curves (green curve and red curve), respectively. Both families of limit cycles approach their respective homoclinic cycles. See Fig. 1a, b for details, where the solid and dotted

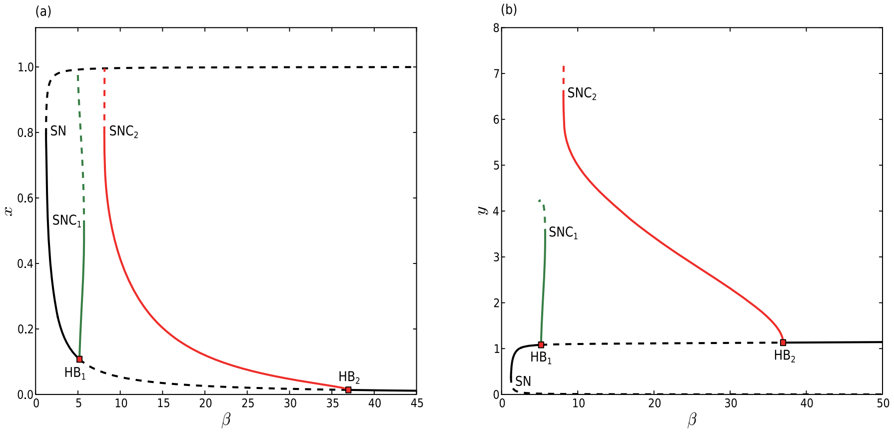

Fig. 1 One-parameter bifurcation diagram of system (4) with respect to β. **a** β versus x; **b** β versus y

curves represent the branches of stable and unstable solutions for equilibrium or limit cycle, respectively.

4.1.1 *β* and *h* as the Primary Bifurcation Parameters

Next, using β and h as the primary bifurcation parameters, and the first set of parameter values in (34), we construct a two-parameter bifurcation diagram. This diagram includes the supercritical Hopf bifurcation curve H_s (red), saddle-node bifurcation curve SN (blue), homoclinic bifurcation curve Hom (green), and the saddle-node bifurcation curve of limit cycles SNL (black), see Fig. 2 for details. We observe one Bogdanov–Takens (BT) bifurcation point $BT(5.69222 \times 10^{-1}, 1.66517 \times 10^{-1})$ at $\beta = 1.85972, h = 1.88424 \times 10^{-1}$. Additionally, there is one codimension-2 cusp of limit cycles $CPL(6.60981 \times 10^{-1}, 1.48855)$ at $\beta = 2.39576, h = 2.88069 \times 10^{-2}$, with a period of 1.52956×10^1.

The saddle-node bifurcation curve of the limit cycles SNL is presented separately in Fig. 3. It's worth noting that there is a saddle-node point $SNLC(0.6816, 4.667)$ when $\beta = 6.44311, h = -0.0004778$. In this case, we can observe that the cusp of the limit cycles arises from the transition of SNC_1 rather than from the transition of SNC_2 as $h = -0.0004$, where SNC_1 and SNC_2 are shown in Fig. 1. This implies that three limit cycles will bifurcate from the Hopf bifurcation point HB_1 as the parameter h or β vary.

The whole bifurcation diagram is divided into eight regions: I–VIII. The corresponding phase portraits are described in Table 3 and given by Fig. 4 (I–VIII).

4.2 *h* as the Primary Bifurcation Parameter

Now, taking the initial parameter values as follows

$$\alpha = 1, \ \beta = 2.62, \ \delta = 0.43, \ h = 0.01, \ H = 0.034, \tag{35}$$

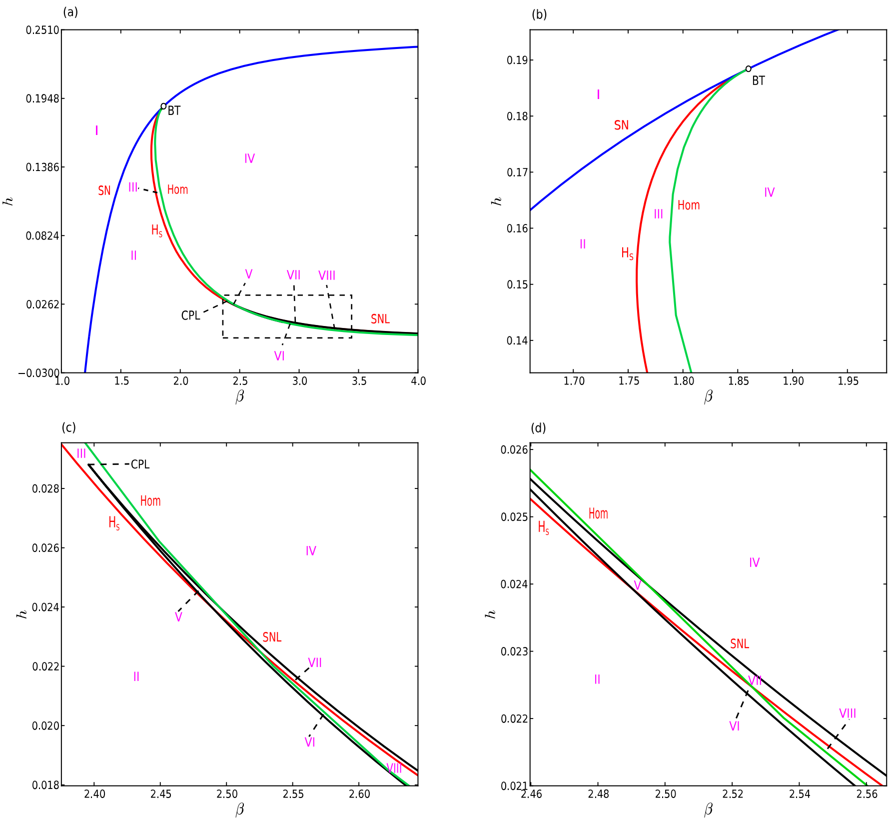

Fig. 2 Two-parameter bifurcation diagram of system (4) with respect to β and h. **a** β versus h; **b** The first zoomed diagram of (**a**); **c** The second zoomed diagram of (**a**); **d** The third zoomed diagram of (**a**). Here H_s, SN, Hom, SNL, CPL, BT denote Hopf bifurcation curve, saddle-node bifurcation curve, homoclinic bifurcation curve, saddle-node bifurcation curve of limit cycles, cusp of limit cycles, and Bogdanov–Takens bifurcation point, respectively

and selecting parameter h as the primary bifurcation parameter, with the remaining parameter values fixed as given in (35), we observe the following bifurcation points: one supercritical Hopf bifurcation point $HB(2.17493 \times 10^{-1}, 8.93871 \times 10^{-1})$ at $h = 1.05086 \times 10^{-2}$; one saddle-node bifurcation point $SN(5.15086 \times 10^{-1}, 7.37967 \times 10^{-2})$ at $h = 2.24684 \times 10^{-1}$; two saddle-node bifurcation points of limit cycles $SNC_1(6.36188 \times 10^{-1}, 1.73750)$ at $h = 1.15925 \times 10^{-2}$ with a period of 1.40662×10^1 and $SNC_2(8.96509 \times 10^{-1}, 1.93445)$ at $h = 1.14295 \times 10^{-2}$ with a period of 1.80842×10^1. Finally, we find that a family of limit cycles approaches a homoclinic cycle. There exists a bistability region $1.14295 \times 10^{-2} < h < 1.14467 \times 10^{-2}$, which indicates the presence of three coexistent limit cycles, with the innermost and the outermost limit cycles being stable, while the middle one being unstable. See Fig. 5a–c for details.

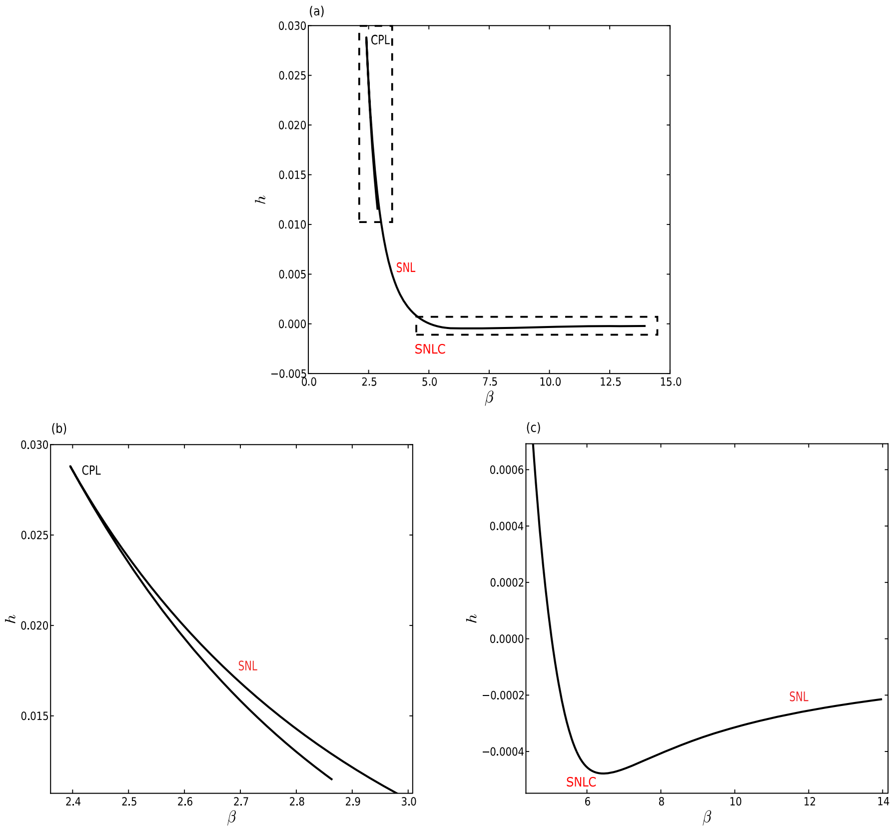

Fig. 3 **a** The saddle-node bifurcation curve of limit cycles SNL of system (4) with respect to β and h; **b** Zoomed diagram of (**a**); **c** Zoomed diagram of (**a**). Here $SNLC$ and CPL denote the saddle-node bifurcation point of limit cycles and cusp of limit cycles, respectively

Table 3 The distribution of phase portraits when β and h as the primary bifurcation parameters

Regions	Existence of equilibria	Steady state
I	No equilibrium	–
II	A saddle and a stable focus	Monostabillity
III	A saddle and a stable limit cycle contains an unstable focus	Monostabillity
IV	A saddle and an unstable focus	–
V	Saddle and three limit cycles (a big stable limit cycle contains A middle unstable limit cycle enclosing a small stable limit cycle) Contain an unstable focus	Bistability
VI	A saddle and a big stable limit cycle contains a small unstable limit Cycle enclosing a stable hyperbolic positive equilibrium	Bistability
VII	A saddle and a big unstable limit cycle contains a small Stable limit cycle enclosing an unstable focus	Monostabillity
VIII	A Saddle and an unstable limit cycle contains a stable focus	Monostabillity

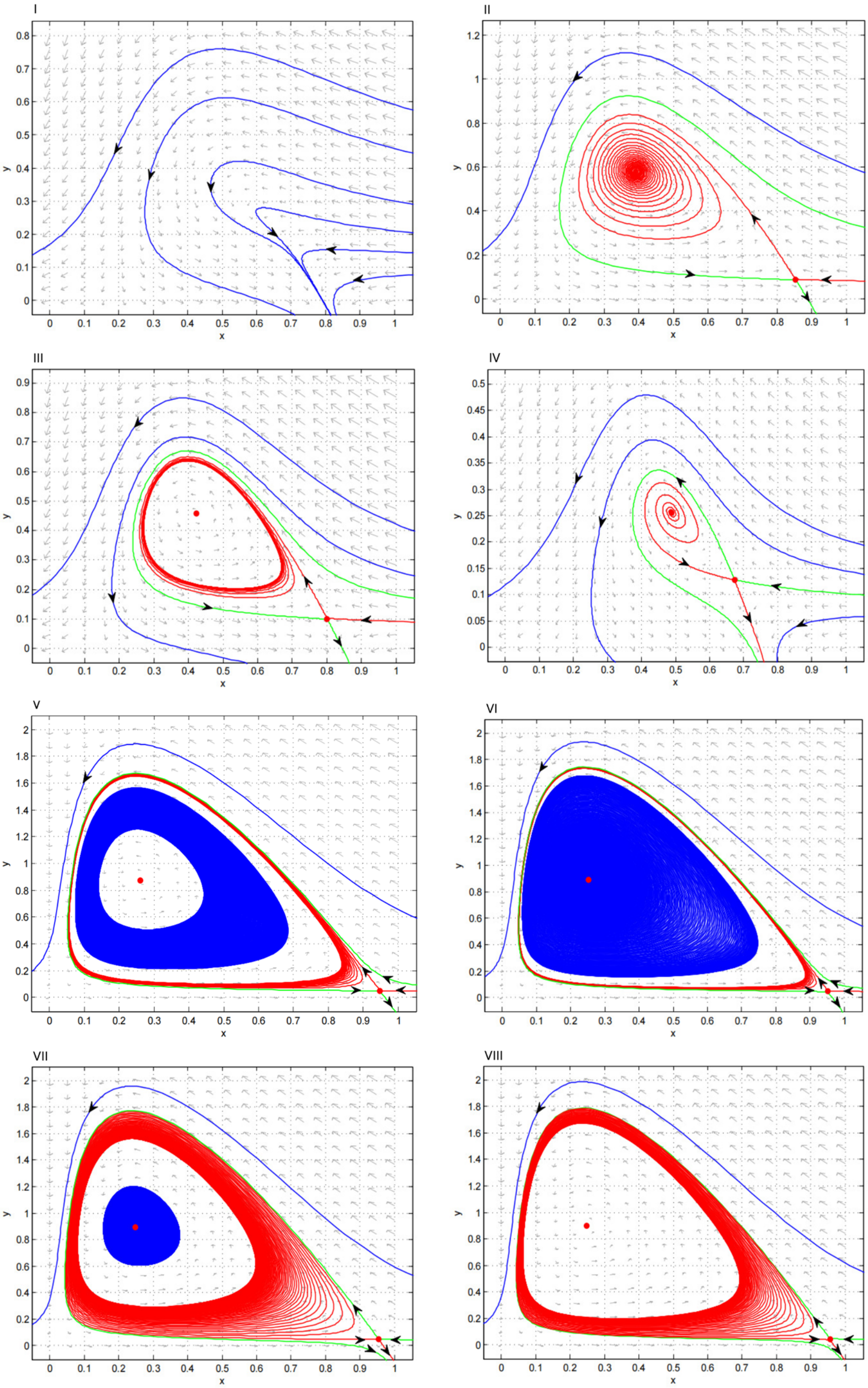

Fig. 4 Phase portraits of eight regions: I–VIII in Fig. 2

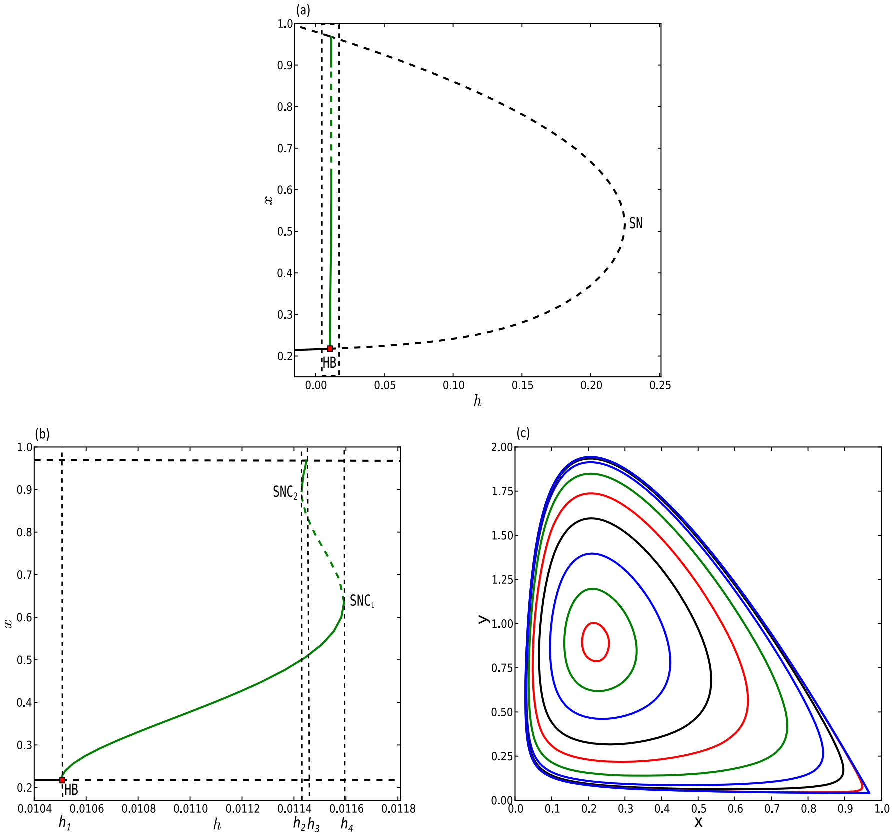

Fig. 5 One-parameter bifurcation diagram of system (4) with respect to h. **a** h versus x; **b** Zoomed diagram of (**a**). **c** A family of limit cycles approaches a homoclinic cycle

4.2.1 *h* and *H* as the Primary Bifurcation Parameters

Finally, considering h and H as the primary bifurcation parameters, we construct a two-parameter bifurcation diagram, which includes both supercritical H_s and subcritical H_u Hopf bifurcation curves (red), saddle-node bifurcation curve SN (blue), homoclinic bifurcation curves Hom_1 and Hom_2 (green), and saddle-node bifurcation curve of limit cycles SNL (black), see Fig. 6 for details. Within this diagram, we identify the following bifurcation points: two Bogdanov–Takens bifurcation points $BT_1(5.66449\times10^{-2}, 6.66732\times10^{-1})$ at $h = 1.76939\times10^{-2}$, $H = -1.93050\times10^{-1}$ and $BT_2(6.23087\times10^{-1}, 8.68350\times10^{-1})$ at $h = -9.85011\times10^{-2}$, $H = 4.99989\times10^{-1}$; one cusp point $CP(3.98346 \times 10^{-1}, -2.92479 \times 10^{-1})$ at $h = 3.22985 \times 10^{-1}$, $H = -9.25278 \times 10^{-2}$; and a codimension-2 cusp point of limit cycles $CPL(7.82190 \times 10^{-1}, 1.86995)$ on the curve SNL at $h = 1.12393 \times 10^{-2}$, $H = 3.88902 \times 10^{-2}$, with a period of 1.55612×10^{1}, which indicates that there exists an acute parameter region of three coexistent limit cycles as $h > 0$ and $H > 0$. The branch SNL also tells us that there is one triangle parameter region consists of gener-

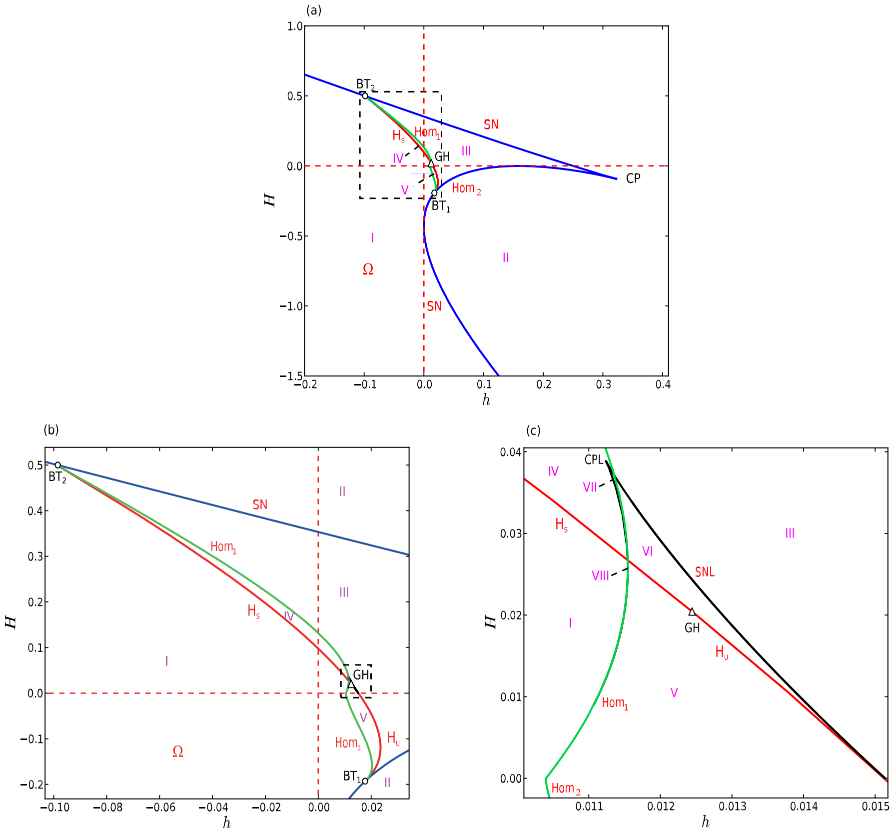

Fig. 6 Two-parameter bifurcation diagram of system (4) with respect to h and H. **a** h versus H; **b** First zoomed part in (**a**). **c** Second zoomed part in (**b**). Here $SN, H_i(i = s, u), Hom_i(i = 1, 2), SNL, CPL, BT_i(i = 1, 2), GH, \Omega$ denote the saddle-node bifurcation curve, supercritical (or subcritical) Hopf bifurcation curve, homoclinic cycle bifurcation curve, saddle-node bifurcation curve of limit cycle, cusp of limit cycle, Bogdanov–Takens bifurcation point, and degenerate Hopf bifurcation point, the double stocking region, respectively

alized Hopf bifurcation point GH, Hopf bifurcation curve H, degenerate homolinic bifurcation point, must have at least two limit cycles.

Note that, the homoclinic cycle curves Hom_1 and Hom_2 bifurcate from two BT bifurcation points BT_1 and BT_2, respectively. However, homolcinic cycles on both branches approach the same heteroclinic cycle connecting two boundary equilibria $E_0^1(\frac{1-\sqrt{1-4h}}{2}, 0)$ and $E_0^2(\frac{1+\sqrt{1-4h}}{2}, 0)$ as $H = 0$. For a more detaild visualization, please refer to Fig. 6b.

The entire bifurcation diagram is divided into eight regions: I-VIII. The corresponding phase portraits are described in Table 4 and given in Fig. 7. Here, Ω is included in the region I representing the region of double stocking for both species. It is crucial to exercise caution in selecting suitable initial conditions to prevent the collapse of the entire system.

Table 4 The distribution of phase portraits when h and H as the primary bifurcation parameters

Regions	Existence of equilibria	Steady state
I	A saddle and a stable focus	Monostabillity
II	No equilibrium	–
III	A saddle and a stable focus	Monostabillity
IV	A saddle and a stable limit cycle contains an unstable focus	Monostabillity
V	A saddle and an unstable limit cycle contains a stable focus	Monostabillity
VI	A saddle and a big unstable limit cycle contains a small Stable limit cycle enclosing an unstable focus	Monostabillity
VII	A saddle and three limit cycles (a big stable limit cycle Contains a middle unstable limit cycle enclosing a small stable limit cycle) Contain an unstable focus	Bistability
VIII	A saddle and a homoclinic cycle contains An unstable limit cycle enclosing a stable focus	Monostabillity

Remark 2 It is evident that system (4) exhibits two boundary equilibria, namely $E_0^1(\frac{1-\sqrt{1-4h}}{2}, 0)$ and $E_0^2(\frac{1+\sqrt{1-4h}}{2}, 0)$, if the special case $H = 0$ holds. The homoclinic cycle curve is discontinuous, presenting a new and interesting phenomenon. This is in stark contrast to many established predator–prey models, where both Bagdanov-Takens bifurcation points are consistently linked by an uninterrupted homoclinic cycle curve [12, 38].

However, all the homoclinic cycles bifurcating from BT_1 and BT_2 converge towards a heteroclinic cycle that connects the two boundary equilibria, E_0^1 and E_0^2. The significant distinction lies in the fact that homoclinic cycles originating from BT_2 converge towards equilibria approaching E_0^1, while those originating from BT_1 converge towards equilibria approaching E_0^2. In other words, the homoclinic cycles emerge from a perturbation of a heteroclinic cycle.

5 Biological Interpretations

In this section, we provide latent biological interpretations for the related bifurcation diagrams and their corresponding phase portraits.

Firstly, we give a detailed explanation of Fig. 5 from a biological perspective. The solid and dotted curves represent the stable and unstable branches of equilibria or limit cycles, respectively. We obtain four important parameter values h_1, h_2, h_3, h_4 that divide the horizontal axis into five subintervals. For $h < h_1$, system (4) possesses two equilibria, one stable and the other unstable, which implies that predators and prey can achieve a state of coexistence by selecting the right initial condition within this region; For $h_1 < h < h_2$ and $h_3 < h < h_4$, system (4) has a stable limit cycle, showing that predators and prey can be kept in a sustained oscillation state, further, the amplitude of the prey oscillation increases as h increases; For $h_2 < h < h_3$, system (4) has multiple solutions consisting of three limit cycles (a big stable limit cycle

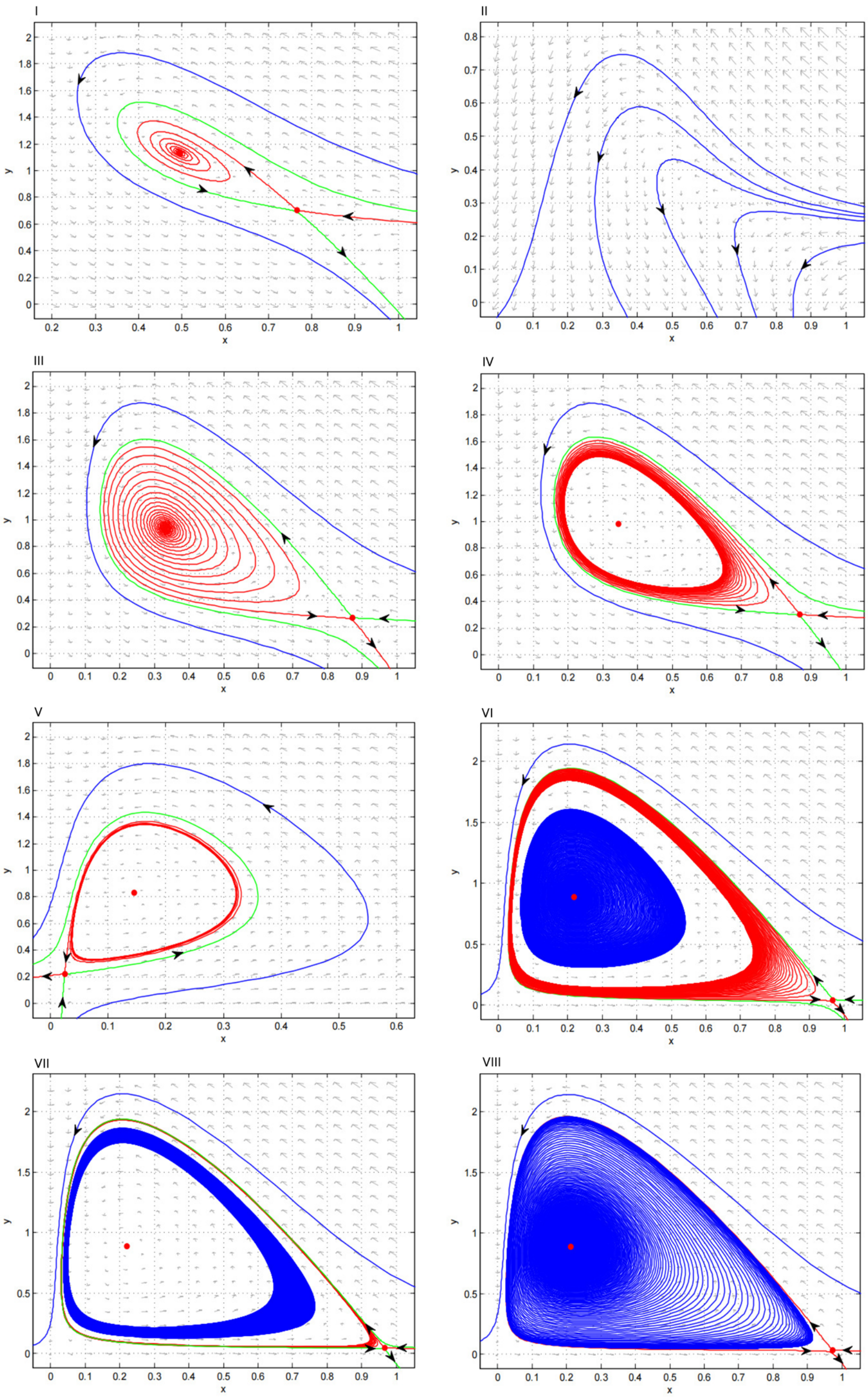

Fig. 7 Phase portraits of eight regions: I–VIII in Fig. 6

contains a middle unstable limit cycle including a small stable limit cycle) and two unstable equilibria, with the unstable limit cycle acting as the separatrix between the basins of these two stable limit cycles. This shows that we can control the system to a relatively large or small sustained oscillation by selecting appropriate initial value; For $h > h_4$, there are two unstable equilibria indicating that the prey population may face extinction for all initial states under large constant-rate harvesting.

Next, we give the biological interpretations of Figs. 2 and 6. In regions I, IV of Fig. 2 and region II of Fig. 6, no positive equilibria remain, indicating that the system will eventually collapse due to the extinction of predators or prey; In regions II, VIII of Fig. 2 and regions I, III, V, VIII of Fig. 6, there is a stable positive equilibrium, which shows that predators and prey can reach a stable coexistence; In regions III, VII of Fig. 2 and regions IV, VI of Fig. 6, there is a stable limit cycle, indicating that a stable coexistence of two species in the form of persistent oscillations can be achieved; In region V of Fig. 2 and region VII of Fig. 6, there are two stable limit cycles. This indicates that we can control the system to a relatively large or small sustained oscillation; In region VI of Fig. 2, there exist a stable limit cycle and a stable positve equilibrium, showing that we can stabilize the predators and prey to a coexistence oscillating state or a equilibrium.

It's worth noting that system (4) may exhibit a heteroclinic cycle with zero predator harvest rate (i.e., $H = 0$) and a threshold value for prey harvest rate, which presents an intriguing new phenomenon and suggests that the predator population may face extinction, leading to the system to collapse. Hence, the harvesting and stocking of predators is important and necessary.

Moreover, as shown in Fig. 6, it is crucial to identify an optimal opportunity to harvest or stock both populations. An ill-timed measure may ultimately cause the system to collapse. Interestingly, we find that simultaneous stocking for predators and prey may not necessarily benefit the coexistence of both species. The initial state may ultimately determine the fate of the predator.

In a word, even small parameter perturbations will have great influence on the dynamical behaviors of system, indicating the vital roles of these parameters (harvesting rate, conversion rate from prey to predators and stocking rate). Of particular importance, by selecting parameter values within different regions and choosing different initial states, we can stabilize the system to a coexistence oscillating state or an equilibrium. This work promotes to a deeper understanding of the dynamics of ecosystems when harvesting and stocking occurs simultaneously, which does benefit to the effective management of ecological systems through harvesting and stocking.

6 Conclusion

In this paper, we conduct a comprehensive study on the dynamics of a predator–prey system including harvesting and stocking of both species. Our detailed bifurcation analysis encompasses saddle-node bifurcation, cusp of high codimension, Bogdanov–Takens bifurcation of codimensions 2 and 3, as well as degenerate Hopf bifurcation of codimension 2. The transitions between different regimes are also depicted in the bifurcation diagrams. Of note, we identify a codimension-2 cusp of limit cycles, a phenomenon first observed in this harvesting and stocking model, signifying the potential for the coexistence of three limit cycles. In particular, we observe a intriguing phenomenon that two BT bifurcation points are not always connected by a continuous homoclinic bifurcation curve.

From a biological perspective, the prey population may face extinction for all initial states under large constant-rate harvesting. Further, simultaneous stocking ($h < 0, H < 0$) may not contribute to the coexistence of both populations; Conversely, the cases ($h > 0, H > 0$) and ($h > 0, H < 0$) promote such coexistence, while the case ($h > 0, H > 0$) may lead to the occurrence of multiple limit cycles. Additionally, if the special case $H = 0$ holds, there's a possibility of coextinction for both predators and prey. The most meaningful thing is that we can stabilize the system to different coexistence states (stable equilibria or persistent oscillations) by selecting parameter values within different regions and choosing different initial states, which fully indicates the important roles of harvesting and stocking, benefiting to the management of predator–prey systems.

It's worth noting that although we have identified the potential for the system to exhibit three limit cycles, we have not been able to eliminate the possibility of more limit cycles coexisting. In addition, the boundary between two limit cycles and three limit cycles warrants further investigation. Taken together, this work provides valuable insights into understanding the dynamics of ecological systems when harvesting and stocking occur simultaneously.

Appendix A Coefficients in the Proof of Theorem 1

$$
\begin{aligned}
\hat{a}_{10} &= \frac{\beta x^*(1-2x^*)}{\delta(\alpha+x^*)},\ \hat{a}_{01} = -\frac{x^*}{\alpha+x^*},\ \hat{a}_{20} = \frac{(2x^*-1)(\beta x^*-\delta(\alpha+x^*))}{\delta(\alpha+x^*)^2} - 1,\\
\hat{a}_{11} &= -\frac{\alpha}{(\alpha+x^*)^2},\ \hat{a}_{30} = \frac{(2x^*-1)(\delta(\alpha+x^*)-\beta x^*)}{\delta(\alpha+x^*)^3},\ \hat{a}_{21} = \frac{\alpha}{(\alpha+x^*)^3};\\
\hat{b}_{10} &= \frac{\beta(2x^*-1)(\beta x^*-\delta(\alpha+x^*))}{\delta(\alpha+x^*)},\ \hat{b}_{01} = \frac{\beta x^*}{\alpha+x^*} - \delta,\\
\hat{b}_{20} &= -\frac{\beta(2x^*-1)(\beta x^*-\delta(\alpha+x^*))}{\delta(\alpha+x^*)^2},\ \hat{b}_{11} = \frac{\alpha\beta}{(\alpha+x^*)^2},\\
\hat{b}_{30} &= \frac{\beta(2x^*-1)(\beta x^*-\delta(\alpha+x^*))}{\delta(\alpha+x^*)^3},\ \hat{b}_{21} = -\frac{\alpha\beta}{(\alpha+x^*)^3};\\
\hat{c}_{11} &= \frac{-2\hat{a}_{01}^2\hat{b}_{20} + \hat{a}_{01}\hat{b}_{01}(2\hat{a}_{20}-\hat{b}_{11}) + \hat{a}_{11}\hat{b}_{01}^2 + \hat{a}_{10}(\hat{a}_{01}\hat{b}_{11}-\hat{a}_{11}\hat{b}_{01})}{\hat{a}_{01}(\hat{a}_{10}+\hat{b}_{01})},\\
\hat{c}_{02} &= \frac{-\hat{a}_{01}^2\hat{b}_{20} + \hat{a}_{01}\hat{b}_{01}(\hat{a}_{20}-\hat{b}_{11}) + \hat{a}_{11}\hat{b}_{01}^2}{\hat{a}_{01}(\hat{a}_{10}+\hat{b}_{01})},\\
\hat{c}_{30} &= \frac{\hat{a}_{10}(\hat{a}_{01}\hat{b}_{21}-\hat{a}_{21}\hat{b}_{01}) + \hat{a}_{01}(\hat{a}_{30}\hat{b}_{01}-\hat{a}_{01}\hat{b}_{30})}{\hat{a}_{01}(\hat{a}_{10}+\hat{b}_{01})},
\end{aligned}
$$

$$\hat{c}_{21} = \frac{-3\hat{a}_{01}^2\hat{b}_{30} + \hat{a}_{01}\hat{b}_{01}(3\hat{a}_{30} - \hat{b}_{21}) + \hat{a}_{21}\hat{b}_{01}^2 + \hat{a}_{10}(2\hat{a}_{01}\hat{b}_{21} - 2\hat{a}_{21}\hat{b}_{01})}{\hat{a}_{01}(\hat{a}_{10} + \hat{b}_{01})},$$

$$\hat{c}_{12} = \frac{-3\hat{a}_{01}^2\hat{b}_{30} + \hat{a}_{01}\hat{b}_{01}(3\hat{a}_{30} - 2\hat{b}_{21}) + 2\hat{a}_{21}\hat{b}_{01}^2 + \hat{a}_{10}(\hat{a}_{01}\hat{b}_{21} - \hat{a}_{21}\hat{b}_{01})}{\hat{a}_{01}(\hat{a}_{10} + \hat{b}_{01})},$$

$$\hat{c}_{03} = \frac{-\hat{a}_{01}^2\hat{b}_{30} + \hat{a}_{01}\hat{b}_{01}(\hat{a}_{30} - \hat{b}_{21}) + \hat{a}_{21}\hat{b}_{01}^2}{\hat{a}_{01}(\hat{a}_{10} + \hat{b}_{01})};$$

$$\hat{d}_{20} = \frac{\hat{a}_{01}^2\hat{b}_{20} + \hat{a}_{10}\hat{a}_{01}(\hat{a}_{20} - \hat{b}_{11}) - \hat{a}_{10}^2\hat{a}_{11}}{\hat{a}_{01}(\hat{a}_{10} + \hat{b}_{01})},$$

$$\hat{d}_{11} = \frac{\hat{a}_{10}(\hat{a}_{11}\hat{b}_{01} + \hat{a}_{01}(2\hat{a}_{20} - \hat{b}_{11})) + \hat{a}_{01}(2\hat{a}_{01}\hat{b}_{20} + \hat{b}_{01}\hat{b}_{11}) - \hat{a}_{10}^2\hat{a}_{11}}{\hat{a}_{01}(\hat{a}_{10} + \hat{b}_{01})},$$

$$\hat{d}_{02} = \frac{\hat{a}_{01}^2\hat{b}_{20} + \hat{a}_{01}(\hat{a}_{10}\hat{a}_{20} + \hat{b}_{01}\hat{b}_{11}) + \hat{a}_{10}\hat{a}_{11}\hat{b}_{01}}{\hat{a}_{01}(\hat{a}_{10} + \hat{b}_{01})},$$

$$\hat{d}_{30} = \frac{\hat{a}_{01}^2\hat{b}_{30} + \hat{a}_{10}\hat{a}_{01}(\hat{a}_{30} - \hat{b}_{21}) - \hat{a}_{10}^2\hat{a}_{21}}{\hat{a}_{01}(\hat{a}_{10} + \hat{b}_{01})},$$

$$\hat{d}_{21} = \frac{\hat{a}_{10}(\hat{a}_{21}\hat{b}_{01} + \hat{a}_{01}(3\hat{a}_{30} - 2\hat{b}_{21})) + \hat{a}_{01}(3\hat{a}_{01}\hat{b}_{30} + \hat{b}_{01}\hat{b}_{21}) - 2\hat{a}_{21}\hat{a}_{10}^2}{\hat{a}_{01}(\hat{a}_{10} + \hat{b}_{01})},$$

$$\hat{d}_{12} = \frac{\hat{a}_{10}(2\hat{a}_{21}\hat{b}_{01} + \hat{a}_{01}(3\hat{a}_{30} - \hat{b}_{21})) + \hat{a}_{01}(3\hat{a}_{01}\hat{b}_{30} + 2\hat{b}_{01}\hat{b}_{21}) - \hat{a}_{10}^2\hat{a}_{21}}{\hat{a}_{01}(\hat{a}_{10} + \hat{b}_{01})},$$

$$\hat{d}_{03} = \frac{\hat{a}_{01}^2\hat{b}_{30} + \hat{a}_{01}(\hat{a}_{10}\hat{a}_{30} + \hat{b}_{01}\hat{b}_{21}) + \hat{a}_{10}\hat{a}_{21}\hat{b}_{01}}{\hat{a}_{01}(\hat{a}_{10} + \hat{b}_{01})}.$$

$$a_{10}^* = -\frac{\delta(2x^* - 1)}{\delta - 2x^* + 1}, \; a_{01}^* = -\frac{(\delta - 1)\delta + 2x^*(2x^* - 1)}{(\delta - 2x^* + 1)(\delta + 4x^* - 2)},$$

$$a_{20}^* = \frac{\delta(\delta + 2x^* - 1)(6x^{*2} - (\delta + 5)x^* + 1)}{x^*(\delta - 2x^* + 1)^2(\delta + 4x^* - 2)},$$

$$a_{11}^* = \frac{2(x^*(6x^* - \delta - 5) + 1)((\delta - 1)\delta + 2x^*(2x^* - 1))}{x^*(\delta - 2x^* + 1)^2(\delta + 4x^* - 2)^2},$$

$$a_{30}^* = -\frac{(1 - 2x^*)^2((\delta - 1)\delta + 2x^*(2x^* - 1))^2}{x^{*2}(\delta - 2x^* + 1)^3(\delta + 4x^* - 2)^2},$$

$$a_{21}^* = -\frac{2(x^*(6x^* - \delta - 5) + 1)((\delta - 1)\delta + 2x^*(2x^* - 1))^2}{x^{*2}(\delta - 2x^* + 1)^3(\delta + 4x^* - 2)^3},$$

$$a_{40}^* = \frac{(1 - 2x^*)^2((\delta - 1)\delta + 2x^*(2x^* - 1))^3}{x^{*3}(\delta - 2x^* + 1)^4(\delta + 4x^* - 2)^3},$$

$$a_{31}^* = \frac{2(x^*(6x^* - \delta - 5) + 1)((\delta - 1)\delta + 2x^*(2x^* - 1))^3}{x^{*3}(\delta - 2x^* + 1)^4(\delta + 4x^* - 2)^4};$$

$$b_{10}^* = \frac{(\delta + 4x^* - 2)(\delta - 2\delta x^*)^2}{(\delta - 2x^* + 1)((\delta - 1)\delta + 2x^*(2x^* - 1))}, \; b_{01}^* = \frac{\delta(2x^* - 1)}{\delta - 2x^* + 1},$$

$$b^*_{20} = -\frac{(\delta - 2\delta x^*)^2}{x^*(\delta - 2x^* + 1)^2},\ b^*_{11} = \frac{2\delta^2(x^*(\delta - 6x^* + 5) - 1)}{x^*(\delta - 2x^* + 1)^2(\delta + 4x^* - 2)},$$

$$b^*_{30} = \frac{(\delta - 2\delta x^*)^2((\delta - 1)\delta + 2x^*(2x^* - 1))}{x^{*2}(\delta - 2x^* + 1)^3(\delta + 4x^* - 2)},$$

$$b_{21} = -\frac{2\delta^2((\delta - 1)\delta + 2x^*(2x^* - 1))(x^*(\delta - 6x^* + 5) - 1)}{x^{*2}(\delta - 2x^* + 1)^3(\delta + 4x^* - 2)^2},$$

$$b^*_{40} = -\frac{\delta^2(1 - 2x^*)^2((\delta - 1)\delta + 2x^*(2x^* - 1))^2}{x^{*3}(\delta - 2x^* + 1)^4(\delta + 4x^* - 2)^2},$$

$$b^*_{31} = \frac{2\delta^2((\delta - 1)\delta + 2x^*(2x^* - 1))^2(x^*(\delta - 6x^* + 5) - 1)}{x^{*3}(\delta - 2x^* + 1)^4(\delta + 4x^* - 2)^3};$$

$$c^*_{20} = -a^*_{20}b^*_{01} + a^*_{11}b^*_{10} - a^*_{10}b^*_{11} + a^*_{01}b^*_{20},\ c^*_{02} = \frac{a^*_{11}}{a^*_{01}},$$

$$c^*_{30} = -a^*_{30}b^*_{01} + a^*_{21}b^*_{10} - a^*_{20}b^*_{11} + a^*_{11}b^*_{20} - a^*_{10}b^*_{21} + a^*_{01}b^*_{30},$$

$$c^*_{21} = \frac{a^*_{10}(a^{*2}_{11} - 2a^*_{01}a^*_{21}) - a^*_{01}a^*_{11}a^*_{20}}{a^{*2}_{01}} + 3a^*_{30} + b^*_{21},\ c^*_{12} = \frac{2a^*_{01}a^*_{21} - a^{*2}_{11}}{a^{*2}_{01}},$$

$$c^*_{40} = -a^*_{40}b^*_{01} + a^*_{31}b^*_{10} - a^*_{30}b^*_{11} + a^*_{21}b^*_{20} - a^*_{20}b^*_{21} + a^*_{11}b^*_{30} - a^*_{10}b^*_{31} + a^*_{01}b^*_{40},$$

$$c^*_{31} = \frac{1}{a^{*3}_{01}}\{a^*_{01}(a^*_{01}(a^*_{01}(4a^*_{40} + b^*_{31}) - 2a^*_{20}a^*_{21}) + a^*_{20}a^{*2}_{11} - a^*_{01}a^*_{30}a^*_{11})$$
$$-a^*_{10}(a^{*3}_{11} - 3a^*_{01}a^*_{21}a^*_{11}$$
$$+3a^{*2}_{01}a^*_{31})\},$$

$$c^*_{22} = \frac{a^{*3}_{11} - 3a^*_{01}a^*_{21}a^*_{11} + 3a^{*2}_{01}a^*_{31}}{a^{*3}_{01}}.$$

Appendix B Coefficients in the Proof of Theorem 2 and Theorem 3

$$\bar{a}_{00} = -\lambda_1,\ \bar{a}_{10} = \frac{\delta(1 - 2x^*)}{\delta - 2x^* + 1},\ \bar{a}_{01} = -\frac{x^*}{\alpha + x^*},\ \bar{a}_{20} = \frac{(1 - 2x^*)^2}{(\alpha + x^*)(\delta - 2x^* + 1)} - 1,$$

$$\bar{a}_{11} = -\frac{\alpha}{(\alpha + x^*)^2};\ \bar{b}_{00} = -\lambda_2,\ \bar{b}_{10} = \frac{(\alpha + x^*)(\delta - 2\delta x^*)^2}{x^*(\delta - 2x^* + 1)^2},\ \bar{b}_{01} = \frac{\delta(2x^* - 1)}{\delta - 2x^* + 1},$$

$$\bar{b}_{20} = -\frac{(\delta - 2\delta x^*)^2}{x^*(\delta - 2x^* + 1)^2},\ \bar{b}_{11} = \frac{\alpha\delta^2}{x^*(\alpha + x^*)(\delta - 2x^* + 1)};$$

$$\bar{c}_{00} = \bar{a}_{01}\bar{b}_{00} - \bar{a}_{00}\bar{b}_{01},\ \bar{c}_{10} = \bar{a}_{11}\bar{b}_{00} - \bar{a}_{10}\bar{b}_{01} + \bar{a}_{01}\bar{b}_{10} - \bar{a}_{00}\bar{b}_{11},$$

$$\bar{c}_{01} = \bar{a}_{10} - \frac{\bar{a}_{00}\bar{a}_{11}}{\bar{a}_{01}} + \bar{b}_{01},\ \bar{c}_{20} = -\bar{a}_{20}\bar{b}_{01} + \bar{a}_{11}\bar{b}_{10} - \bar{a}_{10}\bar{b}_{11} + \bar{a}_{01}\bar{b}_{20},$$

$$\bar{c}_{11} = \frac{\bar{a}_{11}(\bar{a}_{00}\bar{a}_{11} - \bar{a}_{01}\bar{a}_{10})}{\bar{a}^2_{01}} + 2\bar{a}_{20} + \bar{b}_{11},\ \bar{c}_{02} = \frac{\bar{a}_{11}}{\bar{a}_{01}};\ \bar{d}_{00} = \bar{c}_{00},\ \bar{d}_{10} = \bar{c}_{10} - 2\bar{c}_{00}\bar{c}_{02},$$

$$\bar{d}_{01} = \bar{c}_{01},\ \bar{d}_{20} = \bar{c}_{00}\bar{c}^2_{02} - 2\bar{c}_{10}\bar{c}_{02} + \bar{c}_{20},$$

$$\bar{d}_{11} = \bar{c}_{11} - \bar{c}_{01}\bar{c}_{02};\ \bar{e}_{00} = \bar{d}_{00} - \frac{\bar{d}^2_{10}}{4\bar{d}_{20}},\ \bar{e}_{01} = \bar{d}_{01} - \frac{\bar{d}_{10}\bar{d}_{11}}{2\bar{d}_{20}},\ \bar{e}_{20} = \bar{d}_{20}, \bar{e}_{11} = \bar{d}_{11}.$$

$$\bar{a}^*_{00} = -\epsilon_1,\ \bar{a}^*_{10} = \frac{\delta(1-2x^*)}{\delta - 2x^* + 1},\ \bar{a}^*_{01} = -\frac{(\delta-1)\delta + 2x^*(2x^*-1)}{(\delta - 2x^* + 1)(\delta + 4x^* - 2)},$$

$$\bar{a}^*_{20} = \frac{\delta(\delta + 2x^* - 1)(6x^{*2} - (\delta+5)x^* + 1)}{x^*(\delta - 2x^* + 1)^2(\delta + 4x^* - 2)},$$

$$\bar{a}^*_{11} = \frac{2(x^*(6x^* - \delta - 5) + 1)((\delta-1)\delta + 2x^*(2x^*-1))}{x^*(\delta - 2x^* + 1)^2(\delta + 4x^* - 2)^2},$$

$$\bar{a}^*_{30} = -\frac{(1-2x^*)^2((\delta-1)\delta + 2x^*(2x^*-1))^2}{x^{*2}(\delta - 2x^* + 1)^3(\delta + 4x^* - 2)^2},$$

$$\bar{a}^*_{21} = -\frac{2(x^*(6x^* - \delta - 5) + 1)((\delta-1)\delta + 2x^*(2x^*-1))^2}{x^{*2}(\delta - 2x^* + 1)^3(\delta + 4x^* - 2)^3},$$

$$\bar{a}^*_{40} = \frac{(1-2x^*)^2((\delta-1)\delta + 2x^*(2x^*-1))^3}{x^{*3}(\delta - 2x^* + 1)^4(\delta + 4x^* - 2)^3},$$

$$\bar{a}^*_{31} = \frac{2(x^*(6x^* - \delta - 5) + 1)((\delta-1)\delta + 2x^*(2x^*-1))^3}{x^{*3}(\delta - 2x^* + 1)^4(\delta + 4x^* - 2)^4};$$

$$\bar{b}^*_{00} = -\frac{x^*(1-2x^*)^2(\delta + 4x^* - 2)}{12x^{*2} - 2(\delta+5)x^* + 2}\epsilon_2 - \epsilon_3,$$

$$\bar{b}^*_{10} = -\frac{(1-2x^*)^2(\epsilon_2((\delta-1)\delta + 4x^{*2} - 2x^*) + \delta^2(\delta + 4x^* - 2))}{(-\delta + 2x^* - 1)((\delta-1)\delta + 4x^{*2} - 2x^*)},$$

$$\bar{b}^*_{01} = \frac{\epsilon_2((\delta-1)\delta + 2x^*(2x^*-1)) + \delta(2x^*-1)(\delta + 4x^* - 2)}{(\delta - 2x^* + 1)(\delta + 4x^* - 2)},$$

$$\bar{b}^*_{20} = -\frac{(1-2x^*)^2(\epsilon_2((\delta-1)\delta + (2x^*-1)2x^*) + \delta^2(\delta + 4x^* - 2))}{x^*(\delta - 2x^* + 1)^2(\delta + 4x^* - 2)},$$

$$\bar{b}^*_{11} = -\frac{2(x^*(6x^* - \delta - 5) + 1)(\delta^2(\delta + 4x^* - 2) + \epsilon_2((\delta-1)\delta + 2x^*(2x^*-1)))}{x^*(\delta - 2x^* + 1)^2(\delta + 4x^* - 2)^2},$$

$$\bar{b}^*_{30} = \frac{(1-2x^*)^2((\delta-1)\delta + 2x^*(2x^*-1))(\delta^2(\delta + 4x^*-2) + \epsilon_2((\delta-1)\delta + 2x^*(2x^*-1)))}{x^{*2}(\delta - 2x^* + 1)^3(\delta + 4x^* - 2)^2},$$

$$\bar{b}^*_{21} = \frac{1}{x^{*2}(\delta - 2x^* + 1)^3(\delta + 4x^* - 2)^3}(2(x^*(6x^* - \delta - 5) + 1))((\delta-1)\delta$$
$$+2x^*(2x^*-1))(\delta^2(\delta + 4x^* - 2) + \epsilon_2((\delta-1)\delta + 2x^*(2x^*-1))),$$

$$\bar{b}^*_{40} = \frac{(1-2x^*)^2((\delta-1)\delta + 2x^*(2x^*-1))^2(\epsilon_2((\delta-1)\delta + 2x^*(2x^*-1)) + \delta^2(\delta + 4x^* - 2))}{x^{*3}(\delta - 2x^* + 1)^4(2 - \delta - 4x^*)^3},$$

$$\bar{b}^*_{31} = -\frac{1}{x^{*3}(\delta - 2x^* + 1)^4(\delta + 4x^* - 2)^4}2((\delta-1)\delta + 2x^*(2x^*-1))^2$$
$$(6x^{*2} - (\delta+5)x^* + 1)(\epsilon_2((\delta-1)\delta + 2x^*(2x^*-1)) + \delta^2(\delta + 4x^* - 2));$$

$$\bar{c}^*_{00} = \bar{a}^*_{01}\bar{b}^*_{00} - \bar{a}^*_{00}\bar{b}^*_{01},\ \bar{c}^*_{10} = \bar{a}^*_{11}\bar{b}^*_{00} - \bar{a}^*_{10}\bar{b}^*_{01} + \bar{a}^*_{01}\bar{b}^*_{10} - \bar{a}^*_{00}\bar{b}^*_{11},$$

$$\bar{c}^*_{01} = \frac{\bar{a}^*_{01}(\bar{a}^*_{10} + \bar{b}^*_{01}) - \bar{a}^*_{00}\bar{a}^*_{11}}{\bar{a}^*_{01}},$$

$$\bar{c}^*_{20} = \bar{a}^*_{21}\bar{b}^*_{00} - \bar{a}^*_{20}\bar{b}^*_{01} + \bar{a}^*_{11}\bar{b}^*_{10} - \bar{a}^*_{10}\bar{b}^*_{11} + \bar{a}^*_{01}\bar{b}^*_{20} - \bar{a}^*_{00}\bar{b}^*_{21},$$

$$\bar{c}^*_{11} = 2\bar{a}^*_{20} + \frac{\bar{a}^*_{00}\bar{a}^{*2}_{11} - \bar{a}^*_{01}(\bar{a}^*_{10}\bar{a}^*_{11} + 2\bar{a}^*_{00}\bar{a}^*_{21})}{\bar{a}^{*2}_{01}} + \bar{b}^*_{11},\ \bar{c}^*_{02} = \frac{\bar{a}^*_{11}}{\bar{a}^*_{01}},$$

$$\bar{c}^*_{30} = \bar{a}^*_{31}\bar{b}^*_{00} - \bar{a}^*_{30}\bar{b}^*_{01} + \bar{a}^*_{21}\bar{b}^*_{10} - \bar{a}^*_{20}\bar{b}^*_{11} + \bar{a}^*_{11}\bar{b}^*_{20} - \bar{a}^*_{10}\bar{b}^*_{21} + \bar{a}^*_{01}\bar{b}^*_{30} - \bar{a}^*_{00}\bar{b}^*_{31},$$

$$
\begin{aligned}
\bar{c}^*_{21} &= 3\bar{a}^*_{30} + \frac{-\bar{a}^*_{00}\bar{a}^{*3}_{11} + \bar{a}^*_{01}(\bar{a}^*_{10}\bar{a}^*_{11} + 3\bar{a}^*_{00}\bar{a}^*_{21})\bar{a}^*_{11} - \bar{a}^{*2}_{01}(\bar{a}^*_{11}\bar{a}^*_{20} + 2\bar{a}^*_{10}\bar{a}^*_{21} + 3\bar{a}^*_{00}\bar{a}^*_{31})}{\bar{a}^{*3}_{01}} \\
&\quad + \bar{b}^*_{21}, \\
\bar{c}^*_{12} &= \frac{2\bar{a}^*_{01}\bar{a}^*_{21} - \bar{a}^{*2}_{11}}{\bar{a}^{*2}_{01}}, \\
\bar{c}^*_{40} &= -\bar{a}^*_{40}\bar{b}^*_{01} + \bar{a}^*_{31}\bar{b}^*_{10} - \bar{a}^*_{30}\bar{b}^*_{11} + \bar{a}^*_{21}\bar{b}^*_{20} - \bar{a}^*_{20}\bar{b}^*_{21} + \bar{a}^*_{11}\bar{b}^*_{30} - \bar{a}^*_{10}\bar{b}^*_{31} + \bar{a}^*_{01}\bar{b}^*_{40}, \\
\bar{c}^*_{31} &= \frac{1}{\bar{a}^{*4}_{01}}(\bar{a}^*_{00}\bar{a}^{*4}_{11} - \bar{a}^*_{01}\bar{a}^{*2}_{11}(\bar{a}^*_{10}\bar{a}^*_{11} + 4\bar{a}^*_{00}\bar{a}^*_{21}) - \bar{a}^{*3}_{01}(2\bar{a}^*_{20}\bar{a}^*_{21} + \bar{a}^*_{11}\bar{a}^*_{30} + 3\bar{a}^*_{10}\bar{a}^*_{31}) \\
&\quad + \bar{a}^{*2}_{01}(\bar{a}^*_{20}\bar{a}^{*2}_{11} + (3\bar{a}^*_{10}\bar{a}^*_{21} + 4\bar{a}^*_{00}\bar{a}^*_{31})\bar{a}^*_{11} + 2\bar{a}^*_{00}\bar{a}^{*2}_{21})) + 4\bar{a}^*_{40} + \bar{b}^*_{31}, \\
\bar{c}^*_{22} &= \frac{\bar{a}^{*3}_{11} - 3\bar{a}^*_{01}\bar{a}^*_{21}\bar{a}^*_{11} + 3\bar{a}^{*2}_{01}\bar{a}^*_{31}}{\bar{a}^{*3}_{01}};\ \bar{d}^*_{10} = \bar{c}^*_{10} - \bar{c}^*_{00}\bar{c}^*_{02},\ \bar{d}^*_{01} = \bar{c}^*_{01}, \\
\bar{d}^*_{20} &= \bar{c}^*_{20} + \bar{c}^*_{00}\bar{c}^{*2}_{02} - \frac{\bar{c}^*_{10}\bar{c}^*_{02}}{2},\ \bar{d}^*_{11} = \bar{c}^*_{11},\ \bar{d}^*_{30} = \bar{c}^*_{30} + \frac{1}{2}(\bar{c}^*_{10} - 2\bar{c}^*_{00}\bar{c}^*_{02})\bar{c}^{*2}_{02}, \\
\bar{d}^*_{21} &= \bar{c}^*_{21} + \frac{\bar{c}^*_{02}\bar{c}^*_{11}}{2},\ \bar{d}^*_{12} = \bar{c}^*_{12} + 2\bar{c}^{*2}_{02},\ \bar{d}^*_{40} = \bar{c}^*_{40} + \bar{c}^*_{00}\bar{c}^{*4}_{02} + \frac{1}{4}(\bar{c}^*_{02}(\bar{c}^*_{20} \\
&\quad - 2\bar{c}^*_{02}\bar{c}^*_{10}) + 2\bar{c}^*_{30})\bar{c}^*_{02}, \\
\bar{d}^*_{31} &= \bar{c}^*_{31} + \bar{c}^*_{02}\bar{c}^*_{21},\ \bar{d}^*_{22} = \bar{c}^*_{22} - \bar{c}^{*3}_{02} + \frac{3\bar{c}^*_{12}\bar{c}^*_{02}}{2};\ \bar{e}^*_{00} = \bar{d}^*_{00},\ \bar{e}^*_{10} = \bar{d}^*_{10},\ \bar{e}^*_{01} = \bar{d}^*_{01}, \\
\bar{e}^*_{20} &= \bar{d}^*_{20} - \frac{\bar{d}^*_{00}\bar{d}^*_{12}}{2},\ \bar{e}^*_{11} = \bar{d}^*_{11},\ \bar{e}^*_{30} = \bar{d}^*_{30} - \frac{\bar{d}^*_{10}\bar{d}^*_{12}}{3},\ \bar{e}^*_{21} = \bar{d}^*_{21}, \\
\bar{e}^*_{40} &= \bar{d}^*_{40} + \frac{\bar{d}^*_{00}\bar{d}^{*2}_{12}}{4} - \frac{\bar{d}^*_{12}\bar{d}^*_{20}}{6},\ \bar{e}^*_{31} = \bar{d}^*_{31} + \frac{\bar{d}^*_{11}\bar{d}^*_{12}}{6},\ \bar{e}^*_{22} = \bar{d}^*_{22};\ \bar{f}^*_{00} = \bar{e}^*_{00},\ \bar{f}^*_{10} = \bar{e}^*_{10}, \\
\bar{f}^*_{01} &= \bar{e}^*_{01},\ \bar{f}^*_{20} = \bar{e}^*_{20},\ \bar{f}^*_{11} = \bar{e}^*_{11},\ \bar{f}^*_{30} = \bar{e}^*_{30} - \frac{\bar{e}^*_{00}\bar{e}^*_{22}}{3},\ \bar{f}^*_{21} = \bar{e}^*_{21},\ \bar{f}^*_{40} = \bar{e}^*_{40} - \frac{\bar{e}^*_{10}\bar{e}^*_{22}}{4}, \\
\bar{f}^*_{31} &= \bar{e}^*_{31};\ \bar{g}^*_{00} = \bar{f}^*_{00},\ \bar{g}^*_{10} = \bar{f}^*_{10} - \frac{\bar{f}^*_{00}\bar{f}^*_{30}}{2\bar{f}^*_{20}},\ \bar{g}^*_{01} = \bar{f}^*_{01}, \\
\bar{g}^*_{20} &= \bar{f}^*_{20} + \frac{9\bar{f}^*_{00}\bar{f}^{*2}_{30}}{16\bar{f}^{*2}_{20}} - \frac{3(5\bar{f}^*_{10}\bar{f}^*_{30} + 4\bar{f}^*_{00}\bar{f}^*_{40})}{20\bar{f}^*_{20}},\ \bar{g}^*_{11} = \bar{f}^*_{11} - \frac{\bar{f}^*_{01}\bar{f}^*_{30}}{2\bar{f}^*_{20}}, \\
\bar{g}^*_{30} &= \frac{\bar{f}^*_{10}(35\bar{f}^{*2}_{30} - 32\bar{f}^*_{20}\bar{f}^*_{40})}{40\bar{f}^{*2}_{20}},\ \bar{g}^*_{21} = \bar{f}^*_{21} - \frac{3(20\bar{f}^*_{11}\bar{f}^*_{20}\bar{f}^*_{30} + \bar{f}^*_{01}(16\bar{f}^*_{20}\bar{f}^*_{40} - 15\bar{f}^{*2}_{30}))}{80\bar{f}^{*2}_{20}}, \\
\bar{g}^*_{40} &= \frac{\bar{f}^*_{10}\bar{f}^*_{30}(16\bar{f}^*_{20}\bar{f}^*_{40} - 15\bar{f}^{*2}_{30})}{64\bar{f}^{*3}_{20}},\ \bar{g}^*_{31} = \bar{f}^*_{31} + \frac{7\bar{f}^*_{11}\bar{f}^{*2}_{30}}{8\bar{f}^{*2}_{20}} - \frac{5\bar{f}^*_{21}\bar{f}^*_{30} + 4\bar{f}^*_{11}\bar{f}^*_{40}}{5\bar{f}^*_{20}}; \\
\bar{h}^*_{00} &= \bar{g}^*_{00},\ \bar{h}^*_{10} = \bar{g}^*_{10},\ \bar{h}^*_{01} = \bar{g}^*_{01} - \frac{\bar{g}^*_{00}\bar{g}^*_{21}}{\bar{g}^*_{20}},\ \bar{h}^*_{20} = \bar{g}^*_{20},\ \bar{h}^*_{11} = \bar{g}^*_{11} - \frac{\bar{g}^*_{10}\bar{g}^*_{21}}{\bar{g}^*_{20}}, \\
\bar{h}^*_{31} &= \bar{g}^*_{31} - \frac{\bar{g}^*_{21}\bar{g}^*_{30}}{\bar{g}^*_{20}};\ \bar{j}^*_{00} = \bar{h}^*_{00}\bar{h}^{*\frac{4}{5}}_{31}\bar{h}^{*-\frac{7}{5}}_{20},\ \bar{j}^*_{10} = \bar{h}^*_{10}\bar{h}^{*\frac{2}{5}}_{31}\bar{h}^{*-\frac{6}{5}}_{20},\ \bar{j}^*_{01} = \bar{h}^*_{01}\bar{h}^{*\frac{1}{5}}_{31}\bar{h}^{*-\frac{3}{5}}_{20}, \\
\bar{j}^*_{11} &= \bar{h}^*_{11}\bar{h}^{*-\frac{1}{5}}_{31}\bar{h}^{*-\frac{2}{5}}_{20}.
\end{aligned}
$$

Appendix C Coefficients in the Proof of Theorem 5

$$
\begin{aligned}
\sigma_{22} = {} & 20(\alpha+\tilde{x})^4(\alpha h-\tilde{x}^2(\alpha+2\tilde{x}-1))^7-20\tilde{x}(\alpha+\tilde{x})^3(\alpha h-\tilde{x}^2(\alpha+2\tilde{x}-1))^6\\
& \times(2\alpha h-19\tilde{x}^3+(7-22\alpha)\tilde{x}^2+\alpha\tilde{x}(-5\alpha-2\beta+5))\\
& +(\alpha+\tilde{x})^4(-5\alpha h-2\tilde{x}^3)(\alpha\beta\tilde{x}^2(h+(\tilde{x}-1)\tilde{x})+(\alpha h-\tilde{x}^2(\alpha+2\tilde{x}-1))^2)^3\\
& -180\tilde{x}^3(\alpha+\tilde{x})^3(\alpha+3\tilde{x}-1)(-\alpha h+2\tilde{x}^3+(\alpha-1)\tilde{x}^2)^5\\
& \times(-\alpha h+5\tilde{x}^3+(5\alpha-2)\tilde{x}^2+\alpha\tilde{x}(\alpha+\beta-1))\\
& +20\tilde{x}^3(\alpha+\tilde{x})\left(-\alpha h+2\tilde{x}^3+(\alpha-1)\tilde{x}^2\right)^4(2\alpha h+17\tilde{x}^3+(26\alpha-5)\tilde{x}^2\\
& +\alpha\tilde{x}(7\alpha-2\beta-7))(-\alpha h+5\tilde{x}^3+(5\alpha-2)\tilde{x}^2+\alpha\tilde{x}(\alpha+\beta-1))^2\\
& -20\tilde{x}^4(-\alpha h+2\tilde{x}^3+(\alpha-1)\tilde{x}^2)^3(\alpha h+4\tilde{x}^3+(7\alpha-1)\tilde{x}^2\\
& +\alpha\tilde{x}(2\alpha-\beta-2))(-\alpha h+5\tilde{x}^3+(5\alpha-2)\tilde{x}^2+\alpha\tilde{x}(\alpha+\beta-1))^3\\
& +(\alpha+\tilde{x})^2(\alpha\beta\tilde{x}^2(h+(\tilde{x}-1)\tilde{x})+(\alpha h-\tilde{x}^2(\alpha+2\tilde{x}-1))^2)^2\\
& \times(30(\alpha+\tilde{x})^2(\alpha h-\tilde{x}^2(\alpha+2\tilde{x}-1))^3+10\tilde{x}(\alpha+\tilde{x})\\
& \times(-\alpha h+2\tilde{x}^3+(\alpha-1)\tilde{x}^2)^2(-\alpha h+23\tilde{x}^3+(30\alpha-9)\tilde{x}^2\\
& +\alpha\tilde{x}(8\alpha+\beta-8))+\tilde{x}^3(\alpha+\tilde{x})(\alpha h-\tilde{x}^2(\alpha+2\tilde{x}-1))(-10(\alpha-1)\alpha h\\
& +2\alpha\tilde{x}(5(\alpha-1)(7\alpha+\beta-7)-24h)+510\tilde{x}^4+(821\alpha-407)\tilde{x}^3\\
& +\tilde{x}^2(419\alpha^2+\alpha(48\beta-499)+80))+(1-\alpha)\tilde{x}^4\\
& \times(\alpha h+13\tilde{x}^3+(19\alpha-4)\tilde{x}^2+\alpha\tilde{x}(5\alpha-\beta-5))\\
& \times(\alpha h-14\tilde{x}^3+(5-17\alpha)\tilde{x}^2-\alpha\tilde{x}(4\alpha+\beta-4)))\\
& -(\alpha h-\tilde{x}^2(\alpha+2\tilde{x}-1))(\alpha^2h+3\tilde{x}^4+(2\alpha-1)\tilde{x}^3\\
& +\alpha\beta\tilde{x}^2)(\alpha\beta\tilde{x}^2(h+(\tilde{x}-1)\tilde{x})+(\alpha h-\tilde{x}^2(\alpha+2\tilde{x}-1))^2)(\tilde{x}^2(\alpha+\tilde{x})\\
& \quad\times(\alpha h-\tilde{x}^2(\alpha+2\tilde{x}-1))(3\tilde{x}^2(\alpha+\tilde{x})(7\alpha h-7\tilde{x}(\alpha\beta+\tilde{x}(\alpha+2\tilde{x}-1))\\
& -46\tilde{x}(\alpha+\tilde{x})(\alpha+3\tilde{x}-1))+19\tilde{x}(-\alpha-3\tilde{x}+1)(\alpha+\tilde{x})\\
& \times(\alpha h-\tilde{x}(\alpha\beta+\tilde{x}(\alpha+2\tilde{x}-1)))-5(\alpha h-\tilde{x}(\alpha\beta+\tilde{x}(\alpha+2\tilde{x}-1)))^2\\
& +120\tilde{x}^2(\alpha+\tilde{x})^2(\alpha+3\tilde{x}-1)^2)+45(\alpha+\tilde{x})^3(\alpha h-\tilde{x}^2(\alpha+2\tilde{x}-1))^3\\
& +\tilde{x}^3(\alpha h+4\tilde{x}^3+(7\alpha-1)\tilde{x}^2+\alpha\tilde{x}(2\alpha-\beta-2))\\
& \times(5(\alpha h-\tilde{x}(\alpha\beta+\tilde{x}(\alpha+2\tilde{x}-1)))^2+19\tilde{x}(-\alpha-3\tilde{x}+1)\\
& \times(\alpha+\tilde{x})(\alpha h-\tilde{x}(\alpha\beta+\tilde{x}(\alpha+2\tilde{x}-1)))+20\tilde{x}^2(\alpha+\tilde{x})^2(\alpha+3\tilde{x}-1)^2)\\
& -\tilde{x}(\alpha+\tilde{x})^2(\alpha h-\tilde{x}^2(\alpha+2\tilde{x}-1))^2(5\alpha h-301\tilde{x}^3+(135-426\alpha)\tilde{x}^2\\
& -5\alpha\tilde{x}(26\alpha+\beta-26))).
\end{aligned}
$$

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Declarations

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