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Proof of the simplicity conjecture

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Abstract

In the 1970s, Fathi, having proven that the group of compactly supported volume-preserving homeomorphisms of the n-ball is simple for $n \geq 3$, asked if the same statement holds in dimension two. We show that the group of compactly supported area-preserving homeomorphisms of the two-disc is not simple. This settles what is known as the "simplicity conjecture" in the affirmative. In fact, we prove the a priori stronger statement that this group is not perfect.

Our general strategy is partially inspired by suggestions of Fathi and the approach of Oh towards the simplicity question. In particular, we show that infinite twist maps, studied by Oh, are not finite energy homeomorphisms, which resolves the "infinite twist conjecture" in the affirmative; these twist maps are now the first examples of Hamiltonian homeomorphisms that can be said to have infinite energy. Another consequence of our work is that various forms of fragmentation for volume-preserving homeomorphisms that hold for higher dimensional balls fail in dimension two.

A central role in our arguments is played by spectral invariants defined via periodic Floer homology. We establish many new properties of these invariants that are of independent interest. For example, we prove that these spectral invariants extend continuously to area-preserving homeomorphisms of the disc, and we also verify for certain smooth twist maps a conjecture of Hutchings concerning recovering the Calabi invariant from the asymptotics of these invariants.

Contents

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1. Introduction

Let (S, ω) be a surface equipped with an area form. An *area-preserving homeomorphism* is a homeomorphism that preserves the measure induced by ω. Let $\mathrm{Homeo}_c(\mathbb{D}, \omega)$ denote the group of area-preserving homeomorphisms of the two-disc that are the identity near the boundary. Recall that a group is *simple* if it does not have a non-trivial proper normal subgroup. The following fundamental question was raised in the 1970s:

QUESTION 1.1. *Is the group* $\mathrm{Homeo}_c(\mathbb{D}, \omega)$ *simple*?

Indeed, the algebraic structure of the group of volume-preserving homeomorphisms has been well understood in dimension at least three since the work of Fathi [Fat80a] from the 70s; but, the case of surfaces, and in particular Question 1.1, has long remained mysterious.

Question 1.1 has been the subject of wide interest. For example, it is highlighted in the plenary ICM address of Ghys [Ghy07a, §2.2]; it appears on McDuff and Salamon's list of open problems [MS17, §14.7]; it has been one of the main motivations behind the development of C^0-symplectic topology, which we will further discuss in Appendix B; for other examples, see [Ban78], [Fat80a], [Ghy07b], [Bou08], [LR10a], [LR10b], [EPP12]. It has generally been believed since the early 2000s that the group $\mathrm{Homeo}_c(\mathbb{D},\omega)$ is not simple: McDuff and Salamon refer to this as the *simplicity conjecture*. Our main theorem resolves this conjecture in the affirmative.

THEOREM 1.2. *The group* $\mathrm{Homeo}_c(\mathbb{D},\omega)$ *is not simple.*

In fact, we can obtain an a priori stronger result. Recall that a group G is called *perfect* if its commutator subgroup $[G,G]$ satisfies $[G,G]=G$. The commutator subgroup $[G,G]$ is a normal subgroup of G. Thus, every non-abelian simple group is perfect. However, in the case of certain transformation groups, such as $\mathrm{Homeo}_c(\mathbb{D},\omega)$, a general argument due to Epstein and Higman [Eps70], [Hig54] implies that perfectness and simplicity are equivalent; see Proposition 2.1. Hence, we obtain the following corollary.

COROLLARY 1.3. *The group* $\mathrm{Homeo}_c(\mathbb{D},\omega)$ *is not perfect.*

We remark that in higher dimensions, the analogue of Theorem 1.2 contrasts our main result: by [Fat80a], the group $\mathrm{Homeo}_c(\mathbb{D}^n,\mathrm{Vol})$ of compactly supported volume-preserving homeomorphisms of the n-ball is simple for $n\geq 3$. It also seems that the structure of $\mathrm{Homeo}_c(\mathbb{D},\omega)$ is radically different from that of the group $\mathrm{Diff}_c(\mathbb{D},\omega)$ of compactly supported area-preserving diffeomorphisms, as we will review below.

Spectral invariants defined via "Periodic Floer homology" (PFH) play an essential role in our arguments. These "PFH spectral invariants," which were defined by Hutchings, have not been much studied and much of the paper is devoted to establishing some of their foundational properties. These properties are of independent interest, and we refer the reader to Section 3.3 for their precise statements. As far as we know, the present work represents the first applications of these invariants. Since our paper first appeared, further interesting applications have occurred in [CGHS23], [EH21], [CGPZ21].

Background. To place Theorem 1.2 in its appropriate context, and to summarize what is known about some related transformation groups, we begin by reviewing the long and interesting history surrounding the question of simplicity for various subgroups of homeomorphism groups of manifolds. Our focus will be on compactly supported homeomorphisms/diffeomorphisms of manifolds without boundary in the component of the identity.[1]

In the 1930s, in the "Scottish Book," Ulam asked if the identity component of the group of homeomorphisms of the n-dimensional sphere is simple. In 1947, Ulam and von Neumann announced in an abstract [UvN47] a solution to the question in the Scottish Book in the case $n = 2$. In the 50s, 60s, and 70s, there was a flurry of activity on this question and related ones. First, the works of Anderson [And58], Fisher [Fis60], Chernavski, Edwards and Kirby [EK71] led to the proof of simplicity of the identity component in the group of compactly supported homeomorphisms of any manifold. These developments led Smale to ask if the identity component in the group of compactly supported diffeomorphisms of any manifold is simple [Eps70]. This question was answered affirmatively by Epstein [Eps70], Herman [Her73], Mather [Mat74a], [Mat74b], [Mat75], and Thurston [Thu74].[2]

The connected component of the identity in volume-preserving, and symplectic, diffeomorphisms admits a homomorphism, called *flux*, to a certain abelian group. Hence, these groups are not simple when this homomorphism is non-trivial. Thurston proved, however, that the kernel of flux is simple in the volume-preserving setting for any manifold of dimension at least three; see [Ban97, Ch. 5]. In the symplectic setting, Banyaga [Ban78] then proved that this group is simple when the symplectic manifold is closed; otherwise, it is not simple as it admits a non-trivial homomorphism, called *Calabi*, and Banyaga showed that the kernel of Calabi is always simple. We will recall the definition of Calabi in the case of the disc in Section 3.1.

The simplicity of the identity component in compactly supported volume-preserving homeomorphisms is well understood in dimensions greater than two, thanks to the article [Fat80a], in which Fathi shows that, in all dimensions, the group admits a homomorphism, called "mass-flow"; moreover, the kernel of mass-flow is simple in dimensions greater than two. On simply connected manifolds, the mass-flow homomorphism is trivial, and so the group is indeed simple in dimensions greater than two.

[1]The simplicity question is interesting only for compactly supported maps in the identity component, because this is a normal subgroup of the larger group. The group $\mathrm{Homeo}_c(\mathbb{D}, \omega)$ coincides with its identity component.

[2]More precisely, Epstein, Herman and Thurston settled the question in the case of smooth diffeomorphisms, while Mather resolved the case of C^r diffeomorphisms for $r < \infty$ and $r \neq \dim(\mathrm{M}) + 1$. The case of $r = \dim(\mathrm{M}) + 1$ remains open.

Thus, the following rather simple picture emerges from the above cases of the simplicity question. In the non-conservative setting, the connected component of the identity is simple. In the conservative setting, there always exists a natural homomorphism (flux, Calabi, mass-flow) that obstructs the simplicity of the group. However, the kernel of the homomorphism is always simple.

Despite the clear picture above, established by the end of the 70s, the case of area-preserving homeomorphisms of surfaces has remained unsettled — the simplicity question has remained open for the disc and more generally for the kernel of the mass-flow homomorphism[3] — underscoring the importance of answering Question 1.1. In fact, the case of area-preserving homeomorphisms of the disc does seem drastically different. For example, the natural homomorphisms flux, Calabi, and mass-flow mentioned above that obstruct simplicity are all continuous with respect to a natural topology on the group; however, we will show in Corollary 2.2 that there cannot exist a continuous homomorphism out of $\mathrm{Homeo}_c(\mathbb{D}, \omega)$ with a proper non-trivial kernel, when $\mathrm{Homeo}_c(\mathbb{D}, \omega)$ is equipped with the C^0-topology; we will review the C^0-topology in Section 2.2.

"Lots" of normal subgroups and the failure of fragmentation. Le Roux [LR10a] has previously studied the simplicity question for $\mathrm{Homeo}_c(\mathbb{D}, \omega)$, and it is valuable to combine his conclusions with our Theorem 1.2.

Inspired by Fathi's proof of simplicity in higher-dimensions, Le Roux constructs a family P_ρ, for $0 < \rho \leq 1$, of "quantitative fragmentation properties" for $\mathrm{Homeo}_c(\mathbb{D}, \omega)$. He then establishes the following alternative: if any one of these fragmentation properties holds, then $\mathrm{Homeo}_c(\mathbb{D}, \omega)$ is simple; otherwise, there is a huge number of proper normal subgroups, constructed in terms of "fragmentation norms." Thus, in view of our Theorem 1.2, fragmentation fails in a very strong way in dimension two and we have not just one proper normal subgroup but "lots" of them; for example, combining Le Roux's work [LR10a, Cor. 7.1] with our Theorem 1.2 yields the following.

COROLLARY 1.4. *Every compact*[4] *subset of* $\mathrm{Homeo}_c(\mathbb{D}, \omega)$ *is contained in a proper normal subgroup.*

As Le Roux explains [LR10a, §7], this is "radically" different from the situation for the group $\mathrm{Diff}_c(\mathbb{D}, \omega)$ of compactly supported area-preserving diffeomorphisms of the disc with its usual topology. We refer the reader to [LR10a] for the definition of P_ρ, noting as well that in [EPP12, §5.1] it was previously shown that P_ρ does not hold for $1/2 \leq \rho \leq 1$.

[3] We review the mass-flow homomorphism and discuss more about the simplicity question for its kernel in Appendix B.

[4] As above, we are working in the C^0-topology.

Finite energy homeomorphisms and infinite twists. Our proof of Theorem 1.2 is partially inspired by suggestions of Fathi and the approach of Oh towards the simplicity question. It exploits the interplay between the C^0-topology and the celebrated Hofer metric, which is a bi-invariant distance on the group $\mathrm{Diff}_c(\mathbb{D}^2, \omega)$ of area-preserving diffeomorphisms. Recall that any element of our group $\mathrm{Homeo}_c(\mathbb{D}^2, \omega)$ is a C^0-limit of a sequence in $\mathrm{Diff}_c(\mathbb{D}^2, \omega)$. We call an element of $\mathrm{Homeo}_c(\mathbb{D}^2, \omega)$ a *finite energy homeomorphism* if it is the C^0-limit of a sequence of diffeomorphisms whose Hofer norm is uniformly bounded (see Definition 3.1). We prove that finite energy homeomorphisms form a proper normal subgroup of $\mathrm{Homeo}_c(D^2, \omega)$, implying Theorem 1.2.

The most difficult task consists in proving properness. We prove it by showing that the so-called "infinite twist maps" (see Section 3.2) are not finite energy homeomorphisms. This resolves in particular what McDuff and Salamon refer to as the *Infinite Twist Conjecture*, which is Problem 43 on their list of open problems (see [MS17, §14.7]); see Corollary 3.5 for the precise statement of our result.

Organization of the paper. We now explain the organization of the paper. In Section 2, we review some of the necessary background from symplectic geometry, especially the case of surfaces. Section 3 then proves the Simplicity Conjecture, assuming some new facts about the PFH spectral invariants whose proofs we defer to the next section. The next part of the paper is devoted to proving the needed material about PFH spectral invariants. This starts in Section 4, where we review the construction of periodic Floer homology and the associated spectral invariants and we prove some of the properties of PFH spectral invariants, such as Hofer continuity. The next section proves the key fact that these PFH spectral invariants are C^0 continuous for surface diffeomorphisms. The next section is about computations: we explain some relevant computations of PFH, leading to a proof of a kind of Weyl law for positive monotone twist maps.

As a kind of roadmap for the reader who is interested in the Simplicity Conjecture, but not a Floer homology specialist, we want to emphasize that if one is willing to take the needed properties of the PFH spectral invariants on faith, the proof can entirely be understood after reading Sections 2 and 3.

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2. Preliminaries about the symplectic geometry of surfaces

Here we collect some basic facts, and fix notation, concerning two-dimensional symplectic geometry and diffeomorphism groups.

2.1. *Symplectic form on the disc and sphere.* Let $\mathbb{S}^2 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ and $\mathbb{D} := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 1\}$. We equip the sphere $\mathbb{S}^2$ with the symplectic form $\omega := \frac{1}{4\pi} d\theta \wedge dz$, where (θ, z) are cylindrical coordinates on $\mathbb{R}^3$. Note that with this form, $\mathbb{S}^2$ has area 1. Let $S^+ = \{(x, y, z) \in \mathbb{S}^2 : z \geqslant 0\}$ be the northern hemisphere in $\mathbb{S}^2$. In certain sections of the paper, we will need to identify the disc $\mathbb{D}$ with S^+. To do this, we will take the embedding $\iota : \mathbb{D} \to \mathbb{S}^2$ given by the formula

$$\iota(r, \theta) = (\theta, 1 - r^2), \tag{1}$$

where (r, θ) denotes the standard polar coordinates on $\mathbb{R}^2$. We will equip the disc with the area form given by the pullback of ω under ι; explicitly, this is given by the formula $\frac{1}{2\pi} r dr \wedge d\theta$. We will denote this form by ω as well. Note that this gives the disc a total area of $\frac{1}{2}$.

Any area form on $\mathbb{S}^2$ or $\mathbb{D}$ is equivalent to the above differential forms, up to multiplication by a constant.

2.2. *The C^0 topology.* Here we fix our conventions and notation concerning the C^0 topology.

Denote by $\mathrm{Homeo}(\mathbb{S}^2)$ the group of homeomorphisms of the sphere and by $\mathrm{Homeo}_c(\mathbb{D})$ the group of homeomorphisms of the disc whose support is contained in the interior of $\mathbb{D}$. Let d be a Riemannian distance on $\mathbb{S}^2$. The C^0 distance between two maps $\phi, \psi : \mathbb{S}^2 \to \mathbb{S}^2$, is defined by

$$d_{C^0}(\phi, \psi) = \max_{x \in \mathbb{S}^2} d(\phi(x), \psi(x)).$$

We will say that a sequence of maps $\phi_i : \mathbb{S}^2 \to \mathbb{S}^2$ *converges uniformly*, or C^0-converges, to ϕ, if $d_{C^0}(\phi_i, \phi) \to 0$ as $i \to \infty$. As is well known, the notion of C^0-convergence does not depend on the choice of the Riemannian metric. The topology induced by d_{C^0} on $\mathrm{Homeo}(\mathbb{S}^2)$ is referred to as the *C^0 topology.*

The C^0 topology on $\mathrm{Homeo}_c(\mathbb{D})$ is defined analogously as the topology induced by the distance

$$d_{C^0}(\phi, \psi) = \max_{x \in \mathbb{D}} d(\phi(x), \psi(x)).$$

2.3. *Hamiltonian diffeomorphisms.* Let

$$\mathrm{Diff}(\mathbb{S}^2, \omega) := \{\theta \in \mathrm{Diff}(\mathbb{S}^2) : \theta^* \omega = \omega\}$$

denote the group of area-preserving, in other words symplectic, diffeomorphisms of the sphere. Let $C^\infty(\mathbb{S}^1 \times \mathbb{S}^2)$ denote the set of smooth time-dependent Hamiltonians on $\mathbb{S}^2$. A smooth Hamiltonian $H \in C^\infty(\mathbb{S}^1 \times \mathbb{S}^2)$ gives rise to a time-dependent vector field X_H, called the *Hamiltonian vector field*, defined via the equation

$$\omega(X_{H_t}, \cdot) = dH_t.$$

The *Hamiltonian flow* of H, denoted by φ_H^t, is by definition the flow of X_H. A *Hamiltonian diffeomorphism* is a diffeomorphism that arises as the time-one map of a Hamiltonian flow. It is easy to verify that every Hamiltonian diffeomorphism of $\mathbb{S}^2$ is area-preserving. And, as is well known, every area-preserving diffeomorphism of the sphere is in fact a Hamiltonian diffeomorphism. As for the disc, as mentioned in the introduction, every $\theta \in \mathrm{Diff}_c(\mathbb{D}, \omega)$ is Hamiltonian, in the sense that one can find $H \in C_c^\infty(\mathbb{S}^1 \times \mathbb{D})$ such that $\theta = \varphi_H^1$, where the notation is as in the sphere case. Here, $C_c^\infty(\mathbb{S}^1 \times \mathbb{D})$ denotes the set of Hamiltonians on $\mathbb{D}$ whose support is compactly contained in the interior of $\mathbb{S}^1 \times \mathbb{D}$.

Note that $\mathrm{Diff}(\mathbb{S}^2, \omega) \subset \mathrm{Homeo}_0(\mathbb{S}^2, \omega)$ and $\mathrm{Diff}_c(\mathbb{D}, \omega) \subset \mathrm{Homeo}_c(\mathbb{D}, \omega)$. It is well known that $\mathrm{Diff}(\mathbb{S}^2, \omega)$ and $\mathrm{Diff}_c(\mathbb{D}, \omega)$ are dense, with respect to the C^0 topology, in $\mathrm{Homeo}_0(\mathbb{S}^2, \omega)$ and $\mathrm{Homeo}_c(\mathbb{D}, \omega)$, respectively.

2.4. *The action functional and its spectrum.* Spectral invariants take values in the "action spectrum." We now explain what this spectrum is.

Denote by $\Omega := \{z : \mathbb{S}^1 \to \mathbb{S}^2\}$ the space of all loops in $\mathbb{S}^2$. By a *capping* of a loop $z : \mathbb{S}^1 \to \mathbb{S}^2$, we mean a map

$$u : D^2 \to \mathbb{S}^2,$$

such that $u|_{\partial D^2} = z$. We say two cappings u, u' for a loop z are *equivalent* if u, u' are homotopic rel z. Henceforth, we will only consider cappings up to this equivalence relation. Note that given a capping u of a loop z, all other cappings of z are of the form $u\#A$, where $A \in \pi_2(\mathbb{S}^2)$ and $\#$ denotes the operation of connected sum. A *capped loop* is a pair (z, u), where z is a loop and u is a capping for z. We will denote by $\tilde{\Omega}$ the space of all capped loops in the sphere.

Let $H \in C^\infty(\mathbb{S}^1 \times \mathbb{S}^2)$ denote a smooth Hamiltonian in $\mathbb{S}^2$. Recall that $\mathcal{A}_H : \tilde{\Omega} \to \mathbb{R}$, the *action functional* associated to the Hamiltonian H, is defined by

$$\mathcal{A}_H(z, u) = \int_0^1 H(t, z(t))dt \; + \int_{D^2} u^*\omega. \tag{2}$$

Note that $\mathcal{A}_H(z, u\#A) = \mathcal{A}_H(z, u) + \omega(A)$ for every $A \in \pi_2(\mathbb{S}^2)$.

The set of critical points of $\mathcal{A}_H$, denoted by $\mathrm{Crit}(\mathcal{A}_H)$, consists of capped loops $(z, u) \in \tilde{\Omega}$ such that z is a 1-periodic orbit of the Hamiltonian flow φ^t_H. We will often refer to such (z, u) as a capped 1-periodic orbit of φ^t_H. Given an integer k, we may also define a capped k-periodic orbit of H as a pair (z, u), where z is a k-periodic orbit of H and u is a capping of the loop $t \mapsto z(kt)$. The action of a capped k-periodic orbit (z, u) is then defined by the same formula as (2) except that the first integral should be taken between 0 and k.

The *action spectrum* of H, denoted by $\mathrm{Spec}(H)$, is the set of critical values of $\mathcal{A}_H$; it has Lebesgue measure zero. It turns out that the action spectrum $\mathrm{Spec}(H)$ is independent of H in the following sense: If H' is another Hamiltonian such that $\varphi^1_H = \varphi^1_{H'}$, then there exists a constant $C \in \mathbb{R}$ such that

$$\mathrm{Spec}(H) = \mathrm{Spec}(H') + C,$$

where $\mathrm{Spec}(H') + C$ is the set obtained from $\mathrm{Spec}(H')$ by adding the value C to every element of $\mathrm{Spec}(H')$. Schwarz [Sch00, Lemma 3.3] proves this in the case where ω vanishes on $\pi_2(M)$, and the proof generalizes readily to general symplectic manifolds. Moreover, it follows from the proof of [Sch00, Lemma 3.3] that if H, H' are supported in the northern hemisphere $S^+ \subset \mathbb{S}^2$, then the above constant C is zero and hence

$$\mathrm{Spec}(H) = \mathrm{Spec}(H'). \tag{3}$$

The PFH spectral invariants will take values in a more general set, which we call the *higher order action spectrum*. To define it, let H, G be two Hamiltonians. The *composition* of H and G is the Hamiltonian $H\#G(t, x) := H(t, x) + G(t, (\phi^t_H)^{-1}(x))$. This is defined so that $\phi^t_{H\#G} = \phi^t_H \circ \phi^t_G$; see, for example, [HZ94, §5.1, Prop. 1]. Denote by H^k the k-times composition of H with itself. For any $d > 0$, we now define the *order d spectrum* of H by

$$\mathrm{Spec}_d(H) := \cup_{k_1+\cdots+k_j=d} \ \mathrm{Spec}(H^{k_1}) + \cdots + \mathrm{Spec}(H^{k_j}).$$

Note that $\mathrm{Spec}_d(H)$ may equivalently be described as follows: For every value $a \in \mathrm{Spec}_d(H)$, there exist capped periodic orbits $(z_1, u_1), \ldots, (z_k, u_k)$ of H the sum of whose periods is d and such that

$$a = \sum \mathcal{A}_H(z_i, u_i).$$

We can use the above to define the action spectrum for compactly supported disc maps. Recall from Section 2.1 our convention to identify the northern hemisphere of $\mathbb{S}^2$ with the disc; we will use this to define the action spectrum in the case of the disc.

More precisely, if H, H' are supported in the northern hemisphere $S^+ \subset \mathbb{S}^2$ and generate the same time-1 map ϕ, we in fact have $\mathrm{Spec}_d(H) = \mathrm{Spec}_d(H')$ for all $d > 0$. Indeed, as an immediate consequence of equation (3) we have $\mathrm{Spec}(H^k) = \mathrm{Spec}(H'^k)$ for all $k \in \mathbb{N}$, and so it follows from the definition that $\mathrm{Spec}_d(H) = \mathrm{Spec}_d(H')$ for all $d > 0$. Hence, if $\phi \in \mathrm{Diff}_{S^+}(\mathbb{S}^2, \omega)$, then we can define the action spectra of ϕ without any ambiguity by setting

$$\mathrm{Spec}_d(\phi) = \mathrm{Spec}_d(H), \tag{4}$$

where H is any Hamiltonian in $C^\infty_c(\mathbb{S}^1 \times S^+)$ such that $\phi = \varphi^1_H$.

2.5. *Equivalence of perfectness and simplicity.* The goal of this section is to show that in the case of $\mathrm{Homeo}_c(\mathbb{D}, \omega)$, perfectness and simplicity are equivalent. This is completely independent from the rest of the paper, and not needed to prove the simplicity conjecture itself — it is only used to establish the corollary that the group is not perfect.

PROPOSITION 2.1. *Any non-trivial normal subgroup H of* $\mathrm{Homeo}_c(\mathbb{D}, \omega)$ *contains the commutator subgroup of* $\mathrm{Homeo}_c(\mathbb{D}, \omega)$. *Hence,* $\mathrm{Homeo}_c(\mathbb{D}, \omega)$ *is perfect if and only if it is simple.*

As promised in the introduction, we prove in the next corollary that $\mathrm{Homeo}_c(\mathbb{D}, \omega)$ admits no non-trivial continuous homomorphisms. This fact seems to be well known to the experts, however, we do not know of a published reference for it.

COROLLARY 2.2. *The group* $\mathrm{Homeo}_c(\mathbb{D}, \omega)$ *admits no non-trivial homomorphism that is continuous with respect to the C^0 topology.*

Proof. Let H be a non-trivial normal subgroup of $\mathrm{Homeo}_c(\mathbb{D}, \omega)$. We will show that H is dense with respect to the C^0 topology; this proves the corollary because the kernel of a continuous homomorphism is closed.

By Proposition 2.1, we know that H contains the commutator subgroup of $\mathrm{Diff}_c(\mathbb{D},\omega)$. Consequently, H contains the kernel of the Calabi homomorphism as the commutator subgroup of $\mathrm{Diff}_c(\mathbb{D},\omega)$ coincides with the kernel of the Calabi invariant [Ban78]. (For the interested reader, we review the Calabi homomorphism in Section 3.)

We claim that the kernel of the Calabi invariant is dense in $\mathrm{Diff}_c(\mathbb{D},\omega)$; hence, it is dense in $\mathrm{Homeo}_c(\mathbb{D},\omega)$. Indeed, take any $\psi \in \mathrm{Diff}_c(\mathbb{D},\omega)$ and let a denote $\mathrm{Cal}(\psi)$. Pick Hamiltonians H_n such that

- H_n is supported in a disc of diameter $\frac{1}{n}$;
- $\int_{\mathbb{D}} H_n = -a$ — thus, $\mathrm{Cal}(\varphi^1_{H_n}) = -a$.

Then, $\mathrm{Cal}(\varphi^1_{H_n} \circ \psi) = 0$ and $\varphi^1_{H_n} \circ \psi \xrightarrow{C^0} \psi$. □

The proof of Proposition 2.1 relies on a general argument, due to Epstein [Eps70] and Higman [Hig54], which essentially shows that perfectness implies simplicity for transformation groups satisfying certain assumptions. We will present a version of this argument, which we learned in [Fat80a], in our context.

Proof of Proposition 2.1. Pick $h \in H$ such that $h \neq \mathrm{Id}$. We can find a closed *topological disc* — that is, a set that is homeomorphic to a standard Euclidean disc $\mathbb{D}' \subset \mathbb{D}$ such that $h(\mathbb{D}') \cap \mathbb{D}' = \emptyset$. Denote by $\mathrm{Homeo}_c(\mathbb{D}',\omega)$ the subset of $\mathrm{Homeo}_c(\mathbb{D},\omega)$ consisting of area-preserving homeomorphisms whose supports are contained in the interior of $\mathbb{D}'$. We will first prove the following lemma.

LEMMA 2.3. *The commutator subgroup of* $\mathrm{Homeo}_c(\mathbb{D}',\omega)$ *is contained in* H.

Proof. We must show that for any $f, g \in \mathrm{Homeo}_c(\mathbb{D}',\omega)$, the commutator $[f,g] := fgf^{-1}g^{-1}$ is an element of H.

First, observe that for any $f \in \mathrm{Homeo}_c(\mathbb{D}',\omega)$, we have

$$[f,r] \in H \tag{5}$$

for any $r \in H$. Indeed, by normality, $frf^{-1} \in H$ and hence $frf^{-1}r^{-1} \in H$. Next, one can easily check that for any $f, g \in \mathrm{Homeo}_c(\mathbb{D}',\omega)$,

$$[f,g][g,hfh^{-1}] = f[g,[f^{-1},h]]f^{-1}. \tag{6}$$

Note that g and hfh^{-1} are, respectively, supported in $\mathbb{D}'$ and $h(\mathbb{D}')$, which are disjoint. Thus, $[g,hfh^{-1}] = \mathrm{Id}$. Hence, identity (6) yields

$$[f,g] = f[g,[f^{-1},h]]f^{-1}.$$

Now, (5) implies that $[g,[f^{-1},h]] \in H$ which, by normality of H, implies that $f[g,[f^{-1},h]]f^{-1} \in H$. This gives us the conclusion of the lemma. □

We continue with the proof of Proposition 2.1. Fix a small $\varepsilon > 0$, and let $\mathcal{E}$ be the set consisting of all $g \in \mathrm{Homeo}_c(\mathbb{D},\omega)$ whose supports are contained in some topological disc of area ε. It is a well-known fact that the set $\mathcal{E}$ generates the group $\mathrm{Homeo}_c(\mathbb{D},\omega)$. This is usually referred to as the *fragmentation property*, and it was proven by Fathi; see Theorems 6.6, A.6.2, and A.6.5 in [Fat80a].

We claim that $[f,g]\in H$ for any $f,g\in\mathcal{E}$. Indeed, assuming ε is small enough, we can find a topological disc U that contains the supports of f and g and whose area is less than the area of $\mathbb{D}'$. There exists $r\in\mathrm{Homeo}_c(\mathbb{D},\omega)$ such that $r(U)\subset\mathbb{D}'$. As a consequence, rfr^{-1}, rgr^{-1} are both supported in $\mathbb{D}'$ and hence, by Lemma 2.3, $[rfr^{-1},rgr^{-1}]\in H$. Since H is a normal subgroup of $\mathrm{Homeo}_c(\mathbb{D},\omega)$, and $[rfr^{-1},rgr^{-1}]=r[f,g]r^{-1}$, we conclude that $[f,g]\in H$.

Now, the set $\mathcal{E}$ generates $\mathrm{Homeo}_c(\mathbb{D},\omega)$ and $[f,g]\in H$ for any $f,g\in\mathcal{E}$. Hence, the quotient group $\mathrm{Homeo}_c(\mathbb{D},\omega)/H$ is abelian. Thus, H contains the commutator subgroup of $\mathrm{Homeo}_c(\mathbb{D},\omega)$. □

3. **The proof of the Simplicity Conjecture**

We now give the proof of Theorem 1.2, assuming some facts that we will prove later in the paper. More precisely, this section will explain how to prove Theorem 1.2 given various new properties about "PFH spectral invariants" that we then prove.

3.1. *A proper normal subgroup of* $\mathrm{Homeo}_c(\mathbb{D},\omega)$. To prove Theorem 1.2, we will define below a normal subgroup of $\mathrm{Homeo}_c(\mathbb{D},\omega)$ that is a variation on the construction of Oh-Müller [OM07]. We will show that this normal subgroup is proper.

The energy, or the *Hofer norm*, of a Hamiltonian $H\in C_c^\infty(\mathbb{S}^1\times\mathbb{D})$ is defined by the quantity

$$\|H\|_{(1,\infty)}=\int_0^1\left(\max_{x\in\mathbb{D}}H(t,\cdot)-\min_{x\in\mathbb{D}}H(t,\cdot)\right)dt.$$

Definition 3.1. An element $\phi\in\mathrm{Homeo}_c(\mathbb{D},\omega)$ is a *finite-energy homeomorphism* if there exists a sequence of smooth Hamiltonians $H_i\in C_c^\infty(\mathbb{S}^1\times\mathbb{D})$ such that the sequence $\|H_i\|_{(1,\infty)}$ is bounded; i.e., there exists $C\in\mathbb{R}$ such that $\|H_i\|_{(1,\infty)}\leqslant C$, and the Hamiltonian diffeomorphisms $\varphi^1_{H_i}$ converge uniformly to ϕ. We will denote the set of all finite-energy homeomorphisms by $\mathrm{FHomeo}_c(\mathbb{D},\omega)$.

Theorem 1.2 will follow from the following result, where we show that

THEOREM 3.2. *The set* $\mathrm{FHomeo}_c(\mathbb{D},\omega)$ *is a proper normal subgroup of* $\mathrm{Homeo}_c(\mathbb{D},\omega)$.

We first show that $\mathrm{FHomeo}_c(\mathbb{D},\omega)$ is a normal subgroup. The properness will be proved in Section 3.4.

Proof that $\mathrm{FHomeo}_c(\mathbb{D},\omega)$ *is a normal subgroup.* Consider smooth Hamiltonians $H,G\in C_c^\infty(\mathbb{S}^1\times\mathbb{D})$. As was partly mentioned in Section 2.4, it is well known (and proved, for example, in [HZ94, §5.1, Prop. 1]) that the Hamiltonians

$$H\#G(t,x) := H(t,x) + G(t,(\varphi_H^t)^{-1}(x)), \quad \bar{H}(t,x) := -H(t,\varphi_H^t(x)), \tag{7}$$

generate the Hamiltonian flows $\varphi_H^t \phi_G^t$ and $(\varphi_H^t)^{-1}$ respectively. Furthermore, given $\psi \in \mathrm{Diff}_c(\mathbb{D},\omega)$, the Hamiltonian $H(t,\psi(x))$ generates the flow $\psi^{-1}\varphi_H^t\psi$.

We now show that FHomeo_c is closed under conjugation. Take $\phi \in \mathrm{FHomeo}_c(\mathbb{D},\omega)$, and let H_i and C be as in Definition 3.1. Let $\psi \in \mathrm{Homeo}_c(\mathbb{D},\omega)$, and take a sequence $\psi_i \in \mathrm{Diff}_c(\mathbb{D},\omega)$ that converges uniformly to ψ. Consider the Hamiltonians $K_i(t,x) := H_i(t,\psi_i(x))$. The corresponding Hamiltonian diffeomorphisms are the conjugations $\psi_i^{-1}\varphi^1_{H_i}\psi_i$ that converge uniformly to $\psi^{-1}\phi\psi$. Furthermore,

$$\|K_i\|_{(1,\infty)} = \|H_i\|_{(1,\infty)} \leqslant C,$$

where the inequality follows from the definition of $\mathrm{FHomeo}_c(\mathbb{D},\omega)$.

We will next check that FHomeo_c is a group. Take $\phi,\psi \in \mathrm{FHomeo}_c$, and let $H_i, G_i \in C_c^\infty(\mathbb{S}^1 \times \mathbb{D})$ be two sequences of Hamiltonians such that $\varphi^1_{H_i}, \varphi^1_{G_i}$ converge uniformly to ϕ,ψ, respectively, and $\|H_i\|_{(1,\infty)}, \|G_i\|_{(1,\infty)} \leqslant C$ for some constant C. Then, the sequence $\varphi^{-1}_{H_i} \circ \varphi^1_{G_i}$ converges uniformly to $\phi^{-1} \circ \psi$. Moreover, by the above formulas, we have $\varphi^{-1}_{H_i} \circ \varphi^1_{G_i} = \varphi^1_{\overline{H}_i\#G_i}$. Since $\|\overline{H}_i\#G_i\|_{(1,\infty)} \leqslant \|H_i\|_{(1,\infty)} + \|G_i\|_{(1,\infty)} \leqslant 2C$, this proves that $\phi^{-1}\circ\psi \in \mathrm{FHomeo}_c$, which completes the proof that FHomeo_c is a group. $\square$

Remark 3.3. In defining $\mathrm{FHomeo}_c(\mathbb{D},\omega)$ as above, we were inspired by the article of Oh and Müller [OM07], who defined a normal subgroup of $\mathrm{Homeo}_c(\mathbb{D},\omega)$, denoted by $\mathrm{Hameo}_c(\mathbb{D},\omega)$, which is usually referred as the group of *hameomorphisms*; see Appendix B for its definition. It has been conjectured that $\mathrm{Hameo}_c(\mathbb{D},\omega)$ is a proper normal subgroup of $\mathrm{Homeo}_c(\mathbb{D},\omega)$; see, for example, [OM07, Question 4.3].

It can easily be verified that $\mathrm{Hameo}_c(\mathbb{D},\omega) \subset \mathrm{FHomeo}_c(\mathbb{D},\omega)$. Hence, it follows from the above theorem that $\mathrm{Hameo}_c(\mathbb{D},\omega)$ is a proper normal subgroup of $\mathrm{Homeo}_c(\mathbb{D},\omega)$.

In the next section, we will see explicit examples of ϕ that we will show are in $\mathrm{Homeo}_c(\mathbb{D},\omega) \setminus \mathrm{FHomeo}_c(\mathbb{D},\omega)$.

3.2. *The Calabi invariant and the infinite twist.* The hard part of Theorem 3.2 is to show properness. Here we describe the key example of an area-preserving homeomorphism that is not in $\mathrm{FHomeo}_c(\mathbb{D},\omega)$.

We first summarize some background that will motivate what follows. As mentioned above, for smooth, area-preserving compactly supported two-disc diffeomorphisms, non-simplicity is known, via the Calabi invariant. More precisely, the *Calabi invariant* of $\theta \in \mathrm{Diff}_c(\mathbb{D}, \omega)$ is defined as follows. Pick any Hamiltonian $H \in C_c^\infty(\mathbb{S}^1 \times \mathbb{D})$ such that $\theta = \varphi_H^1$. Then,

$$\mathrm{Cal}(\theta) := \int_{\mathbb{S}^1} \int_{\mathbb{D}} H \, \omega \, dt.$$

It is well known that the above integral does not depend on the choice of H and so $\mathrm{Cal}(\theta)$ is well defined; it is also known that $\mathrm{Cal} : \mathrm{Diff}_c(\mathbb{D}, \omega) \to \mathbb{R}$ is a non-trivial group homomorphism, i.e., $\mathrm{Cal}(\theta_1 \theta_2) = \mathrm{Cal}(\theta_1) + \mathrm{Cal}(\theta_2)$. For further details on the Calabi homomorphism, see [Cal70], [MS17].

We will need to know the value of the Calabi invariant for the following class of area-preserving diffeomorphisms. Let $f : [0, 1] \to \mathbb{R}$ be a smooth function vanishing near 1, and define $\phi_f \in \mathrm{Diff}_c(\mathbb{D}, \omega)$ by $\phi_f(0) := 0$ and $\phi_f(r, \theta) := (r, \theta + 2\pi f(r))$. If the function f is taken to be (positive/negative) monotone, then the map ϕ_f is referred to as a *(positive/negative) monotone twist*. Since we will be working exclusively with positive monotone twists, we will assume monotone twists are all positive, unless otherwise stated.

Now suppose that $\omega = \frac{1}{2\pi} r dr \wedge d\theta$. A simple computation (see our conventions in Section 2) shows that ϕ_f is the time–1 map of the flow of the Hamiltonian defined by

$$F(r, \theta) = \int_r^1 s f(s) ds. \tag{8}$$

From this we compute

$$\mathrm{Cal}(\phi_f) = \int_0^1 \int_r^1 s f(s) ds \; r dr. \tag{9}$$

We can now introduce the element that will not be in $\mathrm{FHomeo}_c(\mathbb{D}, \omega)$. Let $f : (0, 1] \to \mathbb{R}$ be a smooth function that vanishes near 1, is decreasing, and satisfies $\lim_{r \to 0} f(r) = \infty$. Define $\phi_f \in \mathrm{Homeo}_c(\mathbb{D}, \omega)$ by $\phi(0) := 0$ and

$$\phi_f(r, \theta) := (r, \theta + 2\pi f(r)). \tag{10}$$

It is not difficult to see that ϕ_f is indeed an element of $\mathrm{Homeo}_c(\mathbb{D}, \omega)$ that is in fact smooth away from the origin. We will refer to ϕ_f as an *infinite twist*.

We use infinite twists ϕ_f to prove Theorem 3.2 by proving the following result.

THEOREM 3.4. *If*

$$\int_0^1 \int_r^1 s f(s) ds \; r dr = \infty, \tag{11}$$

then $\phi_f \notin \mathrm{FHomeo}_c(\mathbb{D}, \omega)$.

Since, as stated in Remark 3.3, $\mathrm{Hameo}_c(\mathbb{D}, \omega) \subset \mathrm{FHomeo}_c(\mathbb{D}, \omega)$, we obtain the following corollary from Theorem 3.4, which resolves the Infinite Twist conjecture[5] mentioned in the introduction.

COROLLARY 3.5 ("Infinite Twist Conjecture"). *Any infinite twist ϕ_f satisfying (11) is not in* $\mathrm{Hameo}_c(\mathbb{D}, \omega)$.

Infinite twists can be defined on any symplectic manifold, and we discuss them further in Appendix B in the context of future open questions.

3.3. *Spectral invariants from periodic Floer homology.* To prove Theorem 3.4, we use the theory of *periodic Floer homology* (PFH), discussed in Section 4. PFH is a version of Floer homology for area-preserving diffeomorphisms that was introduced by Hutchings [Hut02], [HS05]. As with ordinary Floer homology, PFH can be used to define "spectral invariants." More precisely, in the present context these spectral invariants take the form of a sequence of functions $c_d : \mathrm{Diff}_c(\mathbb{D}, \omega) \to \mathbb{R}$, where $d \in \mathbb{N}$, which we call *PFH spectral invariants* and which satisfy various useful properties. We give the definition of c_d in Section 4.3; see, in particular, Remark 4.6.

The definition of PFH spectral invariants is due to Michael Hutchings [Hut17], but very few properties have been established about these. We will prove in Theorem 4.5 that the PFH spectral invariants satisfy the following properties:

(1) Normalization: $c_d(\mathrm{Id}) = 0$.
(2) Monotonicity: Suppose that $H \leqslant G$ where $H, G \in C^\infty_c(\mathbb{S}^1 \times \mathbb{D})$. Then, $c_d(\varphi^1_H) \leqslant c_d(\varphi^1_G)$ for all $d \in \mathbb{N}$.
(3) Hofer Continuity: $|c_d(\varphi^1_H) - c_d(\varphi^1_G)| \leqslant d\|H - G\|_{(1,\infty)}$.
(4) Spectrality: $c_d(\varphi^1_H) \in \mathrm{Spec}_d(H)$ for any $H \in C^\infty_c(\mathbb{S}^1 \times \mathbb{D})$, where $\mathrm{Spec}_d(H)$ is the order d spectrum of H defined in Section 2.4.

A key property, which allows us to use the PFH spectral invariants for studying homeomorphisms (as opposed to diffeomorphisms), is the following theorem, which we prove in Section 5 via the methods of continuous symplectic topology.

THEOREM 3.6. *The spectral invariant $c_d : \mathrm{Diff}_c(\mathbb{D}, \omega) \to \mathbb{R}$ is continuous with respect to the C^0 topology on $\mathrm{Diff}_c(\mathbb{D}, \omega)$. Furthermore, it extends continuously to* $\mathrm{Homeo}_c(\mathbb{D}, \omega)$.

[5]The actual formulation in [MS17] of the Infinite Twist conjecture is slightly different than this, because it does not include the condition (11). However, without this condition, one can produce infinite twists that lie in $\mathrm{Hameo}_c(\mathbb{D}, \omega)$. The authors of [MS17] have confirmed in private communication with us that imposing condition (11) is consistent with what they intended.

Another key property is the following, which was originally conjectured in greater generality by Hutchings [Hut17].

THEOREM 3.7. *The PFH spectral invariants* $c_d : \mathrm{Diff}_c(\mathbb{D}, \omega) \to \mathbb{R}$ *satisfy the* Calabi property

$$\lim_{d \to \infty} \frac{c_d(\varphi)}{d} = \mathrm{Cal}(\varphi) \tag{12}$$

if φ *is a monotone twist map of the disc.*

Property (12) can be thought of as a kind of analogue of the "Volume Property" for ECH spectral invariants proved in [CGHR15]. Our proof of Theorem 3.7, presented in Section 6, deduces it from computations of PFH for certain twists maps of $\mathbb{S}^2$; this is a topic of interest beyond the Simplicity Conjecture; for example, we used these computations in [CGHS23]. We mention for the interested reader that some newer proofs of Theorem 3.7, proving more general statements via different methods, can be found in [CGPZ21], [EH21].

3.4. *Proofs of the theorems.* We now give the proofs of Theorems 3.2, 3.4 and 1.2, assuming the results about PFH spectral invariants stated above.

Proof. Theorem 1.2 is an immediately consequence of Theorem 3.2, and Theorem 3.2 is an immediate consequence of Theorem 3.4, since we already proved in Section 3.1 that $\mathrm{FHomeo}_c(\mathbb{D}, \omega)$ is a (non-trivial) normal subgroup. Thus, it remains to prove Theorem 3.4.

We start for the benefit of the reader with an outline of how we do this. Theorem 3.6 allows one to define the PFH spectral invariants for any $\psi \in \mathrm{Homeo}_c(\mathbb{D}, \omega)$. We will show, by using the Hofer Continuity property, that if ψ is a finite-energy homeomorphism, then the sequence of PFH spectral invariants $\{c_d(\psi)\}_{d \in \mathbb{N}}$ grows at most linearly. On the other hand, in the case of an infinite twist ϕ_f, satisfying the condition in equation (11), the sequence $\{c_d(\phi_f)\}_{d \in \mathbb{N}}$ has super-linear growth, as a consequence of the Calabi property (12). From this we can conclude that $\phi_f \notin \mathrm{FHomeo}_c(\mathbb{D}, \omega)$, as desired.

The details are as follows. We begin with the following lemma, which tells us that for a finite-energy homeomorphism ψ, the sequence of PFH spectral invariants $\{c_d(\psi)\}_{d \in \mathbb{N}}$ grows at most linearly.

LEMMA 3.8. *Let* $\psi \in \mathrm{FHomeo}_c(\mathbb{D}, \omega)$ *be a finite-energy homeomorphism. Then, there exists a constant* C, *depending on* ψ, *such that*

$$\frac{c_d(\psi)}{d} \leqslant C \forall d \in \mathbb{N}.$$

Proof. By definition, ψ being a finite-energy homeomorphism implies that there exist smooth Hamiltonians $H_i \in C_c^\infty(\mathbb{S}^1 \times \mathbb{D})$ such that the sequence

$\|H_i\|_{(1,\infty)}$ is bounded, i.e., there exists $C \in \mathbb{R}$ such that $\|H_i\|_{(1,\infty)} \leqslant C$, and the Hamiltonian diffeomorphisms $\varphi^1_{H_i}$ converge uniformly to ψ.

The Hofer continuity property and the fact that $c_d(\mathrm{Id}) = 0$ imply that

$$c_d(\varphi^1_{H_i}) \leqslant d\|H_i\|_{(1,\infty)} \leqslant dC$$

for each $d \in \mathbb{N}$.

On the other hand, by Theorem 3.6, $c_d(\psi) = \lim_{i\to\infty} c_d(\varphi^1_{H_i})$. We conclude from the above inequality that $c_d(\psi) \leqslant dC$ for each $d \in \mathbb{N}$. □

We now turn our attention to showing that the PFH spectral invariants of an infinite twist ϕ_f, which satisfies equation (11), violate the inequality from the above lemma. We will need the following.

LEMMA 3.9. *Let ϕ_f be an infinite twist satisfying* (11), *as described in Section* 3.2. *Then there exists a sequence of smooth monotone twists $\phi_{f_i} \in \mathrm{Diff}_c(\mathbb{D},\omega)$ satisfying the following properties*:

(1) *the sequence ϕ_{f_i} converges in the C^0 topology to ϕ_f*;
(2) *there exist Hamiltonians F_i, compactly supported in the interior of the disc $\mathbb{D}$, such that $\varphi^1_{F_i} = \phi_{f_i}$ and $F_i \leqslant F_{i+1}$*;
(3) $\lim\limits_{i\to\infty} \mathrm{Cal}(\phi_{f_i}) = \infty$.

Proof. Recall that f is a decreasing function of r that vanishes near 1 and satisfies $\lim\limits_{r\to 0} f(r) = \infty$. It is not difficult to see that we can pick smooth functions $f_i : [0,1] \to \mathbb{R}$ satisfying the following properties:

(1) $f_i = f$ on $[\frac{1}{i}, 1]$;
(2) $f_i \leqslant f_{i+1}$.

Let us check that the monotone twists ϕ_{f_i} satisfy the requirements of the lemma. To see that they converge to ϕ_f, observe that ϕ_f and ϕ_{f_i} coincide outside the disc of radius $\frac{1}{i}$. Hence, $\phi_f^{-1}\phi_{f_i}$ converges uniformly to Id because it is supported in the disc of radius $\frac{1}{i}$. Next, note that by formula (8), ϕ_{f_i} is the time–1 map of the Hamiltonian flow of $F_i(r,\theta) = \int_r^1 s f_i(s)ds$. Clearly, $F_i \leqslant F_{i+1}$ because $f_i \leqslant f_{i+1}$. Finally, by formula (9) we have

$$\mathrm{Cal}(\phi_{f_i}) = \int_0^1 \int_r^1 s f_i(s) ds\, r dr \geqslant \int_{\frac{1}{i}}^1 \int_r^1 s f_i(s) ds\, r dr = \int_{\frac{1}{i}}^1 \int_r^1 s f(s) ds\, r dr.$$

Recall that f has been picked such that $\int_0^1 \int_r^1 s f(s) ds \; r dr = \infty$; see equation (11). We conclude that $\lim\limits_{i\to\infty} \mathrm{Cal}(\phi_{f_i}) = \infty$. □

We will now use Lemma 3.9 to complete the proof of Theorem 3.4.

By the Monotonicity property, we have $c_d(\phi_{f_i}) \leqslant c_d(\phi_{f_{i+1}})$ for each $d \in \mathbb{N}$. Since ϕ_{f_i} converges in C^0 topology to ϕ_f, we conclude from Theorem 3.6 that

$c_d(\phi_f) = \lim_{i\to\infty} c_d(\phi_{f_i})$. Combining the previous two lines we obtain the following inequality:

$$c_d(\phi_{f_i}) \leqslant c_d(\phi_f) \quad \forall\, d, i \in \mathbb{N}.$$

Now the Calabi property of Theorem 3.7 tells us that $\lim_{d\to\infty} \frac{c_d(\phi_{f_i})}{d} = \mathrm{Cal}(\phi_{f_i})$. Combining this with the previous inequality we get $\mathrm{Cal}(\phi_{f_i}) \leqslant \lim_{d\to\infty} \frac{c_d(\phi_f)}{d}$ for all i. Hence, by the third item in Lemma 3.9,

$$\lim_{d\to\infty} \frac{c_d(\phi_f)}{d} = \infty,$$

and so by Lemma 3.8, ϕ_f is not in $\mathrm{FHomeo}_c(\mathbb{D}, \omega)$. □

Remark 3.10. The proof outlined above does not use the full force of Theorem 3.7; it only uses the fact that $\lim_{d\to\infty} \frac{c_d(\varphi)}{d} \geqslant \mathrm{Cal}(\varphi)$.

4. **Periodic Floer Homology and basic properties of the PFH spectral invariants**

The remainder of the paper is devoted to proving the promised properties of the PFH spectral invariants required to prove Theorem 3.4 and therefore Theorem 1.2.

In this section, we recall the definition of Periodic Floer Homology (PFH), due to Hutchings [Hut02], [HS05], and the construction of the spectral invariants that arise from this theory, also due to Hutchings [Hut17]. We will then prove that PFH spectral invariants satisfy the Monotonicity, Hofer Continuity, and Spectrality properties that we mentioned in the previous section. The spectral invariants appearing in Section 3.3 are defined by identifying area-preserving maps of the disc, $\mathrm{Diff}_c(\mathbb{D}, \omega)$, with area-preserving maps of the sphere, which are supported in the northern hemisphere S^+, and using the PFH of $\mathbb{S}^2$. Thus, the three aforementioned properties will follow from related properties about PFH spectral invariants on $\mathbb{S}^2$; see Theorem 4.5 below.

4.1. *Preliminaries on J-holomorphic curves and stable Hamiltonian structures.* A stable Hamiltonian structure (SHS) on a closed three-manifold Y is a pair (α, Ω), consisting of a 1-form α and a closed two-form Ω, such that

(1) $\alpha \wedge \Omega$ is a volume form on Y;
(2) $\ker(\Omega) \subset \ker(d\alpha)$.

Observe that the first condition implies that Ω is non-vanishing, and as a consequence, the second condition is equivalent to $d\alpha = g\Omega$, where $g : Y \to \mathbb{R}$ is a smooth function.

A stable Hamiltonian structure determines a plane field $\xi := \ker(\alpha)$ and a *Reeb* vector field R on Y given by

$$R \in \ker(\Omega),\ \alpha(R) = 1.$$

Closed integral curves of R are called *Reeb orbits*; we regard Reeb orbits as equivalent if they are equivalent as currents.

Stable Hamiltonian structures were introduced in [BEH+03], [CM05] as a setting in which one can obtain general Gromov-type compactness results, such as the SFT compactness theorem, for pseudo-holomorphic curves in $\mathbb{R}\times Y$. Here are two examples of stable Hamiltonian structures that are relevant to our story.

Example 4.1. A *contact* form on Y is a 1-form λ such that $\lambda \wedge d\lambda$ is a volume form. The pair $(\alpha, \Omega) := (\lambda, d\lambda)$ gives a stable Hamiltonian structure with $g \equiv 1$. The plane field ξ is the associated contact structure, and the Reeb vector field as defined above gives the usual Reeb vector field of a contact form.

The *contact symplectization* of Y is

$$X := \mathbb{R} \times Y_\varphi,$$

which has a standard symplectic form, defined by

$$\Gamma = d(e^s\lambda), \tag{13}$$

where s denotes the coordinate on $\mathbb{R}$.

Example 4.2. Let (S, ω_S) be a closed surface, and denote by φ a smooth area-preserving diffeomorphism of S. Define the mapping torus

$$Y_\varphi := \frac{S \times [0,1]}{(x,1) \sim (\varphi(x), 0)}.$$

Let r be the coordinate on $[0,1]$. Now, Y_φ carries a stable Hamiltonian structure $(\alpha, \Omega) := (dr, \omega_\varphi)$, where ω_φ is the canonical closed two form on Y_φ induced by ω_S. Note that the plane field ξ is given by the vertical tangent space of the fibration $\pi : Y_\varphi \to \mathbb{S}^1$ and the Reeb vector field is given by $R = \partial_r$. Here, $g \equiv 0$. Observe that the Reeb orbits here are in correspondence with the periodic orbits of φ.

We define the *symplectization* of Y_φ by

$$X := \mathbb{R} \times Y_\varphi,$$

which has a standard symplectic form, defined by

$$\Gamma = ds \wedge dr + \omega_\varphi, \tag{14}$$

where s denotes the coordinate on $\mathbb{R}$.

We say an almost complex structure J on $X = \mathbb{R} \times Y$ is *admissible*, for a given SHS (α, Ω), if the following conditions are satisfied:

(1) J is invariant under translation in the $\mathbb{R}$-direction of $\mathbb{R} \times Y$;
(2) $J\partial_s = R$, where s denotes the coordinate on the $\mathbb{R}$-factor of $\mathbb{R} \times Y$;
(3) $J\xi = \xi$, where $\xi := \ker(\alpha)$, and $\Omega(v, Jv) > 0$ for all nonzero $v \in \xi$.

We will denote by $\mathcal{J}(\alpha,\Omega)$ the set of almost complex structures that are admissible for (α,Ω). The space $\mathcal{J}(\alpha,\Omega)$ equipped with the C^∞ topology is path connected, and even contractible. When the SHS is clear from context, we will call J admissible without specifying which SHS we are referring to.

Define a *J-holomorphic map* to be a smooth map $u:(\Sigma,j)\to(X,J)$, satisfying the equation

$$du\circ j = J\circ du, \tag{15}$$

where (Σ,j) is a closed Riemann surface (possibly disconnected), minus a finite number of punctures. A J-holomorphic map $u:(\Sigma,j)\to(X,J)$ is called *somewhere injective* if there exists a point $z\in\Sigma$ such that $u^{-1}(u(z))=\{z\}$ and $du:T_z\Sigma\to T_{u(z)}X$ is injective. An equivalence class of J-holomorphic maps under the relation of biholomorphisms of the domain will be called a *J-holomorphic curve*. In this paper, we will only consider J-holomorphic curves that are asymptotic to nondegenerate Reeb orbits at their punctures, and this will be our standing assumption for the remainder of the paper; see [Wen16] for the precise definition of asymptotic in this context. Such a J-holomorphic curve has the property that it is determined by its image if it is somewhere injective [Wen16]. We will call a J-holomorphic curve *irreducible* when its domain is connected. As is common in the literature on ECH, we will sometimes have to consider J-holomorphic maps up to equivalence of currents, and we call such an equivalence class a *J-holomorphic current*; more precisely, a J-holomorphic current is a finite set $\{(C_i,m_i)\}$, where the C_i are distinct irreducible somewhere injective J-holomorphic curves and the m_i are positive integers. We will call a J-holomorphic current *irreducible* when it consists of just one ordered pair (C_i,m_i).

In the lemma below we state a standard property of J-holomorphic curves that plays a key role in our arguments. For a proof see, for example, the argument in [Wen16, Lemma 9.9].

LEMMA 4.3. *Suppose $J\in\mathcal{J}(\alpha,\Omega)$ where (α,Ω) is a stable Hamiltonian structure on Y. If C is a J-holomorphic curve in $\mathbb{R}\times Y$, then Ω is pointwise non-negative on C. Furthermore, Ω vanishes at a point on C only if C is tangent to the span of ∂_s and R.*

4.2. *PFH spectral invariants.* Periodic Floer homology (PFH) is a version of Floer homology, defined by Hutchings [Hut02], [HS05], for area-preserving maps of surfaces. The construction of PFH is closely related to the better-known embedded contact homology (ECH) and, in fact, predates the construction of ECH. We now review what we need to know about the definition of PFH. For further details on PFH, we refer the reader to [Hut02], [HS05], and for more about the motivation underlying the definitions, we refer the reader to [Hut14].

Let (S, ω_S) be a closed[6] surface with an area form, and let φ be a *nondegenerate* smooth area-preserving diffeomorphism. Non-degeneracy is defined as follows: A periodic point p of ϕ, with period k, is said to be non-degenerate if the derivative of φ^k at the point p does not have 1 as an eigenvalue. We say φ is *d-nondegenerate* if all of its periodic points of period at most d are nondegenerate; if φ is d-nondegenerate for all d, then we say it is non-degenerate. A C^∞-generic area-preserving diffeomorphism is nondegenerate. To define spectral invariants, we will need a "twisted" version of PFH, and we now provide the details of its construction.

Remark 4.4. If we were to carry out the construction outlined below, nearly verbatim, for a contact SHS $(\lambda, d\lambda)$, rather than the SHS (dr, ω_φ), then we would obtain the (twisted) *embedded contact homology* ECH; see [Hut14], [HS06] for further details.

4.2.1. *Definition of twisted PFH.* Assume now and below for simplicity that $S = \mathbb{S}^2$ and that φ is nondegenerate. (For other surfaces, a similar story holds, but we will not need this.) The twisted periodic Floer homology $\widetilde{\mathrm{PFH}}$ is the homology of a chain complex $\widetilde{\mathrm{PFC}}$. To define the twisted PFH chain complex, we begin by defining certain finite sets $\alpha = \{(\alpha_i, m_i)\}$, called *orbit sets*. Specifically, we require that each α_i is an embedded Reeb orbit, the α_i are distinct, and the m_i are positive. An orbit set is called a *PFH generator* if $m_i = 1$ whenever α_i is *hyperbolic*.[7] An orbit set α has an associated class $[\alpha] \in H_1(Y_\varphi; \mathbb{Z})$; in the case $S = \mathbb{S}^2$, $H_1(Y_\varphi; \mathbb{Z})$ is canonically identified with $\mathbb{Z}$, and we call the image of $[\alpha]$ under this identification the *degree* of α.

Choose a reference cycle γ_0 in Y_φ such that $\pi|_{\gamma_0} : \gamma_0 \to \mathbb{S}^1$ is an orientation-preserving diffeomorphism, and fix a trivialization τ_0 of ξ over γ_0. We can now define the $\widetilde{\mathrm{PFH}}$ chain complex $\widetilde{\mathrm{PFC}}(\varphi, d)$. A generator of $\widetilde{\mathrm{PFC}}(\varphi, d)$, called a *twisted PFH generator*, is a pair (α, Z), where α is a PFH generator of degree d, and Z is a relative homology class in $H_2(Y_\varphi, \alpha, d\gamma_0)$. Here, $H_2(Y_\varphi, \alpha, \beta)$ is defined to be the set of equivalence classes of 2-chains Z in Y_φ satisfying $\partial Z = \sum m_i \alpha_i - \sum n_i \beta_i$. The original idea behind the definition of a twisted PFH generator is that we will want to study pseudoholomorphic curves C asymptotic to PFH generators α, and then the relative homology class Z allows us to keep track of the homology class of C: We say that a J-holomorphic current C in $X = \mathbb{R} \times Y_\varphi$, is a current *from* (α, Z) *to* (β, Z') if C is asymptotic to α as

[6] PFH can still be defined if S is not closed, but we will not need this here.

[7] Being hyperbolic means that the eigenvalues at the corresponding periodic point of φ are real. Otherwise, the orbit is called *elliptic*.

$s \to +\infty$, asymptotic to β as $s \to -\infty$, and satisfies

$$Z = [C] + Z';$$

we refer the reader to [HS05, p. 307] for the precise definition of asymptotic in this context. An important motivation for us is that the introduction of the relative homology class Z allows us to define an action, see Section 4.2.2.

The chain complex $\widetilde{\mathrm{PFC}}(\varphi, d)$ is freely generated over[8] $\mathbb{Z}_2$ by twisted PFH generators. The $\mathbb{Z}_2$ vector space $\widetilde{\mathrm{PFC}}(\varphi, d)$ has a canonical $\mathbb{Z}$-grading I given by

$$I(\alpha, Z) = c_{\tau,\tau_0}(Z) + Q_{\tau,\tau_0}(Z) + \sum_i \sum_{k=1}^{m_i} CZ_\tau(\alpha_i^k). \tag{16}$$

Here, τ is (a homotopy class) of a trivialization of the plane field ξ over all Reeb orbits, $c_\tau(Z)$ denotes the relative first Chern class of ξ over Z, $Q_\tau(Z)$ denotes the "relative self-intersection," and $CZ_\tau(\gamma^k)$ denotes the Conley-Zehnder index of the k^{th} iterate of γ; all of these quantities are computed using the trivialization τ, and we refer the reader to [Hut02, §2] or [Hut14, §3] for their definition. Note that the above index depends on the choice of the reference cycle γ_0 and the trivialization τ_0 of ξ over γ_0, though it can be shown that it does not depend on the choice of trivialization τ over Reeb orbits. If C is a J-holomorphic current from (α, Z) to (β, Z'), then we call the quantity $I(\alpha, Z) - I(\beta, Z')$ the *ECH index* of C.

We now define the differential on $\widetilde{\mathrm{PFC}}(\varphi, d)$. Suppose now that $I(\alpha, Z) - I(\beta, Z') = 1$, and let $J \in \mathcal{J}(dr, \omega_\varphi)$. We define

$$\mathcal{M}_J((\alpha, Z), (\beta, Z'))$$

to be the moduli space of J-holomorphic currents C in $X = \mathbb{R} \times Y_\varphi$, modulo translation in the $\mathbb{R}$ direction, that are asymptotic to α as $s \to +\infty$, asymptotic to β as $s \to -\infty$, and satisfy

$$Z = [C] + Z';$$

we refer the reader to [HS05, p. 307] for the precise definition of asymptotic in this context. For generic $J \in \mathcal{J}(dr, \omega_\varphi)$, the above moduli space is a compact 0-dimensional manifold [Hut02, Th. 1.8], and we define the differential by the rule

$$\langle \partial(\alpha, Z), (\beta, Z') \rangle = \#\mathcal{M}_J((\alpha, Z), (\beta, Z')),$$

where $\#$ denotes mod 2 cardinality. Although the chain complex $\widetilde{\mathrm{PFC}}(\varphi, d)$ is infinite dimensional, the differential is well defined for the following reason: for a fixed (α, Z), the set of all (β, Z') such that $I(\beta, Z') = I(\alpha, Z) - 1$ is finite

[8] We could also define PFH over $\mathbb{Z}$, but we do not need this here.

because φ is non-degenerate, and so there are only finitely many Reeb orbit sets of degree d, and hence only finitely many pairs (β, Z') in any fixed grading. It is known that $\partial^2 = 0$ by [HT07], [HT09], and so the homology $\widetilde{\mathrm{PFH}}(\varphi, d)$ is well defined. Lee and Taubes [LT12] proved that the homology of this chain complex does not depend on the choice of J; in fact, they show that for the case $S = \mathbb{S}^2$, it depends only on d.

For future motivation, we note that the Lee-Taubes invariance results discussed here come from an isomorphism of PFH and a version of the Seiberg-Witten Floer theory from [KM07].

Importantly, for the applications to this paper, in computing $\widetilde{\mathrm{PFH}}(\varphi, d)$, we can relax the assumption that φ is nondegenerate to requiring only that φ is d-nondegenerate.

By a direct computation in the case where φ is an irrational rotation of the sphere, i.e., $\varphi(z, \theta) = (z, \theta + \alpha)$ with α being irrational, we obtain

$$\widetilde{\mathrm{PFH}}_*(\varphi, d) = \begin{cases} \mathbb{Z}_2 & \text{if } * = d \bmod 2, \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

Here is a brief outline of the computation leading to the above identity. The Reeb vector field in Y_φ has two simple Reeb orbits γ_+, γ_- corresponding to the north and the south poles. Both of these orbits are elliptic and so the orbit sets of $\widetilde{\mathrm{PFC}}(\varphi, d)$ consist entirely of elliptic Reeb orbits. This implies that the difference in index between any two generators of $\widetilde{\mathrm{PFC}}(\varphi, d)$ chain complex is an even integer; see [Hut02, Prop. 1.6.d]. Thus, the PFH differential vanishes. Now, the above identity follows from the fact that for each index k, satisfying $k = d \bmod 2$, there exists a unique generator of index k in $\widetilde{\mathrm{PFC}}(\varphi, d)$.

4.2.2. *Definition of the spectral invariants.* The vector space $\widetilde{\mathrm{PFC}}(\varphi, d)$ carries a filtration, called the *action filtration*,[9] induced by

$$\mathcal{A}(\alpha, Z) = \int_Z \omega_\varphi.$$

We define $\widetilde{\mathrm{PFC}}^L(\varphi, d)$ to be the $\mathbb{Z}/2$ vector space spanned by generators (α, Z) with $\mathcal{A}(\alpha, Z) \leqslant L$.

By Lemma 4.3, ω_φ is pointwise non-negative along any J-holomorphic curve C, and so $\int_C \omega_\varphi \geqslant 0$. This implies that the differential does not increase the action filtration, i.e.,

$$\partial(\widetilde{\mathrm{PFC}}^L(\varphi, d)) \subset \widetilde{\mathrm{PFC}}^L(\varphi, d).$$

[9]The relation between the quantity $\mathcal{A}(\alpha, Z)$ and the Hamiltonian action functional discussed in Section 2.4 will be clarified in Lemma 4.10.

Hence, it makes sense to define $\widetilde{\mathrm{PFH}}^L(\varphi, d)$ to be the homology of the subcomplex $\widetilde{\mathrm{PFC}}^L(\varphi, d)$.

We are now in position to define the PFH spectral invariants. There is an inclusion induced map

$$\widetilde{\mathrm{PFH}}^L(\varphi, d) \to \widetilde{\mathrm{PFH}}(\varphi, d). \tag{18}$$

If $0 \neq \sigma \in \widetilde{\mathrm{PFH}}(\varphi, d)$ is any nonzero class, then we define the *PFH spectral invariant*

$$c_\sigma(\varphi)$$

to be the infimum, over L, such that σ is in the image of the inclusion induced map (18) above. The number $c_\sigma(\varphi)$ is finite, because, as explained above, there are only finitely many pairs $(\alpha, Z) \in \widetilde{\mathrm{PFC}}(\varphi, d)$ of a fixed grading. We remark that $c_\sigma(\varphi)$ is given by the action of some (α, Z). Indeed, this can be deduced from the following two observations:

(1) If $L < L'$ are such that there exists no (α, Z) with $L \leqslant \mathcal{A}(\alpha, Z) \leqslant L'$, then the two vector spaces $\widetilde{\mathrm{PFC}}^L(\varphi, d)$ and $\widetilde{\mathrm{PFC}}^{L'}(\varphi, d)$ coincide and so $\widetilde{\mathrm{PFH}}^L(\varphi, d) \to \widetilde{\mathrm{PFH}}(\varphi, d)$ and $\widetilde{\mathrm{PFH}}^{L'}(\varphi, d) \to \widetilde{\mathrm{PFH}}(\varphi, d)$ have the same image.

(2) The set of action values $\{\mathcal{A}(\alpha, Z) : (\alpha, Z) \in \widetilde{\mathrm{PFC}}^L(\varphi, d)\}$ forms a discrete subset of $\mathbb{R}$. This is a consequence of the fact that, as stated above, there are only finitely many Reeb orbit sets of degree d.

In Remark 4.8 below we show that $c_\sigma(\varphi)$ does not depend on the choice of the admissible almost complex structure J. Note, however, that it does depend on the choice of the reference cycle γ_0.

4.3. *Initial properties of PFH spectral invariants.* Let $p_- = (0, 0, -1) \in \mathbb{S}^2$. We set

$$\mathcal{S} := \{\varphi \in \mathrm{Diff}(\mathbb{S}^2, \omega) : \varphi(p_-) = p_-,\ -\tfrac{1}{4} < \mathrm{rot}(\varphi, p_-) < \tfrac{1}{4}\},$$

where $\mathrm{rot}(\varphi, p_-)$ denotes the rotation number of φ at p_-; see [KH95] for the definition of rotation number. We remark that our choice of the constant $\frac{1}{4}$ is arbitrary; any other constant in $(0, \frac{1}{2})$ would be suitable for us; we just need to slightly enlarge the class of diffeomorphisms arising from $\mathrm{Diff}_c(\mathbb{D}^2, \omega)$, so as to facilitate computations.

Recall from the previous section that the spectral invariant c_σ depends on the choice of reference cycle $\gamma_0 \in Y_\varphi$. For $\varphi \in \mathcal{S}$, there is a unique embedded Reeb orbit through p_-, and we set this to be the reference cycle γ_0.

The grading on $\widetilde{\mathrm{PFH}}$ depends on the choice of trivialization τ_0 over γ_0; our convention in this paper is that we always choose τ_0 such that the rotation number θ of the linearized Reeb[10] flow along γ_0 with respect to τ_0 satisfies $-\frac{1}{4} < \theta < \frac{1}{4}$. This determines τ_0 uniquely.

We will want to single out some particular spectral invariants for $\varphi \in \mathcal{S}$ and show that they have various convenient properties; we will use these to define the spectral invariants for $\varphi \in \mathrm{Diff}_c(\mathbb{D}, \omega)$.

Having set the above conventions, we do this as follows. Suppose that $\varphi \in \mathcal{S}$ is non-degenerate. According to equation (17), for every pair (d, k) with $k = d$ mod 2, we have a distinguished nonzero class $\sigma_{d,k}$ with degree d and grading k, and so we can define

$$c_{d,k}(\varphi) := c_{\sigma_{d,k}}(\varphi).$$

Lastly, we also define[11]

$$c_d(\varphi) := c_{d,-d}(\varphi).$$

We will see in the proof of Theorem 4.5 that the $c_{d,k}(\varphi)$ for nondegenerate φ determine $c_{d,k}(\varphi)$ for all φ by continuity.

To prepare for what is coming, we identify a class of Hamiltonians $\mathcal{H}$ with the key property, among others, that $\mathcal{S} = \{\varphi^1_H : H \in \mathcal{H}\}$. We define

$$\begin{aligned}\mathcal{H} := \{H \in C^\infty(\mathbb{S}^1 \times \mathbb{S}^2) : \varphi^t_H(p_-) = p_-, H(t, p_-) = 0,\ \ \forall\, t \in [0,1],\\ -\tfrac{1}{4} < \mathrm{rot}(\{\varphi^t_H\}, p_-) < \tfrac{1}{4}\},\end{aligned}$$

where $\mathrm{rot}(\{\varphi^t_H\}, p_-)$ is the rotation number of the isotopy $\{\varphi^t_H\}_{t\in[0,1]}$ at p_-. Observe that $\mathcal{S} = \{\varphi^1_H : H \in \mathcal{H}\}$.

The theorem below, which is the main result of this section, establishes some of the key properties of the PFH spectral invariants and furthermore allows us to extend the definition of these invariants to all, possibly degenerate, $\varphi \in \mathcal{S}$. In the statement below, $\|\cdot\|_{(1,\infty)}$ denotes the energy, or the *Hofer norm*, on $C^\infty(\mathbb{S}^1 \times \mathbb{S}^2)$, which is defined as follows:

$$\|H\|_{(1,\infty)} = \int_0^1 \left(\max_{x\in\mathbb{S}^2} H(t,x) - \min_{x\in\mathbb{S}^2} H(t,x)\right) dt.$$

THEOREM 4.5. *The PFH spectral invariants $c_{d,k}(\varphi)$ admit a unique extension to all $\varphi \in \mathcal{S}$ satisfying the following properties*:

[10] Following [Hut14, §3.2], we define the rotation number θ as follows: Let $\{\psi_t\}_{t\in\mathbb{R}}$ denote the 1-parameter group of diffeomorphisms of Y_φ given by the flow of the Reeb vector field. Then, $D\psi_t : T_{\gamma_0(0)}Y_\varphi \to T_{\gamma_0(t)}Y_\varphi$ induces a symplectic linear map $\phi_t : \xi_{\gamma_0(0)} \to \xi_{\gamma_0(t)}$, which using the trivialization τ_0 we regard as a symplectic linear transformation of $\mathbb{R}^2$. We define θ to be the rotation number of the isotopy $\{\phi_t\}_{t\in[0,1]}$.

[11] Alternatively, one may define $c_d(\varphi) := c_{d,k}(\varphi)$ for any $-d \le k \le d$ satisfying $k = d$ mod 2. These alternative definitions are all suitable for our purposes in this article.

(1) *Monotonicity*: *Suppose that* $H \leqslant G$, *where* $H, G \in \mathcal{H}$. *Then,*

$$c_{d,k}(\varphi_H^1) \leqslant c_{d,k}(\varphi_G^1).$$

(2) *Hofer Continuity*: *For any* $H, G \in \mathcal{H}$, *we have*

$$|c_{d,k}(\varphi_H^1) - c_{d,k}(\varphi_G^1)| \leqslant d\|H - G\|_{(1,\infty)}.$$

(3) *Spectrality*: $c_{d,k}(\varphi_H^1) \in \operatorname{Spec}_d(H)$ *for any* $H \in \mathcal{H}$.
(4) *Normalization*: $c_{d,-d}(\mathrm{Id}) = 0$.

Remark 4.6. To define the PFH spectral invariant $c_{d,k}$ for $\varphi \in \operatorname{Diff}_c(\mathbb{D}, \omega)$, we use equation (1) to identify $\operatorname{Diff}_c(\mathbb{D}, \omega)$ with area-preserving diffeomorphisms of the sphere that are supported in the interior of the northern hemisphere S^+.

We similarly define $c_d : \operatorname{Diff}_c(\mathbb{D}, \omega) \to \mathbb{R}$, which was introduced in Section 3.3. It follows from Theorem 4.5 that $c_d : \operatorname{Diff}_c(\mathbb{D}, \omega) \to \mathbb{R}$ satisfies properties (1)–(4) in Section 3.3.

The rest of this section is dedicated to the proof of the above theorem. The proof requires certain preliminaries. First, it will be convenient to explicitly identify Y_φ with $\mathbb{S}^1 \times \mathbb{S}^2$. To do so, pick $H \in \mathcal{H}$ such that $\varphi = \varphi_H^1$.[12] We define

$$\begin{aligned} \mathbb{S}^1 \times \mathbb{S}^2 &\to Y_\varphi, \\ (t, x) &\mapsto \left((\varphi_H^t)^{-1}(x), t\right), \end{aligned} \tag{19}$$

where t denotes the variable on $\mathbb{S}^1$. For future reference, note that this identifies the Reeb vector field on Y_φ with the vector field

$$\partial_t + X_H \tag{20}$$

on $\mathbb{S}^1 \times \mathbb{S}^2$. The 2-form ω_φ pulls back under this map to the form

$$\omega + dH \wedge dt,$$

where ω is the area form on $\mathbb{S}^2$.

The Reeb orbit γ_0 maps under (19) to the preimage of p_- under the map $\mathbb{S}^1 \times \mathbb{S}^2 \to \mathbb{S}^2$; we will continue to denote it by γ_0. Moreover, the trivialization τ_0 from above agrees (up to homotopy) under this identification with the trivialization over γ_0 given by pulling back a fixed frame of $T_{p_-}\mathbb{S}^2$ under the map $\mathbb{S}^1 \times \mathbb{S}^2 \to \mathbb{S}^2$.

[12] We remark that the choice of $H \in \mathcal{H}$ such that $\varphi = \varphi_H^1$ is unique up to homotopy of Hamiltonian isotopies rel endpoints. This fact, which is not used in our arguments, may be deduced from properties of the rotation number.

The map (19) allows us to identify $\mathbb{R} \times \mathbb{S}^1 \times \mathbb{S}^2$ with the symplectization X via

$$\begin{aligned} &\mathbb{R} \times \mathbb{S}^1 \times \mathbb{S}^2 \to X, \\ &\qquad (s, t, x) \mapsto (s, (\varphi_H^t)^{-1}(x), t). \end{aligned}$$

The symplectic form Γ on X then pulls back to

$$\omega_H = ds \wedge dt + \omega + dH \wedge dt. \tag{21}$$

Let H, G be two Hamiltonians in $\mathcal{H}$. As mentioned earlier, $\widetilde{\mathrm{PFH}}(\varphi_H^1, d)$ is isomorphic to $\widetilde{\mathrm{PFH}}(\varphi_G^1, d)$. The proof of this uses Seiberg-Witten theory and is carried out in [LT12, Cor. 6.1]. This isomorphism is canonical with a choice of reference cycle in $H_2(\mathbb{S}^1 \times \mathbb{S}^2, \gamma_0, \gamma_0)$; we say more about this in Remark 4.9 below. We take this reference cycle to be the constant cycle[13] over γ_0. In this case, we will see below that the canonical isomorphism

$$\widetilde{\mathrm{PFH}}(\varphi_H^1, d) \to \widetilde{\mathrm{PFH}}(\varphi_G^1, d), \tag{22}$$

preserves the $\mathbb{Z}$-grading.

As is generally the case with related invariants, one might expect this isomorphism to be induced by a chain map counting certain ECH index zero J-holomorphic curves. In fact, it is not currently known how to define the map (22) this way; the construction uses Seiberg-Witten theory. Nevertheless, the map in (22) does satisfy a "holomorphic curve" axiom that was proven by Chen [Che21] using variants of Taubes' "Seiberg-Witten to Gromov" arguments in [Tau96]. A similar "holomorphic curve" axiom was proven in the context of embedded contact homology by Hutchings-Taubes.

To state what we will need to know about this holomorphic curve axiom in our context, given Hamiltonians $H, G \in \mathcal{H}$, define

$$K = G + \beta(s) \cdot (H - G)$$

for $s \in \mathbb{R}$, where $\beta : \mathbb{R} \to [0, 1]$ is some non-decreasing function that is 0 for s sufficiently negative, 1 for s sufficiently positive, and satisfies $1 + \beta'(s) \cdot (H - G) > 0$. We can think of K as above as a function on $\mathbb{R} \times \mathbb{S}^1 \times \mathbb{S}^2$. Now consider the form

$$\omega_X = ds \wedge dt + \omega + d(K dt).$$

This is a symplectic form on $\mathbb{R} \times \mathbb{S}^1 \times \mathbb{S}^2$. Observe that, for $s \gg 0$, the form ω_X agrees with the symplectization form ω_H, and for $s \ll 0$, it agrees with

[13] This is the projection $\gamma_0 \times I \to \gamma_0$.

the symplectization form ω_G. Let J_X be any ω_X-compatible[14] almost complex structure that agrees with a generic (dt, ω_H) admissible almost complex structure J_+ for $s \gg 0$ and with a generic (dt, ω_G) admissible almost complex structure J_- for $s \ll 0$.

Then, the holomorphic curve axiom implies that (22) is induced by a (non-canonical) chain map

$$\Psi_{J_X,H,G} : \widetilde{\mathrm{PFC}}(\varphi_H^1, d, J_+) \to \widetilde{\mathrm{PFC}}(\varphi_G^1, d, J_-), \tag{23}$$

with the property that if $\langle \Psi_{J_X,H,G}(\alpha, Z), (\beta, Z') \rangle \neq 0$, then there is an ECH index 0 J_X-*holomorphic building* C from α to β such that

$$Z' + [C] = Z, \tag{24}$$

as elements of $H_2(\mathbb{S}^1 \times \mathbb{S}^2, \alpha, d\gamma_0)$; we say more about this in Remark 4.9 below. Here, by a J_X-holomorphic building from α to β, we mean a sequence of J_i-holomorphic curves

$$(C_0, \ldots, C_i, \ldots, C_k),$$

such that the negative asymptotics of C_i agree with the positive asymptotics of C_{i+1}, the curve C_0 is asymptotic to α at $+\infty$, and the curve C_k is asymptotic to β at $-\infty$; we refer the reader to [Hut14, §5.3] for more details. We remark for future reference that the C_i are called *levels*, and each J_i is either[15] J_X, J_+ or J_-. The condition that the ECH index of the building is zero means that the sum of the ECH indices of the levels add up to zero. In particular, this index condition, together with (24), implies the earlier claim that the map (22) preserves the $\mathbb{Z}$-grading by additivity of c_τ and Q_τ, since the trivializations over γ_0 required to define the grading on $\widetilde{\mathrm{PFC}}(\varphi_H^1)$ and $\widetilde{\mathrm{PFC}}(\varphi_G^1)$ are the same.

We will want to assume that J_X is *compatible with the fibration* $\mathbb{R} \times \mathbb{S}^1 \times \mathbb{S}^2 \to \mathbb{R} \times \mathbb{S}^1$ in the following sense: Let $\mathbb{V}$ be the vertical tangent bundle of this fibration, and denote by $\mathbb{H}$ the ω_X-orthogonal complement of $\mathbb{V}$; observe that $\mathbb{H}$ is spanned by the vector fields ∂_s and $\partial_t + X_K$. Then, we will want J_X to preserve $\mathbb{V}$ and $\mathbb{H}$. Given any admissible $J_\pm$ on the ends, we can achieve this as follows. On the horizontal tangent bundle $\mathbb{H}$, we always demand that J_X sends ∂_s to $\partial_t + X_K$. On the vertical tangent bundle, we observe that $\omega_X|_\mathbb{V} = \omega$ and, in particular, $\omega_X|_\mathbb{V}$ is independent of s. We can then connect $J_+|_\mathbb{V}$ to $J_-|_\mathbb{V}$ through a path of ω-tamed almost complex structures on $\mathbb{V}$.

We can now prove Theorem 4.5. We break the proof up into two parts, namely we first prove all of the properties except for Spectrality, and then we prove Spectrality.

[14] Recall that an almost complex structure J is *compatible* with a symplectic form ω if $g(u, v) := \omega(u, Jv)$ defines a Riemannian metric.

[15] More can be said, but we will not need this additional information

Proof of Theorem 4.5: Monotonicity, Hofer continuity, and normalization. We begin by first supposing that the monotonicity and Hofer continuity properties hold when φ^1_G, φ^1_H are nondegenerate and explain how this implies the rest of the theorem. To that end, let $H \in \mathcal{H}$, not necessarily nondegenerate, and take a sequence $H_i \in \mathcal{H}$ that C^2 converges to H and such that $\varphi^1_{H_i}$ is nondegenerate. Then, we define

$$c_{d,k}(\varphi^1_H) = \lim_{i\to\infty} c_{d,k}(\varphi^1_{H_i}).$$

This limit exists thanks to the inequality $|c_{d,k}(\varphi^1_{H_i}) - c_{d,k}(\varphi^1_{H_j})| \leqslant d\|H_i - H_j\|_{(1,\infty)}$. Moreover, the same inequality implies that the limit value does not depend on the choice of the sequence H_i and so $c_{d,k}(\varphi^1_H)$ is well defined for all $H \in \mathcal{H}$. Thus, we obtain a well-defined mapping

$$c_{d,k} : \mathcal{S} \to \mathbb{R}.$$

It can be seen that $c_{d,k}$ continues to satisfy the monotonicity and Hofer continuity properties for degenerate φ^1_G, φ^1_H. Moreover, note that, by the Hofer continuity property, the mapping $c_{d,k} : \mathcal{S} \to \mathbb{R}$ is uniquely determined by its restriction to the set of all non-degenerate $\varphi \in \mathcal{S}$.

To prove that $c_{d,-d}(\mathrm{Id}) = 0$, it is sufficient to show that $c_{d,-d}(\varphi) = 0$ in the case where φ is a positive irrational rotation of the sphere; that is, $\varphi(z,\theta) = (z, \theta + \alpha)$ with α being a small and positive irrational number. As in the explanation for equation (17), the chain complex $\widetilde{\mathrm{PFC}}(\varphi, d)$ has a unique generator in indices k such that $k = d \bmod 2$ and it is zero for other indices. The unique generator of index $-d$ is of the form (α, Z), where $\alpha = \{(\gamma_0, d)\}$ and Z is the trivial class in $H_2(Y_\varphi, d\gamma_0, d\gamma_0)$. The action $\mathcal{A}(\alpha, Z)$ is zero. This proves that $c_{d,-d}(\varphi) = 0 = c_{d,-d}(\mathrm{Id})$.[16]

For the rest of the proof, we will suppose that φ^1_H, φ^1_G are nondegenerate. We will now prove the monotonicity and Hofer continuity properties. Let J_+, J_- be any generic admissible almost complex structures for φ^1_H and φ^1_G respectively, and let $(\alpha_1, Z_1) + \cdots + (\alpha_m, Z_m)$ be a cycle in $\widetilde{\mathrm{PFC}}(\varphi^1_H, d, J_+)$ representing $\sigma_{d,k}$, with

$$c_{\sigma_{d,k}}(\varphi^1_H) = \mathcal{A}(\alpha_1, Z_1) \geqslant \cdots \geqslant \mathcal{A}(\alpha_m, Z_m).$$

Fix an almost complex structure J_X that is compatible with the fibration and agrees with J_+ for s sufficiently positive and J_- for s sufficiently negative. Let (β, Z') be a generator in $\widetilde{\mathrm{PFC}}(\varphi^1_G, d)$ that has maximal action among generators that appear with a non-zero coefficient in

$$\Psi_{J_X,H,G}\left((\alpha_1, Z_1) + \cdots + (\alpha_m, Z_m)\right).$$

[16] With a similar argument one can prove that $c_{d,k}(\mathrm{Id}) = 0$ for every $-d \leq k \leq d$ with $k = d \bmod 2$.

Then, by the aforementioned holomorphic curve axiom there is a J_X-holomorphic building from some (α_i, Z_i) to (β, Z'). For the rest of the proof, we will write $(\alpha_i, Z_i) = (\alpha, Z)$ and will denote the J_X-holomorphic building by C.

For the arguments below, which only involve energy and index arguments, we can assume that C consists of a single J_X-holomorphic level — in other words, is an actual J_X-holomorphic curve, rather than a building — so to simplify the notation, we assume this.

For the remainder of the proof we will need the following lemma.

LEMMA 4.7. *The following identity holds*:

$$\mathcal{A}(\alpha, Z) - \mathcal{A}(\beta, Z') = \int_C \omega + dK \wedge dt + K' ds \wedge dt.$$

Furthermore, we have

$$\int_C \omega + dK \wedge dt \geqslant 0.$$

In the above statement, K' denotes $\frac{\partial K}{\partial s}$ and, for the rest of this section, dK denotes the derivatives in the $\mathbb{S}^2$ directions.

Proof of Lemma 4.7. We will begin by proving that

$$\mathcal{A}(\alpha, Z) - \mathcal{A}(\beta, Z') = \int_C \omega + d(K dt), \tag{25}$$

which establishes the first item because $\omega + d(Kdt) = \omega + dK \wedge dt + K' ds \wedge dt$. Note that we can write

$$\mathcal{A}(\alpha, Z) = \int_Z \omega + d(H dt), \quad \mathcal{A}(\beta, Z') = \int_{Z'} \omega + d(G dt).$$

Hence, equation (25) will follow if we show that

$$\int_C \omega = \int_Z \omega - \int_{Z'} \omega, \text{ and } \int_C d(K dt) = \int_Z d(H dt) - \int_{Z'} d(G dt).$$

The first identity holds because all of these integrals are determined by the homology classes, and we have $[C] = Z - Z'$. The second identity follows from the following chain of identities:

$$\begin{aligned}\int_C d(K dt) &= \int_\alpha K dt - \int_\beta K dt \\ &= \int_\alpha H dt - \int_\beta G dt \\ &= \int_Z d(H dt) - \int_{Z'} d(G dt),\end{aligned}$$

where the first equality holds by Stokes' theorem, the second follows from the definition of K, and the third is a consequence of Stokes' theorem combined with the fact that H, G both belong to $\mathcal{H}$ and so vanish on γ_0. This completes the proof of the first item in the lemma.

Now, we will show that $\int_C(\omega + dK \wedge dt) \geqslant 0$ by showing that the form $\omega + dK \wedge dt$ is pointwise non-negative along C. Indeed, at any point $p \in X$, we can write any vector as $v + h$, where $v \in \mathbb{V}$ and $h \in \mathbb{H}$ are vertical and horizontal tangent vectors as described in the paragraph before the proof of Theorem 4.5. Since C is J_X-holomorphic, it is sufficient to show that $(\omega + dK \wedge dt)(v + h, J_X v + J_X h) \geqslant 0$. We will show that

$$(\omega + dK \wedge dt)(v + h, J_X v + J_X h) = \omega_X(v, J_X v), \tag{26}$$

which proves the inequality because J_X is ω_X-tame. Now, to simplify our notation Ω will denote $\omega + dK \wedge dt$ for the rest of the proof. Expanding the left-hand side of the above equation we get

$$\Omega(v + h, J_X v + J_X h) = \Omega(v, J_X v) + \Omega(h, J_X h) + \Omega(v, J_X h) + \Omega(h, J_X v).$$

We will now show that $\Omega(v, J_X v) = \omega_X(v, J_X v)$ and $\Omega(h, J_X h) = \Omega(v, J_X h) = \Omega(h, J_X v) = 0$, which clearly implies equation (26). To see this, note that v and $J_X v$ are in the kernel of $ds \wedge dt$, hence

$$\Omega(v, J_X v) = \omega_X(v, J_X v),$$

$$\Omega(v, J_X h) = \omega_X(v, J_X h) = 0, \quad \Omega(h, J_X v) = \omega_X(h, J_X v) = 0.$$

It remains to show that $\Omega(h, J_X h) = 0$, that is, $\Omega|_{\mathbb{H}} = 0$. This follows from the fact that $\mathbb{H}$ is spanned by $\{\partial_s, \partial_t + X_K\}$ and ∂_s is in the kernel of Ω. Indeed, a 2-form on a 2-dimensional vector space with non-trivial kernel is identically zero. $\square$

Note that $c_{d,k}(\varphi^1_H) \geqslant \mathcal{A}(\alpha, Z)$ and $c_{d,k}(\varphi^1_G) \leqslant \mathcal{A}(\beta, Z')$. Hence,

$$c_{d,k}(\varphi^1_H) - c_{d,k}(\varphi^1_G) \geqslant \mathcal{A}(\alpha, Z) - \mathcal{A}(\beta, Z'). \tag{27}$$

As a consequence of this inequality, Monotonicity would follow from proving that if $H \geqslant G$, then $\mathcal{A}(\alpha, Z) - \mathcal{A}(\beta, Z') \geqslant 0$. By the above lemma we have

$$\mathcal{A}(\alpha, Z) - \mathcal{A}(\beta, Z') \geqslant \int_C K' \, ds \wedge dt. \tag{28}$$

If $H \geqslant G$, then $K' \geqslant 0$. Moreover, $ds \wedge dt$ is pointwise non-negative on C. Indeed, continuing with the notation as above,

$$ds \wedge dt(v + h, J_X v + J_X h) = ds \wedge dt(h, J_X h),$$

since v and $J_X v$ are in the kernel of $ds \wedge dt$; on the other hand, we saw in the proof of the previous lemma that $\Omega|_{\mathbb{H}} = 0$, so

$$\big(1 + \beta'(H - G)\big) \, ds \wedge dt(h, J_X h) = \omega_X(h, J_X h) \geq 0.$$

Hence $\int_C K' ds \wedge dt \geqslant 0$, which proves Monotonicity.

As for Hofer Continuity, it is sufficient to show that

$$\left|\int_C K'\, ds \wedge dt\right| \leqslant d\|H-G\|_{(1,\infty)}. \tag{29}$$

Indeed, this inequality combined with inequalities (27) and (28) implies that $c_{d,k}(\varphi^1_G) - c_{d,k}(\varphi^1_H) \leqslant d\|H-G\|_{(1,\infty)}$. Similarly, by switching the role of H and G, one gets $c_{d,k}(\varphi^1_H) - c_{d,k}(\varphi^1_G) \leqslant d\|H-G\|_{(1,\infty)}$, which then implies Hofer Continuity.

It remains to prove inequality (29). Since as above $ds \wedge dt$ is pointwise non-negative on C, we have

$$\left|\int_C K' ds \wedge dt\right| = \left|\int_C \beta'(s)(H-G) ds \wedge dt\right| \leqslant \int_C \beta'(s)|H-G| ds \wedge dt.$$

Note that because H, G both vanish at the point p_-, for all t, x, we have

$$|H(t,x) - G(t,x)| \leqslant \max_{\mathbb{S}^2}(H_t - G_t) - \min_{\mathbb{S}^2}(H_t - G_t).$$

Hence, we get

$$\left|\int_C K' ds \wedge dt\right| \leqslant \int_C \beta'(s)\left(\max_{\mathbb{S}^2}(H_t - G_t) - \min_{\mathbb{S}^2}(H_t - G_t)\right) ds \wedge dt.$$

We can evaluate the second integral by projecting C to the (s,t) plane; this projection has degree d, and since $\int_{-\infty}^{+\infty} \beta' = 1$, the second integral evaluates to

$$d\|H-G\|_{(1,\infty)}.$$

This completes the proof of Hofer Continuity. □

Remark 4.8. In the special case where $H = G$, but the two J_i are different, the Monotonicity argument above, applied first to $H \geqslant G$ and next to $G \geqslant H$, gives that the spectral invariant does not depend on J.

Remark 4.9. On the Seiberg-Witten side, the twisted theory corresponds to a version of the Floer homology where, instead of taking the quotient of solutions by the full gauge group $\mathcal{G} = C^\infty(M, \mathbb{S}^1)$, one only takes the quotient by the subgroup $\mathcal{G}^0 \subset \mathcal{G}$ of gauge transformations in the connected component of the identity. This has an $H^1(Y)$ action, induced by the action via gauge transformations, which corresponds to the $H_2(Y)$ action on twisted PFH given by adding a homology class.

As mentioned above, it was remarked by Taubes [Tau10, §1] that the twisted invariant on the PFH/ECH side depends on a choice of reference cycle, and there is an isomorphism between the invariants for different choices of reference cycles that is canonical only up to a choice of element of $H_2(Y, \rho, \rho')$, where ρ, ρ' are two reference cycles. Implicit in this assertion is that after a choice of reference cycle R, the isomorphism (23) satisfies a holomorphic curve axiom for buildings C satisfying

$$Z + R = [C] + Z'. \tag{30}$$

This is the best way to think about (24); this corresponds to the case where our reference cycle is constant over ρ.

For more about the connection between the twisted theory and the relevant Seiberg-Witten Floer homology, we refer the reader to [Tau10, §§1, 2], where Taubes is writing about twisted ECH; we have adapted what is written there to the PFH context, as suggested by [LT12, Cor. 6.1].

It remains to prove Spectrality. As stated in Section 4.2.2, the spectral invariant $c_{d,k}(\varphi^1_H)$ is the action of a twisted PFH generator (α, Z) of degree d. Spectrality, hence Theorem 4.5, is then a consequence of the following lemma.

LEMMA 4.10. *Let (α, Z) be a twisted PFH generator of degree d for $\varphi = \varphi^1_H$ with $H \in \mathcal{H}$. Then, $\mathcal{A}(\alpha, Z)$ belongs to* $\mathrm{Spec}_d(H)$, *as defined in Section* 2.4.

Before giving the proof, we describe a construction that will be used in the proof and also later in the paper.

4.3.1. *The class Z_α.* Let α be an orbit set. We will construct a specific relative homology class $Z_\alpha \in H_2(Y_\varphi, \alpha, \gamma_0)$, for $\varphi = \varphi^1_H$ with $H \in \mathcal{H}$, as follows. A key input in the construction of this class is a certain map $u_\alpha : D^2 \to \mathbb{S}^2$ that we will also want to refer to later in the paper.

We first construct Z_α in the case $d = 1$. Let $q \in \mathrm{Fix}(\varphi)$, and suppose that α is the Reeb orbit in the mapping cylinder corresponding to q. The relative cycle Z_α will be of the form $Z_\alpha = Z_0 + Z_1 + Z_2$. We begin by choosing a path η in $\mathbb{S}^2 \times \{0\} \subset Y_{\varphi^1_H}$ such that $\partial\eta = (q, 0) - (p_-, 0)$. We parametrize this curve with a variable $x \in [0, 1]$. We define Z_0 to be the chain induced by the map

$$[0,1]^2 \to Y_{\varphi^1_H}, \ \ (x,t) \mapsto (\eta(x), t).$$

Its boundary is given by $\partial Z_0 = \alpha - \gamma_0 + (\eta, 0) - (\varphi(\eta), 0)$. Next we define Z_1 to be the chain induced by the map

$$[0,1]^2 \to Y_{\varphi^1_H}, \ \ (t,x) \mapsto (\varphi^t_H(\eta(x)), 0).$$

Then, $\partial Z_1 = (\varphi(\eta), 0) - (\eta, 0) - (\varphi^t_H(q), 0)$. Finally, we define Z_2 to be the chain induced by a map $(u_\alpha, 0)$, where $u_\alpha : D^2 \to \mathbb{S}^2$ is such that $u_\alpha|_{\partial D^2}$ is the Hamiltonian orbit $t \mapsto \varphi^t_H(q)$. There is some ambiguity in the choice of u_α here, but to resolve this we select u_α according to the following rules:

(i) If $\alpha = \gamma_0$, the Reeb orbit corresponding to p_-, then we take u_α to be the constant disc with image p_-.

(ii) If $\alpha \neq \gamma_0$, then we take u_α such that its image does not contain p_-.

Then Z_2 does not depend on the choice of u_α satisfying the above two conditions, and we now define $Z_\alpha := Z_0 + Z_1 + Z_2$. The key point of the definition is that $\partial Z_\alpha = \alpha - \gamma_0$.

Next we consider the case where α is an orbit set of degree m consisting of only one periodic orbit. Let q be a periodic point, of (not necessarily minimal) period $m \in \mathbb{N}$, and suppose that α is the Reeb orbit in the mapping cylinder corresponding to q. Then, q is a fixed point of φ_H^m. Consider the mapping torus $Y_{\varphi_H^m}$. There is a map $c : Y_{\varphi_H^m} \to Y_{\varphi_H^1}$, pulling back $\omega_{\varphi_H^1}$ to $\omega_{\varphi_H^m}$, given by mapping each interval $\mathbb{S}^2 \times [\frac{k}{m}, \frac{k+1}{m}]$ onto Y_φ via the map $(x,t) \to (\varphi_H^k(x), m \cdot t - k)$. Now repeat the construction from above to produce a relative cycle Z' in Y_{φ^m} and define Z_α to be the pushforward of Z' under the map c

Finally, let $\alpha = \{(\alpha_i, m_i)\}$, where the α_i are simple closed Reeb orbits. So, each (α_i, m_i) corresponds to a (not necessarily simple) orbit of a periodic point q_i of φ_H^1. By using the construction in the previous paragraph, we can associate a relative cycle to each (α_i, m_i); the sum, over i, of all of these cycles gives a relative cycle from α to $d\gamma_0$, where d is the sum of the periods of the periodic points q_i.

4.3.2. *Proof of Spectrality.*

Proof of Lemma 4.10. As in the proof of the Monotonicity and Hofer continuity properties above, we may assume that φ is nondegenerate. Note that it is sufficient to prove that $\mathcal{A}(\alpha, Z_\alpha) \in \mathrm{Spec}_d(H)$, where Z_α is the class constructed in Section 4.3.1, since any other $Z \in H_2(Y_\varphi, \alpha, \gamma_0)$ is of the form $Z_\alpha + k[\mathbb{S}^2]$ where $k \in \mathbb{Z}$ and so $\mathcal{A}(\alpha, Z) = \mathcal{A}(\alpha, Z_\alpha) + k$.

To prove this, it suffices by the definition of Spec_d to prove this in the $d = 1$ case, since an m-periodic point of φ is a fixed point of φ^m and this is generated by H^m. So, assume this, and let Z_0, Z_1, Z_2 and u_α be as in the definition of Z_α in Section 4.3.1. We have $\int_{Z_0} \omega_\varphi = 0$. As for Z_1, we have

$$
\begin{aligned}
\int_{Z_1} \omega_\varphi &= \int\int_{[0,1]^2} \omega\langle \partial_t \varphi_H^t(\eta(x)), \partial_x \varphi_H^t(\eta(x))\rangle \\
&= \int\int_{[0,1]^2} \omega\langle X_{H_t}(\varphi_H^t(\eta(x))), \partial_x \varphi_H^t(\eta(x))\rangle \\
&= \int\int_{[0,1]^2} dH_t(\partial_x \varphi_H^t(\eta(x))) = \int\int_{[0,1]^2} \partial_x H_t(\varphi_H^t(\eta(x))) \\
&= \int_0^1 H_t(\varphi_H^t(q)) - H_t(\varphi^t(p_-))dt = \int_0^1 H_t(\varphi_H^t(q))dt.
\end{aligned}
$$

Finally, $\int_{Z_2} \omega_\varphi = \int_{D^2} u_\alpha^* \omega$. Hence, $\int_{Z_\alpha} \omega_\varphi \in \mathrm{Spec}(H)$, and so the lemma is proved. $\square$

Remark 4.11. We note for future reference that as φ_H^m can be viewed as the time 1-map of the Hamiltonian $F_t(x) = mH_{mt}(x)$, we have by the above argument that for any periodic point q of period m, corresponding to an orbit set α,

$$\int_{Z_\alpha} \omega_\varphi = \int_{D^2} u_\alpha^* \omega + \int_0^m H_t(\varphi_H^t(q))dt. \tag{31}$$

5. C^0 continuity

Here we prove Theorem 3.6, using Theorem 4.5 from Section 4.

The central objects of Theorem 3.6 are the maps $c_d : \mathrm{Diff}_c(\mathbb{D}^2, \omega) \to \mathbb{R}$. Remember from Section 4.3 and Remark 4.6 that these maps are defined from the spectral invariants $c_d : \mathcal{S} \to \mathbb{R}$, by identifying $\mathrm{Diff}_c(\mathbb{D}^2, \omega)$ with the group $\mathrm{Diff}_{S^+}(\mathbb{S}^2, \omega)$ consisting of symplectic diffeomorphisms of $\mathbb{S}^2$, which are supported in the interior of the northern hemisphere S^+. In the present section, we directly work in the group $\mathrm{Diff}_{S^+}(\mathbb{S}^2, \omega)$.

More generally, given an open subset $U \subset \mathbb{S}^2$, we will denote by $\mathrm{Diff}_U(\mathbb{S}^2, \omega)$ the set of all Hamiltonian diffeomorphisms compactly supported in an open subset U.

Our proof is inspired by the proof of the C^0-continuity of barcodes (hence, of spectral invariants) arising from Hamiltonian Floer theory presented in [LRSV21]. However, our case is complicated by the fact that we are working with periodic points while [LRSV21] only deals with fixed points. Other existing proofs of C^0-continuity of spectral invariants make use[17] of the product structure on Hamiltonian Floer homology. It might be possible to define a "quantum product" on PFH (see [HS05]), however at the time of the writing of this article, such structures do not exist.

Let d be a positive integer. As in [LRSV21], we treat separately the C^0-continuity of c_d at the identity and elsewhere. Theorem 3.6 will be a consequence of the following two propositions.

PROPOSITION 5.1. *The map $c_d : \mathrm{Diff}_{S^+}(\mathbb{S}^2, \omega) \to \mathbb{R}$ is continuous at* Id *with respect to the C^0-topology on $\mathrm{Diff}_{S^+}(\mathbb{S}^2, \omega)$.*

PROPOSITION 5.2. *Every $\eta \in \mathrm{Homeo}_{S^+}(\mathbb{S}^2, \omega)$ with $\eta \neq \mathrm{Id}$ admits a C^0-neighborhood $\mathcal{V}$ such that the restriction of c_d to $\mathcal{V} \cap \mathrm{Diff}_{S^+}(\mathbb{S}^2, \omega)$ is uniformly continuous with respect to the C^0-distance.*

This last proposition readily implies that any $\eta \in \mathrm{Homeo}_{S^+}(\mathbb{S}^2, \omega) \setminus \{\mathrm{Id}\}$ admits a C^0-neighborhood $\mathcal{V}$ to which c_d extends continuously. In particular, it

[17]The product is usually used to deduce continuity everywhere from continuity at Id. Without a product, we need another argument to prove continuity in the complement of the identity.

extends continuously at η. Since this holds for any such homeomorphism η, this shows together with Proposition 5.1 that c_d extends to a map $\mathrm{Homeo}_{S^+}(\mathbb{S}^2,\omega) \to \mathbb{R}$ continuous with respect to C^0-topology, hence Theorem 3.6.

Proposition 5.2 can be rephrased as follows. Any homeomorphism $\eta \in \mathrm{Homeo}_{S^+}(\mathbb{S}^2,\omega)$, $\eta \neq \mathrm{Id}$, admits a neighborhood $\mathcal{V}$ in $\mathrm{Homeo}_{S^+}(\mathbb{S}^2,\omega)$ such that for all $\varepsilon > 0$, there exists $\delta > 0$ satisfying

$$\forall\, \phi, \psi \in \mathcal{V} \cap \mathrm{Diff}_{S^+}(\mathbb{S}^2,\omega),\ \text{if } d_{C^0}(\phi,\psi) < \delta,\ \text{then } |c_d(\phi) - c_d(\psi)| < \varepsilon. \tag{32}$$

The Hofer norm. Our proofs will make intensive use of the Hofer norm for Hamiltonian diffeomorphisms. We now recall its definition and basic properties. We refer the reader to [Pol01] and the references therein for a general introduction to the material presented here.

We have seen earlier in the paper the definition of the Hofer norm of a Hamiltonian on the sphere and the disc. On a general symplectic manifold, the *Hofer norm* of a compactly supported Hamiltonian diffeomorphism ϕ is defined as

$$\|\phi\| = \inf\{\|H\|_{(1,\infty)}\},$$

where the infimum runs over all compactly supported Hamiltonians H whose time-1 map is ϕ. It satisfies a triangle inequality

$$\|\phi \circ \psi\| \leqslant \|\phi\| + \|\psi\|$$

for all Hamiltonian diffeomorphisms ϕ, ψ, it is conjugation invariant and, moreover, we have $\|\phi^{-1}\| = \|\phi\|$ for all Hamiltonian diffeomorphisms ϕ.

The displacement energy of a subset A of the ambient symplectic manifold is by definition the quantity

$$e(A) := \inf\{\|\phi\| : \phi(\overline{A}) \cap \overline{A} = \emptyset\}.$$

On a surface, it is known that for a disjoint union of closed discs, with each disc having area a, and whose union covers less than half the area of the surface, the displacement energy is a.

Important note. We will use the Hofer norm on the symplectic manifold $\mathbb{S}^2 \setminus \{p_-\}$. Thus, all the Hamiltonians considered in this section will be compactly supported in the complement of the south pole p_-; in particular, they belong to $\mathcal{H}$.

Note that the second item of Theorem 4.5 can be reformulated as

$$|c_d(\psi) - c_d(\phi)| \leqslant d \cdot \|\psi^{-1} \circ \phi\| \tag{33}$$

for all Hamiltonian diffeomorphisms $\phi, \psi \in \mathrm{Diff}_{\mathbb{S}^2 \setminus \{p_-\}}(\mathbb{S}^2,\omega)$.

5.1. *Continuity at the identity.* We first prove Proposition 5.1. The case $d = 1$ can be proved with the same proof as [Sey13a], using the so-called ε-shift technique. We will generalize this idea to make the proof work for all $d \geqslant 1$.

Let us start our proof with a lemma.

LEMMA 5.3. *Let $d \geqslant 1$, and let F be a time-independent Hamiltonian, compactly supported in $\mathbb{S}^2 \setminus \{p_-\}$. Let $f = \phi_F^1$ be the time-one map it generates. Assume that the next two conditions are satisfied*:

(a) *for all $k \in \{1, \dots, d\}$, the k-periodic points of f are precisely the critical points of F;*

(b) *none of the critical points of F are in the closure of S^+.*

Then, there exists $\delta > 0$ such that $c_d(\phi \circ f) = c_d(f)$ for any $\phi \in \mathrm{Diff}_{S^+}(\mathbb{S}^2, \omega)$ with $d_{C^0}(\phi, \mathrm{Id}) < \delta$.

Postponing the proof of this lemma, we now explain how it implies Proposition 5.1.

Proof of Proposition 5.1. Let $\varepsilon > 0$. Let F be a function on $\mathbb{S}^2$ satisfying the assumptions of Lemma 5.3, and assume furthermore that

$$\max F - \min F \leqslant \frac{\varepsilon}{2d}.$$

For instance a C^2-small function supported in the complement of p_- all of whose critical points are in the southern hemisphere is appropriate. Then let δ be as provided by Lemma 5.3.

Let $\phi \in \mathrm{Diff}_{S^+}(\mathbb{S}^2, \omega)$ be such that $d_{C^0}(\phi, \mathrm{Id}) < \delta$. Then, we have $c_d(\phi \circ f) = c_d(f)$. Using inequality (33) twice and the fact that $c_d(\mathrm{Id}) = 0$, we obtain

$$|c_d(\phi)| \leqslant |c_d(\phi \circ f)| + d\|f\| = |c_d(f) - c_d(\mathrm{Id})| + d\|f\| \leqslant 2d\,\|f\|.$$

Now, by definition of the Hofer norm, $\|f\| \leqslant \max F - \min F$. Thus we get

$$c_d(\phi) \leqslant \varepsilon.$$

This show the C^0-continuity of c_d at Id. □

We will now prove the lemma.

Proof of Lemma 5.3. Let F be as in the statement of the lemma and $f = \phi_F^1$. We want to prove that $c_d(f)$ remains unchanged when we C^0-perturb f with a Hamiltonian diffeomorphism supported in the northern hemisphere. To obtain this, we will first prove that the entire spectrum remains unchanged under such perturbations.

Let $k \in \{1, \dots, d\}$; we begin by showing that the set of k-periodic points is unchanged by these perturbations. By assumption, there exists $c > 0$ such that

$$d(f^k(x), x) > c$$

for all x in the closure of S^+. Now note that the diffeomorphism $(\phi f)^k$ converges to f^k uniformly when ϕ tends uniformly to Id. Thus, there exists $\delta > 0$

such for $d_{C^0}(\phi, \mathrm{Id}) < \delta$, the inequality $d((\phi f)^k(x), x) > 0$ holds for all x in the closure of S^+. In other words, $\phi \circ f$ has no k-periodic points in the closure of S^+. Since ϕ coincides with the identity outside S^+, this implies that $\phi \circ f$ and f have the same k-periodic points, which are in turn the critical points of F. For the rest of the proof, we pick δ such that the above holds for all $k \in \{1, \dots, d\}$, and ϕ such that $d_{C^0}(\phi, \mathrm{Id}) < \delta$.

We next show that the actions of these k-periodic points, i.e., the critical points of F, agree when computed with respect to f and $\phi \circ f$.

To compute these actions, let H be a Hamiltonian supported in S^+ such that $\varphi_H^1 = \phi$. By (7), the isotopy $\varphi_H^t \varphi_F^t$ (whose time one map is $\phi \circ f$) is generated by the Hamiltonian $H \# F(t, x) = H(t, x) + F((\varphi_H^t)^{-1}(x))$.

Let y be a critical point of F. Then, $\varphi_F^t(y) = y$ for all $t \in [0, 1]$, and since $y \notin S^+$, we also have $\varphi_H^t(y) = y$ for all $t \in [0, 1]$. Thus y remains fixed along the whole isotopy. A capping of such an orbit is a trivial capping to which is attached $\ell[\mathbb{S}^2]$ for some $\ell \in \mathbb{Z}$. Also note that since H is supported in S^+,

$$H \# F(t, \varphi_H^t \varphi_F^t(y)) = H(t, y) + F(y) = F(y).$$

Applying formula (2) we obtain

$$\begin{aligned}\mathcal{A}_{H\#F}(y, \ell[\mathbb{S}^2]) &= \int_0^1 H\#F(t, \varphi_H^t \varphi_F^t(y))dt + \ell \mathrm{Area}(\mathbb{S}^2).\\ &= F(y) + \ell\\ &= \mathcal{A}_F(y, \ell[\mathbb{S}^2]).\end{aligned}$$

This shows that $\mathrm{Spec}(H\#F) = \mathrm{Spec}(F)$. A similar argument shows that $\mathrm{Spec}((H\#F)^k) = \mathrm{Spec}(F^k)$ for all $k \in \{1, \dots, d\}$, thus $\mathrm{Spec}_d(H\#F) = \mathrm{Spec}_d(F)$.

By (4), we have proved

$$\mathrm{Spec}_d(\phi \circ f) = \mathrm{Spec}_d(f) \tag{34}$$

for all $\phi \in \mathrm{Diff}_{S^+}(\mathbb{S}^2, \omega)$ such that $d_{C^0}(\phi, \mathrm{Id}) < \delta$.

There remains the step of deducing $c_d(\phi \circ f) = c_d(f)$ from this equality of spectrums.

Given $\phi \in \mathrm{Diff}_{S^+}(\mathbb{S}^2, \omega)$ such that $d_{C^0}(\phi, \mathrm{Id}) < \delta$, one can construct, using the Alexander isotopy, a Hamiltonian isotopy $(\varphi_K^t)_{t \in [0,1]}$ in $\mathrm{Diff}_{S^+}(\mathbb{S}^2, \omega)$, such that $d_{C^0}(\varphi_K^s, \mathrm{Id}) < \delta$ for all $s \in [0, 1]$ and $\varphi_K^1 = \phi$; we refer the reader to [Sey13a, Lemma 3.2] for the details.

Equation (34) then implies $\mathrm{Spec}_d(\varphi_K^s \circ f) = \mathrm{Spec}_d(f)$ for all $s \in [0, 1]$. Now, by Theorem 4.5 the function $s \mapsto c_d(\varphi_K^s \circ f)$ is continuous and takes its values in the measure 0 subset $\mathrm{Spec}_d(f) \subset \mathbb{R}$. As a consequence, it is constant. This shows $c_d(\phi \circ f) = c_d(f)$ and concludes our proof. $\square$

5.2. *Continuity away from the identity.* We now turn our attention to Proposition 5.2. We want to prove (32), i.e., that any $\eta \in \mathrm{Homeo}_{S^+}(\mathbb{S}^2, \omega)$, $\eta \neq \mathrm{Id}$, admits an open neighborhood $\mathcal{V}$ such that for all $\varepsilon > 0$, there exists $\delta > 0$ satisfying

$$\begin{aligned} (35) \qquad & \forall f \in \mathcal{V} \cap \mathrm{Diff}_{S^+}(\mathbb{S}^2, \omega) \quad \forall g \in \mathrm{Diff}_{S^+}(\mathbb{S}^2, \omega), \\ & \qquad \text{if } d_{C^0}(g, \mathrm{Id}) < \delta \text{ then } |c_d(gf) - c_d(f)| < \varepsilon. \end{aligned}$$

Our proof will follow from three lemmas, which we now introduce.

To state the first, let us introduce some terminology. We will say that a diffeomorphism f *d-displaces* a subset U if the subsets U, $f(U), \dots, f^d(U)$ are pairwise disjoint. Our first lemma states that for g supported in an open subset d-displaced by f, an even stronger version of (35) holds. It is adapted from [Ush10, Lemma 3.2], which can be regarded as the analogue for the $d = 1$ case.

LEMMA 5.4. *Let $f \in \mathrm{Diff}_{S^+}(\mathbb{S}^2, \omega)$, and let B be an open topological disc whose closure is included in $\mathbb{S}^2 \setminus \{p_-\}$ and that is d-displaced by f. Then, for all $\phi \in \mathrm{Diff}_B(\mathbb{S}^2, \omega)$, we have $c_d(\phi \circ f) = c_d(f)$.*

We will prove this lemma in Section 5.3. To apply it, we need there to exist an appropriate open disc B. The next lemma is the key ingredient for this.

LEMMA 5.5. *Let $\eta \in \mathrm{Homeo}_c(\mathbb{D}, \omega)$ with $\eta \neq \mathrm{Id}$. Then, there exists $x \in \mathbb{D}$ such that $x, \eta(x), \eta^2(x), \dots, \eta^d(x)$ are pairwise distinct points.*

In particular, by the lemma, there exists an open topological disc B whose closure is d-displaced by η.

If we then let $\mathcal{V}$ be the C^0 open neighborhood of η given by the set of all $f \in \mathrm{Homeo}_{S^+}(\mathbb{S}^2, \omega)$ that d-displace the closure of the disc B, then by Lemma 5.4, we have $c_d(\phi \circ f) = c_d(f)$ for all $\phi \in \mathrm{Diff}_B(\mathbb{S}^2, \omega)$ and $f \in \mathcal{V} \cap \mathrm{Diff}_{S^+}(\mathbb{S}^2, \omega)$. Now it turns out that every map g that is sufficiently C^0 close to Id is close in Hofer distance to an element in $\mathrm{Diff}_B(\mathbb{S}^2, \omega)$. This is the content of the next lemma.

LEMMA 5.6. *Let B be an open topological disc whose closure is included in $\mathbb{S}^2 \setminus \{p_-\}$. For all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $g \in \mathrm{Diff}_{S^+}(\mathbb{S}^2, \omega)$ with $d_{C^0}(g, \mathrm{Id}) < \delta$, there is $\phi \in \mathrm{Diff}_B(\mathbb{S}^2, \omega)$ such that $\|\phi^{-1} g\| \leqslant \varepsilon$.*

We will prove Lemma 5.6 at the end of Section 5.3. Assuming this, we can achieve the proof of (35) and hence of Proposition 5.2, as we now explain.

Proof of Proposition 5.2. Let $\varepsilon > 0$, and let $\delta > 0$ be as provided by Lemma 5.6. Also let $f \in \mathcal{V} \cap \mathrm{Diff}_{S^+}(\mathbb{S}^2, \omega)$. Then, for all g satisfying $d_{C^0}(g, \mathrm{Id}) < \delta$, there exists $\phi \in \mathrm{Diff}_B(\mathbb{S}^2, \omega)$ such that $\|\phi^{-1} g\| < \varepsilon$. Thus, using Lemma 5.4, Hofer continuity (33) and the conjugation invariance of the Hofer norm, we have

$$|c_d(gf) - c_d(f)| = |c_d(gf) - c_d(\phi f)| \leqslant d \|f^{-1} \phi^{-1} g f\| = d \|\phi^{-1} g\| < d\varepsilon.$$

Since ε is arbitrary, this concludes the proof of (35) and of Proposition 5.2, modulo the proofs of Lemmas 5.4, 5.5 and 5.6. $\square$

5.3. *Proofs of Lemmas 5.4, 5.5 and 5.6.* We now provide the proofs of the lemmas stated in the preceding section. We start with the simplest one.

Proof of Lemma 5.5. Let $\eta \in \mathrm{Homeo}_c(\mathbb{D}, \omega)$ with $\eta \neq \mathrm{Id}$, and let d be a positive integer. It is known (see [CK94]) that for any positive integer N, $\eta^N \neq \mathrm{Id}$. Thus, there exists a point $x \in \mathbb{D}$ such that $\eta^{d!}(x) \neq x$. For such a point, $x, \eta(x), \eta^2(x), \ldots, \eta^d(x)$ are pairwise distinct. Indeed, otherwise, there would be integers $0 \leqslant k < \ell \leqslant d$ such that $\eta^k(x) = \eta^\ell(x)$, and we would get $\eta^{\ell-k}(x) = x$, in contradiction with $\eta^{d!}(x) \neq x$ since $d!$ is evenly divided by $\ell - k$. $\square$

We now turn our attention to Lemma 5.4.

Proof of Lemma 5.4. Let F be a Hamiltonian supported in S^+ with $\varphi_F^1 = f$, and let G be a Hamiltonian supported in B with $\varphi_G^1 = \phi$. We will prove that for all $s \in [0,1]$, $\mathrm{Spec}_d(\varphi_G^s f) = \mathrm{Spec}_d(f)$. This implies, as in the proof of Lemma 5.3, that the map $s \mapsto c_d(\varphi_G^s f)$ is constant, hence the lemma.

Let $s \in [0,1]$. We will first verify that the diffeomorphism $\varphi_G^s f$ admits the same k-periodic points as f for all $k \in \{1, \ldots, d\}$.

For all $\ell \in \{1, \ldots, d\}$, we have $B \cap f^\ell(B) = \emptyset$ and $B \cap f^{-\ell}(B) = \emptyset$. It follows that $\varphi_G^s(f^\ell(B)) = f^\ell(B)$ for all $\ell \in \{-d, \ldots, d\}$, hence

$$(\varphi_G^s f)^k(f^{-\ell}(B)) = f^{k-\ell}(B) \quad \forall k \in \{1, \ldots, d\}, \quad \forall \ell \in \{0, \ldots, d\}.$$

Since $f^{-\ell}(B) \cap f^{k-\ell}(B) = \emptyset$ for such k, ℓ, this implies $\varphi_G^s f$ has no k-periodic points with $1 \leqslant k \leqslant d$ in $\bigcup_{\ell=0}^d f^{-\ell}(B)$.

We now fix $k \in \{1, \ldots, d\}$. If $x \notin \bigcup_{\ell=0}^d f^{-\ell}(B)$, then $(\varphi_G^s \circ f)^k(x) = f^k(x)$. As a consequence, the k-periodic points of $\varphi_G^s \circ f$ are exactly those of f.

We will now prove that the corresponding action values coincide as well. For that purpose, it is convenient to use an isotopy generating the $(\varphi_G^s \circ f)^k$ obtained by concatenation of isotopies rather than composition. Namely, the map $(\varphi_G^s \circ f)^k$ is the time-1 map of the isotopy ψ^t, which at time $t \in [\frac{\ell}{2k}, \frac{\ell+1}{2k}]$ for $\ell \in \{0, \ldots, 2k-1\}$ is given by

$$\psi^t = \begin{cases} \varphi_F^{\rho(2kt-\ell)} \circ (\varphi_G^s \circ f)^{\frac{\ell}{2}} & \text{if } \ell \text{ is even,} \\ \varphi_G^{s\rho(2kt-\ell)} \circ f \circ (\varphi_G^s \circ f)^{\frac{\ell-1}{2}} & \text{if } \ell \text{ is odd.} \end{cases}$$

Here, $\rho : [0,1] \to [0,1]$ is a non-decreasing smooth function which is equal to 0 near 0 and equal to 1 near 1. The role of the time-reparametrization ρ is simply to make the isotopy smooth at the gluing times.

This isotopy is generated by the Hamiltonian K given by the formula

$$K_t(x) = \begin{cases} 2k\rho'(2kt-\ell)F_{\rho(2kt-\ell)}(x) & \text{if } \ell \text{ is even,} \\ 2ks\rho'(2kt-\ell)G_{s\rho(2kt-\ell)}(x) & \text{if } \ell \text{ is odd} \end{cases}$$

for $\ell \in \{0, \dots, 2k-1\}$ and $t \in [\frac{\ell}{2k}, \frac{\ell+1}{2k}]$.

We will compute the spectrum of $\varphi_G^s \circ f$ with the help of this particular Hamiltonian. The action of a capped 1-periodic orbit (z, u) of K, with $z = \varphi_K^t(x)$, is given by

$$\begin{aligned} \mathcal{A}_K(z,u) &= \int_{\mathbb{D}^2} u^*\omega + \int_0^1 K_t(\psi^t(x))\,dt = \int_{\mathbb{D}^2} u^*\omega + \sum_{\ell=0}^{2k-1} \int_{\frac{\ell}{2k}}^{\frac{\ell+1}{2k}} K_t(\psi^t(x))\,dt \\ &= \int_{\mathbb{D}^2} u^*\omega + \sum_{j=0}^{k-1} \Bigg(\int_0^1 F_t(\varphi_F^t \circ (\varphi_G^s \circ f)^j(x))\,dt \\ &\qquad + \int_0^1 sG_{st}(\varphi_G^{st} \circ f \circ (\varphi_G^s \circ f)^j(x))\,dt \Bigg), \end{aligned}$$

after suitable change of variable. Since we showed above that $\varphi_G^s \circ f$ has no k-periodic points in $f^{-1}(B)$, we know that $f \circ (\varphi_G^s \circ f)^j(x)$ does not belong to B, hence to the support of G. It follows that the integrand for the third integral above has to vanish and the integrand for the second integral above can be simplified, so that we get

$$\mathcal{A}_K(z,u) = \int_{\mathbb{D}^2} u^*\omega + \sum_{j=0}^{k-1} \int_0^1 F_t(\varphi_F^t \circ f^j(x))\,dt.$$

We see that this action does not depend on s. As a consequence, we get that $\mathrm{Spec}_d(\varphi_G^s f) = \mathrm{Spec}_d(f)$ for all $s \in [0,1]$. $\square$

The proof of Lemma 5.6 remains. Its proof will consist of two steps. First, by Lemma 5.7 below, a diffeomorphism that is sufficiently C^0-close to identity can be appropriately fragmented into maps supported in balls of small area. Second, we observe that moving these maps with small support into B can be achieved with small Hofer norm; this is the content of Lemma 5.8 below.

Before starting the proof of Lemma 5.6, let us state the first of these two lemmas.

Lemma 5.7 ([LRSV21, Lemma 47]). *Let ω_0 denote the standard area form on $\mathbb{R}^2$. Let m be a positive integer and ρ a positive real number. For $i = 1, \dots, m$, denote by U_i the open rectangle $(0,1) \times (\frac{i-1}{m}, \frac{i}{m})$. Then, there exists $\delta > 0$, such that for every $g \in \mathrm{Diff}_c((0,1)\times(0,1), \omega_0)$ with $d_{C^0}(g, \mathrm{Id}) < \delta$, there exist $g_1 \in \mathrm{Diff}_c(U_1, \omega_0), \dots, g_m \in \mathrm{Diff}_c(U_m, \omega_0)$ and $\theta \in \mathrm{Diff}_c((0,1)\times(0,1), \omega_0)$ supported in a disjoint union of topological discs whose total area is less than ρ, such that $g = g_1 \circ \cdots \circ g_m \circ \theta$.*

We can now give the promised proof.

Proof of Lemma 5.6. Let $\varepsilon > 0$. We pick an integer N satisfying $\frac{1}{2N} < \mathrm{area}(B)$, m a positive multiple of N such that $2\frac{N+1}{m} < \varepsilon$, and $\rho = \frac{1}{m}$.

Let δ be as provided by Lemma 5.7, and let $g \in \mathrm{Diff}_{S^+}(\mathbb{S}^2, \omega)$ be such that $d_{C^0}(g, \mathrm{Id}) < \delta$. The map g admits a fragmentation into the form $g = g_1 \circ \cdots \circ g_m \circ \theta$, with all the g_i supported in pairwise disjoint topological discs U_i of area $\frac{1}{2m}$ and θ supported in a disjoint union of discs of total area less than $\frac{1}{2m}$. (Here, the factor $\frac{1}{2}$ comes from the fact that the northern hemisphere S^+ has area $\frac{1}{2}$, while Lemma 5.7 is stated on $(0,1) \times (0,1)$ which has area 1.)

For $j = 1, \ldots, N$, denote by f_j the composition $f_j := \prod_{i=0}^{\frac{m}{N}-1} g_{j+iN}$. Also write $f_{N+1} := \theta$, so that noting that the g_i commute, we have the following formula:

$$g = \prod_{j=1}^{N+1} f_j.$$

Each f_j for $j = 1, \ldots, N$ is supported in $V_j = \bigcup_{i=0}^{\frac{m}{N}-1} U_{j+iN}$ whose area is $\frac{1}{2N} < \mathrm{area}(B)$. Note that V_j is a disjoint union of discs, each of area $\frac{1}{2m}$. By assumption, the support of $f_{N+1} = \theta$, which we denote by V_{N+1}, is also included in a disjoint union of discs of total area smaller than $\frac{1}{2m}$.

Let us now state our second lemma, whose proof we postpone to the end of this section.

LEMMA 5.8. *Let $a \in (0, \frac{1}{2})$, let $B_1, \ldots, B_k \subset S^+$ be pairwise disjoint open topological discs each of area smaller than a, and let $B \subset S^+$ be a topological disc with* $\mathrm{area}(B) > ka$. *Then, there exists a Hamiltonian diffeomorphism $h \in \mathrm{Diff}_{\mathbb{S}^2 \setminus \{p_-\}}(\mathbb{S}^2, \omega)$ that maps $B_1 \cup \cdots \cup B_k$ into B and satisfies $\|h\| \leqslant 2a$.*

In our situation, this lemma implies that for any $j = 1, \ldots, N+1$, there exists a Hamiltonian diffeomorphism $h_j \in \mathrm{Diff}_{\mathbb{S}^2 \setminus \{p_-\}}(\mathbb{S}^2, \omega)$, such that $h_j(V_j) \subset B$ and $\|h_j\| \leqslant \frac{1}{m}$.

Consider the diffeomorphism

$$\phi = \prod_{j=1}^{N+1} h_j \circ f_j \circ h_j^{-1}.$$

By construction, ϕ is supported in B. We will show that $\|\phi^{-1} g\| < \varepsilon$, which will achieve the proof of Lemma 5.6.

To prove that $\|\phi^{-1} g\| < \varepsilon$, let us introduce $\tilde{g}_k = f_1 \circ f_2 \circ \cdots \circ f_k$ and $\phi_k = (h_1 \circ f_1 \circ h_1^{-1}) \circ (h_2 \circ f_2 \circ h_2^{-1}) \circ \cdots \circ (h_k \circ f_k \circ h_k^{-1})$ for $k = 1, \ldots, N+1$. In particular, $\tilde{g}_{N+1} = g$ and $\phi_{N+1} = \phi$. Then, for all $k = 1, \ldots, N$, we have

$$\phi_{k+1}^{-1} \circ \tilde{g}_{k+1} = h_{k+1} \circ (f_{k+1}^{-1} \circ h_{k+1}^{-1} \circ f_{k+1}) \circ (f_{k+1}^{-1} \circ (\phi_k^{-1} \circ \tilde{g}_k) \circ f_{k+1}).$$

Thus, by the triangle inequality and the conjugation invariance,

$$\|\phi_{k+1}^{-1} \circ \tilde{g}_{k+1}\| \leqslant \|h_{k+1}\| + \|h_{k+1}^{-1}\| + \|\phi_k^{-1} \circ \tilde{g}_k\| \leqslant \frac{2}{m} + \|\phi_k^{-1} \circ \tilde{g}_k\|.$$

Hence, by induction, $\|\phi_i^{-1} \circ \tilde{g}_i\| \leq \frac{2i}{m}$, and so $\|\phi^{-1} \circ g\| \leqslant 2\frac{N+1}{m} < \varepsilon$, as wished. □

The last remaining proof is now the following.

Proof of Lemma 5.8. Let $U := B_1 \cup \cdots \cup B_k$. Since U has smaller area than B, there exists a Hamiltonian diffeomorphism $\psi \in \mathrm{Diff}_{S^+}(\mathbb{S}^2, \omega)$ such that $\psi(U) \subset B$. The Hofer norm of ψ may not be small, but we will replace ψ with an appropriate commutator of ψ whose Hofer norm will be controlled.

Since the displacement energy of U is smaller than $a < \frac{1}{2}$, there exists a Hamiltonian diffeomorphism $\ell \in \mathrm{Diff}_{\mathbb{S}^2 \setminus \{p_-\}}(\mathbb{S}^2, \omega)$ such that $\ell(\overline{U}) \cap \overline{U} = \emptyset$ and $\|\ell\| \leqslant a$.

Since $\ell(U)$ has area smaller than $\frac{1}{2}$, there exists $\chi \in \mathrm{Diff}_{\mathbb{S}^2 \setminus \{p_-\}}(\mathbb{S}^2, \omega)$ that fixes U and such that $\chi(\ell(U)) \cap S^+ = \emptyset$, which in particular implies that $\chi(\ell(U)) \cap \mathrm{supp}(\psi) = \emptyset$. Then, $\ell' = \chi \circ \ell \circ \chi^{-1}$ satisfies the following properties:

(i) $\ell'(U) \cap U = \emptyset$,

(ii) $\|\ell'\| \leqslant a$,

(iii) $\ell'(U) \cap \mathrm{supp}(\psi) = \emptyset$.

To prove Property (i), start from $\ell(U) \cap U = \emptyset$ and compose with χ, to get $\chi \circ \ell(U) \cap \chi(U) = \emptyset$. Since $U = \chi^{-1}(U)$, we obtain $\chi \circ \ell \circ \chi^{-1}(U) \cap U = \emptyset$, hence Property (i). Property (ii) follows from the conjugation invariance of the Hofer norm. Property (iii) is a consequence of Property (i), since χ fixes U.

Now, set $h := \psi \circ \ell'^{-1} \circ \psi^{-1} \circ \ell'$. By Property (iii), $\ell'^{-1} \circ \psi^{-1} \circ \ell'(U) = U$. Thus, $h(U) = \psi(U) \subset B$. Moreover,

$$\|h\| \leqslant \|\psi \circ \ell'^{-1} \circ \psi^{-1}\| + \|\ell'\| = 2\|\ell'\| \leqslant 2a.$$

This concludes our proof. □

6. **The periodic Floer homology of a monotone twist**

In this section we explain how to compute PFH for certain twist maps; more precisely, we give a combinatorial model of the PFH chain complex for such maps. As we explain, this leads to a combinatorial formula for computing PFH spectral invariants. We then use this formula to deduce Theorem 3.7. As mentioned above, the formula has also had further applications; see [CGHS23]

6.1. *Perturbations of rotation invariant Hamiltonian flows.* Throughout this section, we consider Hamiltonian flows on $(\mathbb{S}^2, \omega = \frac{1}{4\pi} d\theta \wedge dz)$ for an autonomous Hamiltonian

$$H = \frac{1}{2} h(z),$$

where h is some function of z. We have

$$X_H = 2\pi h'(z)\partial_\theta. \tag{36}$$

Hence,

$$\varphi_H^1(\theta, z) = (\theta + 2\pi h'(z), z)$$

We further restrict h to satisfy

$$h' > 0, h'' > 0, h(-1) = 0.$$

Furthermore, we demand that $h'(-1), h'(1)$ are irrational numbers satisfying $h'(-1) \leqslant \frac{\varepsilon_0}{d}$ and $\lceil h'(1) \rceil - h'(1) \leqslant \frac{\varepsilon_0}{d}$, where ε_0 is a small positive number and $\lceil \cdot \rceil$ denotes the ceiling function. Let $\mathcal{D}$ denote the set of Hamiltonians H that satisfy all of these conditions, and observe that $\mathcal{D} \subset \mathcal{H}$, where $\mathcal{H}$ was defined in Section 4.3. As a consequence of Theorem 4.5, we have well-defined PFH spectral invariants $c_{d,k}(\varphi_H^1)$ for all $H \in \mathcal{D}$.

The periodic orbits of φ_H^1 are then as follows:

(1) There are elliptic orbits p_+ and p_-, corresponding to the north and south poles, respectively.
(2) For each p/q in lowest terms such that $h' = p/q$ is rational, there is a circle of periodic orbits, all of which have period q.

These circles of periodic orbits are familiar from Morse-Bott theory, and are sometimes referred to as "Morse-Bott circles." There is also a standard φ_H^1-admissible[18] almost complex structure J_{std} respecting this symmetry; its action on $\xi = T(\mathbb{S}^2 \times \{pt\}) = T\mathbb{S}^2$ is given by the standard almost complex structure on $\mathbb{S}^2$.

As is familiar in this context (see [HS05, §3.1]), we can perform a C^2-small perturbation of H, to split such a circle corresponding to the locus where $h' = p/q$ into two periodic orbits, one elliptic and one hyperbolic, such that the elliptic one $e_{p,q}$ has slightly negative monodromy angle, and the eigenvalues for the hyperbolic one $h_{p,q}$ are positive. Furthermore, the C^2-small perturbation can be taken to be supported in an arbitrarily small neighborhood of the circle where $h' = p/q$. More precisely, given a φ_H^1 such as above, we can find a perturbation of φ_H^1 satisfying the properties listed in the definition below.

Definition 6.1. Consider φ_H^1 as above, and fix any positive d and $\varepsilon > 0$. We call an area-preserving diffeomorphism φ_0 of $\mathbb{S}^2$ a *nice perturbation* of φ_H^1 if it satisfies the following properties:

[18] This means that the almost complex structure is compatible with the standard SHS on the mapping torus for φ_H^1.

(1) The only periodic orbits of φ_0 that are of degree at most d are p_+, p_-, and the orbits $e_{p,q}$ and $h_{p,q}$ from above such that $q \leqslant d$. Furthermore, all of these orbits are non-degenerate.
(2) The eigenvalues of the linearized return map for $e_{p,q}$ are within ε of 1.
(3) $\varphi_0(\theta, z) = \varphi_H^1(\theta, z)$ as long as z is not within ε of a value such that $h' = p/q$ where $q \leqslant d$.
(4) φ_0 is chosen so that "Double Rounding" cannot occur for generic J close to J_{std}. See Section 6.2 for the definition of Double Rounding.

Observe that, for a given nice perturbation φ_0, we can pick a time-dependent Hamiltonian $\tilde{H}$ such that $\varphi^1_{\tilde{H}} = \varphi_0$ and $\tilde{H} = H$ as long as z is not within ε of a value such that $h' = p/q$ with $q \leqslant d$.

6.2. *The combinatorial model.* We now aim to describe the promised combinatorial model of $\widetilde{\text{PFH}}$ for the Hamiltonians described in the previous section. For the remainder of this section, fix $d \in \mathbb{N}$ and φ_H^1, where $H \in \mathcal{D}$.

To begin, define a *concave lattice path* to be a piecewise linear, continuous path P, in the xy-plane, such that P starts and ends on integer lattice points, its starting point is on the y-axis, the nonsmooth points of P are also at integer lattice points, and P is concave, in the sense that it always lies above any of the tangent lines at its smooth points. Moreover, every edge of P has slope between zero and $\lceil h'(1) \rceil$. We say a concave lattice path is *labeled* if each of its edges is labeled by either e or h, and an edge whose slope is either zero or $\lceil h'(1) \rceil$ is labeled e. Below, we will establish a bijection between labeled concave lattice paths as defined in the previous paragraph and $\widetilde{\text{PFH}}$ generators (α, Z).

Let $\alpha = \{(\alpha_i, m_i)\}$ be an orbit set of degree d for a nice perturbation φ_0 of φ_H^1. First of all, recall that the simple Reeb orbits for Y_{φ_0}, with degree no more than d are

(1) the Reeb orbits $\gamma_\pm$ corresponding to $p_\pm$;
(2) for each z such that $h'(z) = p/q$ in lowest terms, where $q \leqslant d$, there are Reeb orbits of degree q corresponding to the periodic orbits $e_{p,q}$ and $h_{p,q}$, which we will also denote by $e_{p,q}$ and $h_{p,q}$.

We will now associate to the PFH generator $\alpha = \{(\alpha_i, m_i)\}$ a labeled concave lattice path P_α whose starting point we require to be $(0,0)$. If $(\gamma_-, m_-) \in \alpha$, we set $v_- = m_-(1,0)$ and label it by e. If $(\gamma_+, m_+) \in \alpha$, we set $v_+ = m_+(1, \lceil h'(1) \rceil)$ and label it by e. Next, consider the orbits in α corresponding to $z = z_{p,q}$ such that $h'(z) = p/q$; note that there are at most two such entries in α: one corresponding to $e_{p/q}$ and another corresponding to $h_{p/q}$. To these entries, we associate the labeled vector $v_{p,q} = m_{p,q}(q,p)$, where $m_{p,q}$ is the sum of multiplicities of $e_{p/q}$ and $h_{p/q}$; the vector is labeled h if $(h_{p/q}, 1) \in \alpha$, and e otherwise. (For motivation, note that by the conditions on the twisted PFH

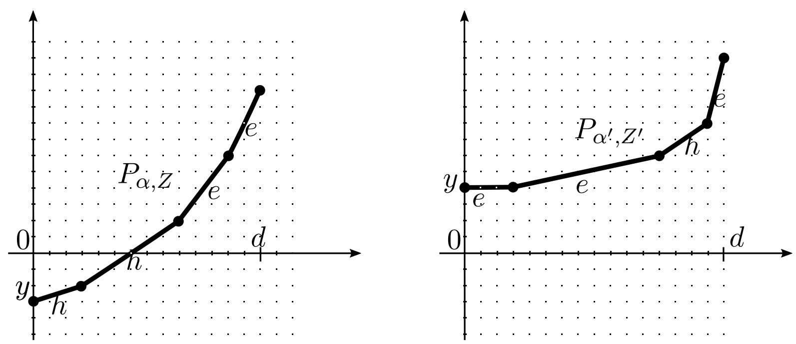

Figure 1. The lattice path $P_{\alpha,Z}$ for $\alpha = \{(h_{1/3}, 1), (e_{2/3}, 1), (h_{2/3}, 1), (e_{4/3}, 1), (e_2, 2)\}$, $Z = Z_\alpha - 3[\mathbb{S}^2]$, and $P_{\alpha',Z'}$ for $\alpha' = \{(\gamma_-, 3), (e_{2/9}, 1), (h_{2/3}, 1), (\gamma_+, 1)\}$, $Z' = Z_\alpha + 4[\mathbb{S}^2]$ (assuming $\lceil h'(1) \rceil = 4$).

chain complex, an m_i corresponding to a hyperbolic orbit must equal 1.) To build P_α from all of the data in α, we simply concatenate the vectors $v_-, v_{p,q}, v_+$ into a concave lattice path. Note that there is a unique way to do this: the path must begin with v_-, it must end with v_+, and the vectors $v_{p/q}$ must be concatenated in increasing order with respect to the ratios p/q.

Now, given a twisted PFH generator (α, Z) for $\widetilde{\mathrm{PFC}}$, we define a mapping

$$(\alpha, Z) \mapsto P_{\alpha,Z},$$

which associates a labeled concave lattice path $P_{\alpha,Z}$ to (α, Z). More specifically, when $Z = Z_\alpha$, where Z_α is the class defined in Section 4.3.1, we define P_{α,Z_α} to be the labeled concave lattice path P_α. Since $H_2(Y_\varphi) = \mathbb{Z}$, generated by the class of $\mathbb{S}^2 \times \{\mathrm{pt}\}$, for any other (α, Z), we have $Z = Z_\alpha + y[\mathbb{S}^2]$. We then define $P_{\alpha,Z}$ to be P_α shifted by the vector $(0, y)$. We leave to the reader to check that the mapping defined here is a bijection between generators $(\alpha, Z) \in \widetilde{\mathrm{PFC}}$ of degree d and labeled concave lattice paths with horizontal displacement d; for simplicity, we sometimes call the horizontal displacement the *degree* of the lattice path. Figure 1 shows two examples of such concave lattice paths.

We now state some of the key properties of the above bijection

$$(\alpha, Z) \mapsto P_{\alpha,Z}.$$

Action: Define the *action* $\mathcal{A}(P_{\alpha,Z})$ as follows. We first define the actions of the edges of $P_{\alpha,Z}$ by the following formulae:

$$\mathcal{A}(v_-) = 0, \quad \mathcal{A}(v_+) = m_+ \frac{h(1)}{2},$$
$$\mathcal{A}(v_{p,q}) = \frac{m_{p,q}}{2}(p(1 - z_{p,q}) + qh(z_{p,q})),$$

where v_-, v_+, and $v_{p,q}$ are as above. We then define the action of $P_{\alpha,z}$ to be

$$\mathcal{A}(P_{\alpha,Z}) = y + \mathcal{A}(v_+) + \sum_{v_{p,q}} \mathcal{A}(v_{p,q}), \tag{37}$$

where y is such that $P_{\alpha,Z}$ begins at $(0,y)$.

We claim that by picking the nice perturbation φ_0 to be sufficiently close to φ^1_H we can arrange for $\mathcal{A}(\alpha, Z)$ to be as close to $\mathcal{A}(P_{\alpha,Z})$ as we wish. To show this it is sufficient to prove it when α is a simple Reeb orbit and $Z = Z_\alpha$, where Z_α is the relative class from Section 4.3.1. We have to consider the following three cases:

- If $\alpha = \gamma_-$, then $\mathcal{A}(\alpha, Z_\alpha) = 0$, by equation (31), which coincides with $\mathcal{A}(1, 0)$.
- If $\alpha = \gamma_+$, then $\mathcal{A}(\alpha, Z_\alpha) = \frac{h(1)}{2}$, by equation (31), which coincides with $\mathcal{A}(1, \lceil h'(1) \rceil)$. Note that in equation (31), the term $\int_{\mathbb{D}^2} u_\alpha^* \omega$ is zero.
- The remaining case is when $\alpha = e_{p,q}$ or $h_{p,q}$; here, it is sufficient to show that the action of the Reeb orbits at $z_{p,q}$, for the unperturbed diffeomorphism φ^1_H, is exactly the quantity $\frac{1}{2}\left(p(1 - z_{p,q}) + qh(z_{p,q})\right)$. This follows from equation (31): the term $\int_{\mathbb{D}^2} u_\alpha^* \omega$ is exactly $\frac{1}{2}p(1 - z_{p,q})$ and the term $\int_0^q H_t(\varphi^t_H(q))dt$ is exactly $\frac{1}{2}qh(z_{p,q})$.

Index: Next, we associate an *index* to a concave lattice path P that begins at a point $(0, y)$, on the y-axis, and has degree d.

First, we form (possibly empty) regions $R_\pm$, where R_- is the closed region bounded by the x-axis, the y-axis, and the part of P below the x-axis, while R_+ is the closed region bounded by the x-axis, the line $x = d$, and the part of P above the x-axis. Let j_+ denote the number of lattice points in the region R_+, not including lattice points on P, and let j_- denote the number of lattice points in the region R_-, not including the lattice points on the x-axis; see Figure 2 below. We now define

$$j(P) := j_+(P) - j_-(P). \tag{38}$$

This definition of j is such that if one shifts P vertically by 1, then $j(P)$ increases by $d + 1$

Given a path $P_{\alpha,Z}$, associated to a $\widetilde{\mathrm{PFH}}$ generator (α, Z), we define its index by

$$I(P_{\alpha,Z}) := 2j(P_{\alpha,Z}) - d + h, \tag{39}$$

where h denotes the number of edges in $P_{\alpha,Z}$ labeled by h. See Figure 2 for an example of a computation of this combinatorial index. It will turn out that $I(P_{\alpha,Z})$ coincides with the PFH index of $I(\alpha, Z)$ as defined in equation (16).

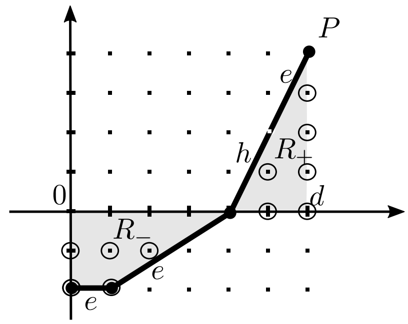

Figure 2. The lattice points included in the count of $j(P)$ are circled. On this example, $j_+(P) = 6$, $j_-(P) = 5$, $d = 6$, $h = 1$, thus $j(P) = 1$ and $I(P) = -3$.

Corner rounding and the differential: Lastly, we define a combinatorial process that corresponds to the PFH differential. Let P_β be a concave lattice path of degree d that begins on the y-axis; note that for the moment we do not suppose that P_β is labeled. Then, if we attach vertical rays to the beginning and end of P_β, in the positive y direction, we obtain a closed convex subset R_β of the plane; see Figure 3. For any given corner of P_β, where we include the initial and final endpoints of P_β as corners, we can define a *corner rounding* operation by removing this corner, taking the convex hull of the remaining integer lattice points in R_β, and taking the lower boundary of this region, namely the part of the boundary that does not consist of vertical lines. Note that the newly obtained path is of degree d.

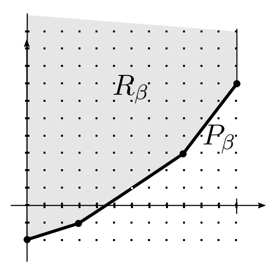

Figure 3. The region R_β.

Now suppose that P_α and P_β are two labeled concave lattice paths. We say that P_α is obtained from P_β by *rounding a corner and locally losing one h* if P_α is obtained from P_β by a corner rounding such that the following conditions are satisfied; see the examples in Figure 4:

(i) Let k denote the number of edges in P_β, with an endpoint at the rounded corner, which are labeled h. We require that $k > 0$; so $k = 1$ or $k = 2$.

(ii) Of the new edges in P_α, created by the corner rounding operation, exactly $k-1$ are labelled h.

(iii) The edges in P_α that coincide with an edge of P_β, or are contained in an edge of P_β, keep the same labels as in P_β.

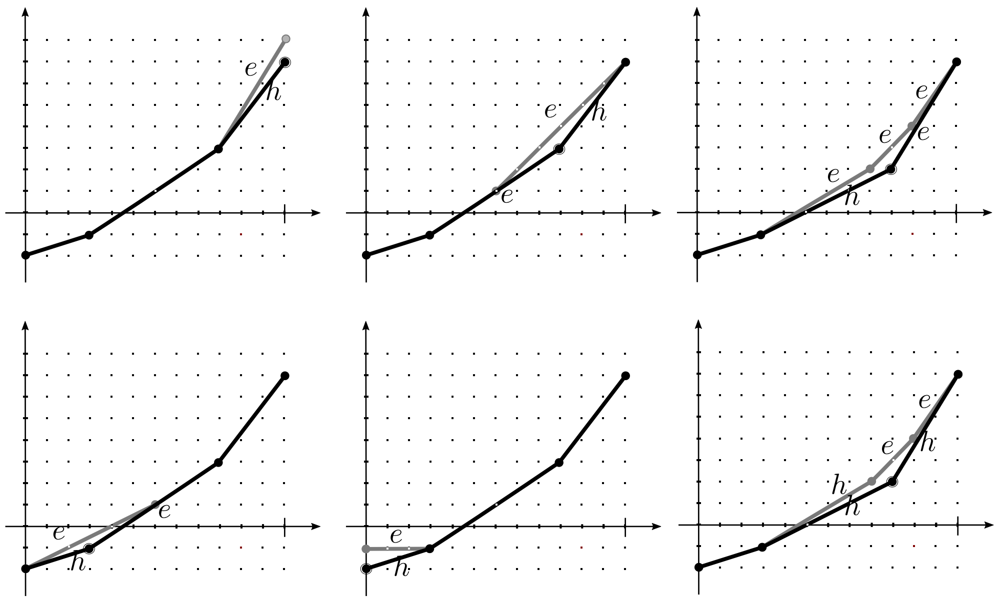

Figure 4. Some examples for the "rounding corner and locally losing one h" operation. The path P_β is in black, the new edges in P_α are in grey. The rounded corner is circled. We only label the relevant edges.

Similarly, we can give the promised definition of the "Double Rounding," which has already been introduced in Section 6.1. Namely, if $P_{\beta,Z'}$ has three consecutive edges, all labeled by h, we say that $P_{\alpha,Z}$ is obtained from $P_{\beta,Z'}$ by *double rounding* if we remove both interior lattice points for these three edges, take the convex hull of the remaining lattice points (in the region formed by adding vertical lines, as above), and label all new edges by e.

The notions introduced above and the proposition below give a complete combinatorial interpretation of the twisted PFH chain complex:

Proposition 6.2. *Fix $d > 0$, and let φ_0 be a nice perturbation of φ^1_H, where $H \in \mathcal{D}$. Then, for generic φ_0-admissible almost complex structure J close to J_{std}, the bijection*

$$(\alpha, Z) \mapsto P_{\alpha,Z}$$

between the set of $\widetilde{\mathrm{PFH}}$ generators of degree d and the set of concave lattice paths of degree d has the following properties:

(1) $\mathcal{A}(\alpha, Z) \sim \mathcal{A}(P_{\alpha,Z})$;
(2) $I(\alpha, Z) = I(P_{\alpha,Z})$;
(3) $\langle \partial(\alpha, Z), (\beta, Z') \rangle \neq 0$ *if and only if* $P_{\alpha,Z}$ *is obtained from* $P_{\beta,Z'}$ *by rounding a corner and locally losing one* h.

Here, by $\mathcal{A}(\alpha, Z) \sim \mathcal{A}(P_{\alpha,Z})$, *we mean that by choosing our nice perturbation* φ_0 *sufficiently close to* φ^1_H, *we can arrange for* $\mathcal{A}(\alpha, Z)$ *to be as close to* $\mathcal{A}(P_{\alpha,Z})$ *as we wish.*

Proof. We have already proven the first of the three listed properties in the above proposition. The other two properties follow from adapting the arguments in [HS06], [HS05] to our setting, so for brevity we will not provide the details. For the interested reader, we have provided an outline[19] of the necessary modifications in Appendix A. □

6.3. *Computation of the spectral invariants.* With the combinatorial model behind us, we now explain how to compute some relevant PFH spectral invariants via Theorem 6.3 below. This will be used in our proof of Theorem 3.7.

We will need to introduce some notation and conventions before stating, and proving, the main result of this section. Throughout this section, we fix φ to be a (smooth) positive monotone twist map of the disc. Recall from Remark 4.6 that we define PFH spectral invariants for maps of the disc by identifying $\mathrm{Diff}_c(\mathbb{D}, \omega)$ with maps of the sphere supported in the northern hemisphere $S^+ \subset \mathbb{S}^2$, where the sphere $\mathbb{S}^2$ is equipped with the symplectic form $\omega = \frac{1}{4\pi} d\theta \wedge dz$.

Every monotone twist map of the disc φ can be written as the time–1 map of the flow of an autonomous Hamiltonian

$$H = \frac{1}{2} h(z),$$

where $h : \mathbb{S}^2 \to \mathbb{R}$ is a function of z satisfying

$$h' \geqslant 0, h'' \geqslant 0, h(-1) = 0, h'(-1) = 0.$$

This will be our standing assumption throughout this section. For the main result of this section, Theorem 6.3, we will need to impose the additional assumption that

$$h'(1) \in \mathbb{N}. \tag{40}$$

The reason for imposing the above assumption is that in our combinatorial model, Proposition 6.2, $h'(1)$ is assumed to be sufficiently close to $\lceil h'(1) \rceil$.

[19] For a *very* interested reader, an extremely detailed exposition can be found in [CGHS20, §5].

Observe that every Hamiltonian H as above can be C^∞ approximated by the Hamiltonians considered in Section 6.1.

Although φ is degenerate, we can still define the notion of a *concave lattice path for* φ as any lattice path obtained from a starting point $(0,y)$, with $y \in \mathbb{Z}$, and a finite sequence of consecutive edges v_{p_i,q_i}, $i = 0,\dots,\ell$, such that

- $v_{p_i,q_i} = m_{p_i,q_i}(q_i,p_i)$ with q_i, p_i coprime;
- the slopes p_i/q_i are in increasing order;
- we have $0 \leqslant p_0/q_0$ and p_ℓ/q_ℓ is at most $h'(1)$.

If $p_0 = 0$, as in Section 6.2 we will define $v_- = m_-(1,0) = v_{p_0,q_0}$. If $p_\ell/q_\ell = h'(1)$, we will define $v_+ = m_+(1,h'(1)) = v_{p_l,q_l}$. We also let $z_{p,q}$ be such that $h'(z_{p,q}) = p/q$.

We can also define the action of such a path just as in Section 6.2: We first define

$$\mathcal{A}(v_-) = 0, \quad \mathcal{A}(v_+) = m_+\frac{h(1)}{2}, \quad \mathcal{A}(v_{p,q}) = \frac{m_{p,q}}{2}(p(1-z_{p,q})+qh(z_{p,q})). \tag{41}$$

We then define the action of a concave lattice path P to be

$$\mathcal{A}(P) = y + \mathcal{A}(v_+) + \sum_{v_{p,q}} \mathcal{A}(v_{p,q}). \tag{42}$$

The definition of $j(P)$ from Section 6.2 (see equation (38)) is still valid here. With this in mind, we have the following:

THEOREM 6.3. *Let* $\varphi \in \mathrm{Diff}_c(\mathbb{D},\omega)$ *be a monotone twist satisfying the assumption* (40). *Then, for all integers* $d > 0$ *and* $k = d \mod 2$,

$$c_{d,k}(\varphi) = \max\{\mathcal{A}(P) : 2j(P) - d = k\}, \tag{43}$$

where the max is over all concave lattice paths P *for* φ *of horizontal displacement* d.

We remark that there exist similar formulas for ECH capacities of concave [CCGF+14, Th. 1.2.1] and convex [CG19, Cor. A.12] toric domains; see also [Hut11] for earlier related results.

Proof. We can take a C^∞ small perturbation of φ to a d-nondegenerate Hamiltonian diffeomorphism φ_0, which itself is a *nice perturbation* of some φ^1_H, where $H \in \mathcal{D}$ as in Section 6.1.

Since $c_{d,k}(\varphi)$ is the limit of $c_{d,k}(\varphi_0)$, as we take smaller and smaller perturbations, to prove (43) it suffices to show that an analogous formula holds for $c_{d,k}(\varphi_0)$. We will achieve this by proving that the spectral invariant $c_{d,k}(\varphi_0)$ is carried by the element σ of the PFH chain complex for φ_0 given by

$$\sigma := \sum(\alpha, Z),$$

where the sum is over all twisted PFH generators (α, Z) where α consists of only elliptic orbits, is of degree d and $I(\alpha, Z) = k$. Equivalently, the corresponding concave lattice path $P_{\alpha,Z}$ has edges that are all labeled e, and it has degree d and index k. Note that here, since α consists of only elliptic orbits, $I(\alpha, Z) = 2j(P_{\alpha,Z}) - d$. To see why the above sum defining σ is finite, note that since φ_0 is non-degenerate, there are only finitely many twisted PFH generators (α, Z) with degree d and index k.

We first claim that σ is in the kernel of the PFH differential. Indeed, by Proposition 6.2, the differential is the mod 2 sum over every (β, Z') such that $P_{\alpha,Z}$ can be obtained from $P_{\beta,Z'}$ by rounding a corner and locally losing one h. Fix one such $P_{\beta,Z'}$. It has exactly one edge labelled h, and so there are exactly two concave paths, say $P_{\alpha,Z}$ and $P_{\tilde{\alpha},\tilde{Z}}$, which are obtained from $P_{\beta,Z'}$ by rounding a corner and locally losing one h. The two paths $P_{\alpha,Z}$ and $P_{\tilde{\alpha},\tilde{Z}}$ are different because, for example, when you round the two corners for an edge, one rounding contains one of the corners, and the other contains the other corner. Now, (α, Z) and $(\tilde{\alpha}, \tilde{Z})$ both contribute to σ and thus, (β, Z') appears exactly twice in the differential of σ; hence, its mod 2 contribution to the differential is zero. Consequently, we see that σ is in the kernel of the PFH differential.

Now, by Proposition 6.2, no concave path with all edges labeled e is ever in the image of the differential, because the concave path corresponding to the negative end of a holomorphic curve counted by the differential has more edges labeled h than the concave path corresponding to the positive end, and in particular has at least one edge labeled h. So, $[\sigma] \neq 0$ in homology. In fact, σ must carry the spectral invariant for similar reasons. Specifically, if there is some other chain complex element σ' homologous to σ, then $\sigma + \sigma'$ must be in the image of the differential. Nothing in the image of the differential has a path with all edges labeled by e, so σ' must contain all possible paths of degree d and index k with all edges labeled by e, and so its action must be at least as much as σ. This shows that $c_{d,k}(\varphi_0)$ is given by the action of σ, which completes the proof. ☐

Corollary 6.4. *Under the assumptions of Theorem 6.3,*[20] *we have*

$$c_{d,k}(\varphi) \leqslant c_{d,k+2}(\varphi)$$

for any (d, k).

Proof. If φ is the identity, then the corollary holds by direct computation, as in our proof of the Normalization property in Theorem 4.5. Otherwise, if we

[20]For more general φ, (46) can still be established, by using the PFH "U-map," but we will not need this in the present work.

take an action maximizing path of index k, with all edges labeled e as above, we can always round a corner, and then the grading increases by 2, and the action does not decrease. □

6.4. *Proof of Theorem 3.7*. We now prove Theorem 3.7, which establishes the Calabi property for monotone twist maps of the disc that were introduced in Section 3.2.

Theorem 3.7 will follow from the theorem below for the invariants $c_{d,k}$, which was originally conjectured in greater generality by Hutchings [Hut17].

THEOREM 6.5. *Let (k_d), $d = 1, 2, \dots$ be a sequence of integers, with $k_d = d \bmod 2$ for any d. Then, for any positive monotone twist φ, we have*

$$\mathrm{Cal}(\varphi) = \lim_{d\to\infty} \left(\frac{c_{d,k_d}(\varphi)}{d} - \frac{k_d}{2(d^2+d)} \right). \tag{44}$$

A first observation, concerning equation (44), which is also due to Hutchings, is that it suffices to establish (44) for a single such sequence (d, k_d) with $d = 1, 2, \dots$ ranging over the positive integers. Indeed, for d-nondegenerate φ, there is an automorphism of the twisted PFH chain complex given by

$$(\alpha, Z) \mapsto (\alpha, Z + [S^2]),$$

where $[S^2]$ denotes the class of the sphere. By [Hut02, Prop. 1.6], this increases the grading by $2d + 2$. It also increases the action by 1. So, we have

$$c_{d,k+2d+2}(\varphi) = c_{d,k}(\varphi) + 1 \tag{45}$$

for all φ. Now, the right-hand side of equation (44) is invariant under increasing the numerator of the first fraction by one, and increasing the numerator of the second fraction by $2d + 2$. Moreover, by Corollary 6.4 we obtain

$$c_{d,k}(\varphi) \leqslant c_{d,k'}(\varphi), \tag{46}$$

when $k' \geqslant k$, with $k = k' = d \bmod 2$ and φ a positive monotone twist. Thus, given an arbitrary sequence $\tilde{k}_d$, we can assume by the above analysis that $\tilde{k}_d$ is within $2d + 2$ of k_d, and $|c_{d,\tilde{k}_d} - c_{d,k_d}| \leqslant 1$; the limit on the right-hand side of (44) is then the same for c_{d,k_d} and $c_{d,\tilde{k}_d}$.

We now give a proof of Theorem 6.5. It is sufficient to prove Theorem 6.5 for monotone twists φ, which can be written as φ_H^1 with the Hamiltonian H satisfying (40). This is because the left- and right-hand sides of equation 44, i.e., the Calabi invariant and the PFH spectral invariants, are (Lipschitz) continuous with respect to the Hofer norm and, moreover, every monotone twist can be approximated, in the Hofer norm, by monotone twists satisfying (40). Hence, we will suppose for the rest of this section that our monotone twists φ satisfy (40). This allows us to apply Theorem 6.3.

Our proof relies on a version of the isoperimetric inequality for non-standard norms, which we now recall; the idea of using this inequality comes from Hutchings' proof in [Hut11, §8] of the "Volume Property" for ECH spectral invariants for certain toric contact forms.

Let $\Omega \subset \mathbb{R}^2$ be a convex compact connected subset. Using the standard Euclidean inner product, the dual norm associated to Ω, denoted $\|\cdot\|^*_\Omega$, is defined for any $v \in \mathbb{R}^2$ by

$$\|v\|^*_\Omega = \max\{v \cdot w : w \in \partial\Omega\}. \tag{47}$$

Let $\Lambda \subset \mathbb{R}^2$ be an oriented, piecewise smooth curve and denote by $\ell_\Omega(\Lambda)$ its length measured with respect to $\|\cdot\|^*_\Omega$. When Λ is closed, its length remains unchanged under translation of Ω.

For our proof, we will suppose that Ω is the region bounded by the graph of h, the horizontal line through $(1, h(1))$, and the vertical line through $(-1, 0)$. Denote by $\hat{\Omega}$ the region obtained by rotating Ω clockwise by ninety degrees; see Figure 5. We orient the boundary $\partial\hat{\Omega}$ counterclockwise with respect to any point in its interior.

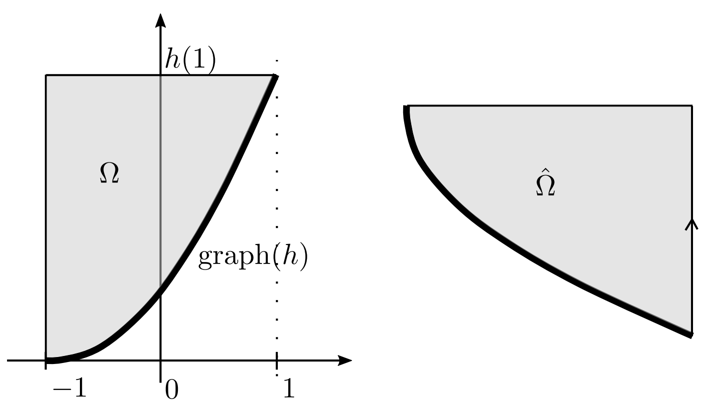

Figure 5. The convex subsets Ω, $\hat{\Omega}$.

Proof of Theorem 6.5. Let P be a concave lattice path of horizontal displacement d for φ. Complete the path P to a closed, convex polygon by adding a vertical edge at the beginning of P and a horizontal edge at the end; orient this polygon counterclockwise, relative to any point in its interior; and, rotate it clockwise by ninety degrees. Call the resulting path Λ; see Figure 6. We will need the following lemma.

Lemma 6.6. *The following identities hold*:

(1) $\ell_\Omega(\partial\hat{\Omega}) = 2(2h(1) - I)$, *where* $I := \int_{-1}^{1} h(z)dz$;

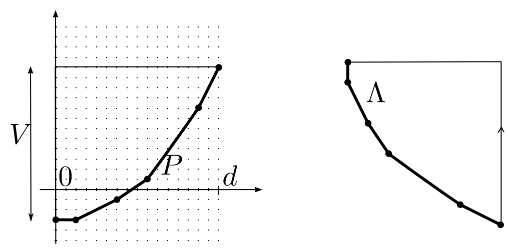

Figure 6. The path P and the closed path Λ.

(2) $\ell_\Omega(\Lambda) = dh(1)+2y+2V-2\mathcal{A}(P)$, *where* V *denotes the vertical displacement of the path* P.

Proof of Lemma 6.6. According to the Isoperimetric Theorem [BM94], for any simple closed curve Γ, we have

$$\ell^2_\Omega(\Gamma) \geqslant 4A(\Omega)A(\Gamma), \tag{48}$$

where $A(\Omega)$ and $A(\Gamma)$ denote the Euclidean areas of Ω and the region bounded by Γ, respectively. Moreover, equality holds when Γ is a scaling of a ninety degree clockwise rotation of $\partial\Omega$; see [Hut11, Example 8.3]. The first item follows immediately from the equality case of the theorem applied to $\Gamma = \partial\hat{\Omega}$ because $A(\Gamma) = A(\Omega) = 2h(1) - I$. Alternatively, item (1) could be obtained via direct computation.

We now prove the second item. The length of the polygon Λ is given by the sum $\sum_{e\in\Lambda} \|e\|^*_\Omega$, where the sum is taken over the edge vectors e of Λ. It follows from the method of Lagrange multipliers that

$$\|e\|^*_\Omega = e \cdot p_e$$

for some point $p_e \in \partial\Omega$, where e points in the direction of the outward normal cone at p_e. Hence, we can write

$$\ell_\Omega(\Lambda) = \sum_{e\in\Lambda} e \cdot p_e = \sum_{e\in\Lambda} e \cdot (p_e - m), \tag{49}$$

where the second equality holds, for any $m \in \mathbb{R}^2$, because Λ is closed. We will calculate $\ell_\Omega(\Lambda)$ using the choice $m = (1, 0)$.

Let e denote one of the edges of Λ corresponding to a vector $v_{p,q} = m_{p,q}(q, p)$ in P. Now, we have $e = m_{p,q}(p, -q)$, since we are taking a ninety degree clockwise rotation; moreover, $p_e - m = (z_{p,q} - 1, h(z_{p,q}))$. Thus,

$$\begin{aligned} e \cdot (p_e - m) &= m_{p,q}(p, -q) \cdot (z_{p,q} - 1, h(z_{p,q})) \\ &= m_{p,q} \left(p(z_{p,q} - 1) - qh(z_{p,q})\right) \\ &= -2\mathcal{A}(v_{p,q}), \end{aligned}$$

where the final equation follows from (41).

If e is an edge of Λ corresponding to either of the vectors $v = v_- = m_-(1,0)$ or $v = v_+ = m_+(1, \lceil h'(1) \rceil)$, then a similar computation to the above yields $e \cdot (p_e - m) = -2\mathcal{A}(v)$. Summing over all of the edges e of Λ, corresponding to vectors in P, we obtain the quantity

$$2y - 2\mathcal{A}(P). \tag{50}$$

The remaining two edges of Λ are the vectors $e_1 = (0, d)$ and $e_2 = (-V, 0)$ for which we have

$$\begin{aligned} e_1 \cdot (p_{e_1} - m) &= (0, d) \cdot (-1, h(1)) = dh(1), \\ e_2 \cdot (p_{e_2} - m) &= (-V, 0) \cdot (-2, 0) = 2V. \end{aligned} \tag{51}$$

We obtain from equations (49), (50), and (51) that $\ell_\Omega(\Lambda) = dh(1) + 2y + 2V - 2\mathcal{A}(P)$. □

Step 1: *Calabi gives the lower bound.* Here, we will prove the lower bound needed for establishing equation (44). In other words, we will show that for any sequence (k_d), we have

$$\mathrm{Cal}(\varphi) \leqslant \liminf_{d \to \infty} \left(\frac{c_{d,k_d}(\varphi)}{d} - \frac{1}{2} \frac{k_d}{d^2 + d} \right). \tag{52}$$

To prove the above, fix $\varepsilon > 0$. We will show that for all sufficiently large positive integers d, there exists a sequence of concave lattice paths $\{P_{\varepsilon,d}\}$, such that

$$\left| \mathrm{Cal}(\varphi) - \left(\frac{\mathcal{A}(P_{\varepsilon,d})}{d} - \frac{1}{2} \frac{k_d}{d^2 + d} \right) \right| \leqslant \varepsilon, \tag{53}$$

where $k_d = 2j(P_{\varepsilon,d}) - d$ denotes the combinatorial index of $P_{\varepsilon,d}$. By Theorem 6.3, we have $\mathcal{A}(P_{\varepsilon,d}) \leqslant c_{d,k_d}(\varphi)$ and, by the argument we explained in Section 6.4 (see the discussion after Theorem 6.5), proving (52) for one sequence k_d with d ranging across all sufficiently large positive integers proves it for all such sequences, and so we conclude (52) from the above, since ε was arbitrary.

We now turn to the description of the concave paths $P_{\varepsilon,d}$. Let P be a concave path approximating the graph of h such that it begins at $(-1, 0)$, ends on the line $x = 1$, is piecewise linear, and its vertices are rationals with numerator an even integer and denominator d. Let Λ be the convex polygon obtained as follows: Add a vertical edge at the beginning of P and a horizontal edge at the end of it; orient this polygon counterclockwise, relative to any point in its interior; and, rotate it clockwise by ninety degrees. The convex polygon Λ approximates $\partial\hat{\Omega}$. More precisely, given ε, we pick, for all sufficiently large positive integers d, paths P, subject to the conditions above, and such that

(A) P is within ε of the graph of h;

(B) $|\ell_\Omega(\Lambda) - \ell_\Omega(\partial\hat{\Omega})| \leqslant \varepsilon$, which by Lemma 6.6 is equivalent to

$$|\ell_\Omega(\Lambda) - 2(2h(1) - I)| \leqslant \varepsilon;$$

(C) the area of the region under the path P, and above the x-axis, is within ε of I.

Let $P_{\varepsilon,d}, \Lambda_{\varepsilon,d}$ be the images of P, Λ, respectively, under the mapping

$$(x, y) \mapsto \frac{d}{2}(x+1, y).$$

The path $P_{\varepsilon,d}$ is a concave lattice path of degree d. Recall that $\mathrm{Cal}(\varphi) = \frac{1}{4}I$. We will prove the two inequalities below, which will imply equation (53):

$$\begin{aligned} &\left|\frac{\mathcal{A}(P_{\varepsilon,d})}{d} - \frac{I}{2}\right| \leqslant \frac{3\varepsilon}{4}, \\ &\left|\frac{1}{2}\frac{k_d}{d^2+d} - \frac{I}{4}\right| \leqslant \frac{\varepsilon}{4}. \end{aligned} \tag{54}$$

We first examine the term $\frac{\mathcal{A}(P_{\varepsilon,d})}{d}$. By Lemma 6.6, and using the fact that $\ell_\Omega(\Lambda_{\varepsilon,d}) = \frac{d}{2}\ell_\Omega(\Lambda)$, we obtain

$$\begin{aligned} \frac{\mathcal{A}(P_{\varepsilon,d})}{d} &= \frac{dh(1) + 2V - \ell_\Omega(\Lambda_{\varepsilon,d})}{2d} \\ &= \frac{h(1)}{2} + \frac{V}{d} - \frac{\ell_\Omega(\Lambda)}{4}. \end{aligned}$$

By item (A) above, the term $\frac{V}{d}$ is within $\frac{\varepsilon}{2}$ of $\frac{h(1)}{2}$. By item (B) above, the term $\ell_\Omega(\Lambda)$ is within ε of $2(2h(1) - I)$, hence the first inequality in (54).

As for the second inequality, we know that, up to an error of $O(d)$, the index k_d is twice the area between the x-axis and the path $P_{\varepsilon,d}$. Because $P_{\varepsilon,d}$ is a scaling of P by a factor of $\frac{d}{2}$, item (C) above implies

$$-\frac{d^2}{2}\varepsilon + O(d) \leqslant k_d - \frac{d^2}{2}I \leqslant \frac{d^2}{2}\varepsilon + O(d),$$

which for sufficiently large d yields the second inequality in (54). □

Step 2: *Calabi gives the upper bound.* We now complete the proof of Theorem 6.5. We emphasize again, for the convenience of the reader, that as mentioned in Remark 3.10, we do not actually need this step of the proof for the proof of our main result Theorem 1.2.

To complete the proof, we need to show that

$$\mathrm{Cal}(\varphi) \geqslant \limsup_{d\to\infty}\left(\frac{c_{d,k}(\varphi)}{d} - \frac{1}{2}\frac{k}{d^2+d}\right). \tag{55}$$

To do this, we will show that

$$\mathrm{Cal}(\varphi) \geqslant \limsup_{d\to\infty}\left(\frac{\mathcal{A}(P)}{d} - \frac{1}{2}\frac{k}{d^2+d}\right) \tag{56}$$

for all degree d concave lattice paths P of combinatorial index k.

Let P be a concave lattice path of degree d and combinatorial index k. Let E be a real number with $E > h(1)$ and $E > \frac{2V}{d}$, and let Ω_E be the compact convex subset of $\mathbb{R}^2$ bounded by the graph of h, the vertical segments $\{-1\} \times [0, E]$ and $\{1\} \times [h(1), E]$, and the horizontal segment $[0, d] \times \{E\}$. For example, with the notation of Lemma 6.6, we have $\Omega = \Omega_{h(1)}$. Inequality (56) will follow from letting E tend to ∞ after applying the isoperimetric inequality (48) to the domain Ω_E and to the following curve Λ_E.

To define Λ_E, consider the region delimited by our lattice path P, the vertical segments $\{0\} \times [y, y + \frac{dE}{2}]$ and $\{d\} \times [y + V, y + \frac{dE}{2}]$, and the horizontal segment $[0, d] \times \{y + \frac{dE}{2}\}$. Our curve Λ_E is the boundary of this region, rotated clockwise by ninety degrees; see Figure 7.

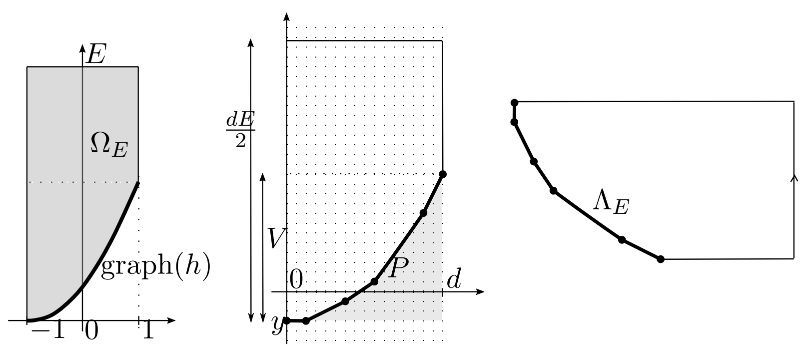

Figure 7. The convex subset Ω_E and the path Λ_E.

The isoperimetric inequality (48) gives

$$\ell^2_{\Omega_E}(\Lambda_E) \geqslant 4A(\Omega_E)A(\Lambda_E). \tag{57}$$

The area factors are easily computed. We have

$$A(\Omega_E) = 2E - I, \quad A(\Lambda_E) = \tfrac{1}{2}d^2E - a(P),$$

where $a(P)$ denote the area of the region between P, the horizontal segment $[0, d] \times \{y\}$ and the vertical segment $\{d\} \times [y, y + V]$ (in grey on Figure 7). Moreover, a computation similar to that of item (2) in Lemma 6.6, gives

$$\ell_{\Omega_E}(\Lambda_E) = 2dE + 2y - 2\mathcal{A}(P).$$

Thus, (57) gives

$$(2dE + 2y - 2\mathcal{A}(P))^2 \geqslant 4(2E - I)(\tfrac{1}{2}d^2E - a(P)).$$

After simplification, we obtain

$$-2dE\mathcal{A}(P) + \mathcal{A}(P)^2 + 2dEy - 2y\mathcal{A}(P) + y^2 \geqslant -2a(P)E - \tfrac{1}{2}d^2EI + Ia(P).$$

Dividing by $2d^2E$ and letting E go to $+\infty$ then yields:

$$\frac{1}{4}I \geqslant \frac{\mathcal{A}(P)}{d} - \frac{a(P)+dy}{d^2}.$$

Now $2a(P)+2dy$ corresponds to k up to an error $O(d)$. Thus, for all sequence of paths P_d of degree d and index k_d,

$$\frac{1}{4}I \geqslant \limsup_{d\to\infty}\left(\frac{\mathcal{A}(P_d)}{d} - \frac{1}{2}\frac{k_d}{d^2+d}\right),$$

from which (56) follows.

Appendix A. More about combinatorial PFH and ECH

The purpose of this appendix is to give an outline of the argument from [HS06], [HS05], adapted to our setting, that was promised in Proposition 6.2.

Preliminaries: *Comparison with previous results.* We first review the setups in [HS06], [HS05], [Cho16] to clarify for the reader the ways in which our setting differs from the settings for previously known results. The papers [HS06], [Cho16] consider a number of manifolds, including the case of $\mathbb{S}^1\times\mathbb{S}^2$, which is the one relevant to us here, and they give a combinatorial model very much analogous to the one in Proposition 6.2. However, they are about toric contact forms and they give a combinatorial presentation for the corresponding ECH, while what we need for the proposition is about PFH. The paper [HS05] is about PFH and gives a combinatorial model for the chain complex that is analogous to the one in Proposition 6.2; however, it is only for Dehn twists, while we are interested in monotone twist maps of $\mathbb{S}^2$. However, as we will see below, none of these differences are serious for the arguments.

Outline of the proof of Proposition 6.2.

Step 1: *Computation of the index.* We first sketch how to prove the second item in Proposition 6.2, which gives a combinatorial interpretation of the ECH index I. We assume here, and throughout this appendix, that we have trivialized the mapping torus via (19).

Given an orbit set α, we first define a relative homology class $Z'_\alpha \in H_2(\mathbb{S}^2\times\mathbb{S}^1,\alpha,d\gamma_-)$ as follows. Write $\alpha=\{(\gamma_-,m_-)\}\cup\{(\alpha_i,m_i)\}_i\cup\{(\gamma_+,m_+)\}$, where each (α_i,m_i) is either an $(h_{p_i/q_i},1)$ or an $(e_{p_i/q_i},m_{p_i/q_i})$. We define $Z'_\alpha := m_-Z'_- + m_+Z'_+ + \sum_i m_i Z'_{\alpha_i}$, where

- $Z'_- \in H_2(\mathbb{S}^2\times\mathbb{S}^1,\gamma_-,\gamma_-)$ is the trivial class;
- $Z'_+ \in H_2(\mathbb{S}^2\times\mathbb{S}^1,\gamma_+,\gamma_-)$ is represented by the map

  $$S_+ : [0,1]\times[0,q]\to\mathbb{S}^2\times\mathbb{S}^1, \quad (s,t)\mapsto (R_{t\lceil h'(1)\rceil}(\eta(s)),t),$$

  where η is a meridian from the South pole p_- to the North pole p_+, and $R_{t\kappa}$ denotes the rotation on $\mathbb{S}^2$ by the angle $2\pi t\kappa$;

- for $\alpha_i = e_{p,q}$ or $h_{p,q}$, the relative class $Z'_{\alpha_i} \in H_2(\mathbb{S}^2 \times \mathbb{S}^1, \alpha_i, q\gamma_-)$ is represented by the map

$$S_{\alpha_i} : [0,1] \times [0,q] \to \mathbb{S}^2 \times \mathbb{S}^1, \quad (s,t) \mapsto (R_{t\frac{p}{q}}(\eta(s)), t),$$

where η is a portion of the great circle that begins at p_- and ends at $z_{\frac{p}{q}}$.

Recall that we also need to fix trivializations τ of the vertical tangent bundle along the periodic orbits: along the orbit γ_-, the trivialization is given by any frame of $T_{p_-}\mathbb{S}^2$ independent of t; along γ_+, we take a frame that rotates positively with rotation number $\lceil h'(1) \rceil$; along other orbits, we use as trivializing frame $(\partial_\theta, \partial_z) \in T\mathbb{S}^2$. One now computes from the definitions that

$$CZ_\tau(\alpha) = \sum_i \sum_{k=1}^{m_i} CZ(\alpha_i^k) + \sum_{k=1}^{m_-} CZ(\gamma_-^k) + \sum_{k=1}^{m_+} CZ(\gamma_+^k) = -M + h, \tag{58}$$

$$c_\tau(Z'_\alpha) = -\sum m_i p_i - m_+ \lceil h'(1) \rceil = -w_\alpha + y_\alpha, \tag{59}$$

and

$$Z_\alpha = Z'_\alpha + (w_\alpha - y_\alpha)[\mathbb{S}^2],$$

where Z_α is the class from Section 4.3.1. Here, M denotes the total multiplicity of all orbits, h denotes the total number of hyperbolic orbits, and $(0, y_\alpha), (d, w_\alpha)$ denote the endpoints of the path P_{α, Z_α}. Similarly to [HS05, §3.2], one also computes that

$$Q_\tau(Z'_\alpha) = -(w_\alpha - y_\alpha) - 2\mathrm{Area}({\mathcal{R}_\alpha}') - (w_\alpha - y_\alpha)(d-1), \tag{60}$$

where ${\mathcal{R}_\alpha}'$ is the region between $P_{\alpha,Z}$ and the straight line connecting $(0, y_\alpha)$ to (d, w_α).

By combining (58), (59), (60), and the definition of the grading (16), we have

$$I(\alpha, Z'_\alpha) = -M + h - (w_\alpha - y_\alpha)(d+1) - 2\mathrm{Area}(\mathcal{R}'_\alpha).$$

An application of Pick's Theorem, which we leave to the reader, then establishes the second item in the proposition for Z'_α. The second item for a general $Z \in H_2(\mathbb{S}^2 \times \mathbb{S}^1, \alpha, d\gamma_-)$ then follows as another easy exercise using the fact that $H_2(\mathbb{S}^2 \times \mathbb{S}^1, \alpha, d\gamma_-)$ is an affine space over $H_2(\mathbb{S}^2 \times \mathbb{S}^1; \mathbb{Z})$.

Step 2: *Curves correspond to corner rounding.* We now comment on the proof of the third item. We break the proof into two parts. The first is the following lemma, which establishes the "only if" part of the item.

LEMMA A.1. *Let φ_0 be a nice perturbation of φ^1_H, where $H \in \mathcal{D}$. Assume that $I(P_{\alpha,Z}) - I(P_{\beta,Z'}) = 1$. Then, for generic admissible J close to J_{std},*

$$\langle \partial(\alpha, Z), (\beta, Z') \rangle \neq 0$$

only if $P_{\alpha,Z}$ is obtained from $P_{\beta,Z'}$ by rounding a corner and locally losing one h.

The remainder of this step sketches the proof of this lemma. To start, a straightforward adaptation of [HS05, Prop. 3.12] and [HS06, Prop. 10.12], which we skip for brevity, yields the following lemma.

Lemma A.2. *Let φ_0 be a nice perturbation of φ_H^1, where $H \in \mathcal{D}$, let J be any admissible almost complex structure, and let C be a J-holomorphic curve from (α, Z) to (β, Z'). Then*

(a) *$P_{\beta,Z'}$ is never above $P_{\alpha,Z}$.*
(b) *Let $z_0 \in (-1,1)$ be such that C intersects $S_z := \{z = z_0\} \subset \mathbb{R} \times \mathbb{S}^2 \times \mathbb{S}^1$ transversely, and assume that φ_0 has no periodic points of period $\leq d$ on S_z. If this intersection is nullhomologous in S_z, then it is empty.*

Thus by part (a) of this lemma, we know that $P_{\beta,Z'}$ is never above $P_{\alpha,Z}$. Consider the region between $P_{\alpha,Z}$ and $P_{\beta,Z'}$. We can take this region and decompose it into two kinds of subregions: *non-trivial* subregions where $P_{\alpha,Z}$ is *above* $P_{\beta,Z'}$ — meaning that the parts of $P_{\alpha,Z}$ and $P_{\beta,Z'}$ intersect at most at two points in these regions; and, *trivial* subregions where the concave paths (without the labels) coincide. See Figure 8.

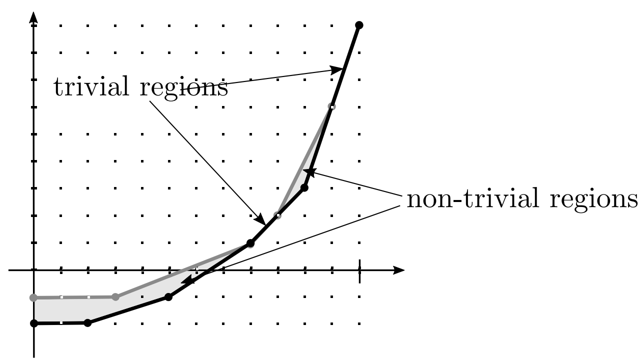

Figure 8. Examples of trivial and non-trivial regions. The path $P_{\beta,Z'}$ is in black, the path $P_{\alpha,Z}$ is in grey were it does not coincide with $P_{\beta,Z'}$.

Now, by general theory as in [Hut14, Prop. 3.7], any curve C counted by the twisted PFH differential can be written in the form $C = C_0 \sqcup C_1$, where C_1 is irreducible and has Fredholm and ECH index one and C_0 is a union of $\mathbb{R}$-invariant cylinders or multiple covers thereof. It suffices to establish the result for the "interesting" component C_1.

One first shows that for the component C_1 of a curve C satisfying the hypotheses of the lemma, there is exactly one non-trivial region and no trivial regions. Here is a sketch of the proof. First, we show that there must be at least one non-trivial region: otherwise, by invoking Lemma A.2(b), one shows that C_1 would have to be "local," in the sense that C_1 is a cylinder from some $h_{p,q}$ to an $e_{p,q}$ arising from the perturbation of the Morse-Bott setup; however, these cylinders cancel in pairs, since, as is familiar in the Morse-Bott picture, they correspond to flow lines from a perfect Morse function on the circle of Reeb orbits. Thus, there is at least one non-trivial region, and an index computation along the lines of Step 1 shows that if there was more than one region, then the Fredholm index of C_1 would be at least two. It therefore remains to argue that there are no trivial regions for C_1. For brevity, we skip this since the argument for this is similar to the kind of arguments we have already presented in this paragraph.

Now assume that C_1 is a curve from (α, Z) to (β, Z'). Then, we know from above that there is exactly one non-trivial region between $P_{\alpha,Z}$ and $P_{\beta,Z'}$. By the second item of Proposition 6.2, we have

$$1 = I(\alpha, Z) - I(\beta, Z') = 2j + h_\alpha - h_\beta,$$

where j is the number of lattice points in the region between $P_{\alpha,Z}$ and $P_{\beta,Z'}$, not including lattice points on $P_{\alpha,Z}$, and h_α, h_β denote the number of edges labeled h in $P_{\alpha,Z}, P_{\beta,Z'}$, respectively. The number of edges in $P_{\beta,Z'}$, which we denote by r_β, satisfies the following inequality: $h_\beta \leqslant r_\beta \leqslant j+1$. Hence, in view of Step 1 we have

$$I(\alpha, Z) - I(\beta, Z') \geqslant 2(r_\beta - 1) - r_\beta = r_\beta - 2,$$

with equality if and only if $P_{\alpha,Z}$ and $P_{\beta,Z'}$ start at the same point, end at the same point, the region between $P_{\alpha,Z}$ and $P_{\beta,Z'}$ contains no interior lattice points, every edge of $P_{\beta,Z'}$ is labelled h, and no edge of $P_{\alpha,Z}$ is labeled h. Since $I(\alpha, Z) - I(\beta, Z') = 1$, we can rewrite the above inequality as $r_\beta \leqslant 3$. Thus, $r_\beta \in \{1, 2, 3\}$ and Lemma A.1 follows from a straightforward combinatorial analysis of these three cases for r_β that we leave to the reader; the case $r_\beta = 3$ is exactly the case of Double Rounding, which is ruled out by the choice of nice perturbation.

Step 3: *Corner rounding corresponds to curves.* To complete the proof of the proposition, we therefore have to show

LEMMA A.3. *Let φ_0 be a nice perturbation of φ^1_H, where $H \in \mathcal{D}$. If $P_{\alpha,Z}$ is obtained from $P_{\beta,Z'}$ by rounding a corner and locally losing one h, then $\langle \partial(\alpha, Z), (\beta, Z')\rangle \neq 0$. In other words, counting mod 2 we have*

$$\#\mathcal{M}_J((\alpha, Z), (\beta, Z')) = 1$$

for generic admissible J.

The remainder of this step is devoted to the sketch of the proof of this lemma.

Define

$$X_1 := \{(t,\theta,z) \in \mathbb{S}^1 \times \mathbb{S}^2 : -1 < z\}, \quad X_1' := \{(t,\theta,z) \in \mathbb{S}^1 \times \mathbb{S}^2 : z < 1\}.$$

We first show that either every $C \in \mathcal{M}_J((\alpha,Z),(\beta,Z'))$ is entirely contained in X_1, or every $C \in \mathcal{M}_J((\alpha,Z),(\beta,Z'))$ is entirely contained in X_1'; the basic idea of the proof is that, otherwise, in view of Lemma A.2(b) and the computations in Step 1, the ECH index I for such a C would be larger than one; for brevity, we say no more here.

The idea is now to relate $\#\mathcal{M}_J((\alpha,Z),(\beta,Z')) = 1$ to a count previously studied in the context of the ECH differential. We explain this in the case where $\mathcal{M}_J((\alpha,Z),(\beta,Z')) = M_J(\alpha,\beta;X_1)$, i.e., the curves are entirely in X_1; the case of X_1' is essentially the same.

We identify X_1 with a subset of the boundary of the "concave toric domain"

$$X_\Omega := \{(z_1,z_2) | (\pi|z_1|^2, \pi|z_2|^2) \in \Omega\},$$

where Ω is the region bounded by the axes and the graph of $f(x)$, where $f(x) := \frac{h(1-2x)}{2}$ for $0 \leqslant x \leqslant 1$. It is well known (and easy to prove) that the boundary ∂X_Ω is a contact manifold, with contact form given by the restriction of the standard one-form on $\mathbb{R}^4$. Consider the subset of ∂X_Ω given by $X_2 := \{(z_1,z_2) | \pi(|z_1|^2,|z_2|^2) \in \partial\Omega - \{(1,0)\}\}$. Note that this is ∂X_Ω with a Reeb orbit removed. Define the mappings

$$\begin{aligned} &\psi : X_1 \to X_2, \quad (t,\theta,z) \mapsto \left(\tfrac{1}{2}(1-z), \theta, \tfrac{1}{2}h(z), 2\pi t\right), \\ &\Psi : \mathbb{R} \times X_1 \to \mathbb{R} \times X_2, \quad (s,t,\theta,z) \mapsto \left(s, \tfrac{1}{2}(1-z), \theta, \tfrac{1}{2}h(z), 2\pi t\right). \end{aligned} \tag{61}$$

Here, we are regarding $\partial X_\Omega \subset \mathbb{C}^2 = \mathbb{R}^4$, and we are equipping $\mathbb{C}^2$ with coordinates $(\rho_1 := \pi|z_1|^2, \theta_1, \rho_2 := \pi|z_2|^2, \theta_2)$. One can check that the above diffeomorphisms have the following properties:

(i) The Reeb vector field R on X_1 pushes forward under ψ to a positive multiple of the contact Reeb vector field $\hat{R}$ on X_2.

(ii) The two-form $d\lambda$ on X_2 pulls back under ψ to ω_φ on X_1. Thus, the SHS $(\lambda, d\lambda)$ on X_2 pulls back under ψ to the SHS $(\psi^*\lambda, \omega_\varphi)$ on X_1.

By property (i), ψ induces a bijection between the Reeb orbit sets of R in X_1 and the Reeb orbit sets of $\hat{R}$ in X_2. We will denote the induced bijection by

$$\alpha \mapsto \hat{\alpha}.$$

Now, suppose $P_{\alpha,Z}$ is obtained from $P_{\beta,Z'}$ via rounding a corner and locally losing one h. Let $\hat{J}$ be a contact admissible almost complex structure on the symplectization $\mathbb{R} \times \partial X_\Omega$, and consider $\mathcal{M}_{\hat{J}}(\hat{\alpha},\hat{\beta})$ as the space of $\hat{J}$-holomorphic currents C, modulo translation in the $\mathbb{R}$ direction, which are asymptotic to $\hat{\alpha}$ as $s \to +\infty$ and $\hat{\beta}$ as $s \to -\infty$. For a generic choice of contact admissible $\hat{J}$, this moduli space is finite and its mod 2 cardinality determines the ECH differential in the sense that

$$\langle \partial_{ECH}\, \hat{\alpha}, \hat{\beta} \rangle = \#\mathcal{M}_{\hat{J}}(\hat{\alpha}, \hat{\beta}).$$

The ECH differential in this case has been worked out in [Cho16],[21] with the conclusion that the following hold:

(A1) The image of every curve in $\mathcal{M}_{\hat{J}}(\hat{\alpha}, \hat{\beta})$ is contained in $\mathbb{R} \times X_2$.

(B1) The mod 2 count of curves in $\mathcal{M}_{\hat{J}}(\hat{\alpha}, \hat{\beta})$ is 1.

Now define J_1 to be the almost complex structure on $\mathbb{R} \times X_1$ given by the pullback under Ψ of the restriction of a generic $\hat{J}$ as above. By (A1) and (B1), the mod 2 count of curves in $\mathcal{M}_{J_1}((\alpha, Z), (\beta, Z'); X_1)$ is one. However, J_1 is not necessarily admissible, because the SHS given by $(\psi^* \lambda, \omega_\varphi)$ does not agree with the standard SHS (dt, ω_φ) on the mapping torus. Nevertheless, these two SHSs are homotopic and one can connect J_1 to an admissible almost complex structure J_0 through a suitable family J_t. One then shows that the counts for J_0 and J_1 agree by constructing a compact cobordism between the relevant moduli spaces via spaces of J_t-holomorphic curves. The argument for this is a standard SFT compactness argument, wherein all possible degenerations into buildings are analyzed, together with an argument that is similar to the argument at the beginning of this step, showing that the J_t-holomorphic curves stay in a compact subset of X_1 via a variant of Lemma A.2(b); we omit the details for brevity. □

Remark A.4. Choi [Cho16] finds his curves by referencing a paper by Taubes [Tau02], which works out various moduli spaces of curves for a particular contact form on $\mathbb{S}^1 \times \mathbb{S}^2$; Choi then does a deformation argument that is analogous to Step 3 above. Choi also uses an inductive argument to reduce to considering moduli spaces of twice and thrice punctured spheres that is nothing like what is in Step 3; the reason he does this is to be able to use the above paper by Taubes, which does not directly address all the curves needed to analyze corner rounding operations, which could lead, for example, to curves with an arbitrary number of ends.[22] These ideas were pioneered by Hutchings-Sullivan in [HS06], [HS05], who use them to find the curves that they need. A strategy like this can also be used instead of citing Choi, but we did not do so for brevity.

We should also note that there is considerably more in Choi's work than what we use here — in particular, Choi analyzes very general contact forms

[21] We also refer the reader to [Yao22b], [Yao22a] for further details about the Morse-Bott arguments used in this computation.

[22] We should note that two other papers [Tau06a], [Tau06b] by Taubes that came after [Tau02] do address these kind of curves.

(for example, the contact form on a toric domain that is neither concave nor convex) for which there could be curves corresponding to regions more general than those that come from rounding a corner and locally losing one h, and for which more complicated arguments are needed — however, we do not need to consider anything like that in this paper.

Appendix B. Discussion and open questions

We discuss here some open questions relating to the main results of our article. We also discuss developments since our article first appeared.

Simplicity on other surfaces. Let M denote a closed manifold equipped with a volume form ω, and denote by $\mathrm{Homeo}_0(M, \omega)$ the identity component in the group of volume-preserving homeomorphisms of M. In [Fat80a], Fathi constructs the *mass-flow homomorphism*

$$\mathcal{F} : \mathrm{Homeo}_0(M, \omega) \to H_1(M)/\Gamma,$$

mentioned above, where $H_1(M)$ denotes the first homology group of M with coefficients in $\mathbb{R}$ and Γ is a discrete subgroup of $H_1(M)$ whose definition we will not need here. Clearly, $\mathrm{Homeo}_0(M, \omega)$ is not simple when the mass-flow homomorphism is non-trivial. This is indeed the case when M is a closed surface other than the sphere. Fathi proved that $\ker(\mathcal{F})$ is simple if the dimension of M is at least three. The following question is posed in [Fat80a, App. A.6].

QUESTION B.1 (Fathi). *Is* $\ker(\mathcal{F})$ *simple in the case of surfaces? In particular, is the group* $\mathrm{Homeo}_0(\mathbb{S}^2, \omega)$ *of area and orientation preserving homeomorphisms of the sphere simple?*

Update: In the original version of this paper, we remarked that one might be able to resolve this question by adapting the methods of the paper, after some further development of the theory of PFH spectral invariants. In fact, we later resolved the question (in the negative) in [CGHM+22a], together with Mak and Smith, using a similar argument to the one given in Section 3; however, we used a different kind of spectral invariant, called "link spectral invariants," instead of PFH ones. After that work, the needed further development of PFH invariants required to resolve Question B.1 via these invariants occurred in [EH21], [CGPZ21], which can be used to give an alternative proof.

C^0-symplectic topology and simplicity in higher dimensions. From a symplectic viewpoint, a natural generalization of area-preserving homeomorphisms to higher dimensions is given by *symplectic homeomorphisms*. These are, by definition, those homeomorphisms that appear as C^0 limits of symplectic diffeomorphisms. By the celebrated rigidity theorem [Eli87], [Gro85] of Eliashberg and Gromov, a smooth symplectic homeomorphism is a symplectic

diffeomorphism. These homeomorphisms form the cornerstone of the field of C^0*-symplectic topology* that explores continuous analogues of smooth symplectic objects; see, for example, [OM07], [Ban08], [HLS15], [HLS16], [BO16].

The connection between C^0-symplectic topology and the simplicity conjecture is formed by the fact that, in dimension two, symplectic homeomorphisms are precisely the area- and orientation-preserving homeomorphisms of surfaces. Indeed, as we mentioned in Section 1, the simplicity conjeture has been one of the driving forces behind the development of C^0-symplectic topology; for example, the articles [OM07], [Oh10], [Vit06], [BS13], [Hum11], [LR10b], [LR10a], [EPP12], [Sey13a], [Sey13b] were, at least partially, motivated by this conjecture.

The connection to C^0-symplectic topology motivates the following generalization of Question 1.1.

Question B.2. *Is* $\overline{\mathrm{Symp}}_c(\mathbb{D}^{2n}, \omega)$, *the group of compactly supported symplectic homeomorphisms of the standard ball, simple*?[23]

As we will now explain, Question B.1 admits a natural generalization to higher dimensions as well. To keep our discussion simple we will suppose that (M, ω) is a closed symplectic manifold. However, this assumption is not necessary and the question below can be reformulated for non-closed manifolds too.

On a symplectic surface (M, ω), the group $\ker(\mathcal{F})$ discussed in the above section is often referred to as the group of *Hamiltonian homeomorphisms* and is denoted by $\overline{\mathrm{Ham}}(M, \omega)$; see, for example, [LC06]. The reason for this terminology is that it can be shown that $\ker(\mathcal{F})$ coincides with the C^0 closure of Hamiltonian diffeomorphisms. Hence, in this language, Question B.1 may be rephrased as the question of whether or not the group of Hamiltonian homeomorphisms is simple. On higher dimensional symplectic manifolds, the elements of the C^0 closure of $\mathrm{Ham}(M, \omega)$ are also called Hamiltonian homeomorphisms and have been studied extensively in C^0-symplectic topology; see, for example, [OM07], [BHS18], [BHS21], [Kaw22].

Question B.3. *Is* $\overline{\mathrm{Ham}}(M, \omega)$ *a simple group*?

In comparison, Banyaga's theorem states that the group of Hamiltonian diffeomorphisms is simple for closed M.

Finite energy homeomorphisms. The group of finite energy homeomorphisms, $\mathrm{FHomeo}(M, \omega)$, can be defined on arbitrary symplectic manifolds; the construction is analogous to what is done in Section 3.1. It forms a normal

[23] An argument involving the Alexander isotopy shows that $\overline{\mathrm{Symp}}_c(\mathbb{D}^{2n}, \omega)$ is connected.

subgroup of $\overline{\mathrm{Ham}}(M,\omega)$. However, we do not know if it is always a proper normal subgroup. Infinite twists can also be constructed on arbitrary symplectic manifolds: the construction of ϕ_f, described in Section 3.2, admits a generalization to $\mathbb{D}^{2n}$. And the analogue of equation (11), the condition for having "infinite" Calabi invariant, can also be formulated in higher dimensions.

QUESTION B.4. *Is it true that infinite twists, which satisfy the higher dimensional analogue of equation (11), are not finite energy homeomorphisms?*

Clearly, a positive answer to this question would settle all of the above simplicity questions. After the first version of this paper appeared, the case of surfaces was resolved in the affirmative in [CGHM+22a]; see the discussion after Question B.1. However, a serious obstacle arises in higher dimensions: here, for example, PFH, and the related Seiberg-Witten theory, have no known generalization.

We now return to the case of the disc, where we know that $\mathrm{FHomeo}_c(\mathbb{D},\omega)$ is a proper normal subgroup of $\mathrm{Homeo}_c(\mathbb{D},\omega)$. This immediately gives rise to several interesting questions about $\mathrm{FHomeo}_c(\mathbb{D},\omega)$.

QUESTION B.5. *Is* $\mathrm{FHomeo}_c(\mathbb{D},\omega)$ *simple?*

As was mentioned in Remark 3.3, the Oh-Müller group $\mathrm{Hameo}_c(\mathbb{D},\omega)$, which we introduce below, is a subgroup of $\mathrm{FHomeo}_c(\mathbb{D},\omega)$, and it can easily be checked that it is a normal subgroup.

Update. We resolved Question B.5 by extending the Calabi invariant to $\mathrm{Hameo}_c(\mathbb{D},\omega)$; see the discussion after Question B.8 below.

QUESTION B.6. *Is the group* $\mathrm{Hameo}_c(\mathbb{D},\omega)$ *a proper normal subgroup of* $\mathrm{FHomeo}_c(\mathbb{D},\omega)$*?*

Update. Buhovsky resolved Question B.6 in the affirmative in [Buh23].

Another interesting direction to explore is the algebraic structure of the quotient $\mathrm{Homeo}_c(\mathbb{D},\omega)/\mathrm{FHomeo}_c(\mathbb{D},\omega)$. At present we are not able to say much beyond the fact that this quotient is abelian; see Proposition 2.1. Here are two sample questions.

QUESTION B.7. *Is the quotient* $\mathrm{Homeo}_c(\mathbb{D},\omega)/\mathrm{FHomeo}_c(\mathbb{D},\omega)$ *torsion-free? Is it divisible?*

Extension of the Calabi invariant. Ghys [Ghy07a], Fathi and Oh [OM07, Conj. 6.8] have asked if the Calabi invariant extends to either of $\mathrm{Hameo}_c(\mathbb{D},\omega)$ or $\mathrm{Homeo}_c(\mathbb{D},\omega)$. It seems natural to add $\mathrm{FHomeo}_c(\mathbb{D},\omega)$ to the list.

QUESTION B.8. *Does the Calabi invariant admit an extension to any of* $\mathrm{Hameo}_c(\mathbb{D},\omega)$, $\mathrm{FHomeo}_c(\mathbb{D},\omega)$, *or* $\mathrm{Homeo}_c(\mathbb{D},\omega)$*?*

We now explain a partial answer to this question, making use of some developments in [CGPZ21], [EH21] that occurred after the first version of this paper appeared.

Theorem B.9. *The Calabi homomorphism*

$$\mathrm{Cal} : \mathrm{Diff}_c(\mathbb{D}, \omega) \to \mathbb{R}$$

extends to $\mathrm{Hameo}_c(\mathbb{D}, \omega)$.

The first complete proof of Theorem B.9 appeared in [CGHM$^+$22a]; we summarize the relevant history below after giving our proof.

Proof. Let us begin with the definition[24] of $\mathrm{Hameo}_c(\mathbb{D}, \omega)$. We say $\phi \in \mathrm{Homeo}_c(\mathbb{D}, \omega)$ is a *hameomorphism* if there exists a sequence of smooth Hamiltonians $H_i \in C^\infty_c(\mathbb{S}^1 \times \mathbb{D})$ and a continuous $H \in C^0_c(\mathbb{S}^1 \times \mathbb{D})$ such that

$$\varphi^1_{H_i} \xrightarrow{C^0} \phi, \text{ and } \|H - H_i\|_{(1,\infty)} \to 0.$$

The set of all hameomorphisms is denoted by $\mathrm{Hameo}_c(\mathbb{D}, \omega)$.[25] It defines a normal subgroup of $\mathrm{Homeo}_c(\mathbb{D}, \omega)$ which is clearly contained in $\mathrm{FHomeo}_c(\mathbb{D}, \omega)$.

Take $\phi \in \mathrm{Hameo}_c(\mathbb{D}, \omega)$ and $H \in C^0_c(\mathbb{S}^1 \times \mathbb{D})$ as in the previous paragraph; we let φ_H denote ϕ. Define

$$\mathrm{Cal}(\varphi_H) := \int_{\mathbb{S}^1} \int_{\mathbb{D}} H \, \omega \, dt. \tag{62}$$

We first show that Cal is well defined. First, note that because Cal is a homomorphism on $\mathrm{Diff}_c(\mathbb{D}, \omega)$, to show this it suffices to show that if $\varphi_H = \mathrm{Id}$, then

$$\int_{\mathbb{S}^1} \int_{\mathbb{D}} H \, \omega \, dt = 0. \tag{63}$$

Suppose that $\varphi_H = \mathrm{Id}$, and fix a sequence $(H_1, H_2, \ldots,)$ for φ_H as in the definition of $\mathrm{Hameo}_c(\mathbb{D}, \omega)$.

Claim B.10. *For all i, we have* $|\frac{c_d}{d}(\varphi^1_{H_i})| \leqslant \|H - H_i\|_{(1,\infty)}$.

Proof. By Hofer continuity of PFH spectral invariants, we have $|\frac{c_d}{d}(\varphi^1_{H_j}) - \frac{c_d}{d}(\varphi^1_{H_i})| \leqslant \|H_j - H_i\|_{(1,\infty)}$ for all i, j. Fixing i and taking the limit of this

[24] The definition we have given here is a slight variation of the one in [OM07]; it can easily be checked that if ϕ is a hameomorphism in the sense of [OM07], then it is also a hameomorphisms in the sense described here.

[25] Oh and Müller use the terminology *Hamiltonian homeomorphisms* for the elements of $\mathrm{Hameo}_c(\mathbb{D}, \omega)$. We have chosen to avoid this terminology because in the surface dynamics literature it is commonly used for referring to homeomorphisms that arise as C^0 limits of Hamiltonian diffeomorphisms.

inequality, as $j \to \infty$, yields the claim, by Theorem 3.6 and item (4) of Theorem 4.5, since $\varphi^1_{H_j} \xrightarrow{C^0} \mathrm{Id}$. $\square$

We now establish (63). For all $i, d \in \mathbb{N}$, we have

$$\begin{aligned}\left|\int_{\mathbb{S}^1}\int_{\mathbb{D}} H\,\omega\,dt\right| &\leqslant \left|\int_{\mathbb{S}^1}\int_{\mathbb{D}} H\,\omega\,dt\ - \mathrm{Cal}(\varphi^1_{H_i})\right| \\ &\quad + \left|\mathrm{Cal}(\varphi^1_{H_i}) - \frac{c_d}{d}(\varphi^1_{H_i})\right| + \left|\frac{c_d}{d}(\varphi^1_{H_i})\right| \\ &\leqslant \|H - H_i\|_{(1,\infty)} + \left|\mathrm{Cal}(\varphi^1_{H_i}) - \frac{c_d}{d}(\varphi^1_{H_i})\right| + \left|\frac{c_d}{d}(\varphi^1_{H_i})\right| \\ &\leqslant 2\|H - H_i\|_{(1,\infty)} + \left|\mathrm{Cal}(\varphi^1_{H_i}) - \frac{c_d}{d}(\varphi^1_{H_i})\right|,\end{aligned}$$

where the last inequality follows from Claim B.10. Now take any $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that if $i \geqslant N$, then $\|H - H_i\|_{(1,\infty)} \leqslant \frac{1}{2}\varepsilon$. Hence, for $i \geqslant N$, we have

$$\left|\int_{\mathbb{S}^1}\int_{\mathbb{D}} H\,\omega\,dt\right| \leqslant \varepsilon + \left|\mathrm{Cal}(\varphi^1_{H_i}) - \frac{c_d}{d}(\varphi^1_{H_i})\right|.$$

By [CGPZ21], [EH21], the conclusion of Theorem 3.7 holds for arbitrary $\varphi \in \mathrm{Diff}_c(\mathbb{D}^2, \omega)$. Hence, $|\int_{\mathbb{S}^1}\int_{\mathbb{D}} H\,\omega\,dt| \leqslant \varepsilon$, hence (63), hence the claimed extension of Cal.

It remains to show that

$$\mathrm{Cal} : \mathrm{Hameo}_c(\mathbb{D}, \omega) \to \mathbb{R}$$

is indeed a homomorphism. Having shown that (62) is well defined, this has in fact already been shown in [Oh10] and so we will only provide a sketch of the argument. Take $\varphi_H, \varphi_G \in \mathrm{Hameo}_c(\mathbb{D}, \omega)$. Without loss of generality, we may suppose that $H(t,x)$ and $G(t,x)$ vanish for t near $0 \in \mathbb{S}^1$; this can be achieved by replacing H with the reparametrization $\rho'(t)H(\rho(t), x)$, where $\rho : [0,1] \to [0,1]$ coincides with 0 near 0 and with 1 near 1; see [Pol01, p. 31] for more details on the reparametrization argument.

It can be checked that $\varphi_H \circ \varphi_G = \varphi_K$, where K is the concatenation of H and G given by the formula

$$K(t,x) = \begin{cases} 2H(2t, x) & \text{if } t \in [0, \frac{1}{2}], \\ 2G(2t-1, x) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

It follows immediately from the above formula, and equation (62), that

$$\mathrm{Cal}(\varphi_H \circ \varphi_G) = \mathrm{Cal}(\varphi_H) + \mathrm{Cal}(\varphi_G).$$

Verification of the fact that $\mathrm{Cal}(\varphi_H^{-1}) = -\mathrm{Cal}(\varphi_H)$ is similar and so we omit it. $\square$

Historical remarks. The above proof of Theorem B.9 was suggested in the first version of this paper, except that at that time, the necessary developments in [CGPZ21], [EH21] had not yet occurred. In fact, prior to [CGPZ21], [EH21], but after the first version of this paper appeared, we showed in [CGHM+22a] that Calabi extends to $\mathrm{Hameo}_c(\mathbb{D}, \omega)$ by using essentially the same argument as above, but replacing the PFH spectral invariants with the aforementioned link spectral invariants, which have similar properties.

Update. We later showed in [CGHM+22b] that Calabi extends to all of $\mathrm{Homeo}_c(\mathbb{D}, \omega)$, though not canonically, via a different kind of argument using link spectral invariants.

New invariants and new applications. A key role in our paper is played by PFH spectral invariants. In the years since our paper first appeared, some new invariants have been discovered that share enough similar properties with the PFH ones that one can adapt the arguments in our paper with minor modifications to obtain the applications to the Simplicity Conjecture, using these other invariants instead of PFH ones.

One such family of invariants is the family of link spectral invariants [CGHM+22a], [PS21] that we already mentioned above. Another is the family of "elementary PFH spectral invariants" defined in [Edt22]. These invariants have various great features. For example, the link spectral invariants are known to be quasimorphisms, and one can use them to deduce very fine information about the normal subgroup structure [CGHM+22b]; and, the elementary PFH spectral invariants can be constructed without making any reference to Floer homology. As for the PFH invariants, some new applications have occurred as well beyond questions about the algebraic structure of homeomorphism groups. We already mentioned the application resolving the Kapovich-Polterovich question; this was a central open problem about the Hofer geometry of surfaces, and it can also be resolved with link spectral invariants [PS21]. Another important application occurred in the series of papers [CGHS23], [EH21], [CGPPZ21], [Pra23], [PP22], proving the C^∞-closing lemma, and various refinements, for area-preserving diffeomorphisms of closed surfaces or compact surfaces with boundary.

References

[And58] R. D. Anderson, The algebraic simplicity of certain groups of homeomorphisms, *Amer. J. Math.* **80** (1958), 955–963. MR 0098145. Zbl 0090.38802. https://doi.org/10.2307/2372842.

[Ban78] A. Banyaga, Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique, *Comment. Math. Helv.* **53** no. 2

(1978), 174–227. MR 0490874. Zbl 0393.58007. https://doi.org/10.1007/BF02566074.

[Ban97] A. Banyaga, *The Structure of Classical Diffeomorphism Groups*, *Math Appl.* **400**, Kluwer Academic Publishers Group, Dordrecht, 1997. MR 1445290. Zbl 0874.58005. https://doi.org/10.1007/978-1-4757-6800-8.

[Ban08] A. Banyaga, On the group of symplectic homeomorphisms, *C. R. Math. Acad. Sci. Paris* **346** no. 15-16 (2008), 867–872. MR 2441923. Zbl 1146.57044. https://doi.org/10.1016/j.crma.2008.06.011.

[Bou08] A. Bounemoura, *Simplicité des Groupes de Transformations de Surfaces*, *Ensaios Mat.* **14**, Sociedade Brasileira de Matemática, Rio de Janeiro, 2008. MR 2458739. Zbl 1161.58007. https://doi.org/10.21711/217504322008/em141.

[BEH+03] F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki, and E. Zehnder, Compactness results in symplectic field theory, *Geom. Topol.* **7** (2003), 799–888. MR 2026549. Zbl 1131.53312. https://doi.org/10.2140/gt.2003.7.799.

[BM94] J. E. Brothers and F. Morgan, The isoperimetric theorem for general integrands, *Michigan Math. J.* **41** no. 3 (1994), 419–431. MR 1297699. Zbl 0923.49019. https://doi.org/10.1307/mmj/1029005070.

[Buh23] L. Buhovsky, On two remarkable groups of area-preserving homeomorphisms, *J. Math. Phys. Anal. Geom.* **19** no. 2 (2023), 339–373. MR 4633987. https://doi.org/10.15407/mag19.02.339.

[BHS18] L. Buhovsky, V. Humilière, and S. Seyfaddini, A C^0 counterexample to the Arnold conjecture, *Invent. Math.* **213** no. 2 (2018), 759–809. MR 3827210. Zbl 1395.37037. https://doi.org/10.1007/s00222-018-0797-x.

[BHS21] L. Buhovsky, V. Humilière, and S. Seyfaddini, The action spectrum and C^0 symplectic topology, *Math. Ann.* **380** no. 1-2 (2021), 293–316. MR 4263685. Zbl 1471.53067. https://doi.org/10.1007/s00208-021-02183-w.

[BO16] L. Buhovsky and E. Opshtein, Some quantitative results in CalC^0 symplectic geometry, *Invent. Math.* **205** no. 1 (2016), 1–56. MR 3514957. Zbl 1348.53074. https://doi.org/10.1007/s00222-015-0626-4.

[BS13] L. Buhovsky and S. Seyfaddini, Uniqueness of generating Hamiltonians for topological Hamiltonian flows, *J. Symplectic Geom.* **11** no. 1 (2013), 37–52. MR 3022920. Zbl 1282.37030. https://doi.org/10.4310/JSG.2013.v11.n1.a3.

[Cal70] E. Calabi, On the group of automorphisms of a symplectic manifold, in *Problems in Analysis* (*Sympos. in Honor of Salomon Bochner*, *Princeton Univ.*, *Princeton*, *N.J.*, 1969), *Princeton Math. Series* **31**, Princeton Univ. Press, Princeton, NJ, 1970, pp. 1–26. MR 0350776. Zbl 0209.25801. https://doi.org/10.1515/9781400869312-002.

[Che21] G. Chen, On cobordism maps on periodic Floer homology, *Algebr. Geom. Topol.* **21** no. 1 (2021), 1–103. MR 4224736. Zbl 1476.57056. https://doi.org/10.2140/agt.2021.21.1.

[Cho16] K. Choi, Combinatorial embedded contact homology for toric contact manifolds, 2016. arXiv 1608.07988.

[CCGF+14] K. Choi, D. Cristofaro-Gardiner, D. Frenkel, M. Hutchings, and V. G. B. Ramos, Symplectic embeddings into four-dimensional concave toric domains, *J. Topol.* **7** no. 4 (2014), 1054–1076. MR 3286897. Zbl 1308.53125. https://doi.org/10.1112/jtopol/jtu008.

[CM05] K. Cieliebak and K. Mohnke, Compactness for punctured holomorphic curves, *J. Symplectic Geom.* **3** no. 4 (2005), 589–654, Conference on Symplectic Topology. MR 2235856. Zbl 1113.53053. https://doi.org/10.4310/JSG.2005.v3.n4.a5.

[CK94] A. Constantin and B. Kolev, The theorem of Kerékjártó on periodic homeomorphisms of the disc and the sphere, *Enseign. Math.* (2) **40** no. 3-4 (1994), 193–204. MR 1309126. Zbl 0852.57012.

[CG19] D. Cristofaro-Gardiner, Symplectic embeddings from concave toric domains into convex ones, *J. Differential Geom.* **112** no. 2 (2019), 199–232, With an appendix by Cristofaro-Gardiner and Keon Choi. MR 3960266. Zbl 1440.53084. https://doi.org/10.4310/jdg/1559786421.

[CGHS20] D. Cristofaro-Gardiner, V. Humilière, and S. Seyfaddini, Proof of the simplicity conjecture, 2020. arXiv 2001.01792v2.

[CGHS23] D. Cristofaro-Gardiner, V. Humilière, and S. Seyfaddini, PFH spectral invariants on the two-sphere and the large scale geometry of Hofer's metric, 2023. https://doi.org/10.4171/JEMS/1351.

[CGHM+22a] D. Cristofaro-Gardiner, V. Humilière, C. Y. Mak, S. Seyfaddini, and I. Smith, Quantitative Heegaard Floer cohomology and the Calabi invariant, *Forum Math. Pi* **10** (2022), Paper No. e27, 59. MR 4524364. Zbl 1508.53090. https://doi.org/10.1017/fmp.2022.18.

[CGHM+22b] D. Cristofaro-Gardiner, V. Humilière, C. Y. Mak, S. Seyfaddini, and I. Smith, Subleading asymptotics of link spectral invariants and homeomorphism groups of surfaces, 2022. arXiv 2206.10749.

[CGHR15] D. Cristofaro-Gardiner, M. Hutchings, and V. G. B. Ramos, The asymptotics of ECH capacities, *Invent. Math.* **199** no. 1 (2015), 187–214. MR 3294959. Zbl 1315.53091. https://doi.org/10.1007/s00222-014-0510-7.

[CGPPZ21] D. Cristofaro-Gardiner, D. Pomerleano, R. Prasad, and B. Zhang, A note on the existence of U-cyclic elements in periodic Floer homology, 2021, *Proc. Amer. Math. Soc.*, *Ser. B*, to appear. arXiv 2110.13844.

[CGPZ21] D. Cristofaro-Gardiner, R. Prasad, and B. Zhang, Periodic Floer homology and the smooth closing lemma for area-preserving surface diffeomorphisms, 2021. arXiv 2110.02925.

[Edt22] O. Edtmair, An elementary alternative to PFH spectral invariants, 2022. arXiv 2207.12553.

[EH21] O. Edtmair and M. Hutchings, PFH spectral invariants and C^∞ closing lemmas, 2021. arXiv 2110.02463.

[EK71] R. D. Edwards and R. C. Kirby, Deformations of spaces of imbeddings, *Ann. of Math.* (2) **93** (1971), 63–88. MR 0283802. Zbl 0214.50303. https://doi.org/10.2307/1970753.

[Eli87] Y. M. Eliashberg, A theorem on the structure of wave fronts and its application in symplectic topology, *Funktsional. Anal. i Prilozhen.* **21** no. 3 (1987), 65–72, 96. MR 0911776. Zbl 0655.58015. https://doi.org/10.1007/BF02577138.

[EPP12] M. Entov, L. Polterovich, and P. Py, On continuity of quasimorphisms for symplectic maps, in *Perspectives in Analysis, Geometry, and Topology*, *Progr. Math.* **296**, Birkhäuser/Springer, New York, 2012, with an appendix by Michael Khanevsky, pp. 169–197. MR 2884036. Zbl 1251.53047. https://doi.org/10.1007/978-0-8176-8277-4_8.

[Eps70] D. B. A. Epstein, The simplicity of certain groups of homeomorphisms, *Compositio Math.* **22** (1970), 165–173. MR 0267589. Zbl 0205.28201.

[Fat80a] A. Fathi, Structure of the group of homeomorphisms preserving a good measure on a compact manifold, *Ann. Sci. École Norm. Sup.* (4) **13** no. 1 (1980), 45–93. MR 0584082. Zbl 0443.58011. https://doi.org/10.24033/asens.1377.

[Fat80b] A. Fathi, Sur l'homomorphisme de Calabi $\mathrm{Diff}_c^\infty(\mathbb{R}^2, m) \to \mathbb{R}$, 1980, in *Transformations et homéomorphismes préservant la mesure. Systèmes dynamiques minimaux.*, Thèse Orsay.

[Fis60] G. M. Fisher, On the group of all homeomorphisms of a manifold, *Trans. Amer. Math. Soc.* **97** (1960), 193–212. MR 0117712. Zbl 0144.22902. https://doi.org/10.2307/1993298.

[Ghy07a] E. Ghys, Knots and dynamics, in *International Congress of Mathematicians. Vol. I*, Eur. Math. Soc., Zürich, 2007, pp. 247–277. MR 2334193. Zbl 1125.37032. https://doi.org/10.4171/022-1/11.

[Ghy07b] E. Ghys, Lecture at the conference "Symplectic topology and measure-preserving dynamical systems", Snowbird, Utah, 2007, 2007, outlined in the book by A. Bounemoura.

[Gro85] M. Gromov, Pseudo holomorphic curves in symplectic manifolds, *Invent. Math.* **82** no. 2 (1985), 307–347. MR 0809718. Zbl 0592.53025. https://doi.org/10.1007/BF01388806.

[Her73] M. R. Herman, Sur le groupe des difféomorphismes du tore, *Ann. Inst. Fourier* (*Grenoble*) **23** no. 2 (1973), 75–86. MR 0391166. Zbl 0269.58004. https://doi.org/10.5802/aif.457.

[Hig54] G. Higman, On infinite simple permutation groups, *Publ. Math. Debrecen* **3** (1954), 221–226. MR 0072136. Zbl 0057.25801. https://doi.org/10.5486/pmd.1954.3.3-4.06.

[HZ94] H. Hofer and E. Zehnder, *Symplectic Invariants and Hamiltonian Dynamics*, *Birkhäuser Adv. Texts: Basel Textbooks*, Birkhäuser Verlag, Basel, 1994. MR 1306732. Zbl 0805.58003. https://doi.org/10.1007/978-3-0348-8540-9.

[Hum11] V. Humilière, The Calabi invariant for some groups of homeomorphisms, *J. Symplectic Geom.* **9** no. 1 (2011), 107–117. MR 2787363. Zbl 1242.53110. https://doi.org/10.4310/JSG.2011.v9.n1.a6.

[HLS15] V. Humilière, R. Leclercq, and S. Seyfaddini, Coisotropic rigidity and C^0-symplectic geometry, *Duke Math. J.* **164** no. 4 (2015), 767–799. MR 3322310. Zbl 1327.53109. https://doi.org/10.1215/00127094-2881701.

[HLS16] V. Humilière, R. Leclercq, and S. Seyfaddini, Reduction of symplectic homeomorphisms, *Ann. Sci. Éc. Norm. Supér.* (4) **49** no. 3 (2016), 633–668. MR 3503828. Zbl 1341.53114. https://doi.org/10.24033/asens.2292.

[Hut02] M. Hutchings, An index inequality for embedded pseudoholomorphic curves in symplectizations, *J. Eur. Math. Soc.* (*JEMS*) **4** no. 4 (2002), 313–361. MR 1941088. Zbl 1017.58005. https://doi.org/10.1007/s100970100041.

[Hut11] M. Hutchings, Quantitative embedded contact homology, *J. Differential Geom.* **88** no. 2 (2011), 231–266. MR 2838266. Zbl 1238.53061. https://doi.org/10.4310/jdg/1320067647.

[Hut14] M. Hutchings, Lecture notes on embedded contact homology, in *Contact and Symplectic Topology*, *Bolyai Soc. Math. Stud.* **26**, János Bolyai Math. Soc., Budapest, 2014, pp. 389–484. MR 3220947. Zbl 1432.53126. https://doi.org/10.1007/978-3-319-02036-5_9.

[Hut17] M. Hutchings, Notes on asymptotics of PFH spectral numbers, 2017, unpublished.

[HS05] M. Hutchings and M. Sullivan, The periodic Floer homology of a Dehn twist, *Algebr. Geom. Topol.* **5** (2005), 301–354. MR 2135555. Zbl 1089.57021. https://doi.org/10.2140/agt.2005.5.301.

[HS06] M. Hutchings and M. Sullivan, Rounding corners of polygons and the embedded contact homology of T^3, *Geom. Topol.* **10** (2006), 169–266. MR 2207793. Zbl 1101.53053. https://doi.org/10.2140/gt.2006.10.169.

[HT07] M. Hutchings and C. H. Taubes, Gluing pseudoholomorphic curves along branched covered cylinders. I, *J. Symplectic Geom.* **5** no. 1 (2007), 43–137. MR 2371184. Zbl 1157.53047. Available at http://projecteuclid.org/euclid.jsg/1197491304.

[HT09] M. Hutchings and C. H. Taubes, Gluing pseudoholomorphic curves along branched covered cylinders. II, *J. Symplectic Geom.* **7** no. 1

(2009), 29–133. MR 2491716. Zbl 1193.53183. Available at http://projecteuclid.org/euclid.jsg/1238592310.

[KH95] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, *Encycl. Math. Appl.* **54**, Cambridge Univ. Press, Cambridge, 1995, with a supplementary chapter by Katok and Leonardo Mendoza. MR 1326374. Zbl 0878.58020. https://doi.org/10.1017/CBO9780511809187.

[Kaw22] Y. Kawamoto, On C^0-continuity of the spectral norm for symplectically non-aspherical manifolds, *Int. Math. Res. Not. IMRN* no. 21 (2022), 17187–17230. MR 4504915. Zbl 1502.57018. https://doi.org/10.1093/imrn/rnab206.

[KM07] P. Kronheimer and T. Mrowka, *Monopoles and Three-Manifolds*, *New Mathematical Monographs* **10**, Cambridge Univ. Press, Cambridge, 2007. MR 2388043. Zbl 1158.57002. https://doi.org/10.1017/CBO9780511543111.

[LC06] P. Le Calvez, Periodic orbits of Hamiltonian homeomorphisms of surfaces, *Duke Math. J.* **133** no. 1 (2006), 125–184. MR 2219272. Zbl 1101.37031. https://doi.org/10.1215/S0012-7094-06-13315-X.

[LR10a] F. Le Roux, Simplicity of Homeo$(\mathbb{D}^2, \partial\mathbb{D}^2, \text{Area})$ and fragmentation of symplectic diffeomorphisms, *J. Symplectic Geom.* **8** no. 1 (2010), 73–93. MR 2609630. Zbl 1231.57030. https://doi.org/10.4310/JSG.2010.v8.n1.a5.

[LR10b] F. Le Roux, Six questions, a proposition and two pictures on Hofer distance for Hamiltonian diffeomorphisms on surfaces, in *Symplectic Topology and Measure Preserving Dynamical Systems*, *Contemp. Math.* **512**, Amer. Math. Soc., Providence, RI, 2010, pp. 33–40. MR 2605313. Zbl 1252.37046. https://doi.org/10.1090/conm/512/10059.

[LRSV21] F. Le Roux, S. Seyfaddini, and C. Viterbo, Barcodes and area-preserving homeomorphisms, *Geom. Topol.* **25** no. 6 (2021), 2713–2825. MR 4347306. Zbl 1494.37040. https://doi.org/10.2140/gt.2021.25.2713.

[LT12] Y.-J. Lee and C. H. Taubes, Periodic Floer homology and Seiberg-Witten-Floer cohomology, *J. Symplectic Geom.* **10** no. 1 (2012), 81–164. MR 2904033. Zbl 1280.57029. https://doi.org/10.4310/JSG.2012.v10.n1.a4.

[Mat74a] J. N. Mather, Commutators of diffeomorphisms, *Comment. Math. Helv.* **49** (1974), 512–528. MR 0356129. Zbl 0289.57014. https://doi.org/10.1007/BF02566746.

[Mat74b] J. N. Mather, Simplicity of certain groups of diffeomorphisms, *Bull. Amer. Math. Soc.* **80** (1974), 271–273. MR 0339268. Zbl 0275.58007. https://doi.org/10.1090/S0002-9904-1974-13456-7.

[Mat75] J. N. Mather, Commutators of diffeomorphisms. II, *Comment. Math. Helv.* **50** (1975), 33–40. MR 0375382. Zbl 0299.58008. https://doi.org/10.1007/BF02565731.

[MS17] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, third ed., *Oxford Grad. Texts in Math.*, Oxford Univ. Press, Oxford, 2017. MR 3674984. Zbl 1380.53003. https://doi.org/10.1093/oso/9780198794899.001.0001.

[Oh10] Y.-G. Oh, The group of Hamiltonian homeomorphisms and continuous Hamiltonian flows, in *Symplectic Topology and Measure Preserving Dynamical Systems*, *Contemp. Math.* **512**, Amer. Math. Soc., Providence, RI, 2010, pp. 149–177. MR 2605316. Zbl 1202.53076. https://doi.org/10.1090/conm/512/10062.

[OM07] Y.-G. Oh and S. Müller, The group of Hamiltonian homeomorphisms and C^0-symplectic topology, *J. Symplectic Geom.* **5** no. 2 (2007), 167–219. MR 2377251. Zbl 1144.37033. Available at http://projecteuclid.org/euclid.jsg/1202004455.

[PP22] A. Pirnapasov and R. Prasad, Generic equidistribution for area-preserving diffeomorphisms of compact surfaces with boundary, 2022. arXiv 2211.07548.

[Pol01] L. Polterovich, *The Geometry of the Group of Symplectic Diffeomorphisms*, *Lectures in Math. ETH Zürich*, Birkhäuser Verlag, Basel, 2001. MR 1826128. Zbl 1197.53003. https://doi.org/10.1007/978-3-0348-8299-6.

[PS21] L. Polterovich and E. Shelukhin, Lagrangian configurations and Hamiltonian maps, 2021. arXiv 2102.06118.

[Pra23] R. Prasad, Generic equidistribution of periodic orbits for area-preserving surface maps, *Int. Math. Res. Not. IMRN* (2023), 02, 1–33. https://doi.org/10.1093/imrn/rnac340.

[Sch00] M. Schwarz, On the action spectrum for closed symplectically aspherical manifolds, *Pacific J. Math.* **193** no. 2 (2000), 419–461. MR 1755825. Zbl 1023.57020. https://doi.org/10.2140/pjm.2000.193.419.

[Sey13a] S. Seyfaddini, C^0-limits of Hamiltonian paths and the Oh-Schwarz spectral invariants, *Int. Math. Res. Not. IMRN* no. 21 (2013), 4920–4960. MR 3123670. Zbl 1378.53101. https://doi.org/10.1093/imrn/rns191.

[Sey13b] S. Seyfaddini, The displaced disks problem via symplectic topology, *C. R. Math. Acad. Sci. Paris* **351** no. 21-22 (2013), 841–843. MR 3128973. Zbl 1395.57036. https://doi.org/10.1016/j.crma.2013.10.007.

[Tau96] C. H. Taubes, SW $\Rightarrow$ Gr: from the Seiberg-Witten equations to pseudo-holomorphic curves, *J. Amer. Math. Soc.* **9** no. 3 (1996), 845–918. MR 1362874. Zbl 0867.53025. https://doi.org/10.1090/S0894-0347-96-00211-1.

[Tau02] C. H. Taubes, A compendium of pseudoholomorphic beasts in $\mathbb{R} \times (S^1 \times S^2)$, *Geom. Topol.* **6** (2002), 657–814. MR 1943381. Zbl 1021.32008. https://doi.org/10.2140/gt.2002.6.657.

[Tau06a] C. H. Taubes, Pseudoholomorphic punctured spheres in $\mathbb{R} \times (S^1 \times S^2)$: properties and existence, *Geom. Topol.* **10** (2006), 785–928. MR 2240906. Zbl 1134.53045. https://doi.org/10.2140/gt.2006.10.785.

[Tau06b] C. H. Taubes, Pseudoholomorphic punctured spheres in $\mathbb{R} \times (S^1 \times S^2)$: moduli space parametrizations, *Geom. Topol.* **10** (2006), 1855–2054. MR 2284051. Zbl 1161.53075. https://doi.org/10.2140/gt.2006.10.1855.

[Tau10] C. H. Taubes, Embedded contact homology and Seiberg-Witten Floer cohomology V, *Geom. Topol.* **14** no. 5 (2010), 2961–3000. MR 2746727. Zbl 1276.57027. https://doi.org/10.2140/gt.2010.14.2961.

[Thu74] W. Thurston, Foliations and groups of diffeomorphisms, *Bull. Amer. Math. Soc.* **80** (1974), 304–307. MR 0339267. Zbl 0295.57014. https://doi.org/10.1090/S0002-9904-1974-13475-0.

[UvN47] S. Ulam and J. von Neumann, On the group of homeomorphisms of the surface of the sphere, *Bull. Amer. Math. Soc.* **53** (1947), 506, (Abstract 283).

[Ush10] M. Usher, The sharp energy-capacity inequality, *Commun. Contemp. Math.* **12** no. 3 (2010), 457–473. MR 2661273. Zbl 1200.53077. https://doi.org/10.1142/S0219199710003889.

[Vit06] C. Viterbo, On the uniqueness of generating Hamiltonian for continuous limits of Hamiltonians flows, *Int. Math. Res. Not.* (2006), Art. ID 34028, 9. MR 2233715. Zbl 1108.53042. https://doi.org/10.1155/IMRN/2006/34028.

[Wen16] C. Wendl, Lectures on symplectic field theory, 2016. arXiv 1612.01009v2.

[Yao22a] Y. Yao, Computing embedded contact homology in Morse-Bott settings, 2022. arXiv 2211.13876.

[Yao22b] Y. Yao, From cascades to J-holomorphic curves and back, 2022. arXiv 2206.04334.

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